Extremal Problems for Multigraphs

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Certificate

This is to certify that this dissertation entitled 'Extremal Problems for Multigraphs ' towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Rik Sarkar at the Indian Institute of Science Education and Research, Pune and Umeå University under the supervision of Dr. Victor Falgas-Ravry, Associate Professor, Department of Mathematics and Mathematical Statistics, Umeå University, and Dr. Soumen Maity, Professor, Department of Mathematics, IISER Pune, during the academic year 2024-2025.

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This thesis is dedicated to
my friends and family

Declaration

I hereby declare that the matter embodied in this report entitled 'Extremal Problems for Multigraphs ' are the results of the work carried out by me at the Department of Mathematics and Mathematical Statistics, Umeå University, and the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Victor Falgas-Ravry and Dr. Soumen Maity, and the same has not been submitted elsewhere for any other degree.

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Abstract

In this thesis, we study the Mubayi-Terry multigraph problem, wherein one seeks to maximise the product of edge multiplicities in a locally sparse multigraph. A multigraph G is called an (s, q) graph if every set of s vertices in G spans at most q edges (counting multiplicities). The problem of determining the maximum sum of edge multiplicities in an n-vertex (s, q) graph is the multigraph analogue of a classical problem in extremal graph theory, which has been studied extensively over the years. More recently, in 2019, Mubayi and Terry introduced the product version of this problem, for which much less is known. The Mubayi-Terry problem is motivated by attempts to develop counting theorems for multigraphs.

Our primary contribution is to resolve the Mubayi-Terry multigraph problem for new infinite families of pairs (s, q). We prove the optimality of a broad class of lower-bound multigraph constructions for this problem. In so doing, we obtain an asymptotic resolution of a conjecture by Day, Falgas-Ravry and Treglown, and vastly generalise previous results on the problem. Our arguments are highly structural, a feature we then leverage to obtain stability results.

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Chapter 1

Introduction

1.1 Background and motivation

Extremal graph theory typically deals with questions of the following kind: How big or small can a parameter of a graph G be, given some constraints G must satisfy? For instance, what is the maximum number of edges in a triangle-free graph on n vertices? This question is answered by Mantel's theorem [19], which states that the maximum number of edges in an n-vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$. Furthermore, the complete bipartite graph with parts of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ is the unique triangle-free graph on n vertices with $\lfloor \frac{n^2}{4} \rfloor$ edges. A graph that attains the extremal (maximal or minimal) value of a given parameter under certain constraints is called an *extremal graph*. In the case of Mantel's theorem, the extremal graph is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, as it maximizes the number of edges among all triangle-free graphs. Mantel's theorem was further generalised in 1941 by Turán's theorem [24], which is a cornerstone of extremal graph theory.

Theorem 1.1.1 (Turán's theorem)**.** *Let G be a K_{r+1}-free graph on n vertices. Then G has at most as many edges as the Turán graph $T(n, r)$, which is the complete r-partite graph on n vertices with part sizes as equal as possible.*

Clearly, $T(n, r)$ is K_{r+1}-free, since any set of $r + 1$ vertices will contain a pair of vertices belonging to the same part. Turán's theorem asserts that $T(n, r)$ attains the maximum number of edges among all n-vertex K_{r+1}-free graphs. Henceforth, we denote by $t_r(n)$ the number of edges in $T(n,r)$.

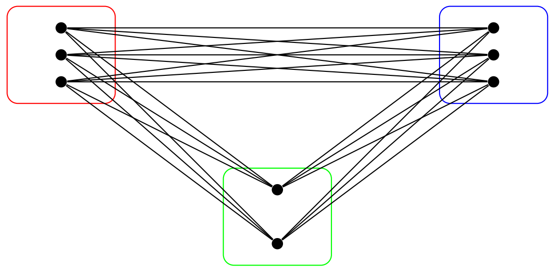

Figure 1.1: The Turán graph $T(8,3)$

Turán's theorem was further generalised by Paul Erdős and Arthur Stone in 1946 (see [10]). Given a graph H, we define $\mathrm{ex}(n,H)$ to be the maximum number of edges in an H-free graph on n vertices. Furthermore, we define the Turán density of H, $\tau(H)$, as follows:

$$\tau(H) := \lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}}.$$

Theorem 1.1.2 (Erdős-Stone theorem)**.**

$$\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2).$$

Consequently, $\tau(H) = \frac{\chi(H)-2}{\chi(H)-1}$*. Here,* $\chi(H)$ *is the chromatic number of* H*, the minimum number of colors needed to color the vertices of* H *such that no pair of adjacent vertices receive the same color.*

Thus, the Erdős-Stone theorem determines $\mathrm{ex}(n,H)$ up to an additive $o(n^2)$ error term. However, for bipartite graphs H (i.e., $\chi(H)=2$), the theorem does not give tight bounds on $\mathrm{ex}(n,H)$. Observe that $\chi(K_{r+1}) = r+1$, and that $t_r(n) \approx \left(\frac{r-1}{r}\right)\frac{n^2}{2}$. Thus, the Erdős-Stone theorem is an asymptotic generalisation of Turán's theorem.

In 1963, Erdős [11, 12] raised the following question: given integers $s \geq 2$ and $0 \leq q \leq \binom{s}{2}$, determine $\mathrm{ex}(n,s,q)$, the maximum number of edges in an n-vertex graph in which every set of s vertices spans at most q edges. This problem has received a significant amount of attention in the literature. When $q = \binom{s}{2} - 1$, this question is answered by Turán's theorem. Erdős [11] resolved the case $s \leq 5$, while a theorem of Dirac [8] dealt with the cases $q \geq \binom{s}{2} - \frac{s}{2}$.

In the range $q \geq \left\lfloor \frac{s^2}{4} \right\rfloor$, $\mathrm{ex}(n, s, q)$ is quadratic in n. Moreover, the asymptotics of $\mathrm{ex}(n, s, q)$ are well understood in this range thanks to the Erdős-Stone theorem, and a number of results are known about the size of the lower order term (see Gol'berg and Gurvich [15], and Griggs, Simonovits, and Thomas [16]). It is also known that $\mathrm{ex}(n, s, q)$ is linear in n for $q < s - 1$, and superlinear but subquadratic in n in the range $s - 1 \leq q < \left\lfloor \frac{s^2}{4} \right\rfloor$ (see [15]). In the case $q < s - 1$, the value of $\mathrm{ex}(n, s, q)$ is known exactly; however, in the "polynomial range" $s - 1 \leq q < \left\lfloor \frac{s^2}{4} \right\rfloor$, the problem is much more difficult. As Erdős himself observed, the case $q = s - 1$ is equivalent to the notoriously difficult and still open problem of determining the Turán number of the even cycle, $\mathrm{ex}(n, C_{2\lfloor s/2 \rfloor})$. Here, C_l denotes the cycle on l vertices.

In the late 1990s, Bondy and Tuza [4] and Kuchenbrod [18] independently began to investigate the integer-weighted version of Erdős' problem. An *integer-weighted graph* is a pair (V, w), where V is a set of vertices and $w : V^{(2)} \to \mathbb{Z}$ assigns an integer weight to each edge (i.e. pair of vertices) in V. Let $\mathrm{ex}_{\mathbb{Z}}(n, s, q)$ denote the maximum sum of edge weights in an integer-weighted graph on n vertices in which the sum of edge-weights inside any set of s vertices is at most q. Bondy and Tuza showed that an analogue of Turán's theorem holds in this setting, with $\mathrm{ex}_{\mathbb{Z}}\left(n, s, \binom{s}{2} - 1\right) = \mathrm{ex}\left(n, s, \binom{s}{2} - 1\right)$. However, unlike the setting of ordinary graphs, there may be several non-isomorphic extremal constructions in the setting of integer-weighted graphs. Kuchenbrod, for his part, determined $\mathrm{ex}_{\mathbb{Z}}(n, s, q)$ for all $s \leq 7$.

A major breakthrough on this problem came in 2002, when Füredi and Kündgen [14] showed that for all integers $s \geq 2$ and $q \geq 0$, we have $\mathrm{ex}_{\mathbb{Z}}(n, s, q) = m(s, q)\binom{n}{2} + O(n)$, where $m(s, q) \in \mathbb{Q}_{\geq 0}$ is defined by

$$m(s, q) := \min \left\{ m : \sum_{i=1}^{s-1} \lfloor 1 + mi \rfloor > q \right\}. \tag{1.1}$$

Thus, Füredi and Kündgen determined the asymptotic growth rate of $\mathrm{ex}_{\mathbb{Z}}(n, s, q)$ for all integers $s \geq 2$ and $q \geq 0$. We now turn our attention to multigraphs, which will be our primary focus in this thesis.

Definition 1.1.1. *A multigraph G is a pair (V, w), where $V = V(G)$ is a set of vertices, and $w = w_G$ is a function $w : V^{(2)} \mapsto \mathbb{Z}_{\geq 0}$ which associates to each pair (or edge) $v_1v_2 \in V^{(2)}$ a non-negative integer weight (or multiplicity) $w(v_1v_2)$.*

Definition 1.1.2. *Given integers $s \geq 2$ and $q \geq 0$, we say a multigraph $G = (V, w)$ is an (s, q)-graph if every s-set of vertices in G spans at most q edges; i.e. $\sum_{v_1v_2 \in X^{(2)}} w(v_1v_2) \leq q$*

for every $X \in V^{(s)}$. Furthermore, we denote by $\mathcal{F}(n,s,q)$ the set of all (s,q)-graphs with vertex set $[n] := \{1,2,...,n\}$.

Definition 1.1.3. *Given a multigraph $G = (V, w)$, we define*

$$e(G) := \sum_{v_1v_2 \in V^{(2)}} w(v_1v_2),$$

$$\mathrm{ex}_\Sigma(n,s,q) := \max\{e(G) : G \in \mathcal{F}(n,s,q)\},$$

$$\mathrm{ex}_\Sigma(s,q) := \lim_{n\to\infty} \frac{\mathrm{ex}_\Sigma(n,s,q)}{\binom{n}{2}}.$$

The existence of the limit $\mathrm{ex}_\Sigma(s,q)$ follows from an easy variant of the averaging argument due to Katona, Nemetz and Simonovits [17], by showing that $\mathrm{ex}_\Sigma(n,s,q)/\binom{n}{2}$ is non-increasing in n and bounded below by 0. This limiting quantity $\mathrm{ex}_\Sigma(s,q)$ can be thought of as the asymptotically maximum arithmetic mean of edge multiplicities in an (s,q)-graph. Clearly, we have $\mathrm{ex}_\Sigma(n,s,q) \le \mathrm{ex}_\mathbb{Z}(n,s,q)$. Füredi and Kündgen observed that for pairs (s,q) with $q > (s-1)\binom{s}{2}$, their constructions of asymptotically optimal integer-weighted graphs did not involve any negative integer weights, and therefore $\mathrm{ex}_\Sigma(s,q) = m(s,q)$, with $m(s,q)$ as defined in Equation (1.1). However, the problem of determining $\mathrm{ex}_\Sigma(s,q)$ when $q \le (s-1)\binom{s}{2}$ is still wide open.

Erdős' problem has also been extensively studied for hypergraphs, where it is much more challenging and is known as the Brown-Erdős-Sós problem [5]. For 3-uniform hypergraphs, the case $(s,q) = (6,3)$ corresponds for instance to the celebrated Ruzsa-Szemerédi $(6,3)$-theorem [22]. We also note that one of the motivations for studying such extremal problems for multigraphs is their application to (largely unsolved) extremal problems for hypergraphs. For instance, Füredi and Kündgen's results were applied by de Caen and Füredi to resolve an old conjecture on the Turán density of the Fano plane (see [7] for more details).

More recently, Mubayi and Terry [20] introduced a product version of Erdős's problem for multigraphs.

Definition 1.1.4. *Given a multigraph $G = (V, w)$, we define*

$$P(G) := \prod_{v_1v_2 \in V^{(2)}} w(v_1v_2),$$

$$\mathrm{ex}_\Pi(n,s,q) := \max\{P(G) : G \in \mathcal{F}(n,s,q)\},$$

$$\mathrm{ex}_\Pi(s,q) := \lim_{n\to\infty} \mathrm{ex}_\Pi(n,s,q)^{\frac{1}{\binom{n}{2}}}.$$

Again, one can show that $\mathrm{ex}_\Pi(n,s,q)^{1/\binom{n}{2}}$ is non-increasing in n and bounded below by 0, and therefore the limit $\mathrm{ex}_\Pi(s,q)$ exists. The limiting quantity $\mathrm{ex}_\Pi(s,q)$ can be thought of as the asymptotically maximum geometric mean of edge multiplicities in an (s,q)-graph.

Problem 1.1.3 (Mubayi-Terry multigraph problem)**.** Given integers $s \geq 2$ and $q \geq 0$, determine $\mathrm{ex}_\Pi(s,q)$.

The motivation for determining the quantity $\mathrm{ex}_\Pi(s,q)$ stems from attempts to develop counting theorems for multigraphs. Erdős, Kleitman and Rothschild [9] showed that the number of K_r-free graphs on the vertex set $[n]$ is $2^{\mathrm{ex}(n,K_r)+o(n^2)}$. Since their foundational result, it has been a major goal in extremal graph theory to prove similar counting results for other classes of graphs. It follows from the Alekseev-Bollobás-Thomason theorem [1, 3] that the number of graphs on the vertex set $[n]$ in which every s-set spans at most q edges is $2^{\mathrm{ex}(n,s,q)+o(n^2)}$. What can be said about multigraphs with the same property?

Using the powerful hypergraph container theories developed by Balogh, Morris and Samotij [2] and Saxton and Thomason [23], Mubayi and Terry [20] showed that for $q > \binom{s}{2}$,

$$\left|\mathcal{F}\left(n,s,q-\binom{s}{2}\right)\right| = \mathrm{ex}_\Pi(s,q)^{\binom{n}{2}+o(n^2)}.$$

Thus, the problem of determining the size of the multigraph family $\mathcal{F}\left(n,s,q-\binom{s}{2}\right)$ is equivalent to determining the product-extremal quantity $\mathrm{ex}_\Pi(s,q)$, making $\mathrm{ex}_\Pi(s,q)$ the "right" analogue of Turán density from the point of view of counting.

Assuming Schanuel's conjecture from number theory, Mubayi and Terry [20] showed that $\mathrm{ex}_\Pi(4,15)$ is a transcendental number. According to them, this is the first explicit (somewhat natural) question in extremal graph theory whose solution is transcendental. Furthermore, they conjectured that $\mathrm{ex}_\Pi(4,6a+3)$ is transcendental for all integers $a \geq 2$. Day, Falgas-Ravry and Treglown [6] proved this (under the assumption of Schanuel's conjecture from number theory) and in so doing introduced a broad class of Turán type lower bound constructions for this problem. Our main contribution is to prove the optimality of these constructions in several cases, and also to obtain corresponding stability results.

1.2 Preliminaries and Notation

We write $[n]$ for the set $\{1, 2, ..., n\}$. Given a set S and $k \in \mathbb{N}$, we denote by $S^{(k)}$ the collection of all subsets of S of size k, hereafter referred to as the k-sets of S. We use $v_1v_2...v_k$ as shorthand for the set $\{v_1, v_2, ...v_k\}$.

A *multigraph* G is a pair (V, w), where $V = V(G)$ is a set of vertices, and $w = w_G$ is a function $w : V^{(2)} \mapsto \mathbb{Z}_{\geq 0}$ which associates to each pair $v_1v_2 \in V^{(2)}$ a non-negative integer *weight* or *multiplicity* $w(v_1v_2)$. Given a subset of vertices $X \subseteq V(G)$, we define

$$e(X) := \sum_{v_1v_2 \in X^{(2)}} w(v_1v_2) \qquad \text{and} \qquad P(X) := \prod_{v_1v_2 \in X^{(2)}} w(v_1v_2)$$

to be the sum and product of edge multiplicities in X respectively. When $X = V(G)$, we simply write $e(G)$ and $P(G)$ for $e(X)$ and $P(X)$ respectively. For a vertex $v \in X \subseteq V(G)$, we define

$$d_X(v) := \sum_{u \in X} w(uv) \qquad \text{and} \qquad p_X(v) := \prod_{u \in X} w(uv)$$

to be the degree and product-degree of v in X respectively. When $X = V(G)$, we simply write $d(v) = d_G(v)$ and $p(v) = p_G(v)$ for $d_X(v)$ and $p_X(v)$ respectively. We say that a multigraph H is a *submultigraph* of G if there exists an injective map $f : V(H) \to V(G)$ such that $w_H(v_1v_2) \leq w_G(f(v_1)f(v_2))$ for all $v_1, v_2 \in V(H)$. We call H an *induced submultigraph* if $w_H(v_1v_2) = w_G(f(v_1)f(v_2))$ for each pair $v_1, v_2 \in V(H)$.

Given two multigraphs $G = ([n], w)$ and $G' = ([n], w')$, we define $\Delta(G, G') := \{v_1v_2 \in [n]^{(2)} : w(v_1v_2) \neq w'(v_1v_2)\}$. We say that G and G' are δ-close if $|\Delta(G, G')| \leq \delta n^2$.

All the extremal constructions we will study arise from blow-ups of patterns, which we formally define as follows:

Definition 1.2.1 (Patterns)**.** *Let X be a set of integers. An X-bounded pattern on N vertices, π, is a function $\pi : [N] \cup [N]^{(2)} \mapsto X$.*

Definition 1.2.2 (Blow-ups)**.** *Given an X-bounded pattern π on N vertices, the family of n-vertex blow-ups of π, denoted by $\mathcal{B}(\pi, n)$, is the collection of multigraphs $G = ([n], w)$, such that there exists an N-partition of the vertex set $[n] = \sqcup_{i=1}^{N} V_i$, with all edges between V_i and*

V_j, $1 \le i < j \le N$, having w_G-multiplicity $\pi(ij)$, and all edges inside V_i, $1 \le i \le N$, having w_G-multiplicity $\pi(i)$.

Furthermore, we define

$$\Sigma(\pi, n) := \max\{e(G) : G \in \mathcal{B}(\pi, n)\} \quad \text{and} \quad \Pi(\pi, n) := \max\{P(G) : G \in \mathcal{B}(\pi, n)\}.$$

Let $\mathcal{S}(\pi, n)$ be the collection of multigraphs $G \in \mathcal{B}(\pi, n)$ with $e(G) = \Sigma(\pi, n)$. Similarly, let $\mathcal{P}(\pi, n)$ be the collection of multigraphs $G \in \mathcal{B}(\pi, n)$ with $P(G) = \Pi(\pi, n)$.

We use standard asymptotic notation. Given any two functions $f, g : \mathbb{Z}_{\ge 0} \to \mathbb{R}$, we say that $f = O(g)$ if there exists an absolute positive constant C such that $|f(n)| \le C|g(n)|$ for all n sufficiently large. We say that $f = o(g)$ if, for every $\epsilon > 0$, $|f(n)| \le \epsilon|g(n)|$ for all n sufficiently large. If $g(n)$ is non-zero for all n, this is equivalent to saying that $f/g \to 0$ as $n \to \infty$.

1.3 Organization of the Thesis

In Chapter 2, we introduce the lower-bound constructions that will be studied throughout the thesis and state our main conjecture. Chapter 3 surveys the known results related to our problem. Chapter 4 introduces some fundamental tools that will be used in our proofs. In Chapter 5, we analyze key properties of the lower-bound construction introduced in Chapter 2.

In Chapter 6, we present an asymptotic resolution of our main conjecture. Chapter 7 establishes an improved bound on an existing result due to Day, Falgas-Ravry, and Treglown, while also providing new examples of “asymptotically flat” intervals. Finally, in Chapter 8, we prove stability results corresponding to some of the extremal results obtained earlier.

Chapters 5, 6, 7, and 8 consist entirely of original work.

Chapter 2

The Main Conjecture

2.1 A lower bound construction and some basic properties

As mentioned in Chapter 1, the constructions we will study arise from blow-ups of certain patterns, which we describe below.

Definition 2.1.1 (Turán Pattern)**.** *For integers $R \geq 2$ and $a \geq 1$, we define the $[a+1]$-bounded pattern on R vertices $\pi_R^a : [R] \cup [R]^{(2)} \mapsto [a+1]$ by:*

$$\pi_R^a(i) = a, \quad \textit{for all } i \in [R],$$
$$\pi_R^a(ij) = a+1, \quad \textit{for all } 1 \leq i < j \leq R.$$

Henceforth, we denote $\Sigma(\pi_R^a, n)$ by $\Sigma^R(a,n)$ and $\Pi(\pi_R^a, n)$ by $\Pi^R(a,n)$.

Definition 2.1.2 (Generalised Turán Pattern)**.** *For integers $r, R, d, a \in \mathbb{N}$ with $a \geq d+1$, we define the $[a+1]$-bounded pattern on $r+R$ vertices $\pi_{r,R}^{a,d} : [r+R] \cup [r+R]^{(2)} \mapsto [a+1]$ by:*

$$\pi_{r,R}^{a,d}(i) = \begin{cases} a-d & \textit{if } i \leq r \\ a & \textit{otherwise} \end{cases}$$

and

$$\pi_{r,R}^{a,d}(ij) = \begin{cases} a-d+1 & \textit{if } 1 \le i < j \le r \\ a+1 & \textit{otherwise} \end{cases}$$

Given a multigraph $G \in \mathcal{B}(\pi_{r,R}^{a,d}, n)$, the vertex set $[n]$ can be partitioned as

$$[n] = (\sqcup_{i=1}^{r} U_i) \bigsqcup (\sqcup_{j=1}^{R} V_j),$$

such that all edges within the sets U_i have multiplicity $a-d$, all edges within the sets V_j have multiplicity a, all edges between distinct U_i have multiplicity $a-d+1$ and all other edges have multiplicity $a+1$. We refer to $(\sqcup_{i=1}^{r} U_i) \bigsqcup (\sqcup_{j=1}^{R} V_j)$ as the canonical partition of G. Furthermore, let $U := \sqcup_{i=1}^{r} U_i$, and $V := \sqcup_{j=1}^{R} V_j$. Henceforth, we denote $\Sigma\left(\pi_{r,R}^{a,d}, n\right)$ by $\Sigma_{r,d}^{R}(a,n)$, and $\Pi\left(\pi_{r,R}^{a,d}, n\right)$ by $\Pi_{r,d}^{R}(a,n)$.

The multigraph construction obtained from blow-ups of the generalised Turán pattern is a special case of Construction 12.1 in [6]. Note that when $r = 1$, this construction is precisely Construction 3.1 in [6].

Let $G \in \mathcal{B}\left(\pi_{r,R}^{a,d}, n\right)$, and let $(\sqcup_{i=1}^{r} U_i) \bigsqcup (\sqcup_{j=1}^{R} V_j)$ be the canonical partition of G. Set $|U_i| = u_i n$ for all $i \in [r]$ and $|V_j| = v_j n$ for all $j \in [R]$. If $G \in \mathcal{S}\left(\pi_{r,R}^{a,d}, n\right)$ or $G \in \mathcal{P}\left(\pi_{r,R}^{a,d}, n\right)$, then by Turán's theorem, there exists $\alpha \in (0,1)$ such that $u_i = \alpha + O(n^{-1})$ for all $i \in [r]$ and $v_j = \frac{1-r\alpha}{R} + O(n^{-1})$ for all $j \in [R]$. Then

$$e(G) = (a+1)\binom{n}{2} - R\binom{\left(\frac{1-r\alpha}{R}\right)n}{2} - d\binom{r\alpha n}{2} - r\binom{\alpha n}{2} + O(n).$$

An easy exercise in optimisation shows that $e(G)$ is asymptotically maximised when

$$\alpha = \frac{1}{r + R(dr+1)}.$$

Similarly,

$$\log(P(G)) = \log(a+1)\binom{n}{2} - R\log\left(\frac{a+1}{a}\right)\binom{\left(\frac{1-r\alpha}{R}\right)n}{2} - \log\left(\frac{a+1}{a-d+1}\right)\binom{r\alpha n}{2}$$
$$- r\log\left(\frac{a-d+1}{a-d}\right)\binom{\alpha n}{2} + O(n).$$

Using calculus, one can see that $P(G)$ is asymptotically maximised when

$$\alpha = \frac{\log\left(\frac{a+1}{a}\right)}{r(R+1)\log\left(\frac{a+1}{a}\right) + R\log\left(\frac{a}{a-d}\right) + R(r-1)\log\left(\frac{a}{a-d+1}\right)}.$$

Thus, in general, the sum-optimal family $\mathcal{S}\left(\pi^{a,d}_{r,R}, n\right)$ and the product-optimal family $\mathcal{P}\left(\pi^{a,d}_{r,R}, n\right)$ are quite different from each other. Indeed, to maximise the sum of edge multiplicities, α must be chosen to be $\frac{1}{r+R(dr+1)}$. On the other hand, to maximise the product of edge multiplicities, α must be strictly less than $\frac{1}{r+R(dr+1)}$ (see Proposition 5.1).

Given integers $r, R, d \in \mathbb{N}$, we define the function $x^d_{r,R} = x^d_{r,R}(a)$ for all integers $a \geq d+1$ by

$$x^d_{r,R}(a) := \frac{\log\left(\frac{a+1}{a}\right)}{r(R+1)\log\left(\frac{a+1}{a}\right) + R\log\left(\frac{a}{a-d}\right) + R(r-1)\log\left(\frac{a}{a-d+1}\right)}.$$

For a multigraph $G \in \mathcal{B}\left(\pi^{a,d}_{r,R}, n\right)$, we observe that $P(G)$ is maximised by making the product-degrees of all the vertices roughly equal. Explicitly, we may obtain $x = x^d_{r,R}(a)$ as the unique solution to the following equation:

$$a\left(\frac{a-d}{a}\right)^{x}\left(\frac{a-d+1}{a}\right)^{(r-1)x}\left(\frac{a+1}{a}\right)^{1-rx} = a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx}{R}}. \tag{2.1}$$

The left-hand side and right-hand side of Equation (2.1) are obtained by considering the product-degree of a vertex in U and V respectively. From this, it is easy to see that

$$\Pi^R_{r,d}(a,n) = \left[a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}}\right]^{\binom{n}{2}+O(n)}, \tag{2.2}$$

where $x^* = x^d_{r,R}(a)$.

2.2 The conjecture

We are now in a position to state our main conjecture, which is a generalisation of Conjecture 3.2 in [6]. Roughly speaking, the conjecture asserts that the constructions arising from blow-ups of the generalised Turán pattern are optimal for the Mubayi-Terry problem in several cases. One of our main contributions is the asymptotic resolution of several instances of this conjecture.

Conjecture 2.2.1. *For all integers $r, R, d, a, s \in \mathbb{N}$ with $a \geq d+1$, $s \geq r + R(dr+1)+1$, and for all n sufficiently large,*

$$\mathrm{ex}_\Pi(n, s, \Sigma^R_{r,d}(a,s)) = \Pi^R_{r,d}(a,n). \tag{2.3}$$

Roughly speaking, the conjecture states that given a sufficiently large integer s, and q equal to the maximum sum of edge multiplicities in an s-vertex blow-up of $\pi^{a,d}_{r,R}$, it is an n-vertex blow-up of $\pi^{a,d}_{r,R}$ which maximises $P(G)$ among all multigraphs $G \in \mathcal{F}(n,s,q)$. Note that the $r = 1$ case of Conjecture 2.2.1 corresponds to Conjecture 3.2 in [6].

For Conjecture 2.2.1, we require $s \geq r + R(dr+1)+1$. For smaller values of s, the sum-optimal family $\mathcal{S}\left(\pi^{a,d}_{r,R}, s\right)$ sees at most r vertices in U (see Proposition 5.2). Thus, the edges of multiplicity $a-d$ are not used at all, and "better" lower bound constructions are available to us. Raising both sides of Equation (2.3) to $1/\binom{n}{2}$ and taking the limit as $n \to \infty$, together with Equation (2.2), yields the following asymptotic version of Conjecture 2.2.1.

Conjecture 2.2.2 (Asymptotic)**.** *For all integers $r, R, d, a, s \in \mathbb{N}$ with $a \geq d+1$ and $s \geq r + R(dr+1)+1$, we have*

$$\mathrm{ex}_\Pi(s, \Sigma^R_{r,d}(a,s)) = a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}},$$

where $x^* = x^d_{r,R}(a)$.

Chapter 3

Known Results

In this chapter, we discuss previous work on the Mubayi-Terry multigraph problem, including current progress on Conjecture 2.2.1.

Theorem 3.1 (Mubayi, Terry [21, Theorem 3])**.** *Let n, s, q, a be integers satisfying $n \geq s \geq 2, a \geq 1$, where $q = a\binom{s}{2} + b$ for some $0 \leq b \leq s-2$. The following hold:*

- *(Extremal) We have $a^{\binom{n}{2}} \leq \mathrm{ex}_{\Pi}(n, s, q) \leq a^{\binom{n}{2}} \left(\frac{a+1}{a}\right)^{\lfloor \frac{b}{b+1} n \rfloor}$. Thus, $\mathrm{ex}_{\Pi}(s, q) = a$.*
- *(Stability) For every $\delta > 0$, there exists $\epsilon > 0$ such that the following holds for all sufficiently large n: if $G \in \mathcal{F}(n, s, q)$, and $P(G) > \mathrm{ex}_{\Pi}(n, s, q)^{1-\epsilon}$, then G is δ-close to the multigraph on $[n]$ with all edge multiplicities equal to a.*

Thus, Mubayi and Terry showed that

$$\mathrm{ex}_{\Pi}\left(s, a\binom{s}{2}\right) = \mathrm{ex}_{\Pi}\left(s, a\binom{s}{2} + 1\right) = \ldots = \mathrm{ex}_{\Pi}\left(s, a\binom{s}{2} + s - 2\right) = a.$$

Moreover, the multigraph with all edge multiplicities equal to a is asymptotically optimal in this interval. This result provides an example of an "asymptotically flat interval", where, despite an increase in q (with s held fixed), $\mathrm{ex}_{\Pi}(n, s, q)$ remains asymptotically unchanged. We will revisit this notion later, as we provide more such examples of asymptotically flat intervals. Mubayi and Terry also proved the following result, generalising Dirac's additive results [8] to the multiplicative setting.

Theorem 3.2 (Mubayi, Terry [21, Theorem 4])**.** *Let n, s, q, a be integers satisfying $a \geq 1$, $n \geq s \geq 2$, with $q = (a+1)\binom{s}{2} - t$ for some $1 \leq t \leq s/2$.*

- *(Extremal) For all $n \geq s$, $ex_\Pi(n, s, q) = a^{\binom{n}{2}} \left(\frac{a+1}{a}\right)^{t_{s-t}(n)}$. Thus, we have $ex_\Pi(s, q) = a\left(\frac{a+1}{a}\right)^{\frac{s-t-1}{s-t}}$. The maximal product is attained by multigraphs in $\mathcal{P}(\pi^a_{s-t}, n)$.*
- *(Stability) For every $\delta > 0$, there exists $\epsilon > 0$ such that the following holds for all sufficiently large n: if $G \in \mathcal{F}(n, s, q)$ and $P(G) > \mathrm{ex}_\Pi(n, s, q)^{1-\epsilon}$, then G is δ-close to an element of $\mathcal{P}(\pi^a_{s-t}, n)$.*

Thus, Theorems 3.1 and 3.2 resolve the problem of determining $\mathrm{ex}_\Pi(s, q)$ whenever the congruence class of q modulo $\binom{s}{2}$ lies within a certain interval of length about $3s/2$. The smallest case (in the lexicographic ordering of pairs (s, q)) not covered by these results is $(s, q) = (4, 9)$. Mubayi and Terry [21, Theorem 5] showed that the quantity $\mathrm{ex}_\Pi(n, 4, 9)$ is related to an old question in extremal graph theory. Let $\mathrm{ex}(n, \{C_3, ..., C_s\})$ denote the maximum number of edges in an n-vertex graph which contains no cycle of length at most s as a subgraph. Then, for all $n \geq 4$, $\mathrm{ex}_\Pi(n, 4, 9) = 2^{\mathrm{ex}(n,\{C_3,C_4\})}$. This result was further generalised by Day, Falgas-Ravry and Treglown [6, Theorem 3.12], who showed that for all $n \geq s \geq 4$, $\mathrm{ex}_\Pi\left(n, s, \binom{s}{2} + s - 1\right) = 2^{\mathrm{ex}(n,\{C_3,...,C_s\})}$.

The next case not covered by Theorems 3.1 and 3.2 is the case $(s, q) = (4, 15)$. This case was resolved by Mubayi and Terry [20].

Theorem 3.3 (Mubayi, Terry [20, Theorem 2])**.** *For all $n \geq 4$,*

$$\mathrm{ex}_\Pi(n, 4, 15) = 2^{\gamma n^2 + O(n)}.$$

Thus,

$$\mathrm{ex}_\Pi(4, 15) = 2^{2\gamma},$$

where $\gamma = \frac{\beta^2}{2} + \beta(1-\beta)\log_2 3$ and $\beta = \frac{\log 3}{2\log 3 - \log 2}$.

Mubayi and Terry showed that $\mathrm{ex}_\Pi(4, 15) = 2^{2\gamma}$ is a transcendental number, assuming Schanuel's conjecture from number theory. According to Mubayi and Terry, this is the first explicit question in extremal graph theory whose solution is transcendental. Furthermore, they conjectured that such transcendental behavior was not an isolated case, and that

$\mathrm{ex}_\Pi(4, 6a+3)$ is transcendental for all integers $a \geq 2$ (see [20, Conjecture 1]). This conjecture was resolved by Day, Falgas-Ravry and Treglown [6], who proved the following more general result.

Theorem 3.4 (Day, Falgas-Ravry, Treglown [6, Theorems 3.5-3.8])**.** *For all integers $a \geq 2, s \in \{4, 5, 6, 7\}$ and n sufficiently large,*

$$\mathrm{ex}_\Pi(n, s, \Sigma^1_{1,1}(a, s)) = \Pi^1_{1,1}(a, n).$$

Remark 3.1. *Theorem 3.4 resolves the $(r, R, d) = (1, 1, 1)$ case of Conjecture 2.2.1 for $s \in \{4, 5, 6, 7\}$. Note that $\Sigma^1_{1,1}(a, 4) = 6a + 3$, and that Theorem 3.4 implies that $\mathrm{ex}_\Pi(4, 6a + 3) = a\left(\frac{a+1}{a}\right)^{x^*}$, where $x^* = x^1_{1,1}(a)$. Assuming Schanuel's conjecture from number theory and using Mihăilescu's theorem (his proof of Catalan's conjecture), this quantity can be shown to be transcendental (see [6, Appendix A]).*

Falgas-Ravry [13] further generalised this result.

Theorem 3.5 (Falgas-Ravry [13, Theorem 1.6])**.** *For all integers $a \geq 2$ and $R \geq 1$, we have*

$$\mathrm{ex}_\Pi\left(2R + 2, a\binom{2R+2}{2} + \mathrm{ex}(2R + 2, K_{R+2}) - 1\right) = a\left(\frac{a+1}{a}\right)^{1-\frac{1-x^*}{R}},$$

where $x^* = x^1_{1,R}(a)$.

Remark 3.2. *Theorem 3.5 resolves the $(r, d) = (1, 1)$ case of Conjecture 2.2.2 for all $R \geq 1$, when $s = 2R+2$. Note that $2R+2$ is smallest value of s for which Conjecture 2.2.2 is valid, and that $\Sigma^R_{1,1}(a, 2R + 2) = a\binom{2R+2}{2} + \mathrm{ex}(2R + 2, K_{R+2}) - 1$.*

As further evidence in favour of the $r = 1$ case of Conjecture 2.2.1, Day, Falgas-Ravry and Treglown [6] proved the following "Step-up" theorem:

Theorem 3.6 (Day, Falgas-Ravry, Treglown [6, Theorem 3.11])**.** *Let $R, d \in \mathbb{N}$ and let $s \geq R(d + 1) + 2$. Suppose there exist natural numbers a_0 and n_0 such that the following holds for all $a \geq a_0$ and $n \geq n_0$:*

$$\mathrm{ex}_\Pi(n, s, \Sigma^R_{1,d}(a, s)) = \Pi^R_{1,d}(a, n).$$

Then, there exist natural numbers $a_1 \geq a_0$ and $n_1 \geq n_0$ such that the following holds for all $a \geq a_1$ and $n \geq n_1$:

$$\mathrm{ex}_{\Pi}(n, s+1, \Sigma^R_{1,d}(a, s+1)) = \Pi^R_{1,d}(a, n).$$

Remark 3.3. *Theorems 3.5 and 3.6 together resolve the $(r, d) = (1, 1)$ case of Conjecture 2.2.2 for all $R \geq 1$, $s \geq 2R + 2$ and a sufficiently large.*

Finally, Day, Falgas-Ravry and Treglown [6] proved the following result related to the optimality of the Turán pattern described in Definition 2.1.1:

Theorem 3.7 (Day, Falgas-Ravry, Treglown [6, Theorem 3.10])**.** *Let a, R, s be integers with $a, R \geq 1$ and $s \geq R + 1$. For all integers $n \geq 2R(s+2) + R(s+2)\sqrt{s-1}$,*

$$\mathrm{ex}_{\Pi}(n, s, \Sigma^R(a, s)) = \Pi^R(a, n).$$

Furthermore, the family of product-maximising multigraphs is precisely $\mathcal{P}(\pi^a_R, n)$.

Remark 3.4. *When $s \leq 2R$, Theorem 3.7 implies Theorem 3.2 (with weaker bounds on n), making Theorem 3.7 a generalization of Theorem 3.2.*

Having now stated all existing results on the Mubayi-Terry multigraph problem, we now discuss some basic tools we will deploy to prove our results.

Chapter 4

Basic Tools

In this chapter, we introduce some tools that are instrumental in proving our main results.

Proposition 4.1 (Integral AM-GM Inequality)**.** *Let $a, n \in \mathbb{N}$ and let $t \in \{0\} \cup [n]$. Suppose $w_1, ..., w_n$ are non-negative integers with $\sum_{i=1}^{n} w_i = an + t$. Then*

$$\prod_{i=1}^{n} w_i \leq a^{n-t}(a+1)^t,$$

with equality if and only if t of the w_i are $a+1$, and the remaining $n-t$ are a.

Proof. Suppose there exist $i, j \in [n]$ with $i \neq j$ such that $w_j \geq w_i + 2$. Then

$$(w_i + 1)(w_j - 1) = w_i w_j + w_j - w_i - 1 \geq w_i w_j + 1.$$

Thus, $\prod_{i=1}^{n} w_i$ is maximised by making the w_i as equal as possible. □

The use of the integral AM-GM inequality is natural, as it allows us to upper bound the product of edge multiplicities in a multigraph, given an upper bound on the sum.

Proposition 4.2 (Weighted geometric averaging)**.** *Let $m \in \mathbb{N}$, and let $\alpha_1, ..., \alpha_m$ be non-negative real numbers summing to 1. Suppose $p_1, ..., p_m$ are non-negative real numbers. Then, there exists $i \in [m]$ such that p_i is at most the $\{\alpha_j\}_{j=1}^{m}$-weighted geometric mean of the quantities $\{p_j\}_{j=1}^{m}$: $p_i \leq \prod_{j=1}^{m} p_j^{\alpha_j}$.*

Proof. Choose $i \in [m]$ such that $p_i = \min_{j \in [m]} p_j$. Then, we have $p_i = \prod_{j=1}^m p_i^{\alpha_j} \le \prod_{j=1}^m p_j^{\alpha_j}$.

□

Proposition 4.3 (Averaging bound). *Let s, q, n be integers with $n \ge s \ge 2$ and $q \ge 0$. Then*

$$\mathrm{ex}_\Sigma(n+1, s, q) \le \left\lfloor \left(\frac{n+1}{n-1}\right) \mathrm{ex}_\Sigma(n, s, q) \right\rfloor.$$

Proof. Let $G \in \mathcal{F}(n+1, s, q)$ be a multigraph with $e(G) = \mathrm{ex}_\Sigma(n+1, s, q)$. Then, since every edge of G is contained in $n-1$ subsets of $V(G)$ of size n,

$$\sum_{\substack{X \subseteq V(G) \\ |X|=n}} e(X) = (n-1)e(G).$$

Since every n-subset of $V(G)$ has at most $\mathrm{ex}_\Sigma(n, s, q)$ edges, we have

$$\sum_{\substack{X \subseteq V(G) \\ |X|=n}} e(X) \le (n+1)\mathrm{ex}_\Sigma(n, s, q).$$

Thus,

$$\mathrm{ex}_\Sigma(n+1, s, q) = e(G) \le \left(\frac{n+1}{n-1}\right) \mathrm{ex}_\Sigma(n, s, q).$$

Since $\mathrm{ex}_\Sigma(n+1, s, q)$ is an integer, the proposition follows. □

We note that the proof of Proposition 4.3 closely follows the averaging argument introduced by Katona, Nemetz, and Simonovits [17]. As a simple illustration of the above tools in action, we quickly show that for all $n \ge s \ge 2$ and $a \ge 1$, we have $\mathrm{ex}_\Pi\left(n, s, a\binom{s}{2}\right) = a^{\binom{n}{2}}$. We first prove by induction on n that $\mathrm{ex}_\Sigma\left(n, s, a\binom{s}{2}\right) \le a\binom{n}{2}$. This clearly holds for $n = s$. Let us assume that $\mathrm{ex}_\Sigma\left(k, s, a\binom{s}{2}\right) \le a\binom{k}{2}$ for some $k \ge s$. Then, by Proposition 4.3, we have $\mathrm{ex}_\Sigma\left(k+1, s, a\binom{s}{2}\right) \le \left(\frac{k+1}{k-1}\right) a\binom{k}{2} = a\binom{k+1}{2}$. Thus, we have $\mathrm{ex}_\Sigma\left(n, s, a\binom{s}{2}\right) \le a\binom{n}{2}$ for all $n \ge s$. By Proposition 4.1, we have $\mathrm{ex}_\Pi\left(n, s, a\binom{s}{2}\right) \le a^{\binom{n}{2}}$. The multigraph G on $[n]$ with all edge multiplicities equal to a is an $\left(s, a\binom{s}{2}\right)$-graph with $P(G) = a^{\binom{n}{2}}$. Thus, $\mathrm{ex}_\Pi\left(n, s, a\binom{s}{2}\right) \ge a^{\binom{n}{2}}$, which proves the claim.

Proposition 4.4 (Degree removal). *Let $s \ge 2$ and $q \ge 0$ be integers, and let c be a positive real number. Suppose the following holds: for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for* *all* $n > n_0$*, every multigraph* $G \in \mathcal{F}(n,s,q)$ *contains a vertex* v *with* $p_G(v) \leq c^{(1+\epsilon)n}$*. Then*

$$\mathrm{ex}_\Pi(s,q) \leq c.$$

Proof. Fix $\epsilon > 0$. Let $n_0 \in \mathbb{N}$ be such that for all $n > n_0$, every multigraph $G \in \mathcal{F}(n,s,q)$ contains a vertex v with $p_G(v) \leq c^{(1+\epsilon)n}$. Let $n > n_0$ and let $G_n \in \mathcal{F}(n,s,q)$ satisfy $P(G_n) = \mathrm{ex}_\Pi(n,s,q)$. Then, by repeatedly removing the vertices of lowest product-degree from G_n until we are left with a multigraph G_{n_0} on n_0 vertices, we have

$$\begin{aligned}\mathrm{ex}_\Pi(n,s,q) = P(G_n) \leq P(G_{n_0}) \prod_{i=n_0+1}^{n} c^{(1+\epsilon)i} &\leq \mathrm{ex}_\Pi(n_0,s,q) \prod_{i=n_0+1}^{n} c^{(1+\epsilon)i} \\ &= c^{(1+\epsilon)\binom{n}{2}} e^{O(n)}.\end{aligned}$$

Raising both sides to $1/\binom{n}{2}$ and letting $n \to \infty$, we get $\mathrm{ex}_\Pi(s,q) \leq c^{1+\epsilon}$. Since $\epsilon > 0$ was arbitrary, the proposition follows. □

With the basic tools in place, we are now in a position to discuss our new contributions.

Chapter 5

Properties of the Construction

In this chapter, we discuss and prove some key properties of the construction arising from blow-ups of the generalised Turán pattern, which will be useful in proving some of our results.

Proposition 5.1. *For any $r, R, d \in \mathbb{N}$, the function $x^d_{r,R} = x^d_{r,R}(a)$ is monotone increasing in a over the interval $[d+1, \infty)$. In particular, for all integers $a \geq d+1$, we have*

$$x^d_{r,R}(d+1) \leq x^d_{r,R}(a) < \lim_{a \to \infty} x^d_{r,R}(a) = \frac{1}{r + R(dr+1)}.$$

Proof. Differentiating $x^d_{r,R}(a)$ with respect to a, we obtain the following expression:

$$\frac{R}{a}\left[f(d) + (r-1)f(d-1)\right],$$

where, for all $t \in \{0\} \cup [a-1]$,

$$f(t) := \frac{t}{a-t}\log\left(\frac{a+1}{a}\right) - \frac{1}{a+1}\log\left(\frac{a}{a-t}\right).$$

Since $\log\left(\frac{a+1}{a}\right) > \frac{1}{a+1}$, for all $t \in \{0\} \cup [a-1]$,

$$f(t) \geq \frac{1}{a+1}\left(\frac{t}{a-t} - \log\left(\frac{a}{a-t}\right)\right) = \frac{1}{a+1}\left(\frac{t}{a-t} - \log\left(1 + \frac{t}{a-t}\right)\right) \geq 0,$$

with equalities only at $t = 0$. This implies that the derivative of $x^d_{r,R}(a)$ with respect to a is strictly positive. Thus, $x^d_{r,R}(a)$ is monotone increasing in a. It is easy to see, using L'Hôpital's rule, for instance, that

$$\lim_{a\to\infty} x^d_{r,R}(a) = \frac{1}{r + R(dr + 1)}.$$

The proposition follows. □

To prove some of the remaining propositions in this chapter, we make use of the following observation. If a multigraph $G \in \mathcal{S}\left(\pi^{a,d}_{r,R}, n\right)$ has canonical partition $(\sqcup_{i=1}^{r} U_i) \bigsqcup (\sqcup_{j=1}^{R} V_j)$ with $U = \sqcup_{i=1}^{r} U_i$, then by Turán's theorem, we may henceforth assume that

$$\left\lfloor \frac{|U|}{r} \right\rfloor = |U_1| \leq ... \leq |U_r| = \left\lceil \frac{|U|}{r} \right\rceil.$$

Similarly, we may assume that

$$\left\lfloor \frac{n - |U|}{R} \right\rfloor = |V_1| \leq ... \leq |V_R| = \left\lceil \frac{n - |U|}{R} \right\rceil.$$

The following proposition describes how vertices must be distributed among various parts to maximize the sum of edge multiplicities in a multigraph $G \in \mathcal{B}(\pi^{a,d}_{r,R}, n)$.

Proposition 5.2 (Partition sizes of $\mathcal{S}\left(\pi^{a,d}_{r,R}, n\right)$). *Let $r, R, d, a, n \in \mathbb{N}$ with $a \geq d + 1$, and let $n = q(Rdr + R + r) + t$, with $0 \leq t \leq Rdr + R + r - 1$. If $G \in \mathcal{S}\left(\pi^{a,d}_{r,R}, n\right)$ has canonical partition $(\sqcup_{i=1}^{r} U_i) \bigsqcup (\sqcup_{j=1}^{R} V_j)$ with $U = \sqcup_{i=1}^{r} U_i$, then*

- *If $r = 1$,*

$$|U| = \begin{cases} q, & \text{if } t = 0, \\ q \text{ or } q + 1, & \text{if } 1 \leq t \leq R, \\ q + 1, & \text{if } R + 1 \leq t \leq Rd + R. \end{cases}$$

- *If* $r \geq 2$,

$$|U| = \begin{cases} qr, & \textit{if } t = 0, \\ qr + k - 1 \textit{ or } qr + k, & \textit{if } (k-1)(Rd+1) + 1 \leq t \leq (k-1)(Rd+1) + R, \\ & \quad 1 \leq k \leq r, \\ qr + k, & \textit{if } (k-1)(Rd+1) + R + 1 \leq t \leq k(Rd+1), \\ & \quad 1 \leq k \leq r-1, \\ (q+1)r, & \textit{if } (r-1)(Rd+1) + R + 1 \leq t \leq Rdr + R + r - 1. \end{cases}$$

Proof. The $r = 1$ case of Proposition 5.2 is covered by Proposition 5.3 in [6]. Therefore, we restrict our attention to the $r \geq 2$ case. The key idea used in this proof is to shift a vertex from U_r to V_1 (if U is too large) or to shift a vertex from V_R to U_1 (if U is too small), while increasing the sum of edge multiplicities in the multigraph.

- **Case I**: $t = 0$.
  If $|U| \leq qr - 1$, then $|U_1| \leq q - 1$. Furthermore, $n - |U| \geq qR(dr+1) + 1$ and $|V_R| \geq q(dr+1) + 1$. Moving a vertex from V_R to U_1 changes $e(G)$ by at least

  $$q(dr+1) - d(qr-1) - (q-1) = d + 1 > 0.$$

  On the other hand, if $|U| \geq qr + 1$, then $|U_r| \geq q + 1$. Furthermore, $n - |U| \leq qR(dr+1) - 1$ and $|V_1| \leq q(dr+1) - 1$. Moving a vertex from U_r to V_1 changes $e(G)$ by at least

  $$d(qr) + q - (q(dr+1) - 1) = 1 > 0.$$

  Thus, $|U| = qr$.

- **Case II**: $(k-1)(Rd+1) + R + 1 \leq t \leq k(Rd+1),$ where $1 \leq k \leq r - 1$.
  If $|U| \leq qr + k - 1$, then $|U_1| \leq q$. Furthermore, $n - |U| \geq q(Rdr + R + r) + (k-1)(Rd+1) + R + 1 - qr - k + 1 = R\left(q(dr+1) + d(k-1) + 1\right) + 1$ and $|V_R| \geq q(dr+1) + d(k-1) + 2$. Moving a vertex from V_R to U_1 changes $e(G)$ by at least

  $$q(dr+1) + d(k-1) + 1 - d(qr + k - 1) - q = 1 > 0.$$

On the other hand, if $|U| \geq qr+k+1$, then $|U_r| \geq q+1$. Furthermore, $n-|U| \leq q(Rdr+R+r)+k(Rd+1)-qr-k-1 = R\left(q(dr+1)+kd\right)-1$ and $|V_1| \leq q(dr+1)+kd-1$. Moving a vertex from U_r to V_1 changes $e(G)$ by at least

$$d(qr+k)+q-q(dr+1)-kd+1 = 1 > 0.$$

Thus, $|U| = qr+k$.

- **Case III**: $(k-1)(Rd+1)+1 \leq t \leq (k-1)(Rd+1)+R,$ where $1 \leq k \leq r$.
  If $|U| \geq qr+k+1$, then we can increase $e(G)$ by moving a vertex from U_r to V_1. The proof for this follows as in Case II, by observing that $(k-1)(Rd+1)+R \leq k(Rd+1)$.
  On the other hand, if $|U| \leq qr+k-2$, then $|U_1| \leq q$. Furthermore, $n-|U| \geq q(Rdr+R+r)+(k-1)(Rd+1)+1-qr-k+2 = R\left(q(dr+1)+d(k-1)\right)+2$ and $|V_R| \geq q(dr+1)+d(k-1)+1$. Moving a vertex from V_R to U_1 changes $e(G)$ by at least

  $$q(dr+1)+d(k-1)-d(qr+k-2)-q = d > 0.$$

  Thus, $|U|$ can be either $qr+k-1$ or $qr+k$. It is easy to check that both of them yield the same value of $e(G)$, so $|U| = qr+k-1$ or $qr+k$.

- **Case IV**: $(r-1)(Rd+1)+R+1 \leq t \leq Rdr+R+r-1$.
  If $|U| \leq (q+1)r-1$, then we can increase $e(G)$ by moving a vertex from V_R to U_1. The proof for this follows similarly as in Case II.
  On the other hand, if $|U| \geq (q+1)r+1$, then $|U_r| \geq q+2$. Furthermore, $n-|U| \leq (q+1)(Rdr+R+r)-1-(q+1)r-1 = R(q+1)(dr+1)-2$ and $|V_1| \leq (q+1)(dr+1)-1$. Moving a vertex from U_r to V_1 changes $e(G)$ by at least

  $$d\left((q+1)r\right)+q+1-(q+1)(dr+1)+1 = 1 > 0.$$

  Thus, $|U| = (q+1)r$.

□

We now use Proposition 5.2 to prove the following proposition on the growth of $\Sigma^R_{r,d}(a,n)$.

Proposition 5.3 (Growth of $\Sigma^R_{r,d}(a,n)$). *Let $r, R, d, a, n \in \mathbb{N}$ with $a \geq d+1$, and let $n = q(Rdr+R+r)+t$, with $0 \leq t \leq Rdr+R+r-1$. Then*

$$\Sigma^R_{r,d}(a,n+1) - \Sigma^R_{r,d}(a,n)$$
$$= \begin{cases} (a+1)n - q(dr+1) - dk, & \text{if } t = k(Rd+1),\ 0 \leq k \leq r-1, \\ (a+1)n - q(dr+1) - dk - l, & \text{if } k(Rd+1)+1+lR \leq t \leq k(Rd+1)+(l+1)R, \\ & \quad 0 \leq k \leq r-1,\ 0 \leq l \leq d-1, \\ (a+1)n - (q+1)(dr+1)+1, & \text{if } r(Rd+1) \leq t \leq Rdr+R+r-1. \end{cases}$$

Proof. We may read off the values of $\Sigma^R_{r,d}(a,n+1) - \Sigma^R_{r,d}(a,n)$ from Proposition 5.2. Let $G \in \mathcal{S}\left(\pi^{a,d}_{r,R}, n\right)$ have canonical partition $(\sqcup_{i=1}^r U_i) \bigsqcup (\sqcup_{j=1}^R V_j)$ and let $U = \sqcup_{i=1}^r U_i$.

- **Case I**: $t = k(Rd+1)$, where $0 \leq k \leq r-1$.
  Consider $G \in \mathcal{S}\left(\pi^{a,d}_{r,R}, n\right)$ with $|U| = qr+k$. We may obtain a multigraph $G' \in \mathcal{S}\left(\pi^{a,d}_{r,R}, n+1\right)$ from G by adding a vertex to U_1. Since $|U_1| = q$, we have

  $$\Sigma^R_{r,d}(a,n+1) - \Sigma^R_{r,d}(a,n) = (a+1)n - d(qr+k) - q = (a+1)n - q(dr+1) - dk.$$

- **Case II**: $k(Rd+1)+lR+1 \leq t \leq k(Rd+1)+(l+1)R$, where $0 \leq k \leq r-1$ and $0 \leq l \leq d-1$.
  Consider $G \in \mathcal{S}\left(\pi^{a,d}_{r,R}, n\right)$ with $|U| = qr+k+1$. We may obtain a multigraph $G' \in \mathcal{S}\left(\pi^{a,d}_{r,R}, n+1\right)$ from G by adding a vertex to V_1. Since

  $$R\left(q(dr+1)+dk+l\right) \leq n - |U| \leq R\left(q(dr+1)+dk+l+1\right) - 1,$$

  we have $|V_1| = q(dr+1)+dk+l$. Thus,

  $$\Sigma^R_{r,d}(a,n+1) - \Sigma^R_{r,d}(a,n) = (a+1)n - q(dr+1) - dk - l.$$

- **Case III**: $r(Rd+1) \leq t \leq Rdr+R+r-1$.
  Consider $G \in \mathcal{S}\left(\pi^{a,d}_{r,R}, n\right)$ with $|U| = (q+1)r$. We may obtain a multigraph $G' \in$

$\mathcal{S}\left(\pi^{a,d}_{r,R}, n+1\right)$ from G by adding a vertex to V_1. Since

$$R\left((q+1)(dr+1)-1\right) \leq n - |U| \leq R\left((q+1)(dr+1)\right) - 1,$$

we have $|V_1| = (q+1)(dr+1)-1$. Thus,

$$\Sigma^R_{r,d}(a, n+1) - \Sigma^R_{r,d}(a,n) = (a+1)n - (q+1)(dr+1)+1.$$

This proves the proposition. $\square$

Proposition 5.4. *Let $r, R, d, s \in \mathbb{N}$. There exists a natural number a_1 with $a_1 \geq d+1$ such that for all $a \geq a_1$,*

$$\Sigma^R_{r,d}(a, s+1) - \Sigma^R_{r,d}(a,s) - 1 < (a+1)s - \left(\frac{1 - r x^d_{r,R}(a)}{R}\right) s.$$

Proof. By Proposition 5.1, $x^d_{r,R}(a)$ increases to $1/(Rdr+R+r)$. Therefore, $(1 - r x^d_{r,R}(a))/R$ decreases to $(dr+1)/(Rdr+R+r)$. In particular, for all a sufficiently large and any $c < (a+1)s - \frac{dr+1}{Rdr+R+r}s$, we have $c < (a+1)s - \left(\frac{1-rx^d_{r,R}(a)}{R}\right)s$. To prove Proposition 5.4, it therefore suffices to show that

$$\Sigma^R_{r,d}(a, s+1) - \Sigma^R_{r,d}(a,s) - 1 < (a+1)s - \frac{dr+1}{Rdr+R+r}s. \tag{5.1}$$

Equation (5.1) can be proved using Proposition 5.3. Let $s = q(Rdr+R+r)+t$, where $0 \leq t \leq Rdr+R+r-1$, and $q = (s-t)/(Rdr+R+r) \in \mathbb{Z}_{\geq 0}$. We consider the following cases:

- **Case I** : $t = k(Rd+1)$, where $0 \leq k \leq r-1$.
  By Proposition 5.3, we have

$$\Sigma^R_{r,d}(a, s+1) - \Sigma^R_{r,d}(a,s) - 1 = (a+1)s - (s-t)\frac{dr+1}{Rdr+R+r} - dk - 1.$$

  Since $t = k(Rd+1)$, it suffices to show that

$$dk+1 > \frac{k(Rd+1)(dr+1)}{Rdr+R+r}.$$

Expanding out the terms, it is easy to see that the above holds for all $0 \leq k \leq r-1$, and the result follows.

- **Case II**: $k(Rd+1)+lR+1 \leq t \leq k(Rd+1)+(l+1)R$, where $0 \leq k \leq r-1$ and $0 \leq l \leq d-1$.
  By Proposition 5.3, we have

$$\Sigma_{r,d}^{R}(a,s+1)-\Sigma_{r,d}^{R}(a,s)-1=(a+1)s-(s-t)\frac{dr+1}{Rdr+R+r}-dk-l-1.$$

  Since $t \leq k(Rd+1)+(l+1)R$, it suffices to show that

$$dk+l+1>\frac{(dr+1)\left(k(Rd+1)+(l+1)R\right)}{Rdr+R+r}.$$

  By rearranging terms, it is easy to see that the above holds whenever $0 \leq k \leq r-1$ and $0 \leq l \leq d-1$. The result follows.

- **Case III**: $r(Rd+1) \leq t \leq Rdr+R+r-1$.
  By Proposition 5.3, we have

$$\Sigma_{r,d}^{R}(a,s+1)-\Sigma_{r,d}^{R}(a,s)-1=(a+1)s-(s-t)\frac{dr+1}{Rdr+R+r}-dr-1.$$

  Since $t \leq Rdr+R+r-1$, we have

$$dr+1>\frac{t(dr+1)}{Rdr+R+r}.$$

  The result follows.

$\square$

Chapter 6

Resolving the Main Conjecture for Large a

In this chapter, we resolve Conjecture 2.2.2 for large a, thereby asymptotically resolving Conjecture 2.2.1 and proving the product-optimality of the generalised Turán pattern in the large a regime.

6.1 Main Results

We asymptotically resolve the "base" case of Conjecture 2.2.1 for sufficiently large a, marking one of the primary contributions of this thesis.

Theorem 6.1.1. *Let $r, R, d \in \mathbb{N}$, and let $s_0 = r + R(dr+1) + 1$. Furthermore, let a be a natural number satisfying*

$$(a+1)^{r(d-1)(2r-1)+2r}(a-d)^{2r-1}(a-d+1)^{(2r-1)(r-1)} \geq a^{rd(2r-1)+2r}. \tag{6.1}$$

Then

$$\mathrm{ex}_\Pi(s_0, \Sigma^R_{r,d}(a, s_0)) = a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}},$$

where $x^* = x^d_{r,R}(a)$.

Remark 6.1.1. *The polynomial inequality given by Equation* (6.1) *holds for all a satisfying*

$$a \geq d(2r-1)\left(d(2r-1)+1\right)+(d-1)(2r-1)(r-1)\left(r(d-1)(2r-1)+2r\right)=O(d^2r^4).$$

See Proposition A4 for a proof.

Next, we prove the following "Step-up" theorem:

Theorem 6.1.2 (Step-up in Conjecture 2.2.1)**.** *Let $r, R, d, s \in \mathbb{N}$ with $s \geq r + R(dr+1)+1$. Suppose there exist natural numbers a_0, n_0 with $a_0 \geq d+1$ such that the following holds for all $a \geq a_0$ and $n \geq n_0$:*

$$\mathrm{ex}_\Pi(n, s, \Sigma^R_{r,d}(a,s)) = \Pi^R_{r,d}(a,n).$$

Then there exist natural numbers a_1, n_1 with $a_1 \geq a_0$ and $n_1 \geq n_0$ such that the following holds for all $a \geq a_1$ and $n \geq n_1$:

$$\mathrm{ex}_\Pi(n, s+1, \Sigma^R_{r,d}(a,s+1)) = \Pi^R_{r,d}(a,n).$$

For fixed natural numbers r, R, d, Theorem 6.1.2 asserts that proving Conjecture 2.2.1 for some value of $s \geq r + R(dr+1)+1$ with a sufficiently large implies Conjecture 2.2.1 for $s+1$ with a sufficiently large. Note that Theorem 6.1.2 is a generalisation of Theorem 3.6 (which corresponds to the $r=1$ case of Theorem 6.1.2). The resolution of the "base" case of Conjecture 2.2.2, together with Theorem 6.1.2, implies the following corollary:

Corollary 6.1.3. *Let $r, R, d \in \mathbb{N}$. For all integers $s \geq r + R(dr+1)+1$, there exists $a_0 = a_0(s)$ such that, for all $a \geq a_0$, we have*

$$\mathrm{ex}_\Pi(s, \Sigma^R_{r,d}(a,s)) = a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}},$$

where $x^* = x^d_{r,R}(a)$.

Thus, Conjecture 2.2.2 holds for all $r, R, d, s \in \mathbb{N}$ with $s \geq r + R(dr+1)+1$ and a sufficiently large. Since the $r=1$ case of Conjecture 2.2.1 corresponds to Conjecture 3.2 in [6], our results provide an asymptotic resolution of Conjecture 3.2 in the large a regime.

As mentioned in Chapter 3, Conjecture 2.2.2 (the asymptotic form of Conjecture 2.2.1) had previously been resolved only for the $(r,d)=(1,1)$ case with a sufficiently large. Our results, therefore, represent a vast generalisation of all previous work and provide the first proof of the product-optimality of a pattern with four different edge multiplicities and arbitrarily large gaps between the smallest and largest multiplicities. At first glance, our results may seem somewhat counterintuitive. The integral AM-GM inequality suggests that, to maximise the product of edge multiplicities in a multigraph under a (local) constraint on their sum, one would expect the edge multiplicities to be roughly equal. However, the generalised Turán pattern, which we have shown to be product-optimal in the large a regime, features an arbitrarily wide spread of edge multiplicities.

6.2 Broad Outline of the Proofs

To prove Theorem 6.1.2, we first show that if $G \in \mathcal{F}(n, s+1, \Sigma^R_{r,d}(a, s+1)) \setminus \mathcal{F}(n, s, \Sigma^R_{r,d}(a, s))$, then for $n \geq N \geq n_0$ with N sufficiently large, and sufficiently large a, G has a vertex of low product-degree.

Next, we consider a multigraph $G \in \mathcal{F}(n, s+1, \Sigma^R_{r,d}(a, s+1))$ and sequentially delete vertices of lowest product-degree to construct a sequence of multigraphs

$$G = G_n, G_{n-1}, G_{n-2}, \ldots,$$

where each G_i is an induced submultigraph of G on i vertices. We continue this process until we obtain a submultigraph $G_{n'}$ of G, where either $n' = N$ or $G_{n'} \in \mathcal{F}(n', s, \Sigma^R_{r,d}(a, s))$.

In the first case, by ensuring that n is sufficiently large compared to N, we guarantee that a large number of vertices of low product-degree have been deleted, allowing us to upper bound $P(G)$. In the second case, we can upper bound $P(G_{n'})$ (and consequently $P(G)$) by assumption.

To prove Theorem 6.1.1, using Proposition 4.4, it suffices to show that for n sufficiently large, every multigraph $G \in \mathcal{F}(n, s_0, \Sigma^R_{r,d}(a, s_0))$ has a vertex of low product-degree, which we hereafter refer to as a *poor* vertex.

We first show that sets of size smaller than s_0 in G must be consistent with our construction (i.e., G is a $(t, \Sigma^R_{r,d}(a,t))$-graph for all $2 \le t \le s_0$); otherwise, G contains a poor vertex. Next, we argue, using weighted geometric averaging (with carefully chosen weights), that it suffices to find "good" partite structures in G to establish the existence of a poor vertex.

To show that such partite structures exist, the key idea is to find a regular submultigraph T on $s_1 > s_0$ vertices in G. Using the regularity of T, we demonstrate that every vertex sending many edges into T induces one part of the "good" partite structure we seek. Finally, by showing that many vertices send a large number of edges into T, we can establish the existence of the required partite structure. This is the novel part of our approach, which allows us to eschew the structural case analysis in previous work (the existence of partite structures had already been used by Falgas-Ravry [13]).

6.3 Proof of Theorem 6.1.2

Proof. By Proposition 5.4, we may fix $a_1 \ge a_0$ such that for all $a \ge a_1$, we have

$$\Sigma^R_{r,d}(a, s+1) - \Sigma^R_{r,d}(a,s) - 1 \le as + \left[1 - \left(\frac{1 - r x^d_{r,R}(a)}{R}\right)\right] s - \epsilon,$$

for some $\epsilon > 0$. Henceforth, we write x^* for $x^d_{r,R}(a)$. Let $N \ge n_0$ be a sufficiently large natural number to be determined later, and let $n_1 = N\left(1 + \frac{s}{\epsilon}\right)$. We claim that for all $a \ge a_1$ and $n \ge n_1$, we have

$$\mathrm{ex}_\Pi(n, s+1, \Sigma^R_{r,d}(a, s+1)) = \Pi^R_{r,d}(a,n).$$

Fix $a \ge a_1$ and $n \ge n_1$. Since every multigraph in $\mathcal{P}(\pi^{a,d}_{r,R}, n)$ is an $(s+1, \Sigma^R_{r,d}(a, s+1))$-graph, we have

$$\mathrm{ex}_\Pi(n, s+1, \Sigma^R_{r,d}(a, s+1)) \ge \Pi^R_{r,d}(a,n),$$

and thus we need only concern ourselves with proving the upper bound. Let $G \in \mathcal{F}(n, s+1, \Sigma^R_{r,d}(a, s+1))$. We sequentially delete vertices of lowest product-degree to obtain a sequence of multigraphs $G = G_n, G_{n-1}, G_{n-2}, ...$, where each G_i is an induced submultigraph of G on i vertices. We stop when we obtain a submultigraph of G, $G_{n'}$, where either $n' = N$ or $G_{n'} \in \mathcal{F}(n', s, \Sigma^R_{r,d}(a,s))$.

Suppose there exists an s-set $U \subseteq V(G_i)$ with $e(U) \geq \Sigma_{r,d}^R(a,s)+1$. Since G_i is an $(s+1, \Sigma_{r,d}^R(a,s+1))$-graph, every vertex outside U sends into U at most

$$\Sigma_{r,d}^R(a,s+1) - \Sigma_{r,d}^R(a,s) - 1 \leq as + \left[1 - \left(\frac{1-rx^*}{R}\right) - \frac{\epsilon}{s}\right] s \quad \text{edges} .$$

Consider $p := \prod_{u \in U} p(u)^{\frac{1}{s}}$. By the integral AM-GM inequality (Proposition 4.1), the contribution of every vertex outside U to p is at most $a\left(\frac{a+1}{a}\right)^{1-\left(\frac{1-rx^*}{R}\right)-\frac{\epsilon}{s}}$. Since all the edges in G_i have bounded multiplicities, the contribution to p due to vertices in U can be bounded above by an absolute constant ($O(1)$). Thus,

$$p \leq a^i \left(\frac{a+1}{a}\right)^{\left[1-\left(\frac{1-rx^*}{R}\right)-\frac{\epsilon}{s}\right]i+O(1)} .$$

From Equation (2.2), we have

$$\frac{\Pi_{r,d}^R(a,i)}{\Pi_{r,d}^R(a,i-1)} = a^i \left(\frac{a+1}{a}\right)^{\left[1-\left(\frac{1-rx^*}{R}\right)\right]i+O(1)} .$$

Thus, for all $i \geq N$, with N sufficiently large, we have

$$p \leq \frac{\Pi_{r,d}^R(a,i)}{\Pi_{r,d}^R(a,i-1)} \left(\frac{a+1}{a}\right)^{-\frac{\epsilon}{2s}i} \leq \frac{\Pi_{r,d}^R(a,i)}{\Pi_{r,d}^R(a,i-1)} \left(\frac{a+1}{a}\right)^{-\frac{\epsilon}{2s}N} .$$

By weighted geometric averaging (Proposition 4.2), G_i contains a vertex $u \in U$ with

$$p_{G_i}(u) \leq \frac{\Pi_{r,d}^R(a,i)}{\Pi_{r,d}^R(a,i-1)} \left(\frac{a+1}{a}\right)^{-\frac{\epsilon}{2s}N} .$$

We now consider the following cases:

- **Case I**: $n' > N$.
  In this case, $G_{n'} \in \mathcal{F}(n', s, \Sigma_{r,d}^R(a,s))$. Since $a \geq a_1 \geq a_0$ and $n' > N \geq n_0$, by assumption, we have

$$P(G_{n'}) \leq \text{ex}_\Pi(n', s, \Sigma_{r,d}^R(a,s)) = \Pi_{r,d}^R(a,n').$$

Then,

$$
\begin{aligned}
P(G) &\le P(G_{n'}) \prod_{i=n'+1}^{n} \frac{\Pi_{r,d}^R(a,i)}{\Pi_{r,d}^R(a,i-1)} \left(\frac{a+1}{a}\right)^{-\frac{\epsilon}{2s}N} \\
&\le \Pi_{r,d}^R(a,n') \prod_{i=n'+1}^{n} \frac{\Pi_{r,d}^R(a,i)}{\Pi_{r,d}^R(a,i-1)} \left(\frac{a+1}{a}\right)^{-\frac{\epsilon}{2s}N} \\
&\le \Pi_{r,d}^R(a,n).
\end{aligned}
$$

- **Case II**: $n' = N$.
  In this case, $G_N \in \mathcal{F}(N, s+1, \Sigma_{r,d}^R(a, s+1))$. By averaging over all sets of size $s+1$, we can argue that the average multiplicity of edges in G_N is at most $a+1$. Thus, $P(G_N) \le (a+1)^{\binom{N}{2}}$. Since $\Pi_{r,d}^R(a,N) \ge a^{\binom{N}{2}}$, we have

$$
P(G_N) \le \Pi_{r,d}^R(a,N) \left(\frac{a+1}{a}\right)^{\binom{N}{2}}.
$$

  Then,

$$
\begin{aligned}
P(G) &\le P(G_N) \prod_{i=N+1}^{n} \frac{\Pi_{r,d}^R(a,i)}{\Pi_{r,d}^R(a,i-1)} \left(\frac{a+1}{a}\right)^{-\frac{\epsilon}{2s}N} \\
&\le \Pi_{r,d}^R(a,N) \left(\frac{a+1}{a}\right)^{\binom{N}{2}} \prod_{i=N+1}^{n} \frac{\Pi_{r,d}^R(a,i)}{\Pi_{r,d}^R(a,i-1)} \left(\frac{a+1}{a}\right)^{-\frac{\epsilon}{2s}N} \\
&= \Pi_{r,d}^R(a,n) \left(\frac{a+1}{a}\right)^{\binom{N}{2} - \frac{\epsilon}{2s}N(n-N)}.
\end{aligned}
$$

  Since $n \ge n_1 \ge N\left(1 + \frac{s}{\epsilon}\right)$, we have $P(G) \le \Pi_{r,d}^R(a,n)$.

Thus, for all $a \ge a_1$ and $n \ge n_1$, $\text{ex}_\Pi(n, s+1, \Sigma_{r,d}^R(a, s+1)) \le \Pi_{r,d}^R(a,n)$. The result follows.

□

6.4 Proof of Theorem 6.1.1

Proof. Since every multigraph in $\mathcal{P}(\pi^{a,d}_{r,R}, n)$ is an $(s_0, \Sigma^R_{r,d}(a, s_0))$-graph, we have

$$\mathrm{ex}_\Pi(n, s_0, \Sigma^R_{r,d}(a, s_0)) \geq \Pi^R_{r,d}(a, n).$$

Raising both sides to $1/\binom{n}{2}$ and taking the limit as $n \to \infty$, combined with Equation (2.2), yields

$$\mathrm{ex}_\Pi(s_0, \Sigma^R_{r,d}(a, s_0)) \geq a \left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}}.$$

Therefore, it suffices to prove the upper bound. By Proposition 4.4, it suffices to show the following: for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, every multigraph $G \in \mathcal{F}(n, s_0, \Sigma^R_{r,d}(a, s_0))$ contains a vertex v with $p_G(v) \leq a^n \left(\frac{a+1}{a}\right)^{\left(1-\frac{1-rx^*}{R}+\epsilon\right)n}$.

Definition 6.4.1. *We define a vertex v in a multigraph $G \in \mathcal{F}(n, s_0, \Sigma^R_{r,d}(a, s_0))$ to be ϵ-poor if $p_G(v) \leq a^n \left(\frac{a+1}{a}\right)^{\left(1-\frac{1-rx^*}{R}+\epsilon\right)n}$.*

Henceforth, we fix $\epsilon > 0$ and write "poor" for "ϵ-poor". We first show that sets of size smaller than s_0 in a multigraph $G \in \mathcal{F}(n, s_0, \Sigma^R_{r,d}(a, s_0))$ must be consistent with our construction (i.e., G is a $(t, \Sigma^R_{r,d}(a, t))$-graph for all $2 \leq t \leq s_0$); otherwise, G contains a poor vertex.

Lemma 6.4.1. *Let $G \in \mathcal{F}(n, s_0, \Sigma^R_{r,d}(a, s_0))$. For n sufficiently large, either G contains a poor vertex or $G \in \bigcap_{2\leq t\leq s_0} \mathcal{F}(n, t, \Sigma^R_{r,d}(a, t))$.*

Proof. Suppose $G \notin \bigcap_{2\leq t\leq s_0} \mathcal{F}(n, t, \Sigma^R_{r,d}(a, t))$. Let t be the largest integer less than s_0 such that $G \notin \mathcal{F}(n, t, \Sigma^R_{r,d}(a, t))$. Observe that $2 \leq t \leq s_0 - 1$. There exists a set of t vertices with at least $\Sigma^R_{r,d}(a, t) + 1$ edges. Let us denote this set by T. By the maximality of t, every vertex outside T sends into T at most

$$\Sigma^R_{r,d}(a, t+1) - \Sigma^R_{r,d}(a, t) - 1 \leq at + \left(1 - \frac{d(r-1)+1}{R[d(r-1)+1]+r-1}\right)t \quad \text{edges.}$$

The last inequality follows from Proposition A2. Consider $p := \prod_{v\in T} p(v)^{\frac{1}{t}}$. By the integral AM-GM inequality (Proposition 4.1), the contribution of every vertex outside T to p is at most $a\left(\frac{a+1}{a}\right)^{1-\frac{d(r-1)+1}{R[d(r-1)+1]+r-1}}$. Since G is an $(s_0, \Sigma^R_{r,d}(a, s_0))$-graph, edges in G have bounded multiplicity. Therefore, the contribution of vertices inside T to p can be bounded above by an absolute constant ($O(1)$), and we have

$$p \le a^n \left(\frac{a+1}{a}\right)^{\left(1-\frac{d(r-1)+1}{R[d(r-1)+1]+r-1}\right)n+O(1)}.$$

Since $\frac{d(r-1)+1}{R[d(r-1)+1]+r-1} > \frac{d(2r-1)+2}{R[d(2r-1)+2]+2r-1}$ (this can be seen by rearranging the terms), it follows from part (3) of Proposition A1 that we have

$$p \le a^n \left(\frac{a+1}{a}\right)^{\left(1-\frac{1-rx^*}{R}\right)n+O(1)}.$$

By weighted geometric averaging (Proposition 4.2), it follows that for n sufficiently large, G contains a poor vertex.

□

Remark 6.4.1. *By Lemma 6.4.1, we may assume that G is a $(2, \Sigma_{r,d}^R(a, 2))$-graph. This means that the multiplicity of every edge is at most $a + 1$.*

Next, we prove that it suffices to find "good" partite structures to establish the existence of a poor vertex. We denote by H_L the multigraph in $\mathcal{B}(\pi_{r,R}^{a,d}, RL + r)$ with $|U_i| = 1$ for all $1 \le i \le r$, and $|V_j| = L$ for all $1 \le j \le R$, where $(\sqcup_{i=1}^r U_i) \bigsqcup (\sqcup_{j=1}^R V_j)$ denotes the canonical partition of H_L.

Lemma 6.4.2. *Let $G = ([n], w) \in \mathcal{F}(n, s_0, \Sigma_{r,d}^R(a, s_0))$. Suppose G contains a copy of H_L, with L satisfying*

$$L \ge \max\left\{\frac{1-rx^*}{Rrx^*}, \frac{rd\log\left(\frac{a+1}{a}\right)}{\log\left(\frac{a+1}{a+1-d}\right)}\left(\frac{1-rx^*}{Rrx^*}\right)\right\}.$$

Then, for n sufficiently large, G contains a poor vertex.

Proof. By Lemma 6.4.1, we may assume that $G \in \bigcap_{2\le t\le s_0} \mathcal{F}(n, t, \Sigma_{r,d}^R(a, t))$, otherwise G contains a poor vertex. Let $(\sqcup_{i=1}^r U_i) \bigsqcup (\sqcup_{j=1}^R V_j)$ be the canonical partition of the copy of H_L in G. Furthermore, let $U := \sqcup_{i=1}^r U_i$ and $V := \sqcup_{j=1}^R V_j$.
Consider $p := \prod_{u\in U} p_G(u)^{x^*} \prod_{v\in V} p_G(v)^{\frac{1-rx^*}{RL}}$. Let y be an arbitrary vertex in G outside H_L, and let $w(y) := \max_{u\in U} w(uy)$. We consider the contribution of y to p in various cases.

- **Case I**: $w(y) = a - d + 1 + k$, where $1 \leq k \leq d$.

  In this case, there exists a part V_j to which y sends at most $d-k$ edges of multiplicity $a+1$. Otherwise, the union of $(d-k+1)$-sets $V_j' \subseteq V_j$ with all edges from y to V_j' having multiplicity $a+1$ and $u \in U$ with $w(uy) = a-d+1+k$, taken together with y forms a set of $R(d-k+1)+2$ vertices with $\Sigma^R_{2,d-k}(a, R(d-k+1)+2) = \Sigma^R_{r,d}(a, R(d-k+1)+2)+1$ edges. This contradicts our assumption that $G \in \bigcap_{2\leq t\leq s_0} \mathcal{F}(n,t,\Sigma^R_{r,d}(a,t))$. Thus, the contribution of y to p is at most

$$(a+1)\left(\frac{a-d+1+k}{a+1}\right)^{rx^*}\left(\frac{a}{a+1}\right)^{\left(1-\frac{d-k}{L}\right)\left(\frac{1-rx^*}{R}\right)}$$
$$= a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}+\left[\left(\frac{d-k}{L}\right)\left(\frac{1-rx^*}{R}\right)-rx^*\frac{\log\left(\frac{a+1}{a-d+1+k}\right)}{\log\left(\frac{a+1}{a}\right)}\right]}.$$

  Claim. For all $1 \leq k \leq d$,

$$\left(\frac{d-k}{L}\right)\left(\frac{1-rx^*}{R}\right) - rx^*\frac{\log\left(\frac{a+1}{a-d+1+k}\right)}{\log\left(\frac{a+1}{a}\right)} \leq 0.$$

  Proof. For $k = d$, equality holds. Hence, we may assume that $1 \leq k \leq d-1$. Let $l = d-k$. To prove the claim, we need to show that for all $1 \leq l \leq d-1$, we have

$$L \geq \frac{l\log\left(\frac{a+1}{a}\right)}{\log\left(\frac{a+1}{a+1-l}\right)}\left(\frac{1-rx^*}{Rrx^*}\right).$$

  By the integral AM-GM inequality (Proposition 4.1), we have $(a+1)^l(a+1-l) \leq a^l(a+1)$, which, in turn, implies that $\log\left(\frac{a+1}{a}\right) \leq \frac{1}{l}\log\left(\frac{a+1}{a+1-l}\right)$. Since $L \geq \frac{1-rx^*}{Rrx^*}$ by assumption, the claim follows. □

  Thus, the contribution of y to p is at most $a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}}$ in this case.

- **Case II**: $w(y) \leq a - d + 1$.

  In this case, unless all edges between y and U have multiplicity $a-d+1$, the contribution of y to p is at most

$$a\left(\frac{a-d}{a}\right)^{x^*}\left(\frac{a-d+1}{a}\right)^{(r-1)x^*}\left(\frac{a+1}{a}\right)^{1-rx^*} = a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}}.$$

  The equality holds by Equation (2.1). On the other hand, if all edges between y and U have multiplicity $a-d+1$, there exists a part V_j to which y sends at most dr edges of multiplicity $a+1$. Otherwise, the union of $(dr+1)$-sets $V_j' \subseteq V_j$ with all edges between y and V_j' having multiplicity $a+1$ and U, taken together with y forms a set of s_0 vertices with $\Sigma_{1,d-1}^R(a,s_0) = \Sigma_{r,d}^R(a,s_0)+1$ edges. This is not possible since G is an $(s_0, \Sigma_{r,d}^R(a,s_0))$-graph. Thus, the contribution of y to p is at most

$$\begin{aligned}&(a+1)\left(\frac{a-d+1}{a+1}\right)^{rx^*}\left(\frac{a}{a+1}\right)^{\left(1-\frac{dr}{L}\right)\left(\frac{1-rx^*}{R}\right)}\\ &= a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}+\left[\frac{dr}{L}\left(\frac{1-rx^*}{R}\right)-rx^*\frac{\log\left(\frac{a+1}{a-d+1}\right)}{\log\left(\frac{a+1}{a}\right)}\right]}.\end{aligned}$$

Since $L \geq \frac{rd\log\left(\frac{a+1}{a}\right)}{\log\left(\frac{a+1}{a+1-d}\right)}\left(\frac{1-rx^*}{Rrx^*}\right)$ by assumption, we have $\frac{dr}{L}\left(\frac{1-rx^*}{R}\right) - rx^*\frac{\log\left(\frac{a+1}{a-d+1}\right)}{\log\left(\frac{a+1}{a}\right)} \leq 0$. Thus, the contribution of y to p is at most $a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}}$ in this case as well.

Therefore, the contribution of every vertex outside H_L to p is at most $a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}}$. The contribution of vertices within H_L to p can be bounded above by an absolute constant $(O(1))$. Hence, we have

$$p \leq a^n\left(\frac{a+1}{a}\right)^{\left(1-\frac{1-rx^*}{R}\right)n+O(1)}.$$

By weighted geometric averaging (Proposition 4.2), for n sufficiently large, G contains a poor vertex. □

Next, we show that all sets of size somewhat larger than s_0 in G must also be consistent with our construction.

Lemma 6.4.3. *For all* $s_0 \leq t \leq 2r + R[d(2r-1)+2]$, $\mathrm{ex}_\Sigma(t, s_0, \Sigma_{r,d}^R(a,s_0)) = \Sigma_{r,d}^R(a,t)$.

Proof. We proceed by induction on t. The lemma clearly holds for $t = s_0$. Suppose $\mathrm{ex}_\Sigma(t,s_0,\Sigma_{r,d}^R(a,s_0)) = \Sigma_{r,d}^R(a,t)$ for some $s_0 \leq t \leq (2r-1)+R[d(2r-1)+2]$. Clearly, $\mathrm{ex}_\Sigma(t+1, s_0, \Sigma_{r,d}^R(a,s_0)) \geq \Sigma_{r,d}^R(a,t+1)$. Therefore, it suffices to show that $\mathrm{ex}_\Sigma(t+1,s_0,\Sigma_{r,d}^R(a,s_0)) \leq \Sigma_{r,d}^R(a,t+1)$. By Proposition 4.3, it suffices to show that

$$\left\lfloor \left(\frac{t+1}{t-1}\right)\Sigma_{r,d}^R(a,t)\right\rfloor \leq \Sigma_{r,d}^R(a,t+1),$$

which is equivalent to showing that

$$\left\lfloor \left(\frac{2}{t-1}\right) \Sigma^R_{r,d}(a,t) \right\rfloor \leq \Sigma^R_{r,d}(a,t+1) - \Sigma^R_{r,d}(a,t).$$

We consider the following cases:

- **Case I**: $t = Rdr + R + r + k(Rd+1)$, where $1 \leq k \leq r-1$.
  From Proposition 5.2, we have

$$\Sigma^R_{r,d}(a,t) = (a+1)\binom{t}{2} - R\binom{d(k+r)+1}{2} - d\binom{k+r}{2} - k.$$

  Furthermore,

$$\left\lfloor \left(\frac{2}{t-1}\right) \Sigma^R_{r,d}(a,t) \right\rfloor = (a+1)t - d(k+r) - \left\lceil \frac{2k}{t-1} \right\rceil.$$

  Since $k \geq 1$,

$$\left\lfloor \left(\frac{2}{t-1}\right) \Sigma^R_{r,d}(a,t) \right\rfloor \leq (a+1)t - d(k+r) - 1 = \Sigma^R_{r,d}(a,t+1) - \Sigma^R_{r,d}(a,t).$$

  The last equality follows from Proposition 5.3.
- **Case II**: $t = Rdr + R + r + k(Rd+1) + 1 + lR + j$, where $0 \leq k \leq r-2$, $0 \leq l \leq d-1$ and $0 \leq j \leq R-1$.
  From Proposition 5.2, we have

$$\begin{aligned}\Sigma^R_{r,d}(a,t) = (a+1)\binom{t}{2} - R\binom{d(k+r)+l+1}{2} - j\left(d(k+r)+l+1\right) \\ - d\binom{k+r+1}{2} - (k+1).\end{aligned}$$

  Furthermore,

$$\begin{aligned}\left\lfloor \left(\frac{2}{t-1}\right) \Sigma^R_{r,d}(a,t) \right\rfloor &= (a+1)t - d(k+r) - l \\ &\quad - \left\lceil \frac{j\left(d(k+r)+l\right) + 2j + (k+r)(d-l) + 2(k+1)}{t-1} \right\rceil.\end{aligned}$$

Since $l \leq d-1$,

$$\left\lfloor \left(\frac{2}{t-1}\right) \Sigma_{r,d}^{R}(a,t) \right\rfloor \leq (a+1)t - d(k+r) - l - 1 = \Sigma_{r,d}^{R}(t+1) - \Sigma_{r,d}^{R}(a,t).$$

The last equality follows from Proposition 5.3.

- **Case III**: $t = Rdr + R + r + (r-1)(Rd+1) + 1 + j$, where $0 \leq j \leq R-1$.
  From Proposition 5.2, we have

$$\Sigma_{r,d}^{R}(a,t) = (a+1)\binom{t}{2} - R\binom{d(2r-1)+1}{2} - j\left(d(2r-1)+1\right) - d\binom{2r}{2} - r.$$

  Furthermore,

$$\left\lfloor \left(\frac{2}{t-1}\right) \Sigma_{r,d}^{R}(a,t) \right\rfloor = (a+1)t - d(2r-1) - \left\lceil \frac{(j+1)d(2r-1) + 2(j+r)}{t-1} \right\rceil.$$

  Since $j \geq 0$,

$$\left\lfloor \left(\frac{2}{t-1}\right) \Sigma_{r,d}^{R}(a,t) \right\rfloor \leq (a+1)t - d(2r-1) - 1 = \Sigma_{r,d}^{R}(a,t+1) - \Sigma_{r,d}^{R}(a,t).$$

  The last equality follows from Proposition 5.3.

The lemma follows. □

Henceforth, we let $s_1 := 2r + R\left(d(2r-1)+2\right)$.

Definition 6.4.2. *Let $G \in \mathcal{F}(n, s_0, \Sigma_{r,d}^{R}(a, s_0))$. For all integers $2 \leq t \leq s_1$, a set of t vertices $T \subseteq V(G)$ is called a heavy set if $e(T) = \Sigma_{r,d}^{R}(a,t)$.*

We now show that we can find a heavy set on s_1 vertices in G. Furthermore, this heavy set must be regular.

Lemma 6.4.4. *Let $G \in \mathcal{F}(n, s_0, \Sigma_{r,d}^{R}(a, s_0))$. For n sufficiently large, either G contains a heavy set on s_1 vertices or G contains a poor vertex.*

Proof. Using Lemmas 6.4.1 and 6.4.3, we may assume that G is a $(t, \Sigma_{r,d}^{R}(a,t))$-graph for all $2 \leq t \leq s_1$. Otherwise, G contains a poor vertex. We may also assume that G contains a heavy set on 2 vertices. Otherwise, all edges in G have multiplicity at most a, and there exists a vertex with product-degree at most a^n.

Suppose G does not contain a heavy set on s_1 vertices. Let $2 \leq t \leq s_1 - 1$ be the largest integer such that G contains a heavy set on t vertices. Let us denote this set by T. By the maximality of t, every vertex outside T sends into T at most

$$\Sigma_{r,d}^R(a, t+1) - \Sigma_{r,d}^R(a, t) - 1 \leq at + \left(1 - \frac{d(2r-1)+2}{R[d(2r-1)+2]+2r-1}\right) t \quad \text{edges.}$$

The last inequality follows from Propositions A2 and A3. Consider $p := \prod_{v \in T} p_G(v)^{\frac{1}{t}}$. By the integral AM-GM inequality (Proposition 4.1), the contribution of every vertex outside T to p is at most

$$a \left(\frac{a+1}{a}\right)^{1 - \frac{d(2r-1)+2}{R[d(2r-1)+2]+2r-1}} \leq a \left(\frac{a+1}{a}\right)^{1 - \frac{1 - rx^*}{R}}.$$

The last inequality follows from part (3) of Proposition A1. The contribution to p due to the vertices within T can be bounded above by an absolute constant ($O(1)$). Thus, we have

$$p \leq a^n \left(\frac{a+1}{a}\right)^{\left(1 - \frac{1 - rx^*}{R}\right) n + O(1)}.$$

By weighted geometric averaging (Proposition 4.2), for n sufficiently large, G contains a poor vertex. □

Lemma 6.4.5. *Let $G \in \mathcal{F}(n, s_0, \Sigma_{r,d}^R(a, s_0))$, and let $T \subseteq V(G)$ be a heavy set on s_1 vertices. Then, for every vertex $v \in T$, $d_T(v) = (a+1)(s_1 - 1) - d(2r-1) - 1$.*

Proof. By Lemma 6.4.3, every set of $s_1 - 1$ vertices in G contains at most $\Sigma_{r,d}^R(a, s_1 - 1)$ edges. Thus, for every vertex $v \in T$, we have

$$d_T(v) \geq \Sigma_{r,d}^R(a, s_1) - \Sigma_{r,d}^R(a, s_1 - 1) = (a+1)(s_1 - 1) - d(2r-1) - 1.$$

Otherwise, $T \setminus \{v\}$ is a set of $s_1 - 1$ vertices with at least $\Sigma_{r,d}^R(a, s_1 - 1) + 1$ edges. (Note that the last equality above follows from Proposition 5.3.) Furthermore, from Proposition

5.2, we have

$$\begin{aligned}\Sigma^R_{r,d}(a,s_1) &= (a+1)\binom{s_1}{2} - R\binom{d(2r-1)+2}{2} - d\binom{2r}{2} - r\\ &= \frac{s_1}{2}\left((a+1)(s_1-1) - d(2r-1) - 1\right).\end{aligned}$$

Since $\sum_{v\in T} d_T(v) = 2e(T) = 2\Sigma^R_{r,d}(a,s_1) = s_1\left((a+1)(s_1-1) - d(2r-1) - 1\right)$, the lemma follows.

□

Finally, we prove that we can indeed find the "good" partite structure we are looking for in G.

Lemma 6.4.6. *Let $G = ([n], w) \in \mathcal{F}(n, s_0, \Sigma^R_{r,d}(a, s_0))$. For n sufficiently large, either G contains a copy of H_L, with $L = d(2r-1)+2$, or G contains a poor vertex.*

Proof. By Lemmas 6.4.1 and 6.4.4, we may assume that $G \in \bigcap_{2\le t\le s_0} \mathcal{F}(n, t, \Sigma^R_{r,d}(a,t))$, and that G contains a heavy set on s_1 vertices. Otherwise, G contains a poor vertex. Let us denote this heavy set on s_1 vertices by T.

Claim. Every vertex outside T sends into T at most $(a+1)s_1 - d(2r-1) - 2$ edges.

Proof. By Lemma 6.4.3, G is an $(s_1, \Sigma^R_{r,d}(a,s_1))$-graph. By Proposition 4.3, every set of s_1+1 vertices in G contains at most $\left\lfloor \binom{s_1+1}{s_1-1} \Sigma^R_{r,d}(a,s_1) \right\rfloor$ edges. Since $e(T) = \Sigma^R_{r,d}(a,s_1)$, every vertex outside T sends into T at most

$$\begin{aligned}\left\lfloor \frac{2}{s_1-1}\Sigma^R_{r,d}(a,s_1)\right\rfloor &= \left\lfloor \frac{s_1}{s_1-1}\left((a+1)(s_1-1) - d(2r-1) - 1\right)\right\rfloor\\ &= (a+1)s_1 - \left\lceil \frac{s_1}{s_1-1}\left(d(2r-1)+1\right)\right\rceil\\ &\le (a+1)s_1 - d(2r-1) - 2 \quad \text{edges.}\end{aligned}$$

□

Let $A := \{v \in [n]\setminus T : v \text{ sends } (a+1)s_1 - d(2r-1) - 2 \text{ edges into } T\}$, and let $v \in A$. We make the following claims:

Claim. $w(vu) \geq a$ for all $u \in T$.

Proof. Suppose there exists a vertex $u \in T$ such that $w(vu) \leq a - 1$. By Lemma 6.4.5, $d_T(u) = (a+1)(s_1 - 1) - d(2r-1) - 1$. Thus,

$$d_{T\cup\{v\}}(u) = d_T(u) + w(vu) \leq (a+1)s_1 - d(2r-1) - 3,$$

and $(T \setminus \{u\}) \cup \{v\}$ is a set of s_1 vertices with

$$e(T) + (a+1)s_1 - d(2r-1) - 2 - d_{T\cup\{v\}}(u) \geq \Sigma^R_{r,d}(a, s_1) + 1 \quad \text{edges.}$$

This contradicts the fact that G is an $(s_1, \Sigma^R_{r,d}(a, s_1))$-graph. □

Therefore, v sends $d(2r-1)+2$ edges of multiplicity a, and $s_1 - d(2r-1) - 2$ edges of multiplicity $a+1$ into T.

Claim. Suppose $w(vu) = a$ for some vertex $u \in T$. Then, for all vertices $y \in T \setminus \{u\}$, $w(vy) = w(uy)$.

Proof. By Lemma 6.4.5, $d_T(u) = (a+1)(s_1-1) - d(2r-1) - 1$. We have

$$d_{T\cup\{v\}}(u) = d_T(u) + w(vu) = (a+1)s_1 - d(2r-1) - 2 = d_{T\cup\{v\}}(v).$$

Thus, deleting either u or v from $T \cup \{v\}$ yields a heavy set on s_1 vertices, wherein every vertex must have the same degree, by Lemma 6.4.5. The claim follows. □

For all vertices $v \in A$, we define $B_v := \{u \in T : w(vu) = a\}$. The claims above imply that $|B_v| = d(2r-1)+2$. Moreover, all edges within B_v have multiplicity a, while all edges from B_v to $T \setminus B_v$ have multiplicity $a+1$.

Claim. Let $v_1, v_2 \in A$. Then, either $B_{v_1} = B_{v_2}$ or $B_{v_1} \cap B_{v_2} = \emptyset$.

Proof. For each $i \in [2]$, all edges within B_{v_i} have multiplicity a, while all edges between B_{v_i} and $T \setminus B_{v_i}$ have multiplicity $a+1$. The claim follows. □

Let m be the largest integer such that there exist vertices $v_1, ..., v_m \in A$ with $B_{v_1}, ..., B_{v_m}$ being pairwise disjoint. Since $|B_{v_i}| = d(2r-1)+2$ for each $i \in [m]$, and $|T| = s_1 = 2r + R[d(2r-1)+2]$, we have $m \leq R$. For each $i \in [m]$, we define

$$A_i := \{v \in A : B_v = B_{v_i}\}.$$

By the maximality of m, we have $A = \cup_{i=1}^m A_i$. Let $n' = |[n] \setminus T| = n - s_1$, and let $|A| = \alpha n'$. Consider $p := \prod_{u \in T} p_G(u)^{\frac{1}{s_1}}$. By the integral AM-GM inequality (Proposition 4.1), the contribution of every vertex in A to p is at most $a\left(\frac{a+1}{a}\right)^{1-\frac{d(2r-1)+2}{s_1}}$, while the contribution of every vertex in $([n] \setminus T) \setminus A$ to p is at most $a\left(\frac{a+1}{a}\right)^{1-\frac{d(2r-1)+3}{s_1}}$. The contribution of vertices within T to p can be bounded above by an absolute constant ($O(1)$). Therefore,

$$p \leq a^n \left(\frac{a+1}{a}\right)^{\left(1-\frac{d(2r-1)+3}{s_1}+\frac{\alpha}{s_1}\right)n+O(1)}.$$

If $1-\frac{d(2r-1)+3}{s_1}+\frac{\alpha}{s_1} \leq 1-\frac{1-rx^*}{R}$, then by geometric averaging (Proposition 4.2), for n sufficiently large, G contains a poor vertex. Thus, we may assume that $1-\frac{d(2r-1)+3}{s_1}+\frac{\alpha}{s_1} > 1-\frac{1-rx^*}{R}$, which is equivalent to $\alpha > \frac{R-2r+rs_1x^*}{R}$.
For each $i \in [m]$, let $|A_i| = \alpha_i n'$. Consider a vertex $u \in B_{v_i}$. We have

$$p(u) \leq a^n \left(\frac{a+1}{a}\right)^{(1-\alpha_i)n+O(1)}.$$

If $1-\alpha_i \leq 1-\frac{1-rx^*}{R}$, then for n sufficiently large, u is a poor vertex. Therefore, we may assume that $1-\alpha_i > 1-\frac{1-rx^*}{R}$, which is equivalent to $\alpha_i < \frac{1-rx^*}{R}$. Since $\alpha = \sum_{i=1}^m \alpha_i$, we have $\alpha < m\left(\frac{1-rx^*}{R}\right)$. Combining the upper and lower bounds on α, we get

$$m > \frac{R-2r+rs_1x^*}{1-rx^*}.$$

We observe that $\frac{R-2r+rs_1x^*}{1-rx^*} \geq R-1$ is equivalent to $d(2r-1)+3 \geq (2r-1)\frac{1-rx^*}{Rrx^*}$, which holds by part (2) of Proposition A1. Thus, $m > R-1$. Since $m \leq R$, this means that $m = R$.
Let $T' := T \setminus (\cup_{i=1}^R B_{v_i})$. Observe that $|T'| = 2r$.

Claim. All edges in T' have multiplicity at most $a-d+1$.

Proof. Suppose there exists an edge y_1y_2 in T' with $w(y_1y_2) = a-d+1+k$ for some $k \in [d]$. Then $\{y_1, y_2\}$, taken along with $d-k+1$ vertices from each B_{v_i} yields a set of $R(d-k+1)+2$ vertices with $\Sigma^R_{2,d-k}(a, R(d-k+1)+2) = \Sigma^R_{r,d}(a, R(d-k+1)+2)+1$ edges. This contradicts the assumption that $G \in \bigcap_{2 \le t \le s_0} \mathcal{F}(n, t, \Sigma^R_{r,d}(a,t))$. □

For every vertex $y \in T'$, we have

$$\begin{aligned} d_{T'}(y) &= d_T(y) - (a+1)\left(R(d(2r-1)+2)\right) \\ &= (a+1)(s_1-1) - d(2r-1) - 1 - (a+1)\left(R(d(2r-1)+2)\right) \\ &= (2r-1)(a-d+1) - 1. \end{aligned}$$

Since every edge in T' has multiplicity at most $a-d+1$, this means that T' consists of a perfect matching of edges of multiplicity $a-d$, with all other edges having multiplicity $a-d+1$. One vertex from each edge of the perfect matching, together with $\cup_{i=1}^R B_{v_i}$ yields a copy of H_L, with $L = d(2r-1)+2$. □

By the integral AM-GM inequality (Proposition 4.1), we have $(a+1)^d(a+1-d) \le a^d(a+1)$. Therefore, $\frac{d\log\left(\frac{a+1}{a}\right)}{\log\left(\frac{a+1}{a+1-d}\right)} \le 1$. Since $r \ge 1$, we have

$$\max\left\{\frac{1-rx^*}{Rrx^*}, \frac{rd\log\left(\frac{a+1}{a}\right)}{\log\left(\frac{a+1}{a+1-d}\right)}\left(\frac{1-rx^*}{Rrx^*}\right)\right\} \le r\left(\frac{1-rx^*}{Rrx^*}\right).$$

By part (2) of Proposition A1, we have

$$L = d(2r-1)+2 \ge (2r-1)\frac{1-rx^*}{Rrx^*} \ge \max\left\{\frac{1-rx^*}{Rrx^*}, \frac{rd\log\left(\frac{a+1}{a}\right)}{\log\left(\frac{a+1}{a+1-d}\right)}\left(\frac{1-rx^*}{Rrx^*}\right)\right\}.$$

Thus, from Lemmas 6.4.2 and 6.4.6, it follows that every multigraph $G \in \mathcal{F}(n, s_0, \Sigma^R_{r,d}(a, s_0))$ contains a poor vertex, for n sufficiently large. This proves Theorem 6.1.1. □

Chapter 7

Results on the Turán Pattern

In this chapter, we present and prove two results related to the optimality of the Turán pattern, as described in Definition 2.1.1.

7.1 An Improved Bound for Theorem 3.7

Day, Falgas-Ravry, and Treglown ([6, Theorem 3.10], stated as Theorem 3.7 in this thesis) proved that for all natural numbers a, R, s, n with $s \geq R+1$ and $n \geq 2R(s+2) + R(s+2)\sqrt{s-1}$, we have

$$\mathrm{ex}_{\Pi}(n, s, \Sigma^R(a, s)) = \Pi^R(a, n).$$

The first open problem raised by Day, Falgas-Ravry, and Treglown in [6, Section 13] was whether the correct bound for this result is simply $n \geq s$. We prove that this is indeed the case.

Theorem 7.1.1 (Multigraph Turán Theorem). *Let $R, a, s \in \mathbb{N}$ with $s \geq R+1$. For all $n \geq s$, we have*

$$\mathrm{ex}_{\Sigma}(n, s, \Sigma^R(a, s)) = \Sigma^R(a, n) = a\binom{n}{2} + t_R(n).$$

Consequently,

$$\mathrm{ex}_\Pi(n, s, \Sigma^R(a, s)) = \Pi^R(a, n) = a^{\binom{n}{2}} \left(\frac{a+1}{a}\right)^{t_R(n)}.$$

Proof. From the lower bound construction given by the family $\mathcal{B}(\pi_R^a, n)$, it is easy to see that for all $n \geq s$, we have $\mathrm{ex}_\Sigma(n, s, \Sigma^R(a, s)) \geq \Sigma^R(a, n)$ and $\mathrm{ex}_\Pi(n, s, \Sigma^R(a, s)) \geq \Pi^R(a, n)$. Therefore, it suffices to prove the upper bounds. We will show by induction on n, that for all $n \geq s$, we have $\mathrm{ex}_\Sigma(n, s, \Sigma^R(a, s)) \leq \Sigma^R(a, n)$. This clearly holds for $n = s$. Let us assume that it holds for some $n = k$, with $k \geq s$. By averaging (Proposition 4.3), we have

$$\begin{aligned}
\mathrm{ex}_\Sigma(k+1, s, \Sigma^R(a, s)) \leq \left\lfloor \left(\frac{k+1}{k-1}\right) \mathrm{ex}_\Sigma(k, s, \Sigma^R(a, s)) \right\rfloor &= \left\lfloor \left(\frac{k+1}{k-1}\right) \Sigma^R(a, k) \right\rfloor \\
&= \left\lfloor \left(\frac{k+1}{k-1}\right) \left(a\binom{k}{2} + t_R(k)\right) \right\rfloor \\
&= a\binom{k+1}{2} + \left\lfloor \left(\frac{k+1}{k-1}\right) t_R(k) \right\rfloor.
\end{aligned}$$

Now,

$$\begin{aligned}
(k+1)t_R(k) - (k-1)t_R(k+1) &= (k-1)(t_R(k) - t_R(k+1)) + 2t_R(k) \\
&\leq (k-1)\left(\left\lfloor \frac{k}{R} \right\rfloor - k\right) + 2\binom{R}{2}\left(\frac{k}{R}\right)^2 \\
&\leq (k-1)\left(\frac{k}{R} - k\right) + 2\binom{R}{2}\left(\frac{k}{R}\right)^2 \\
&= k - \frac{k}{R} < k - 1,
\end{aligned}$$

since $k \geq s \geq R+1$. Rearranging, we get $\left(\frac{k+1}{k-1}\right) t_R(k) < t_R(k+1) + 1$, which implies that $\left\lfloor \left(\frac{k+1}{k-1}\right) t_R(k) \right\rfloor \leq t_R(k+1)$. Thus, we have

$$\begin{aligned}
\mathrm{ex}_\Sigma(k+1, s, \Sigma^R(a, s)) \leq a\binom{k+1}{2} + \left\lfloor \left(\frac{k+1}{k-1}\right) t_R(k) \right\rfloor &\leq a\binom{k+1}{2} + t_R(k+1) \\
&= \Sigma^R(a, k+1).
\end{aligned}$$

So, for all $n \geq s$, we have $\mathrm{ex}_\Sigma(n, s, \Sigma^R(a, s)) = \Sigma^R(a, n) = a\binom{n}{2} + t_R(n)$. By the integral

AM-GM inequality (Proposition 4.1), we then have

$$\mathrm{ex}_\Pi(n,s,\Sigma^R(a,s)) \le a^{\binom{n}{2}} \left(\frac{a+1}{a}\right)^{t_R(n)} = \Pi^R(a,n).$$

This concludes the proof. $\square$

7.2 Asymptotically Flat Intervals

Mubayi and Terry ([21, Theorem 3], stated as Theorem 3.1 in this thesis) showed that for all natural numbers a, s with $s \ge 2$, we have

$$\mathrm{ex}_\Pi\left(s, a\binom{s}{2}\right) = \mathrm{ex}_\Pi\left(s, a\binom{s}{2}+1\right) = \cdots = \mathrm{ex}_\Pi\left(s, a\binom{s}{2}+s-2\right) = a.$$

This result provides an example of an "asymptotically flat interval", where, despite an increase in q (with s held fixed), $\mathrm{ex}_\Pi(s,q)$ remains unchanged. Falgas-Ravry [13] raised the following conjecture, which, if true, generalizes the previous result and provides new examples of asymptotically flat intervals.

Conjecture 7.2.1 (Conjecture 5.1, [13]). *For every $R, a \in \mathbb{N}$ and for every $s \ge 2R+1$, we have*

$$\begin{aligned}\mathrm{ex}_\Pi\left(s, \Sigma^R(a,s)\right) &= \mathrm{ex}_\Pi\left(s, \Sigma^R(a,s)+1\right) = \cdots = \mathrm{ex}_\Pi\left(s, \Sigma^R(a,s) + \left\lfloor\frac{s-1}{R}\right\rfloor - 1\right)\\ &= a^{\frac{1}{R}}(a+1)^{\frac{R-1}{R}}.\end{aligned}$$

Here, we require $s \ge 2R+1$, since this would ensure that $\left\lfloor\frac{s-1}{R}\right\rfloor - 1 \ge 1$, which guarantees at least two distinct values of q in the interval. Note that the $R = 1$ case of Conjecture 7.2.1 corresponds to the result due to Mubayi and Terry. We prove that Conjecture 7.2.1 is true.

Theorem 7.2.2. *Let $R, a, s \in \mathbb{N}$ with $s \ge 2R+1$. We have*

$$\begin{aligned}\mathrm{ex}_\Pi(s, \Sigma^R(a,s)) &= \mathrm{ex}_\Pi(s, \Sigma^R(a,s)+1) = \cdots = \mathrm{ex}_\Pi\left(s, \Sigma^R(a,s) + \left\lfloor\frac{s-1}{R}\right\rfloor - 1\right)\\ &= a^{\frac{1}{R}}(a+1)^{\frac{R-1}{R}}.\end{aligned}$$

Proof. From Theorem 7.1.1, it is easy to see that $\mathrm{ex}_\Pi(s, \Sigma^R(a,s)) = a^{\frac{1}{R}}(a+1)^{\frac{R-1}{R}}$. Therefore, it suffices to show that $\mathrm{ex}_\Pi\left(s, \Sigma^R(a,s) + \left\lfloor \frac{s-1}{R} \right\rfloor - 1\right) \leq a^{\frac{1}{R}}(a+1)^{\frac{R-1}{R}}$. To do so, we will first show by induction on n, that for all $n \geq s$, we have $\mathrm{ex}_\Sigma\left(n, s, \Sigma^R(a,s) + \left\lfloor \frac{s-1}{R} \right\rfloor - 1\right) \leq \Sigma^R(a,n) + \left\lfloor \frac{n-1}{R} \right\rfloor - 1$. This holds trivially for $n = s$. Suppose that it holds for some $n = k$, with $k \geq s$. By averaging (Proposition 4.3), we have

$$\begin{aligned}\mathrm{ex}_\Sigma\left(k+1, s, \Sigma^R(a,s) + \left\lfloor \frac{s-1}{R} \right\rfloor - 1\right) &\leq \left\lfloor \left(\frac{k+1}{k-1}\right) \mathrm{ex}_\Sigma\left(k, s, \Sigma^R(a,s) + \left\lfloor \frac{s-1}{R} \right\rfloor - 1\right) \right\rfloor \\ &\leq \left\lfloor \left(\frac{k+1}{k-1}\right) \left(\Sigma^R(a,k) + \left\lfloor \frac{k-1}{R} \right\rfloor - 1\right) \right\rfloor.\end{aligned}$$

It suffices to show that

$$\left(\frac{k+1}{k-1}\right)\left(\Sigma^R(a,k) + \left\lfloor \frac{k-1}{R} \right\rfloor - 1\right) < \Sigma^R(a,k+1) + \left\lfloor \frac{k}{R} \right\rfloor.$$

Since $\Sigma^R(a,k) = a\binom{k}{2} + t_R(k)$, this is equivalent to

$$2t_R(k) + (k-1)(t_R(k) - t_R(k+1)) + (k+1)\left\lfloor \frac{k-1}{R} \right\rfloor < k + 1 + (k-1)\left\lfloor \frac{k}{R} \right\rfloor.$$

Now, as, $t_R(k+1) - t_R(k) = k - \left\lfloor \frac{k}{R} \right\rfloor$, it is enough to show that

$$2t_R(k) + (k+1)\left\lfloor \frac{k-1}{R} \right\rfloor < k^2 + 1.$$

This holds, since we have

$$2t_R(k) + (k+1)\left\lfloor \frac{k-1}{R} \right\rfloor < 2\binom{R}{2}\left(\frac{k}{R}\right)^2 + \frac{k^2}{R} = k^2 < k^2 + 1.$$

Thus, for all $n \geq s$, we have $\mathrm{ex}_\Sigma\left(n, s, \Sigma^R(a,s) + \left\lfloor \frac{s-1}{R} \right\rfloor - 1\right) \leq \Sigma^R(a,n) + \left\lfloor \frac{n-1}{R} \right\rfloor - 1$. By the integral AM-GM inequality (Proposition 4.1), we have

$$\mathrm{ex}_\Pi\left(n, s, \Sigma^R(a,s) + \left\lfloor \frac{s-1}{R} \right\rfloor - 1\right) \leq a^{\binom{n}{2}} \left(\frac{a+1}{a}\right)^{t_R(n) + \left\lfloor \frac{n-1}{R} \right\rfloor - 1}.$$

Raising both sides to $1/\binom{n}{2}$ and taking the limit as $n \to \infty$ yields the desired result. □

Chapter 8

Stability Results

As mentioned Chapter 1, extremal graph theory is fundamentally concerned with determining how large or small a given parameter of a graph G can be, subject to certain constraints. Beyond optimizing this parameter, one can also investigate whether the problem exhibits *stability* - namely, whether graphs that are nearly optimal must resemble the extremal constructions. In this chapter, we prove stability results corresponding to Theorems 6.1.1 and 7.1.1.

8.1 A Property of Near-Extremal Multigraphs

Given integers $s \geq 2$ and $q \geq 0$, we first show that any multigraph $G \in \mathcal{F}(n,s,q)$ with $P(G)$ close to $\mathrm{ex}_\Pi(n,s,q)$ (for sufficiently large n) contains a large induced submultigraph G' in which every vertex has high product-degree. Using the fact that G' has no low product-degree vertices, we deduce structural properties of G' and use these properties to prove stability results.

Proposition 8.1.1. *Let $s \geq 2$ and $q \geq 0$ be integers such that $\mathrm{ex}_\Pi(s,q) = c > 1$. For all $0 < \delta_1, \delta_2 < 1$, there exists $\epsilon > 0$ such that the following holds for all n sufficiently large: Suppose $G \in \mathcal{F}(n,s,q)$ and $P(G) \geq \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon}$. Then there exists an induced submultigraph $G' \subseteq G$ on n' vertices such that:*

1. *$p_{G'}(v) \geq c^{(1-\delta_1)(n'-1)}$ for every vertex $v \in V(G')$.*

2. $\binom{n}{2} - \binom{n'}{2} \le \delta_2\binom{n}{2}$.

Proof. Fix $0 < \delta_1, \delta_2 < 1$, and let $\epsilon = \frac{\delta_1\delta_2}{2-\delta_2}$. Since $\mathrm{ex}_\Pi(n,s,q)^{\frac{1}{\binom{n}{2}}}$ decreases to c as $n \to \infty$, there exists $N_0 \in \mathbb{N}$ such that the following holds for all $n \ge N_0$:

$$c^{\binom{n}{2}} \le \mathrm{ex}_\Pi(n,s,q) \le c^{(1+\epsilon)\binom{n}{2}}.$$

Suppose $G \in \mathcal{F}(n,s,q)$ and $P(G) \ge \mathrm{ex}_\Pi(n,s,q)^{1-\epsilon}$. This means that $P(G) \ge c^{(1-\epsilon)\binom{n}{2}}$. From G, we sequentially delete the vertices with the lowest product-degree until we obtain a multigraph G' on n' vertices such that $p_{G'}(v) \ge c^{(1-\delta_1)(n'-1)}$ for every vertex $v \in V(G')$. Since G' is an (s,q)-graph, $P(G') \le q^{\binom{n'}{2}}$. We have

$$c^{(1-\epsilon)\binom{n}{2}} \le P(G) \le P(G') \prod_{i=n'+1}^{n} c^{(1-\delta_1)(i-1)} \le q^{\binom{n'}{2}} c^{(1-\delta_1)\binom{n}{2}}.$$

Therefore, $\binom{n'}{2} \ge \frac{(\delta_1-\epsilon)\log c}{\log q}\binom{n}{2}$. Since $\epsilon = \frac{\delta_1\delta_2}{2-\delta_2} < \delta_1$, for n sufficiently large, $n' \ge N_0$. Thus, we have

$$c^{(1-\epsilon)\binom{n}{2}} \le P(G) \le P(G') \prod_{i=n'+1}^{n} c^{(1-\delta_1)(i-1)} \le c^{(1+\epsilon)\binom{n'}{2}} c^{(1-\delta_1)\left[\binom{n}{2}-\binom{n'}{2}\right]}.$$

This means that $\binom{n'}{2} \ge \frac{\delta_1-\epsilon}{\delta_1+\epsilon}\binom{n}{2}$, which in turn implies that $\binom{n}{2} - \binom{n'}{2} \le \frac{2\epsilon}{\delta_1+\epsilon}\binom{n}{2} = \delta_2\binom{n}{2}$. ☐

8.2 Stability for Theorem 6.1.1

Theorem 8.2.1. *Let $r, R, d \in \mathbb{N}$, and let $s_0 = r + R(dr+1) + 1$. Furthermore, let a be a natural number satisfying*

$$(a+1)^{r(d-1)(2r-1)+2r}(a-d)^{2r-1}(a-d+1)^{(2r-1)(r-1)} > a^{rd(2r-1)+2r}. \tag{8.1}$$

For every $\delta > 0$, there exists $\epsilon > 0$ such that the following holds for all n sufficiently large: Suppose $G \in \mathcal{F}(n, s_0, \Sigma^R_{r,d}(a, s_0))$ and $P(G) \ge \mathrm{ex}_\Pi(n, s_0, \Sigma^R_{r,d}(a, s_0))^{1-\epsilon}$. Then G is δ-close to an element of $\mathcal{P}(\pi^{a,d}_{r,R}, n)$.

Remark 8.2.1. *The polynomial inequality given by Equation* (8.1) *holds for all a satisfying*

$$a > d(2r-1)\left(d(2r-1)+1\right)+(d-1)(2r-1)(r-1)\left(r(d-1)(2r-1)+2r\right) = O(d^2r^4).$$

This can easily be seen from the proof of Proposition A4.

Proof. The proof of Theorem 8.2.1 is closely related to the proof of Theorem 6.1.1. We write x^* for $x^d_{r,R}(a)$. Fix $\delta > 0$. Let $\delta_0 > 0$ be sufficiently small. By Proposition 8.1.1, there exists $\epsilon > 0$ such that the following holds for all n sufficiently large:
Suppose $G = ([n], w) \in \mathcal{F}(n, s_0, \Sigma^R_{r,d}(a, s_0))$ and $P(G) \geq \mathrm{ex}_\Pi(n, s_0, \Sigma^R_{r,d}(a, s_0))^{1-\epsilon}$. Then there exists an induced submultigraph $G' \subseteq G$ on n' vertices such that:

1. $p_{G'}(v) \geq a^{n'}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1-rx^*}{R}-\delta_0\right)n'}$ for every vertex $v \in V(G')$.
2. $\binom{n}{2} - \binom{n'}{2} \leq \delta\binom{n}{2}$.

Observe that n' can be made sufficiently large by making n sufficiently large. By part (3) of Proposition A1, we have $\frac{1-rx^*}{R} < \frac{d(2r-1)+2}{R[d(2r-1)+2]+2r-1}$. For δ_0 sufficiently small, we have $\frac{1-rx^*}{R} + 2\delta_0 \leq \frac{d(2r-1)+2}{R[d(2r-1)+2]+2r-1}$. We may assume that $G' \in \bigcap_{2\leq t\leq s_0} \mathcal{F}(n', t, \Sigma^R_{r,d}(a, t))$ and that G' contains a heavy set T on $s_1 = 2r + R\left(d(2r-1)+2\right)$ vertices. Otherwise, by Lemmas 6.4.1 and 6.4.4, G' contains a vertex with product-degree at most

$$a^{n'}\left(\frac{a+1}{a}\right)^{\left(1-\frac{d(2r-1)+2}{R[d(2r-1)+2]+2r-1}\right)n'+O(1)} \leq a^{n'}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1-rx^*}{R}-2\delta_0\right)n'+O(1)}.$$

For n' sufficiently large, this is strictly less than $a^{n'}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1-rx^*}{R}-\delta_0\right)n'}$, a contradiction. Note that G' is a $(2, \Sigma^R_{r,d}(a, 2))$-graph. This implies that every edge in G' has multiplicity at most $a+1$.
As in the proof of Theorem 6.1.1, let $A := \{v \in [n'] \setminus T : v \text{ sends } (a+1)s_1 - d(2r-1) - 2 \text{ edges into } T\}$. For all vertices $v \in A$, we define $B_v := \{u \in T : w(vu) = a\}$. Let m be the largest integer such that there exist vertices $v_1, ..., v_m \in A$ with $B_{v_1}, ..., B_{v_m}$ being pairwise disjoint. For each $i \in [m]$, we define $A_i := \{v \in A : B_v = B_{v_i}\}$. By the maximality of m, we have $A = \cup_{i=1}^m A_i$. Similar to the proof of Lemma 6.4.6, if we can establish that $m > R-1$, this would imply that G' contains a copy of H_L with $L = d(2r-1)+2$.
Let $|A| = \alpha(n' - s_1)$, and for each $i \in [m]$, let $|A_i| = \alpha_i(n' - s_1)$. As in the proof of Lemma

6.4.6, T contains a vertex with product-degree at most $a^{n'}\left(\frac{a+1}{a}\right)^{\left(1-\frac{d(2r-1)+3}{s_1}+\frac{\alpha}{s_1}\right)n'+O(1)}$, and B_{v_i} contains a vertex with product-degree at most $a^{n'}\left(\frac{a+1}{a}\right)^{(1-\alpha_i)n'+O(1)}$. We may assume that

$$1-\frac{d(2r-1)+3}{s_1}+\frac{\alpha}{s_1}>1-\frac{1-rx^*}{R}-2\delta_0 \quad \text{ and } \quad 1-\alpha_i>1-\frac{1-rx^*}{R}-2\delta_0.$$

Otherwise, for n' sufficiently large, G' contains a vertex with product-degree strictly less than $a^{n'}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1-rx^*}{R}-\delta_0\right)n'}$, a contradiction. Therefore, we have

$$\frac{R-2r+rs_1x^*-2\delta_0 s_1 R}{R}<\alpha=\sum_{i=1}^{m}\alpha_i<m\left(\frac{1-rx^*}{R}+2\delta_0\right).$$

This implies that $m>\frac{R-2r+rs_1x^*-2\delta_0 s_1 R}{1-rx^*+2\delta_0 R}$. Since $\frac{R-2r+rs_1x^*}{1-rx^*}>R-1$ is equivalent to $d(2r-1)+3>(2r-1)\left(\frac{1-rx^*}{Rrx^*}\right)$, which holds by part (2) of Proposition A1, for δ_0 sufficiently small, we have $\frac{R-2r+rs_1x^*-2\delta_0 s_1 R}{1-rx^*+2\delta_0 R}\geq R-1$. Thus, $m>R-1$, and G' contains a copy of H_L with $L=d(2r-1)+2$.

Let $(\sqcup_{i=1}^{r}U_i)\bigsqcup(\sqcup_{j=1}^{R}V_j)$ be the canonical partition of the copy of H_L in G', and for all $i\in[r]$, let u_i denote the vertex in U_i. Furthermore, let $U:=\cup_{i=1}^{r}U_i$ and $V:=\cup_{j=1}^{R}V_j$. For all $i\in[r]$, we define $\mathcal{U}_i$ to be the set of vertices $y\in[n']\setminus H_L$ such that $w(yu_i)=a-d$, $w(yu_l)=a-d+1$ for all $l\in[r]\setminus\{i\}$ and $w(yv)=a+1$ for all $v\in V$. For all $j\in[R]$, we define $\mathcal{V}_j$ to be the set of vertices $y\in[n']\setminus H_L$ such that $w(yv)=a$ for all $v\in V_j$ and $w(yv)=a+1$ for all $v\in H_L\setminus V_j$. Finally, we denote by $\mathcal{Z}$ the set of all the remaining vertices in $[n']\setminus H_L$.

We make the following observations:

- Every edge v_1v_2 in $\mathcal{U}_i^{(2)}$ has multiplicity at most $a-d$. Otherwise, $\{v_1,v_2\}$, taken together with $dr+1$ vertices from each V_j, and $U\setminus\{u_i\}$, yields a set of s_0 vertices with strictly more than $\Sigma_{r,d}^{R}(a,s_0)$ edges.

- Every edge v_1v_2 in $\mathcal{U}_{i_1}\times\mathcal{U}_{i_2}$, with $i_1\neq i_2$, has multiplicity at most $a-d+1$. Otherwise, if $w(v_1v_2)=a-d+1+k$ for some $k\in[d]$, $\{v_1,v_2\}$, taken together with $d-k+1$ vertices from each V_j yields a set of $R(d-k+1)+2$ vertices with $\Sigma_{2,d-k}^{R}(a,R(d-k+1)+2)=\Sigma_{r,d}^{R}(a,R(d-k+1)+2)+1$ edges.

- Every edge v_1v_2 in $\mathcal{V}_j^{(2)}$ has multiplicity at most a. Otherwise, $\{v_1,v_2\}$, taken together with u_1, and one vertex from each V_l with $l\neq j$, yields a set of $R+2$ vertices with $\Sigma_{r,d}^{R}(a,R+2)+1$ edges.

Let $|\mathcal{U}_i| = \mu_i(n' - LR - r)$, $|\mathcal{V}_j| = \nu_j(n' - LR - r)$ and $|\mathcal{Z}| = \theta(n' - LR - r)$. Consider $p := \prod_{u \in U} p_{G'}(u)^{x^*} \prod_{v \in V} p_{G'}(v)^{\frac{1-rx^*}{RL}}$. The contribution of all the vertices in $(\cup_{i=1}^r \mathcal{U}_i) \bigcup (\cup_{j=1}^R \mathcal{V}_j)$ to p is precisely $a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}}$. By part (2) of Proposition A1, we have $d(2r-1)+2 > (2r-1)\frac{1-rx^*}{Rrx^*} \geq \max\left\{\frac{1-rx^*}{Rrx^*}, \frac{rd\log\left(\frac{a+1}{a}\right)}{\log\left(\frac{a+1}{a+1-d}\right)}\left(\frac{1-rx^*}{Rrx^*}\right)\right\}$. This, together with the proof of Lemma 6.4.2, implies that the contribution of every vertex in $\mathcal{Z}$ to p is at most $a\left(\frac{a+1}{a}\right)^{1-\frac{1-rx^*}{R}-\delta'}$, for some $\delta' > 0$. Thus, we have $p \leq a^{n'}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1-rx^*}{R}-\delta'\theta\right)n'+O(1)}$. If $1 - \frac{1-rx^*}{R} - \delta'\theta \leq 1 - \frac{1-rx^*}{R} - 2\delta_0$, then for n' sufficiently large, by weighted geometric averaging (Proposition 4.2), G' contains a vertex with product-degree strictly less than $a^{n'}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1-rx^*}{R}-\delta_0\right)n'}$, a contradiction. Therefore, we may assume that $1 - \frac{1-rx^*}{R} - \delta'\theta > 1 - \frac{1-rx^*}{R} - 2\delta_0$, which is equivalent to

$$\theta < \frac{2\delta_0}{\delta'}. \tag{8.2}$$

For each $j \in [R]$, consider a vertex $v \in V_j$. We have $p_{G'}(v) \leq a^{n'}\left(\frac{a+1}{a}\right)^{(1-\nu_j)n'+O(1)}$. For n' sufficiently large, we may assume that $1 - \nu_j > 1 - \frac{1-rx^*}{R} - 2\delta_0$, which implies that for all $j \in [R]$,

$$\nu_j < \frac{1-rx^*}{R} + 2\delta_0. \tag{8.3}$$

Let $\mu = \sum_{i=1}^r \mu_i$, and let $\nu = \sum_{j=1}^R \nu_j$. Since $\mu + \nu + \theta = 1$, from Equations (8.2) and (8.3), we have

$$\mu > rx^* - 2R\delta_0 - \frac{2\delta_0}{\delta'}. \tag{8.4}$$

For each $i \in [r]$, we have

$$p_{G'}(u_i) \leq a^{n'}\left(\frac{a-d+1}{a+1}\right)^{\mu n'}\left(\frac{a-d}{a-d+1}\right)^{\mu_i n'}\left(\frac{a+1}{a}\right)^{n'+O(1)}.$$

For n' sufficiently large, we may assume that

$$a^{n'}\left(\frac{a-d+1}{a+1}\right)^{\mu n'}\left(\frac{a-d}{a-d+1}\right)^{\mu_i n'}\left(\frac{a+1}{a}\right)^{n'} > a^{n'}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1-rx^*}{R}-2\delta_0\right)n'}.$$

Applying Equation (8.4) and simplifying, we have that for all $i \in [r]$,

$$\mu_i < x^* + \frac{2\log\left(\frac{a+1}{a}\right) + \left(2R + \frac{2}{\delta'}\right)\log\left(\frac{a+1}{a-d+1}\right)}{\log\left(\frac{a-d+1}{a-d}\right)}\delta_0. \tag{8.5}$$

Equations (8.2), (8.3) and (8.5), along with the fact that $\mu + \nu + \theta = 1$, imply the existence of a constant c such that

$$0 \le \theta < c\delta_0, \tag{8.6}$$

$$x^* - c\delta_0 < \mu_i < x^* + c\delta_0 \quad \text{for all } i \in [r], \text{ and} \tag{8.7}$$

$$\frac{1 - rx^*}{R} - c\delta_0 < \nu_j < \frac{1 - rx^*}{R} + c\delta_0 \quad \text{for all } j \in [R]. \tag{8.8}$$

We say that an edge v_1v_2 in G' is *light* if:

- $v_1v_2 \in \mathcal{U}_i^{(2)}$ and $w(v_1v_2) < a - d$.
- $v_1v_2 \in \mathcal{U}_{i_1} \times \mathcal{U}_{i_2}$, with $i_1 \ne i_2$, and $w(v_1v_2) < a - d + 1$.
- $v_1v_2 \in \mathcal{V}_j^{(2)}$ and $w(v_1v_2) < a$.
- $v_1v_2 \in \mathcal{V}_{j_1} \times \mathcal{V}_{j_2}$, with $j_1 \ne j_2$, and $w(v_1v_2) < a + 1$.
- $v_1v_2 \in \mathcal{U}_i \times \mathcal{V}_j$ and $w(v_1v_2) < a + 1$.

For every vertex $v \in V(G')$, let f_v denote the number of light edges incident to v. Consider a vertex $v \in \mathcal{V}_j$, for some $j \in [R]$. We have $p_{G'}(v) \le a^{n'}\left(\frac{a+1}{a}\right)^{(1-\nu_j)n' - f_v + O(1)}$. For n' sufficiently large, we may assume that $(1 - \nu_j)n' - f_v > \left(1 - \frac{1-rx^*}{R} - 2\delta_0\right)n'$. Together with Equation (8.8), this yields

$$f_v < (c + 2)\delta_0 n'. \tag{8.9}$$

Finally, consider a vertex $v \in \mathcal{U}_i$, for some $i \in [r]$. We have

$$p_{G'}(v) \le a^{n'}\left(\frac{a-d+1}{a+1}\right)^{\mu n'}\left(\frac{a-d}{a-d+1}\right)^{\mu_i n'}\left(\frac{a+1}{a}\right)^{n' - f_v + O(1)}.$$

For n' sufficiently large, we may assume that

$$a^{n'}\left(\frac{a-d+1}{a+1}\right)^{\mu n'}\left(\frac{a-d}{a-d+1}\right)^{\mu_i n'}\left(\frac{a+1}{a}\right)^{n' - f_v} > a^{n'}\left(\frac{a+1}{a}\right)^{\left(1 - \frac{1-rx^*}{R} - 2\delta_0\right)n'}.$$

Applying Equation (8.7) and simplifying, we get

$$f_v < \frac{2\log\left(\frac{a+1}{a}\right) + c\log\left(\frac{a-d+1}{a-d}\right) + rc\log\left(\frac{a+1}{a-d+1}\right)}{\log\left(\frac{a+1}{a}\right)}\delta_0 n'. \tag{8.10}$$

By Equations (8.9) and (8.10), every vertex in G' has $O(\delta_0 n')$ light edges incident to it. Therefore, G' contains $O(\delta_0 n'^2)$ light edges. By Equations (8.6), (8.7) and (8.8), we can ensure that $\mu_i = x^*$ for all $i \in [r]$, $\nu_j = \frac{1-rx^*}{R}$ for all $j \in [R]$ and $\theta = 0$ by shifting $O(\delta_0 n')$ vertices. We can obtain a multigraph in $\mathcal{P}(\pi^{a,d}_{r,R}, n')$ from G' by appropriately altering the multiplicity of light edges and shifting vertices among various parts. In so doing, we alter at most $O(\delta_0 n'^2)$ edges. Thus, for δ_0 sufficiently small, G' differs from a multigraph in $\mathcal{P}(\pi^{a,d}_{r,R}, n')$ in at most $\frac{\delta}{2}n'^2$ edges, and G differs from a multigraph in $\mathcal{P}(\pi^{a,d}_{r,R}, n)$ in at most $\delta\binom{n}{2} + \frac{\delta}{2}n'^2 < \delta n^2$ edges. This proves Theorem 8.2.1. □

8.3 Stability for Theorem 7.1.1

Theorem 8.3.1. *Let $R, a, s \in \mathbb{N}$ with $R \geq 2$ and $s \geq R+1$. For every $\delta > 0$, there exists $\epsilon > 0$ such that the following holds for all n sufficiently large: Suppose $G \in \mathcal{F}(n, s, \Sigma^R(a, s))$ and $P(G) \geq \Pi^R(a, n)^{1-\epsilon}$. Then, G is δ-close to an element of $\mathcal{P}(\Pi^a_R, n)$.*

Remark 8.3.1. *Theorem 8.3.1 holds for all natural numbers a and all $s \geq R+1$. This contrasts with Theorem 8.2.1, which requires a to be sufficiently large and $s_0 = r + R(dr + 1) + 1$.*

Proof. We first prove the following lemma:

Lemma 8.3.2. *Suppose $G \in \mathcal{F}(n, s, \Sigma^R(a, s))$. Then, either $G \in \mathcal{F}(n, R+1, \Sigma^R(a, R+1))$, or G contains a vertex v with $p_G(v) \leq a^{n-1}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1}{R}-\frac{1}{R(s-1)}+o(1)\right)(n-1)}$.*

Proof. Suppose $G \notin \mathcal{F}(n, R+1, \Sigma^R(a, R+1))$. Let t be the largest integer less than s such that $G \notin \mathcal{F}(n, t, \Sigma^R(a, t))$. Clearly, $R+1 \leq t \leq s-1$. Then, there exists a set of t vertices with at least $\Sigma^R(a, t) + 1$ edges. Let us denote this set by T. By the maximality of t, every vertex outside T sends into T at most

$$\Sigma^R(a,t+1) - \Sigma^R(a,t) - 1 = at + t - \left\lfloor \frac{t}{R} \right\rfloor - 1 \text{ edges.}$$

Consider $p := \prod_{v\in T} p_G(v)^{\frac{1}{t}}$. By the integral AM-GM inequality (Proposition 4.1), the contribution of every vertex outside T to p is at most

$$\left[a^t \left(\frac{a+1}{a}\right)^{t-\left\lfloor \frac{t}{R}\right\rfloor - 1}\right]^{\frac{1}{t}}.$$

Since $\left\lfloor \frac{t}{R}\right\rfloor \geq \frac{t-(R-1)}{R}$, this is at most $a\left(\frac{a+1}{a}\right)^{1-\frac{1}{R}-\frac{1}{Rt}} \leq a\left(\frac{a+1}{a}\right)^{1-\frac{1}{R}-\frac{1}{R(s-1)}}$. The contribution of vertices within T to p can be bounded above by an absolute constant. Thus, we have $p \leq a^{n-1}\left(\frac{a+1}{a}\right)^{(1-\frac{1}{R}-\frac{1}{R(s-1)})(n-1)+O(1)}$. By weighted geometric averaging (Proposition 4.2), the lemma follows. □

Fix $\delta > 0$. Observe that $\Sigma^R(a, R+1) = (a+1)\binom{R+1}{2} - 1$. By Theorem 3.2 (Stability), there exists $\epsilon' > 0$ such that the following holds for all n' sufficiently large: suppose $G' \in \mathcal{F}(n', R+1, \Sigma^R(a, R+1))$ and $P(G') \geq \Pi^R(a, n')^{1-\epsilon'}$. Then, G' is $\frac{\delta}{2}$-close to an element of $\mathcal{P}(\pi_R^a, n')$.

Let $\delta_1 = \min\left\{\frac{1}{2R(s-1)}, \frac{\epsilon' \log\left[a\left(\frac{a+1}{a}\right)^{1-\frac{1}{R}}\right]}{2\log\left(\frac{a+1}{a}\right)}\right\}$. By Proposition 8.1.1, there exists $\epsilon > 0$ such that the following holds for all n sufficiently large: Let $G \in \mathcal{F}(n, s, \Sigma^R(a, s))$ and $P(G) \geq \Pi^R(a, n)^{1-\epsilon}$. Then there exists an induced submultigraph $G' \subseteq G$ on n' vertices such that:

1. $p_{G'}(v) \geq a^{n'-1}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1}{R}-\delta_1\right)(n'-1)}$ for every vertex $v \in V(G')$.

2. $\binom{n}{2} - \binom{n'}{2} \leq \delta\binom{n}{2}$.

By Lemma 8.3.2, $G' \in \mathcal{F}(n', R+1, \Sigma^R(a, R+1))$, for n (and consequently n') sufficiently large. It now suffices to show that $P(G') \geq \Pi^R(a, n')^{1-\epsilon'}$. This would imply that G' is $\frac{\delta}{2}$-close to an element of $\mathcal{P}(\pi_R^a, n')$, and consequently that G differs from an element of $\mathcal{P}(\pi_R^a, n)$ in at most $\delta\binom{n}{2} + \frac{\delta}{2}n'^2 < \delta n^2$ edges. We have $P(G') \geq a^{\binom{n'}{2}}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1}{R}-\delta_1\right)\binom{n'}{2}}$, by the minimum product-degree assumption. Observe that

$$a^{\binom{n'}{2}}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1}{R}\right)\binom{n'}{2}} \leq \Pi^R(a,n') = a^{\binom{n'}{2}}\left(\frac{a+1}{a}\right)^{t_R(n')} \leq a^{\binom{n'}{2}}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1}{R}\right)\left[\binom{n'}{2}+\frac{n'}{2}\right]}. \tag{8.11}$$

It suffices to show that $a^{\binom{n'}{2}}\left(\frac{a+1}{a}\right)^{\left(1-\frac{1}{R}-\delta_1\right)\binom{n'}{2}} \geq \Pi^R(a,n')^{1-\epsilon'}$. Using Equation (8.11), it is enough to show that

$$\left(\frac{a+1}{a}\right)^{\delta_1\binom{n'}{2}+\left(1-\frac{1}{R}\right)\frac{n'}{2}} \leq \left[a\left(\frac{a+1}{a}\right)^{1-\frac{1}{R}}\right]^{\epsilon'\binom{n'}{2}}.$$

For n' sufficiently large, we have $\left(1-\frac{1}{R}\right)\frac{n'}{2} \leq \delta_1\binom{n'}{2}$, and it suffices to show that $\delta_1 \leq \frac{\epsilon' \log\left[a\left(\frac{a+1}{a}\right)^{1-\frac{1}{R}}\right]}{2\log\left(\frac{a+1}{a}\right)}$, which holds by our choice of δ_1. □

Chapter 9

Concluding Remarks

Our primary contribution in this thesis was to resolve Conjecture 2.2.2 for sufficiently large a by proving the optimality of blow-ups of the generalised Turán pattern for the Mubayi-Terry multigraph problem. While we have established the conjecture in the large a regime, it remains open for small values of a, and it would be interesting to obtain a complete resolution of the conjecture.

The generalised Turán pattern can be further extended by considering "iterated" versions of the pattern, as described below.

Definition 9.1 (Iterated Turán pattern)**.** *Given integers $r_1, r_2, \ldots, r_k \in \mathbb{N}$ and $a_1 > a_2 > \ldots > a_k \geq 0$, we define the iterated Turán pattern $\pi_{\mathbf{r}}(\mathbf{a}) := \pi_{r_1,r_2,\ldots,r_k}(a_1, a_2, \ldots a_k)$ as follows. Set $R := \sum_{i=1}^{k} r_i$. For every $v \in [R]$ with $\sum_{i<j} r_i < v \leq \sum_{i \leq j} r_i$ and every v': $v < v' \leq R$, set $\pi_{\mathbf{r}}(\mathbf{a})(v) = a_j$ and $\pi_{\mathbf{r}}(\mathbf{a})(vv') = a_j + 1$.*

Note that the cases $k = 1$ and $k = 2$ of the iterated Turán pattern correspond to the Turán pattern (see Definition 2.1.1) and the generalised Turán pattern (see Definition 2.1.2), respectively, both of which we have shown to be optimal for the Mubayi-Terry problem. Proving the optimality of the iterated Turán pattern for $k \geq 3$ (even for large edge multiplicities) would be a significant generalisation of our results and a major step towards a complete resolution of the Mubayi-Terry problem for general pairs (s, q).

Appendix

Proposition A1. *Let $r, R, d, a \in \mathbb{N}$ with $a \geq d+1$. The following statements are equivalent:*

$$\text{(1)}\quad (a+1)^{r(d-1)(2r-1)+2r}(a-d)^{2r-1}(a-d+1)^{(2r-1)(r-1)} \geq a^{rd(2r-1)+2r},$$

$$\text{(2)}\quad d(2r-1)+2 \geq (2r-1)\frac{1-rx^d_{r,R}(a)}{Rrx^d_{r,R}(a)},$$

$$\text{(3)}\quad \frac{d(2r-1)+2}{R[d(2r-1)+2]+2r-1} \geq \frac{1-rx^d_{r,R}(a)}{R}.$$

Moreover, strict inequality holds in one of these statements if and only if it holds in all.

Proof. The equivalence of (1) and (2) can be shown by observing that

$$\frac{1-rx^d_{r,R}(a)}{Rrx^d_{r,R}(a)} = \frac{r\log\left(\frac{a+1}{a}\right)+\log\left(\frac{a}{a-d}\right)+(r-1)\log\left(\frac{a}{a-d+1}\right)}{r\log\left(\frac{a+1}{a}\right)}.$$

The equivalence of (2) and (3) follows from a simple calculation. □

Proposition A2. *Let $r, R, d, a \in \mathbb{N}$ with $a \geq d+1$. Then*

$$\max_{2\leq t\leq Rdr+R+r} \frac{\Sigma^R_{r,d}(a,t+1)-\Sigma^R_{r,d}(a,t)-1-at}{t} = 1-\frac{d(r-1)+1}{R[d(r-1)+1]+r-1}.$$

Proof. We consider the following cases:

- **Case I**: $t = k(Rd+1)$, where $1 \leq k \leq r-1$.

By Proposition 5.3, we have

$$\frac{\Sigma_{r,d}^R(a,t+1)-\Sigma_{r,d}^R(a,t)-1-at}{t}=1-\frac{dk+1}{k(Rd+1)},$$

and

$$\max_{1\le k\le r-1} 1-\frac{dk+1}{k(Rd+1)}=1-\frac{d(r-1)+1}{Rd(r-1)+r-1}.$$

- **Case II**: $k(Rd+1)+lR+1\le t\le k(Rd+1)+(l+1)R$, where $0\le k\le r-1$ and $0\le l\le d-1$.
  By Proposition 5.3, we have

$$\frac{\Sigma_{r,d}^R(a,t+1)-\Sigma_{r,d}^R(a,t)-1-at}{t}=1-\frac{dk+l+1}{t}\le 1-\frac{dk+l+1}{R[dk+l+1]+k},$$

  and

$$\max_{\substack{0\le k\le r-1\\ 0\le l\le d-1}} 1-\frac{dk+l+1}{R[dk+l+1]+k}=1-\frac{d(r-1)+1}{R[d(r-1)+1]+r-1}.$$

- **Case III**: $r(Rd+1)\le t\le Rdr+R+r-1$.
  By Proposition 5.3, we have

$$\frac{\Sigma_{r,d}^R(a,t+1)-\Sigma_{r,d}^R(a,t)-1-at}{t}=1-\frac{dr+1}{t},$$

  and

$$\max_{r(Rd+1)\le t\le Rdr+R+r-1} 1-\frac{dr+1}{t}=1-\frac{dr+1}{R[dr+1]+r-1}.$$

- **Case IV**: $t=Rdr+R+r$. By Proposition 5.3, we have

$$\frac{\Sigma_{r,d}^R(a,t+1)-\Sigma_{r,d}^R(a,t)-1-at}{t}=1-\frac{dr+2}{Rdr+R+r}.$$

By taking the maximum over all possible cases, the proposition follows. Note that we do not consider Case I when $r=1$. □

Proposition A3. *Let $r, R, d, a \in \mathbb{N}$ with $a \geq d+1$. Furthermore, let $s_0 = Rdr + R + r + 1$ and $s_1 = 2r + R[d(2r-1)+2]$. Then*

$$\max_{s_0 \leq t \leq s_1 - 1} \frac{\Sigma_{r,d}^R(a, t+1) - \Sigma_{r,d}^R(a,t) - 1 - at}{t} = 1 - \frac{d(2r-1)+2}{R[d(2r-1)+2] + 2r - 1}.$$

Proof. We consider the following cases:

- **Case I**: $t = Rdr + R + r + k(Rd+1)$, where $1 \leq k \leq r-1$.
  By Proposition 5.3, we have

$$\frac{\Sigma_{r,d}^R(a, t+1) - \Sigma_{r,d}^R(a,t) - 1 - at}{t} = 1 - \frac{d(k+r)+2}{R[d(k+r)+1] + k + r},$$

  and

$$\max_{1 \leq k \leq r-1} 1 - \frac{d(k+r)+2}{R[d(k+r)+1] + k + r} = 1 - \frac{d(2r-1)+2}{R[d(2r-1)+1] + 2r - 1}.$$

- **Case II**: $Rdr + R + r + k(Rd+1) + lR + 1 \leq t \leq Rdr + R + r + k(Rd+1) + (l+1)R$, where $0 \leq k \leq r-2$ and $0 \leq l \leq d-1$.
  By Proposition 5.3, we have

$$\begin{aligned} \frac{\Sigma_{r,d}^R(a, t+1) - \Sigma_{r,d}^R(a,t) - 1 - at}{t} &= 1 - \frac{d(k+r)+l+2}{t} \\ &\leq 1 - \frac{d(k+r)+l+2}{R[d(k+r)+l+2] + k + r}, \end{aligned}$$

  and

$$\max_{\substack{0 \leq k \leq r-2 \\ 0 \leq l \leq d-1}} 1 - \frac{d(k+r)+l+2}{R[d(k+r)+l+2] + k + r} = 1 - \frac{d(2r-2)+2}{R[d(2r-2)+2] + 2r - 2}.$$

- **Case III**: $Rdr + R + r + (r-1)(Rd+1) + 1 \leq t \leq s_1 - 1$.
  By Proposition 5.3, we have

$$\frac{\Sigma_{r,d}^R(a, t+1) - \Sigma_{r,d}^R(a,t) - 1 - at}{t} = 1 - \frac{d(2r-1)+2}{t},$$

and

$$\max_{\substack{Rdr+R+r+(r-1)(Rd+1)+1\\ \leq t\leq s_1-1}} 1-\frac{d(2r-1)+2}{t} = 1-\frac{d(2r-1)+2}{R[d(2r-1)+2]+2r-1}.$$

By taking the maximum over all possible cases, the proposition follows. Note that we do not consider Cases I and II when $r=1$. □

Proposition A4. *Let $r, d \in \mathbb{N}$. The polynomial inequality given by*

$$(a+1)^{r(d-1)(2r-1)+2r}(a-d)^{2r-1}(a-d+1)^{(2r-1)(r-1)} \geq a^{rd(2r-1)+2r}$$

holds for all a satisfying

$$a \geq d(2r-1)[d(2r-1)+1]+(d-1)(2r-1)(r-1)[r(d-1)(2r-1)+2r].$$

Proof. Bernoulli's inequality states that for all $r \in \mathbb{Z}_{\geq 0}$ and for all $x > -1$, we have

$$(1+x)^r \geq 1+rx.$$

The polynomial inequality in the proposition is equivalent to

$$\left(1+\frac{1}{a}\right)^{r(d-1)(2r-1)+2r}\left(1-\frac{d}{a}\right)^{2r-1}\left(1-\frac{d-1}{a}\right)^{(2r-1)(r-1)} \geq 1.$$

Using Bernoulli's inequality, it suffices to show that

$$(a+r(d-1)(2r-1)+2r)(a-d(2r-1))(a-(2r-1)(r-1)(d-1))-a^3 \geq 0.$$

The left-hand side is a quadratic function of a. The coefficient of a^2 is 1, and the constant term is non-negative. The coefficient of a is

$$-\left(d(2r-1)[d(2r-1)+1]+(d-1)(2r-1)(r-1)[r(d-1)(2r-1)+2r]\right).$$

The proposition follows. □

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