L-functions of Hecke Characters and Cohomology

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by

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Certificate

This is to certify that this dissertation entitled *L*-functions of Hecke Characters and Cohomology towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Manjima Ghosh Hazra at Indian Institute of Science Education and Research(IISER) - Pune under the supervision of Dr. Chandrasheel Bhagwat, Associate Professor, Department of Mathematics , during the academic year 2024-2025.

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To my unwavering pillars of strength - my parents.

Declaration

I hereby declare that the matter embodied in the report entitled L-functions of Hecke Characters and Cohomology are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Chandrasheel Bhagwat and the same has not been submitted elsewhere for any other degree.

Manjima Ghosh Hazra

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Abstract

John Tate in his doctoral dissertation, "Fourier Analysis on Number Fields and Hecke's Zeta Function", established the meromorphic continuation and functional equation of Hecke's Zeta Function over a number field using methods of harmonic analysis on the adèle ring of the number field. The theory in Tate's thesis can be extended to L-functions that are attached to Hecke characters - which are idèle class group characters. In this thesis, we study the necessary background and explore the key concepts to provide a comprehensive exposition of Tate's work. Further, we continue to study Hecke characters - the associated L-functions along with the arithmetic aspects of these L-functions.

Contents

Introduction

It is well known that the Riemann Zeta function, $\zeta(s)$, is defined as the absolutely convergent sum,

$$\zeta(s) = \sum_{n\geqslant 1} \frac{1}{n^s}$$

for $s \in \mathbb{C}$ and $\text{Re}(s) > 1$. This function however, admits a meromorphic continuation to the whole complex plane, except for a simple pole at $s = 1$ and also satisfies a functional equation which can be proved using the Mellin transform(5.1). For a detailed theory on the Riemann Zeta function and other Zeta functions, the reader is advised to refer to [3]. The most famous conjecture which states the existence of all non-trivial zeroes of the Riemann Zeta function within a strip around complex numbers with $\text{Re}(s) = 1/2$ on the complex plane is still unsolved and is an open question.

Over the years, several mathematicians have tried to prove it. A common approach is to look into the generalizations of the Zeta function using Dirichlet series. Two important forms of a Dirichlet series are - (1) $L(s) = \sum_{n\geqslant 1} \chi(n)n^{-s}$; where χ is Dirichlet character and (2) $L(s) = \sum_{n\geqslant 1} \chi(\mathfrak{m})(\mathcal{N}\mathfrak{m})^{-s}$; where $\mathfrak{m}$ are the non-zero ideals and χ is the character of the ideal class group of the number field.

Hecke proved the analytic continuation and functional equation for $L(s, \chi)$ for an idèle class character, χ. However, John Tate made use of the concepts of Fourier analysis on the adèle group to re-prove the same. For a detailed theory on the adèle theoretic concepts, the reader is advised to refer to [3], [1].

The idea behind Tate's work was to choose an appropriate topological ring such as $\mathbb{R}, \mathbb{Q}_p$ or $\mathbb{A}_F$ for a number field F and along with it consider integrals of the form $Z(s) =$

$\int \chi(s)f(x)dx$, where χ is a character of the multiplicative group of the topological ring and $f(x)$ is a nice function on the topological group [1]. The functional equation gives us the duality between (χ, f) and $(\chi^\vee, \hat{f})$ where $\hat{f}$ is the Fourier transform of f and $\chi^\vee$ represents the shifted dual(3.3). These methods can be studied thoroughly and can be used to derive the functional equation for any L-function.(see Chapter 5 for the exposition of Tate's proof for the analytic continuation and functional equation)

André Weil introduced a special type of characters on the idèle class group of a number field [9]. In one of the articles [5], A. Raghuram studied these various notions of Weil's characters, Größencharakter of Hecke characters, cohomological automorphic representation of $GL(1)$ and connected all of them under the theory of Algebraic Hecke Characters. The article mainly studies the arithmetic aspects of the L-functions associated to these characters. This thesis includes the discussion of Hecke characters in the adèle theoretic language along with defining the associated Größencharakter(see Chapter 6). The following section of the thesis includes special values of the Hecke L-functions and mentions Harder's result on the ratio of the critical L-values of such functions [2]. Finally the last part of the thesis discusses how the reciprocity law mentioned by Harder is incorrect for a totally imaginary field and Raghuram's rectification of the reciprocity law by introducing a signature term in it(see Chapter 8).

Chapter 1

Locally Compact Abelian Groups

This chapter includes sections focusing on different concepts. The first section introduces the category of topological groups and its properties. The second section focus on locally compact abelian groups and the third section includes a feature of such groups - the concept of Haar measure.

1.1 Topological Group

Definition 1.1.1. A group G having a topology is called the **topological group** if the following properties are satisfied:

- The group operation $G \times G \mapsto G$ which maps $(g, h) \mapsto gh$ is continuous in the product topology.
- The inversion map $G \mapsto G$ which maps $g \mapsto g^{-1}$ is likewise continuous.

Examples: Here are some examples of topological groups:

1. $\mathbb{R}$ is a topological group with respect to addition.
2. $\mathbb{R}^*$, $\mathbb{C}^*$ are topological groups with respect to multiplication.

3. Any group G with respect to discrete topology is a topological group.

The category of topological groups consists of classes of topological groups having continuous homomorphisms between them and the topology is translation invariant.

Proposition 1.1. (Proposition 1-1, Section 1.1, [1]) Let us denote a topological group as G. Then the following assertions hold:

1. Every neighborhood of the identity, W, contains another neighborhood of the identity, V, such that $VV \subseteq W$. Moreover, every neighborhood of the identity, W, will contain a symmetric neighborhood of the identity, given by V.

2. If H is a subgroup of G, then its closure is also a subgroup. Also, every open subgroup of G is also closed.

3. If K_1 and K_2 are compact subsets of G, then $K_1 K_2$ is also compact.

Proof. The following points provide the proof of the assertions as mentioned in the proposition. This proposition provides very important properties of topological group, regarding its subgroups which can be used further in proofs of other theorems.

1. Let us first prove the first statement of the assertion: We assume that W is open. Then the group operation $\phi : W \times W \to G$ is a continuous map. Therefore $\phi^{-1}(W)$ is open in $W \times W$ containing the point (e, e) where e is the identity of G. Due to the product topology on $W \times W$ we have the neighborhoods V_1, V_2 such that $(e, e) \in V_1 \times V_2 \subseteq W \times W$. Let $V = V_1 \cap V_2$, then V is a neighborhood of e in W such that $VV \subseteq W$.

   Now we will prove the second statement of the assertion: Let $g \in W$ for a $g \in G$. Since $e \in U$ then $gg^{-1} \in W$ and hence $g, g^{-1} \in U$. Therefore, $g \in W \cap W^{-1}$. This shows that $g, g^{-1} \in W \implies g \in W \cap W^{-1}$.

   Similarly, let $g \in W \cap W^{-1}$. Then $g, g^{-1} \in W$ and $gg^{-1} \in W$. This implies that $g \in W \cap W^{-1} \implies g, g^{-1} \in W$. Therefore, $g, g^{-1} \in W \iff g \in W \cap W^{-1}$.

   Hence, $V = W \cap W^{-1}$ is the required symmetric neighborhood of e.

2. Let two convergent nets $\{g_n\}$ and $\{h_n\}$ in H have their limits g and h respectively in $\overline{H}$. Thus the limit of $\{g_n h_n\}$ in H will have their limit gh in $\overline{H}$. Moreover, $\{g_n^{-1}\}$ and

$\{h_n^{-1}\}$ in H will have their limits g^{-1} and h^{-1} respectively in $\overline{H}$. Therefore, we have that $\overline{H}$ is also a subgroup of G.

H being a subgroup of G, the group G is given by a disjoint union of cosets of H. Moreoever, H is also the complement of union of these cosets. If H is open then that implies these translates are open. However since H is also the complement of the union of these open translates, it is closed.

3. K_1, K_2 are compact subsets of G. Therefore, $K_1 K_2$ is the image under the continuous map from compact set $K_1 \times K_2$, $(k_1, k_2) \mapsto k_1 k_2$. Thus image of a compact set under the continuous map is compact and hence $K_1 K_2$ is compact.

□

1.2 Locally Compact Abelian Group

Definition 1.2.1. A abelian topological group G that is both locally compact and Hausdorff is called a **locally compact abelian group**.

Examples: Here are some examples of locally compact abelian groups:

1. $\mathbb{R}$, $\mathbb{C}$ are locally compact abelian groups with respect to addition.

2. The field of p-adic numbers, $\mathbb{Q}_p$, is a locally compact abelian group with respect to addition.

3. The adèle ring of a number field F, $\mathbb{A}_F$ is a locally compact abelian group with respect to addition.

Proposition 1.2. (Proposition 1-6, Section 1.1, [1]) Let G be a topological group which is also Hausdorff. Then a locally compact subgroup H of G is closed. In particular, every discrete subgroup of G is closed.

Proof. This proposition states a important property about the subgroups of a locally compact topological group. The proof goes about by considering a compact neighborhood of identity inside the subgroup mentioned in the proposition and then using the Hausdorff property of the group along with some assertions proved in Proposition 1.1.

Let U be a compact neighborhood of e in the subgroup H. Since H is Hausdorff and U is closed in H, then there exists a neighborhood of the identity W which is closed in G such that $U = W \cap H$. Since $W \cap H$ is compact in H, therefore is compact in G and hence closed.

By Proposition 1.1, part (1), we have a neighborhood V of e in G such that $VV \subseteq W$. Also, $\bar{H}$ is a subgroup of G by Proposition 1.1, part (3). Hence, for $x \in \bar{H}$, every neighborhood of x^{-1} must intersect with H. This implies that there exists some $y \in Vx^{-1} \cap H$.

We claim that $yx \in W \cap H$. Let K be a neighborhood of yx which meets $W \cap H$. Now, $y^{-1}K$ is a neighborhood of x which implies that $y^{-1}K \cap xV$ is a neighborhood of x. Also, $x \in \bar{H}$ hence, there exists some $z \in y^{-1}K \cap xV \cap H$. This gives us that $yz \in K \cap H$. $y \in Vx^{-1}$ and $z \in xV$ are given by construction.

Hence, $yz \in Vx^{-1}xV \implies yz \in VV \subseteq W$. Thus the intersection $K \cap (W \cap H)$ is non-empty. This proves the claim and completes the proof. □

1.3 Haar Measure

A collection $\mathscr{M}$ of a set X is called **σ-algebra** if the following conditions hold:

1. $X \in \mathscr{M}$
2. if $A \in \mathscr{M}$ then the complement $A^c \in \mathscr{M}$.
3. Let $A = \bigcup_{n=1}^{\infty} A_n$ for $A_n \in \mathscr{M}$ for $n \geqslant 1$. Then $A \in \mathscr{M}$.

A set X together with a σ-algebra $\mathscr{M}$ is called a **measurable space**. In case of a topological space X the smallest σ-algebra, $\mathfrak{B}$, is the collection of all open subsets of X. These sets are called the Borel subsets of X.

A **positive measure** μ on a measurable space $(X, \mathscr{M})$ is a function $\mu : \mathscr{M} \to \mathbb{R}_+ \cup \{\infty\}$ that is countably additive, that is $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for any family of disjoint sets

$\{A_n\}$ in $\mathscr{M}$. A positive measure defined on the Borel subsets of a locally compact Hausdorff space X is called the **Borel measure**.

For a Borel measure μ and a Borel subset E we have the following the properties:

- μ is called **outer regular** on E if $\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open}\}$
- μ is called **inner regular** on E if $\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\}$

We define a **Radon measure** on X to be a Borel measure that is finite on compact sets, outer regular property on all Borel sets, and inner regular property on all open sets.

Definition 1.3.1. Let G be a locally compact topological group with μ a Borel measure on G. We say that μ is left translation invariant(similarly right) if for all Borel subsets E of G, $\mu(sE) = \mu(E)$ for all $s \in G$. Then a left or right **Haar measure** on G is a Radon measure μ on G that is left or right translation invariant.

A locally compact abelian group G admits a left(hence right) Haar measure on it which is unique upto scalar multiplication. To see the proof of this we can refer to Theorem 1-8, Section1.2, [1].

Examples: Here are some examples of Haar measure:

1. The Haar measure μ on the unit circle, S^1, is given by a function $f : [0, 2\pi] \to S^1$ given by $f(t) = (\cos(t), \sin(t))$ and $\mu(S) = (1/2\pi)m(f^{-1}(S))$ where m is the Lebesgue measure on $[0, 2\pi]$.
2. For a discrete group G, the Haar measure on G is the counting measure.
3. For the group of non-zero reals with multiplication operation, G, the Haar measure for a Borel subset S of non-zero reals is given by $\mu(S) = \int_S \frac{1}{|x|} dx$.

Chapter 2

Fourier Analysis and Pontryagin Duality

For a locally compact abelian group G, the group of characters is denoted by $\hat{G}$. In this chapter we focus on the characters of G, the structure of the character group $\hat{G}$ and the final section provides the details of the main result called the Pontryagin Duality.

Isomorphism between topological groups preserve both the algebraic as well as the topological structure of the groups. Thus for an isomorphism between two topological groups, the map needs to be bi-continuous.

2.1 Pontryagin Dual

Definition 2.1.1. The continuous homomorphisms from a locally compact group G to the multiplicative group $\mathbb{C}^*$ are called the **characters** of G. If we restrict the co-domain of the map to S^1 then the continuous homomorphisms are called the **unitary characters** of the group G.

Definition 2.1.2. The unitary characters of a locally compact group G make a group under multiplication called the **Pontryagin Dual** $\hat{G}$ of the group G.

Examples: Here are some examples of Pontryagin Dual of a locally compact group G:

1. $\mathbb{R}$, $\mathbb{Q}_p$ - the field of p-adic numbers, the adèle ring of a number field F, $\mathbb{A}_F$, - all these are Pontryagin duals of themselves.

2. The Pontryagin dual of $\mathbb{Z}$ is $\mathbb{R}/\mathbb{Z}$.

The topology on $\hat{G}$ is given by the sets of the form $W(K,V) := \{\chi \in \hat{G} : \chi(K) \subseteq V\}$ for a compact subset $K \subset G$ and an open subset of $V \subset S^1$. This forms the subbase for the topology, named as the **compact-open topology**.

Let us consider the map $\phi : \mathbb{R} \to S^1$ given by $x \mapsto e^{2\pi i x}$. For a real number x contained in $(0,1]$, we define $N(x) = (-x/3, x/3) \in S^1$ as the image under the map ϕ.

Theorem 2.1. (Proposition 3-2, Section 3.1, [1]) Let G be a topological group(abelian). Then the following properties are satisfied:

1. G is discrete implies that $\hat{G}$ is compact.

2. G is compact implies that $\hat{G}$ is discrete.

Proof. This theorem provides a very important result regarding the relation between topological groups and their character groups.

1. Since G is discrete hence the the maps from G to S^1 are continuous and hence a unitary character and we have $\hat{G} = \mathrm{Hom}(G, S^1)$. The compact sets in G are finite sets. Therefore, the compact-open topology on $\hat{G}$ becomes the topology of pointwise convergence. Now, $\mathrm{Hom}(G, S^1) = \hat{G}$ is a closed subset of the space of a compact space(all the maps from $G \to S^1$) which makes it compact.

2. Let us consider the non-trivial unitary character, χ of G. Now we have that $\chi(G)$ is a subgroup of S^1 which is not contained in any set of the form $N(x)$. Therefore, $W(G, N(1))$ is an open in $\hat{G}$, given that G is compact and contains the trivial character only. Thus the singleton set with the trivial character $\{\chi\}$ is open in $\hat{G}$, proving that $\hat{G}$ is discrete.

□

2.2 Fourier Transform and Fourier Inversion Formula

Let us consider a function $f \in L^1(G)$. The **Fourier transform**, $\hat{f} : \hat{G} \to \mathbb{C}$, is defined as

$$\hat{f}(\chi) = \int_G f(y)\overline{\chi}(y)dy \tag{2.1}$$

for $\chi \in \hat{G}$.

Let $V^1(G) = V(G) \cap L^1(G)$ where $V(G)$ spans functions of positive type. A function of positive type is defined as $f \in \mathscr{C}_c(G)$ for which there is a function $\Phi : G \to \mathbb{C}$ in $L_\infty(G)$ such that $\int\int \Phi(x^{-1}y)f(x)dx\overline{f(y)}dt \geqslant 0$. For all such notions we consider G to a locally compact topological group.

There exists a Haar measure $d\chi$ for $\hat{G}$ such that for a function $f \in V^1(G)$ the **Fourier inversion** formula is given by

$$f(y) = \int_{\hat{G}} \hat{f}(\chi)\chi(y)d\chi \tag{2.2}$$

The measure $d\chi$ on $\hat{G}$ is the dual measure to dy of G.

2.3 Pontryagin Duality

Theorem 2.2. (Theorem 3-20, Section 3.4, [1]) The groups G and $\hat{G}$ are mutually dual with the map $\Psi : G \to \hat{\hat{G}}$ as an isomorphism of topological groups.

Proof. We provide an outline of the proof here. Firstly we show that the map Ψ is injective. Next we prove that it is a homeomorphism and lastly that the image of Ψ is dense in $\hat{\hat{G}}$. This will prove the isomorphism as stated in the theorem. The Pontryagin duality theorem establishes a duality between locally compact groups which helps in generalizing the Fourier transform formula for all such locally compact groups. The Fourier inversion theorem is a special case of this theorem.

To show that the map Ψ is injective, it is enough to show that for $y \neq e, y \in G$ there exists a unitary character χ such that $\chi(y) \neq 1$. Suppose such character does not exist.

Define $L_y f(x) := f(yx)$. Then $\hat{f} - \hat{L_y} f \forall \hat{f} \in L^1(G)$. We have that $f = L_y f \forall f \in \mathfrak{B}^1(G)$ using the inversion formula. We have an open neighborhood U of identity such that $U \cap yU = \varnothing$ and a non-zero function of positive type with the support in U. Since $U \cap yU = \varnothing$, it is not possible to find a function f such that $f = L_y f$. This shows that there is such a unitary character and the map Ψ is injective.

Let us consider a compact subset $\hat{K} \subset \hat{G}$, $\hat{K}$ is a neighborhood of identity and an open subset $V \subset S^1$. Now the sets $W(\hat{K}, V) = \{\alpha \in \hat{\hat{G}} : \alpha(\hat{K}) \subseteq V\}$ forms the subbase for the topology on $\hat{\hat{G}}$.

Let the set of elements of $W(\hat{K}, V)$ which arise from G through Ψ be given by $W(\hat{K}, V) \cap \Psi(G)$. Let $W_G(\hat{K}, V) = \{z \in G : \Psi(z)\chi \in V, \chi \in \hat{K}\}$ be a subset of G. This gives us the identity as $\Psi(W_G(\hat{K}, V)) = W(\hat{K}, V) \cap \Psi(G)$. Therefore, Ψ is a homeomorphism.

Now $\Psi(G)$ is locally compact being the homeomorphic image of locally compact G. $\Psi(G)$ is an open subgroup in the closure of $\hat{\hat{G}}$. Since it is an open subgroup of a topological group, hence it is equal to the closure.

We need to show that $\Psi(G)$ is dense in the double dual. Consider a function $\hat{\phi} \in L^1(\hat{G})$ such that $\hat{\phi}$ is non-zero and vanishes on $\Psi(G)$. Let $\chi_0 \in \hat{\hat{G}}$. Then,

$$\hat{\phi}(\hat{\chi_0}) = \int \phi(\chi)\chi_0(\chi^{-1})d\chi$$

Since $\hat{\phi}$ vanishes on $\Psi(G)$ we have that

$$\int \phi(\chi)\chi(z^{-1})d\chi = 0$$

for all $z \in G$. Using Plancherel's theorem we have that $\phi = 0$ almost everywhere hence $\hat{\phi} = 0$ which contradicts our assumption. This completes the proof. □

Chapter 3

Local Fields

This chapter will mainly focus on the characterization of the underlying additive and multiplicative groups of a local field and how the measure is defined on these groups. We start with a brief introduction to the notion of local fields. In the last section we study about the L-factors associated to local fields.

A field is called **local field** if it is complete with respect to a metric induced by a discrete valuation and its residue field is finite. If F_v is a field with respect to an absolute value $| \, . \, |_v$ with $\mathcal{O}_F$ is the valuation ring and π_F is the uniformizer with the residue field as $\mathcal{O}_F/\pi_F\mathcal{O}_F$.

A local field is a locally compact topological field with respect to the non discrete topology. These local fields arise as completions of global fields - if F is a global field then the completion F_v with respect to the absolute value $| \, . \, |_v$ is a local field.

Theorem 3.1. (Theorem 9.9, [6]) Let F_v be a local field. If F_v is Archimedean then it is isomorphic to $\mathbb{R}$ or $\mathbb{C}$; otherwise, F_v is isomorphic to a finite extension of $\mathbb{Q}_{\mathfrak{p}}$ or $\mathbb{F}_{\mathfrak{q}}((t))$.

Proof. Let F_v be a local field with respect to a non-trivial absolute value say $||$ and it is complete with respect to that absolute value. If F_v has characteristic zero then the prime field is $\mathbb{Q}$ and F_v will contain the completion of $\mathbb{Q}$ from the restriction of $||$. By Ostrowski's theorem, the restriction can be equivalent to the standard archimedean absolute value where the completion is $\mathbb{R}$ or the restriction can be $\mathfrak{p}$-adic absolute value where the completion is $\mathbb{Q}_{\mathfrak{p}}$.

If F_v has positive characteristic $\mathfrak{p}$, then the prime field is $\mathbb{F}_{\mathfrak{p}}$ and F_v contains a transcendental element, t'. This gives us that F_v contains $\mathbb{F}_{\mathfrak{p}}(t')$ and also the completion of $\mathbb{F}_{\mathfrak{p}}(t')$ with respect to $||$. Every completion of $\mathbb{F}_{\mathfrak{p}}(t')$ is isomorphic to $\mathbb{F}_{\mathfrak{q}}((t))$ for some $\mathfrak{q}$ a power of $\mathfrak{p}$ and t transcendental over $\mathbb{F}_{\mathfrak{q}}$. Thus F_v contains a subfield isomorphic to $\mathbb{F}_{\mathfrak{q}}((t))$. □

3.1 Characters and Measure on the Additive Group

3.1.1 Characters on the Additive Group of a Local Field

The additive group of F_v is denoted by F_v^+. Let the non-trivial unitary character of F_v be χ. Then for any general element x of F_v^+ the left translate of χ is given by

$$L_y(\chi(x)) = \chi(yx) \text{ for } y \in F_v^+.$$

Theorem 3.2. (Lemma 2.2.1, Section 2.2, [7]) The character χ and its left translates constitute the unitary characters of F_v^+. Moreover, the map

$$\Psi : F_v^+ \to \hat{F_v^+}$$

is an isomorphism of topological groups.

Proof. The proof of this theorem is given in a step-by-step manner. Firstly we prove that Ψ is a homomorphism between the additive group and its character group. Next we prove that it is an injective map. We finally prove the isomorphism after showing that Ψ is bi-continuous and its image is whole of the character group.

The map $L_y(\chi(x))$ is given by $L_y(\chi(x)) : x \to yx \to \chi(yx)$ which is composition of a multiplication map and the unitary character. Both of which are continuous which makes the composition $L_y(\chi(x))$ is continuous.

Now, $\Psi(L_{y_1+y_2}(\chi(x))) = \chi((y_1 + y_2)x) = \chi(y_1x)\chi(y_2x) = \Psi(L_{y_1}(\chi(x)))\Psi(L_{y_2}(\chi(x)))$. Therefore, the map Ψ is a homomorphism from F_v^+ to $\hat{F_v^+}$.

Let us consider that y is in the kernel of Ψ. Therefore, $\chi(yx) = 1$ for all $x \in F_v^+$. Since multiplication by y gives the automorphism on F_v^+ and χ is assumed to be the non-trivial unitary character therefore, therefore $y = 0$ since it is in the kernel. Therefore, Ψ is a injective map.

Let B be the compact set of all $x \in F_v^+$ with $| x | \leqslant M$ for a large enough M. y is close to 0 in F_v^+ which implies yB is close to 0 in F_v^+. Therefore, $\chi(yB)$ is close to the identity character in $\hat{F_V^+}$. Now fix an element y_0 in F_v^+ such that $\chi(y_0) \neq 1$ then $\chi(yB)$ is closer to 1 than $\chi(y_0)$ which implies that $y_0 \notin yB$. Therefore, y is close to 0 in F_v^+. Thus the map Ψ is bi-continuous.

We have a locally compact subgroup of F_v^+ which is $\mathrm{Im}(\Psi)$. Hence, $\mathrm{Im}(\Psi)$ is a closed in $\hat{F_v^+}$. If it is a proper subgroup then there exists a non-zero $x \in F_v^+$ such that image of x is trivial, that is $\chi(yx) = 1$ for all $y \in F_v^+$. Multiplication by x is an automorphism on F_v^+ hence χ is trivial on F_v^+. This is a contradiction. Thus $\mathrm{Im}(\Psi) = \hat{F_v}^+$.

This proves that the map Ψ is an isomorphism. □

Let us now focus on constructing a special non-trivial character of F_v^+. Let p be the rational prime divisor which $\mathfrak{p}$ divides, and R be the completion of the rational field at p.

Define a map $x \to \lambda(x)$ of R into the reals mod 1 as:

1. p is Archimedean, and therefore R is the real numbers. Then, $\lambda(x) = -x \pmod 1$.

2. p is discrete, R is the field of p-adic numbers then. $\lambda(x)$ is defined as follows:

   - $\lambda(x) \in \mathbb{Q}$ with only p powers in the denominator.
   - $\lambda(x) - x$ is a p-adic integer.

We define the map $\Lambda(x) = \lambda(Tr_{F|R}(x))$ and hence $x \to e^{2\pi i \Lambda(yx)}$ becomes a non-trivial character F_v^+. F_v^+ is naturally its own character group if we identify the character $x \to e^{2\pi i \Lambda(yx)}$ with $y \in F_v^+$.

3.1.2 Measure on the Additive Group of a Local Field

F_v^+ is a locally compact abelian group therefore we have that there exists a Haar measure on it unique upto scalar multiplication.

Lemma 3.1. (Lemma 2.2.4, Section 2.2, [7]) Let $\alpha \neq 0$ and $\mathcal{M}$ be a measurable set of F_v^+. Then $\mu'(\mathcal{M}) = \mu(\alpha\mathcal{M})$ is also a Haar measure on F_v^+.

Proof. The map $x \mapsto \alpha x$ for a $x \in F_v^+$ is an algebraic as well as a topological automorphism of F_v^+. If $\mathcal{M}$ is compact in F_v^+ then the image $\alpha\mathcal{M}$ is also compact and finiteness of $\mu(\alpha\mathcal{M})$ gives finiteness of $\mu'(\mathcal{M})$.

For an open measurable subset $\mathcal{M}$ of F_v^+, we have that $\alpha\mathcal{M}$ is also open measurable. For a compact subset $K \subseteq \mathcal{M}$, we have a compact subset $\alpha K \subseteq \alpha\mathcal{M}$. Conversely, for a compact subset $K' \subseteq \alpha\mathcal{M}$, $\alpha^{-1}K'$ is a compact subset of $\mathcal{M}$. Thus $\mu(\alpha\mathcal{M}) = \sup\{\mu(\alpha K) : K \subseteq \mathcal{M}\}$ for K compact. This proves the inner regularity. Same argument works for proving the outer regularity. Now $\mu'(\mathcal{M} + x) = \mu(\alpha\mathcal{M} + \alpha x) = \mu(\alpha\mathcal{M}) = \mu'(\mathcal{M})$ for $x \in F_v^+$. This proves the translation invariance. □

Now we have that μ' and μ both are Haar measures on F_v^+ , then μ' must be a scalar multiple of μ by uniqueness. This scalar is given by $\mu'(M) = \mu(\alpha M) =\mid \alpha \mid \mu(M)$. In case of integration we can use $d\mu(\alpha x) =\mid \alpha \mid dx$. Let us fix a Haar measure dx for F_v^+. Then from our result we can choose a Haar measure which is its own dual when the character group of F_v^+ is F_v^+ itself. This measure is given as follows:

- if $F_v = \mathbb{R}$ then dx is the Lebesgue measure.
- if $F_v = \mathbb{C}$ then dx is twice the Lebesgue measure.
- if F_v is non-archimedean then dx is the measure for which $\mathcal{O}_v$ gets the measure $(\mathcal{N}\mathcal{D})^{-1/2}$ where $\mathcal{D}$ is the different of F.

A character of F_v^+ is of the form $x \to e^{2\pi i\Lambda(yx)}$and identified with $y \in F_v^+$. If the self dual measure is $d\mu(x)$ written as dx the the Fourier transform and the Fourier inversion formula is given by:

$$\hat{f}(y) = \int f(x)e^{-2\pi i\Lambda(yx)}dx \tag{3.1}$$

$$f(x) = \int \hat{f}(y) e^{2\pi i \Lambda(yx)} dy = \int \hat{f}(y) e^{-2\pi i \Lambda(-yx)} dy = \hat{\hat{f}}(-x). \tag{3.2}$$

3.2 Characters and Measure on the Multiplicative Group

3.2.1 Characters on the Multiplicative Group of a Local Field

Let F_v^* be the multiplicative group of F_v and let x be the general element of this group. Let χ be a character of F_v^* which is trivial on $\mathcal{O}_F^*$. The value of $\chi(x)$ depends on the absolute value of x. Let us look into the unramified characters of F_v^* first.

Lemma 3.2. (Lemma 2.3.1, Section 2.3, [7]) The unramified characters of F_v^* looks like $\chi(x) = \mid x \mid^s$ for some $s \in \mathbb{C}$. For an archimedean place, s is determined by χ, otherwise it is determined by $2\pi i / log(\mathcal{N}v)$.

Proof. Let χ be an unramified character of F_v^* the value of which depends on the absolute value of an element x in the group. The set of the absolute values of all elements form the value group of F_v^*.

Define the map $\psi(\mid x \mid) = \chi(x)$ where ψ is a homomorphism with respect to multiplication and it is continuous. hence ψ is a character given by $(\mathbb{R}, .)$ in case of archimedean place and $(\mathcal{N}v^{\mathbb{Z}}, .)$ in case of non-archimedean places.

For archimedean places, ψ has the form $\mid x \mid \mapsto \mid x \mid^s$ for any $s \in \mathbb{C}$. Two different s_1, s_2 gives two different characters and hence s is determined by χ.

For a non-archimedean place, we can find an isomorphism given by $\phi : (\mathcal{N}v^{\mathbb{Z}}, .) \to (\mathbb{Z}, +)$ that maps $\mathcal{N}v^m \mapsto m$. The characters of $\mathbb{Z}$ are of the form z^m where $z \in \mathbb{C}$. z can be written as $re^{i\theta}$ where r and e^{θ} can written as $\mathcal{N}v^a$ and $\mathcal{N}v^b$ for some $a, b \in \mathbb{R}$. Then we have $z = re^{i\theta} = \mathcal{N}v^a \mathcal{N}v^b = \mathcal{N}v^s; s = a + ib \in \mathbb{C}$. The characters of $(\mathcal{N}v^{\mathbb{Z}}, .)$ corresponding to $m \mapsto z^m$ are given by $\mathcal{N}v^m \mapsto (\mathcal{N}v^m)^s$. If s_1, s_2 gives the same character then $\mathcal{N}^{s_1 - s_2} = 1$ and hence $e^{(s_1 - s_2) log \mathcal{N}v} = 1$. Thus $(s_1 - s_2) log \mathcal{N}v = 0$ giving $s_1 \equiv s_2 \pmod{2\pi i \mathcal{N}v}$ and hence s is determined by $2\pi i / log(\mathcal{N}v)$. □

For an archimedean place, an element x of F_v^* can be written uniquely as $x'u$ where

$x' \in \mathcal{O}_F^*$ and $u \in \mathbb{R}$. The map $x \mapsto x'$ is given by $x \mapsto x/\mid x \mid$ and is a homomorphism. For a non-archimedean place, an element x of F_v^* can be written uniquely as $x'u$ where $x' \in \mathcal{O}_F^*$ and u is power of the uniformizer π. The map $x \mapsto x'$ is given by $x \mapsto x\pi^{-\nu_v(x)}$ where $\nu_v(x)$ is the valuation of x and is a homomorphism. The characters of F_v^* take the form of $\chi(x) \mapsto \chi'(x') \mid x \mid^s$ where χ' is an unitary character of $\mathcal{O}_F^*$ and s is determined as in the previous lemma 3.2.

3.2.2 Measure on the Multiplicative Group of a Local Field

We can choose a Haar measure $d\alpha$ on F_v^* by relating it to the measure dx on F_v^+. If we have $f(\alpha) \in L^1(F_v^*)$ then $f(x)|x|^{-1} \in L^1(F_v^+ - 0)$. We define the functional on $L^1(F_v^*)$,

$$\Phi(f) = \int_{F_v^+ - 0} f(x)|x|^{-1} dx \tag{3.3}$$

Now if $g(\alpha) = f(\beta\alpha)$ then

$$\Phi(g) = \int_{F_v^+ - 0} f(\beta x)|x|^{-1} dx = \Phi(f) \tag{3.4}$$

by substituting x by $\beta^{-1}x$. Thus Φ is translation invariant and must come from a Haar measure on F_v^*. Let this measure be $d'\alpha$ and hence

$$\int f(\alpha) d'\alpha = \int_{F_v^+ - 0} f(x)|x|^{-1} dx \tag{3.5}$$

This gives us a 1-1 correspondence between $L^1(F_v^*)$ and $L^1(F_v^+ - 0)$ as $f(\alpha) \mapsto f(x)|x|^{-1}$. Therefore, we choose our standard Haar measure to be $d\alpha = d'\alpha = d\alpha/|\alpha|$ for v archimedean and $d\alpha = \mathcal{N}v/(\mathcal{N}v - 1)d'\alpha = \mathcal{N}v/(\mathcal{N}v - 1)d\alpha/|\alpha|$ for v non-archimedean. In the non-archimedean case we have,

$$\int_{\mathcal{O}_F^*} d\alpha = (\mathcal{N}\mathfrak{d})^{-1/2} \tag{3.6}$$

3.3 L-factors for a Local Field

Let us consider the local field, F_v, with absolute value $|\ .\ |$ and Haar measure dx. We define $d^*x = c.\frac{dx}{|x|}$ for a real number $c > 0$ to be the Haar measure on F_v^*. For an Archimedean field F_v, the scalar is normalized as $c = 1$. For a non-Archimedean field, F_v let us define the following: $\mathcal{O}_F$ the ring of integers, $\mathfrak{p}_F$ the maximal ideal, π_F the uniformizer and F_q the residue field, where $q = \#\mathcal{N}\mathfrak{p}$.

F_v^* is the direct product U_F x V_F, where U_F is the unit group and V_F is the valuation group. For Archimedean fields, V_F is $\mathbb{R}_+^*$ whereas for non-Archimedean fields it is $\mathcal{N}\mathfrak{p}^{\mathbb{Z}}$.

Let $X(F_v) = \mathrm{Hom}(F_v^*, \mathbb{C}^*)$ be the space of continuous group homomorphisms from F_v^* to $\mathbb{C}^*$. The elements $\chi \in X(F_v)$ are called the quasi-characters of F_v^*. Characters in the co-domain S^1 are called **unitary characters**.

Every χ factors into

$$\chi = \mu\ |\ .\ |^s$$

for some $s \in \mathbb{C}$. μ is the restriction of χ on $U_F \subseteq F$ and characters of V_F are of the form $t \mapsto t^s$ for $s \in \mathbb{C}$

We define the real part of s that is $\sigma = \mathrm{Re}(s)$ as the **exponent** of χ.

Consider an arbitrary character χ of F_v^*. We will study the local L-factors for the following cases.

We call the character to unramified if $\chi|_{U_F} = 1$. In case of non-Archimedean F_v, we define the local L-factor to be

$$L(\chi) = \begin{cases} (1 - \chi(\pi_F))^{-1} & \text{unramified} \\ 1 & \text{otherwise} \end{cases} \tag{3.7}$$

In case of $F_v = \mathbb{C}$ we have $U_F = S^1$ and hence χ takes the form,

$$\chi_{s,n} : re^{i\theta} \mapsto r^s e^{in\theta}$$

for some $s \in \mathbb{C}$ and $n \in \mathbb{Z}$. Every character takes the form $\chi = \mu|.|^s$ for some $s \in \mathbb{C}$ and μ is the unitary character(restriction on U_F) and character of V_F are of the form $t \mapsto t^s$. The polar form of the complex number $re^{i\theta}$ includes the absolute value r(radial component) and the argument of the complex number θ(angular component). Under the map $\chi_{s,n}$, we get the image of $re^{i\theta}$ as $r^s e^{in\theta}$. The argument of the complex number gives us that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and these points define a unit circle on the complex pane. Since both $\cos(\theta)$ and $\sin(\theta)$ are periodic functions(with period $2k\pi$) hence the complex numbers $e^{in\theta}$ will again lie on the unit circle and hence belongs to U_F. The radial component, r, gets magnified to r^s and hence belongs to V_F. Therefore, $\chi_{s,n}$ also factors into an element of U_F and V_F.

Then the local L-factor is defined as,

$$L(\chi_{s,n}) = \Gamma_{\mathbb{C}}(s + \tfrac{|n|}{2}) = (2\pi)^{-(s+\frac{|n|}{2})}\Gamma(s + \tfrac{|n|}{2})$$

where

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds \tag{3.8}$$

and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

In case of $F_v = \mathbb{R}$ we have $U_F = \{\pm 1\}$. Let the sign character, sgn, be given by $x \mapsto \frac{x}{|x|}$ The local L-factor is then defined as,

$$L(\chi) = \begin{cases} \Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}\Gamma(\frac{s}{2})} & \mu = 1 \\ \Gamma_{\mathbb{R}}(s+1) & \mu = \text{sgn} \end{cases}$$

Since χ is a character of F_v^* then $\chi \mid . \mid^s$ is also a character of F_v^* and hence $L(s, \chi)$ for $L(\chi \mid . \mid^s)$.

The **shifted dual** of χ is given by $\chi^\vee = \chi^{-1} \mid . \mid$ so that $L(\widetilde{\chi \mid . \mid^s}) = L(1 - s, \chi^{-1})$. The calculation goes like this: since $\chi = \mu \mid . \mid^s$ and $\chi^{-1} = \mu^{-1} \mid . \mid^{-s}$, then $\chi \mid . \mid^s = \mu \mid . \mid^{2s}$ and $\widetilde{\chi \mid . \mid^s} = \mu^{-1} \mid . \mid^{1-2s} = \mu^{-1} \mid . \mid^{-s} \mid . \mid^{1-s}$. Thus $L(\widetilde{\chi \mid . \mid^s}) = L(\mu^{-1} \mid . \mid^{-s} \mid . \mid^{1-s}) = L(1 - s, \chi^{-1})$.

Chapter 4

Global Fields

For performing harmonic analysis on a global field F, we introduce two locally compact abelian groups - $\mathbb{A}_F$, the adèle group, and $\mathbb{I}_F$ the idèle group. The adèle group is moreover is a topological ring defined as the **restricted direct product** of the additive groups of all the completions F_v. The notion of restricted direct product is also used when the adèle ring is replaced by the multiplicative group, $\mathbb{I}_F$, the idèle group. In this chapter we mostly focus on the concepts of restricted direct products - with the adèle ring and the idèle group as its example and their properties. We first start with a brief introduction of global fields.

Global fields are of two types - finite extension of $\mathbb{Q}$ called the number field and finite extensions of $\mathbb{F}_q(t)$ for some prime q called the function field. This whole study focuses only on the number field case.

Let us denote a number field by F. The completion of F at a valuation v, is given by F_v which is a local field. The valuation ring of F_v is given by $\mathcal{O}_F$.

4.1 Restricted Direct Product

Let us fixed a set of indices $J = \{v\}$ and let J_∞ be a fixed subset of it. For each v we define a locally compact group G_v and for all $v \notin J_\infty$ we define compact and open subgroup H_v of G_v. The **restricted direct product** G of G_v with respect to H_v is given by

$$G := \prod'_{v\in J} G_v \;= \{(x_v) : x_v \in G_v \text{ with } x_v \in H_v \text{ for all but finitely many } v\}$$

To define a topology on G we define a subbase which is a neighborhood around identity in the form of $\prod_v N_v$ where N_v is an open neighborhood around 1 in G_v with $N_v = H_v$ for all but finitely many v. Let S be a set of finite indices for which H_v is not defined. Let G_S be a subgroup of G such that it contains all $x \in G$ such that $x_v \in H_v$. It is given by $G_S = \prod_{v\in S} G_v \times \prod_{v\notin S} H_v$. Then G_S is a locally compact group with respect to product topology. Every element of G is contained in some set of G_S for some S. Thus the topology on G is defined as a system of neighborhoods of 1 in G_S and neighborhoods of 1 in G.

4.1.1 Characters on G

Theorem 4.1. (Lemma 5-2,5-3, Section 5.1, [1]) Let $\chi \in \hat{G}$, trivial on all but finitely many H_v. Then we can write χ as $\chi(y) = \prod \chi_v(y_v)$ where $\chi_v = \chi|_{G_v}$. Conversely, given $\chi_v \in \hat{G}_v$ with $\chi|_{H_v} = 1$ for all but finitely many v, we get a character $\chi(y) = \prod \chi_v(y_v)$ in G.

Proof. Choose an open neighborhood U of 1 in the multiplicative group $\mathbb{C}^*$ such that U only the trivial subgroup. $\chi^{-1}(U)$ is open as χ is continuous. Thus we have an open neighborhood $N = \prod N_v$ of the identity such that $\chi(N) \subseteq U$ with $N_v = H_v$ for all v outside the finite set S. For a $v \notin S$, consider the subgroup N as $H_v = \{(\cdots, 1, x, 1, \cdots) : x \in H_v\}$. $\chi(H_v)$ is a subgroup of U and hence trivial. This is true for any $H_v; v \notin S$ and thus χ is trivial on all but finitely many H_v.

To prove the converse statement, consider the finite set of indices S such that $\chi_v|_{H_v} = 1$ for $v \notin S$. Since $\mathbb{C}^*$ is a topological group, we can find an open neighborhood V such that $V^{(m)} \subseteq U$. For each v, we have an open neighborhood N_v of identity in each G_v such that $\chi_v(N_v) \subseteq V$. Then the set $\prod_{v\in S} N_v \times \prod_{v\notin S} H_v$ is an open neighborhood of identity in G. This proves the continuity. ∎

4.1.2 Measure on G

Theorem 4.2. (Proposition 5-5, Section 5.1, [1]) Let us consider G and dg_v be the Haar measure on G_v such that they are normalized on H_v for all but finitely many v,

$$\int_{H_v} dg_v = 1$$

. There exists a unique Haar measure dg on G such that for every finite set of indices S containing J_∞, the restriction dg_S of dg to $G_S = \prod_{v\in S} G_v \times \prod_{v\notin S} H_v$ is the product measure.

Proof. Let S be a finite set of indices containing J_∞. The measure dg_S is the product of the measures dg_v. It is a product measure on G_S and it is a Haar measure on it. The measures dg_v are normalized on H_v for all but finitely many v. By construction of dg_S on G_S, we can define a unique measure on the group G. We have that G is locally compact and hence has a Haar measure, which is the restriction to the Haar measure on G_S. Accordingly, fix any finite set of indices S containing J_∞, and define a Haar measure dg on G such that it restricts to dg_S. This measure is independent of the set S and is unique. □

Lemma 4.1. (Proposition 5-6(iii), Section 5.1, [1]) Let us consider a continuous function $f_v \in L^1(G_v)$ for all places such that $f_v(g_v) = 1$ on H_v for all but finitely many v and $f(g) = \prod_v f_v(g_v)$. Then we have that $f(g)$ is continuous on G. Let S is a finite set of indices such that f_v is trivial on H_v, $\int_{H_v} dg_v = 1$ then $\int_{G_S} f(g)dg = \prod_{v\in S}[\int_{G_v} f_v(g_v)dg_v]$.

Proof. The restriction of f to G_S is given $f|_{G_S} = \prod_{v\in S} f(g_v)$. This is a finite product of continuous functions and hence f is continuous on G_S. This is true for any finite set of indices S. Let $g \in G$ and an open $U \subseteq \mathbb{C}$ such that it contains the image $f(g)$. As g is in one of the G_S there is an open neighborhood of g, N such that $f(N) \subseteq U$. N is an open neighborhood of $g \in G$ and contained in $f^{-1}(U)$. Therefore, f is continuous.

Note that: $\int_{G_S} f(g)dg = \int_{G_S} f(g)dg_S = \prod_{v\in S}\int_{G_v} f(g_v)dg_v \prod_{v\notin S}\int_{H_v} f(g_v)dg_v$.

This equals to $\prod_{v\in S}\int_{G_v} f(g_v)dg_v$. This proves the second part. □

4.2 Adèle Ring and Idèle Group

Let F be a number field and F_v be its completion at place v.

Definition 4.2.1. The **adèle ring** $\mathbb{A}_F$ of F is a topological ring with underlying topological space as the restricted product of the completions F_v with respect to the valuation rings $\mathcal{O}_v$. The addition and multiplication is defined componentwise.

The adèle ring of a global field F is given by $\mathbb{A}_F := \prod_v (F_v, \mathcal{O}_v)$ consisting of the tuples (a_v) where $a_v \in \mathcal{O}_v$ for all but finitely many places v. The adèle ring $\mathbb{A}_F$ of a global field F is locally compact and Hausdorff.

Definition 4.2.2. The **idèle group** $\mathbb{I}_F$ of F is the group of invertible elements of $\mathbb{A}_F$. It is a topological group under the operation of multiplication defined componentwise. It is given as the restricted product of K_v^* with respect to $\mathcal{O}_v^*$ or $\mathbb{I}_F := \prod_v (K_v^*, \mathcal{O}_v^*)$ with restricted product topology rather than subspace topology of $\mathbb{A}_F$.

Theorem 4.3. The adèle ring of F, $\mathbb{A}_F$, is its own character group.

Proof. Let $\hat{\mathbb{A}}_F$ be the character group of the adèle ring, $\mathbb{A}_F$ of F. A character of $\hat{\mathbb{A}}_F$ looks like a tuple with local character at each component $\xi_\mathfrak{p} \mapsto e^{2\pi i \Lambda(\xi_\mathfrak{p}\eta_\mathfrak{p})}$. $\hat{\mathbb{A}}_F$ is a restricted direct product of $\hat{F}_\mathfrak{p}^+$ with respect to $\mathcal{O}_\mathfrak{p}^*$. $\hat{\mathbb{A}}_F$ is identified using $\xi_\mathfrak{p} \mapsto e^{2\pi i \Lambda(\xi_\mathfrak{p}\eta_\mathfrak{p})} \mapsto \eta_\mathfrak{p}$. Therefore a typical element of $\hat{\mathbb{A}}_F$ looks like $\eta = (\cdots, \eta_\mathfrak{p}, \cdots)$ and $\hat{\mathbb{A}}_F$ is a restricted direct product of $F_\mathfrak{p}^+$ with respect to $\mathfrak{D}_\mathfrak{p}^{-1}$. However, we have $\mathfrak{D}_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$ since finitely many primes get ramified. Thus $\hat{\mathbb{A}}_F = \mathbb{A}_F$ with an element like $\eta = (\cdots, \eta_\mathfrak{p}, \cdots)$ being identified with $x = (\cdots, x_\mathfrak{p}, \cdots) \mapsto \prod_\mathfrak{p} e^{2\pi i \Lambda_\mathfrak{p}(\xi_\mathfrak{p}\eta_\mathfrak{p})}$. □

Lemma 4.2. (Lemma 4.1.1, Section 4.1, [7]) The map $\phi : x \mapsto ax$ is an automorphism of the adèle group if and only if $a \in \mathbb{A}_F$ and satisfies the conditions that $a_v \neq 1$ for all v and $| a_v |_v = 1$ for all finitely many v.

Proof. On each place v, the homomorphism is given by $x_v \mapsto a_v x_v$ and this map is continuous since F_v is topological field. Assume the map to be an automorphism and hence, it is surjective. Then there exists a $b \in \mathbb{A}_F$ such that $\phi_a(b) = 1$ which means $a_v b_v = 1$ for all places v. Since $(\cdots, b_v, \cdots) = (\cdots, a_v^{-1}, \cdots)$ is an element of $\mathbb{A}_F$, $a_v^{-1} \in \mathcal{O}_v$ for all but finitely many v giving $| a_v |_v = 1$.

Conversely, taking $b = (\cdots, a_v^{-1}, \cdots)$, we find that the map ϕ_b is the inverse of ϕ_a and has the same form. Therefore, ϕ_a is an automorphism. □

The previous lemma gives us that the map $x \mapsto ax$ is an automorphism if and only if a is an idèle. For an idèle a, $d(ax) = \mid a \mid dx$, where $\mid a \mid = \prod_v \mid a_v \mid_v$, a finite product.

4.3 Idèle Class Group

An idèle-class character or Hecke character is a continuous homomorphism $\chi : \mathbb{I}_F \to \mathbb{C}^*$ such that $\chi|_{F^*} = 1$. The idèle-class characters are identified with **idèle class group** which is given by $C_F := \dfrac{\mathbb{I}_F}{F^*}$.

Let $| \, . \, |_v$ denote the normalized absolute value on the completion F_v. Then the absolute value $| \, . \, |_{\mathbb{A}_F} : \mathbb{I}_F \to \mathbb{R}_+^*$ is given by $\mid x \mid_{\mathbb{A}_F} = \prod_v \mid x_v \mid_v$ where $x = (x_v)$. Let $\mathbb{I}_F^1 = \mathrm{Ker}(| \, . \, |_{\mathbb{A}_F})$. Then the **norm one idèle class group** is given by $C_F^1 := \dfrac{\mathbb{I}_F^1}{F^*}$.

The relation between the idèle class group C_F and the ideal class group Cl_F is described as follows: There is a surjection $\phi : \mathbb{I}_F \to \mathbb{J}_F$, where $\mathbb{J}_F$ is the group fractional ideals of F which maps $x \mapsto (x) = \prod_{\mathfrak{p} \nmid \infty} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})}$. The kernel of this map is given by $\mathbb{I}_F^{S_\infty} = \prod_{\mathfrak{p} | \infty} F_{\mathfrak{p}}^* \times \prod_{\mathfrak{p} \nmid \infty} U_{\mathfrak{p}}$.

Thus there is a surjective homomorphism from $C_F \to Cl_F$ with kernel being $\mathbb{I}_F^{S_\infty} F^*/F^*$.

A basic system of neighborhoods of $1 \in \mathbb{I}_F$ is given by $\prod_{\mathfrak{p} \in S} W_{\mathfrak{p}} \times \prod_{\mathfrak{p} \notin S} U_{\mathfrak{p}}$ for a finite set of places S which include $\mathfrak{p} | \infty$ and $W_{\mathfrak{p}} \subseteq F_{\mathfrak{p}}^*$ is a system of neighborhood of 1. The group $\prod_{\mathfrak{p} \notin S} U_{\mathfrak{p}}$ is compact and if $\prod_{\mathfrak{p} \in S} W_{\mathfrak{p}}$ are bounded then $\prod_{\mathfrak{p} \in S} W_{\mathfrak{p}} \times \prod_{\mathfrak{p} \notin S} U_{\mathfrak{p}}$ is a neighborhood whose closure is compact. Thus $\mathbb{I}_F$ is a locally compact topological group.

Chapter 5

Theory of Zeta Functions

An amazing phenomenon in number theory is the a lot of arithmetic properties of a number field is hidden in an analytic function called the zeta function. A fundamental prototype of a zeta function is the most celebrated Riemann Zeta function. This chapter talks about the Riemann Zeta function, about its functional equation and analytic continuation on the whole complex plane except for a simple pole at $s = 1$ in the first section. In the following section we talk about the functional equation of the local zeta function for a Schwartz-Bruhat function on a local field. The last section gives us the proof of the meromorphic continuation and the functional equation for the global zeta function.

5.1 Riemann Zeta Function

The Riemann Zeta function, $\zeta(s)$, is a function in a complex variable $s = \sigma + i\tau \in \mathbb{C}$. For $\mathrm{Re}(s) = \sigma > 1$ we have $\zeta(s)$ as a converging sum given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{5.1}$$

. The series is absolutely and uniformly convergent for $\mathrm{Re}(s) > 1$ and hence gives an analytic function in the half-plane $\mathrm{Re}(s) > 1$. Euler's identity is given by

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$$

where p runs through the prime numbers. This identity helps us to study more properties.

The Riemann Zeta function is given on the half plane and it admits an analytic continuation to the whole plane with $s = 1$ removed and satisfies the functional equation which relates the argument s to the argument $1 - s$. The integral formula for $\zeta(s)$ is attached to the **Gamma function** which is given by a convergent integral for $\text{Re}(s) > 0$,

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx \tag{5.2}$$

The Gamma function satisfies a number of properties:

1. The Gamma function is analytic and gives a meromorphic continuation to whole complex plane.
2. The Gamma function is nowhere zero and has simple poles at $s = \{0, -1, -2, \cdots\}$.
3. The Gamma function satisfies the functional equations:
   - $\Gamma(s+1) = s\Gamma(s)$
   - $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin\pi s}$
   - $\Gamma(s)\Gamma(s+\dfrac{1}{2}) = \dfrac{2\sqrt{\pi}}{2^{2s}}\Gamma(2s)$
4. $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = 1$, $\Gamma(k+1) = (k!); k = 0, 1, 2, \cdots$

The **theta series** is given by $\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 z}$. Then define $g(x) = (1/2)(\theta(ix) - 1)$. The function,

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \tag{5.3}$$

is the **completed zeta function**. The integral formula for $Z(s)$ is given as,

$$Z(s) = \frac{1}{2}\int_0^\infty (\theta(ix) - 1) x^{s/2-1} dx \tag{5.4}$$

Theorem 5.1. (Theorem 1.6, VII, [3]) The completed zeta function $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ admits analytic continuation on $\mathbb{C} - \{0, 1\}$ and has simple poles at $s = 0$ and $s = 1$ with residues -1 and 1 respectively. It also satisfies the functional equation

$$Z(s) = Z(1 - s) \tag{5.5}$$

Proof. We have that,

$$Z(2s) = \frac{1}{2}\int_0^\infty (\theta(ix) - 1)x^{s-1}dx$$

For a continuous function $f : \mathbb{R}_+^* \to \mathbb{C}$ we define the **Mellin transform** as $L(f, s) = \int_0^\infty (f(x) - f(\infty))x^{s-1}dx$ where $f(\infty) = \lim_{x\to\infty} f(x)$.

Now we have that $Z(2s) = L(f, s)$ for $f(x) = (1/2)\theta(ix)$. Since $\theta(ix) = 1 + 2e^{-\pi x}(1 + \sum_{n=2}^{\infty} e^{-\pi(n^2-1)x})$, we get the transformation formula $f(1/x) = (1/2)\theta(-1/ix) = x^{1/2}f(x)$. $L(f, s)$ has an analytic continuation to $\mathbb{C} - \{0, 1/2\}$ and has simple poles at $s = 0, 1/2$ with residues $-1/2$ and $1/2$ respectively and satisfies $L(f, s) = L(f, 1/2 - s)$.

$Z(s) = L(f, s/2)$ has an analytic continuation to $\mathbb{C} - \{0, 1\}$ and has simple poles at $s = 0, 1$ with residues at -1 and 1 respectively. It satisfies the functional equation,

$$Z(s) = L(f, s/2) = L(f, 1/2 - s/2) = Z(1 - s) \tag{5.6}$$

Hence proved. □

Corollary 5.1. (Corollary 1.7, VII, [3]) The Riemann Zeta function admits an analytic continuation to $\mathbb{C} - \{1\}$ and has simple pole at $s = 1$ with residue 1. $\zeta(s)$ satisfies the functional equation

$$\zeta(1 - s) = 2(2\pi)^{-s}\Gamma(s)\cos(\frac{\pi s}{2})\zeta(s) \tag{5.7}$$

Proof. $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\Gamma(s/2)$ has a simple pole at $s = 0$. Hence $\zeta(s)$ has no pole. $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ has simple pole at $s = 1$ and $\Gamma(1/2) = \sqrt{\pi}$. Hence $\zeta(s)$ has a pole at $s = 1$. The residue turns out to be $\text{Res}_{s=1}\zeta(s) = \pi^{1/2}\Gamma(1/2)^{-1}\text{Res}_{s=1}Z(s) = 1$.

The equation $Z(s) = Z(1-s)$ becomes,

$$\zeta(1-s) = \pi^{1/2-s}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\zeta(s) \tag{5.8}$$

Replacing $(1-s)/2$ and $s/2$ into the formulae $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}$ and $\Gamma(s)\Gamma(s+\frac{1}{2}) = \frac{2\sqrt{\pi}}{2^{2s}}\Gamma(2s)$ respectively,

$$\Gamma(s/2)\Gamma((1+s)/2) = \frac{2\sqrt{\pi}}{2^{2s}}\Gamma(2s)$$

$$\Gamma((1-s)/2)\Gamma((1+s)/2) = \frac{\pi}{\cos\frac{(\pi s)}{2}}$$

Then we have,

$$\frac{\Gamma(s/2)}{\Gamma((1-s)/2)} = \frac{2}{2^s\sqrt{\pi}}\cos\frac{\pi s}{2}\Gamma(s) \tag{5.9}$$

Substituting the value of (5.9) in (5.8) we get,

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos(\tfrac{\pi s}{2})\zeta(s)$$

Hence proved. □

5.2 Local Zeta Functions

Let us consider a local field, F_v, with absolute value $|\ .\ |$ and Haar measure dx. We define $d^*x = c.\frac{dx}{|x|}$ for a real number $c > 0$ to be the Haar measure on the multiplicative group F_v^*.

A complex valued function f is called smooth on F_v if it is $\mathscr{C}^\infty$ for archimedean field F and a locally constant function with compact support for a non-archimdean field. For an archimedean field, a Schwartz function is a smooth function and that satisfies the property that $p(x)f(x) \to 0$ as $x \to \infty$ for all polynomials $p(x)$. A function $f : F_v \to \mathbb{C}$ is called a **Schwartz-Bruhat function** if it is a Schwartz function for an archimedean field or a locally constant function with a compact support for a non-archimedean field. We denote the class of Schwartz-Bruhat functions as $S(F_v)$.

For a Schwartz-Bruhat function $f \in S(F_v)$ and a character χ we define the **the local zeta function** as

$$Z\,(f,\chi) := \int_{F_v^*} f(x)\chi(x)\,d^*x \tag{5.10}$$

Theorem 5.2. (Theorem 7-2, Section 7.1, [1]) Let $f \in S(F_v)$ and $\chi = \mu \mid . \mid^s$, where $s \in \mathbb{C}$, and μ is unitary with exponent $\sigma = \mathrm{Re}(s)$. Then the following statements hold:

1. $Z(f,\chi)$ is absolutely convergent if σ is positive.
2. If $\sigma \in (0,1)$, there is a functional equation

$$Z(\hat{f},\chi^\vee) = \gamma(\chi,\psi,dx)Z(f,\chi)$$

   for some $\gamma(\chi,\psi,dx)$ independent of f, which in fact is meromorphic as a function of s.
3. There exists a factor $\epsilon(\chi,\psi,dx)$ that lies in $\mathbb{C}^*$ for all s and satisfies the relation

$$\gamma(\chi,\psi,dx) = \epsilon(\chi,\psi,dx)\frac{L(\chi^\vee)}{L(\chi)}$$

Proof. We would give an outline of the proof here.

1. Now consider the integral

$$I(f,\sigma) = c\int_{F_v-\{0\}} \mid f(x) \mid . \mid x \mid^{\sigma-1}\ dx \tag{5.11}$$

   which takes finite value for $\sigma > 0$ in case of Archimedean local field since f is a Schwartz function.

   For Non-Archimedean local fields the functions are locally constant with compact support and takes up value of characteristic function of ideals of the form $\pi_F^j\mathcal{O}_F$ the integral takes finite value of $\mathrm{Vol}(\mathcal{O}_F^*,d^*x)\dfrac{\mathcal{N}\mathfrak{p}^{-j\sigma}}{1-\mathcal{N}\mathfrak{p}^{-\sigma}}$ where p_F being the maximal ideal of $\mathcal{O}_F$.

2. Let us choose an auxillary function $g \in S(F_v)$. We first prove a lemma from which the proof of (2) will follow.

Lemma 5.1. For all χ with exponent $\sigma \in (0,1)$ we have that $Z(f,\chi)Z(\hat{g},\chi^\vee) = Z(g,\chi)Z(\hat{f},\chi^\vee)$

Proof. We have that

$$Z(f,\chi)Z(\hat{g},\chi^\vee) = \int\int_{F_v^* \times F_v^*} f(x)g^{-1}(y)\chi(xy^{-1})d^*xd^*y \tag{5.12}$$

d^*xd^*y is product Haar measure on $F_v^* \times F_v^*$ and hence invariant under $(x,y) \to (x,xy)$ then the integral becomes

$$\int\int_{F_v^* \times F_v^*} f(x)g^{-1}(y)\chi(xy^{-1})d^*xd^*y = \int\int_{F_v^* \times F_v^*} f(x)\hat{g}(xy)\chi(y^{-1})|x||y|d^*xd^*y \tag{5.13}$$

Since $c.dx = |x|d^*x$ we have that

$$\int_{F_v^*} f(x)\hat{g}(xy)|x|d^*x = c\int\int_{F_v \times F_v} f(x)g(z)\psi(xyz)dzdx = c\int_{F_v} g(z)\hat{f}(yz)dz \tag{5.14}$$

Therefore, equation (15) becomes

$$\int\int_{F_v^* \times F_v^*} g(x)\hat{f}(xy)\chi(y^{-1})|x||y|d^*xd^*y \tag{5.15}$$

and hence our claim is proved. □

Let $f_0 \in S(F_v)$ and hence, $\gamma(\chi) = \gamma(\chi,\psi,dx) = \dfrac{Z(\hat{f}_0,\chi^\vee)}{Z(f_0,\chi)}$. Now, γ is independent of the function f_0 and we get $Z(\hat{f},\chi^\vee) = \gamma(\chi,\psi,dx)Z(f,\chi)$.

3. Computations are done for a standard measure dx which is self dual with respect to choice of ψ and for some special functions in each of the three cases.

   For $F_v = \mathbb{R}$, dx is the Lebesgue measure with $\psi(x) = e^{-2\pi ix}$, $\chi = |\ .\ |^s$ and $f = e^{-\pi x^2} \in S(\mathbb{R})$. Then,

$$Z(f,\chi) = \int_{\mathbb{R}^*} e^{-\pi x^2}|x|^s d^*x = 2\int_0^\infty e^{-\pi x^2}|x|^{s-1}dx \tag{5.16}$$

   Substitute $u = \pi x^2$ then we get,

$$Z(f,\chi) = \pi^{-s/2}\int_0^\infty e^{-u}u^{s/2-1}dx = \pi^{-s/2}\Gamma(s/2) \tag{5.17}$$

Now from the definition of $L(\chi)$ we have that $Z(f,\chi) = L(\chi)$ for all characters χ. Now,

$$\hat{f}(y) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x y} dx = f(x) \tag{5.18}$$

Then we have,

$$Z(\hat{f}, \chi^\vee) = \int_{\mathbb{R}^*} f(x)\chi^\vee(x) d^*x = L(\chi^\vee) \tag{5.19}$$

So for $\chi = |.|^s$ we have from equations (6.8) and (6.10) that $\gamma(\chi) = L(\chi^\vee)/L(\chi)$ and we can put $\epsilon(\chi) = 1$.

For $\chi = \text{sgn}\, | \,.\, |^S$ we have $f = xe^{\pi x^2}$. Since $\text{sgn}(x) = x/|x|$ we have,

$$Z(f,\chi) = \int_{\mathbb{R}^*} xe^{-\pi x^2}.x/|x|.|x|^s d^*x = \int_{\mathbb{R}^*} e^{-\pi x^2}|x|^{s+1} d^*x = \pi^{-(s+1)/2}\Gamma((s+1)/2) \tag{5.20}$$

Thus we have $Z(f,\chi) = L(\chi)$. We have $\hat{f}(y) = iye^{-\pi y^2}$. Then,

$$Z(\hat{f}, \chi^\vee) = i\int_{\mathbb{R}^*} xe^{-\pi x^2}.x/|x|.|x|^{1-s} d^*x = iL(\chi^\vee). \tag{5.21}$$

Thus we have from equations (6.11) and (6.12) that $\epsilon(\chi) = i$. This completes the proof for the real case.

For $F_v = \mathbb{C}$, dx is twice the ordinary Lebesgue measure with $\psi(x) = e^{-2\pi i(z+\overline{z})}$, $\chi_{s,n} : re^{i\theta} \mapsto r^s e^{in\theta}$ and

$$f_n(z) = \begin{cases} (2\pi)^{-1}(\overline{z})^n e^{-2\pi z\overline{z}} & n \geqslant 0 \\ (2\pi)^{-1} z^{-n} e^{-2\pi z\overline{z}} & n < 0 \end{cases}$$

Then we have the Fourier transform as $\hat{f}_n(z) = (2\pi)^{-1} i^{|n|} f_{-n}(z)$ for all n. Note that $d^*z = (2/r)drd\theta$. For n positive or zero we compute that,

$$Z(f_n, \chi_{s,n}) = \int_{\mathbb{C}^*} f_n(z)\chi_{s,n}(z) d^*z = (1/2\pi)\int_{\mathbb{C}^*} \bar{z}^n e^{-2\pi z\bar{z}}(z\bar{z})^s e^{in\theta(z)} d^*z \tag{5.22}$$

Thus we have,

$$Z(f_n, \chi_{s,n}) = (2\pi)^{-(s+n/2)} \int_0^\infty e^{-2\pi r^2}(2\pi r^2)^{s+n/2-1} 4\pi r dr \tag{5.23}$$

Substituting $t = 2\pi r^2$ we have,

$$Z(f_n, \chi_{s,n}) = (2\pi)^{-(s+n/2)}\Gamma(s + n/2) = L(\chi_{s,n})$$

Having same computation for negative n, we have,

$$Z(f_n, \chi_{s,n}) = (2\pi)^{-(s+|n|/2)}\Gamma(s + |n|/2) = L(\chi_{s,n}) \tag{5.24}$$

From the Fourier transform formula we have that,

$$Z(\hat{f}_n, \chi_{s,n}^\vee) = i^{|n|}(2\pi)^{-(1-s+|n|/2)}\Gamma(1 - s + |n|/2) = i^{|n|}L(\chi_{s,n}^\vee) \tag{5.25}$$

Then we have from equations (6.15) and (6.16), $\gamma(\chi_{s,n}) = i^{|n|}\dfrac{L(\chi_{s,n}^\vee)}{L(\chi_{s,n})}$ and we get $\epsilon(\chi) = i^{|n|}$. This completes the proof for the complex case.

For Non-Archimedean fields with characteristic zero, we define the character as $\psi_{F_v}(x) = \psi_{\mathfrak{p}}(\mathrm{tr}(x))$ for $x \in F_v$. We fix a non-trivial character ψ and a self dual measure. Then we define $m = \inf\{r \in \mathbb{Z} : \psi|_{\mathfrak{p}^r} = 1\}$ for unique prime $\mathfrak{p} \in F_v$. The **conductor** of ψ is defined to be as $\mathfrak{p}^m$ and $\mathfrak{p}^0$ to be $\mathcal{O}_F$. Let χ be the multiplicative character with conductor $\mathfrak{p}^n$ and $\chi_{s,n} : x \mapsto \omega(x/ \mid x \mid)$ for a unitary character ω with conductor $\mathfrak{p}^n$. The function f is defined as

$$f(x) = \begin{cases} \psi(x) & x \in \mathfrak{p}^{m-n} \\ 0 & \text{otherwise} \end{cases}$$

The proof will follow from the computation of $Z(f, \chi_{s,n})$ for $n = 0$ and n positive.

Let us focus on the case where $n = 0$. We know that ψ is trivial $\mathfrak{p}^m$, ω is trivial everywhere and $\mathfrak{p}^m - \{0\}$ is disjoint union of sets $\pi_F^k\mathcal{O}_F^*$ for $k \geqslant m$. Then we compute that,

$$Z(f, \chi_{s,n}) = \int_{F^*} f(x)\chi_{s,n}(x)dx = \int_{\mathfrak{p}^m-\{0\}} |x|^s dx = \mathrm{Vol}(\mathcal{O}_F^*, d^*x)\frac{\mathcal{N}\mathfrak{p}^{-ms}}{1 - \mathcal{N}\mathfrak{p}^{-s}} \tag{5.26}$$

This gives us that

$$Z(f, \chi_{s,n}) = \mathcal{N}\mathfrak{p}^{-ms}\mathrm{Vol}(\mathcal{O}_F^*, d^*x)L(\chi_{s,0}) \tag{5.27}$$

.

Let us focus on the case where $n > 0$. We then have that,

$$Z(f, \chi_{s.n}) = \Sigma_{k \geqslant m-n}\mathcal{N}\mathfrak{p}^{-ks}\int_{\mathcal{O}_F^*} \psi(\pi^k u)\omega(u)d^*u \tag{5.28}$$

For a multiplicative character $\omega : \mathcal{O}_F^* \to S^1$ and an additive character $\Psi : \mathcal{O}_F \to S^1$ we define the **Gauss Sum** to

$$g(\omega, \Psi) = \int_{\mathcal{O}_F^*} \omega(u)\Psi(u)d^*u \tag{5.29}$$

Then,

$$Z(f, \chi_{s,n}) = \Sigma_{k \geqslant m-n} \mathcal{N}\mathfrak{p}^{-ks} g(\omega, \psi_{\pi^k}). \tag{5.30}$$

We use a result on Gauss Sum which states that if ω and Ψ have conductors $\mathfrak{p}^n$ and $\mathfrak{p}^r$ respectively then

(a) if $r < n$, then $g(\omega, \Psi) = 0$

(b) if $r = n$, then $g(\omega, \Psi) = c.\text{Vol}(\mathcal{O}_F, dx)\text{Vol}(U_n, d^*x)$

(c) if $r > n$, then $g(\omega, \Psi) = c.\text{Vol}(\mathcal{O}_F, dx)[\text{Vol}(U_n, d^*x) - \mathcal{N}\mathfrak{p}^{-1}\text{Vol}(U_{r-1}, d^*x)]$

where $U_n = 1 + \mathfrak{p}^n$. Now resuming the computation we have that

$$Z(f, \chi_{s,n}) = \mathcal{N}\mathfrak{p}^{-(m-n)s} g(\omega, \psi_{\pi^{m-n}}) \tag{5.31}$$

.

For computing the Fourier transform of the function f we use another result which states that for $n = 0$ the Fourier transform of f is given by $\text{Vol}(\mathfrak{p}^{m-n}, dx)$ times the characteristic function of $\mathcal{O}_F$ and for $n > 0$ the Fourier transform of f is given by $\text{Vol}(\mathfrak{p}^{m-n}, dx)$ times the characteristic function of $\mathfrak{p}^n - 1$.

Let us focus on the calculation of $Z(\hat{f}, \chi_{s,0}^\vee)$ for $n = 0$.

$$Z(\hat{f}, \chi_{s,0}^\vee) = \text{Vol}(\mathfrak{p}^m, dx) \int_{\mathcal{O}_F - \{0\}} \chi_{s,0}^\vee(y)d^*y = \text{Vol}(\mathfrak{p}^m, dx)\text{Vol}(\mathcal{O}_F^*, d^*x)\frac{1}{1 - \mathcal{N}\mathfrak{p}^{-(1-s)}} \tag{5.32}$$

Thus we have,

$$Z(\hat{f}, \chi_{s,0}^\vee) = \text{Vol}(P^m, dx)\text{Vol}(\mathcal{O}_F^*, d^*x)L(\chi_{s,0}^\vee) \tag{5.33}$$

Therefore, we have from equations (6.18) and (6.24) that,

$$\gamma(\chi_{s,0}) = \mathcal{N}\mathfrak{p}^{ms}\text{Vol}(P^m, dx)\frac{L(\chi_{s,0}^\vee)}{L(\chi_{s,0})} \tag{5.34}$$

and

$$\epsilon(\chi_{s,0}) = \mathcal{N}\mathfrak{p}^{ms}\mathrm{Vol}(\mathfrak{p}^m, dx) \tag{5.35}$$

Let us focus on the calculation of $Z(\hat{f}, \chi_{s,n}^\vee)$ for $n > 0$.

$$Z(\hat{f}, \chi_{s,n}^\vee) = \mathrm{Vol}(\mathfrak{p}^{m-n}, dx)\int_{\mathfrak{p}^n - 1} \bar{\omega}(u)d^*u = \mathrm{Vol}(\mathfrak{p}^{m-n}, dx)\mathrm{Vol}(1+\mathfrak{p}^n, d^*x)\omega(-1) \tag{5.36}$$

since conductor of ω and its conjugate are same. Therefore, from computation we find from equations (6.22) and (6.27) that,

$$\epsilon(\chi_{s,n}) = \gamma(\chi_{s.n}) = \frac{\mathcal{N}\mathfrak{p}^{(m-n)s}\mathrm{Vol}(\mathfrak{p}^{m-n}, dx)\mathrm{Vol}(1+\mathfrak{p}^n, d^*x)\omega(-1)}{g(\omega, \psi_{\pi^{m-n}})} \tag{5.37}$$

Now since the conductor of $\psi_{\pi^{m-n}}$ is n then we have

$$g(\omega, \psi_{\pi^{m-n}}) = c.\mathrm{Vol}(\mathcal{O}_F, dx)\mathrm{Vol}(U_n, d^*x).$$

Substituting $\overline{g(\omega, \psi_{\pi^{m-n}})} = \omega(-1)g(\bar{\omega}, \psi_{\pi^{m-n}})$, $1+\mathfrak{p}^n = U_n$ and $\mathrm{Vol}(\mathfrak{p}^{m-n}) = \mathcal{N}\mathfrak{p}^{-(m-n)}\mathrm{Vol}(\mathcal{O}_F)$ we get that,

$$\epsilon(\chi_{s,n}, \psi, dx) = (1/c)\mathcal{N}\mathfrak{p}^{(m-n)(s-1)}g(\bar{\omega}, \psi_{\pi^{m-n}}) \tag{5.38}$$

This proves the non-archimedean case.

We now see that the poles of $Z(f, \chi)$ are now zeroes of the meromorphic function $\gamma(\chi, \psi, dx)$ which is given by $\epsilon(\chi, \psi, dx)(L(\chi^\vee)/L(\chi))$ since the region of absolute convergence of $Z(f, \chi)$ is $\mathrm{Re}(s) > 0$ and that of $Z(\hat{f}, \chi^\vee)$ is $\mathrm{Re}(s) < 1$. Moreover the zeroes of $\gamma(\chi)$ coincide with the poles of $L(\chi)$.

□

5.3 Global Zeta Functions

Let F be a global field. We define $S(\mathbb{A}_F) = \otimes' S(F_v)$ be the restricted tensor product of Schwartz-Bruhat spaces which contains elements of the form $f = \otimes' f_v : f_v \in S(F_v), f_v|_{\mathcal{O}_F} = 1$ for almost all v. The adélic Schwartz-Bruhat function is given by $f(x) = \prod_v f_v(x_v)$ for all $x = (x_v) \in \mathbb{A}_F$.

At each place v of F, let ψ_v be the standard character and dx_v be the self dual measure. Then $\psi_F(x) = \prod_v \psi_v(x_v)$ for all adèles $x = (x_v) \in \mathbb{A}_F$.

The Haar measure on $\mathbb{A}_F$ is given by the product measure $dx = \prod_v dx_v$. Moreover dx satisfies the relation $d(ax) =| a | dx$ for all idèles $a \in \mathbb{I}_F$. Then $d^*x = \prod_v d^*x_v$ on $\prod_v F_v^*$ and

$$d^*x_v = \begin{cases} \frac{dx_v}{|x_v|_v} & v|\infty \\ (1-\mathcal{N}\mathfrak{p}_v^{-1})^{-1}\frac{dx_v}{|x_v|_v} & v|\mathfrak{p} \end{cases}$$

Then

$$\int_{\mathcal{O}_F^*} d^*x_v = 1$$

for almost all v. Let χ be any $\mathbb{C}^*$-valued character of $\mathbb{I}_F$ that is trivial on F^*.

The adélic Fourier transform for $f \in S(\mathbb{A}_F)$ for a unitary character ψ with $\psi|_F = 1$ is given by,

$$\hat{f}(y) = \int_{\mathbb{A}_F} f(x)\psi(xy)dx \tag{5.39}$$

We will now look for functions on $\mathbb{A}_F$ which are invariant under translation by elements of the field. One such function will be the unitary characters ψ. The other functions can be found by taking average over all elements of F - which is $\widetilde{\Phi}(x) = \Sigma_{\alpha\in F}\Phi(\alpha+x)$ for $\Phi \in S(\mathbb{A}_F)$. When this function is convergent, for all $\delta \in F$ we have $\widetilde{\Phi}(\delta + x) = \Sigma_{\alpha\in F}\Phi(\alpha + \delta + x) = \Sigma_{\alpha'\in F}\Phi(\alpha' + x) = \widetilde{\Phi}(x); \alpha' = \alpha + \delta$. Thus $\widetilde{\Phi}(\delta + x) = \widetilde{\Phi}(x)$.

A complex-valued function f on $\mathbb{A}_F$) is called **admissible** if $\widetilde{f}$ and $\widetilde{\widetilde{f}}$ are both absolutely and uniformly convergent. Every $f \in S(\mathbb{A}_F)$ is admissible.

5.3.1 Poisson Summation Formula

We begin by proving two lemmas which will be needed for the proof of the theorem for Poisson-Summation formula.

Lemma 5.2. (Lemma 7-8, Section 7.2, [1]) For every $f \in S(\mathbb{A}_F)$ we have $\hat{f}|_F = \hat{\hat{f}}|_F$.

Proof. Let us fix $y \in F$. Then,

$$\hat{\tilde{f}}(y) = \int_{\mathbb{A}_F/F} \tilde{f}(x)\psi(xy)\overline{dx} = \int_{\mathbb{A}_F/F} (\Sigma_{\alpha\in K} f(\alpha + x))\psi(xy)\overline{dx} \tag{5.40}$$

where $\overline{dx}$ is the quotient measure on $\mathbb{A}_F/F$ induced by dx on $\mathbb{A}_F$. Now for unitary characters we have $\psi(xy) = \psi((\alpha + x)y)$ for all $\alpha \in F$. Therefore, we have that,

$$\hat{\tilde{f}}(y) = \int_{\mathbb{A}_F/F} (\Sigma_{\alpha\in F} f(\alpha + x)\psi((\alpha + x)y))\overline{dx} = \int_{\mathbb{A}_F} f(x)\psi(xy)dx = \hat{f}(x). \tag{5.41}$$

Thus the proof is completed. □

Lemma 5.3. (Lemma 7-9, Section 7.2, [1]) Let $f \in S(\mathbb{A}_F)$. Then for every $x \in F$ we have $\tilde{f}(x) = \Sigma_{\alpha\in F}\hat{\tilde{f}}(\alpha)\bar{\psi}(\alpha x)$.

Proof. We have that $\hat{f}|_F = \hat{\tilde{f}}|_F$ from the previous lemma which implies that $\Sigma_{\alpha\in F}\hat{\tilde{f}}(\alpha)\bar{\psi}(\alpha x)$ is both uniformly and absolutely convergent. We have that $\Sigma_{\alpha\in F}|\hat{\tilde{f}}(\alpha)| < \infty$ and hence the Fourier inversion formula can also be applied. Since the Pontryagin dual of $\mathbb{A}_F/F$ is itself under discrete topology, the summations correspond to appropriate integrals. □

Theorem 5.3 (Poisson-Summation Formula). (Theorem 7-7, Section 7.2, [1]) Let $f \in S(\mathbb{A}_F)$. Then $\tilde{f} = \tilde{\hat{f}}$.

Proof. For all $y \in F$ we have that

$$\hat{\Phi}(y) = \int_{\mathbb{A}_F/F} \Phi(x)\psi(xy)\overline{dx} \tag{5.42}$$

for a translation invariant Φ on $\mathbb{A}_F$ and $\overline{dx}$ is the quotient measure on $\mathbb{A}_F/F$ induced by dx on $\mathbb{A}_F$. Then $\overline{dx}$ is characterized by

$$\int_{\mathbb{A}_F/F} \tilde{f}(x)\overline{dx} = \int_{\mathbb{A}_F/F} (\Sigma_{\alpha\in K} f(\alpha + x))\overline{dx} = \int_{\mathbb{A}_F/F} f(x)dx \tag{5.43}$$

for all continuous function f on $\mathbb{A}_F$. Putting $x = 0$ in the result of the second lemma and then apply the first one we get, $\tilde{f}(0) = \Sigma_{\alpha\in F}\hat{\tilde{f}}(\alpha) = \Sigma_{\alpha\in F}\hat{f}(\alpha)$. By definition we have that $\tilde{f}(0) = \Sigma_{\alpha\in F}\hat{f}(\alpha)$. Thus we have that $\Sigma_{\alpha\in F}\hat{f}(\alpha) = \Sigma_{\alpha\in F}\hat{f}(\alpha)$ which is same as writing $\tilde{f} = \tilde{\hat{f}}$. This completes the proof. □

5.3.2 Riemann-Roch Theorem

We use this theorem to understand the average $\Sigma_{\alpha\in F} f(\alpha x)$ for an idèle x.

Theorem 5.4. (Theorem 7-10, Section 7.2, [1]) [**Riemann-Roch Theorem**] Let x be an idèle of F and let $f \in S(\mathbb{A}_F)$. Then $\Sigma_{\alpha\in F} f(\alpha x) = \dfrac{1}{|x|}\Sigma_{\alpha\in F}\hat{f}(\alpha x^{-1})$.

Proof. For a fixed $x \in \mathbb{A}_F$ and arbitrary $y \in \mathbb{A}_F$, we define $g(y) = f(yx)$ such that $g \in S(\mathbb{A}_F)$. By Poisson-Summation formula we have, $\Sigma_{\alpha\in F}\hat{g}(\alpha) = \Sigma_{\alpha\in F}\hat{g}(\alpha)$. However we have,

$$\hat{g}(\alpha) = \int_{\mathbb{A}_F} f(yx)\psi(\alpha y)dy = \frac{1}{|x|}\int_{\mathbb{A}_F} f(yx)\psi(\alpha y x^{-1})dy = \frac{1}{|x|}\hat{f}(\alpha x^{-1}). \tag{5.44}$$

Therefore we have, $\Sigma_{\alpha\in F}\hat{g}(\alpha) = \Sigma_{\alpha\in F} f(\alpha x) = \dfrac{1}{|x|}\Sigma_{\alpha\in F}\hat{f}(\alpha x^{-1})$. Hence proved. □

5.3.3 Global Functional Equation

For any Schwartz class function $f \in S(\mathbb{A}_F)$, we define the **the global zeta function**

$$Z(f, \chi) := \int_{\mathbb{I}_F} f(x)\chi(x)\, d^*x \tag{5.45}$$

Theorem 5.5 (**Meromorphic Continuation and Functional Equation**)**.** (Theorem 7-16, Section 7.3, [1]) $Z(f, \chi)$ extends to a meromorphic function of s and satisfies the functional equation

$$Z(f, \chi) = Z(\hat{f}, \chi^{\vee})$$

The extended function $Z(f, \chi)$ is in fact holomorphic everywhere except when $\mu = |\ .\ |^{-i\tau}$, $\tau \in \mathbb{R}$, in which case it has simple poles at $s = i\tau$ and $s = 1 + i\tau$ with corresponding residues given by

$$-\mathrm{Vol}(C_F^1)f(0) \text{ and } \mathrm{Vol}(C_F^1)\hat{f}(0)$$

respectively.(C_F^1 denotes $\mathbb{I}_F^1/F^*$ which is the compact part of the class group $\mathbb{I}_F/F^*$)

Proof. Our focus will be on a number field. Let F be a number field and χ be a character with $\sigma > 1$. We define

$$Z(f,\chi) = \int_0^\infty Z_t(f,\chi)\frac{1}{t}\,dt \tag{5.46}$$

where

$$Z_t(f,\chi) = \int_{\mathbb{I}_F^1} f(tx)\chi(tx)\,d^*x \tag{5.47}$$

We move on to prove the following,

Lemma 5.4. The function $Z_t(f,\chi)$ satisfies

$$Z_t(f,\chi) = Z_{t^{-1}}(\hat{f},\chi^\vee) + \hat{f}(0)\int_{C_F^1}\chi^\vee(x/t)\,d^*x \; - f(0)\int_{C^1{}_F}\chi(tx)\,d^*x \tag{5.48}$$

Proof. Since $C_F^1 = \mathbb{I}_F^1/F^*$, we can write,

$$Z_t(f,\chi) = \int_{C_F^1}(\Sigma_{a\in F*}f(atx))\chi(tx)d^*x = \int_{C_F^1}\chi(tx)d^*x(\Sigma_{a\in F*}f(atx)) \tag{5.49}$$

using the fact that $\chi = 1$ on F^*. Consider the expression,

$$Z_t(f,\chi) + f(0)\int_{C_F^1}\chi(tx)\,d^*x.$$

This is equal to $\int_{C_F^1}\chi(tx)d^*x(\Sigma_{a\in F*}f(atx))$. Applying Riemann-Roch theorem we have that,

$$Z_t(f,\chi) + f(0)\int_{C_F^1}\chi(tx)d^*x = \int_{C_F^1}\frac{\chi(tx)}{|tx|}d^*x(\Sigma_{a\in F}\hat{f}(at^{-1}x^{-1})) \tag{5.50}$$

This equals to by substituting x by x^{-1},

$$Z_t(f,\chi) + f(0)\int_{C_F^1}\chi(tx)\,d^*x = \int_{C_F^1}\chi(tx^{-1})|t^{-1}x|d^*x(\Sigma_{a\in F}\hat{f}(at^{-1}x)). \tag{5.51}$$

We can write equation (7.13) as,

$$\hat{f}(0)\int_{C_F^1}\chi(tx^{-1})|t^{-1}x|d^*x + \int_{C_F^1}\chi(tx^{-1})|t^{-1}x|d^*x(\Sigma_{a\in F*}\hat{f}(at^{-1}x)) \tag{5.52}$$

and since $\chi^\vee = \chi^{-1}\,|\,.\,|$, this equals to

$$Z_{t^{-1}}(\hat{f},\chi^{\vee}) + \hat{f}(0)\int_{C_F^1}\chi^{\vee}(x/t)d^*x$$

This completes the proof of the lemma. □

We can write

$$Z(f,\chi) = \int_0^1 Z_t(f,\chi)(1/t)dt + \int_1^{\infty} Z_t(f,\chi)(1/t)dt \tag{5.53}$$

We have that $\int_1^{\infty} Z_t(f,\chi)(1/t)dt = \int_{x\in\mathbb{I}_F:|x|\geqslant 1} f(x)\chi(x)d^*x$ which converges for all σ. We again have that $\int_0^1 Z_t(f,\chi)(1/t)dt = \int_0^1 Z_{t^{-1}}(\hat{f},\chi^{\vee})(1/t)dt + \eta$ where

$$\eta = \int_0^1 [\hat{f}(0)\chi^{\vee}(t^{-1})\int_{C_F^1}\chi^{\vee}(x)d^*x - f(0)\chi(t)\int_{C_F^1}\chi(x)d^*x](1/t)dt.$$

is the correction term. Substituting t^{-1} by t we get,

$$\int_0^1 Z_{t^{-1}}(\hat{f},\chi^{\vee})(1/t)dt = \int_1^{\infty} Z_t(\hat{f},\chi^{\vee})(1/t)dt$$

which is convergent for all σ.

When χ is non-trivial on C_F^1 we have that both $\int_{C_F^1}\chi^{\vee}(x)d^*x$ and $\int_{C_F^1}\chi(x)d^*x$ are zero and hence $\eta = 0$ likewise. When $\chi = \mu \mid . \mid^s$ trivial on C_F^1, we have $\chi = \mid . \mid^{s-i\tau}$ for $\tau \in \mathbb{R}$. Then we have,

$$\eta = \mathrm{Vol}(C_F^1)[\frac{\hat{f}(0)}{s-(i\tau+1)} - \frac{f(0)}{s-i\tau}] \tag{5.54}$$

This gives us that η is a rational function and thus we get the meromorphic continuation of $Z(f,\chi)$ to the whole plane. We have also shown that $Z(f,\chi)$ is in fact holomorphic everywhere except when $\mu = \mid . \mid^{-i\tau}$, $\tau \in \mathbb{R}$, whence we get simple poles at $s = i\tau$ and $s = 1+i\tau$ with corresponding residues given by $-\mathrm{Vol}(C_F^1)f(0)$ and $\mathrm{Vol}(C_F^1)\hat{f}(0)$ respectively.

The global zeta function looks like,

$$Z(f,\chi) = \int_1^{\infty} Z_t(f,\chi)(1/t)dt + \int_1^{\infty} Z_t(\hat{f},\chi^{\vee})(1/t)dt + \eta(f,\chi) \tag{5.55}$$

which equals to

$$\int_1^\infty \int_{\mathbb{I}_K^1} f(tx)\chi(tx)d^*x(1/t)dt + \int_1^\infty \int_{\mathbb{I}_K^1} \hat{f}(tx)\chi^\vee(tx)d^*x(1/t)dt + \eta(f,\chi) \tag{5.56}$$

Since we have $\hat{\hat{f}}(x) = f(-x)$ and $\widecheck{\chi^\vee} = \chi$, on the other hand we have,

$$Z(\hat{f},\chi^\vee) = \int_1^\infty Z_t(\hat{f},\chi^\vee)(1/t)dt + \int_1^\infty Z_t(\hat{\hat{f}},\chi)(1/t)dt + \eta(\hat{f},\chi^\vee) \tag{5.57}$$

which equals to

$$\int_1^\infty \int_{\mathbb{I}_K^1} f(-tx)\chi(tx)d^*x(1/t)dt + \int_1^\infty \int_{\mathbb{I}_K^1} \hat{f}(tx)\chi^\vee(tx)d^*x(1/t)dt + \eta(\hat{f},\chi^\vee) \tag{5.58}$$

η is invariant under the transformation $(f,\chi) \to (\hat{f},\chi^\vee)$ and χ being an idèle class character is indifferent to sign change and hence $\chi(tx) = \chi(-tx)$. Therefore, we obtain that $Z(f,\chi) = Z(\hat{f},\chi^\vee)$. Hence proved. □

Chapter 6

Hecke Characters

This chapter introduces the notion of Hecke characters which are characters of the idèle class group. We begin by discussing some properties of class groups, and then define Hecke characters. The following sections introduce Dirichlet characters - which are Hecke characters of finite order and also the character at infinity. The final section in this chapter introduces the Größencharaktere and discusses about its correspondence with Hecke characters.

For the remaining part of our study we will consider the following set of notations:

1. $\mathbb{Q}$: field of rational numbers.
2. $\overline{\mathbb{Q}}$: closure of rational numbers inside $\mathbb{C}$.
3. F: a number field.
4. $d_F := [F : \mathbb{Q}]$.
5. $\mathcal{O}_F$: the ring of integers of F.
6. $U_F = \mathcal{O}_F^*$: is the group of units of $\mathcal{O}_F$.
7. $\mathfrak{p}$: finite prime of $\mathcal{O}_F$ or an infinite place.
8. $F_\mathfrak{p}$: the completion of F at $\mathfrak{p}$.
9. $\mathcal{O}_\mathfrak{p}$: the ring of the integers of $F_\mathfrak{p}$ at $\mathfrak{p}$.

10. $\pi_\mathfrak{p}$: uniformizer at $\mathfrak{p}$; $\mathfrak{p}\mathcal{O}_\mathfrak{p} = \pi_\mathfrak{p}\mathcal{O}_\mathfrak{p}$.

11. $U_\mathfrak{p} = \mathcal{O}_\mathfrak{p}^* = \mathcal{O}_\mathfrak{p} - \pi_\mathfrak{p}\mathcal{O}_\mathfrak{p}$ is the group of units of $\mathcal{O}_\mathfrak{p}$.

12. $\Sigma_F = \mathrm{Hom}(F, \mathbb{C})$ - all embeddings of F into $\mathbb{C}$; $\mathrm{Hom}(F, \mathbb{C}) = \mathrm{Hom}(F, \overline{\mathbb{Q}})$.

13. S_∞: set of all Archimedean places of F; Let $r_1 = \#S_r$, $r_2 = \#S_c$ and $d_F = r_1 + 2r_2$.

6.1 Properties of Class Groups

Let $\mathbb{J}_F$ be the group of fractional ideals of F and $\mathbb{P}_F$ be the group of principal fractional ideals of F. $x >> 0$ implies that $x \in F$ is totally positive and that $\rho(x) > 0$ for all $\rho \in \mathrm{Hom}(F, \mathbb{R})$. Let $\mathbb{P}_F^+ = \{(x) \in \mathbb{P}_F : x >> 0\}$; $Cl_F = \mathbb{J}_F/\mathbb{P}_F$ is the class group of F; $Cl_F^+ = \mathbb{J}_F/\mathbb{P}_F^+$ is the **narrow class group** of F.

The narrow class group surjects onto the class group of F and thus we have the exact sequence

$$0 \longrightarrow \frac{\mathbb{P}_F}{\mathbb{P}_F^+} \longrightarrow \frac{\mathbb{J}_F}{\mathbb{P}_F^+} \longrightarrow \frac{\mathbb{J}_F}{\mathbb{P}_F} \longrightarrow 0 \tag{6.1}$$

Now we have that

$$\frac{\mathbb{P}_F}{\mathbb{P}_F^+} \cong \frac{F^*/U_F}{F_+^*/U_F^+} \cong \frac{F^*}{F_+^* U_F}$$

where $F_+^* = \{x \in F^* : x >> 0\}$ and $U_F^+ = U_F \cap F_+^*$.

We have another exact sequence

$$0 \longrightarrow \frac{U_F}{U_F^+} \longrightarrow \frac{F^*}{F_+^*} \longrightarrow \frac{F^*}{F_+^* U_F} \longrightarrow 0 \tag{6.2}$$

Now combining the exact sequences (1) and (2) we get,

$$0 \longrightarrow \frac{U_F}{U_F^+} \longrightarrow \frac{F^*}{F_+^*} \longrightarrow Cl_F^+ \longrightarrow Cl_F \longrightarrow 0 \tag{6.3}$$

Our focus will be on discussing the same properties in terms of the idèle class group for which we will follow the following set of notations:

$\mathbb{A}_F$ is the adèle ring of F; $\mathbb{I}_F$ is the idèle group of F; $C_F = \mathbb{I}_F/F^*$ is the **idèle class group** of F; $U_\mathfrak{p} = \mathcal{O}_\mathfrak{p}^*$ for finite $\mathfrak{p}$, $U_\mathfrak{p} = \mathbb{R}_+^*$ for $\mathfrak{p} \in S_r$, and $U_\mathfrak{p} = \mathbb{C}_+^*$ for $\mathfrak{p} \in S_c$;

$F_\infty := F \otimes \mathbb{R} \cong \prod_{v \in S_r} F_v \times \prod_{w \in S_c} F_w \cong \prod_{v \in S_r} \mathbb{R} \times \prod_{w \in S_c} \mathbb{C}$

$F_\infty^+ = \{x = (x_\lambda) \in F_\infty : x_v > 0\}$

Then we have the following:

$$\frac{\mathbb{I}_F}{F^*(F_\infty^* \prod_{p \notin S_\infty} U_\mathfrak{p})} \cong \frac{\mathbb{J}_F}{\mathbb{P}_F} = Cl_F \tag{6.4}$$

Consider the canonical map $\phi : \mathbb{I}_F \to \mathbb{J}_F$ such that for $x = (x_\mathfrak{p}) \in \mathbb{I}_F$, $\phi(x) := \prod_{p \notin S_\infty} \mathfrak{p}^{ord_\mathfrak{p}(x_\mathfrak{p})}$. The kernel of ϕ will contain all such $x \in \mathbb{I}_F$ such that $\phi(x) = 1$ or those x for which $ord_\mathfrak{p}(x_\mathfrak{p}) = 0$. Therefore, $\mathrm{Ker}(\phi) \subseteq F^*(F_\infty^* \prod_{p \notin S_\infty} U_\mathfrak{p})$. Now for an element $x \in F^*(F_\infty^* \prod_{p \notin S_\infty} U_\mathfrak{p})$ it is clear that with respect to a finite prime $\mathfrak{p}$, $ord_\mathfrak{p}(x) = 0$, which implies that $x \in \mathrm{Ker}(\phi)$. Therefore, $F^*(F_\infty^* \prod_{p \notin S_\infty} U_\mathfrak{p}) \subseteq \mathrm{Ker}(\phi)$. Thus $\mathrm{Ker}(\phi) = F^*(F_\infty^* \prod_{p \notin S_\infty} U_\mathfrak{p})$.

Similarly, for the narrow class group we get:

$$\frac{\mathbb{I}_F}{F_+^*(F_\infty^* \prod_{p \notin S_\infty} U_\mathfrak{p})} \cong \frac{\mathbb{J}_F}{\mathbb{P}_F^+} = Cl_F^+ \tag{6.5}$$

The weak approximation theorem gives us that F^* is dense in F_∞^*; therefore, $F_+^* F_\infty^* = F^*(F_\infty^+)^*$.

Hence,

$$\frac{\mathbb{I}_F}{F^*((F_\infty^+)^*\prod_{p\notin S_\infty}U_\mathfrak{p})} \cong \frac{\mathbb{J}_F}{\mathbb{P}_F^+} = Cl_F^+ \tag{6.6}$$

Now combining (4), (6) and (3) and from the weak approximation theorem we can see that the diagonal inclusion $F^* \hookrightarrow F_\infty^*$ induces the canonical map $F^*/F_+^* \to F_\infty^*/(F_\infty^+)^*$. Thus we get the exact sequence:

$$0 \longrightarrow \frac{U_F}{U_F^+} \longrightarrow \frac{F_\infty^*}{(F_\infty^+)^*} \longrightarrow \frac{\mathbb{I}_F}{F^*((F_\infty^+)^*\prod_{p\notin S_\infty}U_\mathfrak{p})} \longrightarrow \frac{\mathbb{I}_F}{F^*(F_\infty^*\prod_{p\notin S_\infty}U_\mathfrak{p})} \longrightarrow 0 \tag{6.7}$$

6.2 Introduction to Hecke Characters

A **Hecke character** is a continuous homomorphism from the idèle class group to the multiplicative group of complex numbers:

$$\chi : \mathbb{I}_F/F^* \mapsto \mathbb{C}^*$$

Note that we do not require χ to be unitary.
The norm of an idèle $x \in \mathbb{I}_F$ given by the formula $\|x\| := \prod_\mathfrak{p} \mid x_\mathfrak{p} \mid_\mathfrak{p}$; where $\mathfrak{p}$ represents all the valuations of F, each of them is normalized. Thus the map $\|\| : \mathbb{I}_F \to \mathbb{R}_+^*$ is surjective. Let $\mathbb{I}_F^0$ be the kernel of the map.

Now we have that $\|a\| = 1$ if $a \in F^*$ which implies that $F^* \subset \mathbb{I}_F^0$. Therefore we have the split exact sequence

$$0 \longrightarrow \frac{\mathbb{I}_F^0}{F^*} \longrightarrow \frac{\mathbb{I}_F}{F^*} \longrightarrow \mathbb{R}_+^* \longrightarrow 0 \tag{6.8}$$

which gives us

$$\mathbb{I}_F/F^* \cong \mathbb{I}_F^0/F^* \times \mathbb{R}_+^* \tag{6.9}$$

Since $\mathbb{I}_F^0/F^*$ is compact, hence a continuous map from the compact space to $\mathbb{C}^*$ has a compact image, S^1 and a homomorphism from $\mathbb{R}_+^* \to \mathbb{C}^*$ is of the form $x \mapsto \mid x \mid^s$ where $s = (\sigma + i\phi) \in \mathbb{C}$. Therefore any Hecke character χ can be factored into

$$\chi = \chi^0 \otimes \|\|^{\sigma} \tag{6.10}$$

for a unitary Hecke character $\chi^0 : \mathbb{I}_F^0/F^* \to S^1$ and $\sigma \in \mathbb{R}$.

6.3 Dirichlet Characters

A Hecke character $\chi : \mathbb{I}_F/F^* \to \mathbb{C}^*$ with finite image and unramified everywhere gives a character of $\frac{\mathbb{I}_F}{F^*((F_\infty^+)^* \prod_{p \notin S_\infty} U_\mathfrak{p})}$ which implies that it is a character of Cl_F^+. This now requires introducing some **level structure** to give a description of Hecke character of finite order in terms of character of narrow class group with some level structure.

Let us fix some notations: $\mathfrak{m}$ is an integral ideal such that $\mathfrak{m} = \prod_{\mathfrak{p} \notin S_\infty} \mathfrak{p}^{m_\mathfrak{p}}$;

$$U_\mathfrak{p}(m_\mathfrak{p}) = \begin{cases} 1 + \mathfrak{p}^{m_\mathfrak{p}} & \mathfrak{p} \notin S_\infty, \mathfrak{p} \mid \mathfrak{m} \\ U_\mathfrak{p} & \mathfrak{p} \notin S_\infty, \mathfrak{p} \nmid \mathfrak{m} \\ \mathbb{R}_+^* & \mathfrak{p} \in S_r \\ \mathbb{C}^* & \mathfrak{p} \in S_c \end{cases}$$

$\mathbb{U}_F(\mathfrak{m}) := \prod_\mathfrak{p} U_\mathfrak{p}(m_\mathfrak{p})$; $\mathbb{U}_{F,f}(\mathfrak{m}) := \prod_{\mathfrak{p} \notin S_\infty} U_\mathfrak{p}(m_\mathfrak{p})$.

$C_F(\mathfrak{m}) := \mathbb{U}_F(\mathfrak{m})F^*/F^*$ is the congruence subgroup mod $\mathfrak{m}$; $C_F/C_F(\mathfrak{m}) := \mathbb{I}_F/\mathbb{U}_F(\mathfrak{m})F^*$ is the idèle narrow class group mod $\mathfrak{m}$.

$\mathbb{J}_F(\mathfrak{m})$ is the group of fractional ideals relatively prime to $\mathfrak{m}$; $\mathbb{P}_F(\mathfrak{m})$ be the group of principal fractional ideals (x) with $x \equiv 1 \pmod{\mathfrak{m}}$; $\mathbb{P}_F^+(\mathfrak{m})$ is all (x) such that $x >> 0$; $Cl_F^+(\mathfrak{m}) = \mathbb{J}_F(\mathfrak{m})/\mathbb{P}_F^+(\mathfrak{m})$ is the narrow class group mod $\mathfrak{m}$; $Cl_F(\mathfrak{m}) = \mathbb{J}_F(\mathfrak{m})/\mathbb{P}_F(\mathfrak{m})$ is the class group mod $\mathfrak{m}$.

Proposition 6.1. (Proposition 1, Section 2.3, [5]) The canonical homomorphism $w : \mathbb{I}_F \to \mathbb{J}_F$ induces an isomorphism

$$\frac{C_F}{C_F(\mathfrak{m})} \cong \frac{\mathbb{J}_F(\mathfrak{m})}{\mathbb{P}_F^+(\mathfrak{m})}$$

Proof. Let $\mathbb{I}_F^{(\mathfrak{m})} = \{\alpha \in \mathbb{I}_F : \alpha_\mathfrak{p} \in U_\mathfrak{p}(m_\mathfrak{p}), \mathfrak{p} \mid \mathfrak{m}.\infty\}$. Then $\mathbb{I}_F = \mathbb{I}_F^{(\mathfrak{m})} F^*$ since for every $\alpha \in \mathbb{I}_F$ there exists an $a \in F^*$ such that $\alpha_\mathfrak{p} a \equiv 1 \pmod{\mathfrak{p}^{n_\mathfrak{p}}}$ for $\mathfrak{p} \mid \mathfrak{m}$ and $\alpha_\mathfrak{p} a > 0$ for real $\mathfrak{p}$. Therefore, $\beta = (\alpha_\mathfrak{p} a) \in \mathbb{I}_F^{(\mathfrak{m})}$ so that $\alpha = \beta a^{-1} \in \mathbb{I}_F^{(\mathfrak{m})} F^*$. The elements $a \in \mathbb{I}_F^{(\mathfrak{m})} \cap F^*$ generate principal ideals in $P_F^{\mathfrak{m}}$. Therefore the map w defines a surjective homomorphism $C_F = \mathbb{I}_F^{(\mathfrak{m})} F^*/F^* = \mathbb{I}_F^{(\mathfrak{m})}/\mathbb{I}_F^{(\mathfrak{m})} \cap F^* \to Cl_F^+(\mathfrak{m})$.

Since $(\alpha) = 1$ for $\alpha \in \mathbb{I}_F^{(\mathfrak{m})}$, the group $C_F^{(\mathfrak{m})}$ is certainly contained in the kernel. Conversely, if $\alpha \in \mathbb{I}_F^{(\mathfrak{m})}$ is in the kernel then there is an $(a) \in P_F^{\mathfrak{m}}$ with $a \in \mathbb{I}_F^{(\mathfrak{m})} \cap F^*$ such that $(\alpha) = (a)$. For the idèle $\beta = \alpha a^{-1}$, $\beta_\mathfrak{p} \in U_\mathfrak{p}$ for $\mathfrak{p} \nmid \mathfrak{m}.\infty$ and $\beta_\mathfrak{p} \in U_\mathfrak{p}(m_\mathfrak{p})$ for $|\mathfrak{m}.\infty$ which implies that $\beta \in \mathbb{I}_F^{(\mathfrak{m})}$ and so the classes $[\alpha] = [\beta] \in \mathbb{I}_F^{(\mathfrak{m})} F^*/F^*$. Hence, $C_F^{\mathfrak{m}}$ is the kernel of w. Therefore, w is an isomorphism. □

Proposition 6.2. (Proposition 2, Section 2.3, [5]) The congruence subgroup $C_F(\mathfrak{m})$ for any integral ideal $\mathfrak{m}$, is a subgroup of C_F of finite index. The converse is also true.

Proof. $C_F(\mathfrak{m})$ is open in C_F since $\mathbb{U}_F(\mathfrak{m}) = \prod_\mathfrak{p} U_\mathfrak{p}(m_\mathfrak{p})$ is open in $\mathbb{I}_F$. $\mathbb{U}_F(\mathfrak{m})$ is contained in $F_\infty^* \prod_{p \notin S_\infty} U_\mathfrak{p}$ and since $((\mathbb{I}_F/F^*) : (F_\infty^* \prod_{p \notin S_\infty} U_\mathfrak{p} F^*/F^*)) = \#Cl_F = h(\text{say}) < \infty$, we get that:

$C_F : C_F(\mathfrak{m}) = h(F_\infty^* \prod_{p \notin S_\infty} U_\mathfrak{p} F^* : \mathbb{U}_F(\mathfrak{m}) F^*) \leqslant h(F_\infty^* \prod_{p \notin S_\infty} U_\mathfrak{p} : \mathbb{U}_F(\mathfrak{m})) = h \prod_{\mathfrak{p} \nmid \infty} (U_\mathfrak{p} : U_\mathfrak{p}(m_\mathfrak{p})) \prod_{\mathfrak{p} \mid \infty} (U_\mathfrak{p} : U_\mathfrak{p}(m_\mathfrak{p}))$ which is finite.

Conversely, let $\mathcal{N}$ be an arbitrarily closed subgroup of finite index. Then $\mathcal{N}$ is also open, since it is the complement of a finite number of closed cosets. Then the pre-image of $\mathcal{N}$ in $\mathbb{I}_F$ is also open, and contains a subset of the form $W = \prod_{\mathfrak{p} \in S} W_\mathfrak{p} \times \prod_{\mathfrak{p} \notin S} U_\mathfrak{p}$, where S is a finite set of places of F containing the infinite ones, and $W_\mathfrak{p}$ is an open neighborhood of $1 \in F_\mathfrak{p}^*$. If $\mathfrak{p} \in S$ is finite, we have to choose $W_\mathfrak{p} = U_\mathfrak{p}(m_\mathfrak{p})$ since the groups $U_\mathfrak{p}(m_\mathfrak{p})$ form basic neighborhood of $1 \in F_\mathfrak{p}^*$. If $\mathfrak{p}$ is real we have $W_\mathfrak{p} \subseteq \mathbb{R}_+^*$. This will generate the group $\mathbb{R}_+^*$, and $F_\mathfrak{p}^*$ in case of complex place. The subgroup of pre-image generated by W is of the form $\mathbb{U}_F(\mathfrak{m})$, so $\mathcal{N}$ contains the subgroup $C_F(\mathfrak{m})$. □

These show that Hecke characters of finite order are actually the characters of the narrow ray class group. Consider the map $\chi : \mathbb{I}_F/F^* \to \mathbb{C}^*$ which is a continuous homomorphism who has finite order image. Then we have an integral ideal $\mathfrak{m}$ such that:

$$\begin{array}{ccc} \mathbb{I}_F/F^* & \xrightarrow{\chi} & \mathbb{C}^* \\ \downarrow & \nearrow & \\ \dfrac{C_F}{C_F(\mathfrak{m})} & & \end{array}$$

The smallest such $\mathfrak{m}$ is called the conductor of χ, denoted as f_χ.

A **Dirichlet character** of F is given by the homomorphism

$$\chi : \mathbb{J}_F(\mathfrak{m})/\mathbb{P}_F(\mathfrak{m})^+ \to S^1$$

for some integral ideal $\mathfrak{m}$. Generally the algebraic Hecke characters are not of finite order.

6.4 Hecke Character: Character at Infinity

6.4.1 Characters of $\mathbb{R}^*$

A continuous homomorphism $\chi : \mathbb{R}^* \to \mathbb{C}^*$ of the form

$$\chi(x) = \text{sgn}(x)^{n_r} \mid x \mid^w = (\frac{x}{\mid x \mid})^{n_r} \mid x \mid^w \tag{6.11}$$

where $n_r \in \{0, 1\}$ and $w \in \mathbb{C}$ is a character of $\mathbb{R}^*$. This character is unitary if and only if $w = i\phi$.

6.4.2 Characters of $\mathbb{C}^*$

For a complex number $z = x + iy \in \mathbb{C}$, we define $\mid z \mid := \mid z \mid_{\mathbb{R}} := \sqrt{x^2 + y^2}$. $\mid z \mid_{\mathbb{C}} := x^2 + y^2$ is the normalized valuation on $\mathbb{C}$. A continuous homomorphism $\chi : \mathbb{C}^* \to \mathbb{C}^*$ of the form

$$\chi(z) = (\frac{z}{\mid z \mid})^{n_c} \mid z \mid_{\mathbb{C}}^{w} \tag{6.12}$$

with $n_c \in \mathbb{Z}$ and $w \in \mathbb{C}$ is a character of $\mathbb{C}^*$. This character is unitary if and only if $w = i\phi$.

6.4.3 Character at Infinity of Hecke Character

For a Hecke Character, χ, let us denote the **character at infinity** as $\chi_\infty = \chi|_{F_\infty^*}$ where $F_\infty^* \hookrightarrow \mathbb{I}_F$. Let $\lambda \in S_\infty$, $v \in S_r$ and $F_v \cong \mathbb{R}$ canonically, $w \in S_c$ and $F_w \cong \mathbb{C}$ non-canonically and $\mid x_\infty \mid_\infty = \prod_\lambda \mid x_\lambda \mid_\lambda$ for $x_\infty \in F_\infty$.

We can write the character at infinity χ_∞ on $x_\infty \in F_\infty^*$ as

$$\chi_\infty(x_\infty) = (\prod_{\lambda \in S_\infty} (\frac{x_\lambda}{\mid x_\lambda \mid})^{n_\lambda} \mid x_\lambda \mid_\lambda^{i\phi_\lambda}) \mid x_\infty \mid_\infty^{\sigma} \tag{6.13}$$

where $n_v \in \{0, 1\}, n_w \in \mathbb{Z}, \phi_\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{R}$.

6.5 Größencharaktere mod m

Let us consider the following set of notations for the next section: $U_F^1(m) := \{u \in U_F : u \equiv 1 \pmod{m}\}$; $U_F^1(m)^+ := \{u \in U_F^1(m) : u >> 0\}$; $\mathcal{O}_F(m) := \{a \in \mathcal{O}_F : (a, m) = 1\}$; $F(m) := \{x \in F : (x, m) = 1\}$; $F^1(m) := \{x \in F : x \equiv 1 \pmod{m}\}$.

6.5.1 Introduction to Größencharaktere

A Größencharakter mod m is a homomorphism $\psi : \mathbb{J}_F(m) \to \mathbb{C}^*$ for which there exists a pair of characters (ψ_f, ψ_∞) with $\psi_f : (\mathcal{O}_F/m)^* \to \mathbb{C}^*$ and $\psi_\infty : F_\infty^* \to \mathbb{C}^*$ such that for all $a \in \mathcal{O}_F(m)$, we have $\psi((a)) = \psi_f(a \bmod m)\psi_\infty(a)$, (a) is the principal ideal generated by a.

If $u \in U_F = \mathcal{O}_F^*$ is a unit, then $\psi((u)) = 1$ and so $\psi_f(u \bmod m)\psi_\infty(u) = 1$. If $\epsilon \in U_F^1(m) := \{u \in U_F(m) : u \equiv 1 \pmod{m}\}$ then $\psi_\infty(\epsilon) = 1$.

The restriction of Größencharakter mod m, ψ, to the group $\mathbb{P}_F(m)$ uniquely determines ψ_f and ψ_∞ satisfying the compatibility conditions.

Conversely if we have homomorphisms $\Psi_f : (\mathcal{O}_F/m)^* \to \mathbb{C}^*$ and $\Psi_\infty : F_\infty^* \to \mathbb{C}^*$ satisfying compatibility condition that $\Psi_f(u \pmod{m})\Psi_\infty(u) = 1$ for all $u \in U_F$ then there exists a Größencharakter ψ mod m such that $\psi_f = \Psi_f$ and $\psi_\infty = \Psi_\infty$.

6.5.2 Hecke characters and Größencharaktere correspondence

The following proposition shows the surjection between the domain of a Größencharakter and the domain of a Hecke character.

Proposition 6.3. (Proposition 5, Section 3.1, [5]) We have an exact sequence

$$1 \longrightarrow \frac{F(m)}{U_F^1(m)} \xrightarrow{\varkappa} \mathbb{J}_F(m) \times (\mathcal{O}_F/m)^* \times \frac{F_\infty^*}{U_F^1(m)} \xrightarrow{\varrho} \frac{\mathbb{I}_F}{F^*\mathbb{U}_{F,f}(m)} \longrightarrow 1$$

where

1. $\varkappa(a) = ((a)^{-1}, a \pmod{m}, a \pmod{U_F^1(m)})$ for all $a \in F(m)$.
2. $\varrho = \alpha \otimes \beta \otimes \gamma^{-1}$, where
    - $\alpha : \mathbb{J}_F(m) \to \mathbb{I}_F/(F^*\mathbb{U}_{F,f}(m))$. This is induced by $\mathbb{J}_F(m) \to \mathbb{I}_F$ which maps a prime ideal $\mathfrak{p}$ to the idèle $\pi_\mathfrak{p}$ at the place $\mathfrak{p}$ and 1 elsewhere.
    - $\beta : (\mathcal{O}_F/m)^* \to \mathbb{I}_F/(F^*\mathbb{U}_{F,f}(m))$. This map sends $a \in \mathcal{O}_F(m)$ to the idèle containing a at all infinite places, $\mathfrak{p}$ at $\mathfrak{p} \nmid m$ places and 1 at all $\mathfrak{p} \mid m$ places.
    - $\gamma : F_\infty^*/U_F^1(m) \to \mathbb{I}_F/(F^*\mathbb{U}_{F,f}(m))$. This map is induced by the inclusion $F_\infty^* \hookrightarrow \mathbb{I}_F$ and $U_F^1(m)$ maps to $F^*\mathbb{U}_{F,f}(m)$.

Proof. This proposition has immense importance since this clarifies the relation between the Hecke characters and the Größencharakter.

For $a \in F(\mathfrak{m})$ we have that $\varrho(\varkappa(a)) = \alpha((a))^{-1}\beta(a)\gamma(a)$.

The map $\alpha : \mathbb{J}_F(m) \to \mathbb{I}_F/(F^*\mathbb{U}_{F,f}(m))$ sends $\alpha((a)) = \hat{a} \pmod{F^*\mathbb{U}_{F,f}(m)}$.

The map $\beta : (\mathcal{O}_F/m)^* \to \mathbb{I}_F/(F^*\mathbb{U}_{F,f}(m))$ sends $\beta(a) = \hat{a}a_\infty \pmod{F^*\mathbb{U}_{F,f}(m)}$.

The map $\gamma : F^*_\infty/U^1_F(m) \to \mathbb{I}_F/(F^*\mathbb{U}_{F,f}(m))$ sends $\gamma(a) = a^{-1} \pmod{F^*\mathbb{U}_{F,f}(m)}$.

Then $\varrho(\varkappa(a)) = \hat{a}^{-1}\hat{a}a_\infty a_\infty^{-1} \pmod{F^*\mathbb{U}_{F,f}(m)} = 1$ such that $\varrho \circ \varkappa = 1$. Conversely, let $\varrho((\mathfrak{a}), a \pmod{\mathfrak{m}}, b \pmod{U^1_F(\mathfrak{m})}) = \alpha(\mathfrak{a})\beta(a)\gamma(b) = 1$ and let $\mathfrak{a} = \prod_{\mathfrak{p}\nmid\mathfrak{m}.\infty} \mathfrak{p}^{v_\mathfrak{p}}$. Then $\alpha(\mathfrak{a}) = \gamma \pmod{F^*\mathbb{U}_{F,f}(m)}$ for some idèle γ with components $\gamma_\mathfrak{p} - \pi_\mathfrak{p}^{v_\mathfrak{p}}$ for $\mathfrak{p} \nmid \mathfrak{m}.\infty$ and $\gamma_\mathfrak{p} = 1$ for $\mathfrak{p} \mid \mathfrak{m}.\infty$. This gives us $\gamma\hat{a}a_\infty b^{-1} = \xi x$ with $\xi \in \mathbb{U}_{F,f}(m)$ and $x \in F^*$. For $\mathfrak{p} \nmid \mathfrak{m}.\infty$ we have $(\gamma\hat{a}a_\infty b^{-1})_\mathfrak{p} = 1 = \xi_\mathfrak{p} x$ in $F_\mathfrak{p}$. For $\mathfrak{p} \mid \mathfrak{m}$ we have $(\gamma\hat{a}a_\infty b^{-1})_\mathfrak{p} = 1 = \xi_\mathfrak{p} x$ so that $x \in U_\mathfrak{p}(\mathfrak{m}_\mathfrak{p})$ and $v_\mathfrak{p} = \nu_\mathfrak{p}(a^{-1}x) = 0$ since a is relatively prime to $\mathfrak{m}$. Thus $\mathfrak{a} = (ax^{-1})$.

As $x \in U_\mathfrak{p}(\mathfrak{m}_\mathfrak{p})$ then $x \equiv 1 \pmod{\mathfrak{m}}$ hence $\phi(ax^{-1}) = \phi(a)$. For $\mathfrak{p} \mid \infty$ we find $(\gamma\hat{a}a_\infty b^{-1})_\mathfrak{p} = ab_\mathfrak{p}^{-1} = x$ in $F_\mathfrak{p}$ and so $b = a_\infty x^{-1}$ and thus $\psi(ax^{-1}) = \psi(b)$.

So we have that $(\mathfrak{a}, a \pmod{\mathfrak{m}}, b \pmod{U^1_F(\mathfrak{m})}) = ((ax^{-1}), ax^{-1} \pmod{\mathfrak{m}}, ax^{-1} \pmod{U^1_F(\mathfrak{m})})$ and this shows the exactness of the sequence in middle.

Let $\alpha \pmod{F^*\mathbb{U}_{F,f}(m)}$ be a class in $\mathbb{I}_F/(F^*\mathbb{U}_{F,f}(m))$. Then by approximation theorem we can multiply $x \in F^*$ to the idèle α such that $\alpha_\mathfrak{p} \in U_\mathfrak{p}(\mathfrak{m}_\mathfrak{p})$ for $\mathfrak{p} \mid \mathfrak{m}$.

Let $\mathfrak{a} = \prod_{\mathfrak{p}\nmid\mathfrak{m}.\infty} \mathfrak{p}^{v_\mathfrak{p}(\alpha_\mathfrak{p})}$. Then $\alpha(\mathfrak{a}) = \gamma \pmod{\mathbb{I}^\mathfrak{m}_\mathfrak{p} F^*}$ where γ has $\gamma_\mathfrak{p} = \pi_\mathfrak{p}^{v_\mathfrak{p}(\alpha_\mathfrak{p})} = \epsilon_\mathfrak{p}\alpha_\mathfrak{p}, \epsilon_\mathfrak{p} \in U_\mathfrak{p}$ for $\mathfrak{p} \nmid \mathfrak{m}.\infty$ and $\gamma_\mathfrak{p} = 1$ for $\mathfrak{p} \mid \mathfrak{m}.\infty$. Thus $\gamma\alpha^{-1}\alpha_\infty \in \mathbb{I}^\mathfrak{m}_\varrho$ and if we have $b = a_\infty^{-1}$ then $\varrho((\mathfrak{a}, 1 \pmod{\mathfrak{m}}, b \pmod{U^1_F(\mathfrak{m})})) = \gamma b^{-1} \equiv \alpha \pmod{\mathbb{I}^\mathfrak{m}_\varrho F^*}$. Therefore the map ϱ is a surjective homomorphism. □

By the previous proposition, the characters of $\mathbb{I}_F/F^*\mathbb{U}_{F,f}(m)$ correspond to the characters of $\mathbb{J}_F(m) \times (\mathcal{O}_F/m)^* \times F^*_\infty/U^1_F(m)$ that vanish on $\varkappa(F(m)/U^1_F(m))$ and so $\chi, \chi_f, \chi_\infty$ of characters of $\mathbb{J}_F(m)$, $(\mathcal{O}_F/m)^*$ and $F^*_\infty/U^1_F(m)$ respectively such that $\chi((a))^{-1}\chi_f(a \pmod{\mathfrak{m}})\chi_\infty(a \pmod{U^1_F(m)}) = 1$ for $a \in F(\mathfrak{m})$. Thus χ is a Größencharakter mod $\mathfrak{m}$ and χ_f, χ_∞ are uniquely determined by χ.

6.5.3 Infinity type of a Größencharaktere

Given a Größencharaktere $\psi = (\psi_f, \psi_\infty) \pmod{m}$, the character at infinity is given by a continuous map $\psi_\infty : F_\infty^* \to \mathbb{C}^*$ of the form

$$\psi_\infty(x_\infty) = (\prod_{\lambda \in S_\infty} (\frac{x_\lambda}{\mid x_\lambda \mid})^{n_\lambda} \mid x_\lambda \mid_\lambda^{i\phi_\lambda}) \mid x_\infty \mid_\infty^{\sigma} \tag{6.14}$$

where $n_v \in \{0, 1\}, n_w \in \mathbb{Z}, \phi_\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{R}$.

Let ψ be a Größencharaktere mod m with the character at infinity being ψ_∞. Then it as the following properties:

1. There are no restrictions on σ, and n_λ for any $\lambda \in S_\infty$
2. Also for all $v \in S_r$ and all $w \in S_c$ we have $2\phi_v = \phi_w = \phi(\text{say})$.

Therefore, the character at infinity takes the form

$$\psi_\infty(x_\infty) = (\prod_{\lambda \in S_\infty} (\frac{x_\lambda}{\mid x_\lambda \mid})^{n_\lambda}) \mid x_\infty \mid_\infty^{\sigma + i\phi} \tag{6.15}$$

Chapter 7

Algebraic Hecke Characters and L-functions

This chapter focuses on algebraic Hecke characters, their properties and the L-functions attached to them. The first section includes the introduction to algebraic Hecke character. The next two sections discuss about the L-function of an algebraic Hecke character and the existence of critical Hecke characters. Lastly we discuss briefly about the algebraic Hecke characters having coefficients in a finite Galois extension of $\mathbb{Q}$.

7.1 Introduction to Algebraic Hecke Characters

We can get a canoninical map from $\Sigma_F \to S_\infty$ with $\lambda \in S_\infty$. Now for $\lambda = v \in S_r$ we have a real embedding $\tau_v : F \to \mathbb{R}$ and for $\lambda \in S_c$ we have a conjugate pair of complex embeddings $\{\tau_w, \bar{\tau}_w\}$ knowing that the choice of $\tau_w : F \to \mathbb{C}$ is not canonical.

We have $F_\infty = F \otimes \mathbb{R} \cong \prod_{\lambda \in S_\infty} F_\lambda \cong \prod_{v \in S_r} \mathbb{R} \times \prod_{w \in S_c} \mathbb{C}$. Thus for $x_\infty \in F_\infty$ we can write $x_\infty = (x_\lambda)_{\lambda \in S_\infty} = ((x_v)_{v \in S_r}, (z_w)_{w \in S_c})$.

Let χ be a Hecke character of F and χ_∞ be its character at infinity. Such a character is an **algebraic Hecke character** if for every embedding $\tau \in \Sigma_F$, there exists an integer n_τ

such that for $x_\infty \in F_\infty^*$ we have

$$\chi_\infty(x_\infty) = \prod_{v \in S_r} x_v^{n_{\tau w}} \prod_{w \in S_c} z_w^{n_{\tau w}} \bar{z}_w^{n_{\bar{\tau} w}} \tag{7.1}$$

which can be re-written as

$$\chi_\infty(x_\infty) = (\prod_{v \in S_r} (\frac{x_v}{\mid x_v \mid})^{n_{\tau v}} \prod_{w \in S_c} (\frac{z_w}{\mid z_w \mid})^{n_{\tau w} - n_{\bar{\tau} w}})(\prod_{v \in S_r} \mid x_v \mid^{n_{\tau v}} \prod_{w \in S_c} \mid z_w \mid^{(n_{\tau w} + n_{\bar{\tau} w})/2}) \tag{7.2}$$

Comparing with equation (6.13) we get that for $v \in S_r, w \in S_c$ and $\lambda \in S_\infty$

$$n_v \equiv n_{\tau_v} \pmod 2, n_w = n_{\tau_w} - n_{\bar{\tau}_w}, \phi_\lambda = 0, 2\sigma = 2n_{\tau_v} = n_{\tau_w} + n_{\bar{\tau}_w}$$

Let us consider that a Hecke character has modulus m:

Lemma 7.1 (Purity Lemma)**.** (Lemma 7, Section 4.1, [5]) For each $\tau \in \Sigma_F$, suppose we are given $n_\tau \in \mathbb{Z}$. Suppose for some integral ideal m of F we have $\prod_{\tau \in \Sigma_F} \tau(u)^{n_\tau} = 1$ for all $u \in U_F^1(m)$ Then there exists $\mathbf{w} \in \mathbb{Z}$ such that

1. if $S_r \neq \varnothing$, then $n_\tau = \mathbf{w}$ for all $\tau \in \tau_F$
2. if $S_r = \varnothing$, then $n_{\gamma \circ \tau} + n_{\gamma \circ \bar{\tau}} = \mathbf{w}$ for all $\tau \in \Sigma_F$ and $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

Proof. Let the real embeddings of F be $\{v_1, v_2, \cdots, v_r\}$ and the complex pair of embeddings be $\{w_1, \bar{w}_1, w_2, \bar{w}_2, \cdots, w_c, \bar{w}_c\}$. Let us denote $\tau_{v_i} = v_i$ and $\tau_{w_j} = w_j$. Now denote the hyperplane defined by the sum of co-ordinates to be zero by $\mathcal{H} \subset \mathbb{R}^{r+c}$. The Minkowski map $\mathfrak{i} : U_F \to \mathcal{H}$ is given by $\mathfrak{i}(u) = (log|v_1(u)|_\mathbb{R}, \cdots, log|v_r(u)|_\mathbb{R}, log|w_1(u)|_\mathbb{C}, \cdots, log|w_c(u)|_\mathbb{C})$.

We have that $U_F^1(\mathfrak{m})$ has finite index in U_F. Applying $log\ ||_\mathbb{C}$ to $\prod_{\tau \in \Sigma_F} \tau(u)^{n_\tau} = 1$ we get

$$2n_{v_1} log|v_1(u)|_\mathbb{R} + \cdots + 2n_{v_r} log|v_r(u)|_\mathbb{R} + 2n_{w_1} log|w_1(u)|_\mathbb{C} + \cdots + 2n_{w_c} log|w_c(u)|_\mathbb{C} = 0 \tag{7.3}$$

for all $u \in U_F^1(\mathfrak{m})$.

From Dirichlet's unit theorem we have $\Gamma = \mathfrak{i}(U_F)$ is a lattice in the hyperplane $\mathcal{H}$. Therefore, $\Gamma_F^1(\mathfrak{m}) = \mathfrak{i}(U_F^1(\mathfrak{m}))$ is also a lattice in $\mathcal{H}$. Thus there exists $u_1, \cdots, u_{t-1} \in U_F^1(\mathfrak{m})$

such that $\{\mathfrak{i}(u_1), \cdots, \mathfrak{i}(u_{t-1})\}$ makes an $\mathbb{R}$-basis if $\mathcal{H}$ where $t = r+c$. Let $v_i = (a_{i1}, a_{i2}, \cdots, a_{it})$ be a vector in $\mathbb{R}^t$ and consider the matrix, $A = [a_{ij}]$, of order $(t-1) \times t$. The rank and nullity of the same being $(t-1)$ and 1 respectively. We have that $\sum_j a_{ij} = 0$ since each $v_i \in \mathcal{H}$. Therefore for $X = (x_1, x_2, \cdots, x_{t+1}) \in \mathbb{R}^{t+1}$ if $AX = 0$ then we have that all the co-ordinates $(x_1, x_2, \cdots, x_{t+1})$ are equal. Applying (18) to $u_1, \cdots, u_{t-1}$ will give us a solution to $AX = 0$ from which we have $2n_{v_1} = \cdots = 2n_{v_r} = n_{w_1} + n_{\bar{w}_1} = \cdots = n_{w_c} + n_{\bar{w}_c}$. Let this common value be $\mathbf{w}$.

Now for any $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, applying γ^{-1} to $\prod_{\tau \in \Sigma_F} \tau(u)^{n_\tau} = 1$ we get $\prod_{\tau \in \tau_F} \gamma^{-1}\tau(u)^{n_\tau} = \prod_{\tau \in \tau_F} \tau(u)^{n_{\gamma \circ \tau}} = 1$. Same argument goes for $\prod_{\tau \in \tau_F} \tau(u)^{n_{\gamma \circ \bar{\tau}}}$ thus giving $n_{\gamma \circ \tau} + n_{\gamma \circ \bar{\tau}} = \mathbf{w}; \tau \in \tau_F$ and $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Since $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts transitively on Σ_F for $S_r \neq \varnothing$ we get n_τ equal to n_{v_1} for all τ then, n_{v_1} can be assigned the value $\mathbf{w}/2$. If $S_r = \varnothing$ then we can take n_{v_1} can be assigned the value $\mathbf{w}$. □

Let χ be an algebraic Hecke character of F mod m with infinity type $(n_\tau)_{\tau \in \Sigma_F}$ which has **purity weight w** satisfying the conditions in the previous lemma. Now -

1. for both real and complex embeddings, $S_r \neq \varnothing$, then the character $\chi = \chi^o \, \|\|^{\mathbf{w}}$ for χ^o: a Dirichlet character.

2. for no real embeddings, $S_r = \varnothing$, then the character $\chi = \chi^u \, \|\|^{\mathbf{w}/2}$ for χ^u: a unitary Hecke character.

where the character at infinity of χ^u looks like $\chi^u_\infty(x_\infty) = \prod_{w \in S_c} (x_w/\bar{x}_w)^{n_w - \mathbf{w}/2}$.

7.2 Theory of Hecke L-function

7.2.1 Hecke L-function

Let us consider an algebraic Hecke character χ. Let $\psi = (\psi_\infty, \psi_f)$ be the associated Größencharakter and $\mathfrak{m}$ be the conductor of χ, and also of ψ. The finite part of the **Hecke L-function** of χ is defined as:

1. $L_f(s,\chi) = \sum_{\mathfrak{a}} \frac{\psi(\mathfrak{a})}{N_{F/\mathbb{Q}}(\mathfrak{a})^s}$ where $\mathfrak{a}$ are the integral ideals of $\mathcal{O}_F$ that are relatively prime to $\mathfrak{m}$. This is in the form of Dirichlet series.

2. $L_f(s,\chi) = \prod_{\mathfrak{p}}(1 - \chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})N_{F/\mathbb{Q}}(\mathfrak{p})^{-s})^{-1}$ where $\mathfrak{p}$ are prime ideals of $\mathcal{O}_F$ that are relatively prime to $\mathfrak{m}$. This is in the form of Euler product.

7.2.2 The coefficients of the L-function

Proposition 7.1. (Proposition 9, Section 4.2.2, [5]) Let us consider be an algebraic Hecke character, χ. Consider $\psi = (\psi_\infty, \psi_f)$ to be the Größencharakter mod $\mathfrak{m}$ associated with χ. Then the coefficients of the Dirichlet series that represents the finite part of $L_f(s,\chi)$ are contained in a number field.

Proof. For $a \in \mathcal{O}_F(\mathfrak{m})$ take the image of the ideal (a), $\psi((a))$, and we see that it takes values in a finite extension K of $\mathbb{Q}$ inside the closure of $\mathbb{Q}$. This is because $(\mathcal{O}_F/\mathfrak{m})^*$ is a finite group and the infinite part of the Größencharakter, $\psi_\infty(a)$ takes values in the compositum of all the conjugates of F inside $\overline{\mathbb{Q}}$ since ψ_∞ is algebraic. Therefore, the values of ψ on $\mathbb{P}_F(\mathfrak{m})$ lie in K. Since $(\mathbb{J}_F(\mathfrak{m}) : \mathbb{P}_F(\mathfrak{m}))$ is finite, we can get the values of ψ in a finite extension E of K. By the correspondence between the Hecke characters and Größencharaktere we have $\psi(\mathfrak{p}) = \chi_{\mathfrak{p}}(\pi_{\mathfrak{p}})$ for all prime ideals $\mathfrak{p} \nmid \mathfrak{m}$. □

The smallest subfield of $\overline{\mathbb{Q}}$ which contains all the values of the algebraic Hecke character χ is called the **rationality field** denoted by $\mathbb{Q}(\chi)$.

7.2.3 The critical values of the L-function

The completed L-function is defined by:

$$L(s,\chi) := L_\infty(s,\chi)L_f(s,\chi) \tag{7.4}$$

where if χ has an infinity type $(n_\tau)_{\tau\in\Sigma_F}$, $n_\tau \in \mathbb{Z}$, satisfying the purity condition then we define

$$L_\infty(s,\chi) = \prod_{v\in S_r} \Gamma_{\mathbb{R}}(s + n_v + \epsilon_v) \prod_{w\in S_c} \Gamma_{\mathbb{C}}(s + \frac{n_w + n_{\bar{w}}}{2} + \frac{\mid n_w - n_{\bar{w}} \mid}{2}) \tag{7.5}$$

where $\Gamma_{\mathbb{R}} = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}} = 2(2\pi)^{-s}\Gamma(s)$ and $\epsilon_v \in \{0,1\}$ with $\epsilon_v \equiv n_v \pmod 2$.

When χ is a non-trivial character, the completed L-function admits an analytic continuation and the functional equation given by -

$$L(s,\chi) = \epsilon(s,\chi)L(1-s,\chi^{-1}) \tag{7.6}$$

where $\epsilon(s,\chi)$ is an exponential function. The set of all critical integers for $L(s,\chi)$ is denoted as $\text{Crit}(L(s,\chi))$. An integer $\mathbf{m}$ is called critical for the Hecke L-function if both $L(s,\chi)$ and $L(1-s,\chi^{-1})$ are regular at $s=\mathbf{m}$ and do not have poles at $s=\mathbf{m}$.

Proposition 7.2. (Proposition 10, Section 4.2.4, [5]) Let us consider an algebraic Hecke character of F, χ, with the infinity type $(n_\tau)_{\tau\in\Sigma_F}; n_\tau \in \mathbb{Z}$ which satisfies the purity condition. Define the purity weight of χ as $\mathbf{w}$. Then the critical set of integers for $L(s,\chi)$ is given by the following:

1. when the field is totally real($S_c = \varnothing$) then $n_\tau = \mathbf{w}$ for all τ.
   - If there exists $v_1, v_2 \in S_r$ such that $\epsilon_{v_1} \neq \epsilon_{v_2}$, then $\text{Crit}(L(s,\chi)) = \varnothing$
   - If $\epsilon_v = 0$ for all $v \in S_r$, then $\text{Crit}(L(s,\chi))$ is given by

     $\{\cdots, -\mathbf{w}+1-2k, \cdots, -\mathbf{w}-3, -\mathbf{w}-1; -\mathbf{w}+2, -\mathbf{w}+4, \cdots, -\mathbf{w}+2k, \cdots\}$ for $k \in \mathbb{Z}_{\geqslant 0}$.
   - If $\epsilon_v = 1$ for all $v \in S_r$, then $\text{Crit}(L(s,\chi))$ is given by

     $\{\cdots, -\mathbf{w}-2k, \cdots, -\mathbf{w}-2, -\mathbf{w}; -\mathbf{w}+1, -\mathbf{w}+3, \cdots, -\mathbf{w}+1+2k, \cdots\}$ for $k \in \mathbb{Z}_{\geqslant 1}$.
2. F is totally imaginary ($S_r = \varnothing$). Let the width of χ, a non-negative integer, be $\ell = \ell(\chi) = min\{\mid n_w - n_{\bar{w}} \mid : w \in S_c\}$. Then $\mathbf{w} \equiv \ell \pmod 2$ and the $\text{Crit}(L(s,\chi))$ is given by

   $$\{\mathbf{m} \in \mathbb{Z} : 1 - \tfrac{\mathbf{w}}{2} - \tfrac{\ell}{2} \leqslant \mathbf{m} \leqslant -\tfrac{\mathbf{w}}{2} + \tfrac{\ell}{2}\}$$

   The critical set is centered at $\frac{1-\mathbf{w}}{2}$, with cardinality ℓ.
3. F has both real and complex places($S_r \neq \varnothing \neq S_c$), then $\text{Crit}(L(s,\chi)) = \varnothing$.

Proof. We have that the completed L-function has the infinity part defined as $L_\infty(s,\chi) = \prod_{v\in S_r}\Gamma_{\mathbb{R}}(s+n_v+\epsilon_v)\prod_{w\in S_c}\Gamma_{\mathbb{C}}(s+\frac{n_w+n_{\bar{w}}}{2}+\frac{|\,n_w-n_{\bar{w}}\,|}{2})$ where $\Gamma_{\mathbb{R}} = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}} = 2(2\pi)^{-s}\Gamma(s)$ and $\epsilon_v \in \{0,1\}$ with $\epsilon_v \equiv n_v \pmod 2$.

The Gamma function given by $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}ds$ is nowhere zero and has simple poles at $\{s = 0,-1,-2,\cdots\}$. Also the functional equation for the L-function is given by $L(s,\chi) = \epsilon(s,\chi)L(1-s,\chi^{-1})$.

Now let us calculate the critical set of integers for the L-function in the following cases:

1. F is totally real($S_c = \varnothing$) then $n_\tau = \mathbf{w}$ for all τ:

   - $\epsilon_{v_1} \neq \epsilon_{v_2}$ and $n_{v_1} = n_{v_2} = \mathbf{w}$.

     Now for $L(s,\chi)$ we have the critical set of integers as follows,

     (a) $\Gamma_{\mathbb{R}}(s+n_{v_1}+\epsilon_{v_1}) = \Gamma_{\mathbb{R}}(s+\mathbf{w}+0)$ and the critical set of integers would be given by $\frac{s+\mathbf{w}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{-\mathbf{w},-2-\mathbf{w},-4-\mathbf{w},\cdots\}$.

     (b) $\Gamma_{\mathbb{R}}(s+n_{v_2}+\epsilon_{v_2}) = \Gamma_{\mathbb{R}}(s+\mathbf{w}+1)$ and the critical set of integers would be given by $\frac{s+\mathbf{w}+1}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{-1-\mathbf{w},-3-\mathbf{w},-5-\mathbf{w},\cdots\}$.

     Now for $L(1-s,\chi^{-1})$ we have the critical set of integers as follows,

     (a) $\Gamma_{\mathbb{R}}(1-s+n_{v_1}+\epsilon_{v_1}) = \Gamma_{\mathbb{R}}(1-s+\mathbf{w}+0)$ and the critical set of integers would be given by $\frac{1-s+\mathbf{w}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{1-\mathbf{w},3-\mathbf{w},5-\mathbf{w},\cdots\}$.

     (b) $\Gamma_{\mathbb{R}}(1-s+n_{v_2}+\epsilon_{v_2}) = \Gamma_{\mathbb{R}}(1-s+\mathbf{w}+1)$ and the critical set of integers would be given by $\frac{2-s+\mathbf{w}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{2-\mathbf{w},4-\mathbf{w},6-\mathbf{w},\cdots\}$.

     Now combining all the cases for the critical set s we get no intersection among them and hence $\mathrm{Crit}(L(s,\chi)) = \varnothing$.

   - Here we have $\epsilon_v = 0$ and $n_{v_1} = n_{v_2} = \mathbf{w}$.

     Now for $L(s,\chi)$ we have the critical set of integers as follows,

     $\Gamma_{\mathbb{R}}(s+n_v+\epsilon_v) = \Gamma_{\mathbb{R}}(s+\mathbf{w}+0)$ and the critical set of integers would be given by $\frac{s+\mathbf{w}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{-\mathbf{w},-2-\mathbf{w},-4-\mathbf{w},\cdots\}$.

Now for $L(1-s,\chi^{-1})$ we have the critical set of integers as follows,

$\Gamma_{\mathbb{R}}(1-s+n_v+\epsilon_v) = \Gamma_{\mathbb{R}}(1-s+\mathbf{w}+0)$ and the critical set of integers would be given by $\frac{1-s+\mathbf{w}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{1-\mathbf{w}, 3-\mathbf{w}, 5-\mathbf{w},\cdots\}$. Combining all the cases for the critical set s we get no intersection among them and hence $\mathrm{Crit}(L(s,\chi)) = \{\cdots, -\mathbf{w}+1-2k, \cdots, -\mathbf{w}-3, -\mathbf{w}-1; -\mathbf{w}+2, -\mathbf{w}+4, \cdots, -\mathbf{w}+2k, \cdots\}$ for $k \in \mathbb{Z}_{\geqslant 0}$

- Here we have $\epsilon_v = 1$ and $n_{v_1} = n_{v_2} = \mathbf{w}$.

  Now for $L(s,\chi)$ we have the critical set of integers as follows, $\Gamma_{\mathbb{R}}(s+n_v+\epsilon_v) = \Gamma_{\mathbb{R}}(s+\mathbf{w}+1)$ and the critical set of integers would be given by $\frac{s+\mathbf{w}+1}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{-1-\mathbf{w}, -3-\mathbf{w}, -5-\mathbf{w},\cdots\}$.

  Now for $L(1-s,\chi^{-1})$ we have the critical set of integers as follows, $\Gamma_{\mathbb{R}}(s+n_v+\epsilon_v) = \Gamma_{\mathbb{R}}(1-s+\mathbf{w}+1)$ and the critical set of integers would be given by $\frac{2-s+\mathbf{w}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{2-\mathbf{w}, 4-\mathbf{w}, 6-\mathbf{w},\cdots\}$.

  Combining all the cases for the critical set s we get no intersection among them and hence $\mathrm{Crit}(L(s,\chi)) = \{\cdots, -\mathbf{w}-2k, \cdots, -\mathbf{w}-2, -\mathbf{w}; -\mathbf{w}+1, -\mathbf{w}+3, \cdots, -\mathbf{w}+1+2k, \cdots\}$ for $k \in \mathbb{Z}_{\geqslant 1}$.

2. We have that F is totally imaginary ($S_r = \varnothing$) and the width of χ is $\ell = \ell(\chi) = min\{| n_w - n_{\bar{w}} |: w \in S_c\}$ along with $\mathbf{w} \equiv \ell \pmod 2$. We also have that $n_w + n_{\bar{w}} = \mathbf{w}$ and let $| n_w - n_{\bar{w}} | = \ell_{\mathbf{w}}$.

   Now for $L(s,\chi)$ we have the critical set of integers as follows, $\Gamma_{\mathbb{C}}(s+\frac{n_w+n_{\bar{w}}}{2}+\frac{| n_w - n_{\bar{w}} |}{2}) = \Gamma_{\mathbb{C}}(s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2}) = \Gamma(s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2})$ and the critical set of integers would be given by $s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{-\frac{\mathbf{w}}{2}-\frac{\ell_{\mathbf{w}}}{2}, -1-\frac{\mathbf{w}}{2}-\frac{\ell_{\mathbf{w}}}{2}, \cdots\}$.

   Now for $L(1-s,\chi^{-1})$ we have the critical set of integers as follows, $\Gamma_{\mathbb{C}}(1-s+\frac{n_w+n_{\bar{w}}}{2}+\frac{| n_w - n_{\bar{w}} |}{2}) = \Gamma_{\mathbb{C}}(1-s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2}) = \Gamma(1-s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2})$ and the critical set of integers would be given by $1-s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{1-\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2}, 2-\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2}, \cdots\}$.

   Since $\ell = \ell(\chi) = min\{| n_w - n_{\bar{w}} |: w \in S_c\}$, then $\ell < \ell_{\mathbf{w}}$ or $-\ell_{\mathbf{w}}/2 < -\ell/2$ and $-(\mathbf{w}+\ell_{\mathbf{w}})/2 < -(\mathbf{w}+\ell)/2$. Similarly, $1-(\mathbf{w}-\ell_{\mathbf{w}})/2 > 1-(\mathbf{w}-\ell)/2$. Therefore,

$-(\mathbf{w}+\ell)/2 < \mathrm{Crit}(L(s,\chi)) < 1-(\mathbf{w}-\ell)/2$ or $\mathrm{Crit}(L(s,\chi))$ is given by $\{\mathbf{m} \in \mathbb{Z} : 1-\frac{\mathbf{w}}{2}-\frac{\ell}{2} \leqslant \mathbf{m} \leqslant -\frac{\mathbf{w}}{2}+\frac{\ell}{2}\}$.

3. We have that F has both real and complex places($S_r \neq \varnothing \neq S_c$). Therefore, we calculate the critical set for each of the following cases:

   - When $\epsilon_{v_1} \neq \epsilon_{v_2}$ and $n_{v_1} = n_{v_2} = \mathbf{w}$, we already have from part(1) that $\mathrm{Crit}(L(s,\chi)) = \varnothing$.
   - Given that $\epsilon_v = 0$, $n_v = \mathbf{w}$ and $n_w + n_{\bar{w}} = \mathbf{w}$. Then $L(s,\chi) = \Gamma_{\mathbb{R}}(s+\mathbf{w})\Gamma_{\mathbb{C}}(s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2})$.
     Now for $L(s,\chi)$ we have the critical set of integers as follows,
     $\Gamma_{\mathbb{R}}(s+n_v+\epsilon_v) = \Gamma_{\mathbb{R}}(s+\mathbf{w}+0)$ and the critical set of integers would be given by $\frac{s+\mathbf{w}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{-\mathbf{w}, -2-\mathbf{w}, -4-\mathbf{w}, \cdots\}$.
     $\Gamma_{\mathbb{C}}(s+\frac{n_w+n_{\bar{w}}}{2}+\frac{|\,n_w-n_{\bar{w}}\,|}{2}) = \Gamma_{\mathbb{C}}(s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2}) = \Gamma(s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2})$ and the critical set of integers would be given by $s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{-\frac{\mathbf{w}}{2}-\frac{\ell_{\mathbf{w}}}{2}, -1-\frac{\mathbf{w}}{2}-\frac{\ell_{\mathbf{w}}}{2}, \cdots\}$.
     Now for $L(1-s,\chi^{-1})$ we have the critical set of integers as follows,
     $\Gamma_{\mathbb{R}}(1-s+n_{v_1}+\epsilon_{v_1}) = \Gamma_{\mathbb{R}}(1-s+\mathbf{w}+0)$ and the critical set of integers would be given by $\frac{1-s+\mathbf{w}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{1-\mathbf{w}, 3-\mathbf{w}, 5-\mathbf{w}, \cdots\}$.
     $\Gamma_{\mathbb{C}}(1-s+\frac{n_w+n_{\bar{w}}}{2}+\frac{|\,n_w-n_{\bar{w}}\,|}{2}) = \Gamma_{\mathbb{C}}(1-s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2}) = \Gamma(1-s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2})$ and the critical set of integers would be given by $1-s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{1-\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2}, 2-\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2}, \cdots\}$.
     Now combining all the cases for the critical set s we get no intersection among them and hence $\mathrm{Crit}(L(s,\chi)) = \varnothing$.
   - Given that $\epsilon_v = 1$, $n_v = \mathbf{w}$ and $n_w + n_{\bar{w}} = \mathbf{w}$. Then $L(s,\chi) = \Gamma_{\mathbb{R}}(s+\mathbf{w}+1)\Gamma_{\mathbb{C}}(s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2})$.
     Now for $L(s,\chi)$ we have the critical set of integers as follows,
     $\Gamma_{\mathbb{R}}(s+n_v+\epsilon_v) = \Gamma_{\mathbb{R}}(s+\mathbf{w}+1)$ and the critical set of integers would be given by $\frac{s+\mathbf{w}+1}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{-1-\mathbf{w}, -3-\mathbf{w}, -5-\mathbf{w}, \cdots\}$.
     $\Gamma_{\mathbb{C}}(s+\frac{n_w+n_{\bar{w}}}{2}+\frac{|\,n_w-n_{\bar{w}}\,|}{2}) = \Gamma_{\mathbb{C}}(s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2}) = \Gamma(s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2})$ and the critical

set of integers would be given by $s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{-\frac{\mathbf{w}}{2}-\frac{\ell_{\mathbf{w}}}{2}, -1-\frac{\mathbf{w}}{2}-\frac{\ell_{\mathbf{w}}}{2}, \cdots\}$.

Now for $L(1-s,\chi^{-1})$ we have the critical set of integers as follows,

$\Gamma_{\mathbb{R}}(s+n_v+\epsilon_v) = \Gamma_{\mathbb{R}}(1-s+\mathbf{w}+1)$ and the critical set of integers would be given by $\frac{2-s+\mathbf{w}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{2-\mathbf{w}, 4-\mathbf{w}, 6-\mathbf{w},\cdots\}$.

$\Gamma_{\mathbb{C}}(1-s+\frac{n_w+n_{\bar{w}}}{2}+\frac{\mid n_w-n_{\bar{w}}\mid}{2}) = \Gamma_{\mathbb{C}}(1-s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2}) = \Gamma(1-s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2})$ and the critical set of integers would be given by $1-s+\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2} \in \{0,-1,-2,\cdots\}$ which implies that $s \in \{1-\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2}, 2-\frac{\mathbf{w}}{2}+\frac{\ell_{\mathbf{w}}}{2},\cdots\}$.

Now combining all the cases for the critical set s we get no intersection among them and hence $\mathrm{Crit}(L(s,\chi)) = \varnothing$.

Therefore, all the cases are proved. □

The previous proposition introduces the term **critical algebraic Hecke character** which means the algebraic Hecke character which has $\mathrm{Crit}(L(s,\chi)) \neq \varnothing$. The existence of a critical algebraic Hecke character of F implies that F is either totally real or totally imaginary. Moreover, when F is totally real, the parities ϵ_v of the local archimedean characters are all equal. Therefore, we will focus on the existence of such critical algebraic Hecke character with already given infinity type.

7.3 Existence of critical algebraic Hecke character

For a given number field F, with the purity lemma we study the following two conditions - (i) $S_r \neq \varnothing$ and (ii) $S_r = \varnothing$, which will deal with the existence of the algebraic Hecke characters.

7.3.1 When F has a real place

Given that $S_r \neq \varnothing$. In such case an algebraic Hecke character χ necessarily takes the form

$$\chi = \chi^0 \|\|^{\mathbf{w}}$$

for a character χ^0 is of finite order and $\mathbf{w} \in \mathbb{Z}$.

7.3.2 When F is totally imaginary

Given that $S_r = \varnothing$. Consider an infinity type $\mathbf{n} := (n_\tau)_{\tau \in \Sigma_F}$, $n_\tau \in \mathbb{Z}$ satisfying the purity condition then $n_{\gamma\circ\tau} + n_{\gamma\circ\bar{\tau}} = \mathbf{w}$ for all $\tau \in \Sigma_F$ and $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Consider the case when F is a CM field. Then:

Lemma 7.2. (Lemma 11, Section 4.3, [5]) Let F be a CM field. Let $\tau : F \to \overline{\mathbb{Q}}$ and $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and c denotes the complex conjugation in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Then $\gamma \circ c \circ \tau = c \circ \gamma \circ \tau$, complex conjugation and any automorphism of $\overline{\mathbb{Q}}$ commute on the image of a CM field.

Proof. Let us denote $\alpha_1 = \gamma \circ c \circ \tau$ and $\alpha_2 = c \circ \gamma \circ \tau$. Now we can say that their restriction on the totally real field, F^+, are equal. That is $\alpha_1|_{F^+} = \alpha_2|_{F^+}$. This further means that either $\alpha_1 = \alpha_2$ or $\alpha_1 = c \circ \alpha_2$. In the second case we get that, $\gamma \circ c \circ \tau = c \circ c \circ \gamma \circ \tau = \gamma \circ \tau$. This gives us that $c \circ \tau = \tau$ which is not possible. Therefore, $\gamma \circ c \circ \tau = c \circ \gamma \circ \tau$. Hence proved. □

Using this lemma, the purity condition $n_{\gamma\circ\tau} + n_{\gamma\circ\bar{\tau}} = \mathbf{w}$ on F comes down to writing $n_\tau + n_{\bar{\tau}} = \mathbf{w}$ since we can write $n_{\gamma\circ\tau} = n_{\overline{\gamma\circ\tau}}$ for all $\tau \in \Sigma_F$ and $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Proposition 7.3. (Proposition 12, Section 4.3, [5]) Let F be a CM field with the infinity type $\mathbf{n} := (n_\tau)_{\tau \in \Sigma_F}$, $n_\tau \in \mathbb{Z}$ satisfying the purity condition $n_\tau + n_{\bar{\tau}} = \mathbf{w}$ for all $\tau \in \Sigma_F$. Then there exists an algebraic Hecke character with the same infinity type and of some modulus.

Proof. This proposition gives us the existence of an algebraic Hecke character. Such a character if exists will look like

$$\chi_\infty(x_\infty) = (\prod_{w \in S_c} (\frac{x_w}{\bar{x}_w})^{(n_{\tau_w} - n_{\bar{\tau}_w})/2}) \mid x_\infty \mid_\infty^{\mathbf{w}/2}$$

For some integral ideal $\mathfrak{m}$ we have that χ_∞ is trivial on $U_F^1(\mathfrak{m})$. Substituting $s_w = n_{\tau_w} - n_{\bar{\tau}_w}$ and $x_w = r_w e^{i\theta_w}$ we can construct $\chi^u : \mathbb{I}_F/F^* \to S^1$ as $\chi^u_\infty = \prod_w e^{is_w\theta_w}$. Hence we get $\chi = \chi^u \|\|^{\mathbf{w}/2}$. U_F intersects with the compact part of the roots of unity μ_F in F^*. Thus we have $U_F^1(\mathfrak{m}) \cap (S^1, \cdots, S^1) = \{1\}$. Let us then take a unitary character ψ^u_∞ of $F^*_\infty/U_F^1(\mathfrak{m})$ that maps x_∞ to $e^{is_w\theta_w}$ and ψ^u_f a character of $(\mathcal{O}_F/\mathfrak{m})^*$. Then we can get a Größencharaktere, ψ^u mod $\mathfrak{m}$ and let χ^u be the corresponding Hecke character that we can get from the Größencharaktere as mentioned in Proposition 6.3. ☐

Let us consider a totally imaginary field F. Then there exists a maximal CM or totally real subfield, F_1 of F. Let F_0 be the largest totally real subfield of F. Then F_0 can admit at most one totally imaginary quadratic field inside F_0. If F has a CM subfield then denote it as F_1; if not then $F_1 = F_0$.

Proposition 7.4. (Proposition 13, Section 4.3, [5]) Let F be a totally imaginary field. Consider the infinity type $\mathbf{n} := (n_\tau)_{\tau \in \Sigma_F}$, $n_\tau \in \mathbb{Z}$ satisfying the purity condition $n_{\gamma\circ\tau} + n_{\gamma\circ\bar{\tau}} = \mathbf{w}$ for all $\tau \in \Sigma_F$ and $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Then there exists an infinity type $\mathbf{m} := (m_{\tau_1})_{\tau_1 \in \Sigma_{F_1}}$ for F_1 such that if $\tau \in \Sigma_F$ with $\tau|_{F_1} = \tau_1$ then $n_\tau = m_{\tau_1}$.

Proof. A property of a pure infinity type of a totally imaginary field is that it is the base change of the pure infinity type of the maximal CM or totally real subfield F_1 of F. The proposition mentioned above talks about this property of the infinity type.

Since F is totally imaginary field so the purity condition $n_{\gamma\circ\tau} + n_{\gamma\circ\bar{\tau}} = \mathbf{w}$ becomes $n_{\gamma\circ\tau} + n_{\overline{\gamma\circ\tau}} = \mathbf{w}$ from 7.3 for all $\tau \in \Sigma_F$ and $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Then we have $n_{\gamma\circ\tau} = n_{\overline{\gamma\circ\tau}} \forall\tau; \forall\gamma$. Let us consider the function $n : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \times \Sigma_F \to \mathbb{Z}$ which maps $n(\gamma, \tau) = n_{\gamma\circ\tau}$. Therefore for $c \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ we have that $n(\gamma, c\tau) = n(\gamma c, \tau)$. Substituting τ by $\gamma^{-1}\tau$ we get, $n(\gamma, c\gamma^{-1}\tau) = n(\gamma c\gamma^{-1}, \tau) = n(c, \tau)$ and then replacing τ by $c\tau$ we have $n(c, c\tau) = n(\gamma c\gamma^{-1}, c\tau)$ which reduces to $n(1, \tau) = n(\gamma c\gamma^{-1}c, \tau)$.

Now for any $\alpha \in Gal(\overline{\mathbb{Q}}, \mathbb{Q})$ we get,

$$n(\alpha\gamma c\gamma^{-1}c\alpha^{-1}, \tau) = n(\alpha\gamma c\gamma^{-1}\alpha^{-1}\alpha c\alpha^{-1}, \tau) = n((\alpha\tau)c(\alpha\tau)^{-1}, \alpha c\alpha^{-1}\tau) = n(\alpha c\alpha^{-1}c, \tau) = n(1, \tau).$$

Thus for all element $x \in \mathcal{N}$, the normal subgroup of $Gal(\overline{\mathbb{Q}}, \mathbb{Q})$ we have that $n_{x_\tau} = n_\tau$. This normal subgroup is generated by $\{\gamma c\gamma^{-1}c : \gamma \in Gal(\overline{\mathbb{Q}}, \mathbb{Q})\}$. Lemma 7.1 gives us that $Gal(\overline{\mathbb{Q}},\mathbb{Q})$ acts transitively on Σ_F and hence $Gal(\overline{\mathbb{Q}},\mathbb{Q})$ acts transitively on Σ_{F_1} via $Gal(\overline{\mathbb{Q}},\mathbb{Q})/\mathcal{N}$. Now if we have $\tau_1,\tau_2 \in \Sigma_F$ with $\tau_1|_{F_1} = \tau_2|_{F_1} = \tau$ and for $x \in \mathcal{N}$ we have $x\tau_1 = \tau_2$ then $n_{x\tau_1} = n_{\tau_2} = n_\tau$ and this common value is m_{τ_1}. □

Proposition 7.5. (Proposition 14, Section 4.3, [5]) Let F be a totally imaginary field. Consider the infinity type $\mathbf{n} := (n_\tau)_{\tau\in\Sigma_F}$, $n_\tau \in \mathbb{Z}$ satisfying the purity condition $n_{\gamma\circ\tau} + n_{\gamma\circ\bar{\tau}} = \mathbf{w}$ for all $\tau \in \Sigma_F$ and $\gamma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Then there exists an algebraic Hecke character of infinity type $\mathbf{n}$.

1. If $F = F_1$, then there exists $n \in \mathbb{Z}$ such that $n_\tau = n \forall\tau \in \Sigma_F$. Any algebraic Hecke character of infinity type $\mathbf{n}$ looks like $\chi = \chi^0 \|\|^n$ for a Dirichlet character χ^0 of F.
2. If F_1 is a CM field, then the infinity type $\mathbf{n}$ is the base change from an infinity type $\mathbf{m}$ of F_1 and then any algebraic Hecke character of infinity type $\mathbf{n}$ looks like $\chi = \chi_1 \circ N_{F|F_1} \otimes \chi^0$ for some algebraic Hecke character of infinity type $\mathbf{m}$ of F_1 and some Dirichlet character χ^0 of F.

Proof. The above proposition discusses about the existence of algebraic Hecke characters which have pure infinity type from the base change of another infinity type from a totally real subfield and from a maxiaml CM subfield. In both the cases we have the infinity type $\mathbf{m}$ of F_1 which has base change $\mathbf{n}$ of F.

1. For $F_1 = F_0$ the subfield is totally real and hence all m_{τ_1} are equal and hence all n_τ are equal and let their value be equal to n. We can take a character of finite order say χ^0 of F and then put in the formula $\chi = \chi^0 \|\|^n$. If we have χ then $\chi \|\|^{-1}$ is the character of finite order.
2. There exists an algebraic Hecke character χ_1 with the infinity type $\mathbf{m}$. Then considering a finite order character χ^0 of F we can get $\chi = \chi_1 \circ N_{F|F_1} \otimes \chi^0$. If we have such χ, then $\chi\chi_2^{-1}$, where $\chi_2 = \chi_1 \circ N_{F|F_1}$, gives the character of finite order.

□

Let us look into an example: Consider the field $F = \mathbb{Q}(2^{1/3}, \omega)$ where $2^{1/3}$ is the cube root of 2 and ω is the cube root of unity. We have that $\Sigma_F = Gal(F|\mathbb{Q}) = S_3$ which is the permutation group taken to be $\{2^{1/3}, 2^{1/3}\omega, 2^{1/3}\omega^2\}$. Let the elements of S_3 be denoted as $s_1 = e, s_2 = (12), s_3 = (23), s_4 = (13), s_5 = (123), s_6 = (132)$. An element $s \in S_3$ correspond to an embedding $\tau_s : F \to \mathbb{C}$. The embeddings for F are given as following:

$$\tau_{s_1} := 1 : \begin{cases} 2^{1/3} \mapsto 2^{1/3} \\ \omega \mapsto \omega \end{cases} \qquad \tau_{s_3} := \tau : \begin{cases} 2^{1/3} \mapsto 2^{1/3} \\ \omega \mapsto \omega^2 \end{cases}$$

$$\tau_{s_5} := \sigma : \begin{cases} 2^{1/3} \mapsto 2^{1/3}\omega \\ \omega \mapsto \omega \end{cases} \qquad \tau_{s_6} := \sigma^2 : \begin{cases} 2^{1/3} \mapsto 2^{1/3}\omega^2 \\ \omega \mapsto \omega \end{cases}$$

$$\tau_{s_4} := \tau\sigma : \begin{cases} 2^{1/3} \mapsto 2^{1/3}\omega \\ \omega \mapsto \omega^2 \end{cases} \qquad \tau_{s_2} := \tau\sigma^2 : \begin{cases} 2^{1/3} \mapsto 2^{1/3}\omega^2 \\ \omega \mapsto \omega^2 \end{cases}$$

We consider two infinity types - $\mathbf{n}_1 = (n_{\tau_s})_{s \in S_3}$ and $\mathbf{n}_2 = (n'_{\tau_s})_{s \in S_3}$ such that we have,

s	$s_1 = e$	$s_2 = (12)$	$s_3 = (23)$	$s_4 = (13)$	$s_5 = (123)$	$s_6 = (132)$
n'_{τ_s}	a	b	$\mathbf{w} - a$	c	$\mathbf{w} - c$	$\mathbf{w} - b$
n_{τ_s}	a	$\mathbf{w} - a$	$\mathbf{w} - a$	$\mathbf{w} - a$	a	a

where $a, b, c, \mathbf{w} \in \mathbb{Z}$. Now Σ_F is put into pair of complex conjugates as $\{(\tau_{s_1}, \tau_{s_3}), (\tau_{s_2}, \tau_{s_6}), (\tau_{s_4}, \tau_{s_5})\}$ which gives us that $n'_\tau + n'_{\bar{\tau}} = \mathbf{w}$.

Now the infinity type $\mathbf{n}_2 = (n'_{\tau_s})_{s \in S_3}$ is not a pure infinity type. Other possible complex conjugate pairs of Σ_F via automorphisms of $\overline{\mathbb{Q}}$ gives us the pairings like $\{(\tau_{s_1}, \tau_{s_2}), (\tau_{s_3}, \tau_{s_5}), (\tau_{s_4}, \tau_{s_6})\}$ and $\{(\tau_{s_1}, \tau_{s_4}), (\tau_{s_3}, \tau_{s_6}), (\tau_{s_2}, \tau_{s_5})\}$. From these pairings we get that $n_{\tau_{s_1}} + n_{\tau_{s_2}} = n'_{\tau s_1} + n'_{\tau_{s_2}} = \mathbf{w}$ which implies that $b = \mathbf{w} - a$ for $\mathbf{n}_2$ to be pure infinity type. Similarly, we get that $c = \mathbf{w} - a$ for $\mathbf{n}_2$ to be pure infinity type. Therefore, $b, c, \mathbf{w} - a$ all should be equal to each other for $\mathbf{n}_2$ to be pure infinity type. On the other hand $\mathbf{n}_1 = (n_{\tau_s})_{s \in S_3}$ is a pure infinity type and has purity type $\mathbf{w}$. It also a base change from the infinity type $\mathbf{m}$ of $F_1 = \mathbb{Q}(\omega)$ with $m_{\tau_{s_1}} = a$ and $m_{\tau_{s_2}} = \mathbf{w} - a$.

7.4 Algebraic Hecke Characters of F with coefficients in a finite Galois Extension of $\mathbb{Q}$

Let the finite Galois extension of $\mathbb{Q}$ that takes a copy of F inside itself be E, which is further called the field of coefficients. Let us consider the infinity type $\mathbf{n} := (n_\tau)_{\tau:F\to E}$, $n_\tau \in \mathbb{Z}$ or $\mathbf{n} \in \mathbb{Z}[\mathrm{Hom}(F,E)]$. The difference here with the previously mentioned infinity type is that in this case the infinity type is parametrised over $\mathrm{Hom}(F,E)$ instead of $\mathrm{Hom}(F,\overline{\mathbb{Q}})$.

Consider an embedding $\iota : E \to \overline{\mathbb{Q}}$. This gives an identification $\iota^* : \mathrm{Hom}(F,E) \to \mathrm{Hom}(F,\overline{\mathbb{Q}})$ which maps $\tau \mapsto \iota \circ \tau = \iota^*\tau$.

We focus on constructing an algebraic representation $\theta_{\mathbf{n}}$ on a one-dimensional vector space $\mathcal{M}_{\mathbf{n},E}$ over the field E. This representation is given by $\theta_{\mathbf{n}} : G \times E \to GL_1(\mathcal{M}_{\mathbf{n},E})$. G is given as $G = \mathrm{Res}_{F|\mathbb{Q}}(GL_1/F)$ the Weil restriction of scalars of GL_1 over F to $\mathbb{Q}$, the construction of which is discussed in the [article [5], section 5.1].

The τ-component of the representation is given by $a \mapsto \theta_{n^\tau}(a) := a^{n_\tau}$ for $a \in GL_1(E)$. For $x \in F^*$ we have, $\theta_{\mathbf{n}}(x) = \prod_{\tau:F\to E} \theta_{n_\tau}(\tau(x)) = \prod_{\tau:F\to E} \tau(x)^{n_\tau}$. Now we consider an infinity type $\mathbf{n} \in \mathbb{Z}[\mathrm{Hom}(F,E)]$ and an associated representation $(\theta_{\mathbf{n}}, \mathcal{M}_{\mathbf{n},E})$ and a corresponding this there is a sheaf $\overline{\mathcal{M}}_{\mathbf{n},E}$(for details one can refer to [article [5], section 5.2]) on the symmetric space $S^G_{\mathfrak{m}}$, the construction of which is discussed in details in [article [5], section 5.1].

The purity lemma thus can be re-written as:

Lemma 7.3. (Lemma 15, Section 5.2, [5]) Given $\mathbf{n} \in \mathbb{Z}[\mathrm{Hom}(F,E)]$, the sheaf $\overline{\mathcal{M}}_{\mathbf{n},E}$ of vector spaces over E on $S^G_{\mathfrak{m}}$ is non-zero iff there exists $\mathbf{w} \in \mathbb{Z}$ such that

1. if $S_r \neq \emptyset$, then $n_\tau = \mathbf{w} \forall \tau \in \mathrm{Hom}(F,E)$
2. if $S_r = \emptyset$, then $n_{\iota\circ\tau} + n_{\overline{\iota\circ\tau}} = \mathbf{w} \forall \tau \in \mathrm{Hom}(F,E)$ and $\iota \in \mathrm{Hom}(E,\overline{\mathbb{Q}})$.

The details of the proof is discussed in [article [5], section 5.2].

For an integral ideal $\mathfrak{m}$ of F with values in E and infinity type $\mathbf{n} \in \mathbb{Z}[\mathrm{Hom}(F,E)]$, an algebraic Hecke character of F of modulus $\mathfrak{m}$ is defined as $\chi : \mathbb{J}_F(\mathfrak{m}) \to E^*$ such that for $(x) \in \mathbb{P}_F(\mathfrak{m})^+$ we have that $\chi((x)) = \prod_{\tau\in\mathrm{Hom}(F,E)} \tau(x)^{n_\tau}$.

Let χ be an algebraic Hecke character of F with values in E with modulus $\mathfrak{m} = \prod_{\mathfrak{p}|\mathfrak{m}} \mathfrak{p}^{m_\mathfrak{p}}$ with infinity type $\mathbf{n}$. We have a unique continuous homomorphism $\chi_\mathbb{A} : \mathbb{I}_F \to E^*$ with a discrete topology on E^*. This homomorphism follows certain properties:

1. $\chi_\mathbb{A}^{-1}(1) = \prod_{v \in S_\infty} F_v^* \prod_{\mathfrak{p}|\mathfrak{m}} (1 + \mathfrak{p}^{m_\mathfrak{p}} \mathcal{O}_\mathfrak{p}) \prod_{\mathfrak{p}\nmid\mathfrak{m}} \mathcal{O}_\mathfrak{p}^*$ which is open in $\mathbb{I}_F^*$.
2. for $\mathfrak{p} \nmid \mathfrak{m}$, $\chi_\mathfrak{p}$ of $\chi_\mathbb{A}$ satisfies $\chi_\mathfrak{p}(\pi_\mathfrak{p}) = \chi(\mathfrak{p})$.
3. $\chi_\mathbb{A}|_{F^*} = \theta_\mathbf{n}$

Consider the embedding $\iota : E \hookrightarrow \mathbb{C}$ and let the place v_ι be the archimedean place of E. Now we have a continuous homomorphism $\theta_{\mathbf{n},\mathbb{A}} : \mathbb{I}_F \to \mathbb{I}_E$ and composing this with the projection map $\mathbb{I}_E^* \to E_{v_\iota}^* \cong \mathbb{C}^*$ we get $\theta_{\mathbf{n},\iota} : \mathbb{I}_F \to \mathbb{C}^*$.

Thus we have a continuous homomorphism, ${}^\iota\chi : \mathbb{I}_F \to \mathbb{C}^*$ which is defined as ${}^\iota\chi := (\iota \circ \chi_\mathbb{A}).\theta_{\mathbf{n},\iota}^{-1}$.

The following proposition summarises the properties of ${}^\iota\chi$:

Proposition 7.6. (Proposition 16, Section 5.3, [5]) Let χ be an algebraic Hecke character of F with values in E of modulus $\mathfrak{m}$ and infinity type $\mathbf{n}$. For the embedding $\iota : E \to \mathbb{C}$, ${}^\iota\chi$ satisfies the following:

1. ${}^\iota\chi : \mathbb{I}_F/F^* \to \mathbb{C}^*$ is a continuous homomorphism.
2. for all $\mathfrak{p} \notin S_\infty$, ${}^\iota\chi_\mathfrak{p} = \iota \circ \chi_\mathfrak{p}$
3. ${}^\iota\chi_\infty = \theta_{\mathbf{n},\iota}^{-1}|_{F_\infty^*}$ is determined by its values on F^* embedded diagonally in F_∞^* on which, $\theta_{\mathbf{n},\iota}^{-1}|_{F^*} = \iota \circ \theta_\mathbf{n}^{-1} = \theta_{-{}^\iota\mathbf{n}}$, ${}^\iota\chi$ is a Hecke character of F of modulus $\mathfrak{m}$ with infinity type $-{}^\iota\mathbf{n}$.

For $k \in \mathbb{Z}$ the Tate twist ${}^\iota\chi(k) :={}^\iota \chi \otimes \|\|^k$ has infinity type $-{}^\iota\mathbf{n} + k$.

Chapter 8

Theory of Hecke L-functions

This chapter discusses about the critical values of Hecke L-functions and Harder's result on the ratio of critical L-values. The last part of the chapter discusses about the variation of the result in case of a totally imaginary field and re-states the reciprocity law for the critical L-values.

8.1 Critical Values of Hecke L-functions

Let χ be an algebraic Hecke character of F with values in E of modulus $\mathfrak{m}$ and infinity type $\mathbf{n}$. For the embedding $\iota : E \to \mathbb{C}$, let us consider the character ${}^{\iota}\chi$ as in Proposition 7.6. Let us consider the special values of $\mathbb{C}$-valued L-function $L(s, \iota, \chi) := L(s, {}^{\iota}\chi)$.

We can also consider the $E \otimes \mathbb{C}$ valued L-function where $E \otimes \mathbb{C}$ is identified as $\prod_{\iota : E \to \mathbb{C}} \mathbb{C}$ and the L-function is given by $\mathbb{L}(s, \chi) := \{L(s, \iota, \chi)\}_{\iota : E \to \mathbb{C}}$.

We assume that F is totally imaginary number field, then from Proposition 7.2 we have that if the width l of ${}^{\iota}\chi$ satisfies $l \geqslant 2$, then there exists two consecutive integers which are critical for $\mathbb{L}(s, \chi)$.

8.2 Ratio of Critical Values of L-functions

Consider a an extension $F|K$ of number fields and let $\mathrm{Hom}(F, \bar{K}) = \{\sigma_1, \cdots, \sigma_r\}$ and $\{\omega_1, \cdots, \omega_r\}$ is a K-basis of F. The relative discriminant is the given by $\delta_{F|K} := \det([\sigma_i(\omega_j)])^2$. The absolute discriminant of F is given by $\delta_{F|\mathbb{Q}}$. Harder proved a result [2] that states that if m and $m+1$ are critical integers for $\mathbb{L}(s, \chi)$ then

$$\mathrm{Crit}({}^{\iota}\chi) := |\delta_{F|\mathbb{Q}}|^{1/2} \frac{L(m, {}^{\iota}\chi)}{L(m+1, {}^{\iota}\chi)} \in \overline{\mathbb{Q}} \tag{8.1}$$

and for any $\varsigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ we have,

$$\varsigma(\mathrm{Crit}({}^{\iota}\chi)) = \mathrm{Crit}({}^{\varsigma\circ\iota}\chi) \tag{8.2}$$

Also the result stated in the article is for $m = -1$; for any general m one can get the result using Tate twists.

Equation (8.1) shows that the ratio of L-values are algebraic since $|\delta_{F|\mathbb{Q}}|^{1/2}$ is algebraic. Now with the reciprocity law in (8.2) we can get the ratio of L-functions as

$$|\delta_{F|\mathbb{Q}}|^{1/2} \frac{L(m, {}^{\iota}\chi)}{L(m+1, {}^{\iota}\chi)} \in \iota(E) \tag{8.3}$$

The statement in (8.3) turns out to be incorrect in case of F being a totally imaginary field and not of CM type. It needs to be modified by a sign, which is trivial if F is a CM field and non-trivial in case of a totally imaginary field. Let us now focus on an example that shows that the (8.3) is not stable under base change and hence incorrect.

Let us consider the following field extension:

$$\begin{array}{c} F = \mathbb{Q}(i, \sqrt{4+i}) \\ | \\ F_1 = \mathbb{Q}(i) \\ | \\ F_0 = \mathbb{Q} \end{array}$$

Let us calculate the determinant for the extensions: Firstly let us compute $\delta_{F|F_1}$. The minimal polynomial of F over F_1 is given by $x^2-(4+i)$ and since we have integral basis $\{1,\sqrt{4+i}\}$ for the ring of integers we have the discriminant as $\delta_{F|F_1}=4.(4+i)$. The norm is given by $\mathcal{N}_{F_1|\mathbb{Q}}(\delta_{F|F_1})=16.17$. Next we compute $\delta_{F_1|\mathbb{Q}}$. The minimal polynomial of F_1 over $\mathbb{Q}$ is given by x^2+1 and since we have integral basis again $\{1,i\}$ thus we have the discriminant $\delta_{F_1|\mathbb{Q}}=-4$.

Now the discriminant for the whole tower of fields we get, $\delta_{F|\mathbb{Q}}=\delta_{F_1|\mathbb{Q}}^{[F:F_1]}\mathcal{N}_{F_1|\mathbb{Q}}(\delta_{F|F_1})=(-4)^2.16.17=2^8.17$.

Let ψ be an algebraic Hecke character of $F_1=\mathbb{Q}(i)$; $\psi:\mathbb{A}_{F_1}/F_1^*\to\mathbb{C}^*$ with $\psi_\infty:\mathbb{C}^*\to\mathbb{C}^*$ with $\psi_\infty(z)=z^a\bar{z}^b; a,b\in\mathbb{Z}$ and without loss of generality let us assume $a\geqslant b$. The critical set of $L(s,\psi)$ can be computed from Proposition 7.2 with the conditions that $a+b=\mathbf{w}$ and $a-b=\ell$ where $\mathbf{w}$ is the purity weight and ℓ is the width of ψ. The critical set will be given by

$$1-\frac{\mathbf{w}}{2}-\frac{\ell}{2}\leqslant \operatorname{Crit}(L(s,\psi))\leqslant -\frac{\mathbf{w}}{2}+\frac{\ell}{2}$$

On putting the values of $\mathbf{w}$ and ℓ we get that $\operatorname{Crit}(L(s,\psi))=\{1-a,2-a,\cdots,-b\}$.

On assuming $a-b\geqslant 2$ we can see that there are at least two critical points $m,m+1\in\operatorname{Crit}((L(s,\psi))$. Then from equation (8.3) we have that

$$|\delta_{F_1|\mathbb{Q}}|^{1/2}\frac{L(m,\psi)}{L(m+1,\psi)}\in\mathbb{Q}(\psi) \tag{8.4}$$

and $\mathbb{Q}(\psi)$ is generated by the values of ψ_f, the finite part of ψ. Since $|\delta_{F_1|\mathbb{Q}}|^{1/2}\in\mathbb{Q}^*$ we have that,

$$\frac{L(m,\psi)}{L(m+1,\psi)}\in\mathbb{Q}(\psi) \tag{8.5}$$

Let ω be a quadratic Dirichlet character and hence we can apply (8.5) to the character $\psi\omega$ and we get,

$$\frac{L(m,\psi\omega)}{L(m+1,\psi\omega)}\in\mathbb{Q}(\psi) \tag{8.6}$$

Let χ be the base change of ψ which gives a Hecke character of F. It is given by $\chi:=\psi\circ\mathcal{N}_{F|F_1}$. Since $F_\infty=\mathbb{C}\times\mathbb{C}$, then $\chi_\infty(z_1,z_2)=(z_1z_2)^a\overline{(z_1z_2)}^b=z_1^a\bar{z}_1{}^b z_2^a\bar{z}_1{}^b$ which implies that $\chi_\infty = \psi_\infty \times \psi_\infty$ and hence we have $\text{Crit}(L(s,\chi)) = \text{Crit}(L(s,\psi))$.

For a quadratic character ω of F_1 we have that $L(s,\chi) = L(s,\psi)L(s,\psi\omega)$. Hence we have that,

$$\frac{L(m,\chi)}{L(m+1,\chi)} \in \mathbb{Q}(\psi) = \mathbb{Q}(\chi) \tag{8.7}$$

However, applying (8.3) directly to the L-function $L(s,\chi)$ we get that,

$$|\delta_{F|\mathbb{Q}}|^{1/2} \frac{L(m,\chi)}{L(m+1,\chi)} \in \mathbb{Q}(\chi) \tag{8.8}$$

Since $|\delta_{F|\mathbb{Q}}|^{1/2} = (2^8.17)^{1/2} = \sqrt{17} \pmod{\mathbb{Q}^*}$ and hence

$$\sqrt{17}\frac{L(m,\chi)}{L(m+1,\chi)} \in \mathbb{Q}(\chi) \tag{8.9}$$

Now (8.7) and (8.9) are contradictory to each other since $\sqrt{17} \notin \mathbb{Q}(\chi)$. We can see that then the equation (8.3) is not stable under base change.

The question we ponder upon here is that if this example can be generalized for an imaginary quadratic field. For this we consider the tower of fields:

$$\begin{array}{c} F = \mathbb{Q}(\sqrt{d}, \sqrt{d'}) \\ | \\ F_1 = \mathbb{Q}(\sqrt{d}) \\ | \\ F_0 = \mathbb{Q} \end{array}$$

where we assume that $d \in \mathbb{Z}, d' \in \mathbb{C}$ and $d \equiv 2,3 \pmod 4$.

The choice of such d eases the discriminant calculation since we get an integral basis for the ring of integers. Hence we get $\delta_{F_1|\mathbb{Q}} = 4d$. Also, $\delta_{F|F_1} = 4d'$. The norm is given by $\mathcal{N}_{F_1|\mathbb{Q}}(\delta_{F|F_1}) = 16|d'|^2$.

Therefore, $\delta_{F|\mathbb{Q}} = \delta_{F_1|\mathbb{Q}}^{[F:F_1]}\mathcal{N}_{F_1|\mathbb{Q}}(\delta_{F|F_1}) = (4d)^2 16|d'|^2 = 2^8 d^2(a^2+b^2)$ where $d' = a + ib$.

Since $d \in \mathbb{Z}$, then we have $2^8 d^2 \in \mathbb{Q}$. Hence the problem would arise from the factor $(a^2 + b^2)$. The answer to this would come from the existence of the Hecke character which can be derived from its finite part - the Dirichlet character. We can consider the field of coefficients $\mathbb{Q}(\chi)$ of a Dirichlet character $\chi : (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ and find an integer $m \in \mathbb{Z}$ such that $m \notin \mathbb{Z}$.

8.3 Variations in the Result

For the remaining part of the chapter we assume F is totally imaginary field and E is a number field that is Galois over $\mathbb{Q}$ and contains a copy of F. Let the infinity type $\mathbf{n} \in \mathbb{Z}[\mathrm{Hom}(F, E)]$ satisfies the purity condition in Lemma 7.3 and for an embedding $\iota : E \to \mathbb{C}$ let the character ${}^{\iota}\chi$ be as in Proposition 7.6. The width of ${}^{\iota}\chi$ is given by $\ell({}^{\iota}\chi) := \min_{\tau : F \to E}\{|n_{\iota\circ\tau} - n_{\overline{\iota\circ\tau}}|\} = \ell({}^{\iota}\mathbf{n})$. The L-function $L(s, {}^{\iota}\chi)$ has $\ell({}^{\iota}\chi)$ many critical points.

Lemma 8.1. (Lemma 17, Section 6.3, [5]) $\ell({}^{\iota}\chi)$ is independent of ι and depends on $\mathbf{n}$.

Proof. If we have that $F_1 = F_0$ that is F has no CM subfield then the width $\ell({}^{\iota}\chi) = 0$. Thus the lemma is proved in this case. If F is totally imaginary field then $\mathbf{n}$ is the base change from an infinity type $\mathbf{m}$ over F_1. Therefore, $\ell({}^{\iota}\mathbf{n}) = \ell({}^{\iota}\mathbf{m})$. Let $\mathrm{Hom}(F_1, E) = \{\tau_1, \tau_1', \cdots, \tau_k, \tau_k'\}$ and τ_j, τ_j' have same restriction to F_0 and hence $\ell({}^{\iota}\mathbf{m}) = \min_j\{|m_{\tau_j} - m_{\tau_j'}|\}$. ☐

The set of archimedean places of F is given by S_∞. We are interested on the permutation of this set. The set S_∞ is given by $\{v_1, \cdots, v_r\}$; where $r = [F : \mathbb{Q}]/2$. Before going into the permutation set, let us focus on the following lemma:

Lemma 8.2. (Lemma 18, Section 6.3, [5]) The following are equivalent:

1. $s = -1$ and $s = 0$ are critical integers for $L(s, {}^{\iota}\chi)$
2. $-\ell \leqslant \mathbf{w} \leqslant -4 + \ell$
3. For each $\tau : F \to E$, $\iota : E \to \mathbb{C}$ there exists an element from the Weyl group of $GL(2)$, $w_{\iota\circ\tau}$ such that-
    - $l(w_{\iota\circ\tau}) + l(w_{\overline{\iota\circ\tau}}) = 1$

- $w_{\iota\circ\tau}.\begin{pmatrix} n_{\iota\circ\tau} & \\ & 0 \end{pmatrix}$ is dominant.

For the proof and related comments on the lemma, one can refer to Lemma 18 of [5]. This lemma introduces the Weyl group elements of $GL(2)$ which acts on the characters of the torus group of $GL(2)$. $l(w)$ denotes the length of the Weyl group element. The value of this length is 1 if w is non-trivial element and 0 if w is the trivial element. Condition (3) states that for a pair of conjugate embeddings $\{\iota\circ\tau, \overline{\iota\circ\tau}\}$ in Σ_F one of them is trivial Weyl group element and the other one is non-trivial. This defines a **CM-type** for F given by, $\Phi(\mathbf{n},\iota) := \{\tau \in \Sigma_F : l(w_\tau) = 1\}$ and that there is a bijection between this CM-types and the set of archimedean places S_∞ of F.

For each $\varsigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, there is a permutation $\pi_{\mathbf{n},\iota}(\varsigma)$ of S_∞ which acts on the CM-type $\Phi(\mathbf{n},\iota)$. The sign of this permutation is given by

$$\varepsilon_{\mathbf{n},\iota}(\varsigma) := \operatorname{sgn}(\pi_{\mathbf{n},\iota}(\varsigma)).$$

Another CM-type is defined as $\Phi(\tilde{\mathbf{n}},\iota)$ corresponding to a another infinity type ${}^\iota\tilde{\mathbf{n}}$ with respect to an induced representation. Therefore, for this CM-type we have $\varsigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ for which there is a permutation $\pi_{\tilde{\mathbf{n}},\iota}(\varsigma)$ of S_∞. The signature of this permutation is given by

$$\varepsilon_{\tilde{\mathbf{n}},\iota}(\varsigma) := \operatorname{sgn}(\pi_{\tilde{\mathbf{n}},\iota}(\varsigma)).$$

The details of the construction of these permutations can be referred to from section 6.3.2 of [5].

We state the main theorem on L-values now, which is a variation in Harder's result.

Theorem 8.1. (Theorem 19, Section 6.3.3, [5]) Let F be a totally imaginary field and E, a number field, is a Galois extension of $\mathbb{Q}$ which has a copy of F inside it. Consider an infinity type $\mathbf{n} = \Sigma_{\tau:F\to E}(n_\tau\tau) \in \mathbb{Z}[\mathrm{Hom}(F,E)]$ which satisfy the purity condition in Lemma 7.3 and purity weight $\mathbf{w}$. Let χ be an algebraic Hecke character of F with values in E with infinity type $\mathbf{n}$. For an embedding $\iota : E \to \mathbb{C}$ let ${}^\iota\chi$ be a character as defined in Proposition 7.6. Assume the condition $-\ell \leqslant \mathbf{w} \leqslant -4+\ell$ from Lemma 8.2.

Therefore, $\ell \geqslant 2$ and $\mathbf{n}$ is the base change to F from the infinity type of a maximal CM subfield of F. Suppose $m, m+1 \in \mathrm{Crit}(\mathbb{L}(\mathrm{s}, \chi))$ then,

$$|\delta_{F|\mathbb{Q}}|^{1/2} \frac{L(m,{}^{\iota}\chi)}{L(m+1,{}^{\iota}\chi)} \in \iota(E) \tag{8.10}$$

and also, for every $\varsigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ we have the reciprocity law as:

$$\varsigma(|\delta_{F|\mathbb{Q}}|^{1/2} \frac{L(m,{}^{\iota}\chi)}{L(m+1,{}^{\iota}\chi)}) = \varepsilon_{\mathbf{n},\iota}(\varsigma).\varepsilon_{\tilde{\mathbf{n}},\iota}(\varsigma).|\delta_{F|\mathbb{Q}}|^{1/2} \frac{L(m,{}^{\iota\circ\varsigma}\chi)}{L(m+1,{}^{\iota\circ\varsigma}\chi)} \tag{8.11}$$

When we have a CM field F from 7.2 we have that $\varsigma \circ c \circ \tau = c \circ \varsigma \circ \tau$ for $\varsigma, c \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\tau : F \to E$. Thus here we get the permutations $\pi_{\tilde{\mathbf{n}},\iota}(\varsigma) = \pi_{\mathbf{n},\iota}(\varsigma)$ to be equal which in turn gives us that $\varepsilon_{\tilde{\mathbf{n}},\iota}(\varsigma) = \varepsilon_{\mathbf{n},\iota}(\varsigma)$. Particularly, (8.10) is not different from (8.3) as even if we get $c(\Phi(\mathbf{n}, \varsigma \circ \iota)) = \varsigma(c(\Phi(\mathbf{n}, \iota)))$, the permutations can be different with $\varepsilon_{\tilde{\mathbf{n}},\iota}(\varsigma), \varepsilon_{\mathbf{n},\iota}(\varsigma)$ non-trivial.

The article [5] involves vast details about the arithmetic properties of such L-functions and their special values - one can refer to [5] for studying these functions. The idèle class group characters are also an interesting study and the article [9] deals with the same.

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