Spin-10 Spinor Bose-Einstein Condensates

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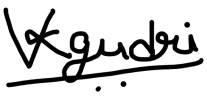
Certificate

This is to certify that this dissertation entitled 'Spin-10 Spinor Bose-Einstein Condensates' towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Vaishakh Chetan Kargudri at Indian Institute of Science Education and Research under the supervision of Dr. Rejish Nath, Associate Professor, Department of Physics , during the academic year 2024-25.



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This thesis is dedicated to Aaji

Declaration

I hereby declare that the matter embodied in the report entitled 'Spin-10 Spinor Bose-Einstein Condensates' are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Rejish Nath and the same has not been submitted elsewhere for any other degree.

Vaishakh Chetan Kargudri

Dr. Rejish Nath

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Abstract

Spinor Bose-Einstein Condensates enable the study of various phenomena such as coherent spin dynamics, quantum phase transitions, topological structures and spin-textures. While there has been significant progress in the theoretical and experimental understanding of spinor BECs with a few sub-levels, there is very little work done for systems with spin $f > 3$. Experimentally, spinor BECs with $f > 3$ have not been obtained. The above fact, along with recent works in ultra-cold Dysprosium atoms, which host a metastable state having total angular momentum $J = 10$, motivated us to take on the spin-10 BEC. Spinor BECs get significantly more complex as we move to higher spins, with additional tensors appearing in the interaction Hamiltonians of the system, and the dynamical equations being dependent on a much larger number of terms. Lots of parallels can be drawn to multi-level atoms due to the high internal spin degrees of freedom. Due to the completely unexplored nature of this system, we attempt to establish foundational knowledge in this thesis, paving the way for in depth analyses of specific aspects in the future. We examine the full Hamiltonian and the Gross-Pitaevskii equations governing the time dynamics of the system. The Bogoliubov spectrum contains $2f + 1$ terms for a spin-f system, and governs the behaviour under elementary excitations in the system. We provide a general method for finding the full energy spectrum for two phases of the spin-f system, namely the ferromagnetic and polar phases. The methods used in this thesis are easily generalizable, and we hope this sets the foundation for additional theoretical work in higher spin condensates in general.

Contents

Chapter 1

Introduction

Bose-Einstein Condensates were predicted in 1924-25 through works by Albert Einstein and Satyendra Nath Bose. It was after these two pioneering scientists that the predicted state of matter was named. However, it would take 70 more years for the first gaseous condensate to be prepared in the laboratory at the University of Colarado Boulder using a gas of Rubidium atoms. Since then various different types of BECs have been experimentally realized. Along with these successes in the laboratory, came a host of theoretical and computational work in the field as well.

Bose-Einstein condensates fall under the regime of ultra-cold physics, where bosonic particles macroscopically occupy a certain quantum state. This type of behaviour is only possible for bosonic species, since simultaneous occupation of a single state is prohibited for fermions under Pauli's Exclusion Principle. Bosonic species are those with integer spins, and include photons, magnons, polaritons as well as certain atoms and molecules. Bose Einstein condensates can allow us to study unique quantum behaviours, such as superfluidity[1, 2, 3] and coherence over macroscopic distances[4, 5, 6]. After 1995,there were rapid experimental realizations and theoretical work on spinor BECs [7, 8, 9, 10, 11], dipolar BECS[12, 13, 14, 15], spin-orbit coupled BECs[16, 17, 18, 19], and binary BEC systems [20, 21, 22, 23].

Spinor BECs refer to the condensates which have spin internal degrees of freedom, due to the hyperfine spin of the particles. These studies have led to the uncovering of interesting phenomena such as magnetic phases[24, 25], quantum phase transitions [26, 27], Faraday patterns [28, 29, 30,

31, 32], topological structures [33, 34, 35, 36] and spin textures [37, 38, 39, 40, 41, 42]. Spinor BECs have been experimentally realized [43] for various species like ^{7}Li, ^{23}Na, ^{39}K, ^{52}Cr etc. Increasing the spin makes the system increasingly complex, increasing the number of magnetic sub-levels, introducing new possible ground state phases, topological excitations, new tensorial terms in the dynamical equations of the system, and so on. There is a considerable lack of existing literature for higher spin systems[44, 45], potentially due to the difficulty of working with such systems. Furthermore, there have been no experimental realizations of BECs with spin $f > 3$.

In this thesis, we look at the properties of homogenous high-spin BECs, particularly spin-10 BECs. The main motivation for this study comes from a recent publication which reported the observation of a metastable state [46, 47, 48, 49] having a total angular momentum of $J = 10$ in a system of Dysprosium atoms. We have attempted to present our results in a way that should make it easy to generalize to other systems and phases of interest. Therefore, we expect the results to have applications in numerous future works on general high-spin systems. We shall attempt to motivate this study from the point of view of multi-level atoms [50, 51, 52]. Our spin-10 system has magnetic sub-levels ranging from $m = +10$ to $m = -10$ giving us 21 total sub-levels, with potential for population dynamics to occur across all sub-levels, with different couplings across sub-levels based on the terms comprising the Hamiltonian of the system. This is very reminiscent of behaviour and dynamics in multi-level atoms, where a specific level with total angular momentum J can split into $2J+1$ levels upon the application of a magnetic field due to the Zeeman splitting of the energy levels. This can provide a complex system to observe couplings and population transfer between the different levels based on system parameters and conditions. Such dynamics and properties have been studied extensively in recent times. A connection can be drawn between the many internal degrees of freedom in our system arising from the hyperfine spin of the particles and the multiple Zeeman sublevels arising in multi-level atoms, and further comparisons may be possible between these two systems in the future.

In this thesis, we describe many basic properties relevant to the system, mainly through the use of analytical and theoretical techniques. To maintain simplicity, in this thesis we consider only contact interactions. We ignore the long-range dipole-dipole interactions between particles. The

thesis is structured as follows,

- **Chapter 2** : We examine important and relevant topics in the field of Spinor Bose-Einstein Condensates, and describe all the formalisms used in this thesis.

- **Chapter 3** : We derive the full Hamiltonian for our spin-10 system in two choices of basis. We further use this Hamiltonian and energy functional to derive the non-linear Schroedinger-like Gross Pitaveskii Equations(GPEs), which govern the full dynamics of the system. We also attempt to demonstrate some of the differences between high and low spin systems, and the challenges associated with working with systems having so many internal spin degrees of freedom.

- **Chapter 4**: This chapter focuses on the Bogoliubov energy spectrum which governs elementary excitations in the system. We perform analytical calculations for the Ferromagnetic and Polar Phases of the general spin-f system, and find the full spectrum. We plot and attempt to examine the derived spectrum for the spin-10 system.

- **Chapter 5**: In the final chapter, we summarize the obtained results, and make a few comments about the future outlook and potential utility of the work done in this thesis.

Chapter 2

Spinor Bose-Einstein Condensates

Spinor Bose-Einstein Condensates refer to the BECs having spin internal degrees of freedom arising from the hyperfine spin of the condensate particles, and atoms can coherently occupy multiple magnetic sub-levels. A spin-f boson will have $2f+1$ magnetic sub-levels ranging from $m=-f$ to $m=+f$. Originally, magnetic traps were used to confine BECs, which resulted in the atoms being confined to only one hyperfine state. However, it was later found that it was possible to retain the spin degrees of freedom through the use of an optical trap, which led to the creation of spinor BECs. A spinor BEC can host many different ground state phases as a result of the interplay between the interparticle interactions and the linear and quadratic Zeeman shifts due to external magnetic fields. Spinor BECs can host unique topological excitations, which are otherwise difficult to find elsewhere in nature. Overall, spinor BECs are a platform to study exotic physical phenomena.

Typically we have both short range contact interactions in such systems, as well as much longer range interactions arising from the dipole moments of the constituent particles. However these long range dipole-dipole interactions are notoriously difficult to incorporate into studies of spinor condensates. We shall work only with the contact interactions in this thesis. We shall work within the mean-field approximation, assuming that the system is dilute enough to consider only two particle interactions. In the mean-field limit, the spin-f condensate order parameter will be represented as a complex $2f+1$ vector.

However, before we jump too far ahead, we need to establish some pre-requisites. In this first section we shall develop all the necessary formalism needed to understand spinor condensates. We shall start with the Hamiltonian of the system, and discuss the observable operators that the Hamiltonian is commonly written in term of in literature. The full description can be found in [8]. Following this we look at the Bogoliubov energy spectrum for low spin systems, specifically the polar phase for spin-1 and spin-2 BECs. The contents of this chapter shall be referenced repeatedly in the remainder of this thesis.

Consider a weakly interacting gas of spin-f bosons. Within the weakly interacting regime, we can approximate all interparticle interactions as contact interactions. The second quantized Hamiltonian can be written as follows

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{2.1}$$

where $\hat{H}_0$ is the single-particle Hamiltonian, while $\hat{V}$ covers all of the interactions.

2.1 Single Particle Hamiltonian

For a general spin-f BEC, $\hat{H}_0$ is given as:

$$\hat{H}_0 = \int d\boldsymbol{r} \sum_{m=-f}^{f} \hat{\psi}_m^\dagger \left[-\frac{\hbar^2 \nabla^2}{2M} - pm + qm^2 + V_{trap}(\boldsymbol{r}) \right] \hat{\psi}_m \tag{2.2}$$

$\hat{\psi}_m$ and $\hat{\psi}_m^\dagger$ denote the field operators that annihilates and create a boson in the mth Zeeman level respectively. $V_{trap}(\boldsymbol{r})$ is the external trapping potential(considered equal here across magnetic sub-levels). p is the linear Zeeman shift, and correspondingly, q is the quadratic Zeeman shift. These cause energy shifts in the magnetic sub-levels due to an external magnetic field. They are expressed in terms of the external magnetic field as:

$$p = -g\mu_B B \quad , \quad q = \frac{(g\mu_B B)^2}{\Delta E_{hf}} \tag{2.3}$$

where g is the Lande hyperfine g-factor, μ_B is the Bohr magneton, B is the external magnetic field, applied in the z-direction, and ΔE_{hf} is the hyperfine energy splitting.

2.2 Interaction Hamiltonian

We consider only binary interactions since the system is assumed to be very dilute. Furthermore, in the very dilute and low temperature regime, only the s-wave scattering between particles becomes important. Since spinor BECs have internal spin-degrees of freedom, there are different ways a scattering can take place between two spin-f particles. This introduces us to the concept of spin collision channels. The collision will conserve the total spin of the colliding particles, and therefore the channels can be labelled by the total spin of the interacting particles.

Let us consider two particle exchange in a spin-f system [8] . We know that the wave function of the many body system shall change by $(-1)^{2f}$. However we can also decompose this into the spin and orbital parts, which would give factors of $(-1)^{\mathscr{F}+2f}$ and $(-1)^{\mathscr{L}}$, where $\mathscr{F}$ and $\mathscr{L}$ are the total spin, and the relative orbital angular momentum of the two particles respectively. Now for consistency we must have, $(-1)^{\mathscr{F}+\mathscr{L}+2f} = (-1)^{2f}$, meaning $(-1)^{\mathscr{F}+\mathscr{L}} = 1$. However, we are only considering s-wave scattering in our system, meaning $\mathscr{L} = 0$, which tells us that $\mathscr{F}$ must be even. Our conclusion from this is that interactions can only take place in certain spin channels, namely, when the total spin of the interacting particles is even. Furthermore, the total spin $\mathscr{F}$ must be conserved in a two particle interaction. Now, we can write the interaction Hamiltonian as:

$$\hat{V} = \sum_{\mathscr{F}=0,2\cdots,2f} \hat{V}^{(\mathscr{F})} \tag{2.4}$$

where we are splitting the interaction Hamiltonian into the allowed interaction channels of the system. Therefore, when considering a spin-10 system, we will have $f = 10$ giving us 11 allowed spin interaction channels, going from $\mathscr{F} = 0$ to $\mathscr{F} = 20$ in increments of 2 (only even values).

We now introduce the operator $\hat{A}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')$, which is the annihilation operator for a boson pair at positions $\boldsymbol{r}$ and $\boldsymbol{r}'$. $\hat{A}^{\dagger}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')$ correspondingly creates a boson pair at positions $\boldsymbol{r}$ and $\boldsymbol{r}'$. This is

defined as:

$$\hat{A}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') = \sum_{m,m'=-f}^{f} \langle \mathscr{F},\mathscr{M}|f,m;f,m'\rangle \hat{\psi}_m(\boldsymbol{r})\hat{\psi}_{m'}(\boldsymbol{r}') \tag{2.5}$$

where $\langle \mathscr{F},\mathscr{M}|f,m;f,m'\rangle$ are the Clebsch Gordan coefficients. The interaction Hamiltonian can be expressed in terms of these operators in the following form:

$$\hat{V}^{(\mathscr{F})} = \frac{1}{2}\int d\boldsymbol{r}\int d\boldsymbol{r}'\, \mathscr{V}^{(\mathscr{F})}(\boldsymbol{r},\boldsymbol{r}') \sum_{\mathscr{M}=-\mathscr{F}}^{\mathscr{F}} \hat{A}^{\dagger}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \tag{2.6}$$

We are working under the mean field approximation. If the system is sufficiently dilute, we can assume that $\mathscr{V}^{(\mathscr{F})}$ can be replaced by a Dirac delta function, $\mathscr{V}^{(\mathscr{F})} = g_{\mathscr{F}}\delta(\boldsymbol{r}-\boldsymbol{r}')$ where $g_{\mathscr{F}}$ is a coupling constant which is directly proportional to the spin- $\mathscr{F}$ channel s-wave scattering length, $g_{\mathscr{F}} = 4\pi\hbar^2 a_{\mathscr{F}}/M$, where M is the mass of the particle.

2.3 Observable Operators

We start by giving the expression for the elements of the spin matrices

$$(f_x)_{mm'} = \frac{1}{2}\left[\sqrt{(f+m+1)(f-m)}\delta_{m+1,m'} + \sqrt{(f-m+1)(f+m)}\delta_{m-1,m'}\right] \tag{2.7}$$

$$(f_y)_{mm'} = \frac{1}{2i}\left[-\sqrt{(f+m+1)(f-m)}\delta_{m+1,m'} + \sqrt{(f-m+1)(f+m)}\delta_{m-1,m'}\right] \tag{2.8}$$

$$(f_z)_{mm'} = m\delta_{mm'} \tag{2.9}$$

We now define the required observable operators. The total density operator is defined as

$$\hat{n}(\boldsymbol{r}) = \sum_{m=-f}^{f} \hat{\psi}^{\dagger}_m(\boldsymbol{r})\hat{\psi}_m(\boldsymbol{r}) \tag{2.10}$$

The spin singlet-pair operator is given by

$$\hat{A}_{00}(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{\sqrt{2f+1}}\sum_{m=-f}^{f}(-1)^{f-m}\hat{\psi}_m(\boldsymbol{r})\hat{\psi}_{-m}(\boldsymbol{r}') \tag{2.11}$$

Similarly we can define the spin-density operator as follows

$$\hat{F}_\nu(\boldsymbol{r}) = \sum_{m,m'=-f}^{f} (f_\nu)_{mm'} \hat{\psi}^\dagger_m(\boldsymbol{r}) \hat{\psi}_{m'}(\boldsymbol{r}) \tag{2.12}$$

Finally, the above spin density operator can be generalized to give the rank-k spin nematic tensor operator ($k = 2,3,4\cdots,(f-1)$) as follows

$$\hat{N}^{(k)}_{\nu_1,\nu_2,\cdots,\nu_k}(\boldsymbol{r}) = \sum_{m,m'=-f}^{f} (f_{\nu_1} f_{\nu_2} \cdots f_{\nu_k})_{mm'} \hat{\psi}^\dagger_m(\boldsymbol{r}) \hat{\psi}_{m'}(\boldsymbol{r}) \tag{2.13}$$

where $(\nu_1,\nu_2,\cdots,\nu_k) = x,y,z$, and $(f_\nu)_{mm'}$ are the spin matrix elements, with $\nu = x,y,z$.

We introduce the projection operator $\hat{\mathscr{P}}_{\mathscr{F}}$, where the total spin angular momentum of the two body state is denoted by $\mathscr{F}$. As the name suggests, $\hat{\mathscr{P}}_{\mathscr{F}}$ projects onto the total spin $\mathscr{F}$ two body state. It is represented as

$$\hat{\mathscr{P}}_{\mathscr{F}} = \sum_{\mathscr{M}=-\mathscr{F}}^{\mathscr{F}} |\mathscr{F},\mathscr{M}\rangle\langle\mathscr{F},\mathscr{M}| \tag{2.14}$$

We can begin with the very commonly known completeness relation, which is given by

$$\hat{I} = \sum_{\mathscr{F}} \hat{\mathscr{P}}_{\mathscr{F}} \tag{2.15}$$

where $\hat{I}$ is the identity operator.

We use this equation above, and (2.5) and (2.10) to simplify and get

$$:\hat{n}(\boldsymbol{r})\hat{n}(\boldsymbol{r}'): = \sum_{\mathscr{F}=0,2,\cdots,2f} \sum_{\mathscr{M}=-\mathscr{F}}^{\mathscr{F}} \hat{A}^\dagger_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \hat{A}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \tag{2.16}$$

where the : : denotes the normal ordering of the operators, which places all the annihilation operators to the left, and the creation operators to the right. We can now also further use the composition of angular momentum to derive another very useful relation

$$\hat{\boldsymbol{f}}_1 \cdot \hat{\boldsymbol{f}}_2 = \frac{1}{2}\left[(\hat{\boldsymbol{f}}_1 + \hat{\boldsymbol{f}}_2)^2 - \hat{\boldsymbol{f}}_1{}^2 - \hat{\boldsymbol{f}}_2{}^2\right] = \frac{1}{2}\hat{\boldsymbol{f}}^2{}_{tot} - f(f+1) \tag{2.17}$$

Now we shall use (2.5), (2.12) and (2.15) to get the following

$$:\hat{F}(\boldsymbol{r})\cdot\hat{F}(\boldsymbol{r}'):=\sum_{\mathscr{F}=0,2,\cdots,2f}\left[\frac{1}{2}\mathscr{F}(\mathscr{F}+1)-f(f+1)\right]\sum_{\mathscr{M}=-\mathscr{F}}^{\mathscr{F}}\hat{A}^{\dagger}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \quad (2.18)$$

We can extend this further to include terms of the form $(\hat{\boldsymbol{f}}_1\cdot\hat{\boldsymbol{f}}_2)^k$ as follows:

$$(\hat{\boldsymbol{f}}_1\cdot\hat{\boldsymbol{f}}_2)^k=\sum_{\mathscr{F}}\left[\frac{1}{2}\mathscr{F}(\mathscr{F}+1)-f(f+1)\right]^k\hat{\mathscr{P}}_{\mathscr{F}} \quad (2.19)$$

where $k=2,3,4\cdots$. Using (2.5) and (2.13), we get the following:

$$\sum_{\nu_1,\nu_2,\cdots\nu_k=x,y,z}:\hat{N}^{(k)}_{\nu_1,\nu_2,\cdots\nu_k}\hat{N}^{(k)}_{\nu_1,\nu_2,\cdots\nu_k}(\boldsymbol{r}'):=\sum_{\mathscr{F}=0,2,\cdots,2f}\left[\frac{1}{2}\mathscr{F}(\mathscr{F}+1)-f(f+1)\right]^k \times\sum_{\mathscr{M}=-\mathscr{F}}^{\mathscr{F}}\hat{A}^{\dagger}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \quad (2.20)$$

This gives us three main equations (2.16), (2.18) and (2.20), which we shall use to express the interaction Hamiltonian (2.6) in terms of these $(f+1)$ observable operators. We will use the results from this section in Chapter(3).

2.4 Bogoliubov Spectrum for Low Spin BECs

Bogoliubov theory describes the behaviour of a BEC under elementary excitations in the ground state order parameter, and the behaviour associated with these excitations is governed by the Bogoliubov energy spectrum. The Bogoliubov spectrum for low spin systems can readily be found in literature [8, 9, 53, 54, 55]. Here, we shall attempt to provide an overview of the polar phase (where all particles occupy the $m=0$ component) spectrum of the spin-1 and spin-2 systems. This will help better motivate our calculations in Chapter(4), where we will perform a general analytical calculation for the polar phase and ferromagnetic phase of a general spin-f system.

2.4.1 Spin-1

The polar phase has the ground state order parameter(OP) of $\zeta = (0, \sqrt{n}, 0)^T$. This state is stable for large values of q, and as the condensate only occupies the $m = 0$ state, the spin density vector has zero magnitude.

The chemical potential for this system is $\mu = c_0 n$. We write a perturbed solution of this polar state,

$$\Psi_m = (\zeta_m + \delta\psi_m) e^{-i\mu t/\hbar} \tag{2.21}$$

and the perturbation is given by the ansatz

$$\delta\psi_m = u_m e^{-i\omega t} + v_m^* e^{+i\omega t} \tag{2.22}$$

These perturbations are very small, which means we shall discard all quadratic terms (and higher) in u_m, v_m.

The GPEs governing the dynamics of the spin-1 condensate can commonly be found in literature, and are given as,

$$i\hbar \frac{\partial \Psi_{\pm 1}}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2M} + q \mp p + c_0 n \pm c_1 F_z \right] \Psi_{\pm 1} + \left[\frac{c_1 F_\mp}{\sqrt{2}} \right] \Psi_0 \tag{2.23}$$

$$i\hbar \frac{\partial \Psi_0}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2M} + c_0 n \right] \Psi_0 + \left[\frac{c_1 F_+}{\sqrt{2}} \right] \Psi_1 + \left[\frac{c_1 F_-}{\sqrt{2}} \right] \Psi_{-1} \tag{2.24}$$

We shall insert our ansatz (4.3), and simplify by retaining only terms up to linear in u_m and v_m. After simplifying and equating the coefficients of $\exp\left(\frac{-i\omega t}{\hbar}\right)$ and $\exp\left(\frac{i\omega t}{\hbar}\right)$ on either side of the equation, (2.23) will give us the BdG equations governing dynamics of excitations in the $m = \pm 1$

components,

$$\omega u_1 = \left(-\frac{\hbar^2\nabla^2}{2M} - p + q + c_1 n\right) u_1 + c_1 n v_{-1} \tag{2.25}$$

$$-\omega v_1 = \left(-\frac{\hbar^2\nabla^2}{2M} - p + q + c_1 n\right) v_1 + c_1 n u_{-1} \tag{2.26}$$

$$\omega u_{-1} = \left(-\frac{\hbar^2\nabla^2}{2M} u_{-1} - p u_{-1} + q u_{-1} + c_1 n u_{-1} + c_1 n v_1\right) \tag{2.27}$$

$$-\omega v_{-1} = \left(-\frac{\hbar^2\nabla^2}{2M} v_{-1} - p v_{-1} + q v_{-1} + c_1 n v_{-1} + c_1 n u_1\right). \tag{2.28}$$

Similarly (2.24) gives the equation for the excitation in the $m = 0$ component,

$$\omega u_0 = \left(-\frac{\hbar^2\nabla^2}{2M} u_0 + c_0 n u_0 + c_0 n v_0\right) \tag{2.29}$$

$$-\omega v_0 = \left(-\frac{\hbar^2\nabla^2}{2M} v_0 + c_0 n v_0 + c_0 n u_0\right) \tag{2.30}$$

Since our system is homogeneous, we can write the perturbation in the plane wave basis, which has the effect of replacing the free particle energy with $\varepsilon_k = \hbar^2 k^2/2M$. This has given us 6 total equations which can be written as an eigenvalue equation, where the energy spectrum is given by the eigenvalues of the following matrix,

$$\mathbb{M}_1 = \begin{pmatrix} \tilde{\varepsilon}_{k,+} + c_1 n & 0 & 0 & 0 & 0 & c_1 n \\ 0 & -(\tilde{\varepsilon}_{k,+} + c_1 n) & 0 & 0 & -c_1 n & 0 \\ 0 & 0 & (\varepsilon_k + c_0 n) & (c_0 n) & 0 & 0 \\ 0 & 0 & -(c_0 n) & -(\varepsilon_k + c_0 n) & 0 & 0 \\ 0 & c_1 n & 0 & 0 & \tilde{\varepsilon}_{k,-} + c_1 n & 0 \\ -c_1 n & 0 & 0 & 0 & 0 & -(\tilde{\varepsilon}_{k,-} + c_1 n) \end{pmatrix} \tag{2.31}$$

where $\tilde{\varepsilon}_{k,\pm} = \varepsilon_k \mp p + q$. This 6×6 matrix can now be written as block diagonal matrix consisting of three 2×2 matrices. The eigenvalues of the full matrix are simply given by the eigenvalues of the individual blocks. The BdG matrix obtained for the higher spin systems can also be handled this way, and we shall use use this fact later as well. The eigenvalues give the energy spectrum of

the spin-1 Polar phase, which finally comes out to be:

$$E_{k,\mp 1} = \sqrt{(\varepsilon_k + q)(\varepsilon_k + q + 2c_1 n)} \pm p \tag{2.32}$$

$$E_{k,0} = \sqrt{\varepsilon_k(\varepsilon_k + 2nc_0)} \tag{2.33}$$

where the $E_{k,0}$ mode is the gapless mode (meaning $E_{k,0} = 0$, when $k = 0$), corresponding to density excitations in the system, and the remaining 2 modes are the gapped modes($E_{k,m} \neq 0$ when $k = 0$, for $m \neq 0$) which are tied to spin excitations in the system.

2.4.2 Spin-2

We now move onto the spin-2 polar phase, which has the ground state order parameter of $\zeta = (0, 0, \sqrt{n}, 0, 0)^T$. We also easily find that $\mu = c_0 n + c_2 n/5$, where μ is the chemical potential of the system. The spin-2 GPEs governing the dynamics, are given as

$$i\hbar \frac{\partial \psi_{\pm 2}}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2M} + 4q \mp 2p + c_0 n \pm 2c_1 F_z \right] \psi_{\pm 2} + c_1 F_{\mp} \Psi_{\pm 1} + \frac{c_2}{\sqrt{5}} A_{00} \Psi^*_{\mp 2} \tag{2.34}$$

$$i\hbar \frac{\partial \psi_{\pm 1}}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2M} + q \mp p + c_0 n \pm c_1 F_z \right] \psi_{\pm 1} + c_1 \left(\frac{\sqrt{6}}{2} F_{\mp} \Psi_0 + F_{\pm} \Psi_{\pm 2} \right) - \frac{c_2}{\sqrt{5}} A_{00} \Psi^*_{\mp 1} \tag{2.35}$$

$$i\hbar \frac{\partial \psi_0}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2M} + c_0 n \right] \psi_0 + \frac{\sqrt{6}}{2} c_1 \left(F_- \Psi_1 + F_+ \Psi_{-1} \right) + \frac{c_2}{\sqrt{5}} A_{00} \Psi^*_0 \tag{2.36}$$

As in the spin-1 case, we consider a perturbed state of the polar phase order parameter

$$\Psi_m = (\zeta_m + \delta \psi_m) e^{-i\mu t/\hbar} \tag{2.37}$$

and the perturbation takes the form of the following ansatz

$$\delta \psi_m = u_m e^{-i\omega t} + v^*_m e^{+i\omega t} \tag{2.38}$$

and insert the above ansatz into all 5 spin-2 GPEs, neglecting the terms that are $\mathcal{O}(u_m^2)$ and $\mathcal{O}(v_m^2)$, or higher. After simplifications and some elementary transformations, we get an eigenvalue equa-

tion, where the relevant matrix can be diagonalized to get

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} (\varepsilon_k + c_0 n + c_2 n/5) & (c_0 n + c_2 n/5) \\ -(c_0 n + c_2 n/5) & -(\varepsilon_k + c_0 n + c_2 n/5) \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \tag{2.39}$$

$$\begin{pmatrix} u_{\pm 2} \\ v_{\mp 2} \end{pmatrix} = \begin{pmatrix} (\varepsilon_k + 4q \mp 2p - c_2 n/5) & (c_2 n/5) \\ -(c_2 n/5) & -(\varepsilon_k + 4q \pm 2p - c_2 n/5) \end{pmatrix} \begin{pmatrix} u_{\pm 1} \\ v_{\mp 1} \end{pmatrix} \tag{2.40}$$

and finally

$$\begin{pmatrix} u_{\pm 1} \\ v_{\mp 1} \end{pmatrix} = \begin{pmatrix} (\varepsilon_k + q \mp p - c_2 n/5 + 3nc_1) & (3nc_1 - c_2 n/5) \\ -(3nc_1 - c_2 n/5) & -(\varepsilon_k + q \pm p - c_2 n/5 + 3nc_1) \end{pmatrix} \begin{pmatrix} u_{\pm 1} \\ v_{\mp 1} \end{pmatrix} \tag{2.41}$$

The final energy spectrum comes out to be:

$$E_{k,\mp 2} = \sqrt{(\varepsilon_k + 4q)\left(\varepsilon_k + 4q - \frac{2c_2 n}{5}\right)} \pm 2p \tag{2.42}$$

$$E_{k,\mp 1} = \sqrt{(\varepsilon_k + q)\left(\varepsilon_k + q + 6nc_1 - \frac{2c_2 n}{5}\right)} \pm p \tag{2.43}$$

$$E_{k,0} = \sqrt{\varepsilon_k \left(\varepsilon_k + 2nc_0 + \frac{2nc_2}{5}\right)} \tag{2.44}$$

where the $E_{k,0}$ mode is the gapless mode(meaning $E_{k,0} = 0$, when $k = 0$), corresponding to density excitations in the system, and the remaining 4 modes are the gapped modes($E_{k,m} \neq 0$ when $k = 0$, for $m \neq 0$) which are tied to spin excitations.

In this chapter, we have established the basic formalism for the theoretical description of spinor BECs. We started by describing the general formalism for Spinor BECs in general, which will prove vital in Chapter(3), when we derive the Hamiltonian and Gross-Pitaevskii equations for the spin-10 system. We also provided the calculations pertaining to the Bogoliubov spectrum for the polar phase of spin-1 and spin-2 BECs. We will expand upon these calculations in Chapter(4) where we work with general spin-f systems.

Chapter 3

Spin-10 Bose-Einstein Condensate

The main motivation for picking the spin-10 system comes from a series of works on doubly dipolar dysprosium atoms[46, 47, 48, 49]. In these works, they chiefly speak about the total angular momemtum $J_a = 9$ and $J_b = 10$ in dysprosium. They couple these two states using an uniform electric field, and since dysprosium has a magnetic diople moment, this creates a doubly dipolar system. However, the chief point of interest for us is the $J_b = 10$, whose linewidth under the electric dipole approximation has been reported to be close to zero, meaning it is a metastable state. A metastable state with total angular momentum 10 has a similar sub-level structure to our system of interest, which is the spin-10 BEC. This above result was the main reason we started working on the properties of the spin-10 system. Perhaps the work done in this thesis could also have implications in future works in this field.

We shall use the formalism described in the previous chapter, and apply it to our system of interest. We saw that a spin-f system has $f+1$ linearly independent observable operators that we can use to write the Hamiltonian of the system. For a spin-10 system, we have 11 independent operators, namely the spin singlet-pair operator, the total density operator, the spin density operator, and then 8 spin nematic tensor operators ranging from rank 2 to rank 9. We also correspondingly have 11 total allowed spin interaction channels in the spin-10 Hamiltonian, ranging from $\mathscr{F} = 0, 2, 4, \cdots, 20$.

There are two different, but physically equivalent ways to write the full Hamiltonian. We can ei-

ther choose to write it in terms of the creation and annhilation operators and the scattering lengths, or in terms of the observable operators and the corresponding interaction coefficients. We shall derive and examine the full Hamiltonian for the spin-10 system in this chapter. We shall then use the derived Hamiltonian to find the general form of the Gross-Pitaevskii equations for the system. These are a set of $2f+1$ non-linear Schroedinger like partial differential equations which describe the complete time evolution of a spin-f condensate.

3.1 Hamiltonian

We have 11 independent operators for the spin-10 system. Let us describe the relation between the creation and annihilation operators, and the 11 observable operators in the form of a matrix equation, which looks as follows

$$\begin{pmatrix} \hat{A}^{\dagger}_{00}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{00}(\boldsymbol{r},\boldsymbol{r}') \\ :\hat{n}(\boldsymbol{r})\hat{n}(\boldsymbol{r}'): \\ :\hat{F}(\boldsymbol{r})\cdot\hat{F}(\boldsymbol{r}'): \\ \sum_{\nu_1,\nu_2=x,y,z}:\hat{N}^{(2)}_{\nu_1,\nu_2}(\boldsymbol{r})\hat{N}^{(2)}_{\nu_1,\nu_2}(\boldsymbol{r}'): \\ \vdots \\ \sum_{\nu_1,\nu_2,\cdots\nu_9=x,y,z}:\hat{N}^{(9)}_{\nu_1,\nu_2,\cdots\nu_9}(\boldsymbol{r})\hat{N}^{(9)}_{\nu_1,\nu_2,\cdots\nu_9}(\boldsymbol{r}'): \end{pmatrix} = \mathscr{G} \begin{pmatrix} \hat{A}^{\dagger}_{00}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{00}(\boldsymbol{r},\boldsymbol{r}') \\ \sum^{2}_{\mathscr{M}=-2}\hat{A}^{\dagger}_{2\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{2\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \\ \sum^{4}_{\mathscr{M}=-4}\hat{A}^{\dagger}_{4\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{4\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \\ \sum^{6}_{\mathscr{M}=-6}\hat{A}^{\dagger}_{6\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{6\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \\ \vdots \\ \sum^{20}_{\mathscr{M}=-20}\hat{A}^{\dagger}_{20\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{20\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \end{pmatrix} \tag{3.1}$$

where $\mathscr{G}$ is a 11×11 matrix whose entries will be given by (2.16), (2.18) and (2.20).Calculating out the matrix elements leads to the following 11×11 matrix

$$\mathscr{G} = \left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
-110 & -107 & -100 & -89 & -74 & -55 \\
12100 & 11449 & 10000 & 7921 & 5476 & 3025 \\
-1.331\times10^{6} & -1.225\times10^{6} & -1\times10^{6} & -7.0496\times10^{5} & -4.052\times10^{5} & -1.663\times10^{5} \\
1.464\times10^{8} & 1.311\times10^{8} & 1\times10^{8} & 6.274\times10^{7} & 2.999\times10^{7} & 9.151\times10^{6} \\
-1.611\times10^{10} & -1.403\times10^{10} & -1\times10^{10} & -5.584\times10^{9} & -2.219\times10^{9} & -5.033\times10^{8} \\
1.771\times10^{12} & 1.501\times10^{12} & 1\times10^{12} & 4.970\times10^{11} & 1.642\times10^{11} & 2.768\times10^{10} \\
-1.949\times10^{14} & -1.606\times10^{14} & -1\times10^{14} & -4.423\times10^{13} & -1.215\times10^{13} & -1.522\times10^{12} \\
2.144\times10^{16} & 1.718\times10^{16} & 1\times10^{16} & 3.937\times10^{15} & 8.992\times10^{14} & 8.373\times10^{13} \\
-2.358\times10^{18} & -1.838\times10^{18} & -1\times10^{18} & -3.504\times10^{17} & -6.654\times10^{16} & -4.605\times10^{15}
\end{array}\right.$$

$$\left.\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
-32 & -5 & 26 & 61 & 100 \\
1024 & 25 & 676 & 3721 & 10000 \\
-3.276\times10^{4} & -1.250\times10^{2} & 1.757\times10^{4} & 2.269\times10^{5} & 1\times10^{6} \\
1.049\times10^{6} & 6.250\times10^{2} & 4.570\times10^{5} & 1.385\times10^{7} & 1\times10^{8} \\
-3.355\times10^{7} & -3.125\times10^{3} & 1.188\times10^{7} & 8.446\times10^{8} & 1\times10^{10} \\
1.074\times10^{9} & 1.562\times10^{4} & 3.089\times10^{8} & 5.152\times10^{10} & 1\times10^{12} \\
-3.436\times10^{10} & -7.813\times10^{4} & 8.032\times10^{9} & 3.143\times10^{12} & 1\times10^{14} \\
1.100\times10^{12} & 3.906\times10^{5} & 2.088\times10^{11} & 1.917\times10^{14} & 1\times10^{16} \\
-3.518\times10^{13} & -1.953\times10^{6} & 5.430\times10^{12} & 1.169\times10^{16} & 1\times10^{18}
\end{array}\right] \tag{3.2}$$

We can invert (3.1) to get

$$\begin{pmatrix}
\hat{A}^{\dagger}_{00}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{00}(\boldsymbol{r},\boldsymbol{r}') \\
\sum^{2}_{\mathscr{M}=-2}\hat{A}^{\dagger}_{2\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{2\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \\
\sum^{4}_{\mathscr{M}=-4}\hat{A}^{\dagger}_{4\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{4\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \\
\sum^{6}_{\mathscr{M}=-6}\hat{A}^{\dagger}_{6\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{6\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \\
\vdots \\
\sum^{20}_{\mathscr{M}=-20}\hat{A}^{\dagger}_{20\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{20\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')
\end{pmatrix} = \mathscr{G}^{-1}
\begin{pmatrix}
\hat{A}^{\dagger}_{00}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{00}(\boldsymbol{r},\boldsymbol{r}') \\
:\hat{n}(\boldsymbol{r})\hat{n}(\boldsymbol{r}'): \\
:\hat{F}(\boldsymbol{r})\cdot\hat{F}(\boldsymbol{r}'): \\
\sum_{\nu_1,\nu_2=x,y,z}:\hat{N}^{(2)}_{\nu_1,\nu_2}(\boldsymbol{r})\hat{N}^{(2)}_{\nu_1,\nu_2}(\boldsymbol{r}'): \\
\vdots \\
\sum_{\nu_1,\nu_2,\cdots\nu_9=x,y,z}:\hat{N}^{(9)}_{\nu_1,\nu_2,\cdots\nu_9}(\boldsymbol{r})\hat{N}^{(9)}_{\nu_1,\nu_2,\cdots\nu_9}(\boldsymbol{r}'):
\end{pmatrix} \tag{3.3}$$

Inverting matrix $\mathscr{G}$ given in (3.2) gives us the following matrix

$$\mathscr{G}^{-1} = \begin{bmatrix} 1 & -7.722\times10^{-15} & -5.157\times10^{-16} & -3.798\times10^{-17} & 2.120\times10^{-19} & 7.952\times10^{-20} \\ -2.174 & 1.202\times10^{-1} & 2.635\times10^{-2} & 2.727\times10^{-4} & -4.316\times10^{-5} & -9.326\times10^{-7} \\ 2.113 & -4.165\times10^{-1} & -9.109\times10^{-2} & -8.885\times10^{-4} & 1.497\times10^{-4} & 3.135\times10^{-6} \\ -1.507 & 7.011\times10^{-1} & 1.524\times10^{-1} & 1.316\times10^{-3} & -2.518\times10^{-4} & -4.966\times10^{-6} \\ 8.326\times10^{-1} & -7.984\times10^{-1} & -1.718\times10^{-1} & -1.128\times10^{-3} & 2.851\times10^{-4} & 5.025\times10^{-6} \\ -3.583\times10^{-1} & 7.063\times10^{-1} & 1.487\times10^{-1} & 3.480\times10^{-4} & -2.451\times10^{-4} & -3.398\times10^{-6} \\ 1.185\times10^{-1} & -5.693\times10^{-1} & -1.124\times10^{-1} & 1.053\times10^{-3} & 1.595\times10^{-4} & 1.344\times10^{-6} \\ -2.917\times10^{-2} & 1.208 & 3.463\times10^{-2} & -1.708\times10^{-3} & -6.655\times10^{-5} & -1.179\times10^{-7} \\ 5.047\times10^{-3} & 5.204\times10^{-2} & 1.390\times10^{-2} & 7.595\times10^{-4} & 1.162\times10^{-5} & -1.317\times10^{-7} \\ -5.481\times10^{-4} & -3.029\times10^{-3} & -7.423\times10^{-4} & -2.525\times10^{-5} & 6.099\times10^{-7} & 4.368\times10^{-8} \\ 2.815\times10^{-5} & 1.165\times10^{-4} & 2.781\times10^{-5} & 7.814\times10^{-7} & -3.157\times10^{-8} & -1.611\times10^{-9} \end{bmatrix}$$

$$\begin{bmatrix} 2.184\times10^{-23} & 2.730\times10^{-24} & -1.787\times10^{-25} & -1.521\times10^{-27} & -7.329\times10^{-30} \\ 8.207\times10^{-9} & 3.100\times10^{-10} & 8.917\times10^{-13} & -2.196\times10^{-14} & -1.307\times10^{-16} \\ -2.958\times10^{-8} & -1.045\times10^{-9} & -2.688\times10^{-12} & 7.411\times10^{-14} & 4.235\times10^{-16} \\ 5.283\times10^{-8} & 1.663\times10^{-9} & 3.426\times10^{-12} & -1.180\times10^{-13} & -6.344\times10^{-16} \\ -6.452\times10^{-8} & -1.698\times10^{-9} & -2.235\times10^{-12} & 1.207\times10^{-13} & 6.007\times10^{-16} \\ 5.878\times10^{-8} & 1.205\times10^{-9} & 3.737\times10^{-13} & -8.690\times10^{-14} & -3.950\times10^{-16} \\ -3.961\times10^{-8} & -5.946\times10^{-10} & 6.258\times10^{-13} & 4.501\times10^{-14} & 1.852\times10^{-16} \\ 1.847\times10^{-8} & 1.934\times10^{-10} & -6.019\times10^{-13} & -1.658\times10^{-14} & -6.139\times10^{-17} \\ -5.288\times10^{-9} & -3.588\times10^{-11} & 2.611\times10^{-13} & 4.141\times10^{-15} & 1.376\times10^{-17} \\ 7.288\times10^{-10} & 2.200\times10^{-12} & -5.946\times10^{-14} & -6.312\times10^{-16} & -1.879\times10^{-18} \\ -1.661\times10^{-11} & 2.089\times10^{-13} & 5.764\times10^{-15} & 4.443\times10^{-17} & 1.185\times10^{-19} \end{bmatrix} \tag{3.4}$$

Therefore to summarize, we have the following interacting part of the Hamiltonian for the spin-$\mathscr{F}$ channel in a spin-10 system

$$\hat{V}^{(\mathscr{F})} = \frac{1}{2}\int d\boldsymbol{r}\int d\boldsymbol{r}'\, \mathscr{V}^{(\mathscr{F})}(\boldsymbol{r},\boldsymbol{r}') \sum_{\mathscr{M}=-\mathscr{F}}^{\mathscr{F}} \hat{A}^{\dagger}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \tag{3.5}$$

and then summing over all 11 possible spin channels gives us the total interaction Hamiltonian. Finally, we can replace the $\sum_{\mathscr{M}=-\mathscr{F}}^{\mathscr{F}} \hat{A}^{\dagger}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')$ in (3.5) using the matrix given in (3.4) to write everything in terms of the 11 independent observable operators to get the complete spin-10 Hamiltonian.

Looking back to the spin-1 system, we do a similar transformation to write the interaction Hamiltonian in terms of the commonly used c_0 and c_2 coefficients. For the spin-10 system, since there are 11 independent operators in the interaction Hamiltonian, there will be 11 interaction coefficients,

ranging from c_0 to c_{10}. We can write the total Hamiltonian for a spin-10 BEC as follows

$$\begin{aligned}\hat{H} = \int d\boldsymbol{r} \Bigg[\sum_{m,m'=-10}^{10} \hat{\psi}_m^\dagger \left\{ -\frac{\hbar^2\nabla^2}{2M} - pm + qm^2 + V_{trap}(\boldsymbol{r}) \right\} \hat{\psi}_{m'} \\ + \frac{1}{2}\Big\{ c_1 \hat{A}_{00}^\dagger(\boldsymbol{r},\boldsymbol{r})\hat{A}_{00}(\boldsymbol{r},\boldsymbol{r}) + c_0 : \hat{n}(\boldsymbol{r})^2 : + c_2 : \hat{F}(\boldsymbol{r})^2 : \\ + c_3 \sum_{\nu_1,\nu_2=x,y,z} : \left(\hat{N}^{(2)}_{\nu_1,\nu_2}(\boldsymbol{r})\right)^2 : + \cdots + c_{10} \sum_{\nu_1,\nu_2,\cdots\nu_9=x,y,z} : \left(\hat{N}^{(9)}_{\nu_1,\nu_2,\cdots\nu_9}(\boldsymbol{r})\right)^2 : \Big\} \Bigg] \end{aligned} \tag{3.6}$$

where all of the interaction coefficients are explicitly determined by the entries of the matrix given in (3.4). We write down the first few explicitly below

- $c_1 : g_0 - 2.174g_2 + 2.113g_4 - 1.507g_6 + 0.8326g_8 - 0.3583g_{10} + 0.1185g_{12} - 0.02917g_{14} + 0.005047g_{16} - 0.0005481g_{18} + 0.00002815g_{20}$
- c_0: $-7.722\times10^{-15}g_0 + 0.1202g_2 - 0.4165g_4 + 0.7011g_6 - 0.7984g_8 + 0.7063g_{10} - 0.5693g_{12} + 1.208g_{14} + 0.05204g_{16} - 0.003029g_{18} + 0.0001165g_{20}$
- c_2: $-5.157\times10^{-16}g_0 + 0.02635g_2 - 0.09109g_4 + 0.1524g_6 - 0.1718g_8 + 0.1487g_{10} - 0.1124g_{12} + 0.03463g_{14} + 0.01390g_{16} - 0.0007423g_{18} + 0.00002781g_{20}$

and so on.

As we can clearly see from (3.4), the later coefficients will be much smaller. The c_n coefficients fall in value as n increases.

3.2 Spin Nematic Tensor Magnitudes

Now that we have derived the full Hamiltonian, it might be worthwhile to have a look at how some of the terms in the Hamiltonian behave, both within the spin-10 system, as well as across different spin-f systems. We are already fairly familiar with the behaviour of the number density, spin density and spin singlet terms since these are the terms that appear very often in literature for spin-f systems when $f \leq 3$. Therefore we shall focus on spin nematic tensors.

The spin nematic tensors first appear in the spin-3 system and add a unique feature to the dynamics of the system. While we are familiar with the spin singlet operator which couples the $\pm m$ components, and the spin density operator which couples the mth sub-level to itself and to $m \pm 1$, as is evident from the structure of the spin matrices (2.7), (2.8), (2.9). However, the spin-nematic tensor couples sub-levels that are even further apart, which can be discerned from their definition (2.13). Therefore, it can cause jumps in the population across sub-levels, instead of the sequential dynamics that would be needed if the nematic tensors were absent. This has been described for the spin-3 system in [56]. These high rank nematic tensors can therefore make the couplings between components and the dynamics between them significantly more complex as we move to higher spins. However, there is limited existing work on the potential physics arising due to the higher rank nematic tensors.

3.2.1 Spin-10 Nematic Tensors

Let us start with the spin-10 system we have studied so far in this chapter. We know from (3.4) that the interaction coefficients decrease in magnitude as their subscript increases. Essentially, in general, we expect $c_{n+1} < c_n$, although of course, their exact values will depend on the values of all the scattering lengths. Now we know that we have 8 nematic tensors in the Hamiltonian of the spin-10 system, ranging from rank-2 all the way up to rank-9. It might therefore be tempting to ignore these higher ranked spin-nematic tensors, because their respective interaction coefficients are negligible compared to the initial few coefficients. However, we have found that this may not be as straightforward as it appears. To demonstrate this, let us consider two very commonly used order parameters:

- Polar Phase: Complete occupation of only the $m = 0$ component.
- Ferromagnetic Phase: Complete occupation of the $m = +f$ component.

Using these two order parameters, we calculated the magnitudes of all the terms appearing in the Hamiltonian given in (3.6). The results are shown in fig(3.1) and fig(3.2) for the polar and the ferromagnetic phases respectively. The magnitudes of the rank-k nematic tensors increase incredi-

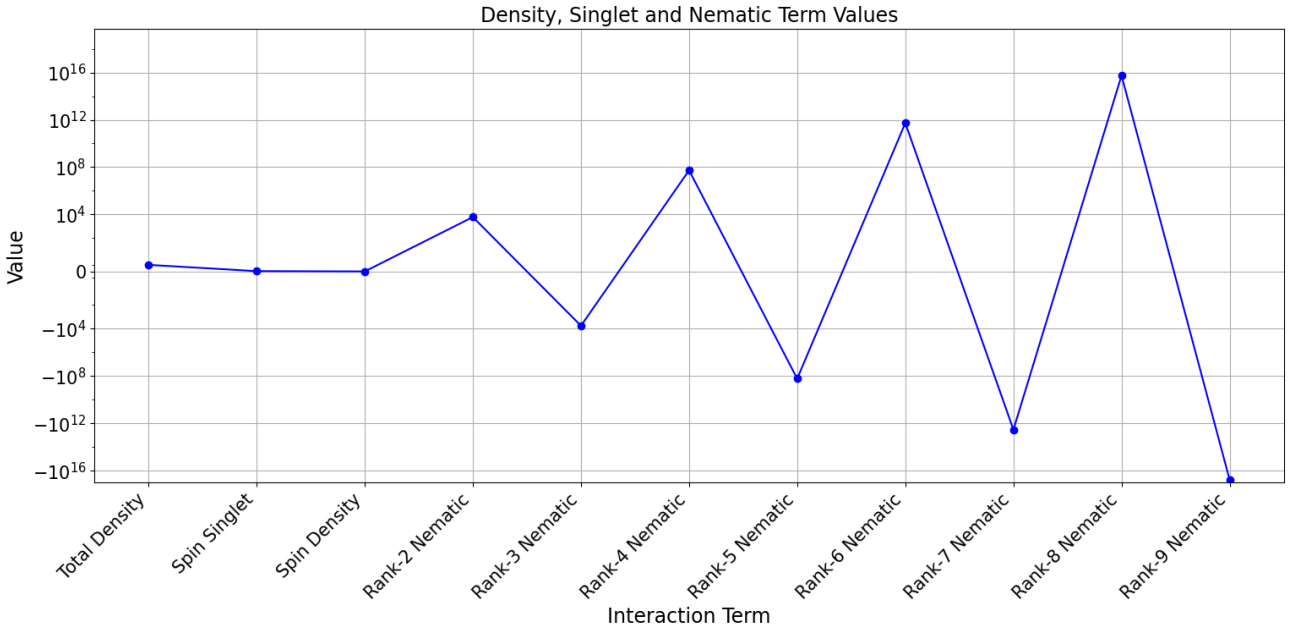

Figure 3.1: Using the Polar Phase order parameter, values of the density, singlet, and nematic terms for the spin-10 system, with the y-axis on a symmetric logarithmic scale

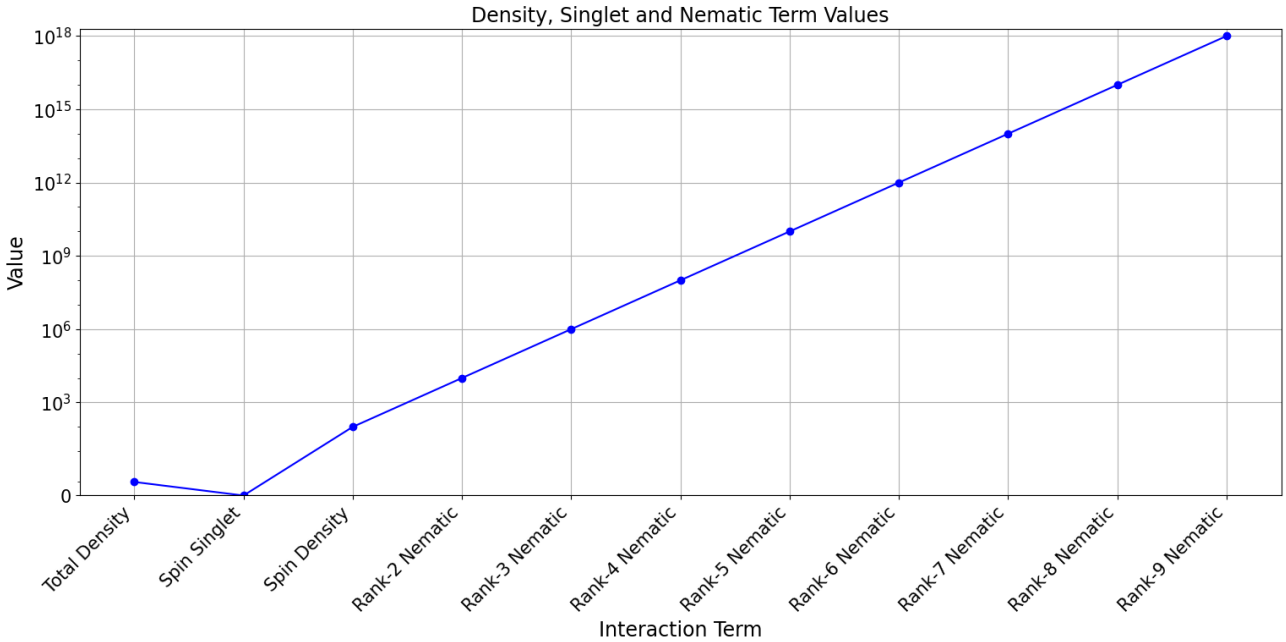

Figure 3.2: Using the Ferromagnetic Phase order parameter, values of the density, singlet, and nematic terms for the spin-10 system, with the y-axis on a logarithmic scale

bly quickly as we increase the tensor rank for both order order parameters. This is why we cannot simply throw away the later nematic tensors. When trying to simulate the system, one will have to pay attention to whether these tensors grow in magnitude faster than the interaction coefficients fall.

The values of the nematic tensors are independent of the scattering lengths of the system. The interplay between scattering length and the interaction coefficient values is an interesting problem to consider. Clearly, we do not have any experimental values to rely on. However, if we were to consider all scattering lengths to be sampled from a uniform random distribution with a certain mean and standard deviation, we find that as we decrease the standard deviation, all interaction coefficients(except c_0) also fall in value. This means that theoretically, if the scattering length values are extremely close together, the interaction coefficients will fall in value fast enough to safely ignore higher order nematic terms.

This also has implications when it comes to approximation techniques like the spatial Single Mode Approximation(SMA), which is commonly used to simplify numerical simulations of systems. The asymmetric part of the Hamiltonian should be a perturbation to the symmetric part of the Hamiltonian for the SMA to be valid. Here, symmetry is referring to whether that part of the Hamiltonian remains unchanged under the exchange of indices corresponding to the different spin components. Clearly since the nematic tensors are part of the asymmetric portion of the Hamiltonian, we need their energy contribution to be small. Therefore this interplay between the growth of the nematic tensors and the fall of the interaction coefficients needs to be studied very carefully before attempting to use the SMA for such high spin systems.

3.2.2 Comparison Between Spin-f tensors

In the previous section, we had a close look at the spin nematic tensors that appear in the spin-10 Hamiltonian. Now we shall look at these tensors as a function of the spin of the system. We consider the same two order parameters as before: ferromagnetic and polar. The rank-2 nematic tensor first appears in the spin-3 Hamiltonian. Let us consider the magnitude of this quantity plotted as a function of spin of the system. The results are given in fig(3.3) and fig(3.4). We can see that

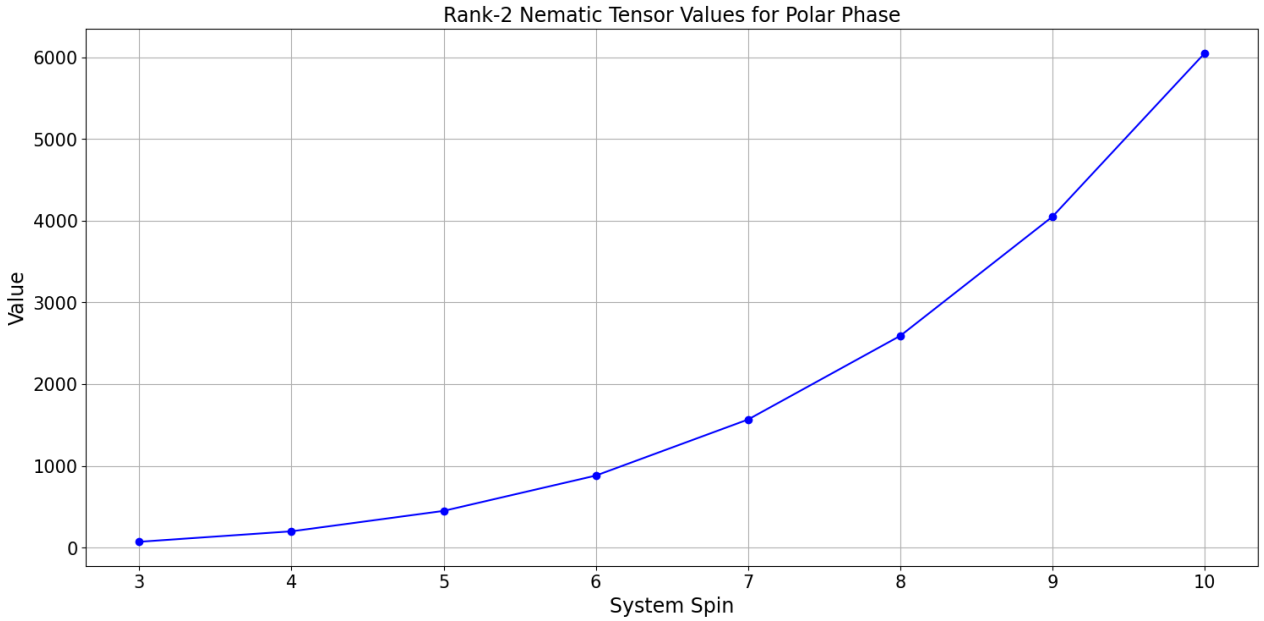

Figure 3.3: Values of the rank-2 nematic tensor for different system spins, using the Polar Phase order parameter.

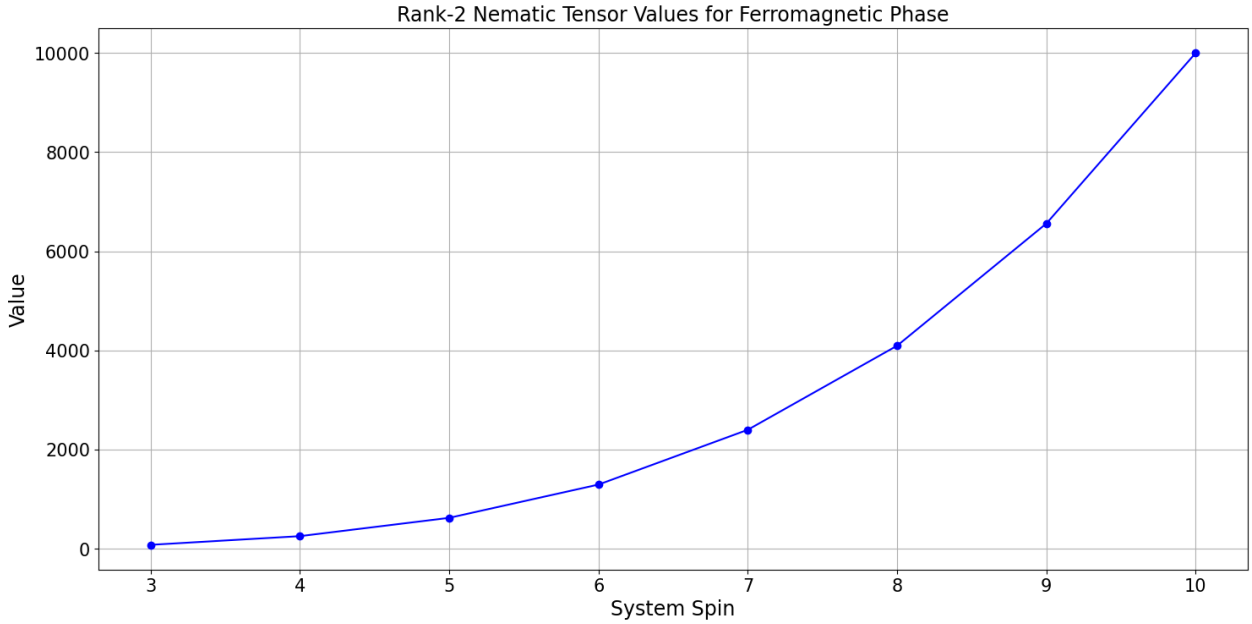

Figure 3.4: Values of the rank-2 nematic tensor for different system spins, using the Ferromagnetic Phase order parameter.

even just for the rank-2 nematic tensor, we have a significant increase in magnitude with increase in the spin of the BEC. We see very similar behaviour for both the Polar and Ferromagnetic order parameters, but the figures indicate that the values of the tensor for a given spin, are larger for the ferromagnetic order parameter compared to the polar order parameter, which is to be expected given the structure of the spin-matrices.

3.3 Gross-Pitaevskii Equations

In the previous section, we fully derived the Hamiltonian for a spin-10 BEC under the mean-field approximation. In this section, we shall focus on finding the non-linear Gross Pitaevskii equations. These are very commonly used in literature to describe the time evolution of the condensate, which can reveal some interesting population dynamics, as well as spin-textures, phase transitions, and topological structures. The GPEs are a set of $2f+1$ equations for a spin-f system. We have 1 GPE corresponding to each magnetic sub-level of the system. The GPEs are given by the following general equation:

$$i\hbar\frac{\partial\psi_m(\boldsymbol{r})}{\partial t} = \frac{\delta E}{\delta\psi_m^*(\boldsymbol{r})} \tag{3.7}$$

where E is the mean field energy functional given by $E[\psi] = \langle\hat{H}\rangle_0$, where $\langle\cdots\rangle_0 = \langle\boldsymbol{\zeta}|\cdots|\boldsymbol{\zeta}\rangle$. Here the state $|\boldsymbol{\zeta}\rangle$ is the state vector under the mean field approximation where all the condensed bosons are occupying a single spatial mode(which we can say is the i=0 mode), and a single spin state. this state vector can be denoted as follows:

$$|\boldsymbol{\zeta}\rangle = \frac{1}{\sqrt{N!}}\left(\sum_{m=-f}^{f}\zeta_m\hat{a}_{m0}^\dagger\right)^N|\text{vac}\rangle \tag{3.8}$$

Here $|\text{vac}\rangle$ is the vacuum state, and ζ_m is normalized as follows

$$\sum_{m=-f}^{f}|\zeta_m|^2 = 1 \tag{3.9}$$

We shall also subsequently need to make use of the following identities:

$$\langle \hat{\psi}_m(\boldsymbol{r})\rangle_0 = \langle \hat{\psi}_m^\dagger(\boldsymbol{r})\rangle_0 = 0 \tag{3.10}$$

$$\langle \hat{\psi}_m^\dagger(\boldsymbol{r})\hat{\psi}_{m'}(\boldsymbol{r}')\rangle_0 = \psi_m^*(\boldsymbol{r})\psi_{m'}(\boldsymbol{r}') \tag{3.11}$$

$$\langle \hat{\psi}_{m_1}^\dagger(\boldsymbol{r})\hat{\psi}_{m_2}^\dagger(\boldsymbol{r}')\hat{\psi}_{m_2'}(\boldsymbol{r}'')\hat{\psi}_{m_1'}(\boldsymbol{r}''')\rangle_0 = \left(1-\frac{1}{N}\right)\psi_{m_1}^*(\boldsymbol{r})\psi_{m_2}^*(\boldsymbol{r}')\psi_{m_2'}(\boldsymbol{r}'')\psi_{m_1'}(\boldsymbol{r}''') \tag{3.12}$$

and we have

$$\psi_m(\boldsymbol{r}) = \sqrt{N}\zeta_m\phi_{m0}(\boldsymbol{r}) \tag{3.13}$$

Here ϕ_{mi} is a orthonormal, complete basis, where m is the magnetic quantum number and i is the spatial mode. We already have the expression for the non-interacting part of the Hamiltonian, and we have (2.5) and (2.6) for the interacting part of the Hamiltonian. This relates the annihilation operator to the Clebsch Gordan coefficients as follows

$$\hat{V}^{(\mathscr{F})} = \frac{1}{2}\int d\boldsymbol{r}\int d\boldsymbol{r}'\,\mathscr{V}^{(\mathscr{F})}(\boldsymbol{r},\boldsymbol{r}')\sum_{\mathscr{M}=-\mathscr{F}}^{\mathscr{F}}\hat{A}^\dagger_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}')\hat{A}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') \tag{3.14}$$

$$\hat{A}_{\mathscr{F}\mathscr{M}}(\boldsymbol{r},\boldsymbol{r}') = \sum_{m,m'=-f}^{f}\langle \mathscr{F},\mathscr{M}|f,m;f,m'\rangle\psi_m(\boldsymbol{r})\psi_{m'}(\boldsymbol{r}') \tag{3.15}$$

We now express $C^{\mathscr{F}\mathscr{M}}_{f,m;f,m'} = \langle \mathscr{F},\mathscr{M}|f,m;f,m'\rangle$, where $C^{\mathscr{F}\mathscr{M}}_{f,m;f,m'}$ is the Clebsch Gordan coefficient. Since we are working under the mean-field approximation, we approximate $\mathscr{V}^{(\mathscr{F})} = g_{\mathscr{F}}\delta(\boldsymbol{r}-\boldsymbol{r}')$ as before. We can then write the full general Hamiltonian for the spin-10 system as follows:

$$\begin{aligned}\hat{H} = \int d\boldsymbol{r}\Bigg[&\sum_{m,m'=-f}^{f}\hat{\psi}_m^\dagger\left(-\frac{\hbar^2\nabla^2}{2M} - p(f_z)_{mm'} + q(f_z^2)_{mm'} + V_{trap}(\boldsymbol{r})\right)\hat{\psi}_{m'} \\ &+\frac{1}{2}\left(g_0\hat{A}^\dagger_{00}(\boldsymbol{r})\hat{A}_{00}(\boldsymbol{r}) + g_2\sum_{\mathscr{M}=-2}^{2}\hat{A}^\dagger_{2\mathscr{M}}(\boldsymbol{r})\hat{A}_{2\mathscr{M}}(\boldsymbol{r}) + \cdots + g_{20}\sum_{\mathscr{M}=-20}^{20}\hat{A}^\dagger_{20\mathscr{M}}(\boldsymbol{r})\hat{A}_{20\mathscr{M}}(\boldsymbol{r})\right)\Bigg]\end{aligned} \tag{3.16}$$

Now we use (3.15) to write the full Hamiltonian as,

$$\hat{H} = \int d\boldsymbol{r} \left[\sum_{m,m'=-10}^{10} \hat{\psi}_m^\dagger \left(-\frac{\hbar^2\nabla^2}{2M} - p(f_z)_{mm'} + q(f_z^2)_{mm'} + V_{trap}(\boldsymbol{r}) \right) \hat{\psi}_{m'} \right.$$
$$\left. + \frac{1}{2} \sum_{\mathscr{F}=0,2,\cdots,20} \sum_{\mathscr{M}=-\mathscr{F}}^{\mathscr{F}} \sum_{\substack{m_1,m_1',\\ m_2,m_2'=-10}}^{10} g_{\mathscr{F}} C^{\mathscr{F}\mathscr{M}}_{10,m_1;10,m_1'} C^{\mathscr{F}\mathscr{M}}_{10,m_2;10,m_2'} \hat{\psi}_{m_1}(\boldsymbol{r}) \hat{\psi}_{m_1'}(\boldsymbol{r}) \hat{\psi}^\dagger_{m_2}(\boldsymbol{r}) \hat{\psi}^\dagger_{m_2'}(\boldsymbol{r}) \right] \tag{3.17}$$

From here we find the expectation value of this Hamiltonian with respect to the state vector defined in (3.8). We use the identities(3.10), (3.11) and (3.12). We shall also use $(f_z)_{mm'} = m\delta_{mm'}$, meaning $(f_z^2)_{mm'} = m^2\delta_{mm'}$. This gives us

$$E[\psi] = \langle \hat{H} \rangle_0 = \int d\boldsymbol{r} \left[\sum_{m=-10}^{10} \psi_m^* \left(-\frac{\hbar^2\nabla^2}{2M} - pm + qm^2 + V_{trap}(\boldsymbol{r}) \right) \psi_m \right.$$
$$\left. + \frac{1}{2} \sum_{\mathscr{F}=0,2,\cdots,20} \left(\sum_{\mathscr{M}=-\mathscr{F}}^{\mathscr{F}} \left(\sum_{m_1,m_1',m_2,m_2'=-10}^{10} g_{\mathscr{F}} C^{\mathscr{F}\mathscr{M}}_{10,m_1;10,m_1'} C^{\mathscr{F}\mathscr{M}}_{10,m_2;10,m_2'} \psi_{m_1}(\boldsymbol{r}) \psi_{m_1'}(\boldsymbol{r}) \psi^*_{m_2}(\boldsymbol{r}) \psi^*_{m_2'}(\boldsymbol{r}) \right) \right) \right] \tag{3.18}$$

Where we are ignoring the term that goes as $(1/N)$ since it will be very small compared to the the remaining terms in the equation. Now we use (3.7) to find the non-linear Gross-Pitaevskii equations. This will give us the general form of the GPEs:

$$i\hbar \frac{\partial \psi_m(\boldsymbol{r})}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2M} - pm + qm^2 + V_{trap}(\boldsymbol{r}) \right) \psi_m$$
$$+ \frac{1}{2} \sum_{\mathscr{F}=0,2,\cdots,20} \left(\sum_{\mathscr{M}=-\mathscr{F}}^{\mathscr{F}} \left(\sum_{m_1,m_1',m_2=-10}^{10} g_{\mathscr{F}} C^{\mathscr{F}\mathscr{M}}_{10,m_1;10,m_1'} C^{\mathscr{F}\mathscr{M}}_{10,m;10,m_2} \psi_{m_1}(\boldsymbol{r}) \psi_{m_1'}(\boldsymbol{r}) \psi^*_{m_2}(\boldsymbol{r}) \right) \right) \tag{3.19}$$

However, this can be simplified a little further. For the Clebsch Gordan(CG) coefficient determined by the parameters $\langle j_1\ m_1\ j_2\ m_2|J\ M\rangle$, the explicit expression of the CG coefficient has the term $\delta(m_1+m_2,M)$ multiplying with the rest of the expression, where δ represents the dirac-delta function. Therefore the CG coefficient will simply be 0 unless we have $m_1+m_2=M$. Furthermore, we can see from (3.19) that we have the term $C^{\mathscr{F}\mathscr{M}}_{f,m_1;f,m_1'}C^{\mathscr{F}\mathscr{M}}_{f,m;f,m_2}$. Since $\mathscr{M}$ is common for both CG terms, we must have the following condition for the entire term to be non-zero

$$m_1+m_1'=m+m_2=\mathscr{M} \tag{3.20}$$

Now we look at the second term in (3.19) more closely, specifically the multiple summations we have to work with. For a specific GPE, the parameter m will always be fixed. Now if we fix m_1 and m_1', then $\mathscr{M}$ can only take one value if the expression has to remain non-zero, which is $\mathscr{M}=m_1+m_1'$. Therefore the summation over $\mathscr{M}$ is actually unnecessary, and can be eliminated since only one term in the summation leads to a non-zero value. Furthermore, we notice that the summation over $\mathscr{M}$ goes from $-\mathscr{F}$ to $\mathscr{F}$, we must also have $\mathscr{F}\geq|m_1+m_1'|$ to ensure that $\mathscr{M}$ takes the value m_1+m_1'. Now all of this means that we can rewrite (3.19) as follows:

$$\begin{aligned} i\hbar\frac{\partial\psi_m(\boldsymbol{r})}{\partial t} &= \left(-\frac{\hbar^2\nabla^2}{2M}-pm+qm^2+V_{trap}(\boldsymbol{r})\right)\psi_m \\ &+\frac{1}{2}\sum_{\mathscr{F}=|m_1+m_1'|,\cdots,2f}\left(\sum_{m_1,m_1',m_2=-f}^{f} g_{\mathscr{F}}\,C^{\mathscr{F}\ (m_1+m_1')}_{f,m_1;f,m_1'}C^{\mathscr{F}\ (m_1+m_1')}_{f,m;f,m_2}\,\psi_{m_1}(\boldsymbol{r})\psi_{m_1'}(\boldsymbol{r})\psi^*_{m_2}(\boldsymbol{r})\right) \end{aligned} \tag{3.21}$$

The first summation in the above equation will go from $|m_1+m_1'|$ to $2f$ in steps of 2 if $|m_1+m_1'|$ is even, or $|m_1+m_1'|+1$ to $2f$ in steps of 2, if $|m_1+m_1'|$ is odd. This is particles can only interact in spin channels having even $\mathscr{F}$ as discussed in Chapter(2). There are $(2f+1)$ magnetic sub-levels for a spin-f system, and there will be 1 GPE equation for every sub-level, which we can explicitly determine by plugging in the value of m, and the relevant Clebsch Gordan coefficients into (4.1).

Alternatively, we can also write the general GPE in terms of the interaction coefficients using the Hamiltonian described in (3.6). The derivation is fairly straightforward and proceeds similarly to the previous, so we shall directly state the final form here. The alternative form of the GPEs is

given as

$$i\hbar\frac{\partial\psi_m}{\partial t}=\left[-\frac{\hbar^2\nabla^2}{2M}-mp+m^2q+V_{trap}(\boldsymbol{r})+c_0n\right]\psi_m+\frac{(-1)^{f-m}c_1}{\sqrt{2f+1}}A_{00}\psi^*_{-m}+c_2\sum_{m'=-f}^{f}\boldsymbol{F}\cdot\boldsymbol{f}_{mm'}\psi_{m'}$$
$$+c_3\sum_{\nu,\nu'}\left(N^{(2)}_{\nu,\nu'}\sum_{m'=-f}^{f}(f_\nu f_{\nu'})_{mm'}\psi_{m'}\right)+\cdots+c_9\sum_{\nu_1,\nu_2,\cdots,\nu_9}\left(N^{(9)}_{\nu_1,\nu_2,\cdots,\nu_9}\sum_{m'=-f}^{f}(f_{\nu_1}f_{\nu_2}\cdots f_{\nu_9})_{mm'}\psi_{m'}\right) \tag{3.22}$$

These are both perfectly valid ways to describe the GPEs for the spin-10 system, and both have their benefits depending on the context of the problem.

Therefore, in this chapter, we have found the full Hamiltonian for the spin-10 BEC. The entire method can easily be generalized to find the Hamiltonian of a general spin-f system. We also looked at the nematic terms in more detail, specifically how their magnitude depends on the rank of the tensor, and the spin of the system and showed how this can have some interesting implications. We also derived the non-linear Gross-Pitaevskii equations for the system, in two different forms. We shall use the results from this section, specifically the results from Section(3.3), to find the energy spectrum of a general spin-f system for certain phases.

Chapter 4

Bogoliubov Spectrum

Small perturbations in the system, such as quantum or thermal fluctuations, or even external perturbations, can often highlight collective behaviour often observed in such interacting systems, and are often described as elementary excitations of the ground state of the system. These elementary excitations in a Bose-Einstein condensate are described by the Bogoliubov theory, and the behaviour associated with these excitations is governed by the Bogoliubov energy spectrum of the system. The spectrum for a scalar BEC with no interparticle interactions, is given simply by that of a single particle($\sim k^2$). However, the introduction of interactions changes the nature of the spectrum, which can now be given by linear behaviour initially, followed by quadratic single particle like behaviour at larger values of the wavevector k.

For spinor BECs, the spectrum becomes much more complicated for higher spins, owing to a larger number of tensorial interaction terms, as well as more sub-levels in the system. Furthermore, the possible ground state phases can also be much larger in number, and significantly more complicated to work with. We will have multiple branches of excitation in spinor BECs, namely $2f+1$ terms in the energy spectrum for a spin-f BEC, and the spectrum will change depending on the considered phase of the system. Since different phases will have a different spectrum, modes can develop instabilities when crossing a phase transition. Phase transitions can be understood in terms of the instabilities in the energy spectrum, and the spectrum can therefore provide valuable microscopic information about the behaviour of the system.

To find the spectrum, we start with the field operator for BEC and consider the ansatz where we introduce a small perturbation to this field operator. We then use the GPEs derived previously, and discard all terms that are quadratic or higher in the introduced perturbation. Simplifying further will lead us to an eigenvalue equation, where the eigenvalues are the modes of the energy spectrum for the system. We provided an introduction to this topic in Section(2.4), where we found the polar phase spectrum of spin-1 and spin-2 BECs. In this section, We shall perform a full analytical calculation of the spectrum for the polar and ferromagnetic phases of the general spin-f system. Following this, we shall look at the full spectrum of a spin-10 system for the above phases.

4.1 General Spin-f System

Having looked at lower spin-systems, we are ready to move onto the general spin-f system. As the spin degrees of freedom increase, the complexity of the system increases rapidly, for example, in terms of the number of ground state phases observed. For our discussion, we shall consider two important states, the state where the entire population is in the $m = 0$ sub-level(which we may term as the polar phase, analogous to the lower spin systems we discussed), and the ferromagnetic state where the entire population is in the $m = +f$ sub-level (maximal magnetization in the system). The general form of the non-linear Gross-Pitaevskii equations that we derived in Section(3.3) for a spin-f system is given as,

$$i\hbar\frac{\partial\psi_m(\boldsymbol{r})}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2M} - pm + qm^2 + V_{trap}(\boldsymbol{r})\right)\psi_m$$
$$\frac{1}{2}\sum_{\mathscr{F}=|m_1+m_1'|,\cdots,2f}\left(\sum_{m_1,m_1',m_2=-f}^{f} g_{\mathscr{F}}\, C^{\mathscr{F}\,(m_1+m_1')}_{f,m_1;f,m_1'} C^{\mathscr{F}\,(m_1+m_1')}_{f,m;f,m_2}\,\psi_{m_1}(\boldsymbol{r})\psi_{m_1'}(\boldsymbol{r})\psi^*_{m_2}(\boldsymbol{r})\right) \quad (4.1)$$

Here $C^{\mathscr{F}\,(m_1+m_1')}_{f,m_1;f,m_1'} = \langle \mathscr{F}, \mathscr{M}|f,m;f,m'\rangle$ are the Clebsch Gordan coefficients, and $g_{\mathscr{F}} = 4\pi\hbar^2 a_{\mathscr{F}}/M$, where $a_{\mathscr{F}}$ is the total spin-$\mathscr{F}$ channel scattering length

We consider a uniform stationary state described by the $2f+1$ component order-parameter $\boldsymbol{\zeta}$, and

then add the perturbation

$$\Psi_m = (\zeta_m + \delta\psi_m)e^{-i\mu t/\hbar} \tag{4.2}$$

with the perturbation given by the ansatz

$$\delta\psi_m = u_m e^{-i\omega t} + v_m^* e^{+i\omega t} \tag{4.3}$$

Inserting the ansatz in (4.1), and linearising in u_m, v_m, we obtain the coupled differential equation describing the dynamics of these perturbations. The equations, in general, couple all the states. But we can show that for ferromagnetic and polar phases, the components couple in such a way that the resultant matrix can always be reduced to a block diagonal form. Solving the eigenvalue equation will give the energy spectrum for the phase considered. The key is to analyze the GPEs and find out how different components couple to each other after inserting our ansatz and maintaining linearity in the perturbation. We shall prove this for both phases below.

4.1.1 Polar Phase

Proof of a Claim

We use the zero magnetization phase, having the ground state order parameter $\zeta_0 = 1$ and $\zeta_{m\neq 0} = 0$. This corresponds to the phase commonly referred to as the "polar phase" in lower spin systems.

Upon inserting our ansatz (4.3) into the GPE (4.1), we get the Bogoliubov-de-Gennes(BdG) equations, which couples excitations in different components. Let us examine in detail how different component excitations couple to each other.

The first part of the right hand side of the GPE (4.1) consists of the term,

$$\left(-\frac{\hbar^2\nabla^2}{2M} + U_{trap}(\boldsymbol{r}) - pm + qm^2\right)\psi_m \tag{4.4}$$

which couples each component only to itself. The main portion of interest will be the second

portion of the equation

$$\frac{1}{2}\sum_{\mathscr{F}=|m_1+m_1'|,\cdots,2f}\left(\sum_{m_1,m_1',m_2=-f}^{f} g_{\mathscr{F}}\, C^{\mathscr{F}\ (m_1+m_1')}_{f,m_1;f,m_1'} C^{\mathscr{F}\ (m_1+m_1')}_{f,m;f,m_2}\, \psi_{m_1}(\boldsymbol{r})\psi_{m_1'}(\boldsymbol{r})\psi^*_{m_2}(\boldsymbol{r})\right) \tag{4.5}$$

Through this term, the evolution of a component is coupled to the other ones. For further simplification, we look at the ψ_0 GPE equation and the rest of the equations separately. In Section(3.3), we discussed the properties of the Clebsch Gordan coefficients, which gave us the following condition

$$m_1 + m_1' = m + m_2 \tag{4.6}$$

We also recall that in (4.1), there is a summation over m_1, m_1', m_2, so all possible combinations of these three parameters will need to be considered. We divide our problem into two cases.

$\mathbf{m = 0}$

From (4.5), we see that the non-linear portion of the coupled GPE has a general form containing the term $\psi_{m_1}\psi_{m_1'}\psi^*_{m_2}$. The perturbed order parameter is given by

$$\psi_m = \left(\zeta_m + u_m \exp\left(\frac{-i\omega t}{\hbar}\right) + v^*_m \exp\left(\frac{i\omega t}{\hbar}\right)\right) e^{-i\mu t/\hbar} \tag{4.7}$$

Therefore, linearity of the expanded terms in u_m, v_m require two out of three $\psi_{\tilde{m}}$ terms to be ψ_0. We therefore know that two out of m_1, m_1' and m_2 must equal m to maintain the required linearity. However, this immediately follows from (4.6) that the third parameter must always equal m. It follows that $m_1 = m_1' = m_2 = m = 0$. This implies that the excitation in $m = 0$ phase does not couple to other components and can be separately solved for the polar phase.

$\mathbf{m \neq 0}$

If $m \neq 0$, we still need two two of the terms in $\psi_{m_1}\psi_{m_1'}\psi^*_{m_2}$ to be ψ_0 for it to be linear in u_m, v_m. We also recall (4.6). There are only three possibilities to consider, and we shall go through all of

them.

1. $m_1 = m_1' = 0$ and $m_2 \neq 0$. In this case, for $m_1 + m_1' = m + m_2$ to hold, we would need $m_2 = -m$.

2. $m_1 = m_2 = 0$ and $m_1' \neq 0$. From (4.6), we must have $m_1' = m$.

3. $m_1' = m_2 = 0$ and $m_1 \neq 0$, clearly gives us $m_1 = m$.

Finally let us check the couplings between the m^{th} component and the zeroth component. We shall consider the first of the three conditions mentioned above

$$\begin{aligned}\psi_0\psi_0\psi_{-m}^* &= \left(\sqrt{n} + u_0 \exp\left(\frac{-i\omega t}{\hbar}\right) + v_0^* \exp\left(\frac{i\omega t}{\hbar}\right)\right)^2 \left(u_{-m}^* \exp\left(\frac{i\omega t}{\hbar}\right) + v_{-m} \exp\left(\frac{-i\omega t}{\hbar}\right)\right) \\ &= n\left(u_{-m}^* \exp\left(\frac{i\omega t}{\hbar}\right) + v_{-m} \exp\left(\frac{-i\omega t}{\hbar}\right)\right) \end{aligned} \tag{4.8}$$

where we have neglected higher order terms in u, mv_m. There is clearly no dependence on u_0 or v_0 on the RHS, which proves that there is no dependence of the u_m and v_m parameters on u_0 or v_0 in the coupled equations. The other two conditions will clearly also yield the same conclusion.
We have therefore proven that for any value of m, u_m and v_m will only be coupled to $u_{\pm m}$ and $v_{\pm m}$. This shows us that the eventual matrix(that needs to be diagonalized) can always be brought into a block diagonal form. This result will significantly simplify our calculation.

Energy Spectrum Calculation

In this section, we shall perform the explicit analytical calculation to derive the Bogoliubov spectrum. We start by finding the chemical potential, which we denote by μ. We take the unperturbed order parameter given by $\Psi_0 = \sqrt{n}e^{-i\mu t/\hbar}$. When inserted into (4.1) with $m = 0$, we get,

$$\mu = \frac{1}{2}n \sum_{\mathscr{F}=0,2,\cdots,2f} \left[g_{\mathscr{F}} \left(C^{\mathscr{F}\,0}_{f,0;f,0} \right)^2 \right] \tag{4.9}$$

In order to solve for the BdG spectrum, Let us again divide our problem into two cases

$\mathbf{m = 0}$

Inserting our ansatz (4.3) into (4.1) gives us the following LHS,

$$i\hbar\frac{\partial\Psi_0}{\partial t} = \left(\omega u_0 \exp\left(\frac{-i\omega t}{\hbar}\right) - \omega v_0^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) e^{-i\mu t/\hbar} + \mu\left(\sqrt{n} + u_0 \exp\left(\frac{-i\omega t}{\hbar}\right) + v_0^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) e^{-i\mu t/\hbar} \tag{4.10}$$

The RHS will be given as

$$\begin{aligned} i\hbar\frac{\partial\Psi_0}{\partial t} &= \left[-\frac{\hbar^2\nabla^2}{2M}\right]\Psi_0 + \frac{1}{2}\sum_{\mathscr{F}=0,2,\cdots,20} g_{\mathscr{F}}(C^{\mathscr{F}\,0}_{10,0;10,0})^2\Psi_0\Psi_0\Psi_0^* \\ &= \left[-\frac{\hbar^2\nabla^2}{2M}\right]\left(\sqrt{n} + u_0 \exp\left(\frac{-i\omega t}{\hbar}\right) + v_0^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \frac{1}{2}\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}(C^{\mathscr{F}\,0}_{f,0;f,0})^2 \\ &\left(n^{3/2} + nu_0^* \exp\left(\frac{i\omega t}{\hbar}\right) + nv_0 \exp\left(\frac{-i\omega t}{\hbar}\right) + 2nu_0 \exp\left(\frac{-i\omega t}{\hbar}\right) + 2nv_0^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) \end{aligned} \tag{4.11}$$

However, from the ground state Polar phase order parameter, we know that we must have:

$$\left[-\frac{\hbar^2\nabla^2}{2M}\right](\sqrt{n}) + \frac{1}{2}\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}(C^{\mathscr{F}\,0}_{f,0;f,0})^2(n^{3/2}) = \mu\sqrt{n} \tag{4.12}$$

We substitute this back into the previous equation, Canceling out the common $\mu\sqrt{n}$ term reduces the equation to

$$\begin{aligned} &\left(\omega u_0 \exp\left(\frac{-i\omega t}{\hbar}\right) - \omega v_0^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \mu\left(u_0 \exp\left(\frac{-i\omega t}{\hbar}\right) + v_0^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) \\ &= \left[-\frac{\hbar^2\nabla^2}{2M}\right]\left(u_0 \exp\left(\frac{-i\omega t}{\hbar}\right) + v_0^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \frac{1}{2}\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}(C^{\mathscr{F}\,0}_{f,0;f,0})^2 \\ &\left(nu_0^* \exp\left(\frac{i\omega t}{\hbar}\right) + nv_0 \exp\left(\frac{-i\omega t}{\hbar}\right) + 2nu_0 \exp\left(\frac{-i\omega t}{\hbar}\right) + 2nv_0^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) \end{aligned} \tag{4.13}$$

And finally equating the coefficients of $\exp\left(\frac{i\omega t}{\hbar}\right)$ and $\exp\left(\frac{-i\omega t}{\hbar}\right)$ gives us the following two equations

$$\omega u_0 = \left[\varepsilon_k + \frac{n}{2}\left(\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}\left(C^{\mathscr{F}\,0}_{f,0;f,0}\right)^2\right)\right]u_0 + \frac{n}{2}\left(\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}(C^{\mathscr{F}\,0}_{f,0;f,0})^2\right)v_0 \quad (4.14)$$

$$-\omega v_0 = \left[\varepsilon_k + \frac{n}{2}\left(\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}\left(C^{\mathscr{F}\,0}_{f,0;f,0}\right)^2\right)\right]v_0 + \frac{n}{2}\left(\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}(C^{\mathscr{F}\,0}_{f,0;f,0})^2\right)u_0 \quad (4.15)$$

This can be written as

$$\omega\begin{pmatrix}u_0\\ v_0\end{pmatrix} = \mathscr{M}_0\begin{pmatrix}u_0\\ v_0\end{pmatrix} \quad (4.16)$$

where the matrix, is given by

$$\mathscr{M}_0 = \begin{pmatrix}\left[\varepsilon_k + (n/2)\left(\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}\left(C^{\mathscr{F}\,0}_{f,0;f,0}\right)^2\right)\right] & n/2\left(\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}(C^{\mathscr{F}\,0}_{f,0;f,0})^2\right)\\ -(n/2)\left(\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}(C^{\mathscr{F}\,0}_{f,0;f,0})^2\right) & -\left[\varepsilon_k + (n/2)\left(\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}\left(C^{\mathscr{F}\,0}_{f,0;f,0}\right)^2\right)\right]\end{pmatrix} \quad (4.17)$$

where writing in the plane wave basis has given $\varepsilon_k = -\hbar^2k^2/2M$. The eigenvalues of this matrix give us the first term in the energy spectrum

$$E_{k,0} = \sqrt{\varepsilon_k\left[\varepsilon_k + n\left(\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}\left(C^{\mathscr{F}\,0}_{f,0;f,0}\right)^2\right)\right]} \quad (4.18)$$

$\mathbf{m \neq 0}$

In the previous section, we were able to prove that u_m, v_m couple only to themselves, and to u_{-m}, v_{-m}, meaning our coupled equations take the form

$$i\hbar\frac{\partial\Psi_m}{\partial t} = \mathscr{L}_m\Psi_m + \mathscr{L}'_m\Psi^*_{-m} \quad (4.19)$$

To proceed, we need the forms of $\mathscr{L}_m$ and $\mathscr{L}'_m$. When $m \neq 0$, we have 3 cases as discussed in the previous section. We shall consider them one at a time

- $m_1 = m'_1 = 0$

This means $m_2 = -m$. Now $(m_1 + m_1') = 0$, and therefore (4.5) will give us

$$\sum_{\mathscr{F}=0,2,\cdots,2f} \left(g_{\mathscr{F}} C^{\mathscr{F}\ 0}_{f,0;f,0} C^{\mathscr{F}\ 0}_{f,m;f,-m} \Psi_0 \Psi_0 \Psi^*_{-m} \right) \tag{4.20}$$

Inserting our ansatz will mean $\Psi_0 \Psi_0 \Psi^*_{-m}$ will simply give us $n\Psi^*_{-m} + \mathscr{O}(w^2)$. Therefore, we shall get

$$\mathscr{L}'_m = \frac{1}{2} n \left[\sum_{\mathscr{F}=0,2,\cdots,2f} \left(g_{\mathscr{F}} C^{\mathscr{F}\ 0}_{f,0;f,0} C^{\mathscr{F}\ 0}_{f,m;f,-m} \right) \right] \tag{4.21}$$

- $m_1 = m_2 = 0$

  Here $m_1' = m$. The term we get from (4.5) is

$$\tilde{\mathscr{L}}_m = \frac{1}{2} \sum_{\mathscr{F}} \left(g_{\mathscr{F}} C^{\mathscr{F}\ m}_{f,0;f,m} C^{\mathscr{F}\ m}_{f,m;f,0} \Psi_0 \Psi_m \Psi^*_0 \right) \tag{4.22}$$

  The summation will go from $|m|$ to $2f$ if m is even, and $|m|+1$ to $2f$ if m is odd. We also have $\Psi_0 \Psi_m \Psi^*_0 = n\Psi_m + \mathscr{O}(w^2)$. Therefore, we get

$$\tilde{\mathscr{L}}_m = \frac{1}{2} \sum_{\mathscr{F}} \left(n\, g_{\mathscr{F}} C^{\mathscr{F}\ m}_{f,0;f,m} C^{\mathscr{F}\ m}_{f,m;f,0} \right) \Psi_m \tag{4.23}$$

  However, we also recall that the Clebsch Gordan coefficients are symmetric under particle exchange, meaning

$$C^{\mathscr{F}\ m}_{f,0;f,m} = C^{\mathscr{F}\ m}_{f,m;f,0} \tag{4.24}$$

  giving us

$$\tilde{\mathscr{L}}_m = \frac{1}{2} \sum_{\mathscr{F}} n\, g_{\mathscr{F}} \left((C^{\mathscr{F}\ m}_{f,0;f,m} \right)^2 \Psi_m \tag{4.25}$$

- $m_1' = m_2 = 0$

  Here $m_1 = m$. However, it is almost immediately obvious that this will give us the exact same result as the previous case, giving us the overall result of

$$2\tilde{\mathscr{L}}_m = \sum_{\mathscr{F}} n\, g_{\mathscr{F}} \left((C^{\mathscr{F}\ m}_{f,0;f,m} \right)^2 \Psi_m \tag{4.26}$$

To summarize, if m is even:

$$\mathscr{L}_m = \left[-\frac{\hbar^2\nabla^2}{2M} - pm + qm^2\right] + n \sum_{\mathscr{F}=|m|,|m|+2,\cdots 2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, m}_{f,0;f,m}\right)^2 \tag{4.27}$$

and if m is odd:

$$\mathscr{L}_m = \left[-\frac{\hbar^2\nabla^2}{2M} - pm + qm^2\right] + n \sum_{\mathscr{F}=|m|+1,|m|+3,\cdots 2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, m}_{f,0;f,m}\right)^2 \tag{4.28}$$

Henceforth, we shall signify all summations over $\mathscr{F}$ simply as $\sum_{\mathscr{F}=|m|\cdots 20}$. Moving forward, it shall be assumed that these summation will only take even values. Therefore, we shall in general write

$$\mathscr{L}_m = \left[-\frac{\hbar^2\nabla^2}{2M} - pm + qm^2\right] + n \sum_{\mathscr{F}=|m|\cdots 2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, m}_{f,0;f,m}\right)^2 \tag{4.29}$$

$$\mathscr{L}'_m = \frac{1}{2} n \left[\sum_{\mathscr{F}=0,2,\cdots,2f} \left(g_{\mathscr{F}}\, C^{\mathscr{F}\, 0}_{f,0;f,0} C^{\mathscr{F}\, 0}_{f,m;f,-m}\right)\right] \tag{4.30}$$

The full coupled equations for $m \neq 0$ are given by This gives us:

$$i\hbar\frac{\partial \Psi_m}{\partial t} = \left(\omega u_m \exp\left(\frac{-i\omega t}{\hbar}\right) - \omega v^*_m \exp\left(\frac{i\omega t}{\hbar}\right)\right) e^{-i\mu t/\hbar} + \mu\left(u_m \exp\left(\frac{-i\omega t}{\hbar}\right) + v^*_m \exp\left(\frac{i\omega t}{\hbar}\right)\right) e^{-i\mu t/\hbar} \tag{4.31}$$

$$= \mathscr{L}_m \left(u_m \exp\left(\frac{-i\omega t}{\hbar}\right) + v^*_m \exp\left(\frac{i\omega t}{\hbar}\right)\right) e^{-i\mu t/\hbar} + \mathscr{L}'_m \left(u^*_{-m} \exp\left(\frac{i\omega t}{\hbar}\right) + v_m \exp\left(\frac{-i\omega t}{\hbar}\right)\right) e^{-i\mu t/\hbar} \tag{4.32}$$

Now we can cancel $e^{-i\mu t/\hbar}$, and use (4.29) and (4.30) to get

$$\left(\omega u_m \exp\left(\frac{-i\omega t}{\hbar}\right) - \omega v^*_m \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \mu\left(u_m \exp\left(\frac{-i\omega t}{\hbar}\right) + v^*_m \exp\left(\frac{i\omega t}{\hbar}\right)\right) = \left[\left(-\frac{\hbar^2\nabla^2}{2M} - pm + qm^2\right) + n\left(\sum_{\mathscr{F}=|m|\cdots 2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, m}_{f,0;f,m}\right)^2\right)\right]\left(u_m \exp\left(\frac{-i\omega t}{\hbar}\right) + v^*_m \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \frac{n}{2}\left[\sum_{\mathscr{F}=0\cdots 2f} \left(g_{\mathscr{F}}\, C^{\mathscr{F}\, 0}_{f,0;f,0} C^{\mathscr{F}\, 0}_{f,m;f,-m}\right)\right] \left(u^*_{-m} \exp\left(\frac{i\omega t}{\hbar}\right) + v_{-m} \exp\left(\frac{-i\omega t}{\hbar}\right)\right) \tag{4.33}$$

We finally equate the coefficients of $\exp\left(\frac{-i\omega t}{\hbar}\right)$ and $\exp\left(\frac{i\omega t}{\hbar}\right)$, which both need to be separately equal. We also use (4.9) to get:

$$\omega u_m = \left[\varepsilon_k - pm + qm^2 + n\left(\sum_{\mathscr{F}=|m|\cdots 2f} g_{\mathscr{F}}\,(C^{\mathscr{F}\,m}_{f,0;f,m})^2\right)\right.$$
$$\left. -\frac{1}{2}n\left(\sum_{\mathscr{F}=0\cdots,2f} g_{\mathscr{F}}\left(C^{\mathscr{F}\,0}_{f,0;f,0}\right)^2\right)\right] u_m + \frac{n}{2}\left[\sum_{\mathscr{F}=0\cdots,2f}\left(g_{\mathscr{F}}\,C^{\mathscr{F}\,0}_{f,0;f,0}C^{\mathscr{F}\,0}_{f,m;f,-m}\right)\right] v_{-m} \quad (4.34)$$

$$(-\omega v_m) = \left[\varepsilon_k - pm + qm^2 + n\left(\sum_{\mathscr{F}=|m|\cdots 2f} g_{\mathscr{F}}\,(C^{\mathscr{F}\,m}_{f,0;f,m})^2\right)\right.$$
$$\left. -\frac{1}{2}n\left(\sum_{\mathscr{F}=0,2,\cdots,2f} g_{\mathscr{F}}\left(C^{\mathscr{F}\,0}_{f,0;f,0}\right)^2\right)\right] v_m + \frac{n}{2}\left[\sum_{\mathscr{F}=0,2,\cdots,2f}\left(g_{\mathscr{F}}\,C^{\mathscr{F}\,0}_{f,0;f,0}C^{\mathscr{F}\,0}_{f,m;f,-m}\right)\right] u_{-m} \quad (4.35)$$

where $\varepsilon_k = -\hbar^2k^2/2M$. We can now finally write

$$\omega\begin{pmatrix} u_f \\ v_{-f} \\ u_{f-1} \\ v_{1-f} \\ \vdots \\ u_{-f} \\ v_f \end{pmatrix} = \mathscr{M}\begin{pmatrix} u_f \\ v_{-f} \\ u_{f-1} \\ v_{1-f} \\ \vdots \\ u_{-f} \\ v_f \end{pmatrix} \quad (4.36)$$

excluding the u_0 and v_0 terms since we have already dealt with them previously. We find that the above matrix $\mathscr{M}$ is a block diagonal matrix consisting of $2f$ 2×2 matrices along the diagonal. The general form of each block is

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad (4.37)$$

where we have:

$$S_{11} = \left[\varepsilon_k - mp + m^2 q + n\left(\sum_{\mathscr{F}=|m|,\cdots 2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, m}_{f,0;f,m}\right)^2\right) - \frac{1}{2}n\left(\sum_{\mathscr{F}=0\cdots,2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, 0}_{f,0;f,0}\right)^2\right)\right] \tag{4.38}$$

$$S_{12} = \frac{n}{2}\left[\sum_{\mathscr{F}=0\cdots,2f}\left(g_{\mathscr{F}} C^{\mathscr{F}\, 0}_{f,0;f,0} C^{\mathscr{F}\, 0}_{f,m;f,-m}\right)\right] \tag{4.39}$$

$$S_{21} = -\frac{n}{2}\left[\sum_{\mathscr{F}=0\cdots,2f}\left(g_{\mathscr{F}} C^{\mathscr{F}\, 0}_{f,0;f,0} C^{\mathscr{F}\, 0}_{f,m;f,-m}\right)\right] \tag{4.40}$$

$$S_{22} = -\left[\varepsilon_k + mp + m^2 q + n\left(\sum_{\mathscr{F}=|m|,\cdots 2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, m}_{f,0;f,m}\right)^2\right) - \frac{1}{2}n\left(\sum_{\mathscr{F}=0\cdots,2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, 0}_{f,0;f,0}\right)^2\right)\right] \tag{4.41}$$

Where we have used the property of the Clebsch Gordan coefficients: $C^{\mathscr{F}\, m}_{10,0;10,m} = C^{\mathscr{F}\, -m}_{10,0;10,-m}$, as well as the symmetry of the coefficients under exchange: $C^{\mathscr{F}\, 0}_{10,m;10,-m} = C^{\mathscr{F}\, 0}_{10,-m;10,m}$. Here m varies from $-f$ to f excluding 0, thereby giving us $2f$ such matrices. The eigenvalue of the above matrix is given in general as

$$\begin{aligned} E_{k,m} = \mathbf{sqrt}\Bigg\{\Bigg[\varepsilon_k + m^2 q + n\left(\sum_{\mathscr{F}=|m|,\cdots 2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, m}_{f,0;f,m}\right)^2\right) - \frac{1}{2}n\left(\sum_{\mathscr{F}=0,\cdots,2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, 0}_{f,0;f,0}\right)^2\right) \\ + \frac{n}{2}\left(\sum_{\mathscr{F}=0,\cdots,2f}\left(g_{\mathscr{F}} C^{\mathscr{F}\, 0}_{f,0;f,0} C^{\mathscr{F}\, 0}_{f,m;f,-m}\right)\right)\Bigg]\Bigg[\varepsilon_k + m^2 q + n\left(\sum_{\mathscr{F}=|m|,\cdots 2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, m}_{f,0;f,m}\right)^2\right) - \\ \frac{1}{2}n\left(\sum_{\mathscr{F}=0,\cdots,2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, 0}_{f,0;f,0}\right)^2\right) - \frac{n}{2}\left(\sum_{\mathscr{F}=0,\cdots,2f}\left(g_{\mathscr{F}} C^{\mathscr{F}\, 0}_{f,0;f,0} C^{\mathscr{F}\, 0}_{f,m;f,-m}\right)\right)\Bigg]\Bigg\} \mp mp \end{aligned} \tag{4.42}$$

The above expression gives us $2f$ terms in the energy spectrum, with the final term being calculated before in (4.18) as

$$E_{k,0} = \pm\sqrt{\varepsilon_k\left[\varepsilon_k + n\left(\sum_{\mathscr{F}=0,\cdots,2f} g_{\mathscr{F}} \left(C^{\mathscr{F}\, 0}_{f,0;f,0}\right)^2\right)\right]} \tag{4.43}$$

We have therefore found the expression for all $2f+1$ modes for a general spin-f system.

4.1.2 Ferromagnetic Phase

We shall follow a very similar approach to find the energy spectrum of the ferromagnetic phase, which has the order parameter $(1,\,0,\,0,\,0\cdots,0)^T$, where all the population is in the $m=+f$ component. We shall use a similar proposition to simplify our calculations. The second term in the GPE as before is given by (4.5), with the same condition: $m_1+m_1'=m+m_2$. We once again divide our problem into 2 parts

- $m=f$

  For this GPE, 2 out of the 3 Ψ terms must be Ψ_f to maintain linearity in w, and the condition $m_1+m_1'=m+m_2$ ensures that the remaining term will also be Ψ_f.

- $m\neq f$

  In this case again 2 terms have to be Ψ_f. Now the three possibilities are:

  - $m_1=m_1'=f$, we quickly see that this case will reduce to 0 due to the condition $m_1+m_1'=m+m_2$, since it would require $m_2>f$ which is not possible for the spin-f system.
  - $m_1=m_2=f$, clearly gives us $m_1'=m$
  - $m_1'=m_2=f$ gives $m_1=m$

Also, by making a similar argument as the previous section:

$$\begin{aligned}\psi_f\psi_f\psi_{-m}^* &= \left(\sqrt{n}+u_f\exp\left(\frac{-i\omega t}{\hbar}\right)+v_f^*\exp\left(\frac{i\omega t}{\hbar}\right)\right)^2\left(u_{-m}^*\exp\left(\frac{i\omega t}{\hbar}\right)+v_{-m}\exp\left(\frac{-i\omega t}{\hbar}\right)\right)\\ &= n\left(u_{-m}^*\exp\left(\frac{i\omega t}{\hbar}\right)+v_{-m}\exp\left(\frac{-i\omega t}{\hbar}\right)\right)\end{aligned} \tag{4.44}$$

neglecting the higher order terms as always. This can be easily extended to the other two cases described above. Therefore, we have shown that u_m and v_m are only coupled to themselves in the dynamical equations. This actually differs from the polar phase, since in that case, they were also coupled to the u_{-m} and v_{-m} parameters. We now proceed to the full calculation. We start by finding the chemical potential of the system, for which we will use the $m=f$ GPE and $\Psi_f=\sqrt{n}e^{-i\mu t/\hbar}$,

which gives us:

$$\sqrt{n}\mu e^{-i\mu t/\hbar} = -fp + f^2 q + \frac{1}{2}\left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2 \Psi_f \Psi_f \Psi_f^* \tag{4.45}$$

$$\mu = -fp + f^2 q + \frac{1}{2} n \left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2 \tag{4.46}$$

Now that we have the chemical potential, we can get into the calculation

$\mathbf{m} = \mathbf{f}$

$$i\hbar \frac{\partial \Psi_f}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2M} - fp + f^2 q\right] + \frac{1}{2}\left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2 \Psi_f \Psi_f \Psi_f^* \tag{4.47}$$

This gives us

$$\begin{aligned} i\hbar \frac{\partial \Psi_f}{\partial t} &= \left[\left(\omega u_f \exp\left(\frac{-i\omega t}{\hbar}\right) - \omega v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \mu\left(\sqrt{n} + u_f \exp\left(\frac{-i\omega t}{\hbar}\right) + v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right)\right] e^{-i\mu t/\hbar} \\ &= \left[-\frac{\hbar^2 \nabla^2}{2M} - fp + f^2 q\right] + \frac{1}{2}\left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2 \Psi_f \Psi_f \Psi_f^* \end{aligned} \tag{4.48}$$

Using our ansatz, we can expand this further to give:

$$\begin{aligned} i\hbar \frac{\partial \Psi_f}{\partial t} &= \left[\left(\omega u_f \exp\left(\frac{-i\omega t}{\hbar}\right) - \omega v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \mu\left(\sqrt{n} + u_f \exp\left(\frac{-i\omega t}{\hbar}\right) + v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right)\right] e^{-i\mu t/\hbar} \\ &= \left[-\frac{\hbar^2 \nabla^2}{2M} - fp + f^2 q\right]\left(\sqrt{n} + u_f \exp\left(\frac{-i\omega t}{\hbar}\right) + v_f \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \frac{1}{2}\left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2 \\ &\left(n^{3/2} + n u_f^* \exp\left(\frac{i\omega t}{\hbar}\right) + n v_f \exp\left(\frac{-i\omega t}{\hbar}\right) + 2n u_f \exp\left(\frac{-i\omega t}{\hbar}\right) + 2n v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) \end{aligned} \tag{4.49}$$

But we also know that the following equation must be satisfied for the ferromagnetic order parameter:

$$\left[-\frac{\hbar^2 \nabla^2}{2M} - fp + f^2 q\right](\sqrt{n}) + \frac{1}{2}\left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2 (n^{3/2}) = \mu\sqrt{n} \tag{4.50}$$

Therefore we can write

$$
\begin{aligned}
i\hbar\frac{\partial\Psi_f}{\partial t} &= \left[\left(\omega u_f \exp\left(\frac{-i\omega t}{\hbar}\right) - \omega v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \mu\left(\sqrt{n} + u_f \exp\left(\frac{-i\omega t}{\hbar}\right) + v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right)\right] e^{-i\mu t/\hbar} \\
&= \left[-\frac{\hbar^2\nabla^2}{2M} - fp + f^2 q\right]\left(u_f \exp\left(\frac{-i\omega t}{\hbar}\right) + v_f \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \frac{1}{2}\left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2 \\
&\left(n u_f^* \exp\left(\frac{i\omega t}{\hbar}\right) + n v_f \exp\left(\frac{-i\omega t}{\hbar}\right) + 2n u_f \exp\left(\frac{-i\omega t}{\hbar}\right) + 2n v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \mu\sqrt{n} \quad (4.51)
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
i\hbar\frac{\partial\Psi_f}{\partial t} &= \left(\omega u_f \exp\left(\frac{-i\omega t}{\hbar}\right) - \omega v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) e^{-i\mu t/\hbar} + \mu\left(u_f \exp\left(\frac{-i\omega t}{\hbar}\right) + v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) e^{-i\mu t/\hbar} \\
&= \left[-\frac{\hbar^2\nabla^2}{2M} - fp + f^2 q\right]\left(u_f \exp\left(\frac{-i\omega t}{\hbar}\right) + v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) + \frac{1}{2}\left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2 \\
&\left(n u_f^* \exp\left(\frac{i\omega t}{\hbar}\right) + n v_f \exp\left(\frac{-i\omega t}{\hbar}\right) + 2n u_f \exp\left(\frac{-i\omega t}{\hbar}\right) + 2n v_f^* \exp\left(\frac{i\omega t}{\hbar}\right)\right) \quad (4.52)
\end{aligned}
$$

Now we can separately equate the $\exp\left(\frac{-i\omega t}{\hbar}\right)$ and $\exp\left(\frac{i\omega t}{\hbar}\right)$ parts of the equation, and insert (4.46) for the expression of the chemical potential, to get

$$
\omega u_f = \left[\varepsilon_k + \frac{1}{2}n\left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2\right] u_f + \frac{1}{2}n\left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2 v_f \quad (4.53)
$$

and similarly, we get

$$
-\omega v_f = \left[\varepsilon_k + \frac{1}{2}n\left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2\right] v_f + \frac{1}{2}n\left(g_{2f} C^{2f,2f}_{f,f;f,f}\right)^2 u_f \quad (4.54)
$$

where as before, $\varepsilon_k = -\hbar^2 k^2/2M$.

$\mathbf{m \neq f}$

We have shown that u_m, v_m couple only to themselves, meaning we can write

$$
i\hbar\frac{\partial\Psi_m}{\partial t} = \mathscr{L}_m \Psi_m \quad (4.55)
$$

We have two potential cases to discuss. The first is $m_1 = m_2 = f$, and $m_1' = m$, and the second is $m_1' = m_2 = f$ and $m_1 = m$. It is very obvious that both the possibilities will give the same term

$$\tilde{\mathscr{L}}_m = \frac{1}{2} \sum_{\mathscr{F}=|f+m|,\cdots,2f} \left(g_{\mathscr{F}} C^{\mathscr{F}\ |f+m|}_{f,f;f,m} C^{\mathscr{F}\ |f+m|}_{f,m;f,f}\ \psi_f(\boldsymbol{r}) \psi_m(\boldsymbol{r}') \psi_f^*(\boldsymbol{r}') \right) \tag{4.56}$$

We use the fact that the Clebsch Gordan coefficients are symmetric under particle exchange for integer spins and ignore all terms quadratic in w or higher to give

$$\mathscr{L}_m = \left[-\frac{\hbar^2\nabla^2}{2M} - mp + m^2 q \right] + n \sum_{\mathscr{F}=|f+m|,\cdots,2f} \left(g_{\mathscr{F}} C^{\mathscr{F}\ |f+m|}_{f,f;f,m} \right)^2 \tag{4.57}$$

where the summation over $\mathscr{F}$ takes only even values as always. Therefore the equations are given by

$$\begin{aligned} i\hbar \frac{\partial \Psi_m}{\partial t} &= \left(\omega u_m \exp\left(\frac{-i\omega t}{\hbar}\right) - \omega v_m^* \exp\left(\frac{i\omega t}{\hbar}\right)) e^{-i\mu t/\hbar} + \mu (u_m \exp\left(\frac{-i\omega t}{\hbar}\right) + v_m^* \exp\left(\frac{i\omega t}{\hbar}\right) \right) e^{-i\mu t/\hbar} \\ &= \left[-\frac{\hbar^2\nabla^2}{2M} - mp + m^2 q + n \sum_{\mathscr{F}=|f+m|,\cdots,2f} \left(g_{\mathscr{F}} C^{\mathscr{F}\ (f+m)}_{f,f;f,m} \right)^2 \right] \left(u_m \exp\left(\frac{-i\omega t}{\hbar}\right) + v_m^* \exp\left(\frac{i\omega t}{\hbar}\right) \right) \end{aligned} \tag{4.58}$$

where the $\mathscr{F}$ summation only takes even values. Now we can equate the $\exp\left(\frac{-i\omega t}{\hbar}\right)$ and $\exp\left(\frac{i\omega t}{\hbar}\right)$ coefficients, and use (4.46) for the chemical potential to get

$$\begin{aligned} \omega u_m = \Bigg[\varepsilon_k - (m-f)p + (m^2 - f^2)q + n \sum_{\mathscr{F}=|f+m|,\cdots,2f} \left(g_{\mathscr{F}} C^{\mathscr{F}\ (f+m)}_{f,f;f,m} \right)^2 \\ - \frac{1}{2} n \left(g_{2f} C^{2f,2f}_{f,f;f,f} \right)^2 \Bigg] u_m \end{aligned} \tag{4.59}$$

and similarly for the $\exp\left(\frac{i\omega t}{\hbar}\right)$ coefficients

$$-\omega v_m = \left[\varepsilon_k - (m-f)p + (m^2-f^2)q + n\sum_{\mathscr{F}=|f+m|,\cdots,2f}\left(g_{\mathscr{F}}C^{\mathscr{F}\,(f+m)}_{f,f;f,m}\right)^2 - \frac{1}{2}n\left(g_{2f}C^{2f,2f}_{f,f;f,f}\right)^2\right]v_m \quad (4.60)$$

where The eigenvalue equation now takes the following form:

$$\omega\begin{pmatrix} u_{10} \\ v_{10} \\ u_9 \\ v_9 \\ \vdots \\ u_{-10} \\ v_{-10}\end{pmatrix} = \mathscr{M}\begin{pmatrix} u_{10} \\ v_{10} \\ u_9 \\ v_9 \\ \vdots \\ u_{-10} \\ v_{-10}\end{pmatrix} \quad (4.61)$$

The matrix $\mathscr{M}$ is already block-diagonal. The top left block is a 2×2 matrix, and the rest of the matrix has only diagonal entries. The eigenvalues of $\mathscr{M}$ will be given by the 2 eigenvalues of the topleft 2×2 matrix, and the remaining diagonal entries.

The top left 2×2 matrix is given by (4.53) and (4.54)

$$\begin{pmatrix} \left(\varepsilon_k + \frac{1}{2}n\left(g_{2f}C^{2f,2f}_{f,f;f,f}\right)^2\right) & \frac{1}{2}n\left(g_{2f}C^{2f,2f}_{f,f;f,f}\right)^2 \\ -\frac{1}{2}n\left(g_{2f}C^{2f,2f}_{f,f;f,f}\right)^2 & -\left(\varepsilon_k + \frac{1}{2}n\left(g_{2f}C^{2f,2f}_{f,f;f,f}\right)^2\right)\end{pmatrix} \quad (4.62)$$

which has eigenvalue

$$E_{k,f} = \sqrt{\varepsilon_k\left(\varepsilon_k + n\left(g_{2f}C^{2f,2f}_{f,f;f,f}\right)^2\right)} \quad (4.63)$$

This gives us the first term in the energy spectrum, and the remaining 20 terms can be found directly from (4.59) and (4.60), and are given as

$$E_{k,m} = \left[\varepsilon_k - (m-f)p + (m^2 - f^2)q + n \sum_{\mathscr{F}=|f+m|,\cdots,2f} \left(g_{\mathscr{F}} C^{\mathscr{F}\,(f+m)}_{f,f;f,m} \right)^2 - \frac{1}{2} n \left(g_{2f} C^{2f,2f}_{f,f;f,f} \right)^2 \right] \tag{4.64}$$

for m going from $(-f)$ to $(f-1)$.

4.2 Spin-10 Spectrum

In the previous section we developed a method to derive the energy spectrum for the polar and ferromagnetic phases of any general spin-f system. In this section, we shall consider the spin-10 spectrum

4.2.1 Polar Phase

We shall use the results derived in the previous section, specifically (4.42) and (4.43). The full spectrum is written below

$$\begin{aligned} E_{k,\pm 10} = \Bigg\{ \Bigg[\varepsilon_k + 100q + n \left(\sum_{\mathscr{F}=10,12,\cdots 20} g_{\mathscr{F}} \left(C^{\mathscr{F}\,10}_{10,0;10,10} \right)^2 \right) - \frac{1}{2} n \left(\sum_{\mathscr{F}=0,2,\cdots,20} g_{\mathscr{F}} \left(C^{\mathscr{F}\,0}_{10,0;10,0} \right)^2 \right) \\ + \frac{n}{2} \left(\sum_{\mathscr{F}=0,2,\cdots,20} \left(g_{\mathscr{F}} C^{\mathscr{F}\,0}_{10,0;10,0} C^{\mathscr{F}\,0}_{10,10;10,-10} \right) \right) \Bigg] \Bigg[\varepsilon_k + 100q + n \left(\sum_{\mathscr{F}=10,12,\cdots 20} g_{\mathscr{F}} \left(C^{\mathscr{F}\,10}_{10,0;10,10} \right)^2 \right) - \\ \frac{1}{2} n \left(\sum_{\mathscr{F}=0,2,\cdots,20} g_{\mathscr{F}} \left(C^{\mathscr{F}\,0}_{10,0;10,0} \right)^2 \right) - \frac{n}{2} \left(\sum_{\mathscr{F}=0,2,\cdots,20} \left(g_{\mathscr{F}} C^{\mathscr{F}\,0}_{10,0;10,0} C^{\mathscr{F}\,0}_{10,10;10,-10} \right) \right) \Bigg] \Bigg\}^{1/2} \mp 10p \end{aligned} \tag{4.65}$$

$$E_{k,\pm 9} = \left\{ \left[\varepsilon_k + 81q + n \left(\sum_{\mathscr{F}=10,12,\cdots 20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 9}_{10,0;10,9} \right)^2 \right) - \frac{1}{2} n \left(\sum_{\mathscr{F}=0,2,\cdots,20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 0}_{10,0;10,0} \right)^2 \right) \right.\right.$$
$$\left.\left. + \frac{n}{2} \left(\sum_{\mathscr{F}=0,2,\cdots,20} \left(g_{\mathscr{F}} C^{\mathscr{F}\ 0}_{10,0;10,0} C^{\mathscr{F}\ 0}_{10,9;10,-9} \right) \right) \right] \left[\varepsilon_k + 81q + n \left(\sum_{\mathscr{F}=10,12,\cdots 20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 9}_{10,0;10,9} \right)^2 \right) - \right.\right.$$
$$\left.\left. \frac{1}{2} n \left(\sum_{\mathscr{F}=0,2,\cdots,20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 0}_{10,0;10,0} \right)^2 \right) - \frac{n}{2} \left(\sum_{\mathscr{F}=0,2,\cdots,20} \left(g_{\mathscr{F}} C^{\mathscr{F}\ 0}_{10,0;10,0} C^{\mathscr{F}\ 0}_{10,9;10,-9} \right) \right) \right] \right\}^{1/2} \mp 9p \tag{4.66}$$

$$E_{k,\pm 8} = \left\{ \left[\varepsilon_k + 64q + n \left(\sum_{\mathscr{F}=8,10,\cdots 20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 8}_{10,0;10,8} \right)^2 \right) - \frac{1}{2} n \left(\sum_{\mathscr{F}=0,2,\cdots,20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 0}_{10,0;10,0} \right)^2 \right) \right.\right.$$
$$\left.\left. + \frac{n}{2} \left(\sum_{\mathscr{F}=0,2,\cdots,20} \left(g_{\mathscr{F}} C^{\mathscr{F}\ 0}_{10,0;10,0} C^{\mathscr{F}\ 0}_{10,8;10,-8} \right) \right) \right] \left[\varepsilon_k + 64q + n \left(\sum_{\mathscr{F}=8,10,\cdots 20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 8}_{10,0;10,8} \right)^2 \right) - \right.\right.$$
$$\left.\left. \frac{1}{2} n \left(\sum_{\mathscr{F}=0,2,\cdots,20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 0}_{10,0;10,0} \right)^2 \right) - \frac{n}{2} \left(\sum_{\mathscr{F}=0,2,\cdots,20} \left(g_{\mathscr{F}} C^{\mathscr{F}\ 0}_{10,0;10,0} C^{\mathscr{F}\ 0}_{10,8;10,-8} \right) \right) \right] \right\}^{1/2} \mp 8p \tag{4.67}$$

and so on until we get to:

$$E_{k,\pm 1} = \left\{ \left[\varepsilon_k + q + n \left(\sum_{\mathscr{F}=2,4,\cdots 20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 1}_{10,0;10,1} \right)^2 \right) - \frac{1}{2} n \left(\sum_{\mathscr{F}=0,2,\cdots,20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 0}_{10,0;10,0} \right)^2 \right) \right.\right.$$
$$\left.\left. + \frac{n}{2} \left(\sum_{\mathscr{F}=0,2,\cdots,20} \left(g_{\mathscr{F}} C^{\mathscr{F}\ 0}_{10,0;10,0} C^{\mathscr{F}\ 0}_{10,1;10,-1} \right) \right) \right] \left[\varepsilon_k + q + n \left(\sum_{\mathscr{F}=2,4,\cdots 20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 1}_{10,0;10,1} \right)^2 \right) - \right.\right.$$
$$\left.\left. \frac{1}{2} n \left(\sum_{\mathscr{F}=0,2,\cdots,20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 0}_{10,0;10,0} \right)^2 \right) - \frac{n}{2} \left(\sum_{\mathscr{F}=0,2,\cdots,20} \left(g_{\mathscr{F}} C^{\mathscr{F}\ 0}_{10,0;10,0} C^{\mathscr{F}\ 0}_{10,1;10,-1} \right) \right) \right] \right\}^{1/2} \mp p \tag{4.68}$$

and the final term in the energy spectrum is given by:

$$E_{k,0} = \sqrt{\varepsilon_k \left[\varepsilon_k + n \left(\sum_{\mathscr{F}=0,2,\cdots,20} g_{\mathscr{F}} \left(C^{\mathscr{F}\ 0}_{10,0;10,0} \right)^2 \right) \right]} \tag{4.69}$$

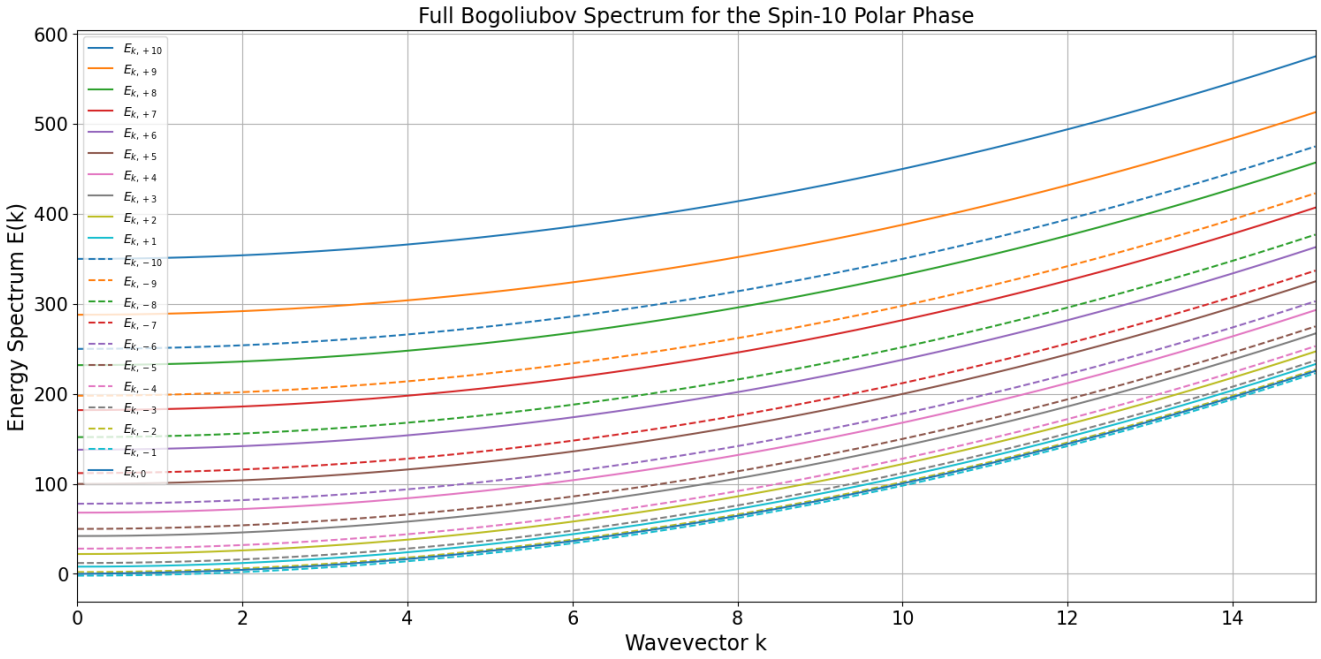

Figure 4.1: Bogoliubov spectrum for the spin-10 Polar Phase. The parameters for this plot are $q = 3$, $p = 5$. The values of $g_{\mathscr{F}}$ have been sampled from a uniform random distribution between $(0, 1)$.

The full spectrum is shown in fig(4.1). As expected, we have 20 gapped modes, represented by $E_{k,m}$ when $m \neq 0$, and 1 gapless mode represented by $E_{k,0}$. This completes our discussion for the Bogoliubov spectrum for the spin-10 Polar phase.

4.2.2 Ferromagnetic Phase

We again use the derived results from the previous section, which is summarized in (4.63) and (4.64). The full spectrum is listed below

$$E_{k,-10} = \left[\varepsilon_k + 20p + n \sum_{\mathscr{F}=0,\cdots,20} \left(g_{\mathscr{F}} C^{\mathscr{F}\,0}_{10,10;10,-10} \right)^2 - \frac{1}{2} n \left(g_{20} C^{20,20}_{10,10;10,10} \right)^2 \right] \tag{4.70}$$

$$E_{k,-9} = \left[\varepsilon_k + 19p - 19q + n \sum_{\mathscr{F}=2,4,\cdots,20} \left(g_{\mathscr{F}} C^{\mathscr{F}\,1}_{10,10;10,-9} \right)^2 - \frac{1}{2} n \left(g_{20} C^{20,20}_{10,10;10,10} \right)^2 \right] \tag{4.71}$$

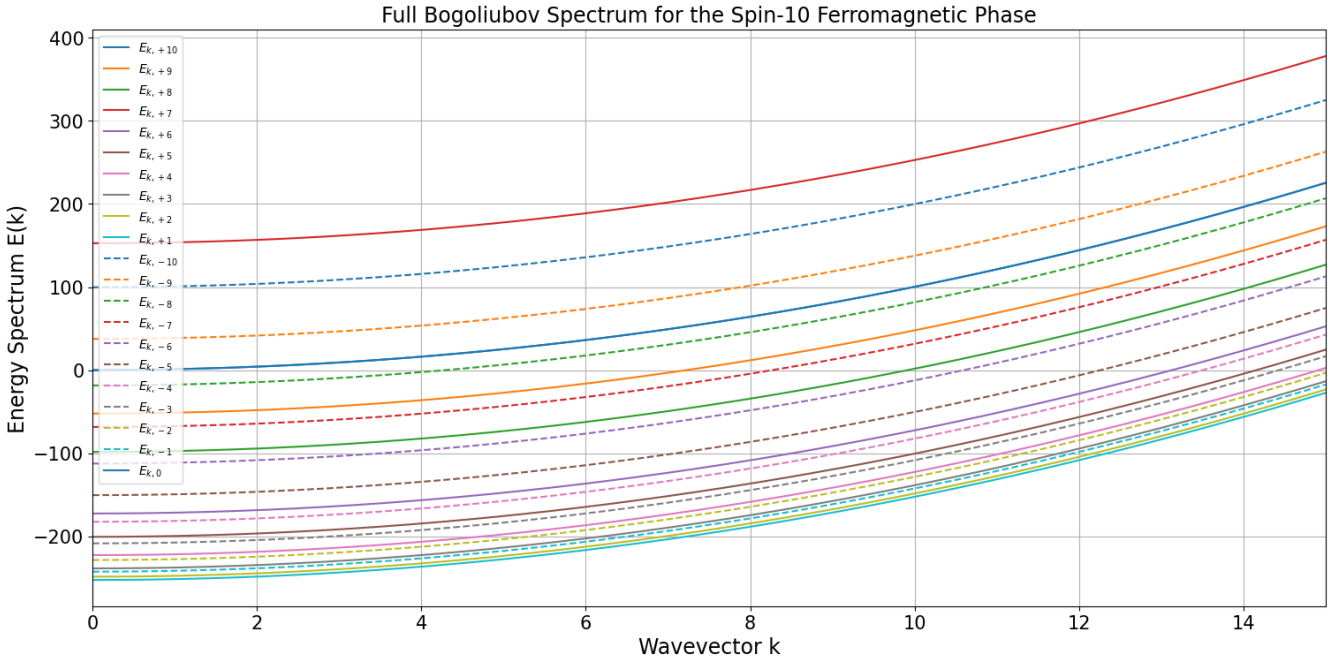

Figure 4.2: Bogoliubov spectrum for the spin-10 Ferromagnetic phase. We can clearly see the gapped $E_{k,0}$ mode, while the other 20 modes are gapped. The parameters for this plot are $q = 3$, $p = 5$. The values of $g_{\mathscr{F}}$ have been sampled from a uniform random distribution between $(0,1)$.

$$E_{k,-8} = \left[\varepsilon_k + 18p - 36q + n \sum_{\mathscr{F}=2,4,\cdots,20} \left(g_{\mathscr{F}} C^{\mathscr{F}\ 2}_{10,10;10,-8}\right)^2 - \frac{1}{2} n \left(g_{20} C^{20,20}_{10,10;10,10}\right)^2\right] \tag{4.72}$$

and so on until

$$E_{k,9} = \left[\varepsilon_k + p - 19q + n \left(g_{20} C^{\mathscr{F}\ 19}_{10,10;10,9}\right)^2 - \frac{1}{2} n \left(g_{20} C^{20,20}_{10,10;10,10}\right)^2\right] \tag{4.73}$$

and finally

$$E_{k,10} = \sqrt{\varepsilon_k \left(\varepsilon_k + n \left(g_{20} C^{20,20}_{10,10;10,10}\right)^2\right)} \tag{4.74}$$

This is shown in fig(4.2), where we have 1 gapless mode($E_{k,10}$), and 20 gapped modes($E_{k,m}$ where $m \neq 10$)

This concludes our discussion for the spin-10 Ferromagnetic phase Bogoliubov spectrum.

To summarize, we performed an analytical calculation for finding the Bogoliubov spectrum

for the polar and ferromagnetic phases of any general spin-f system. This always gives us one gapless mode, and $2f$ gapped modes. We then used our general spin-f results to derive the spin-10 spectrum for the two phases discussed, and plotted each of the modes against the wavevector k.

Chapter 5

Conclusion and Outlook

In this thesis, we examined high spin Bose-Einstein Condensates, particularly the spin-10 BEC. Due to a significant lack of prior literature on BECs with spins higher than 3, we attempted to establish a foundation in this thesis for future studies in high spin systems. We started by finding the full Hamiltonian of the spin-10 system, and the relations between the interaction coefficients c_n and the scattering lengths of the system. We used the Hamiltonian and the corresponding energy functional to find the non-linear Gross-Pitaveskii equations. We also showed how the magnitude of the nematic tensors change in a spin-10 system depending on their rank, and observed that there is a very rapid increase with the rank of the nematic tensor. Furthermore, even for the same rank, the magnitude of the nematic tensor increases with the spin of the system, which can have numerous implications with respect to simulating higher spin systems, or for techniques such as the single mode approximation(SMA). In the last section of our thesis, we analyzed the Bogoliubov energy spectrum that governs elementary excitations in the spin-10 system for two important phases, namely the polar and the ferromagnetic Phases. We first used the derived GPEs to show how different components of the BEC couple under small perturbations in the system, which significantly simplified our calculations. We performed the required analytical calculations to derive full energy spectrum, and represented all the modes in a graph. We expect the work done in this thesis to be useful for higher spin BEC studies, especially in the context of simulating the system, or observing population dynamics akin to those observed in multi-level atoms. Furthermore, the

energy spectrum calculation has been performed using the Clebsch Gordan coefficient representation of the Gross Pitaevskii equation, which makes it fairly straightforward to generalize to any spin-f BEC. Using this, elementary excitations can be studied in a various systems, and the methods used here can be adapted for different ground state order parameters and phases, to further increase the applicability of the work done in this thesis.

Bibliography

[1] Lev Pitaevskii and Sandro Stringari. *Bose-Einstein condensation and superfluidity*. Vol. 164. Oxford University Press, 2016.

[2] Michiel Wouters and Iacopo Carusotto. "Superfluidity and critical velocities in nonequilibrium Bose-Einstein condensates". In: *Physical review letters* 105.2 (2010), p. 020602. DOI: https://doi.org/10.1103/PhysRevLett.105.020602.

[3] Anatoly Kuklov, Nikolay Prokof'ev, and Boris Svistunov. "Superfluid-superfluid phase transitions in a two-component Bose-Einstein condensate". In: *Physical review letters* 92.3 (2004), p. 030403. DOI: https://doi.org/10.1103/PhysRevLett.92.030403.

[4] Edward W Hagley et al. "Measurement of the coherence of a Bose-Einstein condensate". In: *Physical Review Letters* 83.16 (1999), p. 3112. DOI: https://doi.org/10.1103/PhysRevLett.83.3112.

[5] EA Burt et al. "Coherence, correlations, and collisions: What one learns about Bose-Einstein condensates from their decay". In: *Physical Review Letters* 79.3 (1997), p. 337. DOI: https://doi.org/10.1103/PhysRevLett.79.337.

[6] Immanuel Bloch, Theodor W Hänsch, and Tilman Esslinger. "Measurement of the spatial coherence of a trapped Bose gas at the phase transition". In: *Nature* 403.6766 (2000), pp. 166–170. DOI: https://doi.org/10.1038/35003132.

[7] Christopher J Pethick and Henrik Smith. *Bose–Einstein condensation in dilute gases*. Cambridge university press, 2008.

[8] Yuki Kawaguchi and Masahito Ueda. “Spinor Bose–Einstein condensates”. In: *Physics Reports* 520.5 (Nov. 2012), pp. 253–381. ISSN: 0370-1573. DOI: 10.1016/j.physrep.2012.07.005. URL: http://dx.doi.org/10.1016/j.physrep.2012.07.005.

[9] Masahito Ueda and Masato Koashi. “Theory of spin-2 Bose-Einstein condensates: Spin correlations, magnetic response, and excitation spectra”. In: *Physical Review A* 65.6 (2002), p. 063602. DOI: https://doi.org/10.1103/PhysRevA.65.063602.

[10] Mudit Jain, Mustafa A Amin, and Han Pu. “Integrator for general spin-s Gross-Pitaevskii systems”. In: *Physical Review E* 108.5 (2023), p. 055305. DOI: https://doi.org/10.1103/PhysRevE.108.055305.

[11] Dan M Stamper-Kurn and Masahito Ueda. “Spinor Bose gases: Symmetries, magnetism, and quantum dynamics”. In: *Reviews of Modern Physics* 85.3 (2013), pp. 1191–1244. DOI: https://doi.org/10.1103/RevModPhys.85.1191.

[12] Mingwu Lu et al. “Strongly dipolar Bose-Einstein condensate of dysprosium”. In: *Physical review letters* 107.19 (2011), p. 190401. DOI: https://doi.org/10.1103/PhysRevLett.107.190401.

[13] P Pedri and L Santos. “Two-dimensional bright solitons in dipolar Bose-Einstein condensates”. In: *Physical review letters* 95.20 (2005), p. 200404. DOI: https://doi.org/10.1103/PhysRevLett.95.200404.

[14] F Wächtler and L Santos. “Ground-state properties and elementary excitations of quantum droplets in dipolar Bose-Einstein condensates”. In: *Physical Review A* 94.4 (2016), p. 043618. DOI: https://doi.org/10.1103/PhysRevA.94.043618.

[15] L Santos, GV Shlyapnikov, and M Lewenstein. “Roton-maxon spectrum and stability of trapped dipolar Bose-Einstein condensates”. In: *Physical review letters* 90.25 (2003), p. 250403. DOI: https://doi.org/10.1103/PhysRevLett.90.250403.

[16] Y-J Lin, K Jiménez-García, and Ian B Spielman. “Spin–orbit-coupled Bose–Einstein condensates”. In: *Nature* 471.7336 (2011), pp. 83–86. DOI: https://doi.org/10.1038/nature09887.

[17] Tudor D Stanescu, Brandon Anderson, and Victor Galitski. "Spin-orbit coupled Bose-Einstein condensates". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 78.2 (2008), p. 023616. DOI: https://doi.org/10.1103/PhysRevA.78.023616.

[18] Yongping Zhang, Li Mao, and Chuanwei Zhang. "Mean-field dynamics of spin-orbit coupled Bose-Einstein condensates". In: *Physical review letters* 108.3 (2012), p. 035302. DOI: https://doi.org/10.1103/PhysRevLett.108.035302.

[19] J Radić et al. "Vortices in spin-orbit-coupled Bose-Einstein condensates". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 84.6 (2011), p. 063604. DOI: https://doi.org/10.1103/PhysRevA.84.063604.

[20] P Ao and ST Chui. "Binary Bose-Einstein condensate mixtures in weakly and strongly segregated phases". In: *Physical Review A* 58.6 (1998), p. 4836. DOI: https://doi.org/10.1103/PhysRevA.58.4836.

[21] Jacopo Sabbatini, Wojciech H Zurek, and Matthew J Davis. "Phase separation and pattern formation in a binary Bose-Einstein condensate". In: *Physical review letters* 107.23 (2011), p. 230402. DOI: https://doi.org/10.1103/PhysRevLett.107.230402.

[22] KM Mertes et al. "Nonequilibrium dynamics and superfluid ring excitations in binary Bose-Einstein condensates". In: *Physical review letters* 99.19 (2007), p. 190402. DOI: https://doi.org/10.1103/PhysRevLett.99.190402.

[23] DS Hall et al. "Dynamics of component separation in a binary mixture of Bose-Einstein condensates". In: *Physical Review Letters* 81.8 (1998), p. 1539. DOI: https://doi.org/10.1103/PhysRevLett.81.1539.

[24] J Denschlag et al. "Generating solitons by phase engineering of a Bose-Einstein condensate". In: *Science* 287.5450 (2000), pp. 97–101. DOI: 10.1126/science.287.5450.97.

[25] Juha Javanainen and Sung Mi Yoo. "Quantum phase of a Bose-Einstein condensate with an arbitrary number of atoms". In: *Physical review letters* 76.2 (1996), p. 161. DOI: https://doi.org/10.1103/PhysRevLett.76.161.

[26] EM Bookjans, A Vinit, and C Raman. "Quantum phase transition in an antiferromagnetic spinor Bose-Einstein condensate". In: *Physical Review Letters* 107.19 (2011), p. 195306. DOI: https://doi.org/10.1103/PhysRevLett.107.195306.

[27] Rina Kanamoto, Hiroki Saito, and Masahito Ueda. "Quantum phase transition in one-dimensional Bose-Einstein condensates with attractive interactions". In: *Physical Review A* 67.1 (2003), p. 013608. DOI: https://doi.org/10.1103/PhysRevA.67.013608.

[28] Kestutis Staliunas, Stefano Longhi, and Germán J De Valcárcel. "Faraday patterns in Bose-Einstein condensates". In: *Physical review letters* 89.21 (2002), p. 210406. DOI: https://doi.org/10.1103/PhysRevLett.89.210406.

[29] Shreyas Nadiger et al. "Stripe and checkerboard patterns in a stack of driven quasi-one-dimensional dipolar condensates". In: *Physical Review A* 109.3 (2024), p. 033309. DOI: https://doi.org/10.1103/PhysRevA.109.033309.

[30] Alexandru I Nicolin, R Carretero-González, and PG Kevrekidis. "Faraday waves in Bose-Einstein condensates". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 76.6 (2007), p. 063609. DOI: https://doi.org/10.1103/PhysRevA.76.063609.

[31] R Nath and L Santos. "Faraday patterns in two-dimensional dipolar Bose-Einstein condensates". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 81.3 (2010), p. 033626. DOI: https://doi.org/10.1103/PhysRevA.81.033626.

[32] Zhendong Zhang et al. "Pattern formation in a driven Bose–Einstein condensate". In: *Nature Physics* 16.6 (2020), pp. 652–656. DOI: https://doi.org/10.1038/s41567-020-0839-3.

[33] Masahito Ueda. "Topological aspects in spinor Bose–Einstein condensates". In: *Reports on Progress in Physics* 77.12 (2014), p. 122401. DOI: 10.1088/0034-4885/77/12/122401.

[34] Guangyao Li. "Azimuthons and pattern formation in annularly confined exciton-polariton Bose-Einstein condensates". In: *Physical Review A* 93.1 (2016), p. 013837. DOI: https://doi.org/10.1103/PhysRevA.93.013837.

[35] Giuseppe Baio et al. “Topological interfaces crossed by defects and textures of continuous and discrete point group symmetries in spin-2 Bose-Einstein condensates”. In: *Physical Review Research* 6.1 (2024), p. 013046. DOI: https://doi.org/10.1103/PhysRevResearch.6.013046.

[36] Magnus O Borgh, Justin Lovegrove, and Janne Ruostekoski. “Imprinting a topological interface using Zeeman shifts in an atomic spinor Bose–Einstein condensate”. In: *New Journal of Physics* 16.5 (2014), p. 053046. DOI: 10.1088/1367-2630/16/5/053046.

[37] Sandra M Jose, Komal Sah, and Rejish Nath. “Patterns, spin-spin correlations, and competing instabilities in driven quasi-two-dimensional spin-1 Bose-Einstein condensates”. In: *Physical Review A* 108.2 (2023), p. 023308. DOI: https://doi.org/10.1103/PhysRevA.108.023308.

[38] Chad N Weiler et al. “Spontaneous vortices in the formation of Bose–Einstein condensates”. In: *Nature* 455.7215 (2008), pp. 948–951. DOI: https://doi.org/10.1038/nature07334.

[39] Justin Lovegrove, Magnus O Borgh, and Janne Ruostekoski. “Stability and internal structure of vortices in spin-1 Bose-Einstein condensates with conserved magnetization”. In: *Physical Review A* 93.3 (2016), p. 033633. DOI: https://doi.org/10.1103/PhysRevA.93.033633.

[40] Michael Robin Matthews et al. “Vortices in a Bose-Einstein condensate”. In: *Physical Review Letters* 83.13 (1999), p. 2498. DOI: https://doi.org/10.1103/PhysRevLett.83.2498.

[41] Jamil R Abo-Shaeer et al. “Observation of vortex lattices in Bose-Einstein condensates”. In: *Science* 292.5516 (2001), pp. 476–479. DOI: 10.1126/science.1060182.

[42] Hiromitsu Takeuchi. “Phase diagram of vortices in the polar phase of spin-1 Bose-Einstein condensates”. In: *Physical Review A* 104.1 (2021), p. 013316. DOI: https://doi.org/10.1103/PhysRevA.104.013316.

[43] Dallin S Durfee and Wolfgang Ketterle. "Experimental studies of Bose-Einstein condensation". In: *Optics Express* 2.8 (1998), pp. 299–313. DOI: https://doi.org/10.1364/OE.2.000299.

[44] Marcin Fizia and Krzysztof Sacha. "Inert states of spin-5 and spin-6 Bose–Einstein condensates". In: *Journal of Physics A: Mathematical and Theoretical* 45.4 (2012), p. 045103. DOI: 10.1088/1751-8113/45/4/045103.

[45] Shun Uchino, Takaharu Otsuka, and Masahito Ueda. "Dynamical symmetry in spinor Bose-Einstein condensates". In: *Physical Review A—Atomic, Molecular, and Optical Physics* 78.2 (2008), p. 023609. DOI: https://doi.org/10.1103/PhysRevA.78.023609.

[46] Ratheejit Ghosh et al. "Droplet arrays in doubly dipolar Bose-Einstein condensates". In: *Physical Review A* 106.6 (2022), p. 063318. DOI: https://doi.org/10.1103/PhysRevA.106.063318.

[47] Maxence Lepers et al. "Ultracold rare-earth magnetic atoms with an electric dipole moment". In: *Physical Review Letters* 121.6 (2018), p. 063201. DOI: https://doi.org/10.1103/PhysRevLett.121.063201.

[48] Gregor Anich et al. "Comprehensive characterization of an apparatus for cold electromagnetic dysprosium dipoles". In: *Physical Review A* 110.2 (2024), p. 023311. DOI: https://doi.org/10.1103/PhysRevA.110.023311.

[49] Chinmayee Mishra, Luis Santos, and Rejish Nath. "Self-Bound Doubly Dipolar Bose-Einstein Condensates". In: *Phys. Rev. Lett.* 124 (7 Feb. 2020), p. 073402. DOI: 10.1103/PhysRevLett.124.073402.

[50] JH Eberly et al. "Coherent dynamics of N-level atoms and molecules. I. Numerical experiments". In: *Physical Review A* 16.5 (1977), p. 2038. DOI: https://doi.org/10.1103/PhysRevA.16.2038.

[51] Claude Cohen-Tannoudji and Serge Reynaud. "Dressed-atom description of resonance fluorescence and absorption spectra of a multi-level atom in an intense laser beam". In: *Journal*

of Physics B: Atomic and Molecular Physics 10.3 (1977), p. 345. DOI: 10.1088/0022-3700/10/3/005.

[52] Dieter Suter. *The physics of laser-atom interactions*. 19. Cambridge University Press, 1997.

[53] Wenxian Zhang et al. "Dynamical instability and domain formation in a spin-1 Bose-Einstein condensate". In: *Physical review letters* 95.18 (2005), p. 180403. DOI: https://doi.org/10.1103/PhysRevLett.95.180403.

[54] J Steinhauer et al. "Excitation spectrum of a Bose-Einstein condensate". In: *Physical review letters* 88.12 (2002), p. 120407. DOI: https://doi.org/10.1103/PhysRevLett.88.120407.

[55] R Ozeri et al. "Colloquium: bulk Bogoliubov excitations in a Bose-Einstein condensate". In: *Reviews of Modern Physics* 77.1 (2005), pp. 187–205. DOI: https://doi.org/10.1103/RevModPhys.77.187.

[56] L Santos and T Pfau. "Spin-3 chromium bose-einstein condensates". In: *Physical review letters* 96.19 (2006), p. 190404. DOI: https://doi.org/10.1103/PhysRevLett.96.190404.

[57] H Kleinert. "External Sources, Correlations, and Perturbation Theory". In: (). DOI: https://doi.org/10.1142/9789814273572_0003.