Characteristic classes

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by

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Certificate

This is to certify that this dissertation entitled Characteristic classestowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sudhir Kumar at Indian Institute of Science Education and Research under the supervision of Dr. Vivek Mohan Mallick, Assistant Professor, Department of Mathematics , during the academic year 2017-2018.

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This thesis is dedicated to my teachers.

Declaration

I hereby declare that the matter embodied in the report entitled Characteristic classes are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Vivek Mohan Mallick and the same has not been submitted elsewhere for any other degree.

Sudhir Kumar

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Abstract

Many topological spaces exist as the total spaces of real vector bundles over some base spaces. Topological properties like Hausdorffness, connectedness, the first axiom of countability, path connectedness, local connectedness of the total space of a vector bundle can be studied by knowing these topological properties of the base space. We want to classify vector bundles up to vector bundle isomorphism. It is very difficult to classify vector bundles using topological properties. We would be using algebraic topology concepts like singular homology and singular cohomology of base space to classify vector bundles. We have used axioms of Stiefel-Whitney classes to classify some vector bundles.

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Chapter 1

Smooth manifold

1.1 Some problems from smooth manifold

Let M be a smooth manifold. We will denote the set of all smooth functions from M to $\mathbb{R}$ by $C^\infty(M, \mathbb{R})$.

Exercise 1. *Show that $C^\infty(M, \mathbb{R})$ can be made into a ring, and for each $x \in M$, we will get a ring homomorphism $C^\infty(M, \mathbb{R}) \to \mathbb{R}$ whose kernel is a maximal ideal in $C^\infty(M, \mathbb{R})$. If M is compact, show that every maximal ideal in $C^\infty(M, \mathbb{R})$ is the kernel of some homomorphism mentioned above.*

Solution. For any $f, g \in C^\infty(M, \mathbb{R})$, define

$$\begin{aligned} f + g \colon M &\to \mathbb{R} \\ x &\mapsto f(x) + g(x) \end{aligned}$$

and

$$\begin{aligned} fg \colon M &\to \mathbb{R} \\ x &\mapsto f(x)g(x) \end{aligned}$$

With the addition and multiplication defined above, $C^\infty(M, \mathbb{R})$ is a ring.

For x $\in M$, define

$$\phi\colon C^\infty(M,\mathbb{R}) \to \mathbb{R}$$
$$f \mapsto f(x)$$

Then ϕ is a ring homomorphism and is also surjective. Therefore, $C^\infty(M,\mathbb{R})\diagup$ kernel(ϕ) is isomorphic to $\mathbb{R}$.

Since $\mathbb{R}$ is a field, Kernel(ϕ) is a maximal ideal. If ϕ is defined for $x \in M$, we will denote kernel(ϕ) by m_x. Suppose m is a maximal ideal in $C^\infty(M,\mathbb{R})$ such that $m \neq m_x$ for all $x \in M$. Since $m \neq m_x$ for all $x \in M$, there exists a $f_x \in C^\infty(M,\mathbb{R})$ for each $x \in M$ such that $f_x(x) \neq 0$. Since $f_x \neq 0$, there exists a neighborhood U_x of x such that $f_x(y) \neq 0$ for all $y \in U_x$. Since $M = \bigcup\limits_{x\in M} U_x$ and M is compact, $M = \bigcup\limits_{i=1}^{n} U_{x_i}$ for some natural number n. Define $f = f_{x_1}^2 + ... + f_{x_n}^2$. Then $f \in m$ and $f \neq 0$ for all $x \in M$. $f \neq 0$ for all $x \in M$ implies f is invertible. Therefore $m = C^\infty(M,\mathbb{R})$. This is a contradiction.

$\square$

Chapter 2

Vector bundle

2.1 Vector bundle

Let E and B be topological spaces. Let Λ, I and J be index sets. Let $\mathbb{R}$ and $\mathbb{Z}$ denote the real numbers and ring of integers respectively.

Definition 2.1.1. *An n-**dimensional vector bundle** over B is a surjective continuous map $\pi \colon E \to B$ satisfying the following conditions,*

1. *For each $x \in B$, $\pi^{-1}(x)$ is an n-dimensional vector space over $\mathbb{R}$.*
2. *For each $x \in B$, there exists a neighborhood U_α of x and a homeomorphism $h_\alpha \colon U_\alpha \times \mathbb{R}^n \to \pi^{-1}(U_\alpha)$ such that for each $y \in U_\alpha$, the restriction of h_α on $\{y\} \times \mathbb{R}^n$ is a linear isomorphism of $\{y\} \times \mathbb{R}^n$ with $\pi^{-1}(y)$.*

E is known as *total space* of the vector bundle, B is known as its *base space*, π is known as its *projection*, $\pi^{-1}(x)$ is known as *fiber over x* and (U_α, h_α) is known as *local trivialization* at x.

$h_{\alpha y}$ will denote the restriction of h_α on $\{y\} \times \mathbb{R}^n$.

Example 1. *$B \times \mathbb{R}^n$ is an n-dimensional vector bundle over B. It is called trivial bundle. We will denote the n-dimensional trivial vector bundle over B by ε^n*

Example 2. *Let M be an n-dimensional smooth manifold. Then the tangent bundle of M is an n-dimensional vector bundle of M.*

Example 3. *Let E be the tangent bundle of S^n for $n \geq 1$. We have $E = \{(x, v) \in S^n \times \mathbb{R}^{n+1} \mid <x, v>= 0\}$ where $<,>$ is the dot product on $\mathbb{R}^{n+1}$. Here $\pi \colon E \to S^n$ is given by $(x, v) \mapsto x$. Let $U_i = \{x \in S^n \mid x_i \neq 0\}$ for $1 \leq i \leq n+1$. Then $h_i \colon U_i \times \mathbb{R}^n \to \pi^{-1}(U_i)$ is given by $(x, v) \mapsto (x, f_i(v) - <x, f_i(v)> x)$ where $f_i \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ is given by $(x_1, \ldots, x_i, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$. Therefore E is an n-dimensional vector bundle of S^n*

Remark 2.1.1. *Let $\pi \colon E \to B$ be an n-dimensional vector bundle with a local trivialization $\{(U_\alpha, h_\alpha)\}_{\alpha \in \Lambda}$. Define g_α and g_β as the restriction of h_α and h_β respectively on $U_\alpha \cap U_\beta \times \mathbb{R}^n$ whenever $U_\alpha \cap U_\beta \neq \phi$. Then g_α, g_β are homeomorphism and restriction of g_α, g_β on $\{a\} \times \mathbb{R}^n$ is a linear isomorphism of $\{a\} \times \mathbb{R}^n$ with $\pi^{-1}(a)$ for each $a \in U_\alpha \cap U_\beta$. Therefore the following composition $U_\alpha \cap U_\beta \times \mathbb{R}^n \xrightarrow{g_\alpha} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{g_\beta^{-1}} U_\alpha \cap U_\beta \times \mathbb{R}^n$ will give a homeomorphism $g_\beta^{-1} g_\alpha \colon U_\alpha \cap U_\beta \times \mathbb{R}^n \to U_\alpha \cap U_\beta \times \mathbb{R}^n$. We will denote it by $g_{\beta\alpha}$. Since the restriction of $g_{\beta\alpha}$ on $\{a\} \times \mathbb{R}^n$ is a linear isomorphism of $\{a\} \times \mathbb{R}^n$ with itself, we can write $g_{\beta\alpha}$ as*

$$
\begin{aligned}
g_{\beta\alpha} \colon U_\alpha \cap U_\beta \times \mathbb{R}^n &\to U_\alpha \cap U_\beta \times \mathbb{R}^n \\
(a, r) &\mapsto (a, \tau_{\beta\alpha}(a) r)
\end{aligned}
$$

where $\tau_{\beta\alpha} \colon U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$ is a continuous map. If $U_\alpha \cap U_\beta \cap U_\gamma \neq 0$, we get a commutative diagram

$$
\begin{array}{ccccc}
U_\alpha \cap U_\beta \cap U_\gamma \times \mathbb{R}^n & \xrightarrow{g_\alpha} & \pi^{-1}(U_\alpha \cap U_\beta \cap U_\gamma) & \xrightarrow{g_\beta^{-1}} & U_\alpha \cap U_\beta \cap U_\gamma \times \mathbb{R}^n \\
& \searrow^{g_{\gamma\alpha}} & \downarrow g_\gamma^{-1} & \swarrow^{g_{\gamma\beta}} & \\
& & U_\alpha \cap U_\beta \cap U_\gamma \times \mathbb{R}^n & &
\end{array}
$$

This implies that $g_{\gamma\beta} \circ g_{\beta\alpha} = g_{\gamma\alpha}$ and $\tau_{\gamma\beta} \circ \tau_{\beta\alpha} = \tau_{\gamma\alpha}$. $\tau_{\beta\alpha}$ is known as transition function*.*

Exercise 2. *Let B be a topological space. For a given open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of B satisfying the following conditions,*

1. *If $U_\alpha \cap U_\beta \neq \phi$, then there is a homeomorphism $h_{\alpha\beta} \colon U_\alpha \cap U_\beta \times \mathbb{R}^n \to U_\alpha \cap U_\beta \times \mathbb{R}^n$ with $h_{\gamma\beta} \circ h_{\beta\alpha}(x, r) = h_{\gamma\alpha}(x, r)$ for $(x, r) \in U_\alpha \cap U_\beta \cap U_\gamma \times \mathbb{R}^n$.*

2. *$P_1(h_{\alpha\beta}(x,r)) = x$, where $(x,r) \in U_\alpha \cap U_\beta \times \mathbb{R}^n$ and P_1 is the projection map on the first coordinate.*

3. *For each $x \in U_\alpha \cap U_\beta$, the restriction of $h_{\alpha\beta}$ on $\{x\} \times \mathbb{R}^n$ is a linear isomorphism of $\{x\} \times \mathbb{R}^n$ with itself; i.e. there exists a transition function.*

There exists a vector bundle $\pi \colon E \to B$ for which $\{h_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$ are the transition functions.

Solution. Let $F = \bigsqcup_{\alpha\in\Lambda} U_\alpha \times \mathbb{R}^n$. For each U_α, define $h_{\alpha\alpha} = I_\alpha$ where I_α is the identity function on $U_\alpha \times \mathbb{R}^n$. Define an equivalence relation on F by $(x,v) \sim (x,w)$ if and only if there exists an $h_{\alpha\beta}$ such that $h_{\alpha\beta}(x,v) = (x,w)$. Let E be the quotient space resulting from the equivalence relation.

Define

$$\begin{aligned} \pi \colon E &\to B \\ (x,v) &\mapsto x \end{aligned}$$

and

$$\begin{aligned} f_\gamma \colon V_\gamma \times \mathbb{R}^n &\to \pi^{-1}(V_\gamma) \\ (x,v) &\mapsto [x,v] \end{aligned}$$

where V_γ is an element of the open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of B and $[x,v]$ is the equivalence class of (x,v). Then we can define the inverse map of f by

$$\begin{aligned} f_\gamma^{-1} \colon \pi^{-1}(V_\gamma) &\to V_\gamma \times \mathbb{R}^n \\ [x,r] &\mapsto (x,s) \end{aligned}$$

where (x,s) is an element of $V_\gamma \times \mathbb{R}^n$ that belongs to the equivalence class $[x,r]$. If $U_\alpha \cap U_\beta \neq 0$, then the composition $U_\alpha \cap U_\beta \times \mathbb{R}^n \xrightarrow{f_\alpha} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{f_\beta^{-1}} U_\alpha \cap U_\beta \times \mathbb{R}^n$ is the map $h_{\alpha\beta}$. Therefore $\pi \colon E \to B$ is a vector bundle with the transition functions $\{h_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$. □

2.1.1 Bundle map

Definition 2.1.2. *A **bundle map** between two n-dimensional vector bundles $\pi_1 \colon E_1 \to B_1$ and $\pi_2 \colon E_2 \to B_2$ is a continuous map $F \colon E_1 \to E_2$ for which there exist a continuous map $f \colon B_1 \to B_2$ such that the below digram is commutative and restriction of F on $\pi_1^{-1}(b)$ is a linear isomorphism of $\pi_1^{-1}(b)$ with $\pi_2^{-1}(f(b))$.*

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow[f]{} & B_2 \end{array}$$

f is called *a map covered by a bundle map* from E_1 to E_2

Definition 2.1.3. *Two vector bundles $\pi_1 \colon E_1 \to B$ and $\pi_2 \colon E_2 \to B$ are said to be **isomorphic** if there exists a bundle $F \colon E_1 \to E_2$ which is a homeomorphism and f is the identity map of B.*

Example 4. *For $n \geq 1$, let $E = \{(x, v) \in S^n \times \mathbb{R}^n \mid v = rx, r \in \mathbb{R}\}$. Then $\pi \colon E \to S^n$ given by $(x, v) \mapsto x$ is a 1-dimensional vector bundle. It is called normal bundle over S^n. $h \colon E \to S^n \times \mathbb{R}$ given by $(x, v) \mapsto (x, < x, v >)$ is a homeomorphism. $h|_{\pi^{-1}(x)} \colon \pi^{-1}(x) \to \mathbb{R}$ given by $v \mapsto < x, v >$ is a linear isomorphism for all $x \in X$. Therefore normal bundle of S^n is isomorphic to the trivial bundle for all $n \geq 1$.*

Lemma 2.1.1. *Let $\pi_1 \colon E_1 \to B$ and $\pi_2 \colon E_2 \to B$ be two vector bundles. If $f \colon E_1 \to E_2$ is a continuous map which maps $\pi_1^{-1}(b)$ linearly isomorphic to $\pi_2^{-1}(b)$ for each $b \in B$, then f is a homeomorphism.*

Proof. f is a bijective map. Let $f^{-1} \colon E' \to E$ be the inverse of f. We need to show that f^{-1} is continuous. Let $\{(U_\alpha, h_\alpha)\}_{\alpha \in \Lambda}$ and $\{(V_i, g_i)\}_{i \in I}$ be local trivializations of π and π' respectively. For $e \in E$ with $\pi(e) = b$ and $f(e) = e'$, choose U_α and V_i for $\alpha \in \Lambda$ and $i \in I$ such that $b \in U_\alpha \cap V_i$. Define $f' = f|_{\pi^{-1}(U_\alpha \cap V_i)} \colon \pi^{-1}(U_\alpha \cap V_i) \to \pi'^{-1}(U_\alpha \cap V_i)$. f' is continuous and bijective as f maps $\pi^{-1}(b)$ linearly isomorphic to $\pi'^{-1}(b)$. Then we get a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha \cap V_i) & \xrightarrow{f'} & \pi'^{-1}(U_\alpha \cap V_i) \\ {\scriptstyle h_\alpha}\downarrow & & \downarrow{\scriptstyle h_i} \\ U_\alpha \cap V_i \times \mathbb{R}^n & \xrightarrow[h_i \circ f' \circ h_\alpha^{-1}]{} & U_\alpha \cap V_i \times \mathbb{R}^n \end{array}$$

We can write $h_i \circ f' \circ h_\alpha^{-1}$ explicitly as

$$\begin{aligned} h_i \circ f' \circ h_\alpha^{-1} \colon U_\alpha \cap U_i \times \mathbb{R}^n &\to U_\alpha \cap U_i \times \mathbb{R}^n \\ (a, r) &\mapsto (a, \tau_{i\alpha}(a) r) \end{aligned}$$

where $\tau_{i\alpha}(a) \in \mathrm{GL}_n(\mathbb{R})$. Then we can define

$$\begin{aligned} (h_i \circ f' \circ h_\alpha^{-1})^{-1} \colon U_\alpha \cap U_i \times \mathbb{R}^n &\to U_\alpha \cap U_i \times \mathbb{R}^n \\ (a, r) &\mapsto (a, \tau_{i\alpha}(a)^{-1} r) \end{aligned}$$

$(h_i \circ f' \circ h_\alpha^{-1})^{-1}$ is continuous because the inverse map from $\mathrm{GL}_n(\mathbb{R})$ to $\mathrm{GL}_n(\mathbb{R})$ is a continuous map. Therefore $f'^{-1} = h_\alpha^{-1} \circ (h_i \circ f' \circ h_\alpha^{-1})^{-1} \circ h_i$ is continuous. This implies that f^{-1} is continuous on a neighborhood of e' for each $e' \in E'$. Therefore f^{-1} is continuous. □

Corollary 2.1.2. *Let $\pi \colon E \to B$ be an n-dimensional vector bundle with a local trivialization $\{(U_\alpha, h_\alpha)\}_{\alpha\in\Lambda}$. If a vector bundle $\pi' \colon E' \to B$ is constructed with $\{(U_\alpha, h_\alpha)\}_{\alpha\in\Lambda}$ using exercise 2, then the vector bundles $\pi \colon E \to B$ and $\pi' \colon E' \to B$ are isomorphic.*

Proof. Define

$$\begin{aligned} h \colon E &\to E' \\ e &\mapsto [h_\alpha^{-1}(e)] \end{aligned}$$

where $e \in \pi^{-1}(U_\alpha)$ for some $\alpha \in \Lambda$ and $[h_\alpha^{-1}(e)]$ is the equivalence class of $h_\alpha^{-1}(e)$. h is well defined because of the transitivity of transition function. h also maps $\pi^{-1}(b)$ linearly isomorphic to $\pi'^{-1}(b)$ for each $b \in B$. Let $q \colon \bigsqcup_{\alpha\in\Lambda} U_\alpha \times \mathbb{R}^n \to E'$ be the quotient map. For any open set U' of E', $q^{-1}(U')$ is open and $q^{-1}(U') = \bigsqcup_{\alpha\in\Lambda} V_\alpha \times R_\alpha$ with $V_\alpha \times R_\alpha$ open subset $U_\alpha \times \mathbb{R}^n$ for each $\alpha \in \Lambda$. Therefore $h^{-1}(U') = \cup_{\alpha\in\Lambda} h_\alpha(V_\alpha \times R_\alpha)$. $h^{-1}(U')$ is open as each h_α is a homeomorphism. This implies that h is continuous. Using lemma 2.1.1, we get that h is a homeomorphism. □

Corollary 2.1.3. *Let $\pi\colon E \to B$ be an n-dimensional vector bundle with a local trivialization $\{(U_\alpha, h_\alpha)\}_{\alpha\in\Lambda}$. If all the transition functions of $\{(U_\alpha, h_\alpha)\}_{\alpha\in\Lambda}$ map to the identity element of $\mathrm{GL}_n(\mathbb{R})$, then $\pi\colon E \to B$ is isomorphic to the trivial vector bundle.*

Proof. Define $h\colon E \to B$ by $h(e) = h_\alpha^{-1}(e)$ if $e \in \pi^{-1}(U_\alpha)$. Then $h|_{\pi^{-1}(U_\alpha)} = h_\alpha$ and $h_\alpha|_{\pi^{-1}(U_\alpha\cap U_\beta)} = h_\beta|_{\pi^{-1}(U_\alpha\cap U_\beta)}$. Therefore h is continuous. Lemma 2.1.1 implies that h is a vector bundle isomorphism. □

2.1.2 Section of a vector bundle

Definition 2.1.4. *A **section** of a vector bundle $\pi\colon E \to B$ is a continuous map $S\colon B \to E$ with $S(b) \in \pi^{-1}(b)$ for each $b \in B$.*

Section of the tangent bundle of a smooth manifold M is called a *vector field* on M.

Example 5. *$S\colon B \to E$ given by $x \mapsto h_x(x, 0)$ is a section of vector bundle $\pi\colon E \to B$ where h_x is a local trivialization defined for a neighborhood of x. It is called zero section.*

Definition 2.1.5. *A section S of vector bundle $\pi\colon E \to B$ is called **nowhere zero** if $S(b)$ is a non-zero vector of $\pi^{-1}(b)$ for all $b \in B$.*

Definition 2.1.6. *k sections $S_1, \ldots, S_n$ of a vector bundle $\pi\colon E \to B$ is called **nowhere dependent** if $S_1(b), \ldots, S_k(b)$ are linearly independent for each $b \in B$.*

Theorem 2.1.4. *An n-dimensional vector bundle $\pi\colon E \to B$ is isomorphic to the trivial vector bundle if and only if there exist n sections $S_1, \ldots, S_n$ such that the set $\{S_1(b), S_2(b), \ldots, S_n(b)\}$ is a basis of $\pi^{-1}(b)$ for each $b \in B$.*

Proof. An n-dimensional vector bundle $\pi\colon E \to B$ is isomorphic to the trivial vector bundle. Then there exists an isomorphism $h : B \times \mathbb{R}^n \to E$.

Define

$$\begin{aligned} S_i\colon B &\to E \\ b &\mapsto h(b, (0, \ldots, 1, 0, \ldots, 0)) \end{aligned}$$

where 1 is at i^{th} position. Then $S_1, \ldots, S_n$ are nowhere dependent sections.

Conversely, let $S_1, S_2, \ldots, S_n$ be n sections such that the set $\{S_1(b), S_2(b), \ldots$ $, S_n(b)\}$ is a basis of $\pi^{-1}(b)$ for each b $\in$ B.

Define

$$h\colon B \times \mathbb{R}^n \to E$$
$$(b, (x_1, \ldots, x_n)) \mapsto (b, S_1(b)x_1 + \cdots + S_n(b)x_n)$$

h is continuous because s_i's are continuous. From lemma 2.1.1, we get that h is a homeomorphism. Therefore h is a vector bundle isomorphism. □

2.1.3 Subbundle of a vector bundle

Definition 2.1.7. *A vector bundle $\pi_1\colon E_1 \to B$ is called a subbundle of a vector bundle $\pi\colon E \to B$ if $E_1 \subset E$ and $\pi_1^{-1}(b)$ is a vector subspace of $\pi^{-1}(b)$ for each $b \in B$.*

Exercise 3. *For a given vector bundle $\pi\colon E \to B$, show that the projection map $\pi\colon E \to B$ is a homotopy equivalence.*

Solution. We need to show that there exists a map $f\colon B \to E$ such that $\pi \circ f$ is homotopic to I_B and $f \circ \pi$ is homotopic I_E where I_B and I_E are the identity maps of B and E respectively. Let $\{(U_\alpha, h_\alpha)\}_{\alpha \in \Lambda}$ be a local trivialization of $\pi\colon E \to B$. Take f to be the zero section. We will get $\pi \circ f = I_B$. Define

$$H\colon [0,1] \times E \to E$$
$$(t, e) \mapsto h_\alpha(b, ((1-t)v))$$

whenever $\pi(e) = b \in U_\alpha$ and $h_\alpha(b, v) = e$. The function H is defined because $\mathbb{R}^n$ is a convex set. H is continuous because each h_α is a continuous function. Therefore H is a homotopy between I_E and $f \circ \pi$. □

Exercise 4. *If $\pi\colon E \to S^n$ is an 1-dimensional vector bundle over S^1, then it is either isomorphic to Möbius bundle or trivial bundle.*

Solution. Let $\{(U_\alpha, h_\alpha)\}_{\alpha\in\Lambda}$ be a local trivialization of $\pi\colon E \to S^1$. From the open cover $\{U_\alpha\}_{\alpha\in\Lambda}$, we will always get an open cover $\{V_i\}_{i\in I}$ such that V_i's are connected and for each $i \in I$, $V_i \subset U_\alpha$ for some α. If $V_i \subset U_\alpha$ for more that one α, then fix an α and define $g_i = h_\alpha|_{V_i}$. Therefore we get a local trivialization $\{(V_i, g_i)\}_{i\in I}$ of $\pi\colon E \to S^1$. Since S^1 is compact, the open cover $\{V_i\}_{i\in I}$ has a finite subcover. Let $\{V_j\}_{j=1}^n$ covers S^1. Then $\{(V_j, f_j)\}_{j=1}^n$ is a local trivialization of $\pi\colon E \to S^1$. Choose V_k from $\{V_j\}_{j=1}^n$ with $V_k \not\subseteq V_j$ for $k \neq j$. Let $A = \bigcup_{1\geq j\leq n, j\neq k} V_j$. Using exercise 2 and $\{(V_j, f_j)\}_{1\geq j\leq n, j\neq k}$, we get an 1-dimensional vector bundle $\pi_1\colon E_1 \to A$ with the local trivialization $\{(V_j, f_j)\}_{1\geq j\leq n, j\neq k}$. Since A is contractible, $\pi_1\colon E_1 \to A$ is a trivial bundle. Let $h\colon \pi_1^{-1}(A) \to A \times \mathbb{R}$ be a vector bundle isomorphism. Now we have $\{(A, h), (V_k, h_k)\}$ as a local trivialization of $\pi\colon E \to S_1$. $A \bigcap V_k = N_1 \bigcup N_2$ where N_1 and N_2 are disjoint open sets. There are following four possibilities of the transition function $\tau\colon N_1 \bigcup N_2 \to \mathrm{GL}_1(\mathbb{R}) = (\mathbb{R}\backslash\{0\})$

$$\tau(a) = 1 \ \forall\ a \in N_1 \bigcup N_2 \tag{2.1}$$

$$\tau(a) = -1 \ \forall\ a \in N_1 \bigcup N_2 \tag{2.2}$$

$$\tau(a) = \begin{cases} 1 \text{ for } a \in N_1 \\ -1 \text{ for } a \in N_2 \end{cases} \tag{2.3}$$

$$\tau(a) = \begin{cases} -1 \text{ for } a \in N_1 \\ 1 \text{ for } a \in N_2 \end{cases} \tag{2.4}$$

as τ is continuous. The first two cases implies that $\pi\colon E \to S^1$ is trivial and the last two cases implies that $\pi\colon E \to S^1$ is the Möbius bundle. □

2.2 Constructing new vector bundles

2.2.1 Restriction of a vector bundle on a subspace of the base space

Let $\pi\colon E \to B$ be an n-vector bundle and A be a subspace of B. Let $\{(U_\alpha, h_\alpha)\}_{\alpha\in\Lambda}$ be a local trivialization of $\pi\colon E \to B$. Define $E_1 = \pi^{-1}(A), \pi_1 = \pi|_{\pi_1^{-1}(A)}, V_\alpha = A \cap U_\alpha$ and $g_\alpha = h_\alpha|_{V_\alpha\times\mathbb{R}^n}$ for each $\alpha \in \Lambda$. Since the restriction of h_α on $\{a\} \times \mathbb{R}^n$ is isomorphic to $\pi^{-1}(a)$ for each $a \in V_\alpha$, $g_\alpha \colon V_\alpha \times \mathbb{R}^n \to \pi_1^{-1}(V_\alpha)$ is well defined and is also a homeomorphism. Therefore $\pi_1 \colon E_1 \to A$ is an n-dimensional vector bundle with a local trivialization $\{(V_\alpha, g_\alpha)\}_{\alpha\in\Lambda}$.

2.2.2 Induced vector bundle

Let $\pi \colon E \to B$ be an n-dimensional vector bundle and $f \colon A \to B$ be a continuous map. Let $\{(U_\alpha, h_\alpha)\}_{\alpha\in\Lambda}$ be a local trivialization of $\pi \colon E \to B$. Define $E_1 = \{(a, e) \in A \times E \mid f(a) = \pi(e)\}$. Define $\pi_1 \colon E \to B$ as $\pi_1((a, e)) = a$. Let $V_\alpha = f^{-1}(U_\alpha)$. Define

$$
\begin{aligned}
g_\alpha \colon V_\alpha \times \mathbb{R}^n &\to \pi_1^{-1}(V_\alpha) \\
(a, v) &\mapsto (a, h_\alpha(f(a), v))
\end{aligned}
$$

Then g_α^{-1} is given by

$$
\begin{aligned}
g_\alpha^{-1} \colon \pi_1^{-1}(V_\alpha) &\to V_\alpha \times \mathbb{R}^n \\
(a, e) &\mapsto (a, p(h_\alpha^{-1}(e)))
\end{aligned}
$$

where $p \colon U_\alpha \times \mathbb{R}^n \to \mathbb{R}^n$ is defined as $p(b, v) = v$.

g_α and g_α^{-1} are continuous because these maps are compositions of continuous maps. Therefore $\pi_1 \colon E_1 \to A$ is an n-dimensional vector bundle with a local trivialization $\{(V_\alpha, g_\alpha)\}_{\alpha\in\Lambda}$. $f^*\pi \colon f^*E \to A$ will denote the induced bundle $\pi_1 \colon E_1 \to A$. This vector bundle is known as the *vector bundle induced by f*.

Lemma 2.2.1. *Let $\pi_1 \colon E_1 \to A$ and $\pi_2 \colon E_2 \to B$ be two n-dimensional vector bundles and $F \colon E_1 \to E_2$ be a bundle map. If $f \colon A \to B$ be a map covered by the bundle map F, then the induced bundle $f^*\pi_2 \colon f^*E_2 \to A$ and $\pi_1 \colon E_1 \to A$ are isomorphic.*

Proof. Define

$$
\begin{aligned}
\phi \colon E_1 &\to f^*E_2 \\
e &\mapsto (\pi_1(e), F(e))
\end{aligned}
$$

ϕ is continuous because π_1 and F are continuous. Since restriction of ϕ on $\pi_1^{-1}(a)$ is a linear isomorphism of $\pi_1^{-1}(a)$ with $(\{a\} \times \pi_2^{-1}(f(b))) = (f^*\pi_1)^{-1}(a)$ for each $a \in A$, F is a vector bundle isomorphism. The previous statement follows from the lemma 2.1.1. □

2.2.3 Cartesian product of vector bundles

Let $\pi_1\colon E_1 \to A$ and $\pi_2\colon E_2 \to B$ be two vector bundles of dimensions m and n respectively. Let $\{(U_\alpha, h_\alpha)\}_{\alpha\in\Lambda}$ and $\{(V_i, g_i)\}_{i\in I}$ be local trivializations of $\pi_1\colon E_1 \to A$ and $\pi_2\colon E_2 \to B$ respectively. Define

$$\begin{aligned} \pi\colon E_1 \times E_2 &\to A \times B \\ (e_1, e_2) &\mapsto (\pi_1(e_1), \pi_2(e_2)) \end{aligned}$$

and

$$\begin{aligned} H_{\alpha,i}\colon U_\alpha \times V_i \times \mathbb{R}^m \times \mathbb{R}^n &\to \pi_1^{-1}(U_\alpha) \times \pi_2^{-1}(V_i) \\ (a, b, v_1, v_2) &\mapsto (h_\alpha(a, v_1), h_i(b, v_2)) \end{aligned}$$

Then $\pi\colon E_1 \times E_2 \to A \times B$ is an $(m+n)$-dimensional vector bundle with a local trivializations $\{(U_\alpha \times V_i, H_{\alpha,i})\}_{\alpha\in\Lambda, i\in I}$.

Whitney sum

Let $\pi_1\colon E_1 \to B$ and $\pi_2\colon E_2 \to B$ be two vector bundles. Let $\tau = \{(a, b) \in B \times B \mid a = b\}$. Let $\pi'\colon E_1 \times E_2 \to B \times B$ be the Cartesian product of vector bundles $\pi_1\colon E_1 \to B$ and $\pi_2\colon E_2 \to B$. Since $\tau \subset B \times B$, we get the restriction vector bundle $\pi''\colon E' \to \tau$ of $\pi'\colon E_1 \times E_2 \to B \times B$. A map $f\colon \tau \to B$ given by $f(b, b) = b$ is a homeomorphism. Therefore $f \circ \pi''\colon E' \to B$ is a vector bundle. The vector bundle $f \circ \pi''\colon E' \to B$ is known as the **Whitney sum** of $\pi_1\colon E_1 \to B$ and $\pi_2\colon E_2 \to B$ and is denoted by $\pi_1 \oplus \pi_2\colon E_1 \oplus E_2 \to B$. We can write $E_1 \oplus E_2$ and $\pi_1 \oplus \pi_2$ explicitly as $E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 \mid \pi_1(v_1) = \pi_2(v_2)\}$ and

$$\begin{aligned} \pi_1 \oplus \pi_2\colon E_1 \oplus E_2 &\to B \\ (v_1, v_2) &\mapsto \pi_1(v_1) = \pi_2(v_2) \end{aligned}$$

Lemma 2.2.2. *Let $\pi_1\colon E_1 \to B$ and $\pi_2\colon E_2 \to B$ be two subbundles of a vector bundle $\pi\colon E \to B$. If the direct sum of $\pi_1^{-1}(b)$ and $\pi_2^{-1}(b)$ is equal to $\pi^{-1}(b)$ for each $b \in B$, then*

$\pi_1 \oplus \pi_2 \colon E_1 \oplus E_2 \to B$ *is isomorphic to* $\pi \colon E \to B$.

Proof. Define

$$h \colon E_1 \oplus E_2 \to E$$
$$(e_1, e_2) \mapsto e_1 + e_2$$

h is well defined because $\pi_1(e_1) = \pi_2(e_2) = \pi(e_1 + e_2)$. h is also continuous. Lemma 2.1.1 implies that h is a vector bundle isomorphism. □

2.2.4 Euclidean vector bundle

Definition 2.2.1. *Let* $\pi \colon E \to B$ *be a vector bundle. If there exists a continuous map* $\nu \colon E \oplus E \to \mathbb{R}$ *such that restriction of* ν *over* $(\pi \oplus \pi)^{-1}(b)$ *is a symmetric, positive definite, bilinear form for each* $b \in B$*, then* $\pi \colon E \to B$ *is called* ***euclidean vector bundle****.*

ν is called *euclidean metric* on $\pi \colon E \to B$. If B is a smooth manifold, then a euclidean metric on the tangent bundle of B is called *Riemannian metric* and B is called *Riemannian manifold*.

Example 6. *Let* $\pi \colon B \times \mathbb{R}^n \to B$ *be the trivial bundle over* B*. Define*

$$\nu \colon B \times \mathbb{R}^n \oplus B \times \mathbb{R}^n \to \mathbb{R}$$
$$((a, r_1), (a, r_2)) \mapsto < r_1, r_2 >$$

where $<,>$ *is the dot product on* $\mathbb{R}^n$*. Then* $\pi \colon B \times \mathbb{R}^n \to B$ *is a euclidean vector bundle with a euclidean metric* ν*.*

Lemma 2.2.3. *If* $\pi \colon E \to B$ *be an n-dimensional trivial vector bundle with a euclidean metric* ν*, then there are n sections* $\{S_1, \ldots, S_n\}$ *such that* $\nu(S_i(b), S_j(b)) = \delta_{ij}$ *for each* $b \in B$*, where* δ_{ij} *is the Kronecker delta function.*

Proof. From theorem 3.1.3, we know that there are n nowhere dependent sections $s_1, \ldots, s_n$. After applying the Gram-Schmidt process to $\{s_1(b), \ldots, s_n(b)\}$, we will get a normal orthogonal basis $\{S_1(b), \ldots, S_n(b)\}$ of $\pi^{-1}(b)$ for each $b \in B$. Since ν is continuous, $S_1, \ldots, S_n$ are continuous map. □

Lemma 2.2.4. *Let $\pi_1\colon E_1 \to B$ be a subbundle of a euclidean vector bundle $\pi\colon E \to B$ with a euclidean metric ν. Define $(\pi_1^{-1}(b))^{\perp} = \{e \in \pi^{-1}(b) \mid \nu(e, e_1) = 0 \; \forall e_1 \in E_1\}$ and $E_1^{\perp} = \bigsqcup_{b \in B} (\pi_1^{-1}(b))^{\perp}$. Then $\pi_1^{\perp}\colon E_1^{\perp} \to B$ given by $\pi_1^{\perp}(e) = \pi(e)$, is a vector bundle.*

Proof. Let dimensions $\pi_1\colon E_1 \to B$ and $\pi\colon E \to B$ be m and n respectively. We want to construct a local trivialization of $\pi_1^{\perp}\colon E_1^{\perp} \to B$. For $x \in B$, let U be a neighborhood b on which $\pi_1\colon E_1 \to B$ and $\pi\colon E \to B$ are trivial bundle. There are m normal orthogonal local sections $S_1, \ldots, S_m$ and n normal orthogonal local sections $s_1, \ldots, s_n$ of $\pi_1\colon E_1 \to B$ and $\pi\colon E \to B$ respectively. Define an m $\times$ n matrix $T(b) = \Big[\nu(S_i(b)s_j(b))\Big]$. Let $M_{m\times n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with real entries. Define $\phi\colon U \to M_{m\times n}(\mathbb{R})$ given by $\phi(b) = T(b)$. ϕ is a continuous map as S_i's and s_j's are continuous maps. Let M be the set of m $\times$ n matrices with first m columns linearly independent. Then M is open in $M_{m\times n}(\mathbb{R})$. $\phi^{-1}(M)$ is open in U as ϕ is continuous. Since U is open in B, $\phi^{-1}(M)$ is open in B. Then first m columns of $T(b)$ are linearly independent for each $b \in \phi^{-1}(M)$. Then $S_1(b), \ldots, S_m(b), s_{m+1}(b), \ldots, s_n(b)$ are linearly independent for each $b \in \phi^{-1}(M)$ because if not, we can write $S_i(b)$ for some i, in terms of $s_{m+1}, \ldots, s_n$ and the i^{th} column of $T(b)$ will be 0. After applying the GramSchmidt process to $S_1(b), \ldots, S_m(b), s_{m+1}(b), \ldots, s_n(b)$, we will get a normal orthogonal basis $S_1(b), \ldots, S_n(b)$ of $\pi^{-1}(b)$ for each $b \in \phi^{-1}(M)$. Define

$$
\begin{aligned}
h\colon \phi^{-1}(M) \times \mathbb{R}^{(n-m)} &\to (\pi_1^{\perp})^{-1}(\phi^{-1}(M)) \\
(b, (r_{m+1}, \ldots, r_n)) &\mapsto \sum_{k=1}^{(n-m)} r_{m+k} S_{m+k}(b)
\end{aligned}
$$

Then h is a homeomorphism and restriction of h on $\{b\} \times \mathbb{R}^{(n-m)}$ is a linear isomorphism. Therefore $\pi_1^{\perp}\colon E_1^{\perp} \to B$ is a locally trivial bundle at each $x \in B$. □

Corollary 2.2.5. *If $\pi_1\colon E_1 \to B$ is a subbundle of a euclidean vector bundle $\pi\colon E \to B$, then $\pi\colon E \to B$ is isomorphic to $\pi_1 \oplus \pi_1^{\perp}\colon E_1 \oplus E_1^{\perp} \to B$.*

Proof. From lemma 3.2.4, we get that $\pi_1^{\perp}\colon E_1^{\perp} \to B$ is a subbundle of $\pi\colon E \to B$ and the direct sum of $\pi_1^{-1}(b)$ and $(\pi_1^{\perp})^{-1}(b)$ is equal to $\pi^{-1}(b)$ for each $b \in B$. Therefore lemma 3.2.2 implies that $\pi\colon E \to B$ is isomorphic to $\pi_1 \oplus \pi_1^{\perp}\colon E_1 \oplus E_1^{\perp} \to B$. □

Definition 2.2.2. *The vector bundle $\pi_1^{\perp}\colon E_1^{\perp} \to B$ is known as the* ***normal bundle*** *of $\pi_1\colon E_1 \to B$ in $\pi\colon E \to B$.*

2.2.5 Hom-vector bundle and tensor product of vector bundles

Let $\pi_1\colon E_1 \to B$ and $\pi_2\colon E_2 \to B$ be vector bundles. Define $\mathrm{Hom}(E_1, E_2) = \bigsqcup_{b\in B} \mathrm{Hom}(\pi_1^{-1}(b), \pi_2^{-1}(b))$ and $E_1 \otimes E_2 = \bigsqcup_{b\in B} \pi_1^{-1}(b) \otimes \pi_2^{-1}(b)$ where $\mathrm{Hom}(\pi_1^{-1}(b), \pi_2^{-1}(b))$ is the set all linear transformation from $\pi_1^{-1}(b)$ to $\pi_2^{-1}(b)$ and $\pi_1^{-1}(b) \otimes \pi_2^{-1}(b)$ is the tensor product of $\pi_1^{-1}(b)$ and $\pi_2^{-1}(b)$.

Let C be a category in which objects are all finite dimensional vector spaces over $\mathbb{R}$ and morphisms are all isomorphism between such vector spaces. Since $\mathrm{GL}_n(\mathbb{R})$ has a natural topology for $n \geq 0$, the set of all isomorphisms between two finite dimensional vector spaces has a natural topology. A functor $T\colon C \times \ldots \times C \to C$ in m variable is called continuous if T is continuous map of morphisms.

Let $\pi_1\colon E_1 \to B, \ldots, \pi_m\colon E_m \to B$ be m vector bundles. Let $F(b) = T(\pi_1^{-1}(b), \ldots, \pi_m^{-1}(b))$. Let $E = \bigsqcup_{b\in B} F(b)$. Define a map $\pi\colon E \to B$ by $\pi(e) = b$ if $e \in F(b)$.

Theorem 2.2.6. *There exists a topology on E such that $\pi\colon E \to B$ is a vector bundle.*

Proof. For $x \in B$, let $(U, h_1), \ldots, (U, h_m)$ be local trivializations of $\pi_1\colon E_1 \to B, \ldots, \pi_m\colon E_m$ respectively at x. Then $h_{ib}\colon \mathbb{R}^{n_i} \to \pi_1^{-1}(b)$ is linear isomorphism for $1 \leq i \leq m$. Define

$$\begin{aligned} h\colon U \times T(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_m}) &\to \pi^{-1}(U) \\ (b, v) &\mapsto T(h_{1b}, \ldots, h_{mb})(v) \end{aligned}$$

Then h is a bijective map. Define quotient topology on $\pi^{-1}(U)$ induced by h. Let V be an open subset of B with $V \cap U$ nonempty and with local trivialization function $g_i\colon V \times \mathbb{R}^{n_i} \to B$ for $1 \leq i \leq m$. Define a map $g\colon V \times T(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_m}) \to \pi^{-1}(V)$ using $g_1, \ldots, g_m$ same as we defined h. Then $\pi^{-1}(V)$ also has a quotient topology induced by g. We have $\pi^{-1}(U) \cap \pi^{-1}(V) = \pi^{-1}(U \cap V)$. The composition $U \cap V \times T(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_m}) \xrightarrow{h} \pi^{-1}(U \cap V) \xrightarrow{g^{-1}} U \cap V \times T(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_m})$ is continuous because T is a continuous functor. Since $g^{-1} \circ h$ is continuous, the quotient topologies induced by g and h on $\pi^{-1}(U \cap V)$ are same. Now we take these $\pi^{-1}(U)$'s as a basis of a topology of E. With respect to the topology defined on E, π is a continuous map and h is a homeomorphism. Therefore $\pi\colon E \to B$ is a vector bundle. $\square$

Define $\mathrm{Hom}\colon C\times C\to C$ by $(V_1,V_2)\mapsto \mathrm{Hom}(V_1,V_2)$ for finite dimensional vector spaces V_1,V_2. If $f\colon V_1\to V_2$ and $g\colon W_1\to W_2$ are isomorphisms, then $\mathrm{Hom}(f,g)\colon \mathrm{Hom}(V_1,W_1)\to \mathrm{Hom}(V_2,W_2)$ is given by $\phi\mapsto g\circ\phi\circ f^{-1}$. Hom is a continuous functor as $\mathrm{Hom}(f,g)$ is multiplications of matrices. Therefore $\pi\colon \mathrm{Hom}(E_1,E_2)\to B$ is a vector bundle constructed from $\pi_1\colon E_1\to B$ and $\pi_2\colon E_2\to B$. $\pi\colon \mathrm{Hom}(E_1,E_2)\to B$ is known as the *dual vector bundle* of $\pi_1\colon E_1\to B$ and $\pi_2\colon E_2\to B$.

Define the tensor product functor $\otimes\colon C\times C\to C$ by $(V_1,V_2)\mapsto V_1\otimes V_2$ for finite dimensional vector spaces V_1, V_2 and $(f,g)\mapsto f\otimes g$ for isomorphisms f,g. If $f\colon V_1\to V_2$ and $g\colon W_1\to W_2$ are linear maps, then $f\otimes g\colon V_1\times W_1\to V_2\otimes W_2$ is given by $f\otimes g(v_1,w_1)=f(v_1)\otimes g(w_1)$. $\otimes$ is also a continuous functor. Therefore $\pi\colon E_1\otimes E_2\to B$ is a vector bundle constructed from $\pi_1\colon E_1\to B$ and $\pi_2\colon E_2\to B$. $\pi\colon E_1\otimes E_2\to B$ is known as the *tensor product vector bundle* of $\pi_1\colon E_1\to B$ and $\pi_2\colon E_2\to B$. A local trivialization $\{(N_j,f_j)\}_{j\in J}$ for the tensor product vector bundle is constructed from local trivializations $\{(U_\alpha,h_\alpha)\}_{\alpha\in\Lambda}$ and $\{(V_i,g_i)\}_{i\in I}$ of $\pi_1\colon E_1\to B$ and $\pi_2\colon E_2\to B$ respectively. The transition functions of $\{(N_j,f_j)\}_{j\in J}$ are given by $\{\tau_{\alpha_1\alpha_2}\otimes\sigma_{i_1i_2}\}_{\alpha_1,\alpha_2\in\Lambda;i_1,i_2\in I}$ where $\{\tau_{\alpha_1\alpha_2}\}_{\alpha_1,\alpha_2\in\Lambda}$ and $\{\sigma_{i_1i_2}\}_{i_1,i_2\in I}$ are transition functions of $\{(U_\alpha,h_\alpha)\}_{\alpha\in\Lambda}$ and $\{(V_i,g_i)\}_{i\in I}$ respectively.

Exercise 5. *If $\pi\colon E\to B$ is an 1-dimensional vector bundle, then $\pi_1\colon \mathrm{Hom}(E,E)\to B$ is a trivial bundle.*

Solution. We will show that there exists a nowhere zero section. Let $\{(U_\alpha,h_\alpha)\}_{\alpha\in\Lambda}$ be a local trivialization of $\pi\colon E\to B$. A local trivializations of $\pi_1\colon \mathrm{Hom}(E,E)\to B$ is given by $\{(U_\alpha,\mathrm{Hom}(h_\alpha))\}_{\alpha\in\Lambda}$ where

$$\begin{aligned}\mathrm{Hom}(h_\alpha)\colon U_\alpha\times\mathrm{Hom}(\mathbb{R},\mathbb{R})&\to\pi_1^{-1}(U_\alpha)\\ (x,\phi)&\mapsto \mathrm{Hom}(h_{\alpha x})(\phi)=h_\alpha\circ\phi\circ h_\alpha^{-1}\end{aligned}$$

We can observe that $\mathrm{Hom}(h_\alpha)(x,id_{\mathbb{R}})=id_{\pi_1^{-1}(x)}$ where $id_{\mathbb{R}}$ and $id_{\pi_1^{-1}(x)}$ are the identity homomorphisms of $\mathbb{R}$ and $\pi_1^{-1}(x)$ respectively. Define

$$\begin{aligned}s\colon B&\to\mathrm{Hom}(E,E)\\ x&\mapsto id_{\pi_1-1(x)}\end{aligned}$$

and

$$f\colon U_\alpha \to U_\alpha \times \mathrm{Hom}(\mathbb{R},\mathbb{R})$$
$$x \mapsto (x, id_{\mathbb{R}})$$

Then $\mathrm{Hom}(h_\alpha)\circ f = s|_{U_\alpha}$ where $s|_{U_\alpha}$ is restriction of s on U_α. Since $\mathrm{Hom}(h_\alpha)$ and f are continuous, $s|_{U_\alpha}$ is continuous. s is continuous as s is continuous on each U_α for $\alpha \in \Lambda$. Therefore s is a nowhere zero section of the vector bundle $\pi\colon \mathrm{Hom}(E,E) \to B$. □

Exercise 6. *If an n-dimensional vector bundle $\pi\colon E \to B$ has a euclidean metric, then $\pi\colon E \to B$ is isomorphic to the dual bundle $\pi_1\colon \mathrm{Hom}(E,\varepsilon^1) \to B$ where $\pi_2\colon \varepsilon^1 \to B$ is the trivial vector bundle.*

Solution. Let ν be a euclidean metric on $\pi\colon E \to B$. For $v \in \pi^{-1}(b)$, define $\phi_v\colon \pi^{-1}(b) \to \mathbb{R}$ by $\phi_v(u) = \nu(v,u)$. Then ϕ_v is a linear map. Define $\phi\colon \pi^{-1}(b) \to \mathrm{Hom}(\pi^{-1}(b), b\times\mathbb{R})$ by $\phi(v) = (b,\phi_v)$. Then ϕ is also a linear map. ϕ is an isomorphism because ν is positive definite and dimensions of vector spaces $\pi^{-1}(b)$ and $\mathrm{Hom}(\pi^{-1}(b), b\times\mathbb{R})$ are equal. Define

$$h\colon E \to \mathrm{Hom}(E,\varepsilon^1)$$
$$v \mapsto (b,\phi_v)$$

Restriction of h on fibers is a linear isomorphism. Let $\{(U_\alpha,h_\alpha)\}_{\alpha\in\Lambda}$ be a local trivialization of $\pi\colon E \to B$. Since $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R})$ is isomorphic to $\mathbb{R}^n$, we can also give quotient topology on $\pi_1^{-1}(U_\alpha)$ using the map $q\colon U_\alpha \times \mathbb{R}^n \to \pi_1^{-1}(U_\alpha)$ given by $q(b,v) = (b,\phi_{h_\alpha(v)})$. In the topology defined on $\mathrm{Hom}(E,\varepsilon^1)$, $\pi_1\colon \mathrm{Hom}(E,\varepsilon^1) \to B$ is a vector bundle and h a is continuous map. It follows from lemma 2.1.1 that h is a vector bundle isomorphism. □

Exercise 7. *Let A and B be smooth manifolds of dimensions m and n respectively. If $f\colon A \to B$ is a submersion and $K_f = \bigsqcup_{x\in A} \mathrm{kernel}(Df_x)$, then $\pi\colon K \to A$ given by $\pi(e) = x$ if $x \in \mathrm{kernel}(Df_x)$, is an $(m-n)$-dimensional vector bundle.*

Solution. Since $K_f \subset TA$, K has the subspace topology of TA. Using Implicit function theorem, we will get coordinate charts $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in\Lambda}$ and $\{(V_i,\psi_i)\}_{i\in I}$ of A and B respectively such that the composition $\phi_\alpha(U_\alpha) \xrightarrow{\phi_\alpha^{-1}} U_\alpha \xrightarrow{f} V_i \xrightarrow{\psi_i} \psi_i(V_i)$ is given by $\psi_i \circ f \circ \phi_\alpha^{-1}(x_1,\dots,x_n,x_{n+1},\dots,x_m) = (x_1,\dots,x_n)$ for some α and i. Let $g = \psi_i \circ f \circ \phi_\alpha^{-1}$. Then

$Dg_{\phi_\alpha(x)} = (D\psi_i)_{f(x)} Df_x (D\phi_\alpha^{-1})_{\phi_\alpha(x)} = \begin{bmatrix} I_{n\times n} & 0_{(m-n)\times n} \end{bmatrix}$ for each $x \in U_\alpha$, where $I_{n\times n}$ and $0_{(m-n)\times n}$ are the $n \times n$ identity matrix and $(m-n) \times n$ zero matrix respectively. Then $\ker(Dg_{\phi_\alpha(x)}) = \{(0, \ldots, 0, r_{n+1}, \ldots, r_m) \in \mathbb{R}^m\} \cong \mathbb{R}^{(m-n)}$ for each $x \in U_\alpha$. The map

$$\begin{aligned} q\colon U_\alpha \times \mathbb{R}^{(m-n)} &\to \bigsqcup_{x\in U_\alpha} \text{kernel}(Dg_{\phi_\alpha(x)}) \\ (x, (r_{n+1}, \ldots, r_m)) &\to (0, \ldots, 0, r_{n+1}, \ldots, r_m) \end{aligned}$$

is a homeomorphism. Define

$$\begin{aligned} h_\alpha\colon U_\alpha \times \mathbb{R}^{(m-n)} &\to \pi^{-1}(U_\alpha) \\ (x, v) &\mapsto (D\phi_\alpha^{-1})_{\phi_\alpha(x)}(q(x, v)) \end{aligned}$$

and

$$\begin{aligned} h_\alpha^{-1}\colon \pi^{-1}(U_\alpha) &\to U_\alpha \times \mathbb{R}^n \\ e &\mapsto q^{-1}((D\phi_\alpha)_x(e)) \end{aligned}$$

if $e \in \text{kernel}(Dfx)$. h and h^{-1} are well defined because $Df_x = (D\psi_i^{-1})_{f(x)} Dg_{\phi_\alpha(x)} (D\phi_\alpha)_x$ and $Dg_{\phi_\alpha(x)} = (D\psi_i)_{f(x)} Df_x (D\phi_\alpha^{-1})_{\phi_\alpha(x)}$. h and h^{-1} are continuous because h and h^{-1} are composition of continuous functions. Restriction of h_α over $\{x\} \times \mathbb{R}^{(m-n)}$ is a linear isomorphism with $\pi^{-1}(x)$ because $\text{kernel}(Df_x) \cong \{x\} \times \mathbb{R}^{(m-n)}$. Therefore $\pi\colon K_f \to A$ is a vector bundle with a local trivialization $\{(U_\alpha, h_\alpha)\}_{\alpha\in\Lambda}$. □

Chapter 3

Singular homology theory

3.1 Singular theory

Take $e_0 = (0, \ldots, 0, \ldots), e_1 = (1, 0, \ldots, 0, \ldots), \ldots, e_q = (0, \ldots, 1, 0, \ldots, 0, \ldots)$ i.e. for $q > 0$, 1 is at q^{th} place and all other entries are 0.

Definition 3.1.1. *The* ***standard*** n***-simplex*** *is defined as the set* $\triangle_n = \left\{ \sum_{i=0}^{n} a_i e_i \mid \ a_i \geq 0 \ \forall i, \ \sum_{i=0}^{n} a_i = 1 \right\}$.

Definition 3.1.2. *For any topological space X, a continuous map $\sigma \colon \triangle_n \to X$ is defined as a* ***singular*** n***-simplex****.*

For $n > 0$, define

$$F_n^j \colon \triangle_{n-1} \to \triangle_n$$
$$\sum_0^{n-1} a_i e_i \mapsto \sum_0^{n-1} a_i f(e_i)$$

where $f(e_i) = e_i$, $0 \leq i \leq j-1$ and $f(e_i) = e_{i+1}$, $j \leq i \leq n-1$

Definition 3.1.3. *Let X be a topological space and σ be a singular n-simplex in X. The* ***ith-face*** *of σ is defined as $\sigma^{(i)} = \sigma \circ \ F_n^j$.*

It means that $\sigma^{(i)}$ is a singular (n-1)-simplex.

For a commutative ring R with unity, we will denote the free R-module generated by the set of all singular n-simplexes in X by $S_n(X)$.

Definition 3.1.4. *An element of $S_n(X)$ is known as a* ***singular n-chain****.*

Definition 3.1.5. *For $n > 0$, the* ***boundary*** *of a singular n-simplex σ, is defined as $\partial(\sigma) = \sum_{i=0}^{n}(-1)^i\sigma^{(i)}$. For a singular 0-simplex σ, define $\partial(\sigma) = 0$.*

We can also define the boundary of a singular n-chain, $c = \sum_{j=1}^{m} a_j\sigma_j$ by $\partial(\sum_{j=1}^{m} a_j\sigma_j) = \sum_{j=1}^{m} a_j\partial(\sigma_j)$. So, we get a homomorphism

$$\partial_n \colon S_n(X) \to S_{n-1}(X)$$
$$\sum_{j=1}^{m} a_j\sigma_j \mapsto \sum_{j=1}^{m} a_j\partial(\sigma_j)$$

We have a sequence of homomorphisms $\ldots\ S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X)\ \ldots$

Proposition 3.1.1. $\partial_n\partial_{n+1} = 0$

Proof. See proposition 9.2 of [1] □

From above proposition, we will get image$(\partial_{n+1}) \subset$ kernel(∂_n).

Definition 3.1.6. *$Z_n(X) = kernel(\partial_n)$ and $B_n(X) = image(\partial n + 1)$.*

Definition 3.1.7. *An element of $Z_n(X)$ is called n-**cycle** and an element of $B_n(X)$ is called n-**boundary**.*

Since $B_n(X) \subset Z_n(X)$, we can define quotient module $H_n(X) = Z_n(X)/B_n(X)$.

Definition 3.1.8. *$H_n(X)$ is defined as the n^{th}* ***singular homology module*** *of X.*

Example 7. *For a single point x, $H_n(x) = 0$ for all $n > 0$ and $H_0(x) \cong R$. There is a unique singular n-simplex for all $n \geq 0$. Therefore $S_n(x) \cong R$ for all $n \geq 0$. Let x_m denotes the singular m-simplex for all $m \geq 0$. If q is even, $\partial_q(x_q) = x_{q-1} \neq 0$. This implies that $Z_q(x) = 0$. Therefore $H_q(x) = 0$. If n is odd, then $\partial_n(x_n) = 0$. This implies that $Z_n(x) = S_n(x)$. Since $n+1$ is even, we have $\partial_{n+1}(x_{n+1}) = x_n$. This implies that $B_n(x) = S_n(x)$. Therefore $H_n(x) = 0$. Since the boundary of a 0-chain is defined to be 0, $Z_0(x) = S_0(x)$. $\partial_1(x_1) = 0$ implies that $B_0(x) = 0$. Therefore $H_0(x) \cong S_0(x) \cong R$.*

Proposition 3.1.2. *$H_n(X) \cong \oplus_k H_n(X_k)$ where (X_k) is the family of path connected components of X.*

Proof. See proposition 9.5 of [1]. □

Proposition 3.1.3. *If X is path connected, then $H_0(X) \cong R$.*

Proof. See proposition 9.6 of [1]. □

Given a continuous map $f\colon X \to Y$ between two topological spaces X and Y, we get a homomorphism

$$\begin{aligned} S_n(f)\colon S_n(X) &\to S_n(Y) \\ \sum_{j=1}^{m} a_j\sigma_j &\mapsto \sum_{j=1}^{m} a_j f \circ \sigma_j \end{aligned}$$

If $g\colon Y \to Z$ is a map, then $S_n(fg) = S_n(f)S_n(g)$. Since $(f \circ \sigma) \circ F_n^j = f \circ (\sigma \circ F_n^j)$, we will get that $\partial_n S_n(f) = S_{n-1}(f)\partial_n$. If $c \in Z_n(X)$, then $\partial_n S_n(f)(c) = S_{n-1}(f)\partial_n(c) = 0$. This implies that $S_n(f)(c) \in Z_n(Y)$. Therefore we will get a homomorphism

$$\begin{aligned} H_n(f)\colon H_n(X) &\to H_n(Y) \\ \bar{c} &\mapsto \overline{S_n(f)c} \end{aligned}$$

3.2 Chain complexes

Definition 3.2.1. *A **chain complex** over R is a sequence $M = \{M_n, d_n\}$ where $\{M_n\}$ is a sequence of free R-modules and $\{d_n\colon M_n \to M_{n-1}\}$ is a sequence of homomorphisms with*

$d_{n-1}d_n = 0$.

Example 8. *For a topological space X, the sequence $S = \{S_n(X), \partial_n\}$ is a chain complex.*

Define $Z_n(M) = kernel(d_n)$ and $B_n(M) = image(d_{n+1})$. $d_n d_{n+1} = 0$ implies that $B_n(M)$ is a submodule of $Z_n(M)$. Therefore we can define $H_n(M) = Z_n(M)/B_n(M)$.

Definition 3.2.2. *$H_n(M)$ is called n^{th}* ***homology module*** *of M.*

Definition 3.2.3. *A* ***chain map*** *is a sequence $h = \{h_n\}$ where $\{h_n : M_n \to M'_n\}$ is a sequence of homomorphisms between chain complexes $M = \{M_n, d_n\}$ and $M' = \{M'_n, d'_n\}$ with $d'_n h_n = h_{n-1} d_n$.*

Example 9. *If $f : X \to Y$ is a continuous map between topological spaces X and Y, then the sequence $S(f) = \{S_n(f)\}$ is a chain map.*

Since $d'_n h_n = h_{n-1} d_n$, h_n sends $Z_n(M)$ into $Z_n(M')$ and $B_n(M)$ into $B_n(M')$. Therefore we get a homomorphism

$$\begin{aligned} H_n(h) : H_n(M) &\to H_n(M) \\ \overline{m} &\mapsto \overline{h_n(m)} \end{aligned}$$

Definition 3.2.4. *Two chain maps $\{f_n : M_n \to M'_n\}$ and $\{g_n : M_n \to M'_n\}$ are said to be* ***chain homotopic*** *if there exists a sequence of homomorphisms $\{D_n : M_n \to M'_{n+1}\}$ with $d'_{n+1} D_n + D_{n-1} d_n = f_n - g_n$.*

Proposition 3.2.1. *If two chain maps $f = \{f_n\}$ and $g = \{g_n\}$ are chain homotopic, then $H_n(f) = H_n(g)$ for all $n \geq 0$.*

Proof. See proposition 10.6 of [1]. □

Theorem 3.2.2. *For a topological space X, the two chain maps $S(i_0)$ and $S(i_1)$ are chain homotopic where i_0 and i_1 is given by*

$$\begin{aligned} i_0 : X &\to X \times I \\ x &\mapsto (x, 0) \end{aligned}$$

and

$$i_1 \colon X \to X \times I$$
$$x \mapsto (x, 1)$$

Proof. See proposition 11.4 of [1]. $\square$

Theorem 3.2.3. *If f and g are homotopic maps between topological spaces X and Y, then $S(f)$ and $S(g)$ are chain homotopic.*

Proof. Since f and g are homotopic maps, there is a homotopy $H \colon X \times I \to Y$ between f and g. We have $f = H \circ i_0$ and $g = H \circ i_1$ where i_0 and i_1 are the same maps defined in previous theorem. From previous theorem, we get a chain homotopy $\{D_n\}$ between $S(i_0)$ and $S(i_1)$. Define $D'_n = S_{n+1}(H)D_n$. Then $d'_{n+1}D'_n + D'_{n-1}d_n = S_n(H)(d'_{n+1}D_n + D_{n-1}d_n) = S_n(H)(S_n(i_0) - S_n(i_1)) = S_n(H \circ i_0) - S_n(H \circ i_1) = S_n(f) - S_n(g)$. Therefore the sequence $\{D'_n\}$ is a chain homotopy between $S(f)$ and $S(g)$. $\square$

Definition 3.2.5. *A topological space X is* ***aspherical*** *if every continuous map $f \colon S^n \to X$ can be extended to $F \colon E^{n+1} \to X$ for all $n \geq 0$. S^n is the unit sphere in $\mathbb{R}^{n+1}$ and E^{n+1} is the unit ball in $\mathbb{R}^{n+1}$.*

If X is aspherical, then X is path connected. We have $S^0 = \{-1, 1\}$ and $E^1 = [-1, 1]$. For $x, y \in X$, define

$$f \colon S^0 \to X$$
$$-1 \mapsto x$$
$$1 \mapsto y$$

Then f is continuous and therefore it can be extended to continuous $F \colon [-1, 1] \to X$ with $F(-1) = x$ and $F(1) = y$.

Example 10. *A convex subset of $\mathbb{R}^{n+1}$ is aspherical. A contractible space is also aspherical.*

Theorem 3.2.4. *If X is aspherical, then $H_n(X) = 0$ for all $n > 0$ and $H_0(X) \cong R$.*

Proof. See theorem 10.13 of [1]. $\square$

Theorem 3.2.5. *If X is path connected, then $H_1(X, \mathbb{Z})$ is the Abelianization of $\pi_1(X)$.*

Proof. See theorem 12.1 of [1]. □

3.3 Relative homology

Let X be a topological space and A be a subspace of X. We see that $S_q(A)$ is a submodule of $S_q(X)$ $\forall q \geq 0$. We get a chain complex $\{C_q = \ S_q(X)/S_q(A), \ \bar{\partial}_q\}$ where

$$\begin{aligned} \bar{\partial}_q \colon \ S_q(X)/S_q(A) \to \ & S_{q-1}(X)/S_{q-1}(A) \\ \bar{z} \to \ & \partial_q z \ mod \ S_{q-1(A)} \end{aligned}$$

Definition 3.3.1. *q^{th} **relative homology module** of X mod A, $H_q(X, A)$ is defined as $kernel(\bar{\partial}_q)/\ image(\bar{\partial}_{q+1})$.*

If $\partial_q c \in S_{q-1}(A)$ for c $\in S_q(X)$, then $\bar{c} \in$ kernel$(\bar{\partial}_q)$. Define $Z_q(X, A) = \{c \in S_q(X) \mid \partial_q c \ \in \ S_{q-1}(A)\}$. Elements of $Z_q(X)$ are called *relative q-cycles on X mod A*. Define $B_q(x, A) = \{c \in S_q(X) \mid c - c_a = \partial_{q+1}(z)$ for some $c_a \in S_q(A)$ and $z \in S_{q+1}(X)\}$. An element of $B_q(X, A)$ is called *relative q-boundary on X mod A*.

Lemma 3.3.1. $H_q(X, A) \cong Z_q(X, A)/B_q(X, A)$

Proof. $Kernel(\bar{\partial}_q) = Z_q(X, A)/S_q(A)$ and $Image(\bar{\partial}_q) = B_q(X, A)/S_q(A)$. By the third isomorphism theorem, $H_q(X, A) \cong Z_q(X, A)/B_q(X, A)$.

□

Proposition 3.3.2. *If X is path connected and A is nonempty subset of X, then $H_0(X, A) = 0$.*

Proof. If $c = \sum v_x x \in S_0(X)$, then $\partial_1(\sum v_x \sigma_x) = c - \sum v_x x_{0.}$ for $x_0 \in A$ and σ_x is a path joining x and x_0. Therefore $c \in B_0(X, A)$. □

Let $A \subset X$ and $A' \subset X'$. We will denote a continuous map $f\colon X \to X'$ with $f(A) \subset A'$ by a map $f\colon (X, A) \to (X', A')$. Given a map $f\colon (X, A) \to (X', A')$, the chain map $S_q(f)\colon S_q(X) \to S_q(X')$ takes $Z_q(X, A)$ to $Z_q(X', A')$ and $B_q(X, A)$ to $B_q(X', A')$. Therefore we will get a homomorphism $H_q(f)\colon H_q(X, A) \to H_q(X', A')$.

3.4 The exact homology sequence

Let A be a subspace of a topological space X, $i\colon A \to X$ be the inclusion map and $i_X\colon X \to X$ be the identity map.

Corollary 3.4.1. $\bar{\partial}_q\colon H_q(X, A) \to H_{q-1}(A)$ *is a homomorphism.*

Proof. If $\bar{z} \in H_q(X, A)$, then $z \in Z_q(X, A)$. From definition of $Z_q(X, A)$, $\partial_q z \in S_{q-1}(A)$. $\partial_{q-1}\partial_q = 0$ implies $\partial_q z \in Z_q(A)$ and $\bar{\partial}_q \bar{z} \in H_{q-1}(A)$. If $\bar{z}_1 = \bar{z}_2$, then $\bar{z}_1 - \bar{z}_2 = 0$. We have $z_1 - z_2 \in B_q(X, A)$. From definition of $B_q(X, A)$, $z_1 - z_2 = c_a + \partial_{q+1}c$ for some $c_a \in S_q(A)$ and $c \in S_{q+1}(X)$. $\partial_q(z_1 - z_2) = \partial_q c_a \in B_q(A)$ implies $\bar{\partial}_q \bar{z}_1 = \bar{\partial}_q \bar{z}_2$. Therefore $\bar{\partial}_q$ is well defined and $\bar{\partial}_q$ is a homomorphism because ∂_q is a homomorphism. □

We get an infinite sequence of homomorphisms

$$\cdots \longrightarrow H_q(A) \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(i_X)} H_q(X, A) \xrightarrow{\bar{\partial}_q} H_{q-1}(A) \longrightarrow \cdots$$

Theorem 3.4.2.

$$\cdots \longrightarrow H_q(A) \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(i_X)} H_q(X, A) \xrightarrow{\bar{\partial}_q} H_{q-1}(A) \longrightarrow \cdots$$

is an exact sequence.

Proof. Since the composition $H_q(i_X)H_q(i) = H_q(i_X i)\colon H_q(A) \to H_q(X, A)$ is induced by the inclusion map and $Z_q(A) \subset S_q(A) \subset B_q(X, A)$, $H_q(i_X i)$ is the zero homomorphism. It gives $image(H_q(i)) \subset kernel(H_q(i_X))$. For $\bar{z} \in kernel(H_q(i_X))$, $z \in Z_q(X)$ and $z \in B_q(X, A)$. We have $z = c_a + \partial_{q+1}c$ for some $c_a \in S_q(A)$ and $c \in S_{q+1}(X)$. Since $\partial_{q+1}c \in B_q(X)$ and $\partial_q z = 0$, $\bar{z}$ is the image of $\bar{c}_a$. We have $\bar{z} \in image(H_q(i))$. Therefore $kernel(H_q(i_X)) \subset image(H_q(i))$. It implies that $image(H_q(i)) = kernel(H_q(i_X))$. The sequence is exact at $H_q(X)$.

For $\bar{\partial}_q H_q(i_X)\colon H_q(X) \to H_{q-1}(A)$, $\partial_q z = 0$ for all $\bar{z} \in H_q(X)$. Therefore $\bar{\partial}_q H_q(i_X) = 0$. It gives $image(H_q(i_X)) \subset kernel(\bar{\partial}_q)$. If $\bar{z} \in kernel(\bar{\partial}_q)$, then $z \in Z_q(X, A)$ and $\partial_q z \in B_{q-1}(A)$. Therefore $\partial_q z = \partial_q c_a$ for some $c_a \in S_q(A)$. Since $\partial_q(z - c_a) = 0$, $z - c_a \in Z_q(X)$. $c_a \in S_q(A)$ implies $c_a \in B_q(X, A)$. Therefore $\bar{\partial}_q H_q(i_X)\bar{c}_a = 0$. It implies that $\bar{z}$ is the image of $\bar{z} - \bar{c}_a$ under the map $H_q(i_X)$. It gives $kernel(\bar{\partial}_q) \subset image(H_q(i_X))$. Therefore $kernel(\bar{\partial}_q) = image(H_q(i_X))$. It is exact at $H_q(X, A)$.

For $H_{q-1}(i)\bar{\partial}_q\colon H_q(X, A) \to H_{q-1}(X)$, it is the zero homomorphism because ∂_q takes elements of $S_q(X)$ to $B_{q-1}(X)$. We have $image(\bar{\partial}_q) \subset kernel(H_{q-1}(i))$. If $\bar{z} \in kernel(H_{q-1}(i))$, then $z \in Z_{q-1}(A)$ and $z \in B_{q-1}(X)$. Therefore $z = \partial_q c$ for some $c \in S_q(X)$. $\bar{z}$ is the image of $\bar{c}$ under the map $\bar{\partial}_q$. It gives $kernel(H_{q-1}(i)) \subset image(\bar{\partial}_q)$. Therefore $image(\bar{\partial}_q) = kernel(H_{q-1}(i))$. It is also exact at $H_{q-1}(A)$. Hence the sequence of homomorphisms is exact.

□

***Five lemma* 3.4.3.** *The diagram given below is a diagram of R-modules and homomorphisms with all rectangles commutative.*

$$\begin{array}{ccccccccc}
M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & M_4 & \xrightarrow{f_4} & M_5 \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \varepsilon} \\
N_1 & \xrightarrow[h_1]{} & N_2 & \xrightarrow[h_2]{} & N_3 & \xrightarrow[h_3]{} & N_4 & \xrightarrow[h_4]{} & N_5
\end{array}$$

If the rows are exact at joints 2, 3, 4 and α, β, δ, ε are isomorphism, then γ is an isomorphism.

Proof. We will show that γ is injective. Take $a \in kernel(\gamma)$. Then $\gamma(a) = 0$. Since rectangles are commutative, $\delta f_3(a) = h_3\gamma(a) = 0$. Since δ is injective, $f_3(a) = 0$. Therefore $a \in kernel(f_3) = image(f_2)$. We have $a = f_2(b)$ for some $b \in M_2$. Now $h_2\beta(b) = \gamma f_2(b) = \gamma(a) = 0$ implies that $\beta(b) \in kernel(h_2) = image(h_1)$. We have $\beta(b) = h_1(c)$ for some $c \in N_1$. Since α is surjectve, $c = \alpha(a')$ for some $a' \in M_1$. Now we have $\beta(b) = h_1(c) = h_1\alpha(a') = \beta f_1(a')$. Therefore $\beta(b - f_1(a')) = 0$. β is injective implies that $b - f_1(a') = 0$. $f_2(b) = a$, $f_2 f_1 = 0$ and $f_2(b - f_1(a')) = 0$, implies that $a = 0$. Therefore $kernel(\gamma) = 0$.

Now we will show that γ is surjective. Take $m \in N_3$. $h_3(m) \in N_4$ and δ is surjective implies that $h_3(m) = \delta(m')$ for some $m' \in M_4$. We have $0 = h_4h_3(m) = h_4\delta(m') = \varepsilon f_4(m')$.

Since ε is injective, $f_4(m') = 0$. Then $m' \in kernel(f_4) = image(f_3)$. Therefore $m' = f_3(m'')$ for some $m'' \in M_3$. Applying δ to previous equation, $\delta(m') = \delta f_3(m'')$. $h_3(m) = \delta(m')$ and $\delta f_3 = h_3\gamma$ implies that $h_3(m) = h_3\gamma(m'')$. Since $m - \gamma(m'') \in kernel(h_3) = image(h_2)$, $m - \gamma(m'') = h_2(m''')$ for some $m''' \in N_2$. Since β is surjective, $m''' = \beta(u)$ for some $u \in M_2$. Therefore $m - \gamma(m'') = h_2(m''') = h_2\beta(u) = \gamma f_2(u)$. We have $m = \gamma(m'' - f_2(u))$ where $m'' - f_2(u) \in M_3$. Therefore γ is surjective. □

Definition 3.4.1. *A* ***short exact sequence*** *is an exact sequence of R-modules of the form* $0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{j} M_3 \longrightarrow 0$.

Proposition 3.4.4. *If* $0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{j} M_3 \longrightarrow 0$ *is a short exact sequence, then the following statements are equivalent:*

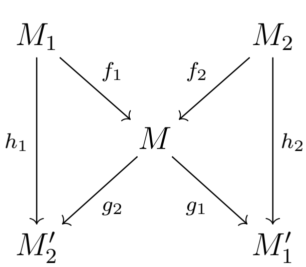
1. *There is a homomorphism* $p\colon M_2 \to M_1$ *such that* $pi = id_{M_1}$.

2. *There is a homomorphism* $q\colon M_3 \to M_2$ *such that* $jq = id_{M_3}$.

Proof. See proposition 14.11 of [1]. □

Definition 3.4.2. *A short exact sequence* $0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{j} M_3 \longrightarrow 0$ *is* ***split*** *if it satisfies either statement 1 or statement 2 of the previous proposition.*

Direct sum lemma **3.4.5.** *Given below is a diagram of R-modules. All triangles are commutative with* $kernel(f_t) = image(g_t)$ *and* h_t *is an isomorphism for* $t = 1, 2$.



Then the compositions

$$M_1 \oplus M_2 \xrightarrow{f_1 \oplus f_2} M \oplus M \xrightarrow{\phi} M$$

$$M \xrightarrow{\psi} M \oplus M \xrightarrow{g_1 \oplus g_2} M'_1 \oplus M'_2$$

are isomorphisms where $\phi(m, m') = m + m'$ *and* $\psi(m) = (m, m)$.

Proof. If $m_1 \in kernel(f_1)$, then $h_1(m) = g_2 f_1(m_1) = 0$. Since h_1 is an isomorphism, $m = 0$. This implies that $kernel(f_1) = \{0\}$. Therefore f_1 is injective. For $m'_2 \in M'_2$, there is a $m_1 \in M_1$ such that $m'_2 = h_1(m_1) = g_2 f_1(m_1)$. Therefore g_2 is surjective. Similarly, f_2 is injective and g_1 is surjective.

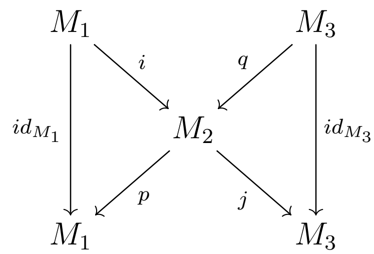
If $(m_1, m_2) \in kernel(\phi(f_1 \oplus f_2))$, then $\phi(f_1 \oplus f_2)(m_1, m_2) = f_1(m_1) + f_2(m_2) = 0$. Applying g_2 to the previous equation, $g_2 f_1(m_1) + g_2 f_2(m_2) = 0$. Since $g_2 f_2 = 0$ and $h_1 = g_2 f_1$, we have $h_1(m_1) = 0$. h_1 is an isomorphism implies that $m_1 = 0$. After applying g_1 to the same equation to which we applied g_2, we will get $m_2 = 0$. Therefore $kernel(\phi(f_1 \oplus f_2)) = \{(0,0)\}$. For $m \in M$, $g_2(m) \in M'_2$. Since h_1 is surjective, $g_2(m) = h_1(m_1) = g_2 f_1(m_1)$ for some $m_1 \in M_1$. Therefore $m - f_1(m_1) \in kernel(g_2) = image(f_2)$. Since $m - f_1(m_1) \in image(f_2)$, $m - f_1(m_1) = f_2(m_2)$ for some $m_2 \in M_2$. Therefore $m = f_1(m_1) + f_2(m_2) =$
$\phi(f_1 \oplus f_2)(m_1, m_2)$. This implies that $\phi(f_1 \oplus f_2)$ is surjective. We showed that the first composition is an isomorphism.

For $m \in kernel((g_1 \oplus g_2)\psi)$, $(g_1(m), g_2(m) = (0,0)$. This implies that $g_1(m) = 0$ and $g_2(m) = 0$. Since $kernel(g_1) = image(f_1)$, $m = f_1(m_1)$ for some $m_1 \in M_1$. We have $0 = g_2(m) = g_2 f_1(m_1) = h_1(m_1)$. Since h_1 is an isomorphism, $m_1 = 0$ and therefore $m = f_1(m_1) = 0$. We have $kernel((g_1 \oplus g_2)\psi) = \{0\}$. Take $(m'_1, m'_2) \in M'_1 \oplus M'_2$. Since $m'_1 \in M'_1$ and g_1 is surjective, $m'_1 = g_1(m')$ for some $m' \in M$. $kernel(g_1) = image(f_1)$ implies that $m'_1 = g_1(m' + f_1(m_1))$ for all $m_1 \in M_1$. Applying g_2 to $m' + f_1(m_1)$, we will get $g_2(m') + g_2 f_1(m_1) = g_2(m') + h_1(m_1)$. Since h_1 is surjective, there is $n_1 \in M_1$ such that $m'_2 = g_2(m') + h_1(n_1)$. Therefore we can write $n_1 = h_1^{-1}(m'_2) - h_1^{-1} g_2(m')$. For $m = m' + f_1(n_1)$, $f_1(m) = m'_1$ and $f_2(m) = m'_2$. We have $(g_1 \oplus g_2)\psi(m) = (m'_1, m'_2)$. Therefore $(g_1 \oplus g_2)\psi(m)$ is surjective. We showed that the second composition is also an isomorphism.

□

Example 11. *Given a split short exact sequence* $0 \longrightarrow M_1 \stackrel{i}{\longrightarrow} M_2 \stackrel{j}{\longrightarrow} M_3 \longrightarrow 0$. *If* $p \colon M_2 \to M_1$ *with* $pi = id_{M_1}$ *is given, then we can construct* $q \colon M_3 \to M_2$ *with* $jq = id_{M_3}$. *From the proof of proposition 3.4.4,* q *is defined as* $q(m_3) = m_2 - ip(m_2)$ *where* $m_3 = j(m_2)$ *for some* $m_2 \in M_2$. *When we apply* p *to* $q(m_3)$, *we will get* $pq(m_3) = p(m_2) - pip(m_2)$. *Since* $pi = id_{M_2}$, *we will get* $pq(m_3) = 0$. *This implies that* $image(q) \subset kernel(p)$. *Take* $m \in kernel(p)$. *Then* $qj(m) = m - ip(m) = m$ *implies that* $m \in image(q)$. *Therefore* $kernel(p) \subset image(q)$. *We have* $kernel(p) = image(q)$. *Similarly given* $q \colon M_3 \to M_2$ *with*

$jq = id_{M3}$, we can construct $p\colon M_2 \to M_3$ with $pi = id_{m_1}$ and $kernel(p) = image(q)$. Therefore we get a diagram satisfying previous proposition.



We have $M_2 \cong M_1 \oplus M_3$ for a split short exact sequence $0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{j} M_3 \longrightarrow 0$.

Proposition 3.4.6. *If A is a retract of X, then $H_n(X) \cong H_n(A) \oplus H_n(X, A)$.*

Proof. We have $ri = id_A$ where i is the inclusion map of A and r is a retraction map. $H_n(r)H_n(i) = H_n(id_A)$ implies that $H_n(i)$ is injective. Therefore the exact sequence

$$\cdots \longrightarrow H_{n+1}(X, A) \xrightarrow{\overline{\partial}_{n+1}} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(i_X)} H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow \cdots$$

gives a split short exact sequence

$$0 \longrightarrow H_n(A) \underset{H_n(r)}{\overset{H_n(i)}{\rightleftarrows}} H_n(X) \xrightarrow{H_n(i_X)} H_n(X, A) \longrightarrow 0$$

for all $n \geq 0$. Using the previous example, we get $H_n(X) = H_n(A) \oplus H_n(X, A)$. □

3.5 The excision theorem

Let $B \subset A \subset X$. We say that U can be excised if the inclusion map $i\colon (X \backslash B, A \backslash B) \to (X, A)$ induces an isomorphism $H_n(i)\colon H_n(X \backslash B, A \backslash B) \to H_n(X, A)$ for all $n \geq 0$.

Theorem 3.5.1. *If the closure of B is contained in the interior A, then A can be excised.*

Proof. See theorem 15.1 of [1]. □

Theorem 3.5.2. *Let $U \subset B \subset A$. If U can be excised and $(X\backslash B, A\backslash B)$ is deformation retract of $(X\backslash U, A\backslash U)$, then B can be excised.*

Proof. See theorem 15.2 of [1]. □

Let $E_n^+ = \{x \in S^n \mid x_{n+1} \geq 0\}$ and $E_n^- = \{x \in S^n \mid x_{n+1} \leq 0\}$.

Theorem 3.5.3. *If $U = \{x \in S^n \mid x_{n+1} < 0\}$, then U can be excised from (S^n, E_n^-) for all $n \geq 1$.*

Proof. See theorem 15.3 of [1]. □

Corollary 3.5.4. *For $n \geq 1$, $H_q(S^n) \cong H_{q-1}(S^{n-1})$ for all $q \geq 2$.*

Proof. From the previous theorem, we have $H_q(E_n^+, S^{n-1}) \cong H_q(S^n, E_n^-)$ for all $q \geq 0$. Since E_n^- is contractible, $H_q(E_n^-) = 0$ for all $q \geq 1$. We get a exact sequence $0 \longrightarrow H_q(S^n) \xrightarrow{H_q(i_n)} H_q(S^n, E_n^-) \longrightarrow 0$ for all $q \geq 2$. Therefore $H_q(S^n) \cong H_q(S^n, E_n^-)$ for all $q \geq 2$. Since the unit ball E^n is a convex set, $H_q(E^n) = 0$ for all $q \geq 1$. The exact sequence $0 \longrightarrow H_q(E^n, S^{n-1}) \xrightarrow{\overline{\partial}_n} H_{q-1}(S^{n-1}) \longrightarrow 0$ gives that $H_q(E^n, S^{n-1}) \cong H_{q-1}(S^{n-1})$ for all $q \geq 2$. (E_n^+, S^{n-1}) is homeomorphic to (E^n, S^{n-1}) implies that $H_q(E_n^+, S^{n-1}) \cong H_q(E^n, S^{n-1})$ for all $q \geq 0$. Therefore we get $H_q(S^n) \cong H_q(S^n, E_n^-) \cong H_q(E_n^+, S^{n-1}) \cong H_q(E^n, S^{n-1}) \cong H_{q-1}(S^{n-1})$ for all $q \geq 2$. □

For $q = 1$ and $n \geq 1$, we have $0 \longrightarrow H_1(E^n, S^{n-1}) \xrightarrow{\overline{\partial}_n} H_0(S^{n-1}) \xrightarrow{H_0(i)} H_0(E^n) \longrightarrow 0 \cdot$ For $n > 1$, S^{n-1} and E^n are path connected. Therefore $H_0(S^{n-1}) \cong R$, $H_0(E^n) \cong R$ and $H_0(i)$ is an isomorphism. We get $H_1(E^n, S^{n-1}) \cong Kernel(H_0(i)) = 0$. For $n = 1$, S^0 has two path components. Therefore $H_0(S^0) \cong R \oplus R$. We get $H_1(E^1, S^0) \cong kernel(H_0(i)) \cong R$.

$$H_1(E^n, S^{n-1}) \cong \begin{cases} 0 & n > 1 \\ R & n = 1 \end{cases}$$

We have $H_q(S^n, E_n^-) \cong H_q(E_n^+, S^{n-1})$ and $H_q(E_{n-1}^+, S^{n-1}) \cong H_q(E^n, S^{n-1})$ for all $q \geq 0$. This implies that $H_q(S^n, E_n^-) \cong H_q(E^n, S^{n-1})$ for all $q \geq 0$. We have the exact sequence

$0 \longrightarrow H_1(S^n) \xrightarrow{H_1(i_{S^n})} H_1(S^n, E_n^-) \xrightarrow{\overline{\partial}_1} H_0(E_n^-) \xrightarrow{H_0(i)} H_0(S^n) \longrightarrow 0$. $H_0(i)$ is isomorphism implies that $\overline{\partial}_1 = 0$. We get $H_1(S^1) \cong H_1(S^n, E_n^-)$. Therefore

$$H_1(S^1) \cong \begin{cases} 0 & n > 1 \\ R & n = 1 \end{cases}$$

Corollary 3.5.5. *For $q \geq 1$ and $n \geq 1$,*

$$H_q(S^n) \cong \begin{cases} R & q = n \\ 0 & q \neq n \end{cases}$$

Proof. It comes from $H_q(S^n) \cong H_{q-1}(S^{n-1}) \cong \cdots \cong H_1(S^{n-(q-1)})$. □

3.6 Mayer-Vietoris sequence

***Barratt-Whitehead Lemma* 3.6.1.**

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & C_{n+1} & \xrightarrow{h_{n+1}} & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow \cdots \\
& \downarrow \gamma_{n+1} & & \downarrow \alpha_n & & \downarrow \beta_n & & \downarrow \gamma_n & \\
\cdots \longrightarrow & C'_{n+1} & \xrightarrow[h'_{n+1}]{} & A'_n & \xrightarrow[f'_n]{} & B'_n & \xrightarrow[g'_n]{} & C'_n & \longrightarrow \cdots
\end{array}$$

If the rows of the given diagram are long exact sequences of R-modules and γ_n are isomorphisms, then there exists a long exact sequence given by

$$\cdots \longrightarrow A_n \xrightarrow{\phi_n} A'_n \oplus B_n \xrightarrow{\psi_n} B'_n \xrightarrow{\delta_n} A_{n-1} \longrightarrow \cdots$$

where $\phi_n(a) = (\alpha_n \oplus f_n)(a, a)$, $\psi_n(a, b) = -f'_n(a) + \beta_n(b)$ and $\delta_n(b) = h_n \circ \gamma_n^{-1} \circ g'_n(b)$

Proof. Firstly we will show the exactness at A'_n. For $b' \in B'_{n+1}$, $\phi_n \circ \delta_{n+1}(b') = \phi_n \circ h_{n+1} \circ \gamma_{n+1}^{-1} \circ g'_{n+1}(b') = (\alpha_n \circ h_{n+1} \circ \gamma_{n+1}^{-1} \circ g'_{n+1}(b'), f_n \circ h_{n+1} \circ \gamma_{n+1}^{-1} \circ g'_{n+1}(b'))$. Since $\alpha_n \circ h_{n+1} = h'_{n+1} \circ \gamma_{n+1}$ and $f_n \circ h_{n+1} = 0$, we get $\phi_n \circ \delta_{n+1}(b') = (0, 0)$. Therefore image$(\delta_{n+1}) \subset$ kernel(ϕ_n). For $a \in$ kernel(ϕ_n), $\alpha_n(a) = 0$ and $f_n(a) = 0$. Since the rows are exact, there exists $c \in C_{n+1}$ such that $h_{n+1}(c) = a$. Commutativity of the diagram implies that $h'_{n+1} \circ \gamma_{n+1}(c) =$

$\alpha_n \circ h_{n+1}(c) = \alpha_n(a) = 0$. $\gamma_{n+1}(c) \in \text{kernel}(h'_{n+1})$ implies that there exists $b' \in B'_{n+1}$ such that $g_{n+1}(b') = \gamma_{n+1}(c)$. Applying $h_{n+1} \circ \gamma_{n+1}^{-1}$ on both side of the previous equation, we get $h_{n+1} \circ \gamma_{n+1}^{-1} \circ g_{n+1} = \alpha_n(c) = a$. Therefore $a \in \text{image}(\delta_{n+1})$. This implies that $\text{kernel}(\phi_n) \subset \text{image}(\delta_{n+1})$. Therefore $\text{image}(\delta_{n+1}) = \text{kernel}(\phi_n)$.

Now we will show the exactness at $A'_n \oplus B_n$. Since $\psi_n \circ \phi_n(a) = -f'_n \circ \alpha_n(a) + \beta_n f_n(a) = 0$, we get $\text{image}(\phi_n) \subset \text{kernel}(\psi_n)$. For $(a', b) \in \text{kernel}(\psi_n)$, $f'_n(a') = \beta_n(b)$. Applying g'_n on the previous equation, $g'_n \circ f'_n(a') = g'_n \circ \beta_n(b) = \gamma_n \circ g_n(b) = 0$. Since γ_n is an isomorphism, we get $g_n(b) = 0$. Therefore there exists $x \in A_n$ such that $f_n(x) = b$. After applying β_n, we get $\beta_n \circ f_n(x) = f'_n \circ \alpha_n(x) = \beta_n(b) = f'_n(a')$. We get $(a' - \alpha_n(x)) \in \text{kernel}(f'_n)$. Therefore $a' - \alpha_n(x) = h'_{n+1}(c')$ for some $c' \in C'_{n+1}$. Since γ_{n+1} is an isomorphism, $c' = \gamma_{n+1}(c)$ for some $c \in C_{n+1}$. Therefore $a' - \alpha_n(x) = h'_{n+1} \circ \gamma_{n+1}(c) = \alpha_n \circ h_{n+1}(c)$. Then for $a = x - h_{n+1}(c)$, $\phi_n(a) = (a', b)$. This implies that $\text{kernel}(\psi_n) \subset \text{image}(\phi_n)$. Therefore $\text{image}(\phi_n) = \text{kernel}(\psi_n)$.

Now we will show the exactness at B'_n. Since $\delta_n \circ \psi_n(a', b) = -h_n \circ \gamma_n^{-1} \circ g'_n \circ f'_n(a') + h_n \circ \gamma_n^{-1} \circ g'_n \circ \beta_n(b) = 0 + h_n \circ \gamma_n^{-1} \circ \gamma'_n \circ g_n(b) = 0$, we get $\text{image}(\psi_n) \subset \text{kernel}(\delta_n)$. For $b' \in \text{kernel}(\delta_n)$, $\gamma_n^{-1} \circ g'_n(b') \in \text{kernel}(h_n)$. Therefore $\gamma_n^{-1} \circ g'_n(b') = g_n(b)$ for some $b \in B_n$. After applying γ_n, we get $g'_n(b') = \gamma_n \circ g_n(b) = g'_n \circ \beta_n(b)$. Since $\beta_n(b) - b' \in \text{kernel}(g'_n)$, $\beta_n(b) - b' = f'_n(a')$ fro some $a' \in A'_n$. This implies that $b' = -f'_n(a') + \beta_n(b) \in \text{image}(\psi_n)$. Therefore $\text{kernel}(\delta_n) \subset \text{image}(\psi_n)$. □

Let X_1 and X_2 be a subspaces of a topological space X. If the homomorphisms of homology modules induced by the inclusion maps $i_1 \colon (X_2, X_1 \cap X_2) \to (X_1 \cup X_2, X_1)$ and $i_2 \colon (X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2)$ are isomorphisms, then (X_1, X_2, X) is called **exact triad**. If a triple (X_1, X_2, X) is an exact triad, then it means that we can excise $X_1 - X_1 \cap X_2$ from $(X_1 \cup X_2, X_1)$ and $X_2 - X_1 \cap X_2$ from $(X_1 \cup X_2, X_2)$. Let $A = X_1 \cap X_2$ and $Y = X_1 \cup X_2$ We know from the theorem 3.4.2 that the rows of the below diagram are exact,

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H_q(A) & \xrightarrow{H_q(i)} & H_q(X_1) & \xrightarrow{H_q(i_{X_1})} & H_q(X_1, A) & \xrightarrow{\bar{\partial}_q} & H_{q-1}(A) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle H_q(i)} & & \downarrow{\scriptstyle H_q(i)} & & \downarrow{\scriptstyle H_q(i_2)} & & \downarrow{\scriptstyle H_{q-1}(i)} & \\
\cdots \longrightarrow & H_q(X_2) & \xrightarrow{H_q(i)} & H_q(Y) & \xrightarrow{H_q(i_Y)} & H_q(Y, X_2) & \xrightarrow{\bar{\partial}_q} & H_{q-1}(X_2) & \longrightarrow \cdots
\end{array}$$

If (X_1, X_2, X) is an exact triad, then $H_q(i_2)$ is an isomorphism for $q \geq 0$. Therefore we will get an exact sequence using Barrat-Whitehead lemma for a given exact triad.

Chapter 4

Cohomology

Let $M = \{M_n, d_n\}$ be a chain complex over R and G be an R-module. Let M_n^* denote $\mathrm{Hom}(M_n, G)$. M_n^* is known as **chain module** We get a homomorphism

$$\begin{aligned} d_n^* \colon M_{n-1}^* &\to M_n^* \\ f &\mapsto f \circ d_n \end{aligned}$$

d_n^* is known as **coboundary map**. d_n^* is a module homomorphism. $d_{n+1}^* \circ d_n^* = 0$ as $d_n \circ d_{n+1} = 0$. We obtain a sequence $\ldots \longrightarrow M_{n-1} \xrightarrow{d_n^*} M_n \xrightarrow{d_{n+1}^*} M_{n+1} \longrightarrow \ldots$ of chain modules and coboundary maps. We will denote the sequence by $M^* = \{M_n^*, d_n^*\}$.

Definition 4.0.1. *The sequence $M^* = \{M_n^*, d_n^*\}$ is called **cochain complex** of the chain complex $M = \{M_n, d_n\}$.*

Definition 4.0.2. $\mathrm{H}^n(M, G)$ *is defined as* $\mathrm{kernel}(d_n^*)/\,\mathrm{image}(d_n^*)$. $\mathrm{H}^n(B, G)$ *is called the n^th **cohomology module** of M.*

For a topological space X, take $M = \{S_n(X, R), \partial_n\}$. Then M^* is denoted by $S^* = \{S^n(X, G), \partial^n\}$ and $\mathrm{H}^n(M, G)$ is denoted by $\mathrm{H}^n(X, G)$. $\mathrm{H}^n(X, G)$ is called n^{th} cohomology module of X.

4.0.1 Cup product

See chapter 24 of [1].

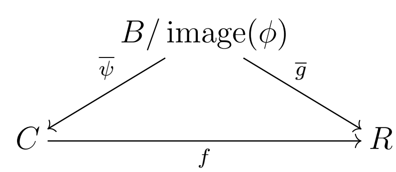
Exercise 8. *If* $A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ *be exact sequence of* R*-modules* A, B, C, *then the dual sequence* $A^* \overset{\phi}{\leftarrow} B^* \overset{\psi}{\leftarrow} C^* \leftarrow 0$ *is also exact.*

Solution. First we will check exactness at C^*. We need to show that $\text{kernel}(\psi^*) = 0$. If $f \in \text{kernel}(\psi^*)$, then $f \circ \psi(b) = 0$ for all $b \in B$. Since ψ is surjective, $f(c) = 0$ for all $c \in C$. This implies that $f = 0$. Therefore, $\text{kernel}(\psi^*) = 0$. We showed that the sequence is exact at C^*. Now we will check exactness at B^*. If $g \in \text{image}(\psi^*)$, then $g = f \circ \psi$ for some $f \in B^*$. Since $\text{kernel}(\psi) = \text{image}(\phi)$, $\phi^*(g) = g \circ \phi = f \circ \psi \circ \phi = 0$. Therefore, $\text{image}(\psi^*) \subset \text{kernel}(\phi^*)$. Finally, we will show that $\text{kernel}(\phi^*) \subset \text{image}(\psi^*)$. For showing this, we will take $g \in \text{kernel}(\phi^*)$ and show that $g = f \circ \psi$ for some $f \in C^*$. Since ψ is surjective and $\text{kernel}(\psi) = \text{image}(\phi)$,

$$\begin{aligned} \overline{\psi} \colon B/\,\text{image}(\phi) &\to C \\ \overline{b} &\mapsto \psi(b) \end{aligned}$$

is an isomorphism. For any $g \in \text{kernel}(\phi^*)$, define

$$\begin{aligned} \overline{g} \colon B/\,\text{image}(\phi) &\to C \\ \overline{b} &\mapsto g(b) \end{aligned}$$



From the above diagram, we got a homomorphism $f = \overline{g} \circ (\overline{\psi})^{-1}$ such that $g = f \circ \psi$ and $f \in C^*$. Therefore $\text{kernel}(\phi^*) \subset \text{image}(\psi^*)$, and hence $\text{kernel}(\phi^*) = \text{image}(\psi^*)$. This implies that the sequence is also exact at B^*. $\square$

Chapter 5

Stiefel-Whitney classes

Let Λ, I and J be index sets. Let $\mathbb{R}$ and $\mathbb{Z}$ denotes the real numbers and ring of integers respectively.

We will first state the four axioms of Stiefel-Whitney classes. Then we will see the consequences and application of the four axioms.

Followings are the four axioms of Stiefel-Whitney classes

Axiom 1 For an n-dimensional vector bundle $\pi\colon E \to B$, there is a sequence of cohomology classes $w_0(\pi), w_1(\pi), \ldots, w_n(\pi), \ldots$ with $w_i(\pi) \in H^i(B, \mathbb{Z}/2\mathbb{Z})$ for $i \geq 0$, $w_0(\pi)$ is the identity element of $H^0(B)$ and $w_k(\pi) = 0$ for $k > n$. The sequence of cohomology classes $w_0(\pi), w_1(\pi), \ldots, w_n(\pi), \ldots$ is called **Stiefel-Whitney classes** of the vector bundle $\pi\colon E \to B$.

Axiom 2 If $f\colon A \to B$ be a map covered by a bundle map from the total space of $\pi'\colon E' \to A$ to the total space of $\pi\colon E \to B$, then $w_i(\pi') = f^* w_i(\pi)$ for $i \geq 0$.

Axiom 3 For vector bundles $\pi_1\colon E_1 \to B$ and $\pi_2\colon E_2 \to B$, $w_k(\pi_1 \oplus \pi_2) = \sum_{i=1}^{k} w_i(\pi_i) \cup w_{k-i}(\pi_2)$ where $w_i(\pi_i) \cup w_{k-i}(\pi_2)$ is the cup product of $w_i(\pi_i)$ and $w_{k-i}(\pi_2)$.

Axiom 4 For the line bundle $\pi_1^1\colon \gamma_1^1 \to \mathbb{R}\mathrm{P}^1$, $w_1(\pi_1^1) \neq 0$.

Proposition 5.0.1. *If vector bundles $\pi_1\colon E_1 \to B$ and $\pi_2\colon E_2 \to B$ are isomorphic, then $w_i(\pi_1) = w_i(\pi_2)$ for $i \geq 0$.*

Proof. Let $h\colon E_1 \to E_2$ be a vector bundle isomorphism. Then the identity map $i_A\colon A \to A$ is covered by h. Therefore $w_i(\pi_1) = i_A^* w_i(\pi_2) = w_i(\pi_2)$ for $i \geq 0$. □

Proposition 5.0.2. *If $\pi\colon E \to B$ is an n-dimensional trivial vector bundle, then $w_i(\pi) = 0$ for $i > 0$.*

Proof. Let $b \in B$. Define a map $h\colon E \to \{b\} \times \mathbb{R}^n$ by $h(x, v) = (b, v)$. Then h is a bundle map and the constant map $f\colon B \to \{b\}$ is covered by h. Since $H^i(\{b\}, \mathbb{Z}/2\mathbb{Z}) = 0$ for $i > 0$, $w_i(\pi) = f^*0 = 0$ for $i > 0$. □

Proposition 5.0.3. *If $\pi\colon E \to B$ is a trivial vector bundle, then $w_k(\pi_1 \oplus \pi) = w_k(\pi_1)$ for a vector bundle $\pi_1\colon E_1 \to B$.*

Proof. $w_k(\pi_1 \oplus \pi) = \displaystyle\sum_{i=1}^{k} w_i(\pi_1) \cup w_{k-i}(\pi) = w_k(\pi_1)$ as $w_i(\pi_1) \cup 0 = 0$ and $w_i(\pi_1) \cup w_0(\pi) = w_i(\pi_1)$. □

Proposition 5.0.4. *If $\pi\colon E \to B$ is an n-dimensional euclidean vector bundle with k nowhere dependent sections, then $w_{n-k+1}(\pi) = \cdots = w_n(\pi) = 0$.*

Proof. Let $S_1, \ldots, S_k$ be k nowhere dependent sections of $\pi\colon E \to B$. Let $F(b)$ be vector subspace of $\pi^{-1}(b)$ spanned by $S_1(b), \ldots, S_k(b)$ for each $b \in B$. Let $E_1 = \displaystyle\bigsqcup_{b \in B} F(b)$. Define a map $\pi_1\colon E_1 \to B$ by $\pi_1(e) = (b)$ if $e \in F(b)$. Then $\pi_1\colon E_1 \to B$ is an k-dimensional trivial subbundle of $\pi\colon E \to B$. Let $\pi_1^\perp\colon E_1^\perp \to B$ be the normal bundle of $\pi_1\colon E_1 \to B$. It follows from proposition 6.0.3 that $w_i(\pi) = w_i(\pi_1 \oplus \pi_1^\perp) = w_i(\pi_1^\perp)$. Since $\pi_1^\perp\colon E_1^\perp \to B$ is $n-k$ dimensional vector bundle, $w_{n-k+1}(\pi) = \cdots = w_n(\pi) = 0$. □

Definition 5.0.1. *Define $\mathrm{H}^\Pi(B; \mathbb{Z}/2\mathbb{Z})$ as the set of all formal infinite series $w_0 + w_1 + \ldots + w_n + \ldots$ with $w_i \in H^i(B; \mathbb{Z}/2\mathbb{Z})$.*

$\mathrm{H}^\Pi(B; \mathbb{Z}/2\mathbb{Z})$ with the additive operation $(w_0 + w_1 + w_2 + \ldots) + (v_0 + v_1 + v_2 + \ldots) = w_0 + v_0 + w_1 + v_1 + \ldots$ and the multiplicative operation $(w_0 + w_1 + w_2 + \ldots)(v_0 + v_1 + v_2 + \ldots) = (w_0 \cup w_0) + (w_0 \cup v_1 + w_1 \cup v_0) + (w_0 \cup v_2 + w_1 \cup v_1 + w_2 \cup v_0) + \ldots$ is a commutative ring.

Definition 5.0.2. *For an n-dimensional vector bundle $\pi\colon E \to B$, the element $w(\pi) = 1 + w_1(\pi) + \cdots + w_n(\pi) + 0 + \ldots$ of $\mathrm{H}^\Pi(B; \mathbb{Z}/2\mathbb{Z})$ is defined as the* ***total Stiefel-Whitney class*** *of the vector bundle $\pi\colon E \to B$.*

Lemma 5.0.5. *The set* $G = \{w_0 + w_1 + w_2 + \ldots \in \mathrm{H}^{\Pi}(B; \mathbb{Z}/2\mathbb{Z}) \mid w_0 = 1\}$ *is an abelian group under multiplication.*

Proof. Since $1 \cup 1 = 1$, G is closed under addition. G is abelian and associative as $\mathrm{H}^{\Pi}(B; \mathbb{Z}/2\mathbb{Z})$ is abelian and associative. For $1+w_1+\ldots \in G$, let $(1+w_1+w_2+\ldots)(1+v_1+v_2+\ldots) = 1$. Then $w_1+v_1 = 0; w_2+w_1 \cup v_1+v_2 = 0; \ldots; w_n+w_{n-1} \cup v_1+\ldots+w_1 \cup v_{n-1}+v_n = 0; \ldots$. Since coefficients are in $\mathbb{Z}/2\mathbb{Z}$, $v_1 = w_1; v_2 = w_2 + w_1 \cup w_1; \ldots; v_n = w_n + w_{n-1} \cup v_1 + \ldots + w_1 \cup v_{n-1}; \ldots$. Therefore $1 + v_1 + \ldots$ is the inverse of $1 + w_1 + \ldots$. □

It is the consequence of the product operation on $\mathrm{H}^{\Pi}(B; \mathbb{Z}/2\mathbb{Z})$ that $w(\pi_1 \oplus \pi_2) = w(\pi_1)w(\pi_2)$ for vector bundles $\pi_1 \colon E_1 \to B$ and $\pi_2 \colon E_2 \to B$.

Lemma 5.0.6. *If* A *is a smooth manifold in* $\mathbb{R}^n$, $\pi \colon TA \to A$ *is the tangent bundle of* A *and* $\pi^{\perp} \colon TA^{\perp} \to A$ *is the normal bundle of* $\pi \colon TA \to A$, *then* $w(\pi^{\perp}) = w(\pi)^{-1}$

Proof. Since $\pi \oplus \pi^{\perp} \colon TA \oplus TA^{\perp} \to A$ is isomorphic to the n-dimensional trivial vector bundle over B, $w(\pi)w(\pi^{\perp}) = w(\pi \oplus \pi^{\perp}) = 1$. Therefore $w(\pi^{\perp}) = w(\pi)^{-1}$. □

Example 12. $w(\pi) = 1$ *for the tangent bundle* $\pi \colon TS^n \to S^n$. *Since* $S^n \subset \mathbb{R}^{n+1}$ *and the normal bundle of* $\pi \colon TS^n \to S^n$ *is the 1-dimensional trivial vector bundle,* $w(\pi) = w(\pi^{\perp})^{-1} = 1$.

Example 13. *We have* $w_1(\pi_1^1) \neq 0$ *for the line bundle* $\pi_1^1 \colon \gamma_1^1 \to \mathbb{R}\mathrm{P}^1$. *Since the inclusion map* $i \colon \gamma_1^1 \to \gamma_n^1$ *is a bundle map, the inclusion map* $f \colon \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^n$ *is covered by the bundle map* i. $f^* w_1(\pi_n^1) = w_1(\pi_1^1) \neq 0$ *implies that* $w_1(\pi_n^1) \neq 0$. *Therefore* $w(\pi_n^1) = 1 + w_1$ *for some non-zero element* w_1 *of* $H^1(B, \mathbb{Z}/2\mathbb{Z})$.

Example 14. *The vector bundle* $\pi_n^1 \colon \gamma_n^1 \to \mathbb{R}\mathrm{P}^n$ *is a subbundle of the trivial bundle* $\pi \colon \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} \to \mathbb{R}\mathrm{P}^n$. $\pi_n^1 \oplus (\pi_n^1)^{\perp} \colon \gamma_n^1 \oplus (\gamma_n^1)^{\perp} \to \mathbb{R}\mathrm{P}^n$ *is isomorphic to the trivial bundle* $\pi \colon \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} \to \mathbb{R}\mathrm{P}^n$. *Therefore* $w((\pi_n^1)^{\perp}) = w(\pi_n^1)^{-1} = (1+w_1)^{-1} = 1+w_1+w_1^2+\ldots+w_1^n$ *where* w_1^n *is the* n*-fold cup product of* w_1.

Lemma 5.0.7. *The tangent bundle* $\pi \colon T\,\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ *and the vector bundle* $\pi' \colon \mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^{\perp}) \to \mathbb{R}\mathrm{P}^n$ *are isomorphic.*

Proof. The canonical map $f \colon S^n \to \mathbb{R}\mathrm{P}^n$ given by $f(x) = \{\pm x\}$ is locally a diffeomorphism. Therefore the tangent spaces of S^n at x and $-x$ map isomorphically to the tangent space of

$\mathbb{R}\mathrm{P}^n$ at $\{\pm x\}$. We can identify the tangent space of $\mathbb{R}\mathrm{P}^n$ at $\{\pm x\}$ with the tangent spaces of S^n at x and x. Therefore the tangent space of $\mathbb{R}\mathrm{P}^n$ at $\{\pm x\}$ is the set of equivalence classes of pairs $\{(x, v), (-x, -v)\}$ with $x \in S^n$ and $< x, v >= 0$. Let $L_{\{\pm x\}}$ be the line passing through x and $-x$ in $\mathbb{R}^{n+1}$. Let $L^{\perp}_{\{\pm x\}}$ be the orthogonal complement of $L_{\{\pm x\}}$ in $\mathbb{R}^{n+1}$. Define

$$l^x \colon L_{\{\pm x\}} \to L^{\perp}_{\{\pm x\}}$$
$$x \mapsto v$$

for a fixed $v \in L^{\perp}_{\{\pm x\}}$. Denote l^x by l^x_v if x maps to v. Then l^x_v is a linear map. Define

$$h \colon T\,\mathbb{R}\mathrm{P}^n \to \mathrm{Hom}(\gamma^1_n, (\gamma^1_n)^{\perp})$$
$$\{(x, v), (-x, -v)\} \mapsto l^x_v$$

Then h maps the tangent space of $\mathbb{R}\mathrm{P}^n$ at $\{\pm x\}$ isomorphically to $\mathrm{Hom}(L_{\{\pm x\}}, L^{\perp}_{\{\pm x\}})$. h is bijective. Since bases of topology on $T\,\mathbb{R}\mathrm{P}^n$ and $\mathrm{Hom}(\gamma^1_n, (\gamma^1_n)^{\perp})$ have quotient topology induce from $U \times \mathbb{R}^n$ where U is an element of coordinate open sets of $\mathbb{R}\mathrm{P}^n$, h is a homeomorphism. Therefore h is a vector bundle isomorphism. □

Theorem 5.0.8. *The Whitney sum of the tangent bundle $\pi \colon T\,\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ and the trivial vector bundle $\pi_1 \colon \varepsilon^1 \to \mathbb{R}\mathrm{P}^n$ is isomorphic to the $(n+1)$-fold Whitney sum $\gamma^1_n \oplus \cdots \oplus \gamma^1_n$.*

Proof. From exercise 5, we get that $\mathrm{Hom}(\gamma^1_n, \gamma^1_n)$ is isomorphic to the trivial vector bundle $\pi_1 \colon \varepsilon^1 \to \mathbb{R}\mathrm{P}^n$. Since the tangent bundle of $\mathbb{R}\mathrm{P}^n$ is isomorphic to $\mathrm{Hom}(\gamma^1_n, (\gamma^1_n)^{\perp})$, $T\,\mathbb{R}\mathrm{P}^n \oplus \varepsilon^1$ is isomorphic to $\mathrm{Hom}(\gamma^1_n, (\gamma^1_n)^{\perp}) \oplus \mathrm{Hom}(\gamma^1_n, \gamma^1_n)$. $\mathrm{Hom}(\gamma^1_n, (\gamma^1_n)^{\perp}) \oplus \mathrm{Hom}(\gamma^1_n, \gamma^1_n)$ is isomorphic to $\mathrm{Hom}(\gamma^1_n, (\gamma^1_n)^{\perp} \oplus \gamma^1_n)$. $\mathrm{Hom}(\gamma^1_n, (\gamma^1_n)^{\perp} \oplus \gamma^1_n)$ is isomorphic to $\mathrm{Hom}(\gamma^1_n, \varepsilon^{n+1})$. $\mathrm{Hom}(\gamma^1_n, \varepsilon^{n+1})$ is isomorphic to $\mathrm{Hom}(\gamma^1_n, \varepsilon^1 \oplus \ldots \oplus \varepsilon^1)$. $\mathrm{Hom}(\gamma^1_n, \varepsilon^1 \oplus \ldots \oplus \varepsilon^1)$ is isomorphic to $\mathrm{Hom}(\gamma^1_n, \varepsilon^1) \oplus \ldots \oplus \mathrm{Hom}(\gamma^1_n, \varepsilon^1)$. From exercise 6, we get that $\mathrm{Hom}(\gamma^1_n, \varepsilon^1)$ is isomorphic to γ^1_n. Therefore $T\,\mathbb{R}\mathrm{P}^n \oplus \varepsilon^1$ is isomorphic to $(n+1)$-fold Whitney sum $\gamma^1_n \oplus \ldots \oplus \gamma^1_n$. □

It follows from the previous theorem that the total Stiefel-Whitney class of the tangent bundle of $\mathbb{R}\mathrm{P}^n$ is $w(\pi^1_n)^{(n+1)} = (1 + w_1)^{(n+1)}$. We will denote the total Stiefel-Whitney class of tangent bundle of $\mathbb{R}\mathrm{P}^n$ by $w(\mathbb{R}\mathrm{P}^n)$.

Corollary 5.0.9. $w(\mathbb{R}\mathrm{P}^n) = 1$ *if and only if* $n + 1 = 2^k$ *for some positive integer* k.

Proof. Assume $w(\mathbb{R}\mathrm{P}^n) = 1$. Suppose $n + 1$ is not a power of 2. If $n + 1$ is a odd positive integer, then $w(\mathbb{R}\mathrm{P}^n) = (1 + w_1)^{n+1} = 1 + (n + 1)w_1 + \ldots \neq 1$ as the coefficient of w_1 is a non-zero modulo 2. If $n + 1$ is an even positive integer, then $n + 1 = 2^k m$ for some odd positive integer m. Since $(1 + w_1)^{2^k} = 1 + w_1^{2^k}$ modulo 2, $w(\mathbb{R}\mathrm{P}^n) = (1 + w_1)^{2^k m} = (1 + w_1^{2^k})^m = 1 + m w_1^{2^k} + \ldots \neq 1$ as m is odd and $2^k < n$. Therefore $n + 1 = 2^k$ for some positive integer k.

Conversely if $n+1 = 2^k$ for some positive integer k, then $w(\mathbb{R}\mathrm{P}^n) = (1+w_1)^{2^k} = 1+w_1^{2^k} = 1 + w_1^{n+1} = 1$ as $T\,\mathbb{R}\mathrm{P}^n$ is an n-dimensional vector bundle. □

It follows from the previous corollary that if the tangent bundle of $\mathbb{R}\mathrm{P}^n$ is the trivial vector bundle, then $n + 1$ must be 2^k for some positive integer k.

Theorem 5.0.10. *If there is a bilinear product operation $\rho\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ without zero divisors, then the tangent bundle of $\mathbb{R}\mathrm{P}^{n-1}$ is the trivial vector bundle.*

Proof. See theorem 4.7 of [2]. □

Exercise 9. *For two vector bundles $\pi_1\colon E_1 \to A$ and $\pi_2\colon E_2 \to B$, $w_k(\pi_1 \times \pi_2) = \sum_{i=0}^{k} w_i \cup w_{k-i}$.*

Solution. Consider the two maps $p_1\colon A \times B \to A$ given by $p_1(a, b) = a$ and $p_2\colon A \times B \to B$ given by $p_2(a, b) = b$. Then $p_1^*\pi_1\colon p_1^*E_1 \to A\times B$ and $p_2^*\pi_2\colon p_2^*E_2 \to A\times B$ are vector bundles induced by p_1 and p_2 respectively. From axiom 2 of Stiefel-Whitney classes, $w_i(p_1^*\pi_1) = w_i(\pi_1)$ and $w_i(p_2^*\pi_2) = w_i(\pi_2)$ for each $i \geq 0$. Consider $p_1^*\pi_1 \oplus p_2^*\pi_2\colon p_1^*E_2 \oplus p_2^*E_2 \to A\times B$, Whitney sum of the two induced vector bundles. We know that

$$\begin{aligned}
p_1^*E_1 &= \{(a, b, e_1) \in A \times B \times E_1 \mid p_1(a, b) = \pi_1(e_1)\}\\
p_2^*E_2 &= \{(a, b, e_2) \in A \times B \times E_2 \mid p_2(a, b) = \pi_2(e_2)\}\\
p_1^*E_1 \oplus p_2^*E_2 &= \{((a_1, b_1, e_1), (a_2, b_2, e_2)) \in p_1^*E_1 \times p_2^*E_2 \mid p_1^*\pi_1((a_1, b_1, e_1)) = p_2^*\pi_2((a_2, b_2, e_2))\}\\
&= \{((a_1, b_1, e_1), (a_2, b_2, e_2)) \in p_1^*E_1 \times p_2^*E_2 \mid a_1 = a_2, b_1 = b_2\}
\end{aligned}$$

Define

$$\begin{aligned}
h\colon p_1^*E_1 \oplus p_2^*E_2 &\to E_1 \times E_2\\
((a, b, e_1), (a, b, e_2)) &\to (e_1, e_2)
\end{aligned}$$

h is continuous and restriction of h on $(p_1^*\pi_1 \oplus p_2^*\pi_2)^{-1}(a,b) = (p_1^*\pi_1)^{-1}(a,b) \times (p_2^*\pi_2)^{-1}(a,b)$ is linear isomorphism of $(p_1^*\pi_1)^{-1}(a,b) \times (p_2^*\pi_2)^{-1}(a,b)$ with $\pi_1^{-1}(a) \times \pi_2^{-1}(b)$. Lemma 3.1.1 implies that h is a vector bundle isomorphism. Therefore $w_k(\pi_1 \times \pi_2) = w_k(p_1^*\pi_1 \oplus p_2^*\pi_2) = \sum_{i=0}^{k} w_i(p_1^*\pi_1) \cup w_{k-i}(p_2^*\pi_2) = \sum_{i=0}^{k} w_i(\pi_1) \cup w_{k-i}(\pi_2)$. □

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