L^2-methods in complex analysis

A Thesis

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by

Aravind R

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

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Supervisor: Dr. Diganta Borah

Certificate

This is to certify that this dissertation entitled L^2-methods in complex analysis towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Aravind R at Indian Institute of Science Education and Research under the supervision of Dr. Diganta Borah, Associate Professor, Department of Mathematics, during the academic year 2024-2025.

Dr. Diganta Borah

Dr. Vivek Mohan Mallick

Committee:

Dr. Diganta Borah

Dr. Vivek Mohan Mallick

To my family, blooded and unblooded alike, who show me true kinship

Declaration

I hereby declare that the matter embodied in the report entitled L^2-methods in complex analysis are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Diganta Borah and the same has not been submitted elsewhere for any other degree.

Aravind R

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Abstract

The main goal of this expository thesis is to study the L^2-technique of Hörmander estimates. Beginning with some elementary considerations such as Poincaré's theorem, domains of holomorphy and the Hartogs theorem, we deal with two interesting applications:

- An elegant solution of the celebrated Levi problem
- Holomorphic extensions in the sense of Ohsawa-Takegoshi

We conclude the thesis by discussing the corresponding analogue of the Hörmander's estimate on compact Kähler manifolds.

Contents

Introduction

Complex analytically i.e, upto biholomorphic equivalence it turns out there is only one topologically simple model domain in $\mathbb{C}$ - the unit disc $\mathbb{D}$. In the several variable scenario, there are many model domains in $\mathbb{C}^n$. Unlike one dimension, some domains are more natural than others; these are called domains of holomorphy. Therefore, we begin with the characterization of these domains using the concept of holomorphic convexity.

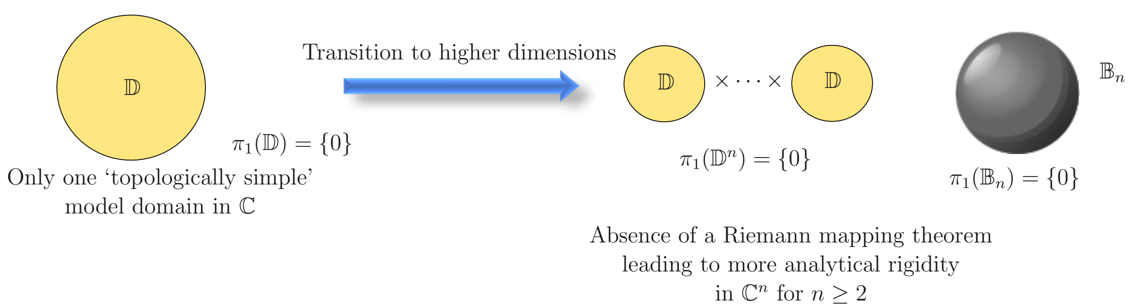


Figure 1: Moving into several dimensions

We show that two of the most basic model domains in $\mathbb{C}^n$: the unit ball and polydisc are biholomorphically inequivalent. We give a proof of this result which is usually attributed to Poincaré. Motivated by this fact we then pursue the study of domains in $\mathbb{C}^n$ further.
The *Kugelsatz* due to Friedrich Hartogs which deals with extension across compact holes is then dealt with, embarking upon how single variable complex analysis is restrictive (in a sense) than the several variable situation. The proof of the Hartogs's extension theorem is done using the $\overline{\partial}$-problem which is essentially a computation of compactly supported Dolbeault cohomology. Non-examples of domains of holomorphy are easily generated using Hartogs's result.
The $\overline{\partial}$-problem is a very central one in the field of several complex variables. When one works with plurisubharmonically weighted Lebesgue spaces, one obtains L^2 estimates for the

solutions to this inhomogeneous Cauchy problem. We first do it in $\mathbb{C}$ and then do it in $\mathbb{C}^n$. As an application, the Hörmander estimate for pseudoconvex domains in $\mathbb{C}^n$ leads to an elegant solution of the Levi problem for domains in $\mathbb{C}^n$, which answers the question: Which domains in $\mathbb{C}^n$ are exactly the domains of holomorphy? The extension theorem due to Ohsawa-Takegoshi is then deduced directly from the Hörmander estimate.

We finally geometrize the stage to a special class of manifolds called 'Kähler' manifolds and prove the Hörmander's estimate.

Chapter 1

Preliminaries

1.1 Basics of several complex variables

Any of the following equivalent definitions can be fixed as one for formalizing the notion of holomorphicity in higher dimensions (see [3, Chapter-1]).

Proposition 1.1.1 (Osgood's theorem)**.** The following notions of holomorphicity for a function $f : U \to \mathbb{C}$ on a domain $U \subset \mathbb{C}^n$ are equivalent:

1. f is ***holomorphic*** along every 'complex' axial direction i.e, $\zeta \mapsto f(\mathbf{z}_0 + \zeta \mathbf{e}_j)$ is holomorphic for every $j = 1, \cdots, n$.

2. f can be expressed locally as a convergent ***multi-variable power series*** $\sum_{\nu \geq 0} a_\nu (\mathbf{z} - \mathbf{z}_0)^\nu$ around every point $\mathbf{z}_0 \in U$.

3. f is ***complex differentiable*** at every point $\mathbf{z}_0 \in U$ i.e,

$$f(\mathbf{z}) = f(\mathbf{z}_0) + T(\mathbf{z} - \mathbf{z}_0) + o(||\mathbf{z} - \mathbf{z}_0||)$$

   for some $T \in \mathcal{L}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$.

The set of holomorphic functions on a domain $U \subset \mathbb{C}^n$ will be denoted by $\mathcal{O}(U)$.

Definition 1.1.1 (Biholomorphic automorphism group)**.** For a bounded domain $U \subset \mathbb{C}^n$, the set of all self biholomorphic maps $\boldsymbol{\varphi} : U \to U$ forms a group under composition which we denote by $(\mathrm{Aut}^{\mathcal{O}}(U), \circ)$. When endowed with the compact-open topology, it is a topological group and this structure will be called the ***biholomorphic automorphism group***.

The topology of locally uniform convergence is actually locally compact and Hausdorff. In fact, Cartan proved that $\mathrm{Aut}^{\mathcal{O}}(U)$ carries the structure of a Lie group (see **[11, Chapter-9]**).

The interesting object of our thesis study is a domain of holomorphy which is formalized by the following notion (see **[3, Chapter-2]**):

Definition 1.1.2. (SINGULAR FUNCTION) A function $f \in \mathcal{O}(U)$ on a domain $U \subset \mathbb{C}^n$, is said to be ***completely singular*** at $\mathbf{z}_0 \in \partial U$ if for every domain $V = V(\mathbf{z}_0) \subset \mathbb{C}^n$ and given a connected component C of $V \cap U$ such that $\mathbf{z}_0 \in \partial C$, there is no $g \in \mathcal{O}(V)$ for which $g|_C = f|_C$.

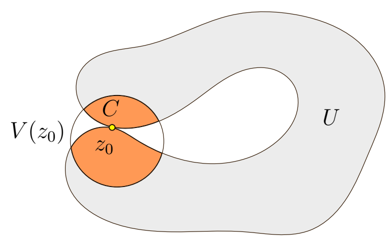


Figure 1.1: Illustrating the definition of singular functions

The crux of the above definition is essentially contained in the idea that we don't want f to be extendible through $\mathbf{z_0}$ even as a multi-valued function, hence qualifies to be *completely singular*.

Definition 1.1.3 (DOMAIN OF HOLOMORPHY)**.** Consider a domain $U \subset \mathbb{C}^n$.

- U is said to be a ***weak domain of holomorphy*** if for every $\mathbf{z}_0 \in \partial U$ there is a $f \in \mathcal{O}(U)$ which is completely singular at $\mathbf{z}_0$.

- U is said to be a ***domain of holomorphy*** if there is a $f \in \mathcal{O}(U)$ which is completely singular on all of ∂U.

1.2 Plurisubharmonic functions

The generalization of subharmonic functions in (complex) dimension 1 to several dimensions are called *plurisubharmonic* functions. They are important in the study of function theory of several complex variables.

Definition 1.2.1 (Plurisubharmonic function). Given $U \subset \mathbb{C}^n$ a domain, an upper semicontinuous function $u : U \to [-\infty, \infty)$ is said to be ***plurisubharmonic*** if for *every* tangent vector $(\mathbf{a}; \boldsymbol{\xi}) \in T_{\mathbf{a}}U$, the mapping:

$$u_{\mathbf{a},\boldsymbol{\xi}} : U(\mathbf{a}, \boldsymbol{\xi}) \to [-\infty, \infty),$$

$$\zeta \to u(\mathbf{a} + \zeta\, \boldsymbol{\xi})$$

is subharmonic, where $U(\mathbf{a}, \boldsymbol{\xi})$ is the connected component of the open set $\{\zeta \in \mathbb{C} : \mathbf{a}+\zeta\, \boldsymbol{\xi} \in U\}$ containing the origin.

For smooth functions, plurisubharmonicity is detected easily with the help of the Levi form.

Definition 1.2.2 (The Levi form). For a $\mathscr{C}^2$ function $u : U \to \mathbb{R}$, the Levi form is defined as:

$$\mathcal{L}ev_u(\boldsymbol{\zeta}; \mathbf{w}) := \sum_{r,s=1}^{n} \frac{\partial^2 u}{\partial z_r \partial \bar{z}_s}(\boldsymbol{\zeta})\; w_r \overline{w_s},$$

for $\boldsymbol{\zeta} \in U$ and $\mathbf{w} \in \mathbb{C}^n$.

A $\mathscr{C}^2$ function $u : U \to \mathbb{R}$ is plurisubharmonic if and only $\mathcal{L}ev_u(\boldsymbol{\zeta}; \mathbf{w}) \geq 0$ for $\boldsymbol{\zeta} \in U$ and $\mathbf{w} \in \mathbb{C}^n$ (see **[3, Chapter-2]**). Using the Levi form, we can get a stronger notion:

Definition 1.2.3 (Strictly plurisubharmonic function). A $\mathscr{C}^2$ function $u : U \to \mathbb{R}$ is said to be strictly plurisubharmonic if $\mathcal{L}ev_u(\boldsymbol{\zeta}; \mathbf{w}) > 0$ for $\boldsymbol{\zeta} \in U$ and $\mathbf{w} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. The collection of all strictly plurisubharmonic functions defined on U will be denoted by $\mathcal{PSH}_{>0}(U)$.

An analogue of the characterization of $\mathcal{PSH}$ functions in terms of the Levi form holds for non-smooth functions as well in terms of currents **[6, Chapter-3]**.

Proposition 1.2.1 (Characterization of $\mathcal{PSH}$ functions). For an upper semicontinuous function $u : U \to [-\infty, \infty)$, the (1,1) current

$$\partial\bar{\partial}u = \sum_{r,s=1}^{n} \frac{\partial^2 u}{\partial z_r \partial \bar{z}_s} \, \mathrm{d}z_r \wedge \mathrm{d}\bar{z}_s,$$

is a positive current i.e, $\partial\bar{\partial}u(\psi) := \partial\bar{\partial}u \wedge \psi \geq 0$ for every non-negative (i.e, coefficients are non-negative functions) $\psi \in \Omega^{(n-1,n-1)}_{\mathrm{c},\mathbb{R}}(U)$ if and only $u \in \mathcal{PSH}(U)$.

Proposition 1.2.2. (Properties of $\mathcal{PSH}$ functions)

1. (Positive Cone) - If $u_1, u_2 \in \mathcal{PSH}(U)$, then $u_1 + u_2 \in \mathcal{PSH}(U)$. For $c > 0$, $cu \in \mathcal{PSH}(U)$ if $u \in \mathcal{PSH}(U)$.

2. (Maximum principle) - If $u \in \mathcal{PSH}(U)$ is not identically constant on U, then u does not attain its global maximum at any point in U. Further, if U is bounded, then

$$u(\mathbf{z}) < \sup_{\boldsymbol{\zeta} \in \partial U} \left\{ \varlimsup_{U \ni \mathbf{z}' \to \boldsymbol{\zeta}} u(\mathbf{z}') \right\} \quad \forall \ \mathbf{z} \in U.$$

3. (Closed under upper envelopes) - If $\mathcal{F} \subset \mathcal{PSH}(U)$ is a non-empty family which is locally bounded above, then the upper envelope $(\sup \mathcal{F})^\star \in \mathcal{PSH}(U)$. Here the upper regularization $u^\star$ of a function $u : U \to \mathbb{R}$ is defined as:

$$u^\star(\boldsymbol{z}) := \varlimsup_{\boldsymbol{\zeta} \to \mathbf{z}} u(\boldsymbol{\zeta}) = \lim_{\varepsilon \to 0} \sup\{u(\boldsymbol{\zeta}) \mid \boldsymbol{\zeta} \in U, |\boldsymbol{\zeta} - \mathbf{z}| < \varepsilon\},$$

4. (No isolated singularites) - Every $u \in \mathcal{PSH}(U \setminus \{\mathbf{z}_0\})$, which is locally bounded near $\mathbf{z}_0 \in U$ can be extended uniquely across $\mathbf{z}_0$.

5. (Separation) - For every open set $U' \subset U$ containing the $\mathcal{PSH}$ hull defined by:

$$\widehat{K}_{\mathcal{PSH}(U)} := \bigcap_{p \in \mathcal{PSH}(U)} \left\{ \mathbf{z} \in U \ : \ p(\mathbf{z}) \leq \sup_K p \right\},$$

   there exists an smooth, exhaustive, strictly plurisubharmonic function $u \in \mathcal{PSH}(U)$ such that $u < 0$ on K and $u \geq 1$ on $U \setminus U'$.

6. (Smooth approximation) - For every $u \in \mathcal{PSH}(U)$, there exists a mollified family, $\{u_\varepsilon\}_{\varepsilon > 0} \subset \mathscr{C}^\infty(U, \mathbb{R}) \cap \mathcal{PSH}(U)$, which decrease pointwise to u as ε decreases to 0.

For a proof see [1, Chapter-1] and [5, Chapter-3].

1.3 The $\bar{\partial}$ operator

The $\bar{\partial}$ operator is a well studied operator by complex analysts. We will need the following regularity result in our study.

1.3.1 A regularity theorem

Let $s \geq 0$ be an integer and $p, q \in \mathbb{N} \cup \{0\}$. The Sobolev space $\mathcal{W}^s_{(p,q)}(U, \text{loc})$ is defined as the vector space of (p, q) forms defined on a domain $U \subset \mathbb{C}^n$ satisfying:

$$\left\{ \sum_{[J,K]:|J|=p,|K|=q} f_{J,K} \, \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K : U \to \mathbb{C} \;\middle|\; \frac{\partial^{(\boldsymbol{\nu},\boldsymbol{\nu}')} f_{J,K}}{\partial z^{\boldsymbol{\nu}} \partial \bar{z}^{\boldsymbol{\nu}'}} \in L^2_{\text{loc},\mathbb{C}}(U) \;\; \forall \;\; |(\boldsymbol{\nu}, \boldsymbol{\nu}')| \leq s \right\}.$$

Theorem 1.3.1. *Let $U \subset \mathbb{C}^n$ be a pseudoconvex domain (no boundedness or smoothness of ∂U is assumed). Let $s \in \mathbb{N}_0 \cup \{\infty\}$, $p, q \in \mathbb{N}_0$, and let $f \in \mathcal{W}^s_{(p,q+1)}(U, \text{loc})$ satisfy $\overline{\partial} f = 0$ weakly. Then the canonical solution u to $\overline{\partial} u = f$ satisfies lies in $u \in \mathcal{W}^{s+1}_{(p,q)}(U, \text{loc})$:*

$$\bar{\partial} \left(\mathcal{W}^s_{(p,q+1)}(U, \text{loc}) \right) = \left\{ v \in \mathcal{W}^{s+1}_{(p,q)}(U, \text{loc}) : \bar{\partial} v = 0 \right\}.$$

As a corollary we obtain:

Corollary 1.3.1.1. If $U \subset \mathbb{C}^n$ is pseudoconvex and $f \in \mathscr{C}^\infty_{(p,q+1)}(U, \mathbb{C})$ such that $\bar{\partial} f = 0$, then there is a $u \in \mathscr{C}^\infty_{(p,q+1)}(U, \mathbb{C})$ satisfying $\bar{\partial} u = f$.

For a proof see [7, Chapter-4] or [5, Chapter-5] .

1.3.2 Dolbeault cohomology with compact support

Dolbeault cohomology is no exception to the fact that cohomology measures the extent of obstruction by showing the failure of exactness of the concerned complex.

Definition 1.3.1 ($\bar{\partial}$ cohomology with compact support)**.** The ***compactly supported Dolbeault cohomology spaces*** are defined as:

$$H^{(p,q)}_{\overline{\partial},c}(\mathbb{C}^n) := \frac{\ker(\overline{\partial} : \mathscr{C}^\infty_{c,(p,q)}(\mathbb{C}^n) \to \mathscr{C}^\infty_{c,(p,q+1)}(\mathbb{C}^n, \mathbb{C}))}{\operatorname{im}(\overline{\partial} : \mathscr{C}^\infty_{c,(p,q-1)}(\mathbb{C}^n, \mathbb{C}) \to \mathscr{C}^\infty_{c,(p,q)}(\mathbb{C}^n, \mathbb{C}))}.$$

for every $p, q \in \mathbb{N} \cup \{0\}$.

In essence, compactly supported Dolbeault cohomology captures the cohomological information as if the ambient space were compact, by working only with compactly supported differential forms.

1.4 Boundary notions

This section collects the prerequisite boundary notions for defining Levi pseudoconvexity (see [7, Chapter-3]).

Definition 1.4.1 (Boundary functions)**.** Let $U \subset \mathbb{C}^n$ be a domain. On an open neighbourhood $U' = U'(\mathbf{z_0})$ of $\mathbf{z_0} \in \partial U$, a function $\varrho \in \mathscr{C}^\infty(U', \mathbb{R})$ such that:

(i) $U' \cap U = \{\mathbf{z} \in U' : \varrho(\mathbf{z}) < 0\}$,

(ii) $d\varrho(\mathbf{z}) \neq 0 \;\forall\; \mathbf{z} \in U'$

is called a (local) ***boundary defining function***.

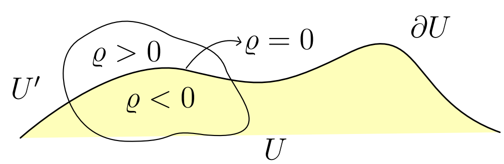


Figure 1.2: Depicting a boundary defining function

The following space will play a key role in determining the (holomorphic) boundary convexity of domains paralleling that of affine convexity.

Definition 1.4.2 (Complex Tangent space)**.** The subspace

$$T^{(1,0)}_{\mathbf{z_0}}(\partial U) := T_{\mathbf{z_0}}(\partial U) \cap J_{\mathbb{C}^n}(T_{\mathbf{z_0}}(\partial U)),$$

is called the ***complex tangent space*** of the boundary at z_0.

It is easily seen that $T^{(1,0)}_{\mathbf{z_0}}(\partial U) = \ker_{\mathbb{C}} \partial\varrho(\mathbf{z_0})$ for any choice of a boundary defining function, therefore it has a natural complex structure of (complex) co-dimension 1.

When these vectors are fed into the Levi form of a boundary defining function, it leads to detections of a type of complex convexity defined in Chapter 6.

Definition 1.4.3 (Analytic discs and the continuity principle)**.** Consider a domain $U \subset \mathbb{C}^n$.

(i) An ***analytic disc*** in U is a continuous map $\mathbf{d} : \overline{\mathbb{D}} \to U$ which is holomorphic in the interior $\mathbb{D}$.

(ii) U is said to obey the ***continuity principle*** if for every family $\{\mathbf{d}_\alpha\}_{\alpha\in\mathcal{A}}$ of analytic discs in U, whose union of images of the boundary $\partial\mathbb{D}$ is relatively compact in U i.e, $\bigcup_{\alpha\in\mathcal{A}} \mathbf{d}_\alpha(\partial\mathbb{D}) \Subset U$, we also have $\bigcup_{\alpha\in\mathcal{A}} \mathbf{d}_\alpha(\mathbb{D}) \Subset U$.

As the diagram below suggests, it is quite plausible that analytic discs can detect complex analytic notions of convexity. In particular, it respects biholomorphic equivalence.

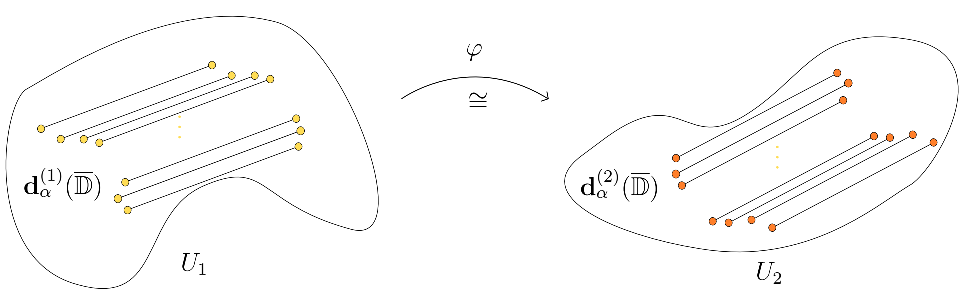


Figure 1.3: Analytic discs can detect complex analytic convexity

The analytic discs $\{\mathbf{d}^{(1)}_\alpha : \overline{\mathbb{D}} \to U_1\}_{\alpha\in\mathcal{A}}$ and $\{\mathbf{d}^{(2)}_\alpha : \overline{\mathbb{D}} \to U_2\}_{\alpha\in\mathcal{A}}$ are denoted by line segments with their boundaries as the end points within the biholomorphically equivalent domains U_1 and U_2 respectively.

1.5 Complex differential geometry

In this section, we give the basic definitions in complex differential geometry (see [10, Chapters-1,3,7,8]) required to discuss the Hörmander's estimate over complex manifolds.
Definition 1.5.1 (Complex manifold)**.** A n-dimensional ***complex manifold*** is a triple $(X, \mathcal{T}_X, \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}})$ consisting of a second countable, Hausdorff topological space $(X, \mathcal{T}_X)$ equipped with a family of holomorphic charts $\varphi_\alpha : U_\alpha \underset{\text{open}}{\subset} X \to V_\alpha \underset{\text{open}}{\subset} \mathbb{C}^n$ i.e, the transition functions (chart changing maps) are all biholomorphisms between open subsets of $\mathbb{C}^n$ for some $n \in \mathbb{N}$.

When the canonical complex structure $J : TX \to TX$ respects a Riemannian metric, we obtain a Hermitian manifold.
Definition 1.5.2 (Hermitian notions)**.**

(i) (Hermitian metric) - A ***Hermitian metric*** on a complex manifold X is a Riemanian metric g which is J invariant i.e, $J^\star g = g$.

(ii) (Fundamental 2-form) - The 2-form $\omega(\cdot,\cdot) := g(J\cdot,\cdot)$ is called the ***fundamental 2-form*** of g.

(iii) (Hermitian manifold) - A complex manifold with a Hermitian metric is called a ***Hermitian manifold*** and will be denoted as (X, ω).

(iv) (Hermitian fiber-metric) - A ***Hermitian fiber-metric*** on a complex vector bundle $E \to X$ is a section $\langle \cdot,\cdot \rangle \in \Gamma(E^\vee \otimes \overline{E^\vee})$ which restricts to a Hermitian inner product on every fiber E_p for $p \in X$. Such a bundle is called a ***Hermitian vector bundle***.

An easy way to obtain an interesting class of complex manifolds is to put a symplectic condition on the fundamental 2-form.
Definition 1.5.3 (Kähler manifold)**.** A Hermitian manifold (X, ω) is said to be ***Kähler*** if ω is d-closed.

We shall need the class of holomorphic vector bundles which are complex vector bundles where the local trivialisations are obtained through holomorphic maps.
Definition 1.5.4 (Holomorphic vector bundle)**.** A ***holomorphic vector bundle*** (E, π, X) of rank k consists of:

(i) complex manifolds E (the total space) and X (the base space)

(ii) a surjective holomorphic map $\pi : E \to X$

(iii) a $\mathbb{C}$-vector space structure on $E_p := \pi^{-1}(p)$ for all $p \in X$

such that there is an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X and a collection of (local) biholomorphisms $\{\Phi_\alpha\}_{\alpha\in\mathcal{A}}$ called local trivialisations satisfying:

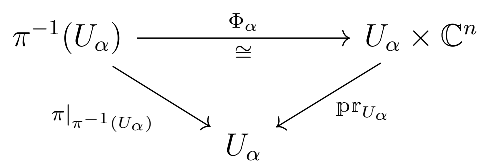


Figure 1.4: Local trivialisations

and $\Phi_\alpha|_{E_q} \in \mathrm{Hom}_{\mathbb{C}}(E_q, \{q\} \times \mathbb{C}^k)$ for every $q \in \pi^{-1}(U_\alpha)$ and $\alpha \in \mathcal{A}$.

Proposition 1.5.1 (The $\bar{\partial}$ operator for bundle valued sections)**.** Suppose $E \to X$ is a holomorphic vector bundle over a complex manifold X. There is a ***family of $\bar{\partial}$ operators*** $\left\{\bar{\partial}_E \colon \Omega^{p,q}(X; E) \to \Omega^{p,q+1}(X; E)\right\}_{p,q\in\mathbb{N}_0^2}$ satisfying the following properties:

(i) For $\sigma \in \Omega^{0,0}(X; E)$, $\bar{\partial}_E\sigma = 0$ if and only if σ is a holomorphic section.

(ii) For $\alpha \in \Omega^{p,q}(X)$ and $\beta \in \Omega^{p',q'}(X; E)$,

$$\bar{\partial}_E(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \bar{\partial}_E\beta.$$

(iii) $\bar{\partial}_E \circ \bar{\partial}_E = 0$.

When we have additional structures, like a Hermitian metric, we can single out a particular connection amongst the affine space of connections on X with some compatibilities called the Chern connection.

Definition 1.5.5 (Fundamental theorem of Hermitian holomorphic geometry)**.** On every Hermitian holomorphic vector bundle, there is a unique connection, called the ***Chern connection***, that is compatible with:

(i) the metric structure - $Z\langle\sigma_1, \sigma_2\rangle = \langle\nabla_Z\sigma_1, \sigma_2\rangle + \langle\sigma_1, \nabla_Z\sigma_2\rangle$ for every $Z \in \mathfrak{X}_{\mathbb{C}}(X)$ and $\sigma_1, \sigma_2 \in \Gamma(E)$.

(ii) the holomorphic structure - $\nabla^{(0,1)} = \bar{\partial}_E$.

We denote the Chern connection by an distinguished overarched left-right arrow $\overset{\leftrightarrow}{\nabla}$, with the arrow $\leftrightarrow$ by emphasizing that the connection has two compatibility features engrained into it.

Definition 1.5.6 (The curvature operator)**.** For a complex vector bundle $E \to X$ with a connection ∇, the curvature operator Θ is defined as the map:

$$\Theta : \Gamma(T_{\mathbb{C}}X) \times \Gamma(T_{\mathbb{C}}X) \times \Gamma(E) \to \Gamma(E)$$

$$(Z_1, Z_2, \sigma) \mapsto \Theta_{(Z_1,Z_2)}\sigma = \nabla_{Z_1}\nabla_{Z_2}\sigma - \nabla_{Z_2}\nabla_{Z_1}\sigma - \nabla_{[Z_1,Z_2]}\sigma.$$

Chapter 2

Domains in $\mathbb{C}^n$

The unit ball and polydisc are biholomorphically inequivalent in dimensions ≥ 2. This result is usually attributed to Poincaré. This shows that in higher dimensions there are a lot of model domains leading to a rich holomorphic geometry and function theory which can take up a very non-trivial character in contrast to one dimension.

2.1 Poincaré's theorem

In this section, we show the inequivalence using Cartan's results (see **[14, Chapter-2]**).

Definition 2.1.1 (Circular domain)**.** A domain $U \subset \mathbb{C}^n$ is said to be ***circular*** if $\mathbb{S}^1 \curvearrowright U$ i.e, $e^{i\theta}\mathbf{z} \in U$ whenever $\mathbf{z} \in U$ and $\theta \in \mathbb{R}$.

The following two results due to Cartan are referred to as the Cartan's uniqueness theorem(s) or lemma(s).

Theorem 2.1.1. *Suppose $U \subset \mathbb{C}^n$ is a bounded domain, and $\mathbf{F} : U \to U$ a holomorphic map such that there exists a $\mathbf{z_0} \in U$, $F(\mathbf{z_0}) = \mathbf{z_0}$ and $D_{\mathbb{C}}\mathbf{F}(\mathbf{z_0}) = \mathbb{id}$. Then $\mathbf{F}(\mathbf{z}) = \mathbf{z}$ for all $\mathbf{z} \in U$.*

PROOF. W.L.O.G, assume $\mathbf{z_0} = \mathbf{0}$. Since U is open and bounded, there exists $r_1, r_2 \in (0, \infty)$ (r_1 small and r_2 large) such that

$$\mathbb{B}_n(r_1) \subset U \subset \mathbb{B}_n(r_2)$$

For $\|\mathbf{z}\|_2 < r_1$, $\mathbf{F}$ has a homogeneous expansion

$$\mathbf{F}(\mathbf{z}) = \mathbf{z} + \sum_{s=2}^{\infty} \mathbf{Q}_s(\mathbf{z})$$

in which each $\mathbf{Q}_s : \mathbb{C}^n \to \mathbb{C}^n$ has components that are homogeneous polynomials of degree s.

Since the domain and codomain of $\mathbf{F}$ are the same, it allows us to consider $\mathbf{F}^{(k)}$ be the k-th iterate of $\mathbf{F}$ defined by

$$\mathbf{F}^{(k)} := \underbrace{\mathbf{F} \circ \cdots \circ \mathbf{F}}_{k \text{ times}}$$

In our notation, $\mathbf{Q}_0 = \mathbf{0}$ and $\mathbf{Q}_1 = \mathbb{id}$ and let $m \geq 2$ be such that $\mathbf{Q}_0 = \mathbf{0}$ for $2 \leq s \leq m-1$. Then $\mathbf{F}^{(k)}$ has the homogeneous expansion

$$\mathbf{F}^{(k)}(\mathbf{z}) = \mathbf{z} + k\,\mathbf{Q}_m(\mathbf{z}) + \cdots$$

in $\mathbb{B}_n(r_1)$, as is easily proved by induction on k. The homogeneity of the maps $\mathbf{Q}_s$ implies that

$$k\,\mathbf{Q}_m(\mathbf{z}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{F}^{(k)}(e^{i\theta}\mathbf{z})\,e^{-im\theta}\,d\theta$$

as a $\mathbb{C}^n$-valued integral for $\|\mathbf{z}\|_2 < r_1$. Since $\mathbf{F}^{(k)}$ maps U into U, we have $|\mathbf{F}^{(k)}(e^{i\theta}\mathbf{z})| < r_2$ for all $\mathbf{z} \in \mathbb{B}_n(r_1)$ and for all $\theta \in \mathbb{R}$. Thus

$$k\,|\mathbf{Q}_m(\mathbf{z})| < r_2$$

for $k \in \mathbb{N}$ and all $\mathbf{z} \in \mathbb{B}_n(r_1)$. Hence $\mathbf{Q}_m = \mathbf{0}$ (by the identity theorem) and by induction we have it for all $m \in \mathbb{N}$. Thus, $\mathbf{F}(\mathbf{z}) = \mathbf{z}$ for all $\mathbf{z} \in \mathbb{B}_n(r_1)$. Since U is connected, by the identity theorem we have that:

$$\boxed{\mathbf{F} \equiv \mathbb{id}_U}$$

□

Theorem 2.1.2. *Let U_1 and U_2 be circular domains in $\mathbb{C}^n$, where U_1 is bounded and both contain the origin. Let $\mathbf{F}$ be a biholomorphic map of U_1 onto U_2, with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$. Then $\mathbf{F}$ is a linear transformation.*

PROOF. Let $\mathbf{G} = \mathbf{F}^{-1}$. Fix a $\theta \in \mathbb{R}$, and define $\mathbf{H} \in \mathcal{O}(U_1, U_1)$ as follows:

$$\mathbf{H}(\mathbf{z}) := \mathbf{G}(e^{-i\theta}\mathbf{F}(e^{i\theta}\mathbf{z})) \quad \forall\ \mathbf{z} \in U_1$$

Since U_1 and U_2 are circular, $\mathbf{H}(\mathbf{z})$ is well-defined, and $\mathbf{H}$ is a holomorphic map of U_1 into U_1 that satisfies $\mathbf{H}(\mathbf{0}) = \mathbf{0}$, $D_{\mathbb{C}}\mathbf{H}(\mathbf{0}) = I$. By Theorem 2.1.1, $\mathbf{H}(\mathbf{z}) = \mathbf{z}$. If we apply $\mathbf{F}$ to this and multiply by $e^{i\theta}$, we obtain

$$\mathbf{F}(e^{i\theta}\mathbf{z}) = e^{i\theta}\mathbf{F}(\mathbf{z}),$$

for all $\mathbf{z} \in U_1$, and for every real θ. The linear term in the homogeneous expansion of $\mathbf{F}$ is therefore the only one that is different from 0. $\square$

Theorem 2.1.3. *When $n > 1$, there is no biholomorphic map of $\mathbb{B}_n$ onto the polydisc $\mathbb{D}^n$.*

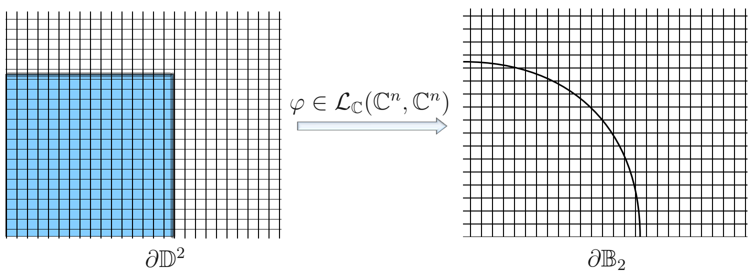


Figure 2.1: The geometry behind the inequivalence

PROOF. Let $\mathbf{a} = \mathbf{F}^{-1}(\mathbf{0})$. Then we can postcompose by an automorphism $\boldsymbol{\varphi}_{\mathbf{a}}$ of Δ_n:

$$(z_1, \dots, z_n) \xrightarrow{\varphi_{\mathbf{a}}} \left(\frac{z_1 - a_1}{1 - \overline{a_1} z_1}, \cdots, \frac{z_n - a_n}{1 - \overline{a_n} z_n} \right)$$

to get $\mathbf{F} \circ \boldsymbol{\varphi}_{\mathbf{a}} \in \mathcal{O}(\mathbb{B}_n, \Delta_n)$ that preserves $\mathbf{0}$. By Theorem 2.1.2, $\mathbf{F} \circ \boldsymbol{\varphi}_{\mathbf{a}}$ is a linear bijection which is defined on all of $\mathbb{C}^n$, hence $\mathbf{F}(\partial\mathbb{B}_n) = \partial\Delta_n$ (this follows from the openness of $\mathbf{F}$). But $\partial\Delta_n$ includes the real segment $[-1, 1] \times \{1\} \times \{0\} \times \cdots \times \{0\}$. Since $\mathbf{F}^{-1}$ is linear, $\partial\mathbb{B}_n = \mathbf{F}^{-1}(\partial\Delta_n)$ must contain a real segment as well, which is absurd as the boundary of an Euclidean ball is not affine convex which is clear as otherwise the quadratic equation

$$\sum_{j=1}^{n} |ta_1^{(j)} + (1-t)a_2^{(j)}|^2 = 1$$

will have every $t \in [0, 1]$ as a root (for any two points $\mathbf{a}_1, \mathbf{a}_2 \in \partial\mathbb{B}_n$) which is impossible! $\square$

2.2 Holomorphic convexity

To understand domains of holomorphy we will study a form of convexity that is invariant under biholomorphic transformations. The following sections aim to characterize these domains using the notion of holomorphic convexity.

Let $U \subset \mathbb{C}^n$ be a domain and take a subset $K \subset U$.
Definition 2.2.1. (Holomorphically convex hull) The subset

$$\widehat{K}_{\mathcal{O}(U)} := \bigcap_{f \in \mathcal{O}(U)} \left\{ \mathbf{z} \in U \ : \ |f(\mathbf{z})| \leq \sup_K |f| \right\}$$

is called the ***holomorphically convex hull*** of K in U.

It is important to note that the holomorphic hull depends on the choice of the ambient domain. The hull gets bigger with the ambient domain. Let's illustrate this with an example (see **[15, Chapter-6]**).
Example 2.2.1. Consider the unit circle $\mathbb{T} \subset \mathbb{C}$.

(i) $\widehat{\mathbb{T}}_{\mathcal{O}(\mathbb{C}^\times)} = \mathbb{T}$.

Consider the two holomorphic functions $\mathrm{id}_{\mathbb{C}^\times}$ and $\dfrac{1}{\mathrm{id}_{\mathbb{C}^\times}}$. The hull $\widehat{\mathbb{T}}_{\mathcal{O}(\mathbb{C}^\times)}$ is therefore contained in $\mathbb{T}$ itself (and contains it).

(ii) $\widehat{\mathbb{T}}_{\mathcal{O}(\mathbb{C})} = \bar{\mathbb{D}}$.

A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ when restricted to $f|_{\bar{\mathbb{D}}}$ attains it's maximum modulus on the boundary $\mathbb{T}$. Therefore, $\bar{\mathbb{D}} \subset \{z \in \mathbb{C} \ : \ |f(z)| \leq \sup_{\mathbb{T}} f\} \ \forall \ f \in \mathcal{O}(\mathbb{C})$. Therefore, we have that $\bar{\mathbb{D}} \subset \widehat{\mathbb{T}}_{\mathcal{O}(\mathbb{C})} \subset \bar{\mathbb{D}}$.

Proposition 2.2.1. (Properties of holomorphically convex hulls)

(i) $\widehat{K}_{\mathcal{O}(U)}$ is closed in U.

(ii) $\widehat{(\widehat{K}_{\mathcal{O}(U)})}_{\mathcal{O}(U)} = \widehat{K}_{\mathcal{O}(U)}$.

(iii) $\widehat{K}_{\mathcal{O}(U)}$ is bounded, if K is bounded.

Definition 2.2.2. (HOLOMORPHIC CONVEXITY) A domain $U \subset \mathbb{C}^n$ is said to be ***holomorphically convex*** if for every subset $K \Subset U$ it is implied that $\widehat{K}_{\mathcal{O}(U)} \Subset U$.

The first important feature of holomorphic convexity is it's biholomorphic invariance.

Theorem 2.2.1. (BIHOLOMORPHISM INVARIANCE OF HOLOMORPHIC CONVEXITY) *If two domains $U_1, U_2 \subset \mathbb{C}^n$ are biholomorphically equivalent, then U_1 is holomorphically convex if and only U_2 is.*

PROOF. Consider a biholomorphism $\varphi : U_1 \to U_2$. Clearly this induces an isomorphism of $\mathbb{C}$-algebras $(\varphi^{-1})^\star : \mathcal{O}(U_1) \cong \mathcal{O}(U_2)$. Suppose that $K \Subset U_2$, we now have to show $\widehat{K}_{\mathcal{O}(U_2)} \Subset U_2$. From the definition we get that

$$\begin{aligned}
\widehat{K}_{\mathcal{O}(U_2)} &:= \bigcap_{\widetilde{f} \in \mathcal{O}(U_2)} \{\mathbf{z} \in U_2 \ : \ |\widetilde{f}(\mathbf{z})| \leq \sup_K \widetilde{f}\} \\
&= \bigcap_{(\varphi^{-1})^\star f \in \mathcal{O}(U_2)} \{\mathbf{z} \in U_2 \ : \ |(\varphi^{-1})^\star f(\mathbf{z})| \leq \sup_K (\varphi^{-1})^\star f\} \\
&= \bigcap_{f \in \mathcal{O}(U_1)} \varphi(\{\mathbf{z} \in U_1 \ : \ |f(\mathbf{z})| \leq \sup_{\varphi^{-1}(K)} f\}) \\
&= \varphi(\widehat{K}_{\mathcal{O}(U_1)}) \Subset U_2.
\end{aligned}$$

□

2.3 Examples of holomorphically convex domains

In this section, we provide examples of holomorphically convex domains from [3, Chapter-2].

2.3.1 In one dimension

In one dimension, the concept of holomorphic convexity is trivial. This is because we can always find a singular holomorphic function whose modulus shoots to infinity as we approach some part of the boundary of the domain.

Proposition 2.3.1. Every domain in $\mathbb{C}$ is holomorphically convex.

PROOF. Consider a domain $U \subset \mathbb{C}$ a subset $K \Subset U$ (in particular it is bounded). We need to show that $\widehat{K}_{\mathcal{O}(U)}$ is relatively compact in U. It is enough to show that the closure of the hull $\widehat{K}_{\mathcal{O}(U)}$ in $\mathbb{C}$ is contained in U as since the hull is bounded, it is enough to show it is closed as a subset of $\mathbb{C}$. Suppose it is not closed; then there is a point $z_0 \in \overline{\widehat{K}_{\mathcal{O}(U)}}^{\mathbb{C}} \setminus U$. Clearly, $z_0 \in \partial\widehat{K}_{\mathcal{O}(U)} \cap \partial U$ and therefore can be approximated by a sequence $z_j \in \widehat{K}_{\mathcal{O}(U)}$ converging to z_0. By the definition of the hull, for every $f \in \mathcal{O}(U)$

$$|f(z_j)| \leq \sup_K |f| \leq \sup_{\overline{K}} |f| < \infty$$

for all $j = 1, 2, \cdots$. The holomorphic function $f : z \mapsto \dfrac{1}{z - z_0}$ clearly is unbounded near z_0 and violates the above deduction. □

2.3.2 In higher dimensions

Holomorphic convexity is a weaker notion of convexity compared to affine convexity for domains in $\mathbb{C}^n$.

Proposition 2.3.2. Every affine convex domain in $\mathbb{C}^n$ is holomorphically convex.

PROOF. We will show that $\widehat{K}_{\mathcal{O}(U)} \subset \widehat{K}_{(\mathbb{R}^{2n})^\vee}$ which readily implies that the proposition. Take a point $\mathbf{z_0} \in U \setminus \widehat{K}_{(\mathbb{R}^{2n})^\vee}$. By definition, there exists a $\mathbb{R}$-linear functional $\varphi : \mathbb{R}^{2n} \to \mathbb{R}$ such that $\varphi(\mathbf{z_0}) > \sup_K \varphi$. Now consider the $\mathbb{R}$-linear map,

$$\mathfrak{Re}(\,.\,) : \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}) \hookrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}).$$

It is easily checked to be injective and hence by the rank-nullity theorem, a linear isomorphism. Therefore, there exists a $\boldsymbol{\alpha} \in \mathbb{C}^n$ such that

$$\varphi(\mathbf{z}) = \mathfrak{Re}(\langle \mathbf{z}, \boldsymbol{\alpha} \rangle) \ \forall \ \mathbf{z} \in \mathbb{C}^n.$$

Define $f \in \mathcal{O}(U)$ as:

$$f : \mathbf{z} \mapsto e^{\langle \mathbf{z},, \boldsymbol{\alpha} \rangle}.$$

One easily obtains, $|e^{\langle \mathbf{z_0}, \boldsymbol{\alpha} \rangle}| > \sup_{\boldsymbol{\zeta} \in K} |e^{\langle \boldsymbol{\zeta}, \boldsymbol{\alpha} \rangle}|$, therefore, $\mathbf{z_0} \in U \setminus \widehat{K}_{\mathcal{O}(U)}$. □

The affine convex hull $\widehat{K}_{(\mathbb{R}^{2n})^\vee}$ can be described with the aid of holomorphic functions as follows:

$$\widehat{K}_{(\mathbb{R}^{2n})^{\vee}} = \{\mathbf{z} \in \mathbb{C}^n : \operatorname{Re}\langle \mathbf{z}, \boldsymbol{\alpha}\rangle \leq \sup_{\boldsymbol{\zeta}\in K} \operatorname{Re}\langle \boldsymbol{\zeta}, \boldsymbol{\alpha}\rangle, \quad \forall\, \boldsymbol{\alpha} \in \mathbb{C}^n\}$$
$$= \{\mathbf{z} \in \mathbb{C}^n : |e^{\langle \mathbf{z}, \boldsymbol{\alpha}\rangle}| \leq \sup_{\boldsymbol{\zeta}\in K} |e^{\langle \boldsymbol{\zeta}, \boldsymbol{\alpha}\rangle}|, \quad \forall\, \boldsymbol{\alpha} \in \mathbb{C}^n\}.$$

Therefore, $\widehat{K}_{(\mathbb{R}^{2n})^{\vee}}$ is described with the aid of the special family of entire functions

$$\{f : \mathbf{z} \mapsto e^{\langle \mathbf{z},, \boldsymbol{\alpha}\rangle} |\ \boldsymbol{\alpha} \in \mathbb{C}^n\}.$$

If one uses the full class of holomorphic functions, one obtains the holomorphic hull providing us with a biholomorphically invariant notion of convexity!

2.4 Characterization of domains of holomorphy

Holomorphic convexiy gains it's importance by allowing for a characterization of the domains of holomorphy via the Cartan-Thullen theorem.

Theorem 2.4.1. *The following statements are all equivalent:*

1. *U is a weak domain of holomorphy.*

2. *U is a holomorphically convex domain.*

3. *For every infinite subset $D \subset U$ without an accumulation point in U, there exists a $f \in \mathcal{O}(U)$ which blows up on D i.e, $\sup_D |f| = \infty$.*

4. *U is a domain of holomorphy.*

PROOF.

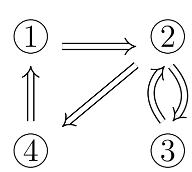


Figure 2.2: A connected digraph illustration

① ⇒ ② The implication follows straightforwardly from the following theorem.

Theorem 2.4.2. (Cartan-Thullen theorem)
For a weak domain of holomorphy U, we have for every $K \Subset U$:

$$\mathbb{d}(K, \partial U) = \mathbb{d}(\widehat{K}_{\mathcal{O}(U)}, \partial U),$$

where $\mathbb{d}$ is the distance in the component wise max. metric. Therefore G is holomorphically convex.

② ⇔ ③

② ⇒ ③ The following lemmas about normal exhaustions (see [3, Chapter-2]) will be used; By a normal exhaustion of a domain $U \subset \mathbb{C}^n$, we mean a sequence $\{K_\nu\}_{\nu\in\mathbb{N}}$ of compact subsets of U such that:

- $K_\nu \Subset (K_{\nu+1})^\circ$, for every $\nu \in \mathbb{N}$.
- $\bigcup_{\nu\in\mathbb{N}} K_\nu = U$.

Lemma 2.4.1. For a holomorphically convex domain $U \subset \mathbb{C}^n$, there exists a normal exhaustion $\{K_\nu\}_{\nu\in\mathbb{N}}$ with $\widehat{(K_\nu)}_{\mathcal{O}(U)} = K_\nu$ for all $\nu \in \mathbb{N}$.

Lemma 2.4.2. Let $\{K_\nu\}_{\nu\in\mathbb{N}}$ be a normal exhaustion of U with $\widehat{(K_\nu)}_{\mathcal{O}(U)} = K_\nu$ for all $\nu \in \mathbb{N}$. Let $\{\eta(\mu)\}_{\mu\in\mathbb{N}}$ a monotonically increasing sequence in $\mathbb{N}$ indexing a sequence of points $\{z_\mu\}_{\mu\in\mathbb{N}}$ such that $z_\mu \in K_{\eta(\mu)+1} - K_{\eta(\mu)}\ \forall\ \mu \in \mathbb{N}$. Then there exists a $f \in \mathcal{O}(U)$ such that $\lim_{\mu\to\infty} |f(z_\mu)| = \infty$.
=

Now let's show the implication in steps:

- Let $\{K_\nu\}_{\nu\in\mathbb{N}}$ be a normal exhaustion with $\widehat{(K_\nu)}_{\mathcal{O}(U)} = K_\nu$ for all ν.
- We construct a sequence on which f is unbounded by induction; let $z_1 \in D - K_1 \neq \varnothing$ and $\eta(1) := \min\{\nu \in \mathbb{N} : z_1 \in K_{\nu+1}\}$.
- Proceeding inductively, let's say we have constructed $z_1, \ldots, z_{\mu-1}$ along with the indices $\eta(1), \ldots \eta(\mu-1)$ such that $z_\nu \in K_{\eta(\nu)+1} - K_{\eta(\nu)}$ for $\nu = 1, \ldots, \mu-1$. At this inductive step, we choose $z_\mu \in D - K_{\nu(\mu-1)+1}$ and $\eta(\mu) := \min\{\nu \in \mathbb{N} : z_\mu \in K_{\nu+1}\}$.
- By the above lemmas 2.4.1 and 2.4.2, we get a $f \in \mathcal{O}(U)$ such that $\sup_{z\in D} |f(z)| \geq \sup_{\mu\in\mathbb{N}} |f(z_\mu)| = \infty$ as required.

② ⇐ ③

- For a sequence $\{z_\nu\}_{\nu\in\mathbb{N}}$ in $\widehat{K}_{\mathcal{O}(U)}$, by definition of the hull, we have $\sup_{\nu\in\mathbb{N}} |f(z_\nu)| \le \sup_K |f| < \infty$ for all $f \in \mathcal{O}(U)$.

- By the forward implication shown just above, $\{z_\nu\}_{\nu\in\mathbb{N}}$ clusters in U .

- Since the hull is closed in U, the cluster point is in the hull itself, $\widehat{K}_{\mathcal{O}(U)}$ is sequentially compact and hence we are done.

② ⇒ ④

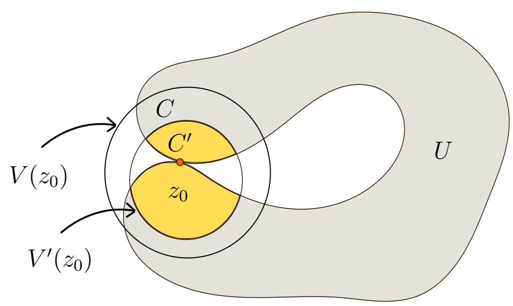


Figure 2.3: Picturing the contradiction

- Considering the sequences $\{z_\mu\}_{\mu\in\mathbb{N}}$ and $\{K_{\eta(\mu)}\}_{\mu\in\mathbb{N}}$ given by the above consideration, we have a $f \in \mathcal{O}(U)$ such that $\lim\limits_{\mu\to\infty} |f(z_\mu)| = \infty$.

- Suppose U is not a domain of holomorphy, then for every $f \in \mathcal{O}(U)$ we have some $z_0 \in \partial U$, a neighbourhood $V = V(z_0)$ and $\widehat{f} \in \mathcal{O}(V)$ such that $\widehat{f}|_C = f|_C$ for some connected component $C \subset V \cap U$.

- Now we consider $V' = V'(z_0) \Subset V$ and the connected component $C' \subset C$ of $V' \cap U$.

- This means:
$$\sup_{C'} |f| = \sup_{C'} |\widehat{f}| \le \sup_{V'} |\widehat{f}| < \infty$$
which contradicts the fact that there are infinitely many members of $\{z_\mu\}_{\mu\in\mathbb{N}}$ in C' and $\lim\limits_{\mu\to\infty} |f(z_\mu)| = \infty$.

④ ⇒ ① Follows trivially from the definitions.

□

2.5 Hartogs's extension theorem

In this section, we have elaborated upon the presentation of Hartogs's extension theorem as in [8, Chapter-4].

2.5.1 The $\overline{\partial}$-problem

The $\bar{\partial}$-problem plays the role of a corrector problem in this context of Cauchy-Riemann system of PDEs. We only need to work with compactly supported smooth differential forms for our purposes.

Theorem 2.5.1. *Let $n > 1$. For every $f \in \mathscr{C}^{\infty}_{c,(0,1)}(\mathbb{C}^n, \mathbb{C})$ such that $\bar{\partial} f = 0$ i.e, $f_1, \ldots, f_n$ satisfy the compatibility conditions,*

$$\frac{\partial f_j}{\partial \bar{z_k}} = \frac{\partial f_k}{\partial \bar{z_j}} \quad \forall\, j, k = 1, \ldots, n$$

we have a unique $u \in \mathscr{C}^{\infty}_{c}(\mathbb{C}^n, \mathbb{C})$ solving:

$$\bar{\partial} u = f, \tag{2.1}$$

given by:

$$u(z_1, \ldots, z_n) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{f_k(z_1, \ldots, \zeta, \ldots, z_n)}{\zeta - z_k}\, d\zeta \wedge d\bar{\zeta}, \tag{2.2}$$

where $k = 1, \ldots, n$ and f_k's are such that $f_k \in \mathscr{C}_c(\mathbb{C}^n, \mathbb{C})$ and:

$$f = f_1 d\bar{z_1} + \ldots + f_n d\bar{z_n}.$$

PROOF.

<u>Existence:</u> The main supporting ingredient will be the Cauchy-Pompieu formula.

<u>STEP-1:</u> We show that the following u (for any $k = 1, \ldots, n$) solves the inhomogeneous

Cauchy problem.

$$\begin{aligned} u(z_1,\dots,z_n) &:= \frac{1}{2\pi i}\iint\limits_{\mathbb{C}} \frac{f_k(z_1,\dots,\zeta,\dots,z_n)}{\zeta - z_k}\, d\zeta\wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\iint\limits_{\mathbb{C}} \frac{f_k(z_1,\dots,\zeta+z_k,\dots,z_n)}{\zeta}\, d\zeta\wedge d\bar{\zeta} \end{aligned}$$

The last step was performed to avoid the singularity while differentiating w.r.t z_k.

<u>Step-2:</u> Take $R>0$ large enough such that $f_k(z_1,\dots,z_n)=0$ when atleast one $z_j\geq R$. Use Cauchy-Pompieu formula on the z_k variable to get:

$$\begin{aligned} f_j(z_1,\dots,z_n) &= \int\limits_{|\zeta|=R} \frac{f_k(z_1,\dots,\zeta,\dots,z_n)}{\zeta - z_k} d\zeta + \frac{1}{2\pi i}\int\limits_{|\zeta|\leq R} \frac{\frac{\partial f_j}{\partial \bar{z}_k}(z_1,\dots,\zeta+z_k,\dots,z_n)}{\zeta} d\zeta\wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\int\limits_{|\zeta|\leq R} \frac{\frac{\partial f_j}{\partial \bar{z}_k}(z_1,\dots,\zeta+z_k,\dots,z_n)}{\zeta}\, d\zeta\wedge d\bar{\zeta}. \end{aligned}$$

<u>Step-3:</u> We compute the partial derivative $\dfrac{\partial u}{\partial \bar{z}_k}$ (note the use of the compatibility conditions):

$$\begin{aligned} \frac{\partial u}{\partial \bar{z}_j}(z_1,\dots,z_n) &= \frac{1}{2\pi i}\frac{\partial}{\partial \bar{z}_j}\iint\limits_{\mathbb{C}} \frac{f_k(z_1,\dots,\zeta,\dots,z_n)}{\zeta - z_k}\, d\zeta\wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\iint\limits_{\mathbb{C}} \frac{\frac{\partial f_k}{\partial \bar{z}_j}(z_1,\dots,\zeta+z_k,\dots,z_n)}{\zeta}\, d\zeta\wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\iint\limits_{\mathbb{C}} \frac{\frac{\partial f_j}{\partial \bar{z}_k}(z_1,\dots,\zeta+z_k,\dots,z_n)}{\zeta}\, d\zeta\wedge d\bar{\zeta} \\ &= f_j(z_1,\dots,z_n). \end{aligned}$$

We will now justify that the order of differentiation and integration could be switched legitimately in the first step. We apply the dominated convergence theorem on the sequence,

$$g_m(\zeta) := \frac{f_k(\mathbf{z} + \zeta \mathbf{e}_k + i\, h_m \mathbf{e}_k) - f_k(\mathbf{z} + \zeta \mathbf{e}_k)}{i\, \zeta h_m},$$

where $\mathbb{R} \ni h_m \to 0$ as $m \to \infty$.

$$|g_m(\zeta)| \leq \sup_{h \in [0,1]} |\zeta^{-1}\, \partial_{\bar{z}_j} f_k(\mathbf{z} + \zeta \mathbf{e}_k + i\, h \mathbf{e}_k)| \leq \frac{C}{|\zeta|}$$

which is integrable because, $d\zeta \wedge d\bar{\zeta}$ is really just the area form (with a constant coefficient) the integration could be carried out with substitution with respect to the polar coordinates, i.e, $\zeta \leftrightarrow re^{i\theta}$ owing to the radial symmetry of the disc yielding:

$$\iint_{[0,\varepsilon) \times [0,2\pi]} \frac{1}{\cancel{r}} \cancel{r} dr d\theta$$

which is integrable. Therefore, because of compact support, the integral on $\{|\zeta| \geq \varepsilon\}$ is zero and we have the integrability as desired.

Uniqueness: The compactly supported solution is unique: if u_1 and u_2 are solutions, then

$$\bar{\partial}(u_1 - u_2) = f - f = 0$$

and so $u_1 - u_2 \in \mathcal{O}_c(\mathbb{C}^n) = \{0\}$, since the only compactly supported entire function is 0, thereby giving us uniqueness. □

Now that we know that there is a unique compactly supported solution of the $\bar{\partial}$-problem, it is fair to ask for a description of the support of the solution. This plays a crucial role in the proof of Hartogs's ***Kugelsatz***.

Proposition 2.5.1. For the $\bar{\partial}$-problem

$$\bar{\partial} u = f_1 d\bar{z}_1 + \ldots + f_n d\bar{z}_n,$$

if $\cup_{j=1}^{n} \text{supp}(f_j) \subset C \Subset \mathbb{C}^n$, where C is compact ($n \geq 2$), then u is supported in the complement of the unbounded component of $\mathbb{C}^n \setminus C$.

PROOF. Decomposition into connected components yields $\mathbb{C}^n \setminus C = U_1 \cup U_2 \cup \cdots$. Now since C is bounded (as it is compact), there exists $R > 0$ large enough such that $C \subset \overline{\mathbb{B}_n(\mathbf{0}, R)}$.

This means that the connected open set $\mathbb{C}^n \setminus \overline{\mathbb{B}_n(\mathbf{0}, R)} \subset \mathbb{C}^n \setminus C$ is contained in a unique component amongst the family $\{U_k\}_{k\in\mathbb{N}}$ which we call U_1 (without loss of generality) since it is a connected component of $\mathbb{C}^n \setminus C$. Now, rest all components are bounded because

$$\overline{\mathbb{B}_n(\mathbf{0}, R)} \setminus C = (\mathbb{C}^n \setminus C) \setminus (\mathbb{C}^n \setminus \overline{\mathbb{B}_n(\mathbf{0}, R)}) = U_2 \cup U_3 \cup \cdots$$

Since $\cup_{j=1}^n \text{supp}(f_j) \subset C \Subset \mathbb{C}^n$, we have that $\bar{\partial}u = 0$ on $\mathbb{C}^n \setminus C$, in particular on it's subset U_1. This means that u is holomorphic away from the support i.e, $u \in \mathcal{O}(U_1)$. Consider the integral representation given in the above theorem:

$$u(z_1, \ldots, z_n) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \frac{f_k(z_1, \ldots, \zeta, \ldots, z_n)}{\zeta - z_k}\, d\zeta \wedge d\bar{\zeta}.$$

Since U_1 is unbounded, it contains a neighbourhood far away such that:

$$|z_1| \geq R' \vee \cdots \vee |z_n| \geq R' \Rightarrow u(z_1, \ldots, z_n) = 0.$$

Therefore, $u|_{U_1} \equiv 0$ implying that $\text{supp}(u) \subset \mathbb{C}^n \setminus U_1$ which in expansive terms is the complement of the unbounded component of $\mathbb{C}^n \setminus C$.

□

Therefore, we have

$$\boxed{H^{(0,1)}_{\bar{\partial},c}(\mathbb{C}^n) = \mathbf{0}}$$

2.5.2 An application-The Hartogs's phenomena

Theorem 2.5.2. (Hartogs's Kugelsatz)
Given a domain $U \subset \mathbb{C}^n$ and a compact subset $K \Subset U$ such that K does not separate the domain i.e, $U \setminus K$ is connected, every holomorphic function $f \in \mathcal{O}(U \setminus K)$ extends uniquely to a $\widetilde{f} \in \mathcal{O}(U)$ i.e, $\widetilde{f}|_{U\setminus K} \equiv f$. In algebraic language, we have:

$$\text{res}\,|_{U\setminus K}\, : \mathcal{O}(U) \xrightarrow{\cong} \mathcal{O}(U \setminus K)$$

is an isomorphism of $\mathbb{C}$-algebras.

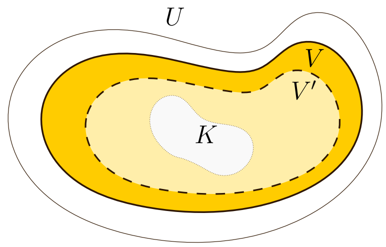


Figure 2.4: The diagrammatic setup of Hartogs's *Kugelsatz*

PROOF.

STEP-1: The idea is to extend the function in a seemingly arbitrary fashion and correct it by the solution of an appropriate $\bar{\partial}$-problem associated with our choice of extension. This will be achieved in the following steps.

STEP-2: In our present context, Proposition 2.5.1 shows that if $\mathrm{supp}(f_1 d\bar{z}_1 + \ldots + f_n d\bar{z}_n) \subset C$ and $\mathbb{C}^n \setminus C$ is connected, then the $\mathrm{supp}(u) \subset C$.

STEP-3: Take a cut-off function $\varphi \in \mathscr{C}^\infty(\mathbb{C}^n, \mathbb{R})$ such that there are open sets V', V such that

$$K \subset V' \Subset V \Subset U$$

and the restrictions

$$\varphi\,|_{V'} \equiv 1 \;, \varphi\,|_{\mathbb{C}^n \setminus \overline{V}} \equiv 0$$

ensuring that φ vanishes on the boundary ∂U.

STEP-4: Consider the 'φ-based' extension $f_\varphi \in \mathscr{C}^\infty(U, \mathbb{C})$ defined as follows:

$$f_\varphi := \begin{cases} (1-\varphi)f & \text{on } U \setminus K\,, \\ 0 & \text{on } V' \end{cases}$$

which is easily checked to be well-defined.

STEP-5: Define $f_1, \ldots, f_n$ as follows:

$$f_j := \begin{cases} \dfrac{\partial f_\varphi}{\partial \bar{z}_j} & \text{on } U \setminus K\,, \\ 0 & \text{on } V' \end{cases} \quad \forall \;\; j = 1, \ldots, n\;.$$

It is easily checked that the f_j's are well-defined and $f_j \in \mathscr{C}_c^\infty(\mathbb{C}^n, \mathbb{C})$.

STEP-6: Consider the $\bar{\partial}$-problem

$$\bar{\partial}u := f_1 d\bar{z}_1 + \ldots + f_n d\bar{z}_n.$$

Using theorem 2.5.1, we know that there exists a solution $u \in \mathscr{C}^{\infty}_{c,(0,1)}(\mathbb{C}^n, \mathbb{C})$ such that $\text{supp}(u) \subset \overline{V} \setminus V' \Subset U$ from STEP-2. Using this, define $\widetilde{f} := f_\varphi - u|_U$ which will be shown to be the desired extension subsequently.

STEP-7: $\widetilde{f} \in \mathcal{O}(U)$ because:

$$\begin{aligned}\bar{\partial}\widetilde{f} &= \bar{\partial}(f_\varphi - u) \\ &= \bar{\partial}f_\varphi - \bar{\partial}u \\ &= \sum_{k=1}^{n} \partial_{\bar{z}_k} f_\varphi \, d\bar{z}_k - f_k d\bar{z}_k \\ &= 0.\end{aligned}$$

STEP-8: Now near ∂U i.e, on $U \setminus \overline{V}$, $\widetilde{f} = (1-\varphi)f - u = f$ as $\varphi|_{U\setminus\overline{V}} \equiv 0$ and $u = 0$ on $\text{supp}(u)^c \supset (\overline{V} \setminus V')^c \supset U \setminus \overline{V}$. Therefore, $\widetilde{f}$ and f agree on the open set $U \setminus \overline{V} \subset U \setminus K$. Invoking the Identity theorem (since $U \setminus K$ is connected), we get

$$\widetilde{f}|_{U\setminus K} \equiv f$$

which means that $\widetilde{f}$ is the unique extension of f to whole of U.

Therefore, the map $\text{res}\,|_{U\setminus K} : \mathcal{O}(U) \to \mathcal{O}(U \setminus K)$ is surjective, hence bijective by the identity theorem. □

Hartogs's theorem strengthens the fact that zero sets of polynomials i.e, elements of $\mathbb{C}[z_1, \cdots, z_n]$ in $\mathbb{C}^n$ are never bounded (equivalently compact) and provides non-examples of domains of holomorphy.

Corollary 2.5.2.1. Consider $n \geq 2$.

(i) For any $f \in \mathcal{O}(U)$ for a domain $U \subset \mathbb{C}^n$, the zero set $\mathcal{Z}_U(f)$ is never compact and f does not possess any isolated singularities.

(ii) Spherical shells defined by $\{\mathbf{z} \in \mathbb{C}^n : a < ||\mathbf{z}||_2 < b\}$ for any positive a, b are ***not*** domains of holomorphy.

Chapter 3

Hörmander estimates

In this chapter, we introduce the Hörmander's method for $\overline{\partial}$-problem in weighted L^2-spaces. We first do it in one dimension for a strictly $\mathcal{PSH}$ weight and later generalize it to higher dimensions with the regularity assumption removed following [1, Chapter-7] and [7, Chapter-4].

3.1 The one dimensional warm-up: On domains of $\mathbb{C}$

Theorem 3.1.1. *Let $U \subset \mathbb{C}$ be an open subset and φ be a $\mathscr{C}^2$ strictly subharmonic function. Then for every $f \in L^2_{(0,1),\mathbb{C}}(U, e^{-\varphi})$ satisfying:*

$$\int_{\zeta \in U} \frac{|f(\zeta)|^2}{\varphi_{z\bar{z}}(\zeta)} e^{-\varphi(\zeta)} d\Lambda_{\mathbb{R}^2}(\zeta) < \infty, \tag{3.1}$$

there exists a unique $u \in L^2_{\mathbb{C}}(U, e^{-\varphi})$ solving:

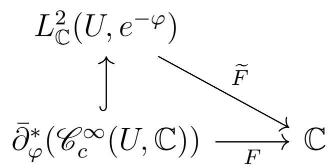
$$\bar{\partial} u = f \tag{3.2}$$

in the sense of distributions, with the L^2 estimate,

$$||u||^2_{L^2_{\mathbb{C}}(U,e^{-\varphi})} = \int_U |u|^2 e^{-\varphi} \leq \int_U \frac{|f|^2}{\varphi_{z\bar{z}}} e^{-\varphi}. \tag{3.3}$$

Note that there is a slight harmless abuse of notation ; f is both seen as $(0,1)$ form and the function appearing when the form is written as $f\,d\bar{z}$.

PROOF. By Hahn-Banach extension theorem, we have the following lifting:



We will derive the result from two lemmas:

<u>Step-1:</u> The following lemma contains the crux of the result in terms of a necessary and sufficient condition.

Lemma 3.1.1. Let $f \in L^2_{\text{loc},\mathbb{C}}(U, e^{-\varphi})$ where $\varphi \in \mathscr{C}(U, \mathbb{C})$. Given $C > 0$, we have that $\exists\, u \in L^2_{\mathbb{C}}(U, e^{-\varphi})$ solving $\bar{\partial}u = f$ with a bound on the L^2-norm given by $||u||^2_{L^2_{\mathbb{C}}(U,e^{-\varphi})} \leq C$ if and only if

$$|\langle f, \beta\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})}| \leq \sqrt{C} \cdot ||\bar{\partial}^*_\varphi \beta||_{L^2_{\mathbb{C}}(U,e^{-\varphi})}$$

PROOF. We will prove the neccesity and sufficiency as follows:

<u>**Necessary:**</u>

Given such a solution u, use Cauchy-Schwarz inequality to get:

$$\begin{aligned}|\langle f, \beta\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})}| &= |{}_{\mathscr{D}'(U)}\langle \bar{\partial}u, \beta\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})}| \\ &\leq ||u||_{L^2_{\mathbb{C}}(U,e^{-\varphi})}\, ||\bar{\partial}^\star_\varphi \beta||_{L^2_{\mathbb{C}}(U,e^{-\varphi})} \\ &\leq \sqrt{C} \cdot ||\bar{\partial}^\star_\varphi \beta||_{L^2_{\mathbb{C}}(U,e^{-\varphi})}.\end{aligned}$$

Here we interpreted $\overline{\partial}u \in \mathscr{D}'(U)$ as the functional:

$$\begin{aligned}\langle\!\langle \beta, \overline{\partial}u \rangle\!\rangle :&= -\int_U \overline{u}\, \partial(\beta e^{-\varphi}) \\ &= \int_U \overline{u}\; \bar{\partial}^\star_\varphi \beta\, e^{-\varphi} \\ &= \langle \bar{\partial}^\star_\varphi \beta, u\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})}.\end{aligned}$$

<u>Sufficient:</u> Consider the functional F defined by:

$$F : \bar{\partial}^{\star}_{\varphi}(\mathscr{C}^{\infty}_{c}(U, \mathbb{C})) \longrightarrow \mathbb{C}$$
$$\bar{\partial}^{\star}_{\varphi}\beta \longmapsto \langle \beta, f \rangle_{L^2_{\mathbb{C}}(U, e^{-\varphi})}.$$

The norm of F is:

$$\begin{aligned} ||F||_{\mathscr{B}(\bar{\partial}^{\star}_{\varphi}(\mathscr{C}^{\infty}_{c}(U,\mathbb{C})))} &= \sup_{\beta \neq 0} \frac{\langle F(\bar{\partial}^{\star}_{\varphi}\beta), F(\bar{\partial}^{\star}_{\varphi}\beta) \rangle^{\frac{1}{2}}_{\mathbb{C}}}{\langle \bar{\partial}^{\star}_{\varphi}\beta, \bar{\partial}^{\star}_{\varphi}\beta \rangle^{\frac{1}{2}}_{L^2_{\mathbb{C}}(U,e^{-\varphi})}} \\ &= \sup_{\beta \neq 0} \frac{\left| \int_U f \bar{\beta} e^{-\varphi} \right|}{\left(\int_U |\bar{\partial}^{\star}_{\varphi}\beta|^2 e^{-\varphi} \right)^{\frac{1}{2}}} \\ &\leq \sqrt{C}. \end{aligned}$$

Therefore, by Hahn-Banach extension theorem, we can extend F to the whole of $L^2_{\mathbb{C}}(U, e^{-\varphi})$ without increasing the norm. Now, we have by Riesz representation theorem an $u \in L^2_{\mathbb{C}}(U, e^{-\varphi})$ such that

$$F(\alpha) = \langle \alpha, u \rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})}$$

for all $\alpha \in L^2_{\mathbb{C}}(U, e^{-\varphi})$. Now,

$$\begin{array}{ccc} F(\bar{\partial}^{\star}_{\varphi}\beta) & = & \langle \bar{\partial}^{\star}_{\varphi}\beta, u \rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})} \\ \shortparallel & & \shortparallel \\ \langle \beta, f \rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})} & & \langle\!\langle \beta, \bar{\partial} u \rangle\!\rangle \end{array}$$

Therefore, $\bar{\partial} u = f$ as distributions. The L^2-estimate follows from:

$$\begin{aligned} |\langle u, u \rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})}|_{\mathbb{C}} &= |F(u)|_{\mathbb{C}} \\ &\leq ||F||_{\mathscr{B}(L^2_{\mathbb{C}}(U,e^{-\varphi}))} \cdot ||u||_{L^2_{\mathbb{C}}(U,e^{-\varphi})} \\ &\leq \sqrt{C}\, ||u||_{L^2_{\mathbb{C}}(U,e^{-\varphi})}. \end{aligned}$$

□

<u>STEP-2:</u> We will require the following identity to complete the proof.

Lemma 3.1.2. For all $\beta \in \mathscr{C}_c^\infty(U,\mathbb{C})$ and $\varphi \in \mathscr{C}^2(U,\mathbb{R})$, we have that:

$$\int_U |\bar\partial_\varphi^\star \beta|^2 e^{-\varphi} = \int_U \left(|\bar\partial\beta|^2 + \varphi_{z\bar z}\,|\beta|^2\right) e^{-\varphi}.$$

PROOF. Observe that the LHS is $\langle \bar\partial_\varphi^\star\beta, \bar\partial_\varphi^\star\beta\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})}$ which by the definition of the adjoint becomes $\langle \bar\partial\bar\partial_\varphi^\star\beta, \beta\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})}$. Now let's calculate $\bar\partial_\varphi^\star\beta$. By definition, we have for all $\alpha \in \mathscr{C}_c^\infty(U,\mathbb{C})$:

$$\begin{aligned}
\langle \bar\partial_\varphi^\star\beta, \alpha\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})} &= \langle \beta, \bar\partial\alpha\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})} \\
&= \int_U \overline{\bar\partial\alpha}\,\beta\; e^{-\varphi} \\
&= \int_{\mathbb{R}^2} \partial\overline{\alpha}\,\beta\; e^{-\varphi} \\
&= \iint\limits_{[-R,R]\times[-R,R]} \left[\partial(\overline{\alpha}\beta\; e^{-\varphi}) - \overline{\alpha}(\partial\beta - \beta\partial\varphi)\; e^{-\varphi}\right] \\
&= -\int_U \overline{\alpha}(\partial\beta - \beta\partial\varphi)\, e^{-\varphi} \\
&= \langle -\partial\beta + \beta\partial\varphi, \alpha\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})},
\end{aligned}$$

where $R > 0$ is big enough to contain the supports of the functions involved. Since it is true for all $\alpha \in \mathscr{C}_c^\infty(U,\mathbb{C})$, one gets

$$\bar\partial_\varphi^\star\beta = -\partial\beta + \beta\,\partial\varphi.$$

Therefore, one straightforwardly solves to get

$$\begin{aligned}
\bar\partial\bar\partial_\varphi^\star\beta &= \bar\partial(-\partial\beta + \beta\,\partial\varphi) \\
&= \underbrace{-\partial\bar\partial\beta + \bar\partial\beta\partial\varphi}_{\bar\partial_\varphi^\star(\bar\partial\beta)} + \beta\,\partial\bar\partial\varphi \\
&= \beta\,\partial\bar\partial\varphi + \bar\partial_\varphi^\star(\bar\partial\beta).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_U |\bar\partial_\varphi^\star\beta|^2 e^{-\varphi} &= \langle \bar\partial_\varphi^\star\beta, \bar\partial_\varphi^\star\beta\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})} \\
&= \langle \bar\partial\bar\partial_\varphi^\star\beta, \beta\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})} \\
&= \langle \beta\,\varphi_{z\bar z} + \bar\partial_\varphi^\star(\bar\partial\beta), \beta\rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})}
\end{aligned}$$

$$
\begin{aligned}
&= \langle \beta, \beta\varphi_{z\bar{z}} \rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})} + \langle \overline{\partial}\beta, \overline{\partial}\beta \rangle_{L^2_{\mathbb{C}}(U,e^{-\varphi})} \\
&= \int_U (|\bar{\partial}\beta|^2 + \varphi_{z\bar{z}}\,|\beta|^2)\, e^{-\varphi}.
\end{aligned}
$$

□

<u>STEP-3:</u> The theorem follows by an application of the Cauchy-Schwarz inequality:

$$
\begin{aligned}
\left| \int_U f\bar{\beta}e^{-\varphi} \right| &\le \left(\int_U \left(\frac{|f|}{\sqrt{\varphi_{z\bar{z}}}} \right)^2 e^{-\varphi} \right)^{\frac{1}{2}} \cdot \left(\int_U |\sqrt{\varphi_{z\bar{z}}}\,\bar{\beta}|^2 e^{-\varphi} \right)^{\frac{1}{2}} \\
&\le \left(\int_U \frac{|f|^2}{\varphi_{z\bar{z}}} e^{-\varphi} \right)^{\frac{1}{2}} \cdot \left(\int_U (|\bar{\partial}\beta|^2 + \varphi_{z\bar{z}}\,|\beta|^2) e^{-\varphi} \right)^{\frac{1}{2}} \\
&\le \left(\int_U \frac{|f|^2}{\varphi_{z\bar{z}}} e^{-\varphi} \right)^{\frac{1}{2}} \cdot \left(\int_U |\bar{\partial}^{\star}_{\varphi}\beta|^2 e^{-\varphi} \right)^{\frac{1}{2}},
\end{aligned}
$$

where we have used the Lemma 3.1.2 obtained in STEP-2. Now, the theorem follows from the lemma 3.1.1 proved in STEP-1. □

3.2 Done and dusted: Domains in $\mathbb{C}^n$

Theorem 3.2.1. (DENSE SUBSETS OF L^2 SPACES) *For a domain $U \subset \mathbb{C}^n$ and a continuous function $\varphi : U \to \mathbb{R}$, we have:*

$$
\mathscr{C}^\infty_c(U, \mathbb{C}) \underset{\text{dense}}{\subset} L^2_{\mathbb{C}}(U, e^{-\varphi}).
$$

PROOF. Consider a normal exhaustion $\{K_j\}_{j=1}^\infty$ of a domain U. Since $K_j^\circ \nearrow U$, from monotone convergence theorem, for every $\varepsilon > 0$, for large $j \ge j_0$ we have

$$
\int_{U \setminus K_j^\circ} |f|^2\, e^{-\varphi} < \frac{\varepsilon^2}{2}
$$

Now, there exists a $\eta \in \mathscr{C}^\infty_c(K_j^\circ, \mathbb{C})$ such that, $||f - \eta||^2_{L^2_{\mathbb{C}}(K_j^\circ)} < \dfrac{\varepsilon^2}{\sup_{K_j} e^{-\varphi}}$ (since φ is continuous on U, it is bounded on K_j). Hence we get that,

$$\begin{aligned}\int_U |f-\eta|^2 e^{-\varphi} &\leq \int_{K_j^\circ} |f-\eta|^2 e^{-\varphi} + \int_{U\setminus K_j^\circ} |f|^2 e^{-\varphi}\\ &< \int_{K_j^\circ} |f-\eta|^2 \cdot \sup_{K_j} e^{-\varphi} + \frac{\varepsilon^2}{2}\\ &< \varepsilon^2.\end{aligned}$$

Therefore, $||f-\eta||_{L^2_{\mathbb{C}}(U,e^{-\varphi})} < \varepsilon$ for a $\eta \in \mathscr{C}_c^\infty(U,\mathbb{C})$. □

Corollary 3.2.1.1. For spaces of L^2 integrable forms, we have that:

$$\mathscr{C}^\infty_{c,(p,q)}(U,\mathbb{C}) \underset{\text{dense}}{\subset} L^2_{(p,q),\mathbb{C}}(U,e^{-\varphi}).$$

PROOF. For every $\sum_{[J,K]} f_{J,K} dz^J \wedge d\bar{z}^K \in L^2_{(p,q),\mathbb{C}}(U,e^{-\varphi})$, choose $\eta_{J,K}$'s in $\mathscr{C}_c^\infty(U,\mathbb{C})$ such that

$$||f_{J,K}-\eta_{J,K}||_{L^2_{\mathbb{C}}(U,e^{-\varphi})} < \frac{\varepsilon}{\binom{n}{p}\binom{n}{q}}$$

by the above proposition. Therefore,

$$\sum_{[J,K]} \int_U |f_{J,K}-\eta_{J,K}|^2 e^{-\varphi} < \varepsilon.$$

□

The above result is general in nature and is not very limited to the present context. The following results (see [7, Chapter-4]) are however contextual to the Hörmander's estimate.

Lemma 3.2.1. (Existence of a $\mathscr{C}^2$ perturbation with specified properties) Consider a compact exhaustion $\{K_j\}_{j=1}^\infty$ of a domain $U \subset \mathbb{C}^n$. For every sequence $\{\eta_j\}_{j=1}^\infty \subset \mathscr{C}_c^\infty(U,[0,1])$ such that $\operatorname{supp}\eta_j \subset K_j$ and $\eta_j \equiv 1$ on K_{j-1}, then we can find a $\psi \in \mathscr{C}^2(U,\mathbb{R})$ such that $|\bar{\partial}\eta_j|^2 \leq e^\psi$ for $j=1,2,\cdots$.

Lemma 3.2.2. (Denseness in graph norm) Given such a ψ from lemma 3.2.1, define $\varphi_j = \varphi + (j-3)\psi$ for $j=1,2,3$ for any $\varphi \in \mathscr{C}^2(U,\mathbb{R})$. We have the following:

$$\mathscr{C}^\infty_{c,(0,1)}(U,\mathbb{C}) \subset \mathcal{D}_{\bar{\partial}^\star_{(0,0\to 1)}} \cap \mathcal{D}_{\bar{\partial}_{(0,1\to 2)}} \underset{\text{dense}}{\subset} L^2_{(0,1),\mathbb{C}}(U,e^{-\varphi_2})$$

in the graph norm given by:

$$|| \cdot ||_{\mathcal{G}_{\overline{\partial}^\star_{(0,0\to1)},\overline{\partial}_{(0,1\to2)}}} := || \, \overline{\partial}^\star_{(0,0\to1)}(\cdot) \, ||_{\varphi_1} + || \cdot ||_{\varphi_2} + || \, \overline{\partial}_{(0,1\to2)}(\cdot) \, ||_{\varphi_3},$$

where the norm subscripts mean:

$$||f||^2_{\varphi_j} = \int_U |f|^2 e^{-\varphi_j}$$

for $j = 1, 2, 3$.

We will require the following identity which will play an instrumental role in the Hörmander estimate.

Proposition 3.2.1. (Bochner-Kodaira identity) For a domain $U \subset \mathbb{C}^n$, $\alpha \in \mathscr{C}^\infty_{c,(0,1)}(U, \mathbb{C})$ and $\varphi, \psi \in \mathscr{C}^2(U, \mathbb{R})$, we have:

$$\int_U \left(\left| e^\psi \overline{\partial}^\star_{(0,0\to1)}\alpha + \sum_{j=1}^n \alpha_j \frac{\partial \psi}{\partial z_j} \right|^2 + |\overline{\partial}_{(0,1\to2)}\alpha|^2 \right) e^{-\varphi} = \int_U \left(\mathcal{L}ev_\varphi(\alpha) + \sum_{j,k=1}^n \left| \frac{\partial \alpha_j}{\partial \overline{z}_k} \right|^2 \right) e^{-\varphi}.$$

PROOF. Let's denote $\overline{\partial}_{(0,0\to1)}$ and $\overline{\partial}_{(0,1\to2)}$ as T and S respectively. To compute the adjoint of T, consider the two inner products:

$$\langle T^\star\alpha, \beta\rangle_{\varphi_1} = \int_U \langle T^\star\alpha, \beta\rangle e^{-\varphi_1}$$
$$\langle \alpha, T\beta\rangle_{\varphi_2} = \int_U \langle \alpha, \overline{\partial}\beta\rangle e^{-\varphi_2} = -\int_U \langle \sum_{j=1}^n \partial_j(\alpha e^{-\varphi_2}), \beta\rangle.$$

If we want $\langle T^\star\alpha, \beta\rangle_{\varphi_1} = \langle \alpha, T\beta\rangle_{\varphi_2}$ for every $\beta \in \mathcal{D}_T$, then

$$\int_U \langle -e^{-\varphi_1} T^\star\alpha, \beta\rangle = \int_U \langle \sum_{j=1}^n -\partial_j(\alpha e^{-\varphi_2}), \beta\rangle.$$

By denseness of $\mathcal{D}_T$, this implies

$$T^\star\alpha = -e^{\varphi_1} \sum_{j=1}^n \partial_j(\alpha e^{-\varphi_2}) = \sum_{j=1}^n -e^{\varphi_1} \frac{\partial(\alpha e^{-\varphi_2})}{\partial z_j}.$$

Consider

$$
\begin{aligned}
|S\alpha|^2 &= \sum_{j<k} \left| \frac{\partial \alpha_j}{\partial \overline{z}_k} - \frac{\partial \alpha_k}{\partial \overline{z}_j} \right|^2 \\
&= \sum_{j,k} \left| \frac{\partial \alpha_j}{\partial \overline{z}_k} \right|^2 - \sum_{j,k} \frac{\partial \alpha_j}{\partial \overline{z}_k} \overline{\frac{\partial \alpha_k}{\partial \overline{z}_j}},
\end{aligned}
$$

and

$$
\begin{aligned}
T^\star \alpha &= -\sum_j e^{\varphi_1} \frac{\partial}{\partial \overline{z}_j} \left(\alpha_j e^{-\varphi_2} \right) \\
&= \sum_j e^{\varphi_1 - \varphi_2} \left(-\frac{\partial \alpha_j}{\partial \overline{z}_j} + \frac{\partial \varphi_2}{\partial \overline{z}_j} \alpha_j \right) \\
&= \sum_j e^{-\psi} \left(-\frac{\partial \alpha_j}{\partial \overline{z}_j} + \frac{\partial \varphi}{\partial \overline{z}_j} \alpha_j - \frac{\partial \varphi}{\partial \overline{z}_j} \alpha_j \right).
\end{aligned}
$$

Now, consider the adjoint of the operator $\overline{\partial}_j$ with respect to $\langle \cdot, \cdot \rangle_\varphi$:

$$
\begin{aligned}
\langle \beta, \overline{\partial}_j \beta' \rangle_\varphi &= \int_U \overline{\overline{\partial}_j \beta'} \cdot \beta \, e^{-\varphi} \\
&= \int_U \partial_j \bar{\beta}' \cdot \beta \, e^{-\varphi} \\
&= \int_U -e^{\varphi} \partial_j (\beta e^{-\varphi}) \bar{\beta}' \cdot e^{-\varphi} \\
&= \langle -e^{\varphi} \partial_j (\beta e^{-\varphi}), \beta' \rangle_\varphi
\end{aligned}
$$

for all $\beta, \beta' \in \mathscr{C}_c^\infty(U, \mathbb{C})$. Therefore, the adjoint takes the form

$$
\begin{aligned}
\overline{\partial}^\star_{j,\varphi}(\beta) &= -e^{\varphi} \partial_j (\beta e^{-\varphi}) \\
&= -\partial_j \beta + \partial_j \varphi \, \beta
\end{aligned}
$$

The commutator is then calculated to be

$$
\begin{aligned}
[\bar{\partial}_k, \bar{\partial}^\star_{j,\varphi}](\beta) &= \bar{\partial}_k \circ \bar{\partial}^\star_{j,\varphi}(\beta) - \bar{\partial}^\star_{j,\varphi} \circ \bar{\partial}_k(\beta) \\
&= \bar{\partial}_k(-\partial_j \beta + \partial_j \varphi \, \beta) - \left(-\partial_j \bar{\partial}_k \beta + \partial_j \varphi \, \bar{\partial}_k \beta \right) \\
&= -\bar{\partial}_k \partial_j \beta + \bar{\partial}_k \partial_j \varphi \, \beta + \partial_j \varphi \, \bar{\partial}_k \beta + \partial_j \bar{\partial}_k \beta - \partial_j \varphi \, \bar{\partial}_k \beta \\
&= \bar{\partial}_k \partial_j \varphi \, \beta.
\end{aligned}
$$

The first term in the LHS of the identity becomes

$$\begin{aligned}
\int_U \left| e^{\psi} T^\star \alpha + \sum_j \alpha_j \partial_j \psi \right|^2 e^{-\varphi} &= \int_U \left| \sum_j \bar{\partial}_j^\star \alpha_j \right|^2 e^{-\varphi} \\
&=: \left\langle \sum_j \overline{\partial}_j^\star \alpha_j, \sum_k \overline{\partial}_k^\star \alpha_k \right\rangle_\varphi \\
&= \sum_{j,k} \left\langle \overline{\partial}_j^\star \alpha_j, \overline{\partial}_k^\star \alpha_k \right\rangle_\varphi \\
&= \sum_{j,k} \left\langle \bar{\partial}_k \bar{\partial}_j^* \alpha_j, \alpha_k \right\rangle_\varphi \\
&= \sum_{j,k} \left\langle \bar{\partial}_j^\star \bar{\partial}_k \alpha_j, \alpha_k \right\rangle_\varphi + \left\langle \bar{\partial}_k \partial_j \varphi \alpha_j, \alpha_k \right\rangle_\varphi \\
&= \sum_{j,k} \left\langle \bar{\partial}_k \alpha_j, \bar{\partial}_j \alpha_k \right\rangle_\varphi + \int_U \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \alpha_j \overline{\alpha}_k e^{-\varphi}.
\end{aligned}$$

Combining everything, we get:

$$\begin{aligned}
\text{LHS} &:= \int_U \left(\left| e^{\psi} T^\star \alpha + \sum_{j=1}^n \alpha_j \frac{\partial \psi}{\partial z_j} \right|^2 + |S\alpha|^2 \right) e^{-\varphi} \\
&= \sum_{j,k} \left\langle \bar{\partial}_k \alpha_j, \bar{\partial}_j \alpha_k \right\rangle_\varphi + \int_U \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \alpha_j \overline{\alpha}_k \, e^{-\varphi} + \int_U \left(\sum_{j,k} \left| \frac{\partial \alpha_j}{\partial \overline{z}_k} \right|^2 - \sum_{j,k} \frac{\partial \alpha_j}{\partial \overline{z}_k} \overline{\frac{\partial \alpha_k}{\partial \overline{z}_j}} \right) e^{-\varphi} \\
&= \int_U \left(\mathcal{L}ev_\varphi(\alpha) + \sum_{j,k=1}^n \left| \frac{\partial \alpha_j}{\partial \overline{z}_k} \right|^2 \right) e^{-\varphi} \\
&=: \text{RHS}
\end{aligned}$$

□

Finally, we arrive at the grand result of Hörmander (see [1, Chapter-7]):

Theorem 3.2.2. (Hörmander's L^2 estimate) *Let $U \subset \mathbb{C}^n$ be a pseudoconvex domain, and let $\varphi \in \mathcal{PSH}(U)$. Suppose $f \in L^2_{(0,1),\mathbb{C}}(U, e^{-\varphi})$, satisfying $\bar{\partial} f = 0$, in the sense of distributions. Then there is a solution $u \in L^2_{\mathbb{C}}(U, e^{-\varphi})$ to the equation $\bar{\partial} u = f$, satisfying the estimate*

$$\int_U |u|^2 \, e^{-\varphi} \leq \int_U h \, e^{-\varphi} \tag{3.4}$$

where $h \in L^{\infty}_{\mathbb{R}}(U)$ is any non-negative function satisfying $i\, f \wedge \bar{f} \leq h\, i\partial\bar{\partial}\varphi$ in the distributional sense and provided the right-hand side is finite. If $\varphi \in \mathscr{C}^{\infty}(U, \mathbb{R})$, then h can be taken to be equal to:

$$h(\boldsymbol{\zeta}) = |f|^2_{i\partial\bar{\partial}\varphi}(\boldsymbol{\zeta}) := \sum_{j,k} [(\mathrm{Hess}_{\mathbb{C},\varphi}(\boldsymbol{\zeta}))^{-1}]_{j,k}\; f_j(\boldsymbol{\zeta})\overline{f_k(\boldsymbol{\zeta})}.$$

PROOF. We will do the proof in stages which first involves showing that the L^2 estimate holds for smooth strictly plurisubharmonic φ and then generalizing to non-smooth weights.

Case - (i): SMOOTH, STRICTLY PLURISUBHARMONIC WEIGHT - φ

Suppose $\varphi \in \mathcal{PSH}_{>0}(U)$,

$$\begin{aligned} h(\boldsymbol{\zeta}) &\geq \sup_{\mathbf{w}\in\mathbb{C}^n\setminus\{\mathbf{0}\}} \frac{i\, f \wedge \bar{f}(\boldsymbol{\zeta})\, \mathbf{w}}{i\, \partial\bar{\partial}\varphi(\boldsymbol{\zeta})\, \mathbf{w}} \\ &= \sup_{\mathbf{w}\in\mathbb{C}^n\setminus\{\mathbf{0}\}} \frac{|\langle f(\boldsymbol{\zeta}), \mathbf{w}\rangle|^2}{\langle \mathbf{w}, \mathbf{w}\rangle_{\varphi}}, \end{aligned}$$

where $\langle \mathbf{w}_1, \mathbf{w}_2\rangle_{\varphi} := \langle \mathbf{w}_1, \mathrm{Hess}_{\mathbb{C},\varphi}(\boldsymbol{\zeta})\mathbf{w}_2\rangle_2$. Thus,

$$\begin{aligned} h(\boldsymbol{\zeta}) &\geq \sup_{\mathbf{w}\in\mathbb{C}^n\setminus\{\mathbf{0}\}} \frac{\left|\langle \mathrm{Hess}_{\mathbb{C},\varphi}(\boldsymbol{\zeta})^{-1} f(\boldsymbol{\zeta}), \mathbf{w}\rangle_{\varphi}\right|^2}{\langle \mathbf{w}, \mathbf{w}\rangle_{\varphi}} \\ &= \sup_{\|\mathbf{w}\|_{\varphi}=1} \left|\langle \mathrm{Hess}_{\mathbb{C},\varphi}(\boldsymbol{\zeta})^{-1} f(\boldsymbol{\zeta}), \mathbf{w}\rangle_{\varphi}\right|^2 \\ &= \left\| \left[\mathrm{Hess}_{\mathbb{C},\varphi}(\boldsymbol{\zeta})^{-1}\right] f(\boldsymbol{\zeta}) \right\|^2_{\varphi} \\ &= \left[\mathrm{Hess}_{\mathbb{C},\varphi}(\boldsymbol{\zeta})^{-1}\right]_{j,k} f_j(\boldsymbol{\zeta})\overline{f_k(\boldsymbol{\zeta})} \\ &= |f|^2_{i\partial\bar{\partial}\varphi}(\boldsymbol{\zeta}). \end{aligned}$$

Clearly, $|f|^2_{i\partial\bar{\partial}\varphi}$ is the minimal function in the case $\varphi \in \mathcal{PSH}_{>0}(U)$.

METHOD OF THREE WEIGHTS:

Since U is pseudoconvex, there exists a smooth, strictly $\mathcal{PSH}$ exhaustion function s on U. From lemma 3.2.1 , we can assume that the cutoff functions equal to 1 on $U_{t+1} := \{s < t+1\}$ by shifting them sufficiently down the sequence and renaming them (recall $\{s < t+1\} \Subset U$). Therefore, we have a $\psi \in \mathscr{C}^2(U, \mathbb{R})$ which vanishes on U_t.

We have the following setup consisting of L^2 spaces and $\bar{\partial}$ operators:

$$L^2_{(0,0),\mathbb{C}}(U, e^{-\varphi_1}) \overset{\bar{\partial}_{(0,0\to 1)}}{\dashrightarrow} L^2_{(0,1),\mathbb{C}}(U, e^{-\varphi_2}) \overset{\bar{\partial}_{(0,1\to 2)}}{\dashrightarrow} L^2_{(0,2),\mathbb{C}}(U, e^{-\varphi_3})$$

For purposes identical to the above preparatory results, we shall nickname the objects as follows:

- $\varphi_j = \varphi + \gamma \circ s + (j-3)\psi$, $H_j := L^2_{(0,j-1),\mathbb{C}}(U, e^{-\varphi_j})$ for $j = 1, 2, 3$ and $\gamma : \mathbb{R} \to \mathbb{R}$ will be defined below.

- $T := \bar{\partial}_{(0,0\to 1)}$; $\mathcal{D}_T := \mathcal{D}_{\bar{\partial}_{(0,0\to 1)}} = \{u \in H_1 \,|\, Tu \underset{\mathscr{D}'(U,\mathbb{C})}{=} v \text{ for some } v \in H_2\}$

  $\mathscr{C}^\infty_{c,(0,0)}(U, \mathbb{C}) \subset \mathcal{D}_T \underset{\text{dense}}{\subset} L^2_{(0,0),\mathbb{C}}(U, e^{-\varphi_1})$

- $S := \bar{\partial}_{(0,1\to 2)}$; $\mathcal{D}_S := \mathcal{D}_{\bar{\partial}_{(0,1\to 2)}} = \{u \in H_2 \,|\, Tu \underset{\mathscr{D}'(U,\mathbb{C})}{=} v \text{ for some } v \in H_3\}$

  $\mathscr{C}^\infty_{c,(0,1)}(U, \mathbb{C}) \subset \mathcal{D}_S \underset{\text{dense}}{\subset} L^2_{(0,1),\mathbb{C}}(U, e^{-\varphi_2})$

We will prove the following symmetric inequality, involving all three L^2 spaces which will then yield the Hörmander's estimate.

$$|\langle f, \alpha\rangle_{\varphi_2}|^2 \leq C \cdot (||T^\star \alpha||^2_{\varphi_1} + ||S\alpha||^2_{\varphi_3}).$$

<u>Closedness of $\ker S$:</u>
First, consider a norm convergent sequence on the graph of S (which we shall denote by $\mathscr{G}_{(S)}$),

$$(u_k, Su_k) \to (u, v)$$

$$\implies u_k \xrightarrow[||\cdot||_{\varphi_2}]{k\to\infty} u \implies u_k \rightharpoonup u \implies Su_k \rightharpoonup Su.$$

But $Su_k \to v \implies Su_k \rightharpoonup v$ implying $Su = v$ in $\mathscr{D}'(U, \mathbb{C})$.

$$\therefore\ u \in \mathcal{D}_S \quad \text{and} \quad Su = v = \lim_{k\to\infty} Su_k = 0 \quad (\because u_k \in \ker S\ \forall k \in \mathbb{N})$$

Therefore, S is a closed (differential) operator. Now, $\{u_k\}_{k=1}^\infty \subset \ker S$ and $u_k \to u$, then $(u_k, Su_k) \to (u, 0)$, therefore by closedness of S, we get that $(u, 0) \in \mathscr{G}_{(S)}$, giving $u \in \mathcal{D}_S$ and $Su = 0$.

$$\therefore\ \overline{\ker S}^{||\cdot||_{\varphi_2}} = \ker S.$$

A key observation to arrive at the estimate (3.4) is getting the following inequality from the

Bochner-Kodaira identity (Proposition 3.2.1):

$$\begin{aligned}
\int_U \mathcal{L}ev_\varphi(\alpha)e^{-\varphi} &\leq \int_U \left(\mathcal{L}ev_\varphi(\alpha) + \sum_{j,k=1}^{n}\left|\frac{\partial \alpha_j}{\partial \bar{z}_k}\right|^2\right) e^{-\varphi} \\
&= \int_U \left(\left|e^{\psi}T^\star\alpha + \sum_{j=1}^{n}\alpha_j \frac{\partial\psi}{\partial z_j}\right|^2 + |S\alpha|^2\right) e^{-\varphi} \\
&\leq \int_U |T^\star\alpha|^2 e^{-\varphi+2\psi} + 2|T^\star\alpha|\,|\langle\alpha,\partial\psi\rangle|e^{-\varphi+\psi} + |\langle\alpha,\partial\psi\rangle|^2 e^{-\varphi} + \int_U |S\alpha|^2 e^{-\varphi} \\
&\leq ||T^\star\alpha||^2_{\varphi_1} + 2\,||T^\star\alpha||^2_{\varphi_1}\cdot \int_U |\alpha|^2|\partial\psi|^2 e^{-\varphi} + \int_U |\alpha|^2|\partial\psi|^2 e^{-\varphi} + ||S\alpha||^2_{\varphi_3} \\
&\leq ||T^\star\alpha||^2_{\varphi_1} + (t^{-1}\,||T^\star\alpha||^2_{\varphi_1} + t\int_U |\alpha|^2|\partial\psi|^2 e^{-\varphi}) + \int_U |\alpha|^2|\partial\psi|^2 e^{-\varphi} + ||S\alpha||^2_{\varphi_3} \\
&= (1+t^{-1})||T^\star\alpha||^2_{\varphi_1} + ||S\alpha||^2_{\varphi_3} + (1+t)\int_U |\bar{\partial}\psi|^2\,|\alpha|^2\,e^{-\varphi}. \qquad (3.5)
\end{aligned}$$

Constructing auxiliary perturbations:

We consider a smooth function $\gamma : \mathbb{R} \to \mathbb{R}$ with the following properties:

- $\gamma \equiv 0$ on $(-\infty, t)$ and γ is increasing on $\mathbb{R}$.
- $\gamma" \geq 0$ i.e, γ is convex.
- $\gamma \circ s \geq 2\psi$.
- $\gamma' \circ s\; i\partial\bar{\partial}s \geq (1+t)\,|\bar{\partial}\psi|^2\, i\partial\bar{\partial}||\mathbf{z}||_2^2$.

The last two criteria for γ can be ensured by considering the sub-level sets:

$$\gamma(t) \geq \sup_{\boldsymbol{\zeta}\in\overline{U}_t} 2\psi \quad \& \quad \gamma'(t) \geq (1+t)\sup_{\boldsymbol{\zeta}\in\overline{U}_t}\ \sup_{||\mathbf{w}||_2=1}\frac{|\bar{\partial}\psi|^2}{\mathcal{L}ev_s(\boldsymbol{\zeta};\mathbf{w})}.$$

By taking a convolution of a increasing step function over integer intervals dominating the monotone, non-negative RHS functions above, with the standard mollifier $\rho \in \mathscr{C}^\infty_c(\mathbb{R}, [0,\infty))$ and considering their integrals, we arrive at such a smooth function with the above specified properties.

Now define $\widetilde{\varphi} := \varphi + \gamma \circ s$. Clearly, $\widetilde{\varphi} \in \mathcal{PSH}_{>0}(U)$. Define the functions $\varphi_j := \widetilde{\varphi} + (j-3)\psi$, for $j = 1, 2, 3$.

$$\begin{aligned}\int_U \mathcal{L}ev_{\widetilde{\varphi}}(\alpha)e^{-\widetilde{\varphi}} &\geq \int_U \mathcal{L}ev_{\varphi}(\alpha)e^{-\widetilde{\varphi}} + \int_U \gamma' \circ s\, \partial\bar{\partial}s(\alpha)\, e^{-\widetilde{\varphi}} \\ &\geq \int_U \mathcal{L}ev_{\varphi}(\alpha)e^{\varphi - 2\varphi_2} + (1+t)\int_U |\bar{\partial}\psi|^2\, \partial\bar{\partial}||\mathbf{z}||_2^2(\alpha)\, e^{-\widetilde{\varphi}}.\end{aligned}$$

By the observation (3.5) applied to $\widetilde{\varphi}$ instead of φ, we get

$$\int_U \mathcal{L}ev_{\widetilde{\varphi}}(\alpha)e^{-\widetilde{\varphi}} \leq (1+t^{-1})||T^\star\alpha||_{\varphi_1}^2 + ||S\alpha||_{\varphi_3}^2 + (1+t)\int_U |\bar{\partial}\psi|^2\, |\alpha|^2\, e^{-\widetilde{\varphi}}.$$

Therefore,

$$\int_U \mathcal{L}ev_{\varphi}(\alpha)e^{\varphi - 2\varphi_2} \leq (1+t^{-1})||T^\star\alpha||_{\varphi_1}^2 + ||S\alpha||_{\varphi_3}^2.$$

<u>Arriving at the solution with a L^2 estimate:</u>

$$\begin{aligned}|\langle f, \alpha\rangle_{\varphi_2}|^2 &= \left| \int_U \sum_{j=1}^n f_j \overline{\alpha}_j e^{-\varphi_2} \right|^2 \\ &\leq \int_U h\, e^{-\varphi} \cdot \int_U \frac{\left| \sum_{j=1}^n f_j \overline{\alpha}_j \right|^2}{h} e^{\varphi - 2\varphi_2} \\ &\leq \int_U h\, e^{-\varphi} \cdot \int_U \mathcal{L}ev_{\varphi}(\alpha)e^{\varphi - 2\varphi_2} \\ &\leq C \cdot \left((1+t^{-1})||T^\star\alpha||_{\varphi_1}^2 + ||S\alpha||_{\varphi_3}^2\right)\end{aligned}$$

for all $\alpha \in \mathcal{D}_{T^\star} \cap \mathcal{D}_S$. Now since $f \in \ker S$, decompose $\alpha = \alpha' + \alpha''$ using $H_2 = \ker S \oplus \ker S^\perp$ to get:

$$\begin{aligned}|\langle f, \alpha\rangle_{\varphi_2}|^2 &= |\langle f, \alpha'\rangle_{\varphi_2}|^2 \\ &\leq C \cdot (1+t^{-1})||T^\star\alpha'||_{\varphi_1}^2.\end{aligned}$$

$\langle T^\star\alpha', \beta\rangle_{\varphi_2} = \langle \alpha', T\beta\rangle_{\varphi_3}$ and since $T\beta \in \ker S \Rightarrow \langle \alpha'', T\beta\rangle_{\varphi_3} = 0$, we have

$$\langle T^\star\alpha', \beta\rangle_{\varphi_2} = \langle T^\star\alpha, \beta\rangle_{\varphi_2}$$

for all $\beta \in H_2$. Therefore, we get the basic estimate:

$$|\langle f, \alpha \rangle_{\varphi_2}|^2 \leq C \cdot (1 + t^{-1}) ||T^\star \alpha||^2_{\varphi_1}. \tag{3.6}$$

Repeating the proof of the Hörmander's estimate (Theorem 3.1.1) in one dimension we get a solution $u_t \in L^2_{\mathbb{C}}(U, e^{-\varphi_1})$ such that:

$$\bar{\partial} u_t = f \text{ and } \int_U |u_t|^2 e^{-\varphi_1} \leq C(1 + t^{-1}).$$

To construct a weak solution with the required L^2 estimate we involve in a series of arguments involving weak convergences. Take a sequence $\{t_j\}_{j=1}^\infty \subset (1, \infty)$ such that $t_j \nearrow \infty$. Since $\gamma \circ s$ and ψ vanish on U_t, we have

$$\int_{U_t} |u_t|^2 e^{-\varphi} = \int_{U_t} |u_t|^2 e^{-\varphi_1} \leq C(1 + t^{-1}).$$

Define $u_j := u_{t_j}$ and $U_j := U_{t_j}$ and for a fixed $k \in \mathbb{N}$, we have that

$$\{u_j : j \geq 1\} \subset \mathbb{B}_{L^2_{\mathbb{C}}(U_1, e^{-\varphi})}(\mathbf{0}, \sqrt{2C})$$

using the weak* compactness of the unit ball, we get a weakly convergent subsequence $\{u_{1,j}\}_{j=1}^\infty$. Now for $k = 2$, we get a further subsequence and by induction for every $k \in \mathbb{N}$, we can assemble them as following:

$$\begin{array}{cccc}
\boxed{u_{1,1}} & u_{1,2} & \cdots & \rightharpoonup u^{(1)} \in L^2_{\mathbb{C}}(U_1, e^{-\varphi}) \\
u_{2,1} & \boxed{u_{2,2}} & \cdots & \rightharpoonup u^{(2)} \in L^2_{\mathbb{C}}(U_2, e^{-\varphi}) \\
\vdots & \vdots & \ddots & \vdots
\end{array}$$

Figure 3.1: Choosing a diagonal sequence-1

Choosing the diagonal subsequence and renaming $u_j := u_{j,j}$ we get that $u_j \rightharpoonup u^{(k)}$ in every $L^2_{\mathbb{C}}(U_k, e^{-\varphi})$ and since weak* topology is Hausdorff, we get that

$$u^{(1)} =_{\text{a.e}} \cdots =_{\text{a.e}} u^{(k)} =_{\text{a.e}} u^{(k+1)} =_{\text{a.e}} \cdots$$

Hence we can patch up these $\{u^{(k)}\}_{k \in \mathbb{N}}$ to get a $u \in \bigcap_{k \in \mathbb{N}} L^2_{\mathbb{C}}(U_k, e^{-\varphi})$ well defined almost everywhere.

$$\langle u, T^\star\alpha\rangle_\varphi = \lim_{j\to\infty}\langle u_j, T^\star\alpha\rangle_\varphi = \langle f, T^\star\alpha\rangle_\varphi$$

Therefore, $\bar\partial u = f$. By Fatou's lemma,

$$\int_U |u|^2 e^{-\varphi} \le \varliminf_{j\to\infty}\int_{U_j}|u_j|^2 e^{-\varphi_1} \le \lim_{j\to\infty} C(1+t_j^{-1}) = C := \int_U he^{-\varphi}.$$

This completes the proof of the Hörmander's estimate in this case.

Case - (ii): For an arbitrary plurisubharmonic weight - φ

For a sequence $\varepsilon_j \searrow 0$, consider the φ-approximating family $\varphi_j := \varphi\star\rho_{\varepsilon_j} + \varepsilon_j||\mathbf{z}||^2$ defined in a neighbourhood of $\overline{U}_j$. Define

$$h_j(\boldsymbol{\zeta}) := |f|^2_{i\partial\bar\partial\varphi_j} := \sup_{\mathbf{w}\in\mathbb{C}^n\setminus\{\mathbf{0}\}}\frac{i\,f\wedge\bar f(\boldsymbol{\zeta})\,\mathbf{w}}{i\,\partial\bar\partial\varphi_j(\boldsymbol{\zeta})\,\mathbf{w}}.$$

By the previous part, we get a family $u_j \in L^2(U, e^{-\varphi_j})$ such that

$$\int_U |u_j|^2 e^{-\varphi_j} \le \int_U h_j e^{-\varphi_j} \le \int_U h_j e^{-\varphi}.$$

Now, consider the (1,1) current $\partial\bar\partial\varphi$ which has distributional coefficients:

$$\partial\bar\partial\varphi = \sum_{r,s=1}^n \underbrace{\frac{\partial^2\varphi}{\partial z_r\partial\bar z_s}}_{\in\mathscr{D}'(U)}\,dz_r\wedge d\bar z_s.$$

By Proposition (1.2.1), we have that by substituting $\psi_{(r,s)} = \bigwedge_{r'\neq r} dz_{r'}\bigwedge_{s'\neq s} d\bar z_{s'}$ into $\partial\bar\partial\varphi$, all the distributions $\varphi_{z_r\bar z_s} := \dfrac{\partial^2\varphi}{\partial z_r\partial\bar z_s}$ are induced by positive Radon measures $\mu_{(r,s)}$ for every $1\le r,s\le n$.

The condition $i\,f\wedge\bar f \le h\,i\partial\bar\partial\varphi$ when expanded gives,

$$\sum_{r,s=1}^n f_r\bar f_s\,dz_r\wedge d\bar z_s \le \sum_{r,s=1}^n \frac{\partial^2\varphi}{\partial z_r\partial\bar z_s}\,dz_r\wedge d\bar z_s.$$

By the Lebesgue-Radon-Nikodym theorem, we can decompose

$$\varphi_{z_r\bar z_s} \equiv \mu_{(r,s)} = \mu'_{(r,s)} + \mu''_{(r,s)} \quad \text{where} \quad \mu'_{(r,s)} \ll \lambda_{\mathbb{R}^{2n}} \ \ \& \ \ \mu''_{(r,s)} \perp \lambda_{\mathbb{R}^{2n}},$$

where there exists a $\beta_{(r,s)} \in L^1_{\text{loc}}(U)$ such that $0 \leq \beta_{(r,s)} \leq \mu_{(r,s)}$ and $\mu'_{(r,s)}(E) = \int_E \beta \, d\lambda_{\mathbb{R}^{2n}}$ for every Lebesgue measurable set E. Since, $\mu''_{(r,s)}$ is supported on a set of Lebesgue measure zero, we easily get $i\, f \wedge \bar{f} \leq i\, h \, \sum_{r,s=1}^{n} \beta_{(r,s)} \, dz_r \wedge d\bar{z}_s$ holds almost everywhere (w.r.t $\lambda_{\mathbb{R}^{2n}}$). Call $\beta_{j,(r,s)} := \beta \star \rho_{\varepsilon_j} \leq (\varphi_j)_{z_r \bar{z}_s}$ and $\beta_{j,(r,s)}$ converges pointwise to $\beta_{(r,s)}$ almost everywhere. Therefore,

$$\overline{\lim_{j\to\infty}} h_j(\boldsymbol{\zeta}) = \overline{\lim_{j\to\infty}} \sup_{\mathbf{w}\in\mathbb{C}^n\setminus\{\mathbf{0}\}} \frac{i\, f \wedge \bar{f}(\boldsymbol{\zeta})\, \mathbf{w}}{i\, \partial\bar{\partial}\varphi_j(\boldsymbol{\zeta})\, \mathbf{w}} \leq \overline{\lim_{j\to\infty}} \sup_{\mathbf{w}\in\mathbb{C}^n\setminus\{\mathbf{0}\}} \frac{i\, f \wedge \bar{f}(\boldsymbol{\zeta})\, \mathbf{w}}{i\, \langle [\beta_{j,(r,s)}(\boldsymbol{\zeta})]\, \mathbf{w}, \mathbf{w}\rangle} \leq h(\boldsymbol{\zeta})$$

and by the Fatou lemma

$$\overline{\lim_{j\to\infty}} \int_U |u_j|^2 e^{-\varphi_j} \leq \int_U \overline{\lim_{j\to\infty}} h_j e^{-\varphi} \leq \int_U h\, e^{-\varphi} =: C. \tag{3.7}$$

First, for every $k, m \in \mathbb{N}$, choose a weakly convergent subsequence $\{u^k_{m,j}\}_{j=1}^{\infty}$. Refer figure (3.2) below:

$$\begin{array}{cccccl}
\boxed{u^{(k)}_{k,1}} & \cdots & u^{(k)}_{k,m-k+1} & \cdots & \rightharpoonup u^{(k)}_k \in L^2_{\mathbb{C}}(U_k, e^{-\varphi_k}) \\
\vdots & \ddots & \vdots & & \vdots \\
u^{(k)}_{m,1} & \cdots & \boxed{u^{(k)}_{m,m-k+1}} & \cdots & \rightharpoonup u^{(k)}_m \in L^2_{\mathbb{C}}(U_k, e^{-\varphi_m}) \\
\vdots & & \vdots & \ddots & \vdots
\end{array}$$

Figure 3.2: Choosing a diagonal sequence-2

Now, assemble all these diagonal subsequences obtained from the above diagram; refer figure (3.3) below:

$$\begin{array}{cccccl}
\boxed{u^{(1)}_{1,1}} & \cdots & u^{(1)}_{k,k} & \cdots & \rightharpoonup u^{(1)} \in \bigcap_{m\geq 1} L^2_{\mathbb{C}}(U_1, e^{-\varphi_m}) \\
\vdots & \ddots & \vdots & & \vdots \\
u^{(k)}_{k,1} & \cdots & \boxed{u^{(k)}_{2k-1,k}} & \cdots & \rightharpoonup u^{(k)} \in \bigcap_{m\geq k} L^2_{\mathbb{C}}(U_k, e^{-\varphi_m}) \\
\vdots & & \vdots & \ddots & \vdots
\end{array}$$

Figure 3.3: Choosing a diagonal sequence-3

By using the diagonal argument, renaming $u_j := u^{(j)}_{2j-1,j}$ we get that:

$$u_j \rightharpoonup u \in \bigcap_{m,k\in\mathbb{N}} L^2(U_k, e^{-\varphi_m}).$$

The inequality (3.7) yields that for every $\delta > 0$ there exists an $N_\delta \in \mathbb{N}$, such that

$$\sup_{j\geq N_\delta} \int_U |u_j|^2 \, e^{-\varphi_j} \leq C + \delta.$$

Now for a fixed k,

$$\int_{U_k} |u_j|^2 \, e^{-\varphi_k} \leq \int_U |u_j|^2 \, e^{-\varphi_j} \leq C + \delta$$

implying that u_j's eventually lie inside the weakly closed set

$$\{v \in L^2(U_k, e^{-\varphi_k}) : ||v||_{\varphi_k} \leq (C+\delta)^{\frac{1}{2}}\}.$$

Hence for a large m, their limit u also lies in it:

$$\int_{U_k} |u|^2 e^{-\varphi_m} \leq \int_{U_m} |u|^2 e^{-\varphi_m} \leq C + \delta.$$

Taking limits, we get

$$\int_U |u|^2 \, e^{-\varphi} = \lim_{k\to\infty} \lim_{m\to\infty} \int_{U_k} |u|^2 e^{-\varphi_m} \leq C + \delta$$

for every $\delta > 0$ implying

$$\int_U |u|^2 \, e^{-\varphi} \leq \int_U h \, e^{-\varphi}.$$

and since weak limits are preserved under differential operators (in the sense of distributions), u satisfies $\bar{\partial} u = f$. □

Chapter 4

Applications of Hörmander's L^2 technique

4.1 The Levi problem - Introduction and background

The Levi problem is a fundamental question in several complex variables which was proposed in the mid 1900s. It asks whether every pseudoconvex domain in $\mathbb{C}^n$ is a domain of holomorphy and is easily seen to be equivalent to examining of suuficiency of pseudoconvexity to guarantee the existence of singular holomorphic functions that cannot be extended past the topological boundary of the domain. The answer is affirmative and was solved by Oka, Bremmerman, Norguet and others in the 1950s. Later in 1965, Hörmander developed a method to obtain solutions to the inhomogeneous $\bar{\partial}$ equation with L^2 bounds employing the pseudoconvexity of the domain. The estimate allows one to construct holomorphic functions with prescribed growth conditions which is the crux of the following proof and the Ohsawa-Takegoshi extension result.

4.1.1 Pseudoconvexity

Let us start wih one of the possible simple definitions of pseudoconvexity (usually referred to as Hartogs pseudoconvexity).

Definition 4.1.1. (Pseudoconvexity)
A domain $U \subset \mathbb{C}^n$ is said to be ***Hartogs pseudoconvex*** if $-\log \delta_U$ is plurisubharmonic on U where $\delta_U : \boldsymbol{\zeta} \mapsto d(\boldsymbol{\zeta}, \partial U)$.

Note that the geometrically defined function $-\log \delta_U : U \to (-\infty, \infty)$ 'exhausts' the domain U i.e, the sublevel sets $\{-\log \delta_U < c\}$ are relatively compact in U for every $c \in \mathbb{R}$. Such a function is called an ***exhaustion function*** for a domain.

The following theorem (see for example [13, Chapters-4,5] or [3, Chapter-2]) captures all the equivalent notions of ***pseudoconvexity***.

Theorem 4.1.1. *The following properties are equivalent for a domain $U \subset \mathbb{C}^n$:*

(i) There is a $\mathscr{C}^2$ strictly plurisubharmonic exhaustion function for U.

(ii) There is a plurisubharmonic exhaustion function for U.

(iii) U is plurisubharmonically convex i.e, convex w.r.t $\mathcal{PSH}(U)$.

(iv) For every analytic disc $\mathrm{D} = \mathbf{d}(\overline{\mathbb{D}})$ in U, one has $\mathbb{d}(\mathrm{D}, \partial U) = \mathbb{d}(\partial \mathrm{D}, \partial U)$.

(v) U satisfies the continuity principle.

(vi) U is Hartogs pseudoconvex.

(vii) U has a $\mathscr{C}^\infty$ strictly plurisubharmonic exhaustion function.

The following boundary notion of pseudoconvexity is needed.

Definition 4.1.2. (Levi pseudoconvex)
A domain $U \subset \mathbb{C}^n$ with $\mathscr{C}^2$ boundary is said to be ***Levi pseudoconvex*** if $\mathcal{L}ev_\varrho$ is positive semi-definite on $T^{(1,0)}_{\mathbf{z_0}}(\partial U)$ i.e,

$$\mathcal{L}ev(\varrho)(\mathbf{z_0}; \mathbf{v}) \geq 0; \;\; \forall \;\; \mathbf{v} \in T^{(1,0)}_{\mathbf{z_0}}(\partial U).$$

If it is positive definite, then U is said to be ***strongly Levi pseudoconvex***.

Proposition 4.1.1. The following are equivalent for a domain $U \subset \mathbb{C}^n$:

(i) U is Hartogs pseudoconvex.

(ii) U can be approximated by a union of bounded, strongly Levi pseudoconvex domains with smooth boundary $U = \bigcup_{j=1}^{\infty} U_j$ and $U_j \Subset U_{j+1}$ for all j.

For a proof, see [7, Chapter-3].

4.2 Solution using the Hörmander's estimate

Let's provide a solution to the Levi problem as in [1, Chapter-8].

Theorem 4.2.1. *For a pseudoconvex domain $U \subset \mathbb{C}^n$, we have:*

$$\widehat{K}_{\mathcal{PSH}(U)} = \widehat{K}_{\mathcal{O}(U)}$$

for every compact subset $K \subset U$ and hence, U is holomorphically convex as well, making it a domain of holomorphy!

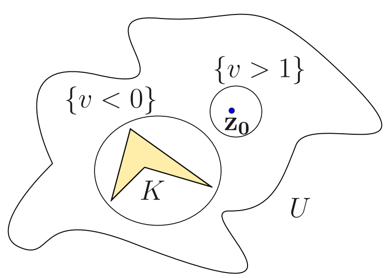


Figure 4.1: Solving the Levi problem

PROOF. Given a point $\mathbf{z} \in \widehat{K}_{\mathcal{PSH}(U)}$, since for every $f \in \mathcal{O}(U)$, it is implied that $|f| \in \mathcal{PSH}(U)$, we get

$$|f(\mathbf{z})| = |f|(\mathbf{z}) \leq \sup_K |f| \Rightarrow \mathbf{z} \in \widehat{K}_{\mathcal{O}(U)}.$$

Therefore, $\widehat{K}_{\mathcal{PSH}(U)} \subset \widehat{K}_{\mathcal{O}(U)}$. To show the converse fix $\mathbf{z_0} \in U \setminus \widehat{K}_{\mathcal{PSH}(U)}$. Let's consider a function $v \in \mathcal{PSH}(U) \cap \mathscr{C}(U, \mathbb{R})$ such that $v < 0$ on K and $v(\mathbf{z_0}) > 1$ (Proposition 1.2.2). Let $\eta \in \mathscr{C}_c^{\infty}(U, \mathbb{R})$ be such that $\eta(\mathbf{z_0}) = 1$ and $\operatorname{supp} \eta \subset \{v > 1\}$. For $t \geq 1$ set

$$v_t := \max\{v, t\}, \quad \text{so that } v_t = v \text{ in } \{v < 0\} \text{ and } v_t > t \text{ on } \operatorname{supp} \eta.$$

Define $\varphi_t \in \mathcal{PSH}(U) \cap \mathscr{C}(U \setminus \{\mathbf{z_0}\}, \mathbb{R})$ as:

$$\varphi_t = ||\mathbf{z} - \mathbf{z_0}||_2^2 + n \log ||\mathbf{z} - \mathbf{z_0}||_2^2 + v_t.$$

By Theorem **3.2.2**, there exists an $u_t \in L^2_{\mathbb{C}}(U, e^{-\varphi_t})$ such that $\bar{\partial} u_t = \bar{\partial}\eta$ and

$$\int_U |u_t|^2 e^{-\varphi_t} \leq \int_U h e^{-\varphi_t},$$

where $0 \leq h \in L^\infty(U)$ is satisfies $i\partial\eta \wedge \bar{\partial}\eta \leq h\ i\partial\bar{\partial}\varphi_t$ which W.L.O.G could be modified to follow $\text{supp}(h) \subset \text{supp}(\eta)$ (by just making it zero outside $\text{supp}(\eta)$).

Since $e^{-\varphi_t}$ is not locally integrable near $\mathbf{z_0}$, we have $u_t(\mathbf{z_0}) = 0$ (since u has to be continuous by the regularity theorem **1.3.1.1** for $\bar{\partial}$). Therefore $f_t := \eta - u_t$ is holomorphic in U, $f_t(\mathbf{z_0}) = 1$, and

$$\begin{aligned}
\int_{\{v<0\}} |f_t|^2 &= \int_{\{v<0\}} |u_t|^2 e^{-\varphi_t}\, e^{\varphi_t} \\
&\leq \sup_{\{v<0\}} e^{\varphi_t} \cdot \int\limits_{\text{supp}(\eta)} h e^{-\varphi_t} \\
&\leq C e^{-t},
\end{aligned}$$

where C is independent of t. Since $|f_t|^2$ is plurisubharmonic, we have for $t \gg 1$

$$\sup_K |f_t| < 1 = |f_t(\mathbf{z_0})|$$

by averaging over polydiscs centred around points in K and we see that $\mathbf{z_0} \notin \widehat{K}_{\mathcal{PSH}(U)}$. $\square$

Hence, pseudoconvex domains are precisely the domains of holomorphy!

4.3 Ohsawa-Takegoshi extension theorem

The interior convexity property of a domain U plays an important role in the extension of holomorphic functions without increasing the $\mathcal{PSH}$-weighted energy. This is captured in the following theorem of Ohsawa and Takegoshi proved in 1987. We have elaborated upon the arXiv paper [2].

Theorem 4.3.1. *Let $U \Subset \mathbb{C}^n$ be a pseudoconvex domain. Suppose $\sup_U |z_n|^2 < e^{-1}$. Then there exists a constant $C_n > 0$ such that for every $\varphi \in \mathcal{PSH}(U)$, every holomorphic function f on $U_0 := U \cap \{z_n = 0\}$ with $\int\limits_{U_0} |f|^2 e^{-\varphi} < \infty$, there exists a holomorphic extension $\widehat{f}$ of f to U such that*

$$\int_U |\widehat{f}|^2 e^{-\varphi} \leq C_n \int_{U_0} |f|^2 e^{-\varphi} \tag{4.1}$$

PROOF. We will do the proof in stages:

<u>DEFINING AN AUXILIARY $\mathcal{PSH}$ FUNCTION:</u>

Let us define a function $\eta \in \mathscr{C}^\infty(U, \mathbb{R})$ as:

$$(\zeta_1, \cdots, \zeta_n) \mapsto -\log(|\zeta_n|^2 + \varepsilon^2) + \log(-\log(|\zeta_n|^2 + \varepsilon^2)).$$

Set $\rho : (\zeta_1, \cdots, \zeta_n) \mapsto \log(|\zeta_n|^2 + \varepsilon^2)$ and $\eta = -\rho + \log(-\rho)$. Clearly, $\rho \in \mathcal{PSH}(U)$:

$$\partial\bar{\partial}\rho = \partial\left(\frac{z_n\, d\bar{z}_n}{|z_n|^2 + \varepsilon^2}\right) = \frac{dz_n \wedge d\bar{z}_n}{|z_n|^2 + \varepsilon^2} - \frac{z_n\, \bar{z}_n}{(|z_n|^2 + \varepsilon^2)^2} dz_n \wedge d\bar{z}_n = \frac{\varepsilon^2\, dz_n \wedge d\bar{z}_n}{(|z_n|^2 + \varepsilon^2)^2}. \tag{4.2}$$

Then set $\psi = -\log \eta$ and

$$\begin{aligned}
\partial\bar{\partial}\psi &= -\frac{\partial\bar{\partial}\eta}{\eta} + \frac{\partial\eta \wedge \bar{\partial}\eta}{\eta^2} \\
&= -\frac{1}{\eta}\left(-\partial\bar{\partial}\rho + \frac{\partial\bar{\partial}\rho}{\rho} - \frac{\partial\rho \wedge \bar{\partial}\rho}{\rho^2}\right) + \frac{\partial\eta \wedge \bar{\partial}\eta}{\eta^2} \\
&= (1 + (-\rho)^{-1})\frac{\partial\bar{\partial}\rho}{\eta} + \frac{\partial\rho \wedge \bar{\partial}\rho}{\eta\rho^2} + \frac{\partial\eta \wedge \bar{\partial}\eta}{\eta^2}.
\end{aligned} \tag{4.3}$$

We can choose $\varepsilon > 0$ small enough so that $-\rho \geq 1$ on U and hence $\psi \in \mathcal{PSH}(U)$ (as $\eta > 0$). Put $\widetilde{\varphi} = \varphi + \log(|z_n|^2 + \delta^2)$ for $0 < \delta < \varepsilon$. Let $\chi \in \mathscr{C}^\infty(\mathbb{R}, [0,1])$ be a cut-off function satisfying $\chi|_{(-\infty,1/2)} = 1$ and $\chi|_{(1,\infty)} = 0$.

<u>A STANDARD REDUCTION:</u>

By Proposition 4.1.1, we have an increasing sequence of bounded, smooth, pseudoconvex domains U_j with $\bigcup_j U_j = U$. Proposition 1.2.2 gives us a sequence of smooth, strictly plurisubharmonic functions φ_j on U such that $\varphi_j \searrow \varphi$. Restricting to $U_j \Subset U$ we may assume that f is holomorphic in some domain V_j such that $U_j \cap \{z_n = 0\} \Subset V_j \underset{\text{open}}{\subset} \{z_n = 0\}$ and if we can find a holomorphic extension F_j of f_j to U_j such that

$$\int_{U_j} |F_j|^2 e^{-\varphi_j} \leq C_n \int_{V_j} |f|^2 e^{-\varphi_j},$$

then we are done. Therefore, the reduced setup consists of the assumptions:

- a bounded, strictly pseudoconvex domain: $U \Subset \mathbb{C}^n$.
- holomorphic data on a co-dim$_{\mathbb{C}} = 1$ slice: $f \in \mathcal{O}(V_0)$ & $V_0 \Supset \overline{U}_0$.
- a smooth, strictly plurisubharmonic weight: $\varphi \in \mathscr{C}^\infty(U, \mathbb{R}) \cap \mathcal{PSH}_{>0}(U)$.

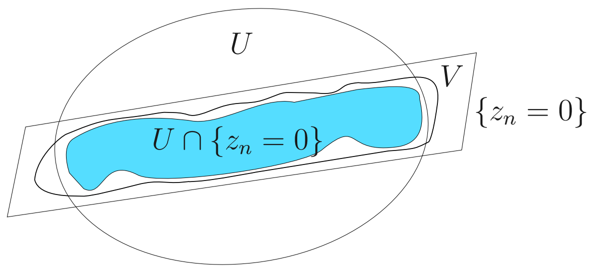


Figure 4.2: A diagram of the reduced extension problem

<u>Using the Hörmander's estimate:</u>

Thus for ε small enough, we have a well-defined smooth $\bar{\partial}$-closed $(0,1)$ form given by $v := f\bar{\partial}\chi(|z_n|^2/\varepsilon^2)$ on U. More elaborately,

$$v_\varepsilon(\zeta_1, \cdots, \zeta_n) := f(\zeta_1, \cdots, \zeta_{n-1})\, \bar{\partial} \left(\chi \circ \frac{|z_n|^2}{\varepsilon^2} \right) (\zeta_1, \cdots, \zeta_n).$$

By Fubini's theorem,

$$\begin{aligned} \int_U |v_\varepsilon|^2 e^{-\widetilde{\varphi}} &= \int_{U_0 \times \left\{ \frac{|z_n|^2}{\varepsilon^2} \leq 1 \right\}} |f|^2\, |\bar{\partial}\chi(|z_n|^2/\varepsilon^2)|^2\, e^{-\widetilde{\varphi}} \\ &= \int_{U_0} |f|^2\, e^{-\varphi} \cdot \int_{\{|z_n| \leq \varepsilon\}} \left| \chi' \circ \tfrac{|z_n|^2}{\varepsilon^2} \right|^2 \\ &< \infty, \end{aligned}$$

hence $v_\varepsilon \in L^2_{(0,1),\mathbb{C}}(U, e^{-\widetilde{\varphi}})$ and there exists a solution $u_{\delta,\varepsilon} := u$ of $\bar{\partial} u = v_\varepsilon$ with minimal L^2-norm in $L^2_{(0,1),\mathbb{C}}(U, e^{-\widetilde{\varphi}})$ by projecting onto $\ker \bar{\partial}^\perp$ implying $u \perp \ker \bar{\partial}$. Since ψ is a bounded function, we have $ue^\psi \perp \ker \bar{\partial}$ in $L^2_{(0,1),\mathbb{C}}(U, e^{-\widetilde{\varphi}-\psi})$. Thus by the Hörmander's L^2-estimate (Theorem 3.2.2),

$$\begin{aligned}\int_U |u|^2 e^{\psi-\widetilde{\varphi}} &\leq \int_U |\bar{\partial}(ue^\psi)|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)} e^{-\psi-\widetilde{\varphi}} \\ &= \int_U |v + u\,\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)} e^{\psi-\widetilde{\varphi}}.\end{aligned}$$

Now,

$$\begin{aligned}|v + u\,\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)} &= \langle v + u\,\bar{\partial}\psi, v + u\,\bar{\partial}\psi\rangle_{(\widetilde{\varphi}+\psi)} \\ &\leq \langle v, v\rangle_{(\widetilde{\varphi}+\psi)} + \langle u\bar{\partial}\psi, v\rangle_{(\widetilde{\varphi}+\psi)} + \langle v, u\bar{\partial}\psi\rangle_{(\widetilde{\varphi}+\psi)} + |u|^2\langle\bar{\partial}\psi, \bar{\partial}\psi\rangle_{(\widetilde{\varphi}+\psi)} \\ &\leq \langle v, v\rangle_{(\widetilde{\varphi}+\psi)} + 2|\langle u\bar{\partial}\psi, v\rangle_{(\widetilde{\varphi}+\psi)}| + |u|^2\langle\bar{\partial}\psi, \bar{\partial}\psi\rangle_{(\widetilde{\varphi}+\psi)} \\ &\leq |v|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)} + \delta^{-1}|v|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)} + \delta\,|\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)}|u|^2\,\mathbb{1}_{\operatorname{supp} v} + |\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)}|u|^2.\end{aligned}$$

Hence, we get

$$\begin{aligned}\int_U |u|^2 e^{\psi-\widetilde{\varphi}} \leq (1+\delta^{-1}) \int_U |v|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)} e^{\psi-\widetilde{\varphi}} + \int_U |\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)}|u|^2 e^{\psi-\widetilde{\varphi}} \\ + \delta \int_{\operatorname{supp} v} |\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)}|u|^2 e^{\psi-\widetilde{\varphi}}.\end{aligned} \tag{4.4}$$

where $\delta > 0$ is a small constant which will be chosen below.

Arriving at the form of the inequality:

Since $\partial\eta \wedge \bar{\partial}\eta = (1 + (-\rho)^{-1})^2 \partial\rho \wedge \bar{\partial}\rho$, we infer from (4.3) that,

$$\partial\bar{\partial}\psi \geq \frac{\partial\rho \wedge \bar{\partial}\rho}{\eta\rho^2} + \frac{\partial\eta \wedge \bar{\partial}\eta}{\eta^2} = \left(\frac{\eta}{(-\rho+1)^2} + 1\right) \frac{\partial\eta \wedge \bar{\partial}\eta}{\eta^2}$$

and

$$\begin{aligned}|\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)}(\boldsymbol{\zeta}) &= \sup_{\mathbf{w}\in T^{1,0}_{\boldsymbol{\zeta}}\mathbb{C}^n\setminus\{0\}} \frac{|\langle\bar{\partial}\psi(\boldsymbol{\zeta}), \mathbf{w}\rangle|^2}{\langle\mathbf{w}, \operatorname{Hess}_{\mathbb{C},\widetilde{\varphi}+\psi}\mathbf{w}\rangle} \\ &\leq \sup_{\mathbf{w}\in T^{1,0}_{\boldsymbol{\zeta}}\mathbb{C}^n\setminus\{0\}} \frac{|\langle\bar{\partial}\psi(\boldsymbol{\zeta}), \mathbf{w}\rangle|^2}{\langle\mathbf{w}, \operatorname{Hess}_{\mathbb{C},\psi}(\boldsymbol{\zeta})\mathbf{w}\rangle}\end{aligned}$$

$$\leq \sup_{\mathbf{w}\in T^{1,0}_{\boldsymbol{\zeta}}\mathbb{C}^n\setminus\{0\}} \frac{\frac{\eta(\boldsymbol{\zeta})^2}{1+\frac{\eta(\boldsymbol{\zeta})}{(-\rho(\boldsymbol{\zeta})+1)^2}}\cdot|\langle\bar{\partial}\psi(\boldsymbol{\zeta}),\mathbf{w}\rangle|^2}{|\langle\bar{\partial}\eta(\boldsymbol{\zeta}),\mathbf{w}\rangle|^2}$$
$$= \frac{1}{1+\frac{\eta(\boldsymbol{\zeta})}{(-\rho(\boldsymbol{\zeta})+1)^2}} \quad (\text{since } \bar{\partial}\psi = -\frac{1}{\eta}\bar{\partial}\eta).$$

Thus

$$\int_U |\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)}|u|^2 e^{\psi-\widetilde{\varphi}} \leq \int_U \frac{|u|^2}{1+\frac{\eta}{(-\rho+1)^2}} e^{\psi-\widetilde{\varphi}}. \tag{4.5}$$

From (4.3), we also get,

$$\partial\bar{\partial}\psi \geq \frac{\partial\bar{\partial}\rho}{\eta} = \frac{\varepsilon^2 dz_n\wedge d\bar{z}_n}{\eta(|z_n|^2+\varepsilon^2)^2}$$

which implies

$$|v|^2_{i\partial\bar{\partial}(\tilde{\varphi}+\psi)}(\boldsymbol{\zeta}) = \sup_{\mathbf{w}\in T^{1,0}_{\boldsymbol{\zeta}}\mathbb{C}^n\setminus\{0\}} \frac{|\langle v(\boldsymbol{\zeta}),\mathbf{w}\rangle|^2}{\langle\mathbf{w},\mathrm{Hess}_{\mathbb{C},\tilde{\varphi}+\psi}\mathbf{w}\rangle}$$
$$\leq \sup_{\mathbf{w}\in T^{1,0}_{\boldsymbol{\zeta}}\mathbb{C}^n\setminus\{0\}} \frac{|\langle v(\boldsymbol{\zeta}),\mathbf{w}\rangle|^2}{\langle\mathbf{w},\mathrm{Hess}_{\mathbb{C},\psi}(\boldsymbol{\zeta})\mathbf{w}\rangle}$$
$$\leq \sup_{\mathbf{w}\in T^{1,0}_{\boldsymbol{\zeta}}\mathbb{C}^n\setminus\{0\}} \frac{\eta(\boldsymbol{\zeta})\,(|\zeta_n|^2+\varepsilon^2)^2}{\varepsilon^2}\cdot\frac{|\langle v(\boldsymbol{\zeta}),\mathbf{w}\rangle|^2}{dz_n\wedge d\bar{z}_n(\boldsymbol{\zeta})(\mathbf{w},\mathbf{w})}$$
$$= \frac{2\eta(\boldsymbol{\zeta})\,(|\zeta_n|^2+\varepsilon^2)^2}{\varepsilon^2}\cdot|f(\boldsymbol{\zeta})|^2\left|\chi'\circ\frac{|\zeta_n|^2}{\varepsilon^2}\right|^2\frac{|\zeta_n|^2}{\varepsilon^4}.$$

Thus, by Fubini's theorem, since $\mathrm{supp}(v)\subset U_0\times\left\{\frac{\varepsilon^2}{2}<|z_n|^2<\varepsilon^2\right\}$, we have

$$\int_U |v|^2_{i\partial\bar{\partial}(\widetilde{\varphi}+\psi)}e^{\psi-\widetilde{\varphi}} \leq \int\limits_{U_0\times\left\{\frac{\varepsilon^2}{2}<|z_n|^2<\varepsilon^2\right\}} \frac{2\eta\,(|z_n|^2+\varepsilon^2)^2}{\varepsilon^2}|f(\boldsymbol{\zeta})|^2\left|\chi'\circ\frac{|z_n|^2}{\varepsilon^2}\right|^2\frac{|z_n|^2}{\varepsilon^4}\,e^{\psi-\widetilde{\varphi}}$$
$$\leq 2\int\limits_{\left\{\frac{\varepsilon^2}{2}<|z_n|^2<\varepsilon^2\right\}} \left|\chi'\circ\frac{|z_n|^2}{\varepsilon^2}\right|^2\frac{(|z_n|^2+\varepsilon^2)^2}{\varepsilon^2}\frac{|z_n|^2}{\varepsilon^4}\frac{1}{|z_n|^2}\cdot\int_{V_0}|f|^2e^{-\varphi}$$
$$\leq C'_n\int_{V_0}|f|^2e^{-\varphi}. \tag{4.6}$$

On the set $\operatorname{supp}(v)$, we have that $-\rho \geq -\log 2 - 2\log\varepsilon > 1$ for $\varepsilon \ll 1$,

$$\partial\psi \wedge \bar{\partial}\psi = \frac{1}{\eta^2}\left(1 + \frac{1}{-\rho}\right)^2 \partial\rho \wedge \bar{\partial}\rho \leq \frac{4}{\eta^2}\partial\rho \wedge \bar{\partial}\rho$$

and

$$\partial\bar{\partial}\psi \geq \frac{\varepsilon^2 dz_n \wedge d\bar{z}_n}{\eta(|z_n|^2 + \varepsilon^2)^2} \geq \frac{|z_n|^2}{\eta(|z_n|^2 + \varepsilon^2)^2} dz_n \wedge d\bar{z}_n = \frac{\partial\rho \wedge \bar{\partial}\rho}{\eta}.$$

Consider

$$\begin{aligned}
|\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\tilde{\varphi}+\psi)}(\boldsymbol{\zeta}) &= \sup_{\mathbf{w}\in T^{1,0}_{\boldsymbol{\zeta}}\mathbb{C}^n\setminus\{0\}} \frac{|\langle\bar{\partial}\psi(\boldsymbol{\zeta}), \mathbf{w}\rangle|^2}{\langle\mathbf{w}, \mathrm{Hess}_{\mathbb{C},\tilde{\varphi}+\psi}\mathbf{w}\rangle} \\
&\leq \sup_{\mathbf{w}\in T^{1,0}_{\boldsymbol{\zeta}}\mathbb{C}^n\setminus\{0\}} \frac{|\langle\bar{\partial}\psi(\boldsymbol{\zeta}), \mathbf{w}\rangle|^2}{\langle\mathbf{w}, \mathrm{Hess}_{\mathbb{C},\psi}(\boldsymbol{\zeta})\mathbf{w}\rangle} \\
&\leq \frac{4}{\eta(\boldsymbol{\zeta})^2} \cdot \eta(\boldsymbol{\zeta}) \sup_{\mathbf{w}\in T^{1,0}_{\boldsymbol{\zeta}}\mathbb{C}^n\setminus\{0\}} \frac{|\langle\bar{\partial}\rho(\boldsymbol{\zeta}), \mathbf{w}\rangle|^2}{|\langle\bar{\partial}\rho(\boldsymbol{\zeta}), \mathbf{w}\rangle|^2}
\end{aligned}$$

and we get

$$\int_{\operatorname{supp} v} |\bar{\partial}\psi|^2_{i\partial\bar{\partial}(\tilde{\varphi}+\psi)}|u|^2 e^{\psi-\tilde{\varphi}} \leq \int_U \frac{4}{\eta}|u|^2 e^{\psi-\tilde{\varphi}}. \tag{4.7}$$

Substituting (4.5), (4.6) & (4.7) into (4.4),

$$\int_U \left(\frac{\frac{\eta}{(-\rho+1)^2}}{1 + \frac{\eta}{(-\rho+1)^2}} - \frac{4\delta}{\eta}\right)|u|^2 e^{\psi-\tilde{\varphi}} \leq \left(1 + \delta^{-1}\right) C'_n \int_{V_0} |f|^2 e^{-\varphi}. \tag{4.8}$$

<u>Passing to the limit to get a holomorphic extension:</u>

Since, $1 < -\rho < \eta < -2\rho$, the following term in the LHS of the inequality (4.8) is bounded below by,

$$\frac{\frac{\eta}{(-\rho+1)^2}}{1 + \frac{\eta}{(-\rho+1)^2}} - \frac{4\delta}{\eta} \geq \frac{\frac{(-\rho)^2}{(-\rho+1)^2}}{1 + \frac{-2\rho}{(-\rho+1)^2}} \frac{1}{\eta} - \frac{4\delta}{\eta} \geq \frac{1}{\eta}\left(\frac{(\rho^2}{(\rho^2 - 4\rho + 1}\right) \geq \frac{\frac{1}{6} - 4\delta}{\eta}$$

and yields

$$\int_U |u|^2 e^{-\tilde{\varphi}} \leq \frac{1 + \delta^{-1}}{\frac{1}{6} - 4\delta} C'_n \int_{V_0} |f|^2 e^{-\varphi} \leq C''_n C'_n \int_{V_0} |f|^2 e^{-\varphi}$$

where $\delta \in \left(0, \frac{1}{24}\right)$ is chosen to minimize $\frac{1+\delta^{-1}}{\frac{1}{6}-4\delta}$. This yields a family of solutions $\{u_{\delta,\varepsilon}\}_{0<\delta,\varepsilon\ll 1}$ which satisfy $\bar{\partial} u_{\delta,\varepsilon} = v_\varepsilon$ Now, extracting a weakly convergent subsequence $u_{\delta,\varepsilon} \to u_\varepsilon$ we get a family $\chi \circ \frac{|z_n|^2}{\varepsilon^2} \cdot f - u_{\delta,\varepsilon}$ which extend f. Passing down to another weakly convergent subsequence yields a holomorphic limit F which satisfies

$$\int_U \frac{|F|^2}{|z_n|^2(-\log|z_n|^2)}\, e^{-\varphi} \le C_n \int_{V_0} |f|^2 e^{-\varphi}.$$

In light of the sequential interior exhaustion $U = \bigcup_j U_j$ by strictly pseudoconvex domains, we get that for every j

$$\int_{U_j} |F_j|^2\, e^{-\varphi_j} \le C_n \int_{V_j^{(0)}} |f|^2 e^{-\varphi}$$

for some $C_n > 0$. $F_j \in L^2_{\mathbb{C}}(U_k, e^{-\varphi_k})$ for $j \ge k$ and has a weakly convergent subsequence converging to $\widehat{f} \in L^2_{\mathbb{C}}(U_k, e^{-\varphi_k})$ for all $k \in \mathbb{N}$ yielding the desired extension as $\bar{\partial}\widehat{f}|_{U_j} = 0$ for all j. □

Chapter 5

L^2 estimate on complex manifolds

Let (X, ω) be a Hermitian manifold. For the purposes of integration, we shall assume our manifolds to be compact. With these features, we aim to sketch the Hörmander's estimate on complex manifolds following [16, Chapters-4,5] closely.

5.1 Constructing the Hilbert spaces

Throughout this section, let $E \to X$ be a holomorphic vector bundle.

Definition 5.1.1. (The L^2 space of bundle valued forms) The module of sections $\Gamma(X, \bigwedge^{(p,q)} T^{\vee}_{\mathbb{C}}X \otimes E)$ can be equipped with an inner product by taking two E-valued, (p, q) forms ψ_1, ψ_2 and defining:

$$\langle\!\langle \psi_1, \psi_2 \rangle\!\rangle_{g,h} := \int_X \langle \psi_1, \psi_2 \rangle_{g,h} \, \mathrm{d}\mathcal{V}_g$$

where the integrand function is obtained by the metric induced by g and h.

The ***L^2 space of bundle valued forms*** denoted by $L^2_{(p,q),E}(X, |\cdot|^2_{g,h}\mathrm{d}\mathcal{V}_g)$ is then defined to be the Hilbert space completion of $\left(\Gamma(X, \bigwedge^{(p,q)} T^{\vee}_{\mathbb{C}}X \otimes E)\,,\, ||\cdot||_{g,h}\right)$.

Definition 5.1.2. ($\bar\partial$ AND WEAK SOLUTIONS) We define the domain of $\bar\partial$ as :

$$\mathrm{dom}(\bar\partial) := \left\{u \mid \bar\partial u \in L^2_{(p,q),E}(X, |\cdot|^2_{g,h}\mathrm{d}\mathcal{V}_g)\right\}.$$

A ***weak solution*** to $\bar\partial u = f$ for $f \in L^2_{(p,q),E}(X, |\cdot|^2_{g,h}\mathrm{d}\mathcal{V}_g)$ (where $q \geq 1$) is then defined as a L^2 integrable $(p, q-1)$ section u satisfying:

$$\langle\!\langle u, \bar\partial^\star \psi\rangle\!\rangle_{g,h} = \langle\!\langle f, \psi\rangle\!\rangle_{g,h}$$

for all $\psi \in \Gamma(X, \bigwedge^{(p,q)} T^\vee_{\mathbb{C}} X \otimes E)$.

5.2 The Bochner-Kodaira identity

We define a 'connection-like' operator as follows:
Definition 5.2.1. (A CONNECTION-LIKE OPERATOR)
Consider the map:

$$\Diamond : \Gamma(X, \bigwedge^{(n,q)} T^\vee_{\mathbb{C}} X \otimes E) \to \Gamma(X, \bigwedge^{(n,q)} T^\vee_{\mathbb{C}} X \otimes T^\vee_{\mathbb{C}} X^{(0,1)} \otimes E),$$

given in coordinates by:

$$\begin{aligned}\varphi = \varphi^\alpha_I dz^1 \wedge \cdots \wedge dz^n \wedge d\bar z^I \otimes e_\alpha \mapsto g_{I\bar L} \frac{\partial}{\partial \bar z^k}(g^{I\bar J}\varphi^\alpha_J)\, d\bar z^k \otimes dz^1 \wedge \cdots dz^n \wedge d\bar z^L \otimes e_\alpha \\ = (\overleftrightarrow{\nabla}_{\bar\partial_k}(\varphi))^\alpha_{\bar I}\, d\bar z^k \otimes dz^1 \wedge \cdots dz^n \wedge d\bar z^I \otimes e_\alpha. \qquad (5.1)\end{aligned}$$

Proposition 5.2.1. (ADJOINT AND LAPLACE-BELTRAMI OPERATORS OF $\Diamond$) Consider $\psi \in \Gamma(X, \bigwedge^{(n,q)} T^\vee_{\mathbb{C}} X \otimes T^\vee_{\mathbb{C}} X^{(0,1)} \otimes E)$ and $\varphi \in \Gamma(X, \bigwedge^{(n,q)} T^\vee_{\mathbb{C}} X \otimes E)$.

(i) The formal adjoint $\Diamond^\star$ of $\Diamond$ is given by:

$$\varphi = \varphi^\alpha_{J,\bar j} d\bar z^j \wedge dz^1 \wedge \cdots \wedge dz^n \wedge d\bar z^I \otimes e_\alpha \mapsto -g^{k\bar j}(\overleftrightarrow{\nabla}_{\partial_k}\varphi)^\alpha_{J,\bar j}\, dz^1 \wedge \cdots \wedge dz^n \wedge d\bar z^I \otimes e_\alpha. \qquad (5.2)$$

(ii) The Laplace-Beltrami operator $\Diamond^\star \Diamond$ of $\Diamond$ is given by:

$$\psi = \psi^\alpha_{\bar{J}} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^I \otimes e_\alpha \mapsto -g^{j\bar{k}} (\overleftrightarrow{\nabla}_{\partial_j} \overleftrightarrow{\nabla}_{\bar{\partial}_k} (\psi))^\alpha_{\bar{J}}\, dz^1 \wedge \cdots dz^n \wedge d\bar{z}^I \otimes e_\alpha. \tag{5.3}$$

Consider the $\bar{\partial} : \Gamma(X, \bigwedge^{(n,q)} T^\vee_{\mathbb{C}} X \otimes E \to \Gamma(X, \bigwedge^{(n,q)} T^\vee_{\mathbb{C}} X \otimes T^\vee_{\mathbb{C}} X^{(0,1)} \otimes E$.

Proposition 5.2.2. (The Bochner-Kodaira identity)

The Laplace-Beltrami operator associated with $\bar{\partial}$ acting on E-valued (n, q) forms given by $\Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^\star + \bar{\partial}^\star\bar{\partial}$ is connected to the curvature of Chern connection compatible with the metric $\mathfrak{h} := \det g \otimes h$ through:

$$\Delta_{\bar{\partial}} = \Diamond^\star \Diamond + \Theta_g(\mathfrak{h}), \tag{5.4}$$

where the curvature term denotes

$$\Theta_g(\mathfrak{h})\varphi := \sum_{k=1}^{q} g^{i\bar{l}} \Theta(\mathfrak{h})^\alpha_{\beta i \bar{j}_k} \varphi^\beta_{j_1 \ldots (\ell)_k \ldots j_q} dz^1 \wedge \cdots \wedge dz^n \wedge dz^{j_1} \wedge \cdots \wedge dz^{j_q} \otimes e_\alpha.$$

5.3 The L^2 estimate on a compact Kähler manifold

Theorem 5.3.1. *Let (X, g) be a compact Kähler manifold and let $F \to X$ be a holomorphic vector bundle with Hermitian metric h. If the curvature of the Chern connection of the metric vector* $\det(g) \otimes h$ *for the vector bundle $\bigwedge^n T^\vee X \otimes F$ satisfies the positivity condition,*

$$\Theta_g(\det g \otimes h) \geq c \cdot \mathbb{id} \tag{5.5}$$

for some constant $c > 0$, then for every F-valued $\bar{\partial}$-closed $(0, 1)$-form f satisfying

$$\int_X |f|^2_{g,h}\, \mathrm{d}\mathcal{V}_g < \infty,$$

there exists an $u \in L^2_{(0,0),E}(X, |\cdot|^2_{g,h}\mathrm{d}\mathcal{V}_g)$ satisfying

$$\bar{\partial} u = f, \quad \text{and} \quad \int_X |u|^2_{g,h} \mathrm{d}\mathcal{V}_g \leq \frac{1}{c} \int_X |f|^2_{g,h} \mathrm{d}\mathcal{V}_g.$$

PROOF. Let $E := \bigwedge^n T^\vee X \otimes F \cong \bigwedge^{(n,0)} T^\vee_{\mathbb{C}} X \otimes F$. Then an F-valued $(0,1)$-form identifies with an E-valued $(n,1)$-form.

Let f be an E-valued $(n,1)$-form with finite L^2 norm. For any smooth, E-valued $(n,1)$-form ψ, by (5.5), we have the estimate

$$\begin{aligned}
\|\psi\|^2 :&= \langle\!\langle \psi, \psi \rangle\!\rangle \leq \frac{1}{c} \langle\!\langle \Theta(\det g \otimes h)\psi, \psi \rangle\!\rangle \\
&\leq \frac{1}{c} \langle\!\langle \Theta(\det g \otimes h)\psi, \psi \rangle\!\rangle + \langle\!\langle \Diamond\psi, \Diamond\psi \rangle\!\rangle \\
&= \frac{1}{c} \langle\!\langle \Delta_{\bar\partial}\psi, \psi \rangle\!\rangle \\
&= \frac{1}{c} \langle\!\langle \bar\partial^\star \bar\partial \psi, \psi \rangle\!\rangle + \langle\!\langle \bar\partial \bar\partial^\star \psi, \psi \rangle\!\rangle.
\end{aligned}$$

Therefore, we get the inequality,

$$\|\psi\|^2 \leq \frac{1}{c} \|\bar\partial \psi\|^2 + \|\bar\partial^* \psi\|^2. \tag{5.6}$$

Now the bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ defined on $\varphi \in \Gamma(X, \bigwedge^{(n,q)} T^\vee_{\mathbb{C}} X \otimes E)$ by

$$(\psi_1, \psi_2)_{\mathcal{H}} := \langle\!\langle \bar\partial\psi_1, \bar\partial\psi_2 \rangle\!\rangle + \langle\!\langle \bar\partial^*\psi_1, \bar\partial^*\psi_2 \rangle\!\rangle$$

is an inner product by (5.6). This ascends to the Hilbert space completion of $\Gamma(X, \bigwedge^{(n,q)} T^\vee_{\mathbb{C}} X \otimes E)$ which we denote by $\mathcal{H}$.

Consider the linear functional $\lambda_f : L^2_{(p,q),E}(X, |\cdot|^2_{g,h} \mathrm{d}\mathcal{V}_g) \to \mathbb{C}$ defined by

$$\lambda_f(\psi) := \langle\!\langle \psi, f \rangle\!\rangle.$$

Cauchy-Schwarz inequality tells us that $|\lambda_f(\psi)|^2 \leq \frac{1}{c} \|f\|^2 \, \|\psi\|^2_{\mathcal{H}}$ implying that $\lambda_f \in \mathcal{H}^\star$. By the Riesz Representation Theorem, there exists $v \in \mathcal{H}$ such that

$$\|v\|^2_{\mathcal{H}} \leq \frac{1}{c} \|f\|^2 \quad \& \quad \langle v, \psi \rangle_{\mathcal{H}} = \lambda_f(\psi), \quad \forall\, \psi \in \mathcal{H}.$$

For smooth ψ, we have

$$\langle\!\langle \bar\partial v, \bar\partial\psi \rangle\!\rangle + \langle\!\langle \bar\partial^\star v, \bar\partial^\star \psi \rangle\!\rangle = \langle\!\langle f, \psi \rangle\!\rangle. \tag{5.7}$$

Now, since $\bar{\partial} f = 0$, we have

$$\begin{aligned} 0 = \langle\langle f, \bar{\partial}^\star \psi \rangle\rangle &= \langle\langle \bar{\partial} v, \bar{\partial}\bar{\partial}^\star \psi \rangle\rangle + \langle\langle \bar{\partial}^\star v, \bar{\partial}^\star \bar{\partial}^\star \psi \rangle\rangle \\ &= \langle\langle \bar{\partial} v, \bar{\partial}\bar{\partial}^\star \psi \rangle\rangle + \langle\langle \bar{\partial}^2 \bar{\partial}^\star v, \psi \rangle\rangle \\ &= \langle\langle \bar{\partial}^\star \bar{\partial} v, \bar{\partial}^\star \psi \rangle\rangle \\ &= \langle\langle \bar{\partial}\bar{\partial}^\star \bar{\partial} v, \psi \rangle\rangle \end{aligned}$$

for every smooth section ψ which implies that $\bar{\partial}\bar{\partial}^\star \bar{\partial} v = 0$ as a current. Therefore,

$$\langle\langle \bar{\partial}^\star \bar{\partial} v, \bar{\partial}^\star \bar{\partial} v \rangle\rangle = \langle\langle \bar{\partial}\bar{\partial}^\star \bar{\partial} v, \bar{\partial} v \rangle\rangle = 0$$

and thus $\bar{\partial}^\star \bar{\partial} v = 0$ further implying that $\langle\langle \bar{\partial} v, \bar{\partial} \psi \rangle\rangle = 0$. Hence, from (5.7), one gets

$$\langle\langle \bar{\partial}^\star v, \bar{\partial}^\star \psi \rangle\rangle = \langle\langle f, \psi \rangle\rangle,$$

which means that $u := \bar{\partial}^\star v$ is a solution of $\bar{\partial} u = f$. The L^2 estimate follows by:

$$\|u\|^2 = \|\bar{\partial}^\star v\|^2 = \|v\|^2_{\mathcal{H}} \leq \frac{\|f\|^2}{c} \quad (\because \|\bar{\partial} v\|^2 = 0).$$

$\square$

Chapter 6

Conclusion

The thesis aimed at exploring the field of several complex variables by adopting the route of L^2 techniques developed from Hörmander's ideas. We first familiarized ourselves with the notion of a domain of holomorphy emphasizing on the first elementary characterization using the concept of holomorphic convexity. The main ideas used here revolve around normal exhaustions and some boundary topology.

The absence of a Riemann mapping theorem in several complex variables fascinates the situation and tells us that there is more scope for non-routine nature in complex analysis of multiple variables as opposed to one. The Poincare's inequivalence formally tells us why.

Post this we realized how the $\bar{\partial}$-problem is a useful tool in complex function theory by studying the *Kugelsatz* of Hartogs.

Equipped with the armoury of functional analysis, the theorems of Riesz representation, Hahn-Banach extension, Banach-Alouglu and distribution theory we then studied the Hörmander estimate on pseudoconvex domains in $\mathbb{C}^n$ starting with the one dimensional consideration.

To apply these ideas, we selected the Levi problem and Ohsawa-Takegoshi extension theorem and saw how the L^2 estimate could be employed in these settings to derive geometrically perceivable reflections of the L^2 technique developed.

The setting of domains in $\mathbb{C}^n$ was abstracted to complex manifolds possessing topological and geometric features i.e, the subclass of compact Kähler manifolds to carry out the L^2 estimate with higher grandeur.

List of Figures

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