K-Theory of Monoid Algebras

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Manish Kumar Singh

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2018

Supervisor: Dr. Rabeya Basu

Certificate

This is to certify that this dissertation entitled K-Theory of Monoid Algebras towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Manish Kumar Singh at Indian Institute of Science Education and Research under the supervision of Dr. Rabeya Basu, Assistant Professor, Department of Mathematics, during the academic year 2017-2018.

Rabeya Basu.

Dr. Rabeya Basu

Committee:

Dr. Rabeya Basu

Dr. Vivek Mohan Mallick

To my family

Declaration

I hereby declare that the matter embodied in the report entitled K-Theory of Monoid Algebras are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Rabeya Basu and the same has not been submitted elsewhere for any other degree.

Manish Kumar Singh

Acknowledgments

I would like to thank my parents, teachers and friends who helped me to reach this stage. I would also like to thank Dr. Rabeya Basu for introducing me to this exciting subject and guiding me through. I am also thankful to Professor Raja Sridharan for providing me a simplified version of Quillen's graded L-G principal (cf: 3.5.1). Finally, I express my deep gratitude to IISER, Pune for giving me an opportunity to learn mathematics.

Abstract

As a generalisation of Serre's problem on projective modules over polynomials ring. in 1980 D. Anderson asked the analogue problem for monoid algebras. In 1988 Joseph Gubeladze proved Anderson's conjecture by geometric and combinatorial methods. Soon after, following his idea R.G Swan came up with algebraic version if Gubeladze's proof. This thesis is an expository article of Swan's paper appeared in 1991.

Contents

Abstract **vi**

1 Preliminaries **4**
1.1 Finitely generated modules 4
1.2 Tensor product 6
1.3 Localisation 8
1.4 Projective modules 10
1.5 Milnor patching property 13

2 Graded rings and modules **16**
2.1 Graded Rings: Basic definition and examples 16
2.2 Graded ideals 18
2.3 Graded prime and maximal ideals 18
2.4 Graded local rings 19
2.5 Graded modules: Basic definitions and examples 20
2.6 Graded free modules 21
2.7 Graded projective modules 21

3 Classical result on Serre's conjecture **25**
3.1 Rank 1 projective modules 25
3.2 Serre's problem for one variable 27
3.3 Serre's problem for graded case 28
3.4 Quillen-Suslin theorem: Outline of the proof 29
3.5 Quillen's graded local-global principal 31

4 Monoids and monoid algebras **34**
4.1 Monoids: Definition and examples 34
4.2 Normal and seminormal monoids 36
4.3 Monoid algebras 38
4.4 Grading of monoid algebras 40

5 Some properties of monoids and monoid algebras **43**
5.1 Extremal submonoids 43
5.2 Homothetic submonoids 45
5.3 Graded Weierstrass preparation theorem 47
5.4 Pyramidal extension 49

6 Swan's proof of Anderson's conjecture **55**
6.1 Preliminary reductions 55
6.2 Non-degenerate pyramidal extension 58
6.3 Admissible sequence 61
6.4 Final proof 65

7 Converese and application of Gubeladze's theorem **66**
7.1 Converse of Anderson's conjecture 66
7.2 A question of Hartmut Lindel 68

8 Appendix **71**
8.1 Leavitt path algebras 71
8.2 Convex geometry and monoids 79
8.3 Definition of $\mathrm{Pic}(R[M])$ and $K_0(R[M])$ 83

Introduction

Let us begin with the following facts in classical algebraic K-theory known as *Serre's problem for projective modules.* In 1955, J.P. Serre asked the following question:

Theorem. *Is every finitely generated projective module over polynomial ring* $K[X_1, \ldots, X_n]$ *over a field* K *free?*

It took two decades and several crucial intermediary milestone before a final resolution was reached due to D. Quillen and A. Suslin independently in 1976. They proved:

Theorem (Quillen-Suslin theorem). *If* R *is a PID and* $A = R[X_1, \ldots, X_n]$ *then every finitely generated projective* A*-module* P *is extended from* R*.*

Quillen's proof relied on two crucial result which were:

Theorem (Horrocks theorem). *Let* $(R, \mathfrak{m})$ *be any local commutative ring, and let* P *be a finitely generated projective* $R[T]$*-module. If* $P\langle t\rangle := R\langle T\rangle \otimes_{R[T]} P$ *is extended from a finitely generated* R*-module, then* P *is extended from* R*.*

Theorem (Quillen's L-G (local-global) principal). *Let* R *be a commutative ring and let* P *be a projective* $R[X_1, \ldots, X_n]$*-module. If* $P_\mathfrak{m}$ *is extended from* $R_\mathfrak{m}$ *for all* $\mathfrak{m} \in max(R_0)$ *if and only if* P *is extended from* R*.*

As we can see, Horrocks thoerem is a result concerning only local rings, it is only due to Quillen L-G principal, that we are able to extend this result for non-local rings.

The aim of this thesis is to study analogue problem in K-theory of monoid algebras from [15].

In 1978, D. Anderson asked the same question for a more generalised class of ring namely the monoid algebras $R[M]$:

Theorem (Anderson's conjecture). *Under what condition on* R *and* M *is every finitely* *generated projective module P over monoid algebra $R[M]$.*

It was finally in 1988 when J. Gubeladze confirmed the Anderson's conjecture.
Theorem (Gubeladze's theorem). *Let P be a finitely generated projective $R[M]$-module. Then P is free if M is affine, finitely generated and seminormal and R is a PID. Gubeladze's theorem in the language of K-theory is equivalent to the following conditions (cf: appendix):*

1. *$Pic(R[M]) = 0$.*

2. *$K_0(R[M]) = \mathbb{Z}$.*

3. *Finitely generated projective $R[M]$-module is free.*

4. *Monoid M is seminormal.*

His proof involved clever interplay of convex -geometry and algebra. In principal Gubeladze's proof relied on the above two theorems (Horrocks and Quillen's L-G) written for monoid algebras and some extra arguments for monoids. In 1991, R. Swan inspired by Gubeladze's geometric ideas, gave an algebraic version of the Gubeladze's theorem.

The layout of the thesis is as follows:

In chapter 1 we recalls the basic commutative algebra required viz few definitions, example and properties of tensor products, localisation tool etc. It also covers projective modules- which is the central object of our study. Finally in the last section we discuss about patching diagrams and its properties. The main references for this chapter is [2]. For definition and properties in commutative algbera cf. [5]. To see the details about Milnor patching property see [11] and [12] .

In chapter 2 we covers the basics of graded ring and modules, which helps us to solve a graded version of Serre's problem. Most of the results can be found in [8]. In chapter 3 we give a proof of Serre's problem by starting with some historical result which includes cases like projective modules of rank 1, polynomial ring in one variable $R[X]$, graded projective modules over graded rings $R = \bigoplus_{i \in \Gamma} R_\gamma$ (cf: [11]) before cumulating in the final proof (only outline) for polynomial rings in n-variables (cf: [13]). In the end we have included a graded-version of Quillen's L-G principal from [4] and [15], one of the two important result for having $K_0(R[X]) = \mathbb{Z}$.

Theorem (Graded Quillen's patching)**.** *Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be a graded ring and P a finitely generated R-module. Then P is extended from R_0 if and only if $P_{\mathfrak{m}}$ is extended from $(R_0)_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset R$.*

The importance of this result is not only restricted to Gubeladze's theorem. As a general result, it is an important tool to study the structure of graded commutative rings.

Chapter 4 and 5 covers technical details of monoids and monoid algebra which would be required in the proof of the Anderson's conjecture. More details can be found out in [4]. In chapter 6, we dicuss the proof of Anderson's conjecture and in chapter 7 we look at the converse of Anderson's conjecture and some its interesting application. Additional details can be found in [4] and [11].

In Appendix we briefly talk about Leavitt-path algebras (LPA). As it turns out that LPA form a non-commutative, non-IBP ring with a $\mathbb{Z}$-grading and serves as an excellent example to test various conjectures. Studying its property is currently an area of active research. It is natural to ask how far can we generalise classical algebraic K-theory problems for this non-commutative graded algebras. More information on this can be found out in [1]. Next, we include a short discussion on convex geometry, which would demonstrate how algebra and convex geometry interact, we also give some geometric intuition behind some of the abstract algebraic constructions and proof. Finally, we define K-theoretic structures: K_0-group and Pic-group.

Chapter 1

Preliminaries

Throughout this thesis let

1. $\mathbb{N}$: set of natural numbers.
2. $\mathbb{Z}$: set of integers.
3. $\mathbb{Z}_+$: set positive integers.
4. $\mathbb{Q}$: set rational numbers
5. $\mathbb{Q}_+$: set positive rational numbers.
6. $\mathbb{R}$: set real numbers and $\mathbb{R}_+$: set positive real numbers.

In this chapter, we would recall certain preliminary results from commutative algebra which would be used throughout this thesis. Also in the end, we discuss patching diagram which again will be useful later.

1.1 Finitely generated modules

Throughout the thesis we will mostly deal with finitely generated modules over a commutative ring R with identity. We now give some standard commutative algebra result whose proof can be found in [2]. Let us begin with the well known *Nakayama Lemma.*

Lemma 1.1.1 (Nakayama Lemma)**.** Let M be finitely generated R-module, $\mathfrak{a}$ an ideal contained in the Jacobson ideal of R, then $\mathfrak{a}M = M$ implies $M = 0$.

Corollary 1.1.2. If M is a finitely generated R-module, N submodule of M and $\mathfrak{a} \subset$ Jacobson radical of R, then $M = \mathfrak{a}M + M$ implies $M = N$.

Definition 1.1.3. A sequence of R-module homomorphism is exact

$$\cdots \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to \cdots$$

if $\operatorname{Im}(f_{i-1}) = \operatorname{Ker}(f_i)$. We denote by $\operatorname{Coker}(f_i) = A_{i+1}\ /\operatorname{Im}\ f_i(A_i)$.

Example 1.1.4. The following sequence is exact, where I is an ideal of the polynomial ring $R[X]$ and f is the ,natural map:

$$0 \to I \to R[X] \to R[X]/I \to 0.$$

Lemma 1.1.5. Consider thr following sequence be an exact sequence of R-modules

$$0 \to A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to 0$$

with P an arbitrary R-module. Let $\operatorname{Hom}_R(P, A_i)$ denote the set of all homomorphism $h : P \to A_i$, then

$$0 \to \operatorname{Hom}_R(P, A_0) \xrightarrow{f_0'} \operatorname{Hom}_R(P, A_1) \xrightarrow{f_1'} \operatorname{Hom}_R(P, A_2)$$

is an exact sequence.

Definition 1.1.6 (Split Exact Sequence)**.** A sequence of R-module homomorphism

$$0 \to A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to 0$$

is called split exact if there exists a R-module homomorphism $g_2 : A_2 \to A_1$ such that $f_2 \circ g_2 = 1_{A_2}$.

Example 1.1.7. The following sequence, where f_i are the inclusion map, is easily verified to be split exact:

$$0 \to 2\mathbb{Z} \xrightarrow{f_0} \mathbb{Z} \xrightarrow{f_1} \mathbb{Z}/2\mathbb{Z} \to 0$$

where the inverse map $({f_1}^{-1})$ is given by $\overline{0} \mapsto 0$ and $\overline{1} \mapsto 1$.

Lemma 1.1.8. If the following sequence of R-modules,

$$0 \to A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to 0$$

is split exact. then

$$A_2 \oplus f_0(A_0) = A_1.$$

1.2 Tensor product

Intuitively $A \otimes B$ is the free R-module generated over $A \times B$ modulo the relations of the type $a \otimes (b + c) = a \otimes n + a \otimes c$.

Definition 1.2.1 (Tensor Product of Modules)**.** Let A and B be a R-module, then there exists a bilinear map $\hat{f}$ between R-modules $A \times B$ and $A \otimes B$ such that whenever their is a bilinear map f between $A \times B$ and T, there exists a linear map π from $A \otimes B$ to T such that $\hat{f} \circ \pi = f$ (it satisfies a universal property).

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[dr, "\hat{f}"] & T \\ & A \otimes B \arrow[u, "\pi"] \end{tikzcd}$$

Few properties of Tensor product of modules:

Lemma 1.2.2.

1. $(M \otimes_R N) \otimes_S P \simeq M \otimes_R (N \otimes_S P)$
2. $M = \bigoplus_i M_i$ and $N = \bigoplus_j N_j$, then $M \otimes N \simeq \bigoplus_{i,j} M_i \otimes N_j$

Proof. For R-modules, we have the following isomorphism.

Let the map

$$\phi : (M \otimes_R N) \times P \to M \otimes_R (N \otimes_S P) \text{ be given by } (x \otimes y, z) \mapsto x \otimes (y \otimes z).$$

Then this map is a bilinear map. Hence, it induces a linear map
$\tilde{\phi}(M \otimes_R N) \otimes_S P \to M \otimes_R (N \otimes_S P)$. This is an isomorphism, since there exists a

inverse map $x \otimes (y \otimes z) \mapsto (x \otimes y) \otimes z$.

2. The map

$$\phi : M \otimes N \to \bigoplus_{i,j} M_i \otimes N_j \text{ given by } (m \otimes n) \mapsto \Sigma(m_i \otimes n_j)$$

is a linear map. Let $\psi_{i,j} : M_i \otimes N_j \to M \otimes N$ be the linear inclusion map. Then $\psi = \Sigma_{i,j}\psi_{i,j}$ (where $\psi_{i,j} = 0$ if it is outside the domain) is also a linear map with $\psi \circ \phi = \mathrm{Id}$ and $\phi \circ \psi = \mathrm{Id}$. Hence ψ is an isomorphism.

□

Lemma 1.2.3. $-\bigotimes R$ is left exact on an exact sequence of R-modules.

Proof. Observe that $\mathrm{Hom}_R(M \otimes N, P) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$. Hence using Lemma 1.1.5 on
$0 \to N_1 \to N_2 \to N_3$ twice (first with P and then with M) we have $\mathrm{Hom}_R(M, , \mathrm{Hom}_R(N_i, P))$ as a left exact sequence and hence $\mathrm{Hom}_R(M \otimes N_i, P)$ is left exact. □

Lemma 1.2.4. Tensor product commutes with direct sum i.e. for R-modules M, N and P

$$(M \oplus N) \otimes P \simeq (M \otimes P) \oplus (N \otimes P).$$

Proof. Consider the map
$f : (M \oplus N) \otimes P \xrightarrow{f} (M \otimes P) \oplus (N \otimes P)$ given by $(m + n) \otimes p \mapsto m \otimes p + n \otimes p$. This is an identity map and hence isomorphism. □

One of the important properties of commutative ring is the invariance of free rank of a free module as illustrated in the following lemma.

Lemma 1.2.5. Let R be a commutative ring, then for finite m, n, $R^n \simeq R^m$ implies $m = n$.

Proof. If $R^m \simeq R^n$, then $(R/\mathfrak{m}) \otimes R^m = (R/\mathfrak{m}) \otimes R^n$. Hence $(R/\mathfrak{m})^m = (R/\mathfrak{m})^n$, but the new modules are vector space and since dimension of vector space is invariant, we have $m = n$. □

Definition 1.2.6. Consider the ring homomorphism $f : A \to B$, then B can be considered as an A-module by defining the scalar multiplication as follows:

For $a \in A, b \in B$, $ab := f(a)b$. We call such B an A-algebra.

Example 1.2.7. The polynomial ring $R[X]$ is an R-algebra with generating set $\langle X, 1\rangle$.

Remark 1.2.8. *As seen from 1.2.7 , $R[X]$ is a finitely generated R-algebra but we know that $R[X]$ is not a finitely generated R-module.*

Definition 1.2.9. Let $R \subset S$ be two commutative rings. Then we say $s \in S$ is *integral* over R if their exists a monic polynomial $f(X) \in R[X]$ such that $f(s) = 0$. The set of all integral element of R over S is called integral closure of R and is denoted by $\tilde{R}$.

Example 1.2.10. Consider $\mathbb{Z} \subsetneq \mathbb{Q}$. Then the integral closure of $\mathbb{Z}$ is $\tilde{\mathbb{Z}} = \mathbb{Z}$.

1.3 Localisation

We now aim to generalise the concept of ring of fraction to an arbitrary commutative ring (not just domain).

Definition 1.3.1. Let R be a ring and S be a multiplicative closed subset of R . Then $S^{-1}R$ denotes the ring $S \times R$ under the equivalence relation that

$$(s_1, r_1) \sim (s_2, r_2)$$

$$\text{if } k(s_1r_2 - s_2r_1) = 0 \text{ for some } k \in S.$$

If $S = (f)$ then $S^{-1}R$ is denoted as R_f. If $S = R - \mathfrak{p}$, $\mathfrak{p}$ a prime ideal, then $S^{-1}R$ is denoted as $R_{\mathfrak{p}}$. If M is an R-module then we can define equivalence class in $S^{-1}M$ as follows:

$$(m_1, r_1) \sim (m_2, r_2)$$

$$\text{if } k(m_1r_2 - m_2r_1) = 0 \text{ for some } k \in S.$$

Example 1.3.2. Consider $S = R - \mathfrak{m}$, where $\mathfrak{m}$ a maximal ideal. Then $R_{\mathfrak{m}}$ is a local ring (and hence the terminology).

Lemma 1.3.3. If $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact sequence of R-modules, then $M'_S \xrightarrow{f_S} M_S \xrightarrow{g_S} M''_S$ is also exact.

Proof. $gf = 0$ implies $g_Sf_S = 0$ and hence $\mathrm{Im}(f_S) \subset \mathrm{Ker}(g_S)$. If $x \in M_S$ such that $g_S(x/s) =$

0, then $kg(x) = 0$ for some $k \in S$, hence $g(kx) = 0$ or $kx \in M'$. Therefore $(x/s) = (kx/ks) \in M'$. $\square$

Corollary 1.3.4. If N is a R-submodule of M, then $(M/N)_S = (M_S/N_S)$.

Corollary 1.3.5. If $\mathfrak{p}$ is a prime ideal, then $(M \otimes_R N)_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes N_{\mathfrak{p}}$.

Lemma 1.3.6. Given R-module M, $S^{-1}R \otimes M \simeq S^{-1}M$ by the map
$f : (a/s) \otimes m \mapsto (am/s)$.

Proof. First we check if the map is well defined or not. If $(a/s) = (b/t)$, then $((a/s) \otimes m) = (am/s)$ and $(bm/t) = f((bm/t))$. Since $(a/s) = (b/t)$ it implies $k(at - bs) = 0$ and hence $km(at - bs) = 0$ or equivalently $k(amt - bms) = 0$ or $(am/s) = (bm/t)$. Hence it is well defined. Now surjectivity of f is obvious $((1/s) \otimes m \mapsto (m/s))$. To check for injectivity of f, let assume to the contrary i.e. $(a/s) \otimes x \neq 0$ but $(a/s)x = 0$ hence $k(ax) = 0$, therefore $(kax)/(kxs) = a/s = 0$, and therefore $(a/s) \otimes x = 0$ - a contradiction. Hence f is injective and therefore f is an isomorphism. $\square$

Definition 1.3.7. Given a R-module M and a ring homomorphism
$f : R \to S$ then $S \otimes_R M$ is a S modules and is called the module obtained by scalar extension.

Remark 1.3.8. *Let P is a projective $R[X_1, \ldots, X_n]$-module, then P is extended from P/X_1P module which is a projective $R[X_2, \ldots, X_n]$-module.*

Example 1.3.9. Let R be a integral domain, and K be its field of fraction, then an R-module M becomes a K-vectors space $M \otimes_R K$.

We call a K-module N extended from R if there exists a R-module M such that $M \otimes K = N$. Note that if N is considered as an R-module (i.e restriction of scalars), then it need not follow that N is extended from R due to next lemma.

Lemma 1.3.10. Let $\psi : R \to K$ be a ring homomorphism and let N be a K-module. If we consider N to be a R-module, then N is direct sum of $N \otimes_R K = N_K$.

Proof. Consider the map $g : N \to N_K$ given by $n \mapsto n \otimes 1$ and the map $f : N_K \to N$ given by $n \otimes k \mapsto kn$, then $f \circ g(n) = f(1 \otimes n) = n$, hence $f \circ g = 1$ and therefore g is injective. It is obvious that f need not be injective, as $b \otimes n = bn = 0$ if b is annihilator of n. To see why it is a direct sum, observe that $N \subset N_K$ due to injectivity of g, hence this homomorphism is a split homomorphism and hence the result follows. $\square$

Remark 1.3.11. *Let $R \subset S$. In general if P is a projective module (see section 1.4) S-module which admits an extension from a R-module Q i.e.*
$P \simeq Q \otimes S$, then P considered as a R-module need not be isomorphic to Q.

Example 1.3.12. If P is a projective $R[X]$ module, then P admits a R-extension P/XP. Clearly P, as an R-module is not isomorphic to P/XP.

Lemma 1.3.13. Let M and N be projective R-module (see next section). If $M_S \xrightarrow{\psi} N_S$ are R_S module isomorphism, then their exists $f \in S$ such that $R_f \xrightarrow{\phi} N_f$ is an R_f-module isomorphism and ϕ localised to ψ.

Proof. cf: ([11] corollary 2.16). □

see

1.4 Projective modules

Definition 1.4.1. P is a projective R-module if it is the direct summand of a free R-module R^n.

Notation 1.4.2. *Let $\mathcal{P}(R)$ denote the set of all projective R module.*

Example 1.4.3. Free Modules are obviously projective. For the case of non-free projective module, consider

$$\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/mn\mathbb{Z}$$

where $(m, n) = 1$ by Chinese remainder theorem. Since $\mathbb{Z}/n\mathbb{Z}$ is direct summand of $\mathbb{Z}/mn\mathbb{Z}$.

Proposition 1.4.4. The following are equivalent definitions for a projective mR-module P.

1. P is direct summand of a free R-module.

2. Short exact of R-modules $0 \to A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to 0$ induces a short exact sequence

$$0 \to \mathrm{Hom}_R(P, A_0) \xrightarrow{f_0'} \mathrm{Hom}_R(P, A_1) \xrightarrow{f_1'} \mathrm{Hom}_R(P, A_2) \to 0.$$

3. Every exact sequence of R-modules $0 \to K \to L \xrightarrow{g} P \to 0$ splits at g.

Proof. (1) $\implies$ (2) : Using results from previous chapter, the only part remained to be proved is

$$\mathrm{Hom}_R(P, A_1) \xrightarrow{f_1'} \mathrm{Hom}_R(P, A_2) \to 0$$

which is same as proving $g \in \mathrm{Hom}(P, A_2)$ implies there exists $h \in \mathrm{Hom}(P, A_1)$.

$$\begin{tikzcd}
\mathcal{F}(P) \arrow[r] & P \arrow[d, "h"] \arrow[dr, "g"] & & \\
& A_1 \arrow[r, "f_1"] & A_2 \arrow[r] & 0
\end{tikzcd}$$

such that $f_1 h = g$.

Since P is projective, let $\mathcal{F}(P)$ denote the free module of which P was direct summand. If $g' \in \mathrm{Hom}(\mathcal{F}(P), A_2)$ then it can easily be lifted to $h' \in \mathrm{Hom}(F(P), A_1)$ by choosing suitable elements

$$f_1(x_i) = y_i$$

$$h'(e_i) = x_i$$

$$g'(e_i) = y_i$$

Since P is direct summand, $h = h'|_P \in \mathrm{Hom}(P, A_1)$. Hence we can successfully lift g to h.

(2) $\implies$ (3) : Substitute A_2 with P in (2) and we get lifting of Id_P -$h \in \mathrm{Hom}(P, M)$ with $f_2 \circ h = \ \mathrm{Id}_P$.

(3) $\implies$ (1) : Consider the exact sequence $0 \to K \to R_n \to P \to 0$ which splits due to (3) which implies $R^n \simeq P \oplus K$. Hence the result. $\square$

Lemma 1.4.5. Let $R = \bigcup_{i \in I} R_i$ ($R_i \subseteq R_{i+1}$) and P be finitely generated projective R-module, then their exists a projective R_i-module Q_i such that $P = R \otimes Q_i$.

Proof. If I is finite, then choose $Q = P$ for some R-module Q. If not, then since P is projective such that $P \oplus Q \simeq R^n$. Hence we get the split exact sequence R, Q $0 \to P \to R^n \to Q \to 0$. Since P is finitely generated by $\langle p_1, \ldots, p_n \rangle$, and each $p_i = \langle b_{i1}, \ldots, b_{in} \rangle \in R^n$, we can choose an index $i \in I$ such that $p_i \in {R_i}^n$. Choosing $Q = P \otimes R_i$, we have our desired projective module. $\square$

Lemma 1.4.6. If $\psi : P \to Q$ is a R-module homomorphism such that $\overline{\psi} : \overline{P} \simeq \bar{Q}$ where bar denotes modulo by $J = \text{rad}(R)$, then ψ is an isomorphism.

Proof. One checks $(P/JP)/(Q/JQ) = (P/Q)/\ J(P/Q) = 0$ hence by Lemma 1.1.1 $P/Q = 0$ or $P \simeq Q$. $\square$

Lemma 1.4.7. Let $(R, \mathfrak{m})$ be a local ring and $P \in \mathcal{P}(R)$, then P is free.

Proof. $F = R/\mathfrak{m}$ is field, hence every module $M/\mathfrak{m}M$ is a vector space over F (and hence free). Therefore M is free using Lemma 1.4.6. $\square$

Definition 1.4.8. Let M be an R-module and $p \in \text{Spec}(R)$. Then M_p is a free R_p-module. By rank of M at p we mean the rank of free R_P-module M_p. We denote it as rk_pM. Then P is free R-modules and $\text{rk}(P) = n$.

Example 1.4.9. Let P be an R-module and. R is a local ring,
then $\text{rk}_p\text{P} = n$ (see [11] where n is the free rank of M).

Lemma 1.4.10. If R has no non-trivial idempotent then $P \in \mathcal{P}(R)$ has a constant rank. More specifically a commutative integral domain has constant rank.

Proof. If R has no idempotent, then $\text{Spec}(R)$ is connected (common fact in commutative algebra). Hence the continuous function rk_p is forced to be constant. A commutative ring has no non-trivial idempotent since $e^2 = e$implies $e(e-1) = 0$ implies $e = 0$ or $e = 1$. $\square$

Proposition 1.4.11. Finitely generated projective modules over PID are free.

Proof. Let P be a projective module over R. Hence by Proposition 1.4.4 there exists an R-module Q such that $P \oplus Q \simeq R^n$. for some $n \geq 0$. Since R is a PID, we have a structure for P and Q i.e.

$$P \simeq R^m \oplus T(P),\ Q \simeq R^t \oplus T(Q)$$

where $T(P)$ and $T(Q)$ represent the torsion part of P and Q respectively. If $T(P) \neq 0$ (or $T(Q) \neq 0$), then $P \oplus Q$ has a torsion element which is a contradiction since $P \oplus Q \simeq R^n$ and R^n has no torsion element. Therefore $T(P) = T(Q) = 0$. Hence P (and Q) are free R-modules. $\square$

1.5 Milnor patching property

Let A, A_1, A_2, A' be commutative rings and i_1, i_2, j_1, j_2 be ring homomorphism such that it satisfies the commutative diagram (1.1).

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"] & A_1 \arrow[d, "j_1"] \\ A_2 \arrow[r, "j_2"] & A' \end{tikzcd} \tag{1.1}$$

Given the ring homomorphsim $i : X \to Y$ the following diagram (1.2) be defined by

$$\mathcal{P}(X) \to \mathcal{P}(Y)$$

$$P \mapsto P \otimes_X Y$$

.

$$\begin{tikzcd} \mathcal{P}(A) \arrow[r] \arrow[d] & \mathcal{P}(A_1) \arrow[d] \\ \mathcal{P}(A_2) \arrow[r] & \mathcal{P}(A') \end{tikzcd} \tag{1.2}$$

Then we say that it satisfies Milnor property if

1. $P_k \in \mathcal{P}(A_k)$; $k = 1, 2$,
2. $h : P_1 \otimes_{A_1} A' \to P_2 \otimes_{A_2} A'$ an isomorphism of A' modules implies,

1. The pull back A-module P is projective i.e. $P \in \mathcal{P}(A)$.
2. $P \otimes_A A_k \simeq P_k$; $k = 1, 2$.

Example 1.5.1 (Type A)**.** Consider the following diagram 1.3 under the condition that j_2 (or j_1) is subjective, then (1.3) satisfies Milnor patching (see [12] for more information).

$$\begin{array}{ccc} A & \xrightarrow{i_1} & A_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ A_2 & \xrightarrow{j_2} & A' \end{array} \tag{1.3}$$

Example 1.5.2 (Type B)**.** Consider the following diagram (1.4), under the condition that $s, t \in A$; $As + At = 1$ and map the being the natural localization map. Then (1.4) satisfies Milnor patching property (see [11] for more informations).

$$\begin{array}{ccc} A & \longrightarrow & A_s \\ \downarrow & & \downarrow \\ A_t & \longrightarrow & A_{st} \end{array} \tag{1.4}$$

Proposition 1.5.3. Consider the following diagram (1.1) and assume it satisfies the Milnor patching property, let $P \in \mathcal{P}(A_1)$ such that $P \otimes A' \in \mathcal{P}^{A_2}(A')$, then $P \in \mathcal{P}^{A}(A_1)$.

Notation 1.5.4. *Here $\mathcal{P}^R S$ means projective modules over S which are extended from R.*

Proof. Since we know $P \otimes_{A_1} A' \in \mathcal{P}^{A_2}(A')$ implies there exists $Q \in \mathcal{P}(A_2)$ such that $P \otimes_{A_1} A' = Q \otimes_{A_2} A'$, Now applying Milnor patching property gives us the pullback $M \in \mathcal{P}(A)$ such that $M \otimes_A A_1 = P$. $\square$

Proposition 1.5.5. Let f be a monic polynomial in $R[X^{-1}]$ and let $P \in \mathcal{P}(R[X, X^{-1}])$ such that $P_f \in \mathcal{P}^R(R[X, X^{-1}]_f)$, then $P \in \mathcal{P}^{R[X]}(R[X, X^{-1}])$.

Proof. Let $g = X^{-n} f$. Then $R[X, X^{-1}]g = R[X, X^{-1}]_f$ Now consider the following digram (1.5):

$$\begin{array}{ccc} R[X] & \longrightarrow & R[X, X^{-1}] \\ \downarrow & & \downarrow \\ R[X]_g & \longrightarrow & R[X, X^{-1}]_f \end{array} \tag{1.5}$$

Since P_f is extended from R, we have $P_f \simeq Q \otimes_R R[X, X^{-1}]$, for some R-module Q. Hence

$$P_f \simeq Q \otimes_R R[X, X^{-1}] = (Q \otimes_R R[X] \otimes_{R[X]} R[X]_g) \otimes_{R[X]_g} (R[X, X^{-1}])$$

or

$$P_f \simeq Q' \otimes_{R[x]_g} R[X, X^{-1}$$

where $Q' = (Q \otimes_R R[X] \otimes_{R[X]} R[X]_g)$.

Now the diagram (1.5) is of the same type as the diagram (1.4) and hence satisfies Milnor patching property. Hence Proposition 1.5.5 follows immediately by applying Proposition 1.5.3. $\square$

Chapter 2

Graded rings and modules

This chapter gives introduction and basic properties of graded rings and modules, which are required because

1. Graded rings are can thought of as a natural generalisation of polynomial rings.
2. The graded L-G principle, which is the key step for the proof of Serre's problem.
3. In the main proof, we give our monoid algebra $R[M]$ a $\mathbb{Z}$-grading and then apply the graded version of L-G principal.

2.1 Graded Rings: Basic definition and examples

Definition 2.1.1. A ring A is Γ-graded if $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, where Γ is an abelian group and each A_γ is an additive subgroup of A satisfying $A_\gamma A_\delta \subseteq A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$.

The set $A^h = \{\bigcup A_\gamma \mid \gamma \in \Gamma\}$ is called the *homogeneous element* of the ring A. The set $\Gamma_A = \{\gamma \in \Gamma \mid A_\gamma \neq 0\}$ is called the *support* of A. If $a \in A_\gamma$, then $\deg(a) = \gamma$.

Example 2.1.2. Let $A = R[X]$ be a polynomial ring with $\mathbb{Z}$-grading i.e ($A = 1 \oplus \langle x \rangle \oplus \langle x^2 \rangle \oplus \cdots$), then $\mathbb{Z}_A = \mathbb{Z}_+ \cup \{0\}$.

Definition 2.1.3. Given Γ-graded ring A and B, a graded ring homomorphism $f : A \to B$

is a ring homomorphism such that $f(A_\gamma) \subset B_\gamma$.

Graded ring homomorphism preserves the degree i.e. index of the homogeneous component of A of element.

Example 2.1.4 (Trivial examples of graded and a non-graded homomorphism)**.**

Let R be a graded ring. Then $1_R : R \to R$ is a graded homomorphism and $0_R : R \to R$ is a non-graded homomorphism.

Now we derive some basic property of graded rings.

Proposition 2.1.5. Let $A = \bigoplus A_\gamma$ be a Γ-graded ring, then

1. 1_A is a homogeneous element of degree 0.

2. A_0 is a subring of A.

3. Each A_γ is an A_0-module.

4. For an invertible element $a \in A_\gamma$, its inverse $a^{-1} \in A_{-\gamma}$.

Proof. 1. We will show that $1_A \in A_0$. Let $1_A = \sum_{\gamma\in\Gamma} a_\gamma$ and b be any homogeneous element of degree δ. Then $1_A b = b = \sum ba_\gamma$. As $ba_\gamma \in A_{\gamma+\delta}$, due to direct sum property, b has unique representation as a sum, and hence $ba_\gamma = 0, \gamma \neq 0$. Since b is any arbitrary term, in general we get $1_A a_\gamma = 0$ which implies $a_\gamma = 0$ for $\gamma \neq 0$. Hence $1_A = a_0$. Therefore $\deg(1_A) = 0$ and hence the result.

2. A_0 is a subgroup from definition and from above, it follows that $1_A \in A_0$. So the only non-trivial part to check is closure under multiplication. But by definition $A_0 A_0 \subseteq A_{0+0} = A_0$. Hence it's a subring.

3. Let $b = \sum b_\delta$ be the inverse element of $a \in A_\gamma$. Then, as before $ab = 1$ implies $ab_\delta = 0$ for $\delta \neq -\gamma$. Since a is invertible, we have $b_\delta = 0$ for $\delta \neq -\gamma$. Hence $a^{-1} = b = b_{-\gamma}$.

$\square$

Example 2.1.6. For an additive abelian group Γ, $\mathbb{Z}[\Gamma]$ has a natural Γ-grading i.e. $\mathbb{Z}[\Gamma] = \bigoplus_{\gamma\in\Gamma} \mathbb{Z}[\Gamma]_\gamma$, where $\mathbb{Z}[\Gamma]_\gamma = \mathbb{Z}\gamma$.

2.2 Graded ideals

Definition 2.2.1. Let A be a Γ-graded ring. An ideal of A is called *graded ideal* if

$$I = \bigoplus_{\gamma \in \Gamma} (I \cap A_\gamma).$$

In other words I is graded ideal if and only if $x \in I, x = \sum x_i$, where x_i are homogeneous elements implies $x_i \in I$.

Definition 2.2.2. Let A be a Γ-graded ring and I be a graded ideal, then the *quotient ring* A/I is Γ-graded as follows:

$$A/I = \bigoplus_{\gamma \in \Gamma} (A/I)_\gamma \text{ , where } (A/I)_\gamma = (A_\gamma + I)/I = A_\gamma/(I \cap A_\gamma).$$

It follows that an ideal is graded if and only if it is generated by homogeneous element.

Example 2.2.3. In light of the above statement, ideal generated by X in the polynomial ring $R[X]$ is a $\mathbb{Z}$-graded ideal.

2.3 Graded prime and maximal ideals

Let P be a graded ideal of Γ-graded ring A. Then its called a *graded prime ideal* of A if $A \neq P$ and for any two graded ideals I and $J \in A$, $IJ \subset P$ implies either I or $J \subset P$. Under commutative setting, we have $x, y \in A^h$, $xy \in P$ implies either x or $y \in P$. In general graded prime ideal need not be a prime ideal.

Definition 2.3.1. A *graded maximal ideal* of a Γ-graded ring A, is the maximal graded ideal among the set of all graded ideals.

Example 2.3.2. Consider the natural $\mathbb{Z}$-grading on a polynomial ring $K[X]$, where K is a field. Then the ideal $\langle X \rangle$ is a graded maximal ideal.

Example 2.3.3. Consider the maximal ideal $\mathfrak{m}$ generated by $\langle X+1 \rangle$ on a polynomial ring $K[X]$, where K is a field. Then it is a maximal ideal but not graded, since neither X nor $1 \in \mathfrak{m}$ (because if $X \in \mathfrak{m}$, then $-X \in \mathfrak{m}$ and hence $1 \in \mathfrak{M}$) but $X + 1 \in \mathfrak{m}$.

2.4 Graded local rings

We recall that a ring A is local if and only if it has one maximal ideal if and only if set of all non-invertible elements forms an ideal. Motivated by this definition, a ring A is a *graded local ring* if the set of all non-invertible elements form a graded ideal. It follows from above that if such an ideal exists, then it should be unique.

Proposition 2.4.1. Let A be a Γ-graded ring. Then A is graded local if and only if A_0 is a local ring.

Proof. Let A be a graded local ring with maximal ideal $\mathfrak{m}$.

Let $\mathcal{M} = \mathfrak{m} \cap A_0$. Then $\mathcal{M}$ is proper ideal of A_0. Suppose $x \in A_0 - \mathcal{M}$, then x is a homogeneous element of degree 0 and is not in $\mathfrak{m}$, hence it is invertible in A. Since $x \in A_0$, its inverse is also in A_0 and hence x in invertible in A_0. As x is an arbitrary element of A_0 we man conclude that $\mathcal{M}$ is a unique maximal ideal of A_0 which make A_0 a local ring.

Let A_0 be a local ring.

Let $\mathcal{M}$ be the ideal generated by non-invertible homogeneous element of A. We will show that $\mathfrak{m}$ is a proper graded ideal and hence A is a graded local ring. Let us assume to the contrary that $\mathfrak{m} = A$. This implies $1 = \sum_i m_i a_i$, where a_i are non-invertible elements in A. Since 1 has degree 0, we have $\deg(m_i a_i) = 0$ and hence each $(m_i a_i) \in \mathcal{M}$ (because if $a_i m_i$ is invertible, then a_i would be invertible; a contradiction) and therefore $1 \in \mathcal{M}$ which is a contradiction, hence the result. $\square$

Example 2.4.2 (Example of a graded local which is not local)**.** Let R be a local ring. Then consider $A = R[X]$. It follows that A is a graded local from Proposition 2.4.1. Clearly A is not local because is has atleast two (infact many) non-identical maximal ideal such as $\mu = \langle x \rangle$ and $\nu = \langle x + 1 \rangle$.

After graded rings, now we discuss the grading on modules.

2.5 Graded modules: Basic definitions and examples

Definition 2.5.1. Let A be a Γ-graded ring. Then a *graded right* module M is an A-module such that $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ and $A_\gamma M_\delta \subseteq M_{\gamma+\delta}$.

Example 2.5.2. $K[X]$ is a $\mathbb{Z}$-graded $K[X]$-module.

If M and N are two γ-graded A-module, then the homomorphism $f : M \to N$ is a *graded homomorphism* if $f(M_\gamma) \subset N_\gamma$ for all $\gamma \in \Gamma$. Similarly we define graded isomorphism.

Example 2.5.3. The 1_M and 0_M is an example of graded and non-graded homomorphism for a graded module M.

Definition 2.5.4. A submodule N of a graded module M is called *graded submodule* if

$$N = \bigoplus_{\gamma\in\Gamma}(N \cap M_\gamma).$$

Remark 2.5.5. *If a Γ-graded ring A is considered a module over itself, then the notion of graded ideal coincide with that of graded submodule.*

Example 2.5.6. Let A be a Γ-graded ring. If $a \in A$ is a homogeneous element of degree α, then the ideal aA is a graded submodule (and ideal) with

$$(aA)_\gamma := aA_{\gamma-\alpha}.$$

It is graded, i.e $aA = \bigoplus(aA \cap A_\gamma)$. Indeed, if $x \in aA$ implies

$$x = a(\cdots + a_{-\gamma} + \cdots + a_0 + \cdots a_\gamma + \cdots)$$

for some a_γ in A_Γ, which in turn implies $aa_{-\gamma} \in A_{-\gamma+\alpha} \cap aA$.

If M is a Γ-graded module and N is its submodule, then we define *graded quotient module* as

$$M/N = \bigoplus_{\gamma\in\Gamma}(M/N)_\gamma \text{ , where } (M/N)_\gamma = (M_\gamma + N)/N.$$

Definition 2.5.7. Let M be Γ-graded right A-module and N be Γ-graded left A-module.

We now defined *Tensor grading* on $M \otimes_R N$ from Lemma 1.2.4 as follows:

$$M \otimes_A N = (\bigoplus_{\gamma \in \Gamma} M_\gamma) \otimes_A (\bigoplus_{\gamma \in \Gamma} N_\gamma) = \bigoplus_{\gamma + \delta \in \Gamma} M_\gamma \otimes_A N_\delta.$$

2.6 Graded free modules

Just as we have the notion of free modules, we have a corresponding notion for graded case and it is called *graded free modules*. In general free module with grading is not same as a graded free module.

Definition 2.6.1 (Graded free modules)**.** Let A be a Γ-graded ring. Then a graded A-module F is *graded free* if it is a free A-module and its basis elements are homogeneous elements of F.

Example 2.6.2. If $A = R[X]$ with $\mathbb{Z}$-grading, then $A \times A$ is a graded free A-module with basis elements $(1, 0)$ and $(0, 1)$.

Example 2.6.3. Consider the matrix ring $M_n(A)$ as $\mathbb{Z}$-graded which is concentrated at degree 0 and consider $M_n(A)$ as module over itself with grading $(M_n(A))_i = e_i M_n(A)$. It follows that all non-zero homogeneous elements are 0 divisors and hence can not constitute a linear independent set. Therefore, the module is not graded-free inspite of it being graded and free.

2.7 Graded projective modules

Just as we have a notion of projective modules, we have a corresponding notion for graded case called *graded free*. Unlike graded free case, here projective modules with grading would imply that the module is graded-projective as we will see later in this section.

Definition 2.7.1. P is called a *graded projective module* if the following

$$\begin{tikzcd} M \arrow[r, "g"] & N \arrow[r] & 0 \\ & P \arrow[u, "j"] \arrow[ul, dashed, "h"] & \end{tikzcd}$$

diagram of graded A-modules and graded A-homomorphism, there is graded A-homomorphism $h : P \to M$ such that $g \circ h = j$.

Lemma 2.7.2. Let the following

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[r, bend left, dashed, "h'"] \arrow[dr, "f"] & M \arrow[d, "g"] \\ & N \end{tikzcd}$$

be a commutative diagram of graded modules M, N, and P, such that $f = gh$, where f is a graded A-homomorphism. If g is a graded homomorphism, then their exists a graded homomorphism $h' : M \to N$ such that $f = gh'$.

Proof. Let $g : M \to N$ be a graded homomorphism. If $p \in P_\alpha$, define $h'(p) = h(p) \cap M_\alpha$ and extend this map linearly to all $p \in P$ as follows:

$$h'(p) = \Sigma_{\alpha \in \Gamma} h(p_\alpha)_\alpha.$$

It follows immediately that h' is a graded A- homomorphism. To check for $gh' = j$, observe that for $p \in P_\alpha$, we have

$$f(p) = g\ \Sigma_{\alpha \in \Gamma}\ h(p)_\alpha = \Sigma_{\alpha \in \Gamma}\ gh(p)_\alpha.$$

Since g and f are graded homomorphism, so it preserves the degree i.e the above relation become

$$f(p) = gh(p)_\alpha = gh'(p).$$

Using the linearity of f, g and h' it follows $f = gh'$. $\square$

Proposition 2.7.3. Let A be a Γ-graded ring and P be a graded A-module, then the followings are equivalent:

1. P is a graded and projective module.

2. P is a graded projective module.

3. $0 \to A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to 0$ implies

$$0 \to \mathrm{Hom}_R(P, A_0) \xrightarrow{f_0'} \mathrm{Hom}_R(P, A_1) \xrightarrow{f_1'} \mathrm{Hom}_R(P, A_2) \to 0$$

where f_0 and f_1 are graded homomorphism.

4. Every short exact sequence

$$0 \to L \to M \xrightarrow{g} P \to 0$$

of graded A homomorphism splits (at g) via a graded map.

5. P is graded isomorphic to direct summand of graded free module.

Proof.

1. (1) $\implies$ (2): Consider the following commutative diagram

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[dr, "f"'] & M \arrow[d, "g"] \\ & N \end{tikzcd}$$

where M and N are graded A-modules and f and g are graded homomorphisms. Since P is a projective module, we have a map h such that $f = gh$. Using Lemma 2.7.2, we have a graded homomorphism h' such that $f = gh'$. Hence P is a graded projective module.

2. (2) $\implies$ (3): Since $\text{Hom}(P, -)$ is always left exact by Lemma 1.1.5, the only non-trivial part remains to be shown is the right exactness. Given $g \in \text{Hom}(P, A_2)$, from the definition of graded projective modules, their exists $h \in \text{Hom}(P, A_1)$. Hence the sequence is right exact.

3. (3) $\implies$ (4): If $h \in \text{Hom}(P, M)$ then $gh \in \text{Hom}(P, P)$. Since $\text{Hom}(P, -)$ is exact, for $1_P \in \text{Hom}(P, P)$ their exists $h' \in \text{Hom}(P, M)$ such that $gh' = 1_P$. Hence g splits.

4. (4) $\implies$ (5): Let $\{p_1, \ldots, p_n\}$ be the homogeneous generators of P (we can choose such a generators by breaking generators of P in homogeneous parts). Let $\deg(p_i) = \delta_i$. Consider the free graded F-module generated by $\{e_1, \ldots, e_n\}$, where $\deg(e_i) = \delta_i$. Then there is a surjective graded-homomorphism $\phi : F \to P$ with $e_i \mapsto p_i$. Since ϕ splits from, we have P as a direct summand of F.

5. (5) $\implies$ (1): Since P is direct summand of a graded free module F and since F is free, we conclude that P is a projective module. From definition P is graded.

□

Example 2.7.4. Due to Proposition 2.7.3, to find examples of graded-free, we have to find a projective module with grading. Consider the ring $A = R[X]$. Then module N (i.e the ideal generated by $\langle X \rangle$) is a graded-projective module.

Chapter 3

Classical result on Serre's conjecture

This chapter deals with classical results from Serre's conjecture. This section assumes importance because for rank 1 normal monoid M, $R[M] \simeq R[X]$ and hence serves as a suitable base for our induction hypothesis for tackling the general problem.

We will first discuss the case of rank 1 projective modules over a polynomial $K[X_1, \ldots, X_n]$ over a field K.

3.1 Rank 1 projective modules

Lemma 3.1.1. Let R be an integral domain, with quotient field K . Let $P \neq 0$ be a R-submodule of K. Then P is projective if and only if there exists R-submodule $Q \subseteq K$, such that $PQ = R$. This would also imply that P is finitely generated.

Proof. Let P be a projective R-module.

Let $F = \bigoplus_{i \in I} Re_i$ be a free R-module for some index set I. Since P is a projective module, we have

$$F \underset{\text{g}}{\overset{\text{f}}{\rightleftarrows}} P$$

such that $f \circ g = 1$.

Now we look at the nature of the map g. Since $g : P \to F$, g must be of the form $g(p) = \Sigma_{i\in I}\ g_i(p)e_i$, for each $g_i = \pi_i \circ g \in \mathrm{Hom}(P, R)$ where π_i is the natural projection. Since both P and R are R-submodule of K, we extend the $g_i : K \to K$ as follows:

$$K \otimes_R P \xrightarrow{1\otimes g_i} K \otimes_R R.$$

Since only endomorphism between K's is multiplication by some element, we now have $g_i(p) = b_i p$ and $g = \sum_{i\in I} b_i p$. Since only finitely many terms are non-zero in the summation, hence we can assume our index set I to be finite and hence without loss of generality let us assume $\{b_1, \dots, b_n\}$ are non-zero.

Let $Q = \Sigma R b_i$, then we have $QP \subseteq R$. So the only part remained to be proved is $QP \supseteq R$. For that observe that $f(e_i) = p_i \in P$. Hence P is generated by $\{p_1, p_2, \dots, p_n\}$. Using the property of split-homomorphism, we have

$$f \circ g(p) = p = f(\Sigma_{i\in I}(b_i p)e_i) = p\Sigma_{i\in I} b_i p_i.$$

That is

$$p = p\Sigma_{i\in I} a_i p_i \implies \Sigma_{i\in I} b_i p_i = 1.$$

Hence $1 \in PQ$ and therefore $PQ = R$.

Conversely let $PQ = R$, to show P is R-projective.

Since $\sum_{i\in I} b_i a_i = 1$, let's define $f : F = \oplus_{i\in I} R e_i \to P$ by $e_i \mapsto a_i$ and similarly define $g : P \to F$ by $p \mapsto \Sigma_{i\in I}(b_i p)e_i$. Now we have split-homomorphism between F and P

$$F \underset{\mathrm{g}}{\overset{\mathrm{f}}{\rightleftarrows}} P.$$

Hence using Proposition1.4.4, we prove that P is projective. □

Proposition 3.1.2. Let R be a UFD, and P is a projective R-module. Then $\mathrm{rk}(P) = 1$ implies $P \simeq R$.

Proof. Assume P to be an ideal of R generated by $\{p_1, p_2, \dots, p_n\}$. Then from Lemma 3.1.1, we have $PQ = R$ and $\sum_1^n b_i p_i = 1$. Since Q in general is not a subset of R, we cannot directly conclude that $P = R$. Let $b_i = c_i/d_i$. Since $b_i p_j \in R$, we have $c_i/d_i \in R$ and hence (due to UFD) we have d_i divides p_j for every possible pair (i,j). Hence $\mathrm{lcm}(d_i) = d$ divides p_j for every j. Therefore

$$\sum_1^n b_i p_i = \sum_1^n (c_i/d_i)p_i = \sum_1^n (\tilde{c}_i/d)p_i = \sum_1^n \tilde{c}_i\tilde{p}_i = 1.$$

Hence $1 \in P$, and therefore $P = R$. □

Corollary 3.1.3. Let A be a UFD, $R = A[X_1, X_2, \dots, X_n]$, and P be a rank 1 projective R-module, then $P \simeq R$.

Proof. By Gauss's lemma on UFD, R is a UFD and hence by Proposition 3.1.2, $P \simeq R$. □

3.2 Serre's problem for one variable

We now consider the case of projective modules over $K[X]$, where K is a field.

Definition 3.2.1. A ring R is called *hereditary* if every ideal of R is projective.

Proposition 3.2.2. [Kaplansky] If R is a hereditary ring, then any submodule A of any free module $F = \oplus Re_i$ is isomorphic to direct sum of R-ideals. In particular A is projective.

Proof. Let F_i denote the submodule of F with basis e_j, and $j \leq i$. Let A be a submodule of F and $A_i = A \cap F_i$. Then $A_{i+1}/A_i \subseteq R$ is a projective R-module. One checks $A_{i+1} = A_i \oplus I_i$. Now we will inductively show that $A = \oplus I_i$.

Suppose not, then there exists a least i such that $a \in A_{i+1} - \sum I_i$, $a \neq 0$. Since $a \in A_{i+1}$ we have $a = \tilde{a} + \bar{a}$, $\tilde{a} \in A_i$ and $\bar{a} \in I_i$. Because we have choose such least i, it follows that $\tilde{a} \in \sum \oplus I_i$, and hence $a \in \oplus I_i$, a contradiction. Therefore it follows $A = \oplus I_i$. □

Corollary 3.2.3. If R is a ring whose ideal are free (like PID), then submodules of free R-modules are free. In particular, all projective modules are free.

Proof. From definition, R is a hereditary ring. Hence by Proposition 3.2.2 all projective modules is a direct sum of ideal which by the above hypothesis is free and hence every projective module is free. □

Remark 3.2.4. *Projective modules over PID's are free.*

3.3 Serre's problem for graded case

We now consider the case where the projective module has a $\mathbb{Z}$-graded structure.

Definition 3.3.1. We say a graded R-module M is *bounded from below* if there exists $r \in \mathbb{Z}$ such that $M_j = 0, j < r$.

Proposition 3.3.2. Any finitely generated graded R-module M is bounded from below.

Proof. Suppose $m_1, \ldots, m_r$ are generators for M. Choose r small enough such that homogeneous component of m_i have degree $\geq r$. Then $M = \sum Rm_i$ and hence the result. □

Proposition 3.3.3. Let R be a graded ring and M be a graded module bounded from below, Then $\overline{M} = M/R^+M = 0$ implies $M = 0$.

Notation 3.3.4. *Let R^+ denote the ideal $R_1 \oplus R_2 \oplus \cdots$.*

Proof. Since M is bounded from below, we have $M = M_r + M_{r+1} + \cdots$, hence $R^+M = M_{r+1} + \cdots$. If $M = R^+M$, then $M_r = 0$ and repeating the same argument, we get $M = 0$.

□

Proposition 3.3.5. Let P, Q be finitely generated R-modules over a graded ring R, with P being a projective R-module. Let $\gamma : Q \to P$ be a graded-ring homomorphism. Then γ is an isomorphism if and only if $\overline{\gamma} : \overline{P} \to \overline{Q}$ is an isomorphism.

Notation 3.3.6. *For an R-module P, by $\overline{P}$ we mean P/R^+P.*

Proof. Assume $\overline{\gamma}$ if isomorphism.

Let $K = \ker(\gamma)$ and $C = \operatorname{coker}(\gamma)$. Then C and K both are graded modules. Since modulo by R^+ is same as $\otimes R/R^+$, and since tensor product is right-exact, we have $\overline{C} = 0$. Also C is finitely generated, and hence by Lemma 1.1.1, $C = 0$ and therefore γ is onto. Since P is projective, $Q = K \oplus P$, where K is kernel of γ and therefore $\overline{Q} = \overline{K} + \overline{P} = \overline{P}$ and hence $\overline{K} = 0$ and since K is finitely generated, again using Lemma 1.1.1, we conclude that $K = 0$. Hence $P \simeq Q$.

Conversely assume γ is isomorphism.

$P \simeq Q$ and hence $P \otimes R/R^+ = Q \otimes R/R^+$ or $\overline{P} = \overline{Q}$. Hence the result follows. $\square$

Theorem 3.3.7. Let $R = R_0 + R_1 + \cdots$ be a graded ring. Let P be a projective R-module. Then P is extended form R_0 or more precisely $R \otimes_{R_0} \overline{P} = P$ where $\overline{P}$ is a graded R_0-module.

Proof. Let $f : P \to \overline{P}$ be a projection map (also a graded R_0-module homomorphism). Since P is projective R-module, $\overline{P}$ is also a projective R_0-module. Hence this homomorphism splits. Let $g : \overline{P} \to P$ be the required split map. Since f is a graded homomorphsim, g can be made into a graded homomorphism. Hence g induces a graded R-module homomorphism $\gamma : Q = R \otimes_{R_0} \overline{P} \xrightarrow{1 \otimes g} P$ given by $(r, \overline{u}) \mapsto ru$. Taking quotient with R^+, we have

$$\overline{\gamma} : R \otimes_{R_0} \overline{P} \otimes R/R^+ \simeq P \otimes_{R_0} R/R^+ \to P \otimes_{R_0} R/R^+ = \overline{P}.$$

Hence $\overline{\gamma}$ as an isomorphism. Hence by Proposition 3.3.5, we have γ as isomorphism. Hence the result. $\square$

Corollary 3.3.8. Let $R = R_0[X_1, \ldots, X_n]$ be the ring with the natural grading. Let P be a graded R-module. Then P is extended from R_0. If R_0 is a PID, then P is free.

3.4 Quillen-Suslin theorem: Outline of the proof

We begin with the following theorem due Horrocks. For more detail refer to [13]

Proposition 3.4.1 (Horrocks theorem)**.** Let $(R, \mathfrak{m})$ be a local ring and let P be a projective $R[X]$-module. If P_f is a free $R[X]_f$-module for some monic polynomial f, then P is free.

One can deduce Proposition 3.4.1 from the following two results. For more details cf [13].

Proposition 3.4.2 (Murthy-Pedrini)**.** Let P and Q be a projective module over $R[X]$ and let P_f be isomorphic to Q_f as a $R[X]_f$-module for some monic element f. Then P and Q are stably isomorphic over $R[X]$.

Proposition 3.4.3. Let $R, \mathfrak{m}$ be a local ring and let P, Q be a projective $R[X]$-modules such that $\mathrm{rk}(Q) < \mathrm{rk}(P)$. Suppose that Q_f is a direct summand of P_f as an $R[X]_f$-module for some monic polynomial $f \in R[X]$. Then Q is a direct summand of P.

Let us now derive Proposition 3.4.1 from Proposition 3.4.2 and Proposition 3.4.3.

Case $n = 1$.

If $P_f \simeq {A_f}^n$, where n is the $\mathrm{rk}(P) = n$ and $A = R[X]$. By Proposition 3.4.2, P is stably isomorphic to A_n. From Proposition 3.1.2, we prove that P is free.

Hence let us assume $n > 1$.

Since ${A_f}^{n-1}$ is isomorphic to direct summand of P_f, from Proposition 3.4.3, A^{n-1} is direct summand of P. Hence $P \simeq A^{n-1} \oplus L$ for some rank 1 projective module L. Hence $L_f \oplus {A_f}^{n-1} \simeq P_f \simeq {A^n}_f$. Since $\mathrm{rk}(L) = 1$, we have $L_f = A$. Hence from $n = 1$ case, we have L as a free A-module. Hence P is free.

Having obtained Proposition 3.4.1, we now use Quillen's localisation to extend Proposition 3.4.1 from local rings to more general rings. We assume that R is any commutative ring with identity.

Proposition 3.4.4 (Quillen's Localisation Theorem)**.** If P is a projective $R[X]$-module such that the $R_{\mathfrak{m}}[X]$-module $P_{\mathfrak{m}}$ is extended from $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R, then P is extended from R.

Now we use the next proposition to arrive at Quillen-Suslin Theorem.

Proposition 3.4.5. Let P be a finitely generated projective module over $R[X]$. If the $R[X]_f$-module P_f is extended from a projective module over R for some monic f, then P is extended from R.

Using the fact that all projective modules over fields are free, we arrive at Quillen-Suslin Theorem.

Proposition 3.4.6 (Quillen -Suslin theorem)**.** Let P be a finitely generated projective module over $R[X]$ and let P_f be $R[X]_f$-free for some monic $f \in R[X]$. Then P is free.

Serre's problem follows from the following proposition:

We use induction on n. Let K be a field and P a finitely generated projective module over $K[X_1, \dots, X_n]$. Consider the field $K(X_1, \dots, X_n)$, then the extended module $P \otimes_{K[X_1,\dots,X_m]} K(X_1, \dots, X_n)$ is a vector over the quotient field and hence free. Since P is finitely generated,

P_f is free over $K[X_1, \dots, X_n]_f$ for some monic polynomial $f(X_1, \dots, X_n)$ (without loss of generality we can assume f to be monic in X_n with coefficient in $K[X_1, \dots, X_{n-1}]$ and use the induction hypothesis). Hence it follows from Proposition 3.4.6 that P is free.

Definition 3.4.7 (Stably Isomorphic)**.** Two R-modules M and N are said to be *stably isomorphic* if $M \oplus R^m \simeq N \oplus R^n$ for some $m, n > 0$.

Horrocks theorem has an alternate formulation due to P. Roberts.

Proposition 3.4.8. Let $(R, \mathfrak{m})$ be a local ring, and A be an R-algebra. let S be multiplicative set of central non zero-divisors in A, and $n \geq 0$ be a fixed integer. Assume that following hypothesis holds:

1. For any $f \in S$, A/fA is finitely generated R-module.

2. $GL_n(\overline{S^{-1}}) = GL_n(S^{-1}A)(GL_n(\overline{A})$.

3. $S^{-1}A$ contains an R-subalgebra B such that $S^{-1}A = A + B$ and $\mathfrak{m}B \subset \mathrm{rad}(B)$.

4. Let P be a finitely generated A-module such that f is regular on P and $\overline{P} \simeq \overline{A^n}$ and $S^{-1}P \simeq (S^{-1}A)^n$.

Then $P \simeq A^n$.

Proof. cf: ([11], 4.1). □

3.5 Quillen's graded local-global principal

A more general version of Proposition 3.4.4 is needed for proving the Anderson's conjecture.

Proposition 3.5.1 (Graded Quillen's Patching)**.** Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be a graded ring and P a finitely generated R-module. Then P is extended from R_0 if and only if $P_\mathfrak{m}$ is extended from $(R_0)_\mathfrak{m}$ for every maximal ideal $\mathfrak{m} \subsetneq R_0$.

Notation 3.5.2. *If $f : A \to B$ is a ring homomorphism, then by $f_*(P)$ we mean the extended module $P \otimes B$, where the module structure is induced by f.*

Proof. Consider the following maps:

1. $i : R_0 \to R$ be the inclusion map (we shall use the same symbol for the map $(R_0)_{\mathfrak{m}} \to R_{\mathfrak{m}}$).

2. $j : R \to R[X]$ be the inclusion map (we shall use the same symbol for the map $R_\nu \to R_\nu[X]$).

3. $w : R \to R[X]$ given by $r_0 + r_1 + \cdots \mapsto r_0 + r_1 X + r_2 X^2 + \cdots$.

4. $\epsilon : R \to R_0$ sending $R_n \mapsto 0$ for $n > 0$.

5. $e_k : R[X] \to R$ for $k = 0, 1$ by sending X to k.

From above we get the following equality:

$$e_0 j = e_1 k = e_1 w = 1_R \text{ and } e_0 w = i\epsilon.$$

Consider a projective R-module P. Let $W = w_*(P)$ i.e. $W = P \otimes_R R[T]$ via w. Let $\mathfrak{m}$ be a maximal ideal in R_0. Hence $P_{\mathfrak{m}}$ is $(R_0)_{\mathfrak{m}}$ extended and hence W_m is $(R_0)_{\mathfrak{m}}$ extended (W is also $R_{\mathfrak{m}}$ extended). Using Lemma 1.3.13, we have $s \in R_0 - \mathfrak{m}$ such that W_s (and P_s) is $(R_0)_s$ extended. Let $\mathfrak{n}$ be a maximal ideal of R such that $\mathfrak{n} \cap R_0 \subset \mathfrak{m}$. Since $\mathfrak{n} \cap R_0 \subset \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R_0 choose $s \in R_0 - \mathfrak{m}$ and the using the fact that W_s is $(R_0)_s$ extended implies $W_{\mathfrak{n}}$ is $R_{\mathfrak{n}}$ extended (note that $s \in R - \mathfrak{n}$). Hence by the usual Quillen's L-G principal, we have W extended from R-module Q.

Now we have

$$P \simeq e_{1*} j_*(Q),$$

since ${e_1}^{-1} = w$ and hence $w_*(P) = W = j_*(Q)$. Therefore,

$$e_{1*} j_*(Q) = e_{0*} j_*(Q) \simeq e_{0*}(W) \simeq e_{0*} w_*(P) \simeq i_* \epsilon_*(P)$$

implying P is extended from R_0.

The other implication is obvious. $\square$

Another proof: (for definition see [3])

Proof. Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ and $v = (a_0, a_1, \ldots, a_m)$ be a unimodular ring in R and

$v_{\mathfrak{m}}$ be locally extended from $(R_0)_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$ of R_0. Hence we can find a $s \in R_0 - \mathfrak{m}$ such that v_s is locally extended from $(R_0)_s$ by Lemma 1.3.13. Consider the ring homomorphism

$$F : R \to R[T] \text{ given by } a = r_0 + r_1 + \cdots \mapsto r_0 + r_1 T + r_2 T^2 + \cdots .$$

Let $F(a_i) = f_i(T)$ and since F is a ring homomorphism, $F(v) = (f_1(T), f_2(T), \ldots, f_m(T))$ is also unimodular. Let $\mathfrak{n}$ be a maximal ideal of R and consider the localisation of $F(v)$ in $R_{\mathfrak{n}}[T]$. Choose $\mathfrak{m} \supset \mathfrak{n} \cap R_0$ and then select s as done before. Since $s \in R_0 - \mathfrak{m}$ we have $s \in R - \mathfrak{n}$. Therefore $F(v)_s$ is R_s extended. Since $F(v)_{\mathfrak{n}}$ is the further localisation of $F(v)_s$, we have

$$F(v)_{\mathfrak{n}} = (f_1(T), f_2(T), \ldots, f_m(T))$$

is extended from $R_{\mathfrak{n}}$. Hence by classical Quillen's L-G principal, we have $F(v)$ is extended from R. This implies $(f_1(0), f_2(0), \ldots, f_m(0))$ and $(f_1(1), f_2(1), \ldots, f_m(1))$ (we substitute $T = 0, 1$ respectively in v) are in the same $GL_m(R)$ orbits. Hence

$$(b_1, \ldots, b_m) \sim_{GL_m(R)} (a_1, \ldots, a_m),$$

where $b_i \in R_0$. Hence v is extended from R_0. $\square$

Chapter 4

Monoids and monoid algebras

In this chapter, we discuss about few definition, and general properties of monoids and monoid algebras.

4.1 Monoids: Definition and examples

A monoid is an algebraic structure which generalises the notion of a group.

Definition 4.1.1 (Monoid)**.** A set $(M, +)$ with a binary operation $M \times M \to M$ which is associative and has an identity element e is called a *monoid*.

Notation 4.1.2. *Throughout this chapter, we would either denote the monoid operation as* $+$ *with identity* 0 *or* $.$ *with identity* 1.

A monoid is said to be commutative if $a + b = b + a$. A subset $N \subseteq M$ is a *submonoid* if it is closed under addition and every element has an inverse. If $m \in M$ has an inverse in M, then m is called a *unit*.

Remark 4.1.3. *A monoid M is said to be an affine monoid if M is finitely generated and a submonoid of $\mathbb{Z}^d$ for some positive integer d. From here onward, unless otherwise mentioned, we would assume our monoid to be affine.*

Example 4.1.4. : Few examples of commutative affine monoids:

1. Any group G is a monoid.

2. $\mathbb{Z}^+$ is a monoid under usual multiplication.

3. Let $R[X_1, \ldots, X_n]$ be a polynomial ring in n variables, then the set

$$X := \{{X_i}^j \mid 1 \leq i \leq n,\ j \geq 0\}$$

forms a monoid under multiplication.

In groups, we have maps (called homomorphism) to compare two groups. Similarly we borrow the concept of homomorphism for monoids as follows:

Definition 4.1.5 (Monoid Homomorphism)**.** Let $\psi : M \to N$ be a map of monoids such that $\psi(a+b) = \psi(a) + \psi(b)$ for all $a, b \in M$, then ψ is said to be a *monoid homomorphism.*

Remark 4.1.6. *Monoid homomorphism between groups turns out to be the group homomorphism and vice-versa.*

Consider the monoid $\mathbb{Z}^+$. Intuitively it is clear that we can complete this monoid to a group $\mathbb{Q}^+$ by adding inverse of non-zero element. We now formalise this argument for arbitrary commutative monoid as follows:

Lemma 4.1.7 (Completion of a monoid)**.** Let M be a monoid, then we can associate a group $\mathrm{gp}(M) = G$ and a monoid homomorphism $\hat{f} : M \to G$ such that the following diagram of monoid homomorphism commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "\hat{f}"] & H \\ & G \arrow[u, dashed, "\pi"] \end{tikzcd}$$

i.e., given any group H, if $f \in \mathrm{Hom}(M,\ H)$ (monoid homomorphism), then there exists a unique $\pi \in \mathrm{Hom}(G,\ H)$ (group homomorphism) such that $\hat{f} \circ \pi = f$.

Outline of proof. *For a monoid M, we construct a new monoid $M \times M$ such that M is identified as $(M, 0)$ and M^{-1} is identified as $(0, M)$. To cover the overlap of elements we go modulo the overlapping elements and finally check $(M \times M)/\sim$ is a group.*

Proof. Let $(x, y) := x - y$ denote an equivalence relation such that $(x, y) \sim (a, b)$ if and only if $x + b = y + a$. This defines an equivalence class on $M \times M$. We define addition on this equivalence class as follows: $(x, y) + (a, b) = (x + a, y + b)$.

We claim that this equivalence class (called G) has a group structure:

It is associative and closed under addition by definition. The inverse of (x, y) is (x, y) for all $(x, y) \in (\frac{M \times M}{\sim})$ (as $(x, y) + (x, y) = (x + y, x + y)$ or $(x + y, x + y) := x + y - (x + y)$ and the identity element is $(z, z) = (0, 0)$ because $(x, y) + (z, z) = (x + z, y + z)$ or $(x + z, y + z) := x + z - (y + z) = x - y := (x, y)$.

We now claim that the map $m \mapsto (m, 0)$ is the monoid homomorphism $\psi : M \to \mathrm{gp}(M)$ and satisfies the universal property and the result follows immediately.

$\psi(m_1 + m_2) = (m_1 + m_2, 0) := m_1 + m_2 - 0$ and $\psi(m_1, 0) + \psi(m_2, 0) := m_1 - 0 + m_2 - 0$. Hence $\psi(m_1 + m_2) = \psi(m_1) + \psi(m_2)$. Now if $f : M \to H$ is monoid-homomorphism such that $m \mapsto h$, then $\pi : G \to H$ is defined as $\pi(m, n) = f(m)f(n)^{-1}$.

$\square$

Definition 4.1.8 (Rank of a monoid)**.** The *rank* of monoid M is defined to be the rank of $M \otimes_{\mathbb{Z}} \mathbb{Q}$ in $\mathbb{Q}$.

Notation 4.1.9. *From here onwards, we will denote rank M as rk(M).*

Example 4.1.10. If $M = \mathbb{Z}^2$, then $\mathbb{Q} \otimes \mathbb{Z}^2 = \mathbb{Q}^2$ and hence $\mathrm{rk}(M) = 2$.

4.2 Normal and seminormal monoids

In this section we recall few standard properties and examples of affine monoids.

Definition 4.2.1. Let M be an affine monoid of $N = \mathrm{gp}(M)$. We define *normalisation* of M in N (in additive notion) to be

$$\widetilde{M_N} := \{x \in N \mid nx \in M \text{ for some } n \in \mathbb{N}\}.$$

Example 4.2.2. Normalisation of $\mathbb{Z}$ over $\mathbb{Q}$ is $\mathbb{Q}$ itself as we can get rid of denominators, by choosing a suitable n to be the value of denominator.

Definition 4.2.3. A monoid M is called *normal* if $\widetilde{M_N} = M$.

We now give an example of a normal monoid.

Example 4.2.4. Consider the monoid $M = \{x\}$ and $G = \{x,\, x^{-1}\}$ under multiplication. Then if $x^{-1} \in \widetilde{M}$ implies $x^{-k} \in M$ for some k. But that is not possible. Hence $x^{-1} \notin \widetilde{M}$. In general every free monoid with no non-trivial units is normal.

Infact, normalisation of M can also be thought of as the intersection of all normal monoids containing M.

Normalisation has a nice geometric picture for affine monoids as the next proposition shows (whose proof can be found in ([4], 2.22):

Proposition 4.2.5. Given M and $N = \mathrm{gp}(M)$, let $C = \mathbb{R}_+M$. Then $\widetilde{M}_N = C \cap M$.

Following is an important consequence of Proposition 4.2.5 which we will use in the later chapter.

Corollary 4.2.6. Let M be integrally closed in N. If $\mathrm{rk}(M) = \mathrm{rk}(N)$, then $\mathrm{gp}(M) = \mathrm{gp}(N)$. In particular, $\mathrm{gp}(N^\star) = \mathrm{gp}(N)$. (see definition 5.1.5).

In a similar manner we define *seminormalisation* of M over $\mathrm{gp}(M)$ as

$$\mathrm{sn}(M) = \{x \in \mathrm{gp}(M) \mid x^2, x^3 \in M\}.$$

Definition 4.2.7. A monoid M is called *seminormal* if $\mathrm{sn}(M) = M$.

Definition 4.2.8. Let M be an affine monoid and $G = \mathrm{gp}(M)$, then

$$\mathrm{sn}\ (M) := \{x \in N \mid x^2, x^3 \in M\}.$$

Example 4.2.9. A free monoid with no non-trivial units is seminormal.

Just like normal monoids, it can be inferred that $\mathrm{sn}(M)$ is the intersection of all seminormal submonoids which contain M.

From definition it follows that every normal monoid is seminormal. However, the converse is not true. There exists seminormal monoid which are not normal. For more details cf. ([4] 2.56a).

Inspite of the previous example, we can still obtain a correspondence between normal and seminormal monoids. Let $M^\star = \mathrm{Int}(M) \cup \{0\}$.

Proposition 4.2.10. An affine monoid M is seminormal if and only if $(M \cap F)_\star$ is a normal monoid for every face F of $\mathbb{R}_+ M$. Hence $\widetilde{M} = \widetilde{M^\star}$.

Proof. If M is seminormal, then $M \cap F$ is seminormal for each face F of $\mathbb{R}_+ M$. Hence it is enough to show that $M^\star$ is normal. By corollary 4.2.6, $\mathrm{gp}(M^\star) = \mathrm{gp}(M)$. If $x \in \mathrm{gp}(M^\star)$, $x \neq 1$ and $w = x^n \in M^\star$, then $w \in \mathrm{Int}(M)$. Rewriting $x = yz^{-1}$ for $y, z \in M^\star$. Since $w \in \mathrm{Int}(M)$, we have $w^m z^{-r} \in M$ for some $m > 0$ and $r = 0, 1, \ldots, n-1$. Then $w^m x^r \in M$ for same range of r. Let $s \geq 0, s = nq + r$, then $x^{nm+s} = w^m x^s = w^q w^m x^r \in M$ and hence $x^t \in M$ for a large t. Since $x \in M$ it is obvious that $x \in \mathrm{Int}(M)$ since $x^n \in \mathrm{Int}(M)$. □

4.3 Monoid algebras

In this section, we discuss about monoid algebras and some its properties. As we know monoid algebras is the ring under consideration in Anderson's conjecture.

Definition 4.3.1. Let R be a commutative ring and M be a monoid, and consider the set

$$R[M] := \{\Sigma_{i \in I}\, r_i m_i \mid r_i \in R, m_i \in M\}.$$

If we define addition and multiplication in a natural way, we see that $R[M]$ forms ring and since $R \subset R[M]$, it is a R-algebra which is also known as *monoid algebra*. It follows that $R[M]$ is an R-module with basis from M.

Example 4.3.2. Some examples of monoid algebras are:

1. Polynomial ring $R[X_1, \ldots, X_n]$ is a monoid algebra where M is a free monoid generated by $\{X_1, \ldots, X_n\}$.

2. Group ring $R[G]$ (like Laurent's polynomial) is a monoid algebra where $M = G$.

Remark 4.3.3. *If M is a free monoid with n generators $\{m_1, \ldots, m_n\}$, then there is a natural isomorphism $R[M] \to R[X_1, \ldots, X_n]$ given by $m_i \mapsto x_i$.*

Lemma 4.3.4. Let M be an affine monoid. Then $R[M]$ is finitely generated as an R-algebra. If R is Noetherian, then using Hilbert basis theorem, we conclude that $R[M]$ is also Noetherian.

Proof. If M is finitely generated (with generators $\{m_1, \ldots, m_n\}$), then $R[M]$ is finitely generated (with same generators) as a R algebra. Let N be a free monoid with n generators e_i. Then from remark 4.3.3 we have $R[N] \simeq R[X_1, \ldots, X_n]$, where $e_i \mapsto X_i$. Since R is Noetherian, it implies $R[X_1, \ldots X_n] \simeq R[N]$ is Noetherian. Let $\psi : N \to M$ be the surjective monoid homomorphism, where $e_i \mapsto m_i$, then this ψ can be extended to a surjective ring homomorphism $\widehat{\phi} : R[N] \to R[M]$ given by $\widehat{\psi}(rm) = r\psi(m)$. Hence it follows that $R[M]$ is a Noetherian ring. $\square$

We know that Laurent polynomial is an example of monoid algebra. Now the next lemma gives a deeper relation between these two.

Lemma 4.3.5. Let M be an affine monoid with $\mathrm{gp}(M) = \mathbb{Z}^r$ for some integer $r \geq 0$, then there is an injective R-algebra homomorphism $R[M] \to R[X_1, X_1^{-1}, \ldots, X_r, X_r^{-1}]$ mapping element of M to monomial in variable $X_1, \ldots, X_r$.

Proof. Clearly $R[\mathbb{Z}^r] \to R[X_1, X_1^{-1}, \ldots, X_r, X_r^{-1}]$ is an isomorphism given by $e_i \mapsto X_i$. So if we show that $R[M] \to R[\mathrm{gp}(M)]$ is an embedding, then we get the required map. We already know that there exists monoid homomorphism $f : M \to \mathrm{gp}(M)$, so we extend f to $f_R : R[M] \to R[\mathrm{gp}(M)]$, where $f_R(r_i m_i) = r_i f(m_i)$ induces a R-algebra homomorphism. $\square$

A simple corollary of the above lemma is the following:

Corollary 4.3.6. Let M be an affine monoid. Then $R[M]$ is integral domain if R is an integral domain.

Proof. Let $\mathrm{gp}(M) = G$, then $M \subseteq G \subset \mathbb{Z}^d$, hence $R[M] \subseteq R[G] \subseteq R[{\mathbb{Z}_+}^d]$. By applying Lemma 4.3.5, we reach our conclusion. $\square$

A result of general interest is the sharpening of the corollary 4.3.6:

Proposition 4.3.7. Let R a domain and M (not affine) a cancellative and torsionfree. Then $R[M]$ is a domain. Conversely, if $R[M]$ is a domain, then R is a domain and M is cancellative and torsionfree.

Proof. Let $f, g \in R[M]$, then $f = \sum_{i=0}^{n} r_i m_i$ and $g = \sum_{i=0}^{k} s_i m_i$ for some $r_i, s_i \in R, m_i \in M$ (assume $n < k$). Consider the submonoid $N = \langle m_1, \ldots, m_k \rangle$. Then $fg \in R[N] \subset R[M]$.

Since N is finitely generated, by corollary 4.3.6, $R[N]$ is a domain, hence $fg \neq 0$ if $f, g \neq 0$. As f and g are arbitrary , result follows.

To prove the converse, observe that R is a domain if $R[M]$ is a domain. If M is not cancellative, then $xy = xz$ would imply $x(y-z) = 0$ which would give contradiction to the fact that $R[M]$ is a domain. If M is not torsionfree, then

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}).$$

Since none of the term is 0 on RHS (because all elements of M are basis for $R[M]$), we again arrive at a contradiction, hence M is torsionfree. $\square$

4.4 Grading of monoid algebras

The aim of these section is show some non-trivial grading on $R[M]$. We begin with a basic lemma.

Lemma 4.4.1. (Gordan's lemma) Let M be finitely generated monoid and let $\psi : M \to \mathbb{Z}$ be a homomorphism. Then $N := \{x \in M \mid \psi(x) \geq 0\}$ is finitely generated.

Outline of proof. *We will show that N is finitely generated by choosing a finitely generated submonoid K and showing $N/K \simeq L$ is finitely generated (in fact a finite set).*

Proof. Let M be generated by $m_1, \dots, m_n$. Then their exists a free monoid F and a surjective monoid homomorphism $f : F \to M$ such that basis of F maps to generators of M. Hence if image of $\psi \circ f \geq 0$ is finitely generated, then image of $\psi \geq 0$ is finitely generated. Hence we may assume that M is free. It follows immediately that N is a submonoid because if $x, y \in N$ then $\psi(x+y) = \psi(x) + \psi(y) \geq 0$ and hence $x + y \in N$.

Let $\{e_i\,, f_j\,, g_k\}$, where $i, j, k \in \mathbb{N}$, are basis of M such that

$$\psi(e_i) \geq 0\ ,\ \psi(f_j) \leq 0 \text{ and } \psi(g_k) = 0.$$

We rearrange the index such that $\{\psi(e_i) = a_i\}$ is in descending order and $\{\psi(f_j) = -b_j\}$ in ascending order. Let K be a submonoid of M generated by

$$K := \langle e_i\,, g_k\,, a_i f_j - b_j e_i \rangle.$$

Observe that $\psi(a_i f_j) - \psi(b_j e_i) = 0$. Since it is closed under multiplication and $\psi(K) \geq 0$ it is a submonoid of N. Let

$$L = N/K.$$

Then we can choose each representative element u such that $\psi(u) < a_1$ (due to presence of e_1 in kernel K). Similarly, if $u = \Sigma x_i e_i + \Sigma y_j f_j$ then either $x_i < b_j$ or $y_j < a_i$ (due to presence of $a_i f_j - b_j e_i$ in K) . Hence only finitely many possible values of u is possible and therefore L is a finite set. Now the result follows. □

We now prove the main proposition of the section, whose conclusion will establish the grading on $R[M]$.

Proposition 4.4.2. Let M be an affine monoid with set of units $U(M) = 1$. Let $G = \mathrm{gp}(M)$. Then their is a group homomorphism $\psi : G \to \mathbb{Z}$ such that $\psi(x) > 0$ for all $x \in M - \{1\}$. Hence $R[M]$ can be graded.

Proof. If rk $(M) = 1$, then $M = \mathbb{Z}^+$ (because if x is the least positive and $-y$ is the highest negative members of M, then $x - y$ violates the either minimality of x or maximality of y, and hence the only way out is to have only positive or negative elements) and hence we have the natural inclusion map to satisfy the above theorem.

We now use induction on rk (G). Let $G = \mathbb{Z} \times H$ (by structure theorem of G) and consider the submonoid $N = M \cap (0 \times H)$ and the projection map $\pi : G \to \mathbb{Z}$ restricted to M. Then N is finitely generated by Lemma 4.4.1 and hence by induction there exists a map $\phi : H \to \mathbb{Z}$ such that $\phi(x) > 0$ for $x \in N - \{1\}$. If $\psi(1, 0) = \lambda$, such that if the map $\psi : G \to \mathbb{Q}$ is well defined in a natural way, then we are done. So the only non-trivial part in the proof is show that such a λ exists. To do so, we divide the generators of G into three parts namely (a, u), $(-b, v)$ and $(0, w)$. So now λ is constrained by these equations

$$a\lambda + \phi(u) > 0 \text{ and } -b\lambda + \phi(v) > 0.$$

Now observe that $b(a,u) + a(-b,v) = (0, bu - av) \in N$. This element cannot be 0 because then $b(a,u) + a(-b,v) = 0$ and hence $U(M) = 1$, a contradiction. Therefore $\phi(bu - av) > 0$ or $b\phi(u) - a\phi(v) > 0$. Since the inequality is strict, the interval is non-empty and hence we can find a λ in between, and therefore the extension map $\psi : G \to \mathbb{Q}$ is well defined. Now we multiply λ by k to get rid of denominator. Hence we finally have $k\psi : G \to \mathbb{Z}$.

Grading: From the above argument, we can grade $R[M]$ as follows, let $(R[M])_n = \psi^{-1}(n)$ for $n \geq 0$ and $(R[M])_n = 0$ for $n < 0$.

$\square$

Proposition 4.4.3. Let G be a finitely generated free group and let M be a submonoid of G. If $\psi : M \to \mathbb{Z}$ is a homomorphism, then we can find a k such that $k\psi : G \to \mathbb{Z}$ is homomorphism.

Proof. We extend the map ψ to $H = \mathrm{gp}(M)$. Let $G = T \times F$ where T/H is the torsion part of G/H. Let k be chosen such that $kT/H = 0$ or $kT \subset H$. Hence $k\psi : T \to \mathbb{Z}$ is the extension of ψ to T. Hence we compose this map with the projection map, to get a map from $G \to \mathbb{Z}$. $\square$

Chapter 5

Some properties of monoids and monoid algebras

5.1 Extremal submonoids

In this section we look at a particular submonoids of M called *extremal* submonoid and list out some of its properties.

Definition 5.1.1 (Extremal submonoids)**.** A submonoid $E \subset M$ is called *extremal* if $xy \in E$ implies $x, y \in E$.

Notation 5.1.2. *The set of all extremal monoids of M will be denoted by $\mathcal{E}_M$.*

Remark 5.1.3. *If M is an affine monoid, then $C = \mathbb{R}_+ M$ is a cone. Let F be a face of C, then $M \cap F$ is a submonoid which is the geometrical realisation of the extremal submonoid.*

Example 5.1.4. In the monoid $M := \langle x_1, x_2 \rangle$, submonoid $\langle x_1 \rangle$ is extremal.

Definition 5.1.5. The interior of M denoted by $\mathrm{Int}(M) := \{x \in M \mid$ for all $y \in M$ we can find $n > 0$ with $x^n = yz, z \in M\}$.

Remark 5.1.6. *If $z \in$ Int(M), then $mz \in$ Int(M) for every $m \in M$.*

Notation 5.1.7. *Let $x \in$ Int(M), then $mx \in$ Int(M) from the above definition. Hence if $M^\star := \{1\} \cup$ Int(M) is a submonoid.*

Remark 5.1.8. *Infact $M^\star$ is a normal monoid because if $x \in gp(M^\star) - \{1\}$ and $x^n \in$*

$M^\star - \{1\}$, then

$$(x^n)^m = yz = x^{mn} = yz$$

and hence $x \in M^\star - \{1\}$.

Remark 5.1.9. *If M is an affine monoid, then $C = \mathbb{R}_+ M$ is a cone. Let Int(C) denote the interior of C in topological sense. Then $M\cap$ Int(C) := is the geometric realisation of Int(M).*

Example 5.1.10. In the monoid $M := \{\langle x_1, x_2\rangle, \mathrm{Int}(M) = x^i y^i \mid i, j \neq 0\}$.

Lemma 5.1.11. If E is a maximal submonoid of $M-$ Int(M), then E is extremal.

Proof. If E is not extremal, then there exists $y \notin E$ such that $xy \in E$. Consider the monoid $\{E, y\}$. This should intersect Int(M). Let $z \in \mathrm{Int}(M)$, such that $z = ey^r$ for some $e \in E$. Now $x^r z \in \mathrm{Int}(M)$ and $x^r z = e(xy)^r \in E$ which implies $E \cap \mathrm{Int}(M) \neq \emptyset$, which is a contradiction. Hence E is an extremal submonoid. □

Lemma 5.1.12. Let E be extremal submonoid of M and let N be any submonoid of M. If E meets Int(N), then $N \subset E$.

Proof. Let $x \in E\cap$ Int(M) and $y \in N$. Therefore we have $x^n = yz$ for some $n > 0$ and some $z \in N$. Since $x \in E$, we have $x^n = yz \in E$. Since E is extremal, it follows that $y, z \in E$. Hence $N \subset E$. □

Corollary 5.1.13. For extremal submonoids E and E' of an affine monoid, if Int$(E)\cap$ Int$(E') \neq \emptyset$, then $E = E'$.

Proof. Using Lemma 5.1.12, we have $E' \subset \mathrm{Int}(E)$ and $E \subset \mathrm{Int}(E')$. Hence $E = E'$. □

Using a combination both Int and extremal submonoids, we have a nice structure of M in terms of the former as illustrated in the next lemma:

Lemma 5.1.14. If M is finitely generated then $M = \bigsqcup_i \mathrm{Int}(E_i), E_i \in \mathcal{E}_M$ i.e. M can be expressed as a disjoint union of extremal submonoids.

Proof. Using 5.1.13, we know that Int(E_i) are disjoint. To see why $\mathcal{E}_\rangle$ covers M, observe that if $x \in M$ is not covered by any Int(E_i), then maximal submonoid containing x of $M-$ Int(E_i) is extremal, a contradiction and hence x is covered by some Int(E_i). □

The following proposition is the generalisation of Proposition 4.4.2.

Proposition 5.1.15. Let M is an affine monoid with no non-trivial units.. Then a submonoid E of M is extremal if and only if their exists a homomorphism $\psi : M \to \mathbb{N}$ with $E = \psi^{-1}(0)$.

Outline of proof. *To prove the converse, we move from monoid homomorphism to group homomorphims, and see M and E as subset of $gp(M)$ and kernel respectively.*

Proof. If such a ψ exists, then if follows that the kernel E is extremal. Indeed, let $a+b \in E$, then $\psi(a+b) = \psi(a) + \psi(b) = 0$ implies $\psi(a) = \psi(b) = 0$.

To prove the converse let $G = \text{gp}(M)$ and let $H' \subset G$ be the subgroup generated by E. Let

$$H = \{g \in G \mid g^n \in H' \text{ for some } n \in \mathbb{N}\}$$

and let $\nu : G \to G/H$ be the natural surjection.

Now we claim $\overline{G}$ is torsionfree because if $\overline{g}^n = 0$ for some n then $\overline{g}^{nm} \in H'$ for some m, which implies $\overline{g}^{nm} \in E$ or $\overline{g}^{-nm} \in E$. Hence $\overline{g} \in E$ implying $\overline{g} = 1$.

Let $\overline{M} = \nu(M)$. If $x \in E$, then $\nu(x) = 1$. Conversely, if $\nu(x) = 1$ implies $x \in H'$ and using the above reasoning $x \in E$. Hence $\overline{M}$ has no non-trivial units. Therefore using Proposition 4.4.2, we have $\mu : \overline{G} \to \mathbb{Z}$ such that $\mu(\overline{M}) \geq 0$. Hence $\mu \circ \nu$ is the required map. $\square$

Corollary 5.1.16. If an extremal submonoid $E \subsetneq M$, then by using Proposition 2.3.1, we can infer that rk $E <$ rk M (strict inequality).

5.2 Homothetic submonoids

We know that if M is finitely generated cancellative torsion free monoid with no non trivial units, then their exists a homomorphism $\psi : M \to \mathbb{N}$ with $\psi^{-1}(0) = \{1\}$ from Proposition 4.4.2. Now for $z \in \text{Int }(M)$ and $m > 0$, we define a map θ_m with centre z as

$$\theta_m(x) = x^m z^{\psi(x)}.$$

If M is normal, define $M^{(m)}$ as normalisation of $\theta_m(M)$ in $\mathrm{gp}(M)$. Now it follows that $M^{(m)} \subset \mathrm{Int}(M) \cup \{1\}$. Indeed, if $y \in M^{(m)}$, then $y^n \in \theta_m(M)$ and since every term of $\theta_m(M)$ has the form $x^m z^{\psi(x)}$, we have $y^n \in M^\star$ (because $z \in M^\star$). Since $M^\star$ is a normal monoid containing M, we have $M^{(m)} \subset M^\star$ by remark 5.1.3.

Definition 5.2.1. Let M be an monoid with no non-trivial units and $z \in \mathrm{Int}(M)$ and ψ as above. Then $M^{(m)}$ is called the *homothetic submonoid of M with centre z.*

Remark 5.2.2. *Homothetic transform corresponds to the homothetic transformation of $\Phi(M)$ in an affine space (see appendix 2).*

Lemma 5.2.3. If θ_m is defined as above, then it is injective.

Proof. Suppose $\theta_m(x) = 1$, then $\theta_m(z)^{\psi(x)} = z^{(m+\psi(z))(\psi(x))} = 1$. Since M ha no non-trivial units, therefore $\psi(x) = 0$. Hence $\theta_m(x) = x^m$ implies $x = 1$ because M is torsionfree. □

Lemma 5.2.4. If $M^{(i)}$ is defined as above, then $M^{(1)} \subset M^{(2)} \subset \ldots$.

Proof. Observe that $\theta_m(x)^{m+1} = \theta_{m+1}(x)^m z^{\psi(x)} \in M^{(m+1)}$, hence $\theta_m(x) \in M^{m+1}$ due to normal property of M^{m+1} and hence $M^m \in M^{m+1}$ again due to normal property of M^{m+1}. □

Lemma 5.2.5. Let M be a monoid. Then $M^\star = \bigcup M^{(m)}$

Proof. Clearly $M^\star \supseteq \cup M^m$ follows immediately because $z \in \mathrm{Int}(M)$. For the converse, consider $x \in \mathrm{Int}(M)$. This implies $x^k = yz$ for some $y, z \in M$. Let $l = \psi(y)$. Now $x^{(kl)} = y^l z^l = y^l z^{\psi(y)}$. Hence $x^{(kl)} \in M^{(l)}$. □

In general if $M \subset N$, no inclusion relation is observed between $\mathrm{Int}(M)$ and $\mathrm{Int}(N)$, but the next lemma show under specific circumstances, we can obtain such a relationship:

Lemma 5.2.6. Let M be a monoid and N a finitely generated submonoid of M. If $\mathrm{rank}(M) = \mathrm{rank}(N)$, then $\mathrm{Int}(M) \supseteq \mathrm{Int}(N)$.

Proof. If $N \cap \mathrm{Int}(M) = \emptyset$, then N is a submonoid of $M - \mathrm{Int}(M)$, hence is contained in some extremal submonoid E. By Proposition 5.1.15 we can find an homomorphism $\phi : M \to \mathbb{Z}$ such that $\phi^{-1}(0) = E$. Hence rank $N \leq$ rank $E <$ rank M which is a contradiction. Hence $N \cap \mathrm{Int}(M) \neq \emptyset$. Let $z \in N \cap \mathrm{Int}(M)$. If $y \in \mathrm{Int}(N)$, then $y^n = zw$. Hence $y^m \in \mathrm{Int}(M)$ which implies $y \in \mathrm{Int}(M)$. □

Lemma 5.2.7. If $z \in \mathrm{Int}(M)$, then $\{z^{-1}, M\} = \mathrm{gp}(M)$.

Proof. Let $x/y \in \mathrm{gp}(M)$. Write $z^r = yt$ with $t \in M$. Then $x/y = tx/z^r$. $\square$

Lemma 5.2.8. Let M be an affine monoid, and N be a finitely generated submonoid of M such that $\mathrm{gp}(M) = \mathrm{gp}(N)$. Then by fixing $z \in \mathrm{Int}(N)$ and $m > 0$, let $\theta_m : M \to M$ be the homothetic transformation. For large enough s, we have $\theta_m^s(M) \subset N^\star$.

Proof. Let $x_1, \ldots, x_n$ generate M. By Lemma 5.2.8, $\langle z^{-1}, N\rangle = \mathrm{gp}(N) = \mathrm{gp}(M)$. Therefore, for some t, $z^{t-1}x_i \in N$ and hence $z^t x_i \in \mathrm{Int}(N)$ for all i. Hence $(\theta_m)^s(x) = x^{a(s)}z^{b(s)\psi(x)}$, where $a(s) = m^s$ and $b(s) = c^{-1}[(m+c)]^s - m^s]$ with $c = \psi(z) > 0$. Therefore, $b(s) \geq sm^{s-1}$. Suppose that $s \geq tm$. Then $b(s) \geq tm^s = ta(s)$ so $(\theta_m)^s(x) = (xz^t)^{a(s)}z^d$ with $d \geq 0$. It follows that $(\theta_m)^s(x_i) \in \mathrm{Int}(N) \subset N^\star$. Since the x_i generate M, we have our result. $\square$

5.3 Graded Weierstrass preparation theorem

This section deals with a technical lemma which will be useful in the next section.

Proposition 5.3.1. Let $d = \delta(v)$ and $A = A_0 \oplus A_1 \oplus \cdots$ be a graded commutative ring and let v be an element of A_d. Let $M_0 \oplus M_1 \oplus \cdots$ be a graded A-module satisfying

$$\nu : M_i \to M_{i+\delta(v)} \text{ given by } m_i \mapsto vm_i \text{ is an isomorphism for } i \geq 0. \tag{5.1}$$

Let $f \in A$ with $f \equiv a_0 + \cdots + a_{nd-1} + v^n \bmod \mathrm{nil}(A)$ (nilpotent elements of A), with $a_i \in A$. Then if $z \in M$, we can write $z = fq + r$ with $q \in M$ and $r_0 \in M_0 + \cdots + M_{nd-1}$. Moreover, r and q are unique.

Proof. Let $f = f_0 + \cdots + f_m$. Then $f_{nd}, \ldots, f_m$ and $f_{nd} - v^n$ are nilpotent and therefore generate a homogeneous nilpotent ideal J. Let $J^h = 0$ for some h. If $h = 0$ then the usual division algorithm applies. Hence we use induction on h.

Let

$$N := \{\, x \in M \mid v^k x \in J^{h-1} \text{ for some } k \geq 0 \,\} = \bigcup_{k\geq 0}(J^{h-1}M : v^k).$$

Then $\nu_1 : N_i \to N_{i+d}$ satisfies the equation (5.1) and hence $\nu_2 : (M/N)_i \to (M/N)_{i+d}$ satisfies (5.1). Also M/N has a A/J^{h-1}-module structure (since $J^{h-1}M \subset N$) and N has a A/J-module structure (since $v^k JM = 0$ and v doesn't annihilate M). Therefore, by induction the proposition holds to the residue class of f in A/J^{h-1} and A/J for the modules M/N and N respectively.

Writing $\overline{z} = \overline{fq} + \overline{r}$ in M/N, and lifting it back to M, we get $z = fq + r + w, w \in N$. Now writing $w = fq' + r'$ in N, we have

$$z = fq + r + fq' + r' = f(q + q') + (r + r').$$

To show uniqueness, let $z = 0$, then we have $\overline{q} = \overline{r} = 0$ in M/N. Hence q and r lie in N and so $q = r = 0$ when induction is applied to N. $\square$

Corollary 5.3.2. Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded ring and let v be an element of A_d satisfying the condition

$$\nu : A_i \to A_{i+d} \text{ given by } x \mapsto vx$$

is an isomorphism for $i \geq 0$.

Let $f \in A$ with $f \equiv f_0 + f_1 + f_{nd-1} + v^n \mod (\mathrm{nil}(A))$. Then
$f = (1 + \mu)(v^n + r)$, where $\mu \in \mathrm{nil}(A)$ and $r \in A_0 + \cdots + A_{nd-1}$.

Proof. Using Proposition 5.3.1 for $M = A$ and $z = v^n$, we have $v^n = fq + r$.

Then Proposition 5.3.1 can also be applied to $M = A/\mathrm{nil}(A)$. To see why observe that surjectivity follows because $v : A_i \to A_{i+d}$ is surjective and injectivity follows because if x is nilpotent only if vx is nilpotent.

Hence we have

$$\overline{v} = \overline{f} - (\overline{f_0} + \overline{f_1} + \overline{f_2} + \cdots + \overline{f_{nd-1}}) = \overline{qf} + \overline{r}.$$

Thus uniqueness of $\overline{q}$ implies $\overline{q} = 1$ or $q = 1 + \mu_1$ with μ_1 being nilpotent. Therefore, q is invertible and $q^{-1} = 1 + \mu$ with $\mu \in \mathrm{nil}(A)$. $\square$

5.4 Pyramidal extension

In this section we study one of the important tools to solve Anderson's conjecture viz. pyramidal extension.

Definition 5.4.1. An extension of monoids $N \subseteq M$ is called *integral* if for every $x \in M$, $x^n \in N$ for some n. An extension is called a *pyramidal extension* if the following conditions hold:

1. M is torsionfree, cancellative, finitely generated, normal and has no non-trivial units.
2. There is a homomorphism (not unique) $\delta : M \to \mathbb{Z}$ such that $N = \{x \in M \mid \delta(x) \leq 0\}$.
3. There is an element $v \in M - N$ such that $\langle v, N\rangle$ is integral over M. The element v is also called as *vertex*. This means that for $x \in M$, there exists $m > 0$ and $y \in N$ such that $x^m = yv^a$.

Remark 5.4.2. *In geometric terms, $N = M \cup \Delta$, where Δ is a pyramid with vertex v and M is a polytope such that they intersect in the facet opposite to v (see Appendix).*

Statement (3) actually says that for $x \in M$, $x^n = v^a y$ for some $y \in N$. Before proving the main proposition on pyramidal extension, we first prove a couple of lemmas.

Lemma 5.4.3. If $x \in M$ and $\delta(x) \geq \delta(v)$, then $x = vy$ for some $y \in M$.

Proof. Using the condition (3) of definition 2.4.1, we have $x^n = v^a y$ for some a. Under the image of δ we have $n\delta(x) = a\delta(v) + \delta(y)$. Since $\delta(y) \leq 0$, we have $n\delta(x) \leq a\delta(v)$. But since $\delta(x) \geq \delta(v)$, we have $n \leq a$. Hence $(v^{-1}x)^n = v^{a-n}y \in M$. Since M is normal, we have $v^{-1}x \in M$ and hence the result. □

Remark 5.4.4. *Let H be the integral closure of $\langle v\rangle$ in gp(M). Then $H = \langle v\rangle$.*

Lemma 5.4.5. If v is the vertex of the pyramidal extension $N \subseteq M$, then $\langle v\rangle$ is extremal in M.

Proof. Let $xy \in \langle v\rangle$, then $xy = v^a$ for some $a > 0$. Using (3) of definition 5.4.1, we have (for common m), $x^m = v^b s$ and $y^m = v^c t$ with $b, c \geq 0$. Now $xy = v^{b+c}st = v^a$ forces $b = c = 1$, as M has no non-trivial units. Since $\langle v\rangle$ is integrally closed it follows $x, y \in \langle v\rangle$. □

Now we move towards the main theorem of this section:

Let $x_1, \ldots, x_d$ generate M. Let us fix $m > 0$ such that ${x_i}^m \in \langle N, v\rangle$ for all i. Hence $x^m = v^a y$ for all $x \in M$.

Definition 5.4.6. Let $\deg(x)$ denote the least a in the expression $x^m = v^a y$ for fixed m.

Remark 5.4.7. *deg*$(v^a) = am$.

Lemma 5.4.8. If $x \in M$ and $\deg(x) \geq \deg(v^a)$, with $a \geq 0$, then $x = v^a y$ for some $y \in N$ with $\deg(y) \leq \deg(x) - \deg(v^a)$.

Proof. Let $x^m = v^a y$ for some $y \in M$ and $b > am$. Hence $(v^{-a}x)^m = v^{b-am}y \in M$ and therefore $v^{-a}x \in M$ as M is normal. $\square$

Definition 5.4.9. Let $f \in A$, we write $\mathrm{def}(f) < d$, if $f = r_1 w_1 + \cdots + r_n w_n$ for some $n > 0$, where $r_i \in L$ and $w_i \in M$ and $\deg(w_i) < d$. An element $g \in A$ is *monic* if $g = v^a + f$ with $\deg(f) < \deg(v^a)$. Then we put $\deg(g) = \deg(v^a) = am$.

Remark 5.4.10. *Let g be a monic polynomial and $f \in A$. Then we can write $f = gq + r$ with deg$(r) <$ deg(g).*

Proposition 5.4.11. Let $N \subseteq M$ be a pyramidal extension with vertex v. Let $(R, \mathcal{M}_R)$ be a local ring and let $\mathcal{M}$ be the maximal ideal of $R[N]$ generated by $\mathcal{M}_R$ and $N - \{1\}$. Let P be a finitely generated projective $R[M]$-module. If P_v is free, then $P_{\mathcal{M}}$ is free.

Proof. Let $L = R[N]_{\mathcal{M}}$ and $A = R[M]_{\mathcal{M}}$. Let

$$S := \{ \text{ set of all monic polynomial } \}$$

and

$$B := \{\alpha + (f/g) : \alpha \in L, g \in S, f \in A, \text{ deg } (f) < \text{ deg } (g)\}.$$

Let P be a projective $R[M]$-module. We now apply Proposition 3.4.8. To do so, we first check the following condition which are:

(1) *A/fA is a finitely generated L-module for all $f \in S$.*

By using Lemma 5.4.3, we have A/fA generated by monomial $m_i v^k$, where m_i is the generating set of M. Since $\deg(m_i z^k) < \deg(f)$, we have only finitely many such monomials and hence A/fA is finitely generated.

(2) $GL_n(\overline{S^{-1}A}) = GL_n(S^{-1}A)/GL_n(\overline{A})$. where GL_n is the set of all $n\times n$ invertible matrices. The proof of this statement is bit lengthy and hence we postpone it to after this proposition.

(3) *There is an L-algebra $B \subset S^{-1}A$ with $S^{-1}A = A + B$ and $\mathfrak{m}B \subset J(B)$, where $\mathfrak{m}$ is the maximal ideal of the local ring R and $J(B)$ denotes the Jacobson radical of the ring B.*

Every element in $S^{-1}A$ has the form f/g, where g is a monic polynomial. Writing $f = gq+r$ and using Lemma 5.4.3, it follows $f/g = q + r/g \in A + B$. Hence $S^{-1}A = A + B$.

If $m_i \in \mathfrak{M}$ and $\alpha_i + f_i/g \in B$, then $u = 1 + \sum m_i(\alpha_i + f_i/g) = e + f/g$, where $f = \sum m_i f_i$. Therefore, $u = e(g + e^{-1}f)/g$ which is a unit in $S^{-1}A$, as $g + e^{-1}f$ is monic (because $\deg(e^{-1}f) < \deg(f) < \deg(g)$) . Hence $\mathfrak{M}B \subset J(B)$.

(4) *$P_S \simeq (A_S)^n$ and $\bar{P} \simeq \bar{A}^n$ for some n.*

Since $A_{\text{red}} = k[v]$ where k is a field, $\bar{P}/\text{nil}(\bar{A})\bar{P}$ if free by Lemma 1.4.5 and so is P. If $v \in S$ is such that P_v is free, the P_S is free . Also since $R[M]$ is a domain, it has a constant rank and therefore rank $P_S = \bar{P}$.

□

Let I be an ideal in $R[M]$ generated by $N - \{1\}$. Then I as an R-module is a free module over R with basis $X = \{yz \mid y \in N - \{1\}, z \in M\}$. Then the R-algebra $C = R[M]/\langle N - \{1\}\rangle$ is a free R-module with basis $Y = M - X$. In C, the multiplication is defined in a natural way i.e. R-linear extension of:

$$x.y = \begin{cases} xy & \text{if } xy \in Y \\ 0 & \text{if } xy \in M - X \end{cases}$$

Let $Y_n := \{\ x \in Y \mid \delta(x) = n\}$ and $C_n = R[Y_n]$. Then $C_n = 0$ for $n < 0$ as $N - \{1\} \subset X$. Hence $C = C_0 \oplus C_1 \oplus \cdots$ is $\mathbb{Z}$-graded (positively graded) with $C_0 = R$.

Lemma 5.4.12. If $i \geq 0$, then $\nu : C_i \to C_{i+\delta(v)}$ given by $c \mapsto vc$ is an isomorphism.

Proof. It is sufficient to prove that vY_i is bijective to $Y_{i+\delta(v)}$. Let $y_i \in Y_i$, suppose $y_i v \notin C_{i+\delta(v)}$, then $y_i v \in N - \{1\}$. Hence $y_i v = xz$ for $x \in N - \{1\}$. Since $\delta(y_i) > 0$ and and $\delta(x) \leq 0$, therefore $\delta(v) < \delta(z)$. From Lemma 5.4.8, we have $z = sv$ for some $s \in N$.

Therefore, by cancellation $y = xs$ implies $y \in N - \{1\}$, a contradiction.

The map is injective because it is a domain. To check for surjectivity, let $y \in Y_{i+\delta(v)}$. Then by Lemma 5.4.3, we have $y = vx$ for some $x \in M$. Hence x must lie in Y_i, otherwise $x \in N - \{1\}$. But then $y \in N - \{1\}$, which is a contradiction. $\square$

Now we tackle the case (2) of Proposition 5.4.11.

Proposition 5.4.13. Let A be a L-algebra and S be a closed set as defined in Proposition 5.4.11, then $GL_n(\overline{S^{-1}A}) = GL_n(S^{-1}A)/GL_n\ (\overline{A})$.

Proof. Let $L/\mathfrak{M} = R/\mathfrak{M}_R = k$ (a field). Then

$$\overline{A} = A/\mathfrak{M}A = R[M]/\langle m, N - \{1\}\rangle = k[M]/\langle N - \{1\}\rangle.$$

If $x \in M$, then $x^m = v^a y$ by 5.4.1 for some $m > 0, y \in N$. Hence either $x \in v^{\mathbb{Z}_+}$, where $v^{\mathbb{Z}_+} := \{1, v, v^2 \ldots\}$ or x is nilpotent modulo $N - \{1\}$. By Lemma 5.4.5, $\langle v\rangle$ is extremal and hence R$[M - \langle v\rangle]$ is an ideal. Therefore $R[M]/\langle M - \langle v\rangle\rangle \simeq R[v]$. Hence

$$(R[M]/\langle N - \{1\}\rangle)_{\text{ red}} \simeq R[v] \text{ and } \overline{A}_{\text{ red}} \simeq k[v]$$

because $\langle M - \langle v\rangle\rangle/\langle N - \{1\}\rangle$ is nilpotent as discussed above.

Since S was a closed set in A, let us look its image in $\overline{A}_{\text{ red}} = k[v]$. If $f \in S$, then $f = v^a + \Sigma s_i m_i$ for some $s_i \in L, m_i \in M$ with $\deg(m_i) < a \deg(v)$. Since we are going modulo $\langle M - v^{\mathbb{Z}_+}\rangle$, only those m_i such that $m_i = v^j$ will survive. Since $\deg(m_i) < a \deg(v)$, we have $j < a$. Furthermore, s_i will go to its residue class in k. Hence S goes to monic polynomials in $k[v]$. Conversely, every monic polynomial can be trivially lifted (identify the coefficient as s_i). Hence

$$(\overline{S^{-1}A})_{\text{ red}} \simeq k(t).$$

Therefore it follows that $\overline{S^{-1}A}_{\text{ red}}$ is a local ring, and hence $SL_n(\overline{S^{-1}A}) = E_n(\overline{S^{-1}A})$ and $SL_n(S^{-1}A) \to SL_n(\overline{S^{-1}A})$ is onto. Hence we only need to prove $U(S^{-1}A) \oplus U(\overline{A}) \to U(\overline{S^{-1}A})$ is surjective (Here SL_n means $n \times n$ matrix of determinant 1 and E_n mean group of elementary matrix generated by $\langle I + \lambda e_{ij}\rangle$, where $i \neq j$ and e_{ij} is the matrix units. See [11] for more details).

From above we know that $C = R[M]/\langle N - \{1\}\rangle$ has a graded structure. Hence

$$\overline{A} = C/\mathfrak{M}C = \overline{A_0} \oplus \overline{A_1} \oplus \cdots$$

where $\overline{A_i} = C_i/\mathfrak{M}C_i$ and a modification of Lemma 5.4.12, we have $\overline{A_i} \to \overline{A_{i+\delta(v)}}$. An unit of $\overline{S^{-1}A}$ has the form f/s where $s \in$ image of $S \subset U(S^{-1}A)$ and $f \in \overline{A}$ divides some element of the image of S. Since $\overline{A}_{\text{red}} = k[v]$, implies that upto a factor of k^*, f maps to a monic polynomial in $k[v]$. Hence by Proposition 5.3.1 $f = (1+\mu)g$, where μ is nilpotent and $g = v^m + a_{m\delta(v)-1} + \cdots + a_0$ with $a_i \in \overline{A_i}$. Since $(1+\mu) \in U(\overline{A})$, it will be sufficient to prove that g lies in image of S. We lift g to h where $h = v^m + b_{m\delta(v)-1} + \cdots + b_0$ where b_i's are linear combination of L and Y_i. Since the element of C_i have smaller degree that t^m for $i < me$, we have lifted g to a monic element (here Y_i is the same Y_i we defined above the Lemma 5.4.12. $\square$

The next lemma will show that our assumption of P_v free is true under certain conditions.

Lemma 5.4.14. Let M be a finitely generated, normal affine monoid with no non-trivial units. let v be an element of M with $\langle v\rangle$ extremal. Then $\langle M, v^{-1}\rangle \simeq \mathbb{Z} \times M_1$ where $\mathbb{Z}$ is generated by v and M_1 has no non-trivial units.

Proof. Let $G = \mathrm{gp}(M)$. If $v = w^n$, with $w \in G$, then by normality of N, $w \in M$. Since $\langle v\rangle$ is extremal, would imply $w \in v$ which would contradict the fact that v is the generator of $\langle v\rangle$. Hence v is an unimodular element of G (see [11] chapter 1 4.12) for the definition of unimodular element). Since G is a free group, we can find a homomorphism $\phi : G \to \mathbb{Z}$ such that $\phi(v) = 1$ (because of unimodularity). Let $M_1 = \{x \in \langle M, v^{-1}\rangle | \phi(x) = 0\}$. To check the surjectivity assume $x \in \langle M, v^{-1}\rangle$ with $\phi(x) = n$. Then $\phi(v^{-n}x) = 0$ which would imply $v^{-n}x \in M_1$. To check for injectivity let $v^a x = v^b y, x, y \in M_1$, then under the image of ϕ, $a = b$ which in turn imply $x = y$. Hence $\langle M, v^{-}1\rangle \simeq M_1 \times \mathbb{Z}$. If $z \in M_1$ is invertible, then $z = v^{-a}x$ and $z^{-1} = v^{-b}y$ with $x, y \in M$. Then $xy = v^{a+b}$ and so $x, y \in \langle v\rangle$ since $\langle v\rangle$ is extremal. Therefore $z = 1$. $\square$

Induction

Let the main theorem (stated in next chapter) holds true for any monoid of rank less than that of rank M.

Corollary 5.4.15. Assume the induction hypothesis. Let $N \subset M$ be a pyramidal extension with vertex v. Let R be a local ring with maximal ideal $\mathcal{M}_R$ and $\mathcal{M}$ be the maximal ideal of $R[N]$ generated by $\mathcal{M}_R$ and $N-\{1\}$. Let P be a finitely generated projective $R[M]$-module, Then $P_{\mathcal{M}}$ is free.

Proof. If we show P_v is free, then by Proposition 2.4.1, we would conclude $P_{\mathcal{M}}$ is free. Now P_v is a $R[M_v]$-module, but by Lemma 5.4.14,

$$R[M_v] = R[M, v^{-1}] = R[\mathbb{Z} \times M_1].$$

By Induction, $R[M_1]$ satisfies the Anderson's conjecture (studied in next chapter), then by Lemma 5.4.14, $R[M \times \mathbb{Z}]$ satisfies the main theorem, hence P_v projective $R[M \times \mathbb{Z}]$-module is free. $\square$

Chapter 6

Swan's proof of Anderson's conjecture

D. Anderson in 1980's conjecture a problem for monoid algebra analogue to the Serre's problem for projective modules. In 1988, the conjecture was proved by J. Gubeladze. His method was based on convex geometry of affine monoids. Following Gubeladze's technique, in 1991 R.G. Swan came up with an algebraic version. This chapter deals with the algebraic version of Anderson's conjecture. To see the original proof of this theorem see [7]. This chapter deals with the resolution of Anderson's conjecture:

Main Theorem

Let R be principal ideal domain (PID) and M be an affine seminormal monoid. Then all finitely generated projective $R[M]$-modules are free.

6.1 Preliminary reductions

In this section, we will simplify the hypothesis of the main theorem. We first start by simplifying hypothesis on M.

Lemma 6.1.1. It is sufficient to prove the main theorem for M with no non-trivial units.

Proof. Let $K' = M - U(M)$ and $K = K' \cup \{1\}$. Now Consider the following commutative diagram:

$$\begin{array}{ccc} R[K] & \longrightarrow & R[M] \\ \downarrow & & \downarrow f \\ R & \xrightarrow{g} & R[U(M)] \end{array} \tag{6.1}$$

under the maps:

1. The vertical map sends K' to 0 and is identity on N.

2. The horizontal map are the inclusion map.

We now claim that:

1. *K is a submonoid and $MK' \subset K'$.*

   Let $x, y \in K$. Since x, y are non-invertible, their product should also be non-invertible, hence its closed under multiplication. Since $\{1\} \in K$, K has an identity element and therefore K is a submonoid of M.

2. *The Diagram 6.1 is a milnor square of type A.*

   In Diagram 6.1, the map $f : R[M] \to R[U(M)]$ is surjective. Since

   $$f(r_0 + r_1k_1 + \cdots + r_nk_n) = f(r_0) = g(r_0),$$

   the pullback of the diagram i.e.

   $$\{(x, y) \in R[M] \times R : f(x) = g(y)\}$$

   is $R[K]$. Hence the diagram satisfies the condition of type A Milnor square

3. *K is seminormal.*

   If $x^2, x^3 \in K$ and $x \in M - K$, then x is invertible, hence x^2, x^3 are invertible, a contradiction. Therefore, $x \in K$ and hence K is seminormal.

Now if P is finitely generated projective module over $R[M]$, then by Proposition 1.5.3 is extended from $R[K]$. Hence without loss of generality, we can assume M to have no non-trivial units. $\square$

Having simplified the assumption on M, we now simplify the assumption on ring R.

Lemma 6.1.2. It is enough to prove the main theorem for the case when R is local.

Proof. Using Proposition 4.4.2 , we can non-trivially grade the ring $R[M]$. Hence we can now apply graded Quillen's patching Theorem 3.5.1, and hence we can assume R to be local. $\square$

Let M be a finitely generated monoid and $M_1, \ldots, M_n$ be the set of extremal submonoids labelled in such a way that $M_i \supseteq M_j$ if $i \leq j$. Hence $M_1 = M$ and $M_n = U(M) = 1$. Let $U_i = \operatorname{Int}(M_i)$ for $i = 1, 2, \ldots, n$ and $W_i = U_1 \cup \cdots \cup U_i$. By Lemma 5.1.14, U_i partitions M. Also, we have $W_0 = \emptyset$ and $W_n = M$. Now this W_i satisfies an important property.

Lemma 6.1.3. Let W_i be defined as above. Then $MW_i \subset W_i$ for each i.

Proof. Let $x \in M$ and $y \in W_i$ (and hence in some U_k, $k \leq i$). Since U_i partitions M, we have $xy \in U_j$. But $U_j \subseteq M_j$, and since M_j is extremal, we have $x, y \in M_j$. Using Lemma 5.1.12, we have $M_k \subseteq M_j$, hence $j \leq q$ and hence $xy \in W_j \subseteq W_i$. $\square$

Having simplified M and R, we now simplify the hypothesis of ring $R[M]$.

Lemma 6.1.4. Let M be the monoid in the hypothesis of the main theorem. If the main theorem holds for all $R[N]$, where N is a submonoid of $\operatorname{rk}(N0 < \operatorname{rk}(M)$, then every projective $R[M]$-module is extended from $R[M^*]$.

Proof. Let J_i be an $R[M]$-submodule generated by W_i. Using Lemma 6.1.3, J_i becomes an ideal. Let $R[M]/J_i = A_i$. Then there is a natural map between $A_{i-1} \to A_i$, whose kernel is $R[U_i]$. Let $M_i^* = \{1\} \cup U_i$ for $i < n$. Then we make the following claim:

Following diagram is a Milnor square of type A:

$$\begin{array}{ccc} R[M_i^*] & \longrightarrow & A_{i-1} \\ \downarrow & & \downarrow \\ R & \longrightarrow & A_i \end{array} \tag{6.2}$$

where horizontal maps are natural and vertical maps takes $M_i^* - \{1\}$ *to* 0.

The natural map $A_{i-1} \to A_i$ is surjective and arguing the same as we did in Lemma 6.1.1, we see that this is indeed a Milnor square of type A.

Since $M_i \subsetneq M$, $\mathrm{rk}(E_i) < \mathrm{rk}(M)$ and M_i^* is filtered union of normal monoids by Lemma 5.2.5, hence all projective $R[M_i^*]$-modules are extended from $R[M]$, which by induction hypothesis are free. Since $A_{n-1} = R$, using Proposition 1.5.3, we have A_{n-1} is extended from $R[M_{n-1}]$ and since $R[M_i]$ is free, A_{n-2} is extended from R. Now using decreasing induction, and noting that $A_0 = R[M]$, we arrive at the conclusion that $R[M]$ is extended from $[M^*]$. $\square$

Lemma 6.1.5. If $\mathrm{rk}(M) = 1$, then the main theorem holds true for $R[M]$.

Proof. If $\mathrm{rk}(M) = 1$, then $R[M] \simeq R[X]$ and from the classical result of Serre's conjecture, we reach our conclusion. $\square$

Combining all the above the result, we have the final simplification as:

Remark 6.1.6. *It is sufficient to prove the main theorem under the following assumption*

1. *R is local and M has no non-trivial units.*

2. *M = Int(N) $\cup\{1\}$, where N is a finitely generated, cancellative, torsionfree, normal monoid with no-non trivial units (see Proposition 4.2.10).*

3. *(Induction hypothesis) Main theorem hold for monoids of rank < rk(M) and is true for rk$(M) = 1$.*

6.2 Non-degenerate pyramidal extension

In the previous chapter, we have discussed the notion of pyramidal extension of N over M. In this section we use pyramidal extension as a tool to prove the main theorem. Since we are using induction to prove our theorem, we now require pyramidal extension to be such that $\delta(w) < 0$ so that this gives $\mathrm{rk}(N) = \mathrm{rk}(M)$ (the next lemma). Such a pyramidal extension is called *non-degenerate extension.* This turns out to be the general case scenario, since we have reduced the size of the monoid without decreasing the rank.

Lemma 6.2.1. If $N \subset M$ be a pyramidal extension with $\delta : M \to \mathbb{Z}$ such that $\delta(w) < 0$, then $\mathrm{gp}(M) = \mathrm{gp}(N)$, where $\mathrm{gp}(M)$ is generated by w and $w^n x$ for large enough n for

$w^n x \in N$ and $x \in \mathrm{gp}(M)$.

Proof. If $x \in \mathrm{gp}(M)$, then $(w^{-1})^n.w^n x = x \in \mathrm{gp}(N)$. Other way inclusion is obvious. $\square$

We start with a lemma which says normalisation preserves pyramidal extension.

Lemma 6.2.2. Let G be a finitely generated free abelian group and let $N \subset M$ be a non-degenerate pyramidal extension of finitely generated normal submonoids of G. Then the normalisation of G also forms a non-degenerate pyramidal extension $\widetilde{N} \subset \widetilde{M}$.

Proof. Let $\delta : M \to \mathbb{Z}$ be the given map. Using Proposition 4.4.2, we can extend this map to $k\delta : G \to \mathbb{Z}$, where $G = \mathrm{gp}(M)$. So, without the loss of generality, we replace δ with $k\delta$. Now let $\overline{N} = \{x \in \widetilde{M} | \delta(x) \leq 0\}$. Thus if $x \in \overline{N}$, then $x^n \in M$ and $x^n \in \overline{N}$, hence $x^n \in N$ and therefore $x \in \widetilde{N}$. Other way inclusion is obvious. $\square$

Let $(R, \mathcal{M}_R)$ be a local ring and M^* denote the submonoid $\mathrm{Int}(M) \cup \{1\}$ and $\mathcal{M}^*$ denote the maximal ideal of $R[N^*]$ generated by $\mathcal{M}_R$ and $N^* - \{1\}$. The next proposition proves a result similar to Proposition 5.4.11.

Lemma 6.2.3. Let R be a local PID. Let $N \subset M$ be a non-degenerate pyramidal extension. If P is a finitely generated projective $R[M^*]$-module, then $P_{\mathcal{M}^*}$ is free.

Proof. Since P is finitely generated $R[M^*]$ module, and M^* is a filtered union of homothetic submodule $M^{(m)}$ with centre at z (which we choose to lie in $\mathrm{Int}(M)$). By Lemma 5.2.5, all generators $m_1, \dots, m_r$ lie in some $M^{(m)}$. So using Lemma 1.4.5, we have a projective $R[M^{(m)}]$-module Q such that $P \simeq Q \otimes R[M]$.

Now we claim the following:

1. *Let θ_m be same entity as defined in 5.2.1, then $\theta_m(N) \subset \theta_m(M)$ is a non-degenerate pyramidal extension and hence $N^{(m)} \subset M^{(m)}$ is a non-degenerate pyramidal extension.*

   Since θ_m is injective by Lemma 5.2.3, $\theta_m(M) \subset \theta_m(N)$ is a pyramidal extension. Using Lemma 6.2.2, it's normalisation is also pyramidal extension, i.e. $M \subset N$ is a pyramidal extension.

2. $\mathcal{M}^* \cap R[N^{(m)}] = \mathcal{M}'$ (Here $\mathcal{M}'$ is the maximal ideal of $R[N^{(m)}]$ generated by $\mathcal{M}_R$ and $N^{(m)} - \{1\}$).

   One way is obvious. For the other way we know $\mathcal{M}^* \cap R[N^{(m)}]$ is an ideal in $R[N^{(m)}]$. Since $\mathcal{M}'$ is a maximal ideal and $\mathcal{M}^* \cap R[N^{(m)}] \subseteq \mathcal{M}'$, it implies $\mathcal{M}^* \cap R[N^{(m)}] = \mathcal{M}'$.

By Proposition 5.4.11, $Q_{\mathcal{M}'}$ is free. Since $N^{(m)} \subset N^*$ and $\mathcal{M}^* \cap R[N^{(m)}] = \mathcal{M}'$, so $P_{\mathcal{M}*}$ is extended from $Q_{\mathcal{M}*}$ and hence is free.

□

Proposition 6.2.4. Let R be a local PID. If $N \subset M$ is a non-degenerate pyramidal extension, then $\mathcal{P}(R[N^*]) \to \mathcal{P}(R[M^*])$ is onto.

Proof. If the diagram

$$\begin{array}{ccc} R[N^*] & \longrightarrow & R[M^*] \\ \downarrow & & \downarrow \\ R[N^*]_{\mathcal{M}^*} & \longrightarrow & R[M^*]_{\mathcal{M}^*} \end{array} \tag{6.3}$$

is Milnor square of type B, then we are done. To show that we take $S = R[N^*] - \mathcal{M}^*$. Now we must show that $R[N^*]/(f) \to R[M^*]/(f)$ isomorphism for all $f \in S$. It is sufficient to check this locally on $R[N^*]$ (using a variant of the result that $M_m = 0$ for all m in max R for a R-module M implies $M = 0$). At $\mathcal{M}^*$, the localisation makes both side 0 (because f, an invertible element becomes a zero-divisor). □

Lemma 6.2.5. Let $\mathfrak{p}$ be the prime ideal in $R[N^*]$ not contained in $\mathcal{M}^*$, then $\mathfrak{p} \cap \mathcal{N}^* = \emptyset$.

Proof. Let $x \in N^* \cap \mathfrak{p}$. As $x \neq 1$, it follows that $x \in \mathrm{Int}(N)$. Let $y \in \mathrm{Int}(N)$. Hence $y^n = xw \in N$ for some q or $y^{2n} = x(xw^2) \in \mathrm{Int}(N)$, and hence $y \in \mathfrak{p}$. Therefore, $\mathfrak{p} \supseteq N^* - \{1\}$. Since $R[N^*]/(N^* - \{1\}) = R$ is local, we have $\mathfrak{p} \subseteq M^*$. □

Observing that $N^* \subset R[N^*]\mathcal{N}$ for some maximal ideal $\mathcal{N}$ in $R[N^*]$. The next lemma says that $R[N^*]$ is locally equal to $R[M^*]$ at all maximal ideals other than $\mathcal{M}'$ and hence the Diagram 6.3 would be Karoubi square or Milnor square of type B.

Lemma 6.2.6. Let $N = \{x \in M | \delta(x) \leq 0\}$ have the same rank as M. Then $N^{-1}N = N^{-1}M = M^{-1}M$.

Proof. One way inclusion is obvious. For other way, let $x/y \in M^{-1}M$. Since N and M have same rank, hence their exists w such that $\delta(w) < 0$. Let $x' = w^n x$ and $y' = w^n y$. Hence $x/y = x'/y'$. If we make n large enough, then $x', y' \in N$. Hence the $N^{-1}N = M^{-1}M$. $\square$

Using Lemma 6.2.6, we have $R[N*]_{\mathcal{N}} \xrightarrow{f} R[M*]_{\mathcal{N}}$ as an isomorphism for a maximal ideal $\mathcal{N}$ in $R[N*]$ because $N^{*-1}N^* = N^{*-1}M$. and hence $N^{*-1}R[N^*] \simeq N^{*-1}R[M^*]$. Further localisation gives $R[N^*]_{\mathcal{N}} \simeq R[M^*]_{\mathcal{N}}$ and hence the result follows by taking quotient with (f).

6.3 Admissible sequence

The previous section relied on the existence of non-degenerate pyramidal extension. In this section we show the existence of such a pyramidal extension.

Definition 6.3.1. A sequence of submonoids $M = M_0, M_1, \ldots, M_n$ is called an admissible sequence if each M_i is torsionfree, cancellative, finitely generated and normal in M and has no non-trivial units, and for each i, either of these happens:

1. $M_i \subset M_{i+1}$
2. $M_i \supset M_{i+1}$ is a non-degenerate pyramidal extension.

It is called *weakly admissible* if (2) is only pyramidal extension.

Since $\text{rank}(M_i) = \text{rk}(M_j)$ for all i, j, by Lemma 5.2.6 we have, $\text{Int}(M_i) = \text{Int}\ (M_j)$ for all i, j.

Now we will try to show that M in the main theorem has an admissible sequence such that M_n is free and $M_n \subset M^*$.

Lemma 6.3.2. If M is an affine monoid and has no non-trivial units, then there is a free monoid $F \subset M^*$ with $\text{gp}(F) = \text{gp}(M) = G$.

Proof. We first claim that:

If $z \in Int(M)$ and $u \in gp(M)$, then for some m, $z^m u \in M$ and hence $z^{m+1}u \in Int(M)$.

Proof of the claim: If $u = x/y$, then take $z^m = yw$, then $z^m u = wx \in M$.

Using this, if G has free generators $t_1, \dots, t_n$, then for some common r, $x_i = z^r t_i \in \mathrm{Int}(M)$. Let $x_0 = z^r$ and N be a submonoid of M generated by $\langle x_0, \dots, x_n \rangle$. Now it easily follows that $\mathrm{gp}(N) = \mathrm{gp}(M)$ (because of x_0, we can invert z^r and hence show that t_i's $\in \mathrm{gp}(N)$). Since $\mathrm{rk}(N) = d$, all the relation between x_i's (since i runs from 0 to d i.e. it has $d+1$ elements), will be of the form

$$a_0 x_0 + a_1 x_1 + \cdots + a_s x_s = a_{s+1} x_{s+1} + \cdots + a_d x_d$$

where all $a_i \geq 0$. If $a_0, a_1 > 0$, and $a_0 \leq a_1$, replace x_0 by $x_0 x_1$ getting a new relation

$$a_0 x_0' + a_1' x_1 + \cdots + a_s x_s = a_{s+1} x_{s+1} + \cdots + a_d x_d$$

where $a_1' = a_1 - a_0$. This decreases the sum of the a_i. Repeating the process (without disturbing the group it generates), we eventually replace $x_0, \dots, x_d$ with $y_0, \dots, y_d$ which generate $\mathrm{gp}(M)$ and satisfy a relation of the form $a y_0 = b y_d$. The normalisation N' of $\langle y_0, y_d \rangle$ is isomorphic to $\mathbb{N}$ (because normal rank 1 monoid is isomorphic to $\mathbb{N}$) and lies in M^*. Let w be the generator of N'. Then $F_1 = \langle y_1, \dots, y_{d-1} \rangle$ is a free monoid as y_i's have no relation. Hence $\langle w, F_1 \rangle$ is also a free monoid with the required properties because there is no relation among generators. Note that $\mathrm{gp}(N) \subset \mathrm{gp}(F) \subset \mathrm{gp}(M)$ so $\mathrm{gp}(F) = \mathrm{gp}(M)$. $\square$

Proposition 6.3.3. Let M be torsionfree, cancellative, finitely generated, normal monoid with no non-trivial units. Let $F \subset M^*$ be a free monoid with $\mathrm{gp}(M) = \mathrm{gp}(F)$. Then there is an admissible sequence $M = M_0, \dots, M_n = F$.

To prove this proposition, we will require the following result. But before that we shall discuss the following consequences of this proposition:

Corollary 6.3.4. Let M be a finitely generated, normal, affine monoid with no non-trivial units. Then there is a weakly admissible sequence $M = M_0, \dots, F, \dots, M_n = \{1\}$.

Proof. We already have sequence till F due to previous proposition. Now we claim that: *there exists a sequence of monoid generated by removing one free basis of F at a time*

So let $x_1, \dots, x_n$ be generators of F and let $\delta : F \to \mathbb{Z}$ by $\delta(x_1) = 1$ and $\delta(x_i) = 0$ for $i \neq 1$. This would show that $\langle x_2, \dots, x_n \rangle$ is pyramidal over F and we can continue this process to reach 1.

□

Lemma 6.3.5. Let M be finitely generated, normal affine monoid with no non-trivial units. Let E be an extremal monoid of M. Let $N \subset M$ be finitely generated and normal in M, and let $N \cap E \subset E' \subset E$ with E' finitely generated and normal in E. Then there is a finitely generated submonoid N', normal in M, with $N \subset N' \subset M$ and $N' \cap E = E'$.

Proof. Let us take N' to be normalisation of $\langle N, E' \rangle$ in M and verify the above claim. Clearly, $E' \subset N \cap E$. Let $x \in N' \cup E$, then for some n, $x^n = yz$, where $y, z \in N$ and E' respectively. Now $x \in E$ and since E is extremal implies $y \in E$ and hence $y \in N \cup E \subset E'$. This shows $x^n \in E'$. Since E' is normal, we have $x \in E'$ and hence the proof. □

Lemma 6.3.6. Let M be finitely generated, normal affine monoid, with no non-trivial units. Let $E \neq M$ be an extremal monoid. Let $E' \subset E$ be a pyramidal extension, then there is a non-degenerate pyramidal extension $N \subset M$ with $N \cap E = E'$.

Proof. Since $E' \subset E$ is pyramidal extension, we have $E' = \{x \in E | \delta(x) \leq 0\}$ and $\delta(v) > 0$ for some $v \in E$ and E integral over $\langle v, E' \rangle$. By Proposition 4.4.3, we can extend this to $\delta' : M \to \mathbb{Z}$. Since E is extremal, we have $\phi : M \to \mathbb{N}$ with $E = \phi^{-1}(0)$. Now define $\delta_k : M \to \mathbb{Z}$ as $\delta_k(x) = \delta'(x) - k\phi(x)$. Hence δ_k also extends E to M. Let T be the generator of M. If k is large enough, then $\delta_k < 0$ for all $T - E$. Hence $N = \{x \in M | \delta_k(x) \leq 0\}$ is the required submonoid which give pyramidal extension. To see it satisfies the integral property, observe that if $x \in M$, then $x = yz$, for $y \in E$ and and $z \in T - E$. Now let $z^m = v^a w$ with $w \in E' \subset N$ (because E' integral over E), hence $x^m = v^a w y^m \in \langle v, N \rangle$. Finally $T - E \neq \emptyset$ as $E \neq M$, hence there exists $u \in T - E$ such that $\delta_k(u) < 0$ which proves the extension is non-degenerate. □

We now use the above lemma's to construct a new admissible sequence for a given weak-admissible sequence.

Corollary 6.3.7. Let M be finitely generated, affine normal monoid with no non-trivial units. Let $E \neq M$ be an extremal submonoid of M. Let $E = E_0, \ldots, E_n$ be a weakly admissible sequence. Then there is an admissible sequence $M = M_0, \ldots, M_n$ with $M_i \cap E = E_i$.

Proof. Suppose inductively, we have constructed $M = M_0, \ldots, M_i$. Now if $E_{i+1} \supset E_i$, then Lemma 6.3.5 gives us the required $M_{i+1} \supset M_i$. Now suppose $E_{i+1} \subset E_i$ is a pyramidal extension, since $M_i \neq E_i$ (because $\mathrm{gp}(M_i) = \mathrm{gp}(M)$ but $\mathrm{gp}(E_i) = \mathrm{gp}(E)$), and $E_i = M \cap E_i$ is extremal in M_i, we can now apply Lemma 6.3.6 to $E_i \subset M_i$, to have a non-degenerate pyramidal extension M_{i+1}. □

Corollary 6.3.8. Let M be a torsionfree, cancellative, finitely generated, normal, with no non-trivial units. Let $E_1, \ldots, E_m$ be proper extremal submonoid of M. Assume Proposition 6.3.3 holds for monoids of rank less than that of M, then there is an admissible sequence $M = M_0, \ldots, M_n$ with $M \cap E_i = \{1\}$ for all i.

Proof. We use induction on m . Since Proposition 6.3.3 holds for E_1, so does the corollary6.3.4, hence by corollary 6.3.7 there is an admissible sequence $M = M_0, \ldots, M_k$ with $M_k \cap E_1 = \{1\}$. By induction hypothesis on M_k and extremal submonoids $M_k \cap E_i$ for $i \geq 2$, so we can extend it. □

In lieu of Corollary 6.3.8 if we take all extremal submonoids, it would follows that we have an admissible sequence $M = M_0, \ldots, M_n$ with $M_n \subset M^*$. Using Lemma 5.2.4, we know M_n will lie in some homothetic submonoid $M^{(m)}$. Here we choose our centre z to lie in $\mathrm{Int}(F)$, which in turn would lie in $\mathrm{Int}(M)$ be Lemma 5.2.6. Hence it follow that there is an admissible sequence from M to M^m by setting $M_{n+1} = M^{(m)}$ (we apply the condition (1) of 6.3.1).

Since as construction θ_m is injective by Lemma 5.2.3, applying Lemma 6.2.2 on the sequence

$$\theta_m M = \theta_m M_0, \ldots, \theta_m M_n$$

gives an admissible sequence $M^{(m)}, \ldots, M_n^{(m)}$ where $M_n^{(m)} \subset (M^{(m)})^{(m)}$. Hence combining this sequence with the original sequence we get a new admissible sequence from M to $(M^{(m)})^m$. Now for any homomorphism θ, $\theta(\widetilde{M})$ is integral over $\theta(M)$, hence normalisation of $\theta(\widetilde{M})$ is same as that of $\theta(M)$. Therefore ${M^{(m)}}^{(m)}$ is the normalisation of ${\theta_m}^2(M)$. Now we repeat this process till normalisation of ${\theta_m}^s(M)$. By Lemma 5.2.8, ${\theta_m}^s(M)$ lies in some F^* for some large s and hence its normalisation too lies in that F^*. Choose the next element of the sequence to be F. Hence we have an admissible sequence from M to F.

6.4 Final proof

To prove the final theorem we need the following lemma.

Lemma 6.4.1. Let M has an admissible sequence $M_0, \dots M_n$, Then

$$\mathcal{P}(R[{M^*}_n]) \to \mathcal{P}(R[M^*])$$

is onto.

Proof. We prove it by induction of i, i.e. $\mathcal{P}(R[{M^*}_i]) \to \mathcal{P}(R[M^*])$. If case (1) of 6.3.1 happens and if $Q \in \mathcal{P}(R[{M^*}_i)]$, then construct Q_{i+1} as $Q_i \otimes R[M^*_{i+1}] \in \mathcal{P}(R[{M^*}_i)]$. This is clearly projective and by induction the map $\mathcal{P}(R[{M^*}_{i+1}]) \to \mathcal{P}(R[M^*))$ is onto. In case (2) of 6.3.1, we use Proposition 6.2.4 to conclude the result. $\square$

Main proof

If $M_n \subset M^*$, then $\mathcal{P}(R[{M^*}_n]) \to \mathcal{P}(R[M^*])$ is onto. Since M_n is free and $M_n \subset M^*$, we have $R[{M^*}_n] \subset R[M_n] \subset R[M^*]$ and since $R[M^*]$ in extended from $R[{M^*}_n]$ it is extended from $R[M_n]$. Since M_n is free, we have $\mathcal{P}(R[M_n]) = \mathbb{Z}$ and therefore $\mathcal{P}(R[M^*]) = \mathbb{Z}$. This completes the final algebraic proof of Gubeladze's theorem due to R.G. Swan (cf. [15]).

Chapter 7

Converese and application of Gubeladze's theorem

In previous chapter we have stated and discussed the Anderson's conjecture. It turns, it admits a converse as follows:

7.1 Converse of Anderson's conjecture

Theorem 7.1.1. Let M be a cancellative monoid. If $\mathrm{Pic}(R[M]) = 1$ for every PID R of characteristics 0, then M is seminormal.

Proposition 7.1.2. Let G be an abelian group. Then the ring $R[G]$ is seminormal for every PID R of characteristics 0 if and only if G is torsionfree.

Using Proposition 7.1.2 we give a proof of 7.1.1.

Proof. From [4] (lemma 8.1) we know that $R(X)$ is a PID if R is so. Since every projective $R[X]$-module P is free, its extension onto $R(X)$ is also free and hence $\mathrm{Pic}(R[X]) \to \mathrm{Pic}(R(X))$ is injective. Hence it follows that

$$\mathrm{Pic}\ (R[M][X]) = \mathrm{Pic}\ (R[X][M]) \to \mathrm{Pic}\ (R(X)[M]) \to \mathrm{Pic}\ (R[M](X)) = 0.$$

Since $R(X)$ is a PID we have $\mathrm{Pic}(R[M]) = \mathrm{Pic}(R(X)[M]) = 0$ and hence $\mathrm{Pic}(R[M][X]) = 0$. By ([4]-theorem 4.74), we have $R[M]$ to be seminormal and hence $R[\,\mathrm{gp}(M)]$ is seminormal. Hence $\mathrm{gp}(M)$ is torsionfree and therefore M is torsionfree as desired in the theorem. $\square$

To prove Proposition 7.1.2, we need the following lemmas:

Lemma 7.1.3. Let R be a commutative ring and G be an abelian group. Let T be the torsion subgroup of G. Then if $R[G]$ is seminormal so is $R[T]$ is seminormal.

Proof. Using the structure theorem of G, we have $G = T \oplus F$, where T and F are torsion and free part respectively. Let $z^2,\ z^3 \in R[T],\ z \in R[G]$. Since there exists a natural homomorphism between $G \xrightarrow{\pi} T$, we assume $w = \pi(z)$. Then w satisfies $w^2 = z^2, w^3 = z^3$ and hence $R[T]$ is seminormal. $\square$

Lemma 7.1.4. Let T be an abelian torsion group with infinite number of distinct residue characteristics. Let H be a finite subgroup of T of order n. Then $R[H]$ is seminormal if $R[T]$ is seminormal.

Proof. Let $H \subset T$ and $z^2,\ z^3 \in R[H],\ z \in R[T]$ and $|H| = n$. Consider $H' = \langle H, z\rangle$ and choose a maximal ideal $\mathfrak{m}$ such that $R/\mathfrak{m}R = k$ has characteristics co-prime to n. Using Maschke's theorem (see [10]), we have $k[H]$ and $k[H']$ as a reduced ring and hence they are seminormal. Now consider the diagram

$$\begin{array}{ccc} R[H] & \longrightarrow & R[H'] \\ \big\downarrow & & \big\downarrow \\ k[H] & \longrightarrow & k[H'] \end{array} \tag{7.1}$$

Let $\overline{z}^2$ and $\overline{z}^3$ be the image of $z^2, z^3 \in k[H]$. Hence $z \in R[H] + \mathfrak{m}R[H']$. Now $R[H'] = R[H] \oplus F$, where $F = H' - H$. Hence $z \in R[H] + \mathfrak{m}F$. Since we have infinite supply of $\mathfrak{m}$ hence $z \in \cap_{\mathfrak{m}}(R[H] \oplus \mathfrak{m}F)$ and because F is finite set, we have $\cap \mathfrak{m}F = \emptyset$ or $z \in \cap_{\mathfrak{m}}(R[H] \oplus 0 = R[H])$. Hence $R[H]$ is seminormal. $\square$

Lemma 7.1.5. Let R be a domain of characteristics 0. Let H be a finite group of order n such that $R[H]$ is seminormal and $n \notin \mathrm{U}(R)$, then R/nR is a reduced ring.

Proof. cf. [14] $\square$

Now we give the proof of Proposition 7.2.2

Proof. Let G be a group with non-zero torsion subgroup T such that $R[G]$ is seminormal. Choose H to be subgroup be of order n and consider the ring of integers $A = \mathbb{Q}[\sqrt{n}]$. Let $R = A[s^{-1}]$ be a PID with $s \in A$ prime to n. Hence $R[T]$ is seminormal, but R/nR is not reduced, hence $R[H]$ is not seminormal, a contradiction to Lemma 7.1.4. Hence G is a torsion‘free group. $\square$

7.2 A question of Hartmut Lindel

As an application of Gubeladze's theorem, we can now answer a question raised by H.Lindel:

Proposition 7.2.1. Let R be a PID and M be a monoid generated by X_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$ following the relation: $X_{ij}X_{kl} = X_{il}X_{kj}$. Then all finitely generated projective module over $R[M]$ is free.

It follows that:

$$R[M] \simeq R[X_1, \ldots, X_{m+n}]/ \sim .$$

If we could prove that M is finitely generated, seminormal and affine monoid, then by Gubeladze's theorem, we could conclude the Proposition 7.2.1. We already know that M is finitely generated by X_{ij}. We will show that M is a submonoid of $\mathbb{Z}^{m+n}$. This submonoid will turn out to be normal (and hence seminormal).

Proposition 7.2.2. Their exists an isomorphism $\psi : M \to N \subset \mathbb{Z}^{m+n}$ such that

$$N := \{(r_1, \ldots, r_m, c_1, \ldots, c_n)\}, \text{ where } r_i, c_j \geq 0 \text{ and } \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j.$$

Corollary 7.2.3. N is a normal monoid.

Proof. Observe that $\mathrm{gp}(N) = \{(r_1, \ldots, r_m, c_1, \ldots, c_n)\}$, where $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$.
Let $x \in \mathrm{gp}(N)$. If $nx \in N$ implies $(nr_1, \ldots, nr_m, nc_1, \ldots nc_n) \in N$ implies $nr_i,\ nc_j \geq 0$ implies $r_i,\ c_j \geq 0$ and hence $x \in N$ therefore N is a normal monoid. $\square$

Now we give a proof of Proposition 7.2.2

Proof. Let $\mathcal{M}$ be a monoid of all $m \times n$ matrices with entries in $\mathbb{N}$ under addition. Let $\nu : \mathcal{M} \to M$ be given by $e_{ij} \mapsto X_{ij}$ where e_{ij} is a unit matrix and $\psi : M \to \mathbb{Z}^{m+n}$ is given by $X_{ij} \mapsto e_i + f_j$. Then we have the following map:

$$\mathcal{M} \xrightarrow{\nu} M \xrightarrow{\phi} \mathbb{Z}^{m+n} \text{ where } e_{ij} \mapsto x_{ij} \mapsto e_i + f_j.$$

1. $\psi \circ \nu$ is surjective.

   If $m = n = 1$. Then inverse image of $(r_1, c_1),\ r_1 = c_1$ is the matrix

   $$\begin{bmatrix} r_1 \end{bmatrix}$$

   We now use induction on m and n. Let $(r_1, \ldots, r_m, c_1, \ldots r_m)$ be a given element of $\mathbb{Z}^{m+n}$. Consider the element $(r_1 - c_1, \ldots, r_m, c_2, \ldots, c_n) \in \mathbb{Z}^{m+n-1}$. By induction their exists a matrix A' as a pre-image. Now consider the matrix

   $$A = \left(\begin{array}{c|c} c_1 & \\ 0 & \\ \vdots & A' \\ 0 & \end{array}\right)$$

   then $\nu(A) = c_1 x_{11} + \nu(A')$ and hence

   $$\psi(c_1 x_{11} + \nu(A')) = \psi(c_1 x_{11}) + (r_1 - c_1, \ldots, r_m, 0, c_2, \ldots, c_n) = (r_1, \ldots, r_m, c_1, c_2, \ldots, c_n).$$

   Hence A is the required map.

2. The map ν is injective.

   If define equivalence relation in $\mathcal{M}$ as follows:

   $$A \sim B \text{ if and only if } A = B + e_{ij} + e_{pq} - e_{iq} - e_{pj} \text{ or } A = B + \epsilon(i, j, p, q).$$

(for $c_{pj}, c_{iq} > 0$). The motivation for such choice of equivalence relation is $\psi(A) = \psi(B)$ or $(\mathcal{M}/\sim) \simeq M$. To show that ν is injective, we will show that two matrices have same image under $\psi \circ \nu$ if and only if they are similar. Let us choose a matrix A in the equivalence class (A) as representation whose (a_{11}) is maximum in its class. Let $(r_1, \dots r_m, c_1, \dots, c_n) \in \mathbb{Z}^{m+n}$. Since $\psi \circ \nu$ s surjective, we have its pre-images $\{A_i\}$. We will say A_i's are similar. For simplicity let us consider two pre-images A and B. Now both (a_{11}) and (b_{11}) are bounded by r_1. If a_{11} is maximal for A, and $a_{i1} \neq 0$ for some i, then $A + \epsilon(1, 1, i, j)$ will exceed A at $(1, 1)$ term, unless $a_{1j} = 0$ for all $j > 1$. Similarly for $a_{1j} \neq 0$ and B. Hence the following happens:

(a) $a_{i1} = 0$ for all $i > 1$. This would imply $r_1 \geq c_1$.

(b) $a_{1j} = 0$ for all $j > 1$. This would imply $r_1 \leq c_1$.

Using the above observation we have

$$A = \left(\begin{array}{c|c} a_{11} & \\ 0 & \\ \vdots & A' \\ 0 & \end{array}\right), \quad B = \left(\begin{array}{c|c} b_{11} & \\ 0 & \\ \vdots & B' \\ 0 & \end{array}\right)$$

Now observe that first $a_{11} = b_{11}$, since no other term from column contributes and their images are equal. Secondly note that by induction we can assume $A' \sim B'$. Since the $\epsilon(i, j, p, q)$ involved doesn't contain the 1 column, we can repeat the same transformation to move from A to B i.e. $A \simeq B$. Repeating this process we will have $A_i \simeq A_j$ and hence ν is injective.

□

Chapter 8

Appendix

In this chapter we discuss about LPA and convex geometry of monoids

8.1 Leavitt path algebras

We first star with Leavitt path algebras

Leavitt path algebras

Definition 8.1.1 (IBN)**.** A ring R is to said to have *IBN* property if $R^m = R^n$ implies $m = n$.

Rings which have IBN property are used quite often and most of rings we encounter have these properties. Some of them are:

Lemma 8.1.2. The following rings have IBN property:

1. R is a commutative ring.
2. R is a local ring.
3. $R \neq 0$ is a noetherian ring.

Proof. 1. Let $R^m \simeq R^n$. Let $\mathfrak{m}$ be a maximal ideal of R. Then $R/\mathfrak{m} = k$ is a field. Using the fact that direct sum commutes with tensor product, we have

$$k^m = (R/\mathfrak{m})^m \simeq (R/\mathfrak{m})^n \simeq k^n.$$

Since isomorphic vector spaces have same dimension, we have $m = n$.

2. Let $R^m \simeq R^n$. If $(R, \mathfrak{m})$ be a local ring, then $R/\mathfrak{m}$ is a division ring D, and since division rings have IBN property, we use the same trick as above to have $D^m \simeq D^n$ and hence $m = n$.

3. Let $\pi : R^m \to R^n$ $(m \geq n)$ be the natural projection. If $R^m \simeq R^n$ under f, then $f \circ \pi : R^m \to R^m$ is a surjective endomorphism, and since R is noetherian Using the fact that surjective Endomorphism is bijective for noetherian rings, $\ker(f \circ \pi) = 0$. This implies $\ker(\pi) = 0$ and hence $m = n$.

$\square$

The above classification may suggest that there might exist rings which are does not have IBN property. Indeed, there are rings which does not have IBN property as the next lemma shows.

Lemma 8.1.3. Let V be a infinite dimension vector space over K, Then the ring B =End(V) is a ring which doesn't have IBN property.

Proof. Let B =End$(\bigoplus_i^\infty e_i K)$. Then $B = \langle f_1, f_2 \rangle$ generate B where $f_1(e_i) = e_{2i}$ and $f_2(e_i) = e_{2i-1}$. Now we will show that f_1, f_2 are free-generators of B and hence $B^2 \simeq B$ will follow. $\square$

In Lemma 8.1.3 we have observed that $B \simeq B^2$ and hence it follows that

$$B^m = B^n \text{ for all } m, n.$$

But this conclusion is not always true for a ring which does not have IBN property. Hence we have the following definition:

Definition 8.1.4. Let R be a ring which does not have IBN property with $m \in \mathbb{N}$ being the minimum for which $R^m \simeq R^{m'}$. For this m, let n denote the minimal such m'. Then R is have a module type (m, n). In Lemma 8.1.3, B is a $(1, 2)$ module type.

For the rest of the appendix, we will be concerned about the properties and structure of rings of module type $(1, n)$. To generate ring of type $(1, n)$, we first analyse the isomorphism $R \simeq R^n$. If $R \simeq R^n$ implies their exists $\phi \in$Hom(R, R^n) and $\psi \in$Hom(R^n, R) such that $\phi \circ \psi = 1_{R^n}$ and $\psi \circ \phi = 1_R$. Writing in term of row and column vectors we have :

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ or } \sum_i^n x_i y_i = 1 \tag{8.1}$$

and,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = (1)_{R^n} \text{ or } y_j x_i = \delta_{ij} 1_R \text{ (for all } 1 \leq i, j \leq n) \tag{8.2}$$

$\sum_1^n x_i y_i = 1_R$ and $Y_j X_i = \delta_{ij} 1_R$ (for all $1 \leq i, j \leq n$). Motivated by above observation we define a free associative K-algebra

$$S = K(X_1, X_2, \dots X_n, Y_1, \dots Y_n),$$

where K is a field and an ideal

$$I = \langle \sum_1^n X_i Y_i - 1, Y_j X_i - \delta_{ij} 1 \rangle.$$

It now follows that $A = S/I$ is a ring of module type $(1, n)$. Such a ring is also denoted as $L_k(1, n)$.

Definition 8.1.5. Let K be any field, and $n > 1$, then the *Leavitt K-algebra of type* $(1, n)$ denoted as $L_k(1, n)$, is the K-algebra

$$S = K(X_1, X_2, \dots X_n, Y_1, \dots Y_n) / \langle \sum_1^n X_i Y_i - 1, Y_j X_i - \delta_{ij} 1 \rangle.$$

Leavitt path algberas

Motivated by the example of $L_k(1,n)$ we generalise it to a broader class of rings called *Leavitt path algebras* also known as LPA. To do so, we first recall directed path and path algebras.

Definition 8.1.6 (Directed graph)**.** A directed graph $E = \langle E^0, E^1, r, s\rangle$ consists of two sets E^0, E^1 along with two function $r, s : E^1 \to E^0$. The element of E^0 are called *vertices* and elements of E^1 are called *edges*.

$$
\begin{tikzcd}
& & & n \\
& b \arrow[r, "g"] & d \arrow[ur, "r"] \arrow[dr, "q"] & \\
a \arrow[loop left, "z"] \arrow[ur, "f"] \arrow[dr, "k"] & & & m \arrow[loop right, "u"] \\
& c \arrow[r, "l"] & e \arrow[ur, "s"] \arrow[dr, "t"] & o \\
& & & p \arrow[loop below, "v"]
\end{tikzcd}
\tag{8.3}
$$

Remark 8.1.7. *Given E^1, we define ghost edges $(E^1)^* = \{e^* \mid e \in E^1\}$ where $r(e^*) = s(e)$ and $s(e^*) = r(e)$.*

Remark 8.1.8. *A path μ is a sequence of edges $\mu = e_1, e_2, \ldots, e_n$ such that $r(e_i) = s(e_{i+1})$.*

Definition 8.1.9 (Path algebras)**.** Let E be an arbitrary graph and K be a field, then the *path algebra KE* is defined as the free associative K-algebra generated by the set $\langle E^0, E^1\rangle$ quotient the following relation:

(V) $vv' = \delta_{v,v'}v$.

(E_1) $\mathrm{s}(e)e = \mathrm{r}(e)e = e$.

Remark 8.1.10. *If we extend the graph to include the ghost edges in a natural way, we get another path algebra $L_K(\hat{E})$ which is the path algebra over $(\langle E^0, E^1, E^{1*}\rangle)$.*

Definition 8.1.11 (Leavitt path algebras)**.** Let E be an arbitrary graph and K be any field. Then we define *Leavitt path algbera* to a free associative K-algebra generated by the set $\langle E^0 \cup E^1 \cup E^{1^*}\rangle$, modulo the following relations:

1. (V) $vv' = \delta_{v,v'}v$.

2. (E_1) $\mathrm{s}(e)e = \mathrm{r}(e)e = e$.

3. (E_2) $\mathrm{r}(e)e^* = e^*\mathrm{s}(e) = e^*$.

4. $(CK1)$ $e^*e' = \delta_{e,e'}$ for all $e, e' \in E^1$.

5. $(CK2)$ $v = \sum_{e \in E^1 | \mathrm{s}(e)=v} ee^*$.

Remark 8.1.12. *$L_K(E)$ is a the quotient of $K\hat{E}$ under the relation $(CK1), (CK2)$.*

Intuitively the operation between symbols in first four relation is concatenation whereas the fifth relation give a weighted value to every vertex v depending upon it role as source.

Example 8.1.13 (Graphicall decription of LPA)**.** Cosider the graph E,

$$\begin{tikzcd} & \mathrm{b} \arrow[r, "\mathrm{g}"] & \mathrm{d} \\ \mathrm{a} \arrow[loop left, "\mathrm{z}"] \arrow[ur, "\mathrm{f}"] \arrow[dr, "\mathrm{k}"] & & \\ & \mathrm{c} \arrow[r, "\mathrm{l}"] & \mathrm{e} \end{tikzcd} \tag{8.4}$$

We now do some computation in $L_K(E)$, and compare it with our intuition of path concatenation and relative value of a vertex as source and emitter.

1. $ab = \delta(a,b)a = 0a = 0$ and $aa = \delta(a,a)a = a$ by (V).
   Remark 8.1.14. *Concatenation of two disjoint point is* 0 *and of same point is the point itself.*

2. $af = f = fb$ by (E_1).
   Remark 8.1.15. *Concatenation of a line and its endpoint is the line itself.*

3. $ck^\star = a = k^\star = k^\star a$ by (E_2).
   Remark 8.1.16. *Since ghost edges are also edges with opposite direction, using 8.1.15 to arrive at the same intuition.*

4. $f^\star f = \delta(f, f^\star r(f) = b$ and $f^\star k = \delta(f,k)r(f) = 0b = 0$ by $(CK1)$.
   Remark 8.1.17. *Concatenation of two path is non-zero if the $r(f) = s(e)$. Since one of the paths is a ghost-path is derived from the other path.*

5. $a = zz^\star + ff^\star + kk^\star$.

   Remark 8.1.18. *This part gives relative weightage to a vertex depending on number of source and edge it contributes to a graph.*

Three examples of LPA

As seen from 8.1.11, the ring $L_K(E)$ depends upon the structure of graph E. In this section we calculate the ring LPA for few simple graph. Before doing, that we introduce those simple graphs.

Example 8.1.19 (Rose with n-petals)**.** Let R_n denote the rose with n-petals as shown:

$$\begin{tikzcd} a \arrow[loop above, "e_n"] \arrow[loop right, "e_1"] \arrow[loop below, "e_2"] \arrow[loop left, dotted, "e_{n-1}"] \end{tikzcd} \tag{8.5}$$

Example 8.1.20 (oriented n-line graph)**.** Let A_n denote the following LPA of the following graph:

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \to v_{n-1} \xrightarrow{e_{n-1}} v_n \tag{8.6}$$

Proposition 8.1.21. Let $n \geq 2$ and K be any field. Then

$$L_K(1, n) \simeq L_K(R_n).$$

Proof. Since $L_K(E)$ is quotient of $L_k(1, n)$, all we need to verify is the the 5 relation (see 8.1.11) are trivial. Identify the elements in the following way:

$$1 \mapsto v, x_i \mapsto e_i, \text{ and } y_i \mapsto {e_i}^\star.$$

Calcualting the 5 relation (see 8.1.11):

1. $vv' = vv = 1 = \delta_{vv'}v = v = 1.$
2. $s(e_i)e_i = 1x_i = x_i 1 = e_i r(e_i) = x_i = e_i.$
3. $r(e_i){e_i}^\star = 1y_i = y_i 1 = {e_i}^\star s(e_i) = y_i.$

4. $e_i{}^\star e_j = y_j x_i = \delta_{i,j}1 = \delta_{e_i,e_j} v$.

5. $v = 1 = \sum_1^n x_i y_i = \sum e_i e_i{}^\star$.

Hence the conclusion follows. □

Corollary 8.1.22. Let K be a field, then

$$K[X, X^{-1}] \simeq L_K(R_1).$$

Proof. For $n = 1$ in Proposition 8.1.21, we have $CK1 : xy = 1$ and $CK2 : yx = 1$ and rest all relations are trivial. Hence $y = x^{-1}$ and therefore $L_K(R_1) \simeq L_K(1,1) \simeq R[x, x^{-1}]$.

□

Proposition 8.1.23. Let K be a field, and $n \geq 1$ any positive number. Then

$$M_n(K) \simeq L_K(A_n).$$

Proof. cf. [1]-Proposition 1.3.5. □

Grading of LPA

Before we will give a grading to LPA, we first introduce another path algebras called *Cohn path algebras* which lies in between KG and $L_K(E)$. More formally:

Definition 8.1.24. Let E be a directed graph and K be a field. Let $C_K(E)$ denote the free associative K-algebra generated by $\langle E^0, E^1, E^{1*}\rangle$ quotient the relation $(V), (E_1), (E_2), (CK1)$.

It follows from the above definition that $C_K(E)/CK2 = L_K(E)$.

Remark 8.1.25. *$C_K(E)$ has an IBN property as proved in [9].*

$\mathbb{Z}$-grading on some specific $L_K(E)$.

Example 8.1.26 ($\mathbb{Z}$-grading on $L_K(A_n)$:)**.** Since we have established that $F : L_K(A_n) \to M_n(K)$, is an isomorphism we instead find a grading on $M_n(K)$ and confirm that the isomorphism F is graded. Consider the subspace

$$A_t = \{A \mid (A)_{i,j} = 0, \text{ for } i - j \neq t\}.$$

Now it's clear that $M_n(K) = \bigoplus A_t$ (set $A_t = 0$ if $t > n$ or $t < 0$). The map F defined in Proposition 8.1.23 turns out to a graded homomorphism (by considering the pre-image of homogeneous component, see Proposition 8.1.30), hence we have grading on $L_K(A_n)$.

Example 8.1.27. $\mathbb{Z}$-grading on $L_K(R_1)$.

Since we have established that
$F : L_K(R_1) \simeq K[X, X^{-1}]$ is an isomorphism and since $K[x, x^{-1}$ has a natural grading, we take the pre-image of homogeneous component and verify that it induced a graded structure (see Proposition 8.1.23).

Remark 8.1.28. *Let A be a graded ideal, if $X \subseteq A_0$, then $I(X)$ is an graded ideal.*

We show that $K\hat{E}$ is a $\mathbb{Z}$-graded K-algebra and $I(CK1)$ is it's graded ideal, hence it will follow that $C_K(E)$ is a $\mathbb{Z}$ graded K-algebra. Using the same argument, we will show that $L_K(E)$ is a graded $\mathbb{Z}$-module.

Definition 8.1.29. Let E be a graph. For any $v \in E^0$, let $\deg(v) = 0$, and $e \in E^1$, $\deg(e = 1)$ and $\deg(e^\star) = -1$. For any monomial $kx_1 \dots x_m$, $\deg(kx_1 \cdots x_m) = \sum_{i=1}^m \deg(x_i)$. Let

$$A_n := \text{span}_K\{x_1 \cdots x_m \mid x_i \in E^0 \cup E^1 \cup E^{1\star} \text{with } \deg(x_1 \cdots x_m) = n\}.$$

Proposition 8.1.30.

1. $K\hat{E} = \bigoplus_{n\in\mathbb{Z}} A_n$ as a K-subspaces and this defines a $\mathbb{Z}$-grading on the path algebra $K\hat{E}$.

2. $C_K E = \bigoplus_{n\in\mathbb{Z}} C_n$, where

$$C_n := \text{ span }_K\{\gamma\lambda^\star \mid \gamma, \lambda \in E^n \text{ and } l(\gamma) - l(\lambda) = n\}.$$

   Hence this defines a $\mathbb{Z}$-grading on $C_K(E)$.

3. $L_K(E) = \bigoplus_{n\in\mathbb{Z}} L_n$, where

$$L_n := \text{ span }_K\{\gamma\lambda^\star \mid \gamma, \lambda \in E^n \text{ and } l(\gamma) - l(\lambda) = n\}.$$

   This defines a $\mathbb{Z}$-grading on $L_K(E)$.

Proof. 1. The free algebra KE, where E is $\langle E^0, E^1, E^{1\star}\rangle$ is $\mathbb{Z}$-graded where homogeneous component is given by deg (see 8.1.29. We now observe that $I(V, E_1, E_2)$ are graded ideals and from 8.1.25 and hence the quotient of free algebra with the relation $(V), (E_1), (E_2)$ is again a graded ring.

2. Using Proposition 8.1.30(1), and observing that $I(CK1)$ is a graded ideal, we infer $K\hat{E}/I(CK1)$ is a graded ring with homogeneous component being its deg.

3. Using Proposition 8.1.30(2), and observing that $I(CK2)$ is a graded ideal, we infer $C_K(E)/I(CK2)$ is a graded ring and homogeneous component being its deg.

□

Motivation for further studies

LPA is a concrete example of a non-commutative graded algebra without IBN property. As it is well known that many problems related to Quillen-Suslin theorem have been studied for graded rings and for non-commutative rings separately, it is natural to ask whether analogue results holds for LPA. For example it will be interesting to deduce analogue of Quillen's L-G principal and Suslin's K_1-analogue of Serre's conjecture for LPA.

8.2 Convex geometry and monoids

The content of this appendix is independent of the thesis. The only aim of this appendix is motivate the abstract algebraic construction through intuitive geometric pictures.The principal object of study here is the property of cone and polytopes. They are defined in terms of halfspaces ${H_\alpha}^+$. For more information see [6] and [4].

Consider a affine space A^n and map $\psi : A^n \to A$ where $\psi(x) = \lambda(x) + c$ where λ is a linear map. The the of $x \in A^n$ such that $\psi(x) \geq 0$ is denoted as ${H_\psi}^+$ and is called *closed halfspaces*. The set $\psi(x) = 0$ is called *hyperplane* and is denoted as H_ψ.

Definition 8.2.1 (Polytopes)**.** Let H_i be a hyperplane and ${H_i}^+$ be its halfspace. Then P is a *polytope* if $P = \bigcap_{i\in I} {H_i}^+$ for finitely many i and is bounded in A^n.

Example 8.2.2. Any polygon in $\mathbb{R}^2$ is a polytope.

Definition 8.2.3 (Cones)**.** If the H_i are all linear affine subspace (i.e. vector subspace), then the polytope is called a *cone*.

Example 8.2.4. A cone shaped figure in 2-dimension is a cone in $\mathbb{R}^2$.

Similar to cones, we call a set X *conical* if it is closed under nonnegative linear combination of its elements. One such set is $\mathbb{R}_+X$ (which is also the smallest conical set containing X). It becomes evident that a cone C is conical. But the converse in also true under a mild condition.

Proposition 8.2.5. Let C be a conical set in V. Then the following are equivalent:

1. C is finitely generated.

2. C is a cone

Proof. cf. ([4] (1.15)). □

Lemma 8.2.6 (Gordan's Lemma). Let C be a rational cone in $\mathbb{R}^d$, then

$$M = \mathbb{Z}^n \cap C$$

$(n \leq d)$ is an affine monoid.

Proof. Since C is a cone, it is finitely generated by $v_1, \ldots, v_n$ (we can assume $v_i \in \mathbb{Z}^d$) over $\mathbb{R}_+$. Let $m \in M$, then $m = \sum_{i=1}^n a_i v_i$, for some $a_i \in \mathbb{R}_+$. Now if we rewrite the sum as

$$m = \sum_{i=1}^n \lfloor a_i \rfloor v_i + \sum_{i=1}^n q_i v_i = m' + m''$$

where $a_i = \lfloor a_i \rfloor + q_i$ ($\lfloor a_i \rfloor$ implies the highest integer less than a_i). Since $m, m' \in M$ we have $m'' \in M$.

Let $B = \{y \mid y = q_i v_i,\ 0 \leq q_i \leq 1\}$. Then B is a bounded subset of $\mathbb{R}^d$. Now $B \cap \mathbb{Z}^d \in M$ and is finite set and $m'' \in B \cap \mathbb{Z}^d$. Therefore $\{v_1, \ldots, v_n\} \cup (B \cap \mathbb{Z}^d)$ is the generator of M and hence M is affine.

□

Corollary 8.2.7. Let M be a affine monoid, $C = \mathbb{R}_+ M$. Then

$$\widehat{M} = C \cap \mathbb{Z}^n$$

is an affine monoid and C is a cone.

Proof. C is cone because C is finitely generated and proposition Proposition 8.2.5. Now C is a cone, then by lemma Lemma 8.2.6 $\widehat{M}$ is an affine monoid.

□

Corollary 8.2.8. Let M and N be an affine monoid, then $M \cap N$ is an affine monoid.

Proof. By corrolary 8.2.8, $\mathbb{R}_+ M$ and $\mathbb{R}_+ N$ are cones, hence $\mathbb{R}_+ M \cap \mathbb{R}_+ N$ is also a rational cone (see [4], 2.11). Now if,

$$\mathbb{R}_+ M \cap \mathbb{R}_+ N = \mathbb{R}_+ (M \cap N)$$

then again using corollary 8.2.7 we conclude that $M \cap N$ is affine monoid.

Now we prove that $\mathbb{R}_+ M \cap \mathbb{R}_+ N = \mathbb{R}_+ (M \cap N)$. Clearly, $\mathbb{R}_+ M \cap \mathbb{R}_+ N \subseteq \mathbb{R}_+ (M \cap N)$ is obvious. For the converse, let x be a ration element in $\mathbb{R}_+ M \cap \mathbb{R}_+ N$. Then their exists $\alpha, \beta \in \mathbb{Z}$ such that $\alpha x \in M$ and $\beta x \in N$. Hence $\alpha\beta x \in M \cap N$, therefore $x \in \mathbb{R}_+ (M \cap N)$. □

Corollary 8.2.9. Let W be a polytope, then $M \mid W = M \cap \mathbb{R}_+ W$ is an affine monoid.

Proof. Using corollary 8.2.8 it follows that $M \mid W$ is an affine monoid. □

Remark 8.2.10. *Given an affine monoid M, we form a cone $C = \mathbb{R}_+ M$. If $\lambda \in C^\star$ (dual cone), then $\Phi(M) := H_{\lambda(x)=a} \cap C$ is a polytope and plays an important role in the structure of $R[M]$ and is called cross-section of M.*

Let P be a polytope and $x \notin \operatorname{aff}(P)$ (smallest affine subspace containing P). Then the $\operatorname{conv}(P, x)$ (smallest convex set containing P and x) is called a *pyramid* over base P and vertex v.

Definition 8.2.11 (Pyramidal Extension- Combinatorial viewpoint)**.** A polytope P have a pyramidal decomposition if $P = \Delta \cup \Gamma$ such that Δ is a pyramid with apex v and Γ meets

Δ in a facet opposite to v. If $\dim(P) = \dim(\Gamma)$ (here dim means the dimension of $\mathrm{aff}(Q)$ for a polytope Q), then the pyramidal decomposition is called non-degenerate decomposition.

We can alternatively proof Gubeladze's theorem using the next two results (Infact, R.G. Swan translated this results in algebraic terms).

Theorem 8.2.12. Let $v \in \Phi(M)$ be a vertex of $\Phi(M) = \Delta \cup \Gamma$- a nondegenerate pyramidal decomposition of $\Phi(M)$. Then every projective module over $R[M]$ is extended from $R[M \mid \Gamma]$.

Remark 8.2.13. *Compare this theorem with proposition Proposition 6.2.4.*

Proof. cf. ([4], 8.6). □

Theorem 8.2.14. Let P be a polytope and $z \in \mathrm{Int}(P)$ a rational point. Then there exists a sequence $(P_i)_{i \in \mathbb{N}}$ of polytopes with the following properties

1. For all $i \in \mathbb{N}$ we have:

    (a) $P_i \subset P$,

    (b) $P_i \subset P_{i+1}$ or $P_{i+1} \subset P_i$,

    (c) if $P_i \subset P_{i+1}$, then P_i is a nondegenerate pyramidal extension of P_{i+1},

2. For every $\epsilon > 0$ there exists an $i \in \mathbb{N}$ such that $P_i \subset U_\epsilon(z) \cap P$.

Remark 8.2.15. *Compare this theorem with proposition Proposition 6.3.3.*

Consequences of Theorem 8.2.12 and Theorem8.2.14 are as follows:

Let Q be a projective $R[M]$-module. The extension $P_1 \subset P$ is pyramidal and hence their exists a projective $R[M \mid P_1]$-module Q_1 such that Q is extended from Q. Now we recursively define $R[M \mid P_i]$-module as follows:

1. If $P_{i+1} \subset P_i$, then $Q_{i+1} = Q_i \otimes R[M \mid P_{i+1}]$.

2. If $P_i \subset P_{i+1}$ be a non-degenerate pyramidal extension, then using theorem 8.2.12 we have Q_{i+1} extended from Q_i.

Since M is noraml if follows that $\mathbb{R}_+M$ has a unimodular triangulation (cf [4], 2.74).

Choose an $\epsilon > 0$. Then there exists $j \in \mathbb{N}$ such that $P_j \subset U_\epsilon(z) \cap P$ for $z \in$ interior of $\Phi(M) \cap D$, where D is a unimodular cone of triangulation. Consider the extension $Q' = Q_k \otimes R[D \cap \mathbb{Z}^d]$. Since D is unimodular $D \cap \mathbb{Z}^d \simeq {\mathbb{Z}_+}^d$ (see [4], Section 2-D for more details) and hence Q' is a free module over $R[D \cap \mathbb{Z}^d]$ by Quillen-Suslin theorem. Since Q is extended from Q', it follows that Q is free $R[M]$-module.

8.3 Definition of Pic($R[M]$) and $K_0(R[M])$

We know discuss the K-theoretic aspect of $R[M]$ in the language of K-theory i.e K_0-group and Picard group to state the original result of J. Gubeladze [7].

Definition 8.3.1. Let R be a commutative ring and let (P) denote the isomorphism class of projective R-module P. Then the *Grothendieck group* K_0R is an additive abelian group generated by (P) under the following relation:

1. Let G be a free abelian group generated by (P).
2. Let $H :=$ subgroup generated by $(P \oplus Q) - (P) - (Q)$.
3. Let $K_0R = G/H$ and $[P]$ image of (P) in K_0R.

Example 8.3.2. Let R be commutative PID. Since every projective R-module P is free, from above we have $K_0(R) = \mathbb{Z}$.

Definition 8.3.3. Pic(R) or *Picard group* is defined as the abelian group whose elements are isomorphic class of rank 1 projective modules $[P]$ and multiplication is defined as $[P].[Q] = [P \otimes Q]$.

Example 8.3.4. Let $R[X]$ be a ring where R is a PID. Since every projective $R[X]$-module is free, we have Pic($R[X]$) $= 1$

Remark 8.3.5. *Anderson's conjecture written in the language of K-theory is essentially equivalent the following statements:*

1. *Pic*$(R[M]) = 1$*,*
2. $K_0(R[M]) = \mathbb{Z}$.
3. *Finitely generated projective $R[M]$-module is free.*

4. Monoid M is seminormal (This is actually the converse of Gubeladze's theorem).

For more information on this section, we refer [11] and [4] .

Bibliography

[1] Gene Abrams, Pere Ara, and Mercedes Siles Molina. *Leavitt path algebras*. Vol. 2191. Springer, 2017.

[2] Michael Atiyah. *Introduction to commutative algebra*. Vol. 361. Westview Press, 1994.

[3] Rabeya Basu. "On Forster's Conjecture and Related Results". In: (2002).

[4] Winfried Bruns and Joseph Gubeladze. *Polytopes, rings, and K-theory*. Vol. 27. Springer, 2009.

[5] N.S. Gopalakrishnan. *Commutative Algebra*. University Press, 2015.

[6] Branko Grünbaum et al. *Convex polytopes*. Vol. 16. Springer, 1967.

[7] I Dzh Gubeladze. "Anderson's conjecture and the maximal monoid class over which projective modules are free". In: *Mathematics of the USSR-Sbornik* 63.1 (1989), p. 165.

[8] Roozbeh Hazrat. *Graded rings and graded Grothendieck groups*. Vol. 435. Cambridge University Press, 2016.

[9] Müge Kanuni and Murad Özaydın. "Cohn-Leavitt Path Algebras and the Invariant Basis Number Property". In: *arXiv preprint arXiv:1606.07998* (2016).

[10] Tsit-Yuen Lam. *A first course in noncommutative rings*. Vol. 131. Springer Science & Business Media, 2013.

[11] Tsit-Yuen Lam. *Serre's problem on projective modules*. Springer Science & Business Media, 2010.

[12] John Milnor. *Introduction to Algebraic K-Theory.(AM-72)*. Vol. 72. Princeton University Press, 2016.

[13] Amit Roy. "Serre's Conjecture". In: *Lectures given at IISc* (1996).

[14] David E Rush. "Picard groups in abelian group rings". In: *Journal of Pure and Applied Algebra* 26.1 (1982), pp. 101–114.

[15] Richard G. Swan. "Gubeladze's proof of Anderson's conjecture". In: *Contemp. Math* 124 (1992), pp. 215–250.