Particle Vortex and Boson Fermion Dualities in 2+1 Dimensions

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by

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Certificate

This is to certify that this dissertation entitled Particle Vortex and Boson Fermion Dualities in 2+1 Dimensions towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Rugved Pund at Indian Institute of Science Education and Research under the supervision of Dr. Sunil Mukhi, Professor, Department of Physics , during the academic year 2017-2018.

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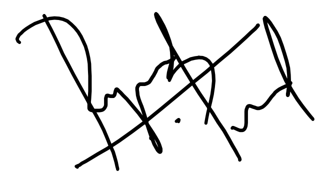
This thesis is dedicated to Mani

Declaration

I hereby declare that the matter embodied in the report entitled Particle Vortex and Boson Fermion Dualities in 2+1 Dimensions are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Sunil Mukhi and the same has not been submitted elsewhere for any other degree.



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Abstract

We motivate the particle/vortex duality and discuss the salient features of Chern-Simons matter theories and use the ideas to help build the boson/fermion dualities in 2+1D quantum field theory. The dynamics of the particle and vortex excitations, phases from renormalization group flows are discussed and shown to be identical on both sides of the duality. Chern-Simons theories and their novel physics in 2+1D is explored to bring about a change in statistics of particles. Using these ideas and after computing the quantum anomalies in 2+1D fermion theories, we propose the statements of abelian and non-abelian bosonization. We motivate the dual physics of the theories by comparing global symmetries, phases from RG flows and gauge-invariant operators on both sides of the duality. Finally, we introduce techniques to manipulate and generalize these dualities to non-abelian quiver theories.

Contents

Introduction

The concept of dualities in physics is surprising and, hence, interesting. To be able to see dualities manifestly and exploit them to understand theories through different perspectives opens up novel physics insights. Dualities like the AdS/CFT in HEP, particle-vortex on the lattice, level/rank in Chern-Simons theories were well established by themselves in the past decades. In recent years however, they have begun to connect in simple but precise ways to form a web that invites curiosity and provides surprisingly non-trivial results. In general, such dual physics derives motivation from previously disconnected fields and introduces new conceptual understanding. Efforts from all possible directions like condensed matter, string theory, supersymmetry, gravity etc. have helped us find different ways to justify and understand these connections fundamentally and manifestly.

In this work, we plan to explore the motivation and central ideas behind establishing such dualities and the duality web, and use those ideas to exploit and extend the web. We shall closely study the particle-vortex duality, the abelian bosonization duality and then finally the non-abelian bosonization duality. Our focus would be on looking at the symmetries, the Renormalization Group flows and the dual degrees of freedom to sufficiently motivate the duality and the underlying dual physics.

The precise statement of the dualities involves a full understanding of the ungauged global symmetries on both sides, renormalization group flows of the theories involved, Chern-Simons terms and their role in flux attachment, quantization of the Chern-Simons levels from gauge invariance, and finally the exact way to couple the global currents to background fields.

The outline of this document is as follows. We shall first see how some simple established dualities in quantum field theories can be derived. Then, we shall introduce some dualities which aren't as easy to derive but can be motivated sufficiently so that we believe them to be true. This shall help us understand the general ways in which one derives confidence in postulating further dualities. We shall then study the particle-vortex duality closely by looking at the individual theories and seeing the dual physics emerge by looking at very similar dynamics that the two theories exhibit. Proceeding along the same lines, we will then propose the abelian bosonization duality and motivate why it should be true. We shall briefly show how the abelian bosonization can be used to derive the particle-vortex duality and a bunch of other well established dualities. Finally we will see the non-abelian version of the bosonization duality and use it to establish dualities between quiver theories.

It shall be helpful to review the interesting physics that Chern-Simons terms introduce in QFTs and how that leads to the phenomenon of flux attachment and statistical transmutation. We shall go through the surprisingly tricky derivation of how the coupling constant of the CS terms is forced to be integer-valued mod 4π after demanding gauge invariance. The full answer involves a detour into Cech cohomology and shall be deferred to the Appendix but the central idea would be presented. Chern-Simons terms that are introduced as a quantum anomaly will also be discussed. Our review of the particle-vortex duality will be aimed at understanding the physics of the full second-quantized quantum field theories and their topological excitations. For convenience, we will also briefly review the concept of topological excitations in QFTs.

Given the lack of secondary literature for most of these topics, we hope this thesis provides a balance between the pedagogical and technical aspects of the field.

Chapter 1

Dualities in Quantum Field Theory

Dualities are ubiquitous in physics. The notion of dualities is primarily useful in helping solve difficult problems by casting them in terms of easier ones. These 'dual' problems are usually easier to solve in the sense that they may involve casting strongly coupled systems in terms of a weakly coupled system amenable to a perturbative expansion, or casting a complicated non-local object in terms of simple local variables, etc.

No matter the way in which dualities are used, the existence of dualities itself is enlightening for a physicist. By looking at the same system using different descriptions, one develops a perspective that can be useful in much wider contexts, possibly in a completely different discipline. Dualities between QFT and gravity, condensed matter and QFT, condensed matter and gravity have proven this and have helped motivate solutions to complex problems in the field.

It is in this context that the recent web [2] of dualities in 2+1 dimensions is important. Chern-Simons theories and matter theories in 2+1D have a neat web of dual descriptions that involve their topological, highly non-local excitations. In the low energy limit, these theories can be analyzed with many techniques and all of those have supported the dualities so far.

Interestingly, the idea of topological excitations behaving like particles is an old one. The 't-Hooft-Polyakov monopole is an example of such a construction in the Georgi-Glashow model. In fact, the magnetic charge of the 't-Hooft monopole being inversely related to the electric charge suggests a clear strong/weak feature of the classic electric-magnetic duality in 4D.

In 2+1 dimensions, Maxwell electromagnetism is dual to a compact scalar theory. This is a baby duality that shall help highlight important features of dualities in general and shall also be useful for building intuition for later.

Let's start by taking the lagrangian

$$\mathcal{L}_{EM} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \tag{1.1}$$

and introducing the dual tensor $\tilde{F}^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda = \frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\nu\lambda}$.

$$\mathcal{L}_{EM} = -\frac{1}{2g^2} \tilde{F}_\mu \tilde{F}^\mu \tag{1.2}$$

Now introduce an auxiliary field B_μ as

$$\mathcal{L}_{EM} = \frac{g^2}{4} B_\mu B^\mu - \frac{1}{2} B_\mu \tilde{F}^\mu \tag{1.3}$$

and integrate out the A_μ using the equation of motion

$$\epsilon^{\mu\nu\lambda}\partial_\nu B_\lambda = 0 \tag{1.4}$$

This implies $B_\mu = \partial_\mu \varphi$ for some real scalar φ, i.e.

$$\mathcal{L}_{EM} = \frac{g^2}{2} \partial_\mu \varphi \, \partial^\mu \varphi \tag{1.5}$$

where φ is called the dual photon. With this we have a duality between a scalar field φ and a vector field A_μ in 2+1D with the correspondence

$$\partial_\mu \varphi = \frac{1}{g^2} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda \tag{1.6}$$

Since the gauge group $U(1)$ is compact, we are allowed non-trivial gauge transformation also known as large gauge transformations. Under such a large gauge transformation, say

$e^{i2\pi\alpha(x)}$, A transforms as $A \to A - d\alpha$ and $\alpha(x)$ winds around an integer number of times,

$$\oint dx^\mu \, \partial_\mu \alpha \in \mathbb{Z} \tag{1.7}$$

Thus, the integral

$$\int dx^\mu \, \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda \to \int dx^\mu \, \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda + \int dx^\mu \, \epsilon_{\mu\nu\lambda} \partial^\nu \partial^\lambda \alpha(x) \tag{1.8}$$

where the last term is equal to $\oint dx^\mu \, \partial_\mu \alpha$ and thus an integer. Thus, under the transformation $\int dx^\mu \, \partial_\mu \varphi$ shift by the integer. This is possible only if the dual photon φ is compact with a radius of 2π.

The Maxwell photon has a non-trivial topological $U(1)$ 'magnetic' symmetry with the current

$$j^\mu_M = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda \tag{1.9}$$

which is conserved because of the Bianchi identity of the field tensor. This is the symmetry under which topological vortices are charged, as we shall soon see. The dual photon, on the other hand, has the conserved current

$$\tilde{j}^\mu = \frac{1}{2\pi} \partial_\mu \varphi \tag{1.10}$$

which is the shift symmetry of the dual photon $\varphi \to \varphi +$ constant. Since φ is compact with radius 2π, this symmetry is only defined modulo 2π and thus corresponds to a $U(1)$ group.

Another thing to notice is that the coupling constant for the gauge field A is inverse of the one for the scalar φ. If these fields were coupled to some other matter fields, then this property would have been useful in calculating quantities in a perturbative expansion. Also, the dual photon is manifestly gauge-invariant and summarizes the dynamics of the Maxwell photon in a non-local way via the relation $\partial_\mu \varphi = \frac{1}{g^2} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda$.

The baby duality is an example of a strong/weak duality as well as local/non-local duality. We shall see this theme recurring in our discussions. However, our main focus for this work will be to understand existing dualities and establish new ones. We shall highlight the dual quantities on both sides and leave the calculations that exploit these features to solve real problems for a later time.

Chapter 2

Particle Vortex Duality

The particle vortex duality has gathered a lot of evidence over the years after it was proposed by Dasgupta and Halperin from many different communities and is now considered one of the most well-established dualities in 2+1 dimensions. Let us discuss the two dual theories and motivate their equivalence.

We shall work with the following two lagrangians, which shall be referred to as the $\lambda\phi^4$ model and the Abelian-Higgs model.

$$\mathcal{L}_1 = |\partial_\mu \phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4 \tag{2.1}$$

$$\mathcal{L}_2 = |(\partial_\mu - iA_\mu)\Phi|^2 - M^2|\Phi|^2 - \Lambda|\Phi|^4 - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} \tag{2.2}$$

Note that we are working in $2+1D$ and hence both the actions have an integral over d^3x. This implies the canonical dimensions of ϕ, Φ are $\frac{1}{2}$, of the gauge field A_μ is 1. The parameters of the theories, as defined, imply that m, $\tilde{m}$ have dimension 1 as usual, λ has dimension 1 and g has dimension $\frac{1}{2}$.

Let's now develop the $\lambda\phi^4$ theory and go over certain key aspects in the context of the particle vortex duality. We shall repeat the exercise for the Abelian-Higgs model and then finally discuss the similarities we should identify as dual physics under the duality.

2.1 $\lambda\phi^4$ model

$$\mathcal{L}_1 = |\partial_\mu \phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4 \tag{2.3}$$

This is a theory of the complex scalar field in $2+1D$ with a global $U(1)$ symmetry and a ϕ^4 interaction vertex. The parameter m identifies the mass of the quanta of the theory, when m^2 is greater than 0. In this case, we have two massive, propagating, field-theoretic degrees of freedom of the ϕ and $\phi^\dagger$ particles that interact via the ϕ^4 interaction. The theory has a conserved current J_μ under the $U(1)$ symmetry, given by $J_\mu = i\phi^\dagger \overleftrightarrow{\partial_\mu} \phi$ and satisfying $\partial_\mu J^\mu = 0$. The conserved charge inside an area S

$$Q = \int_S d^2x \; J^0 \tag{2.4}$$

is the total number of particles minus antiparticles inside S.

The ϕ^4 interaction can spontaneously give rise to a propagating $\phi - \phi^\dagger$ pair. The probability of this happening is governed by the parameter λ which describes the strength of this interaction. In other words, the scattering amplitudes for particles of the theory are all proportional to powers of λ. To first order in λ, if we look at the scattering of a ϕ particle with a $\phi^\dagger$ at rest, one can show that we get an effective delta potential interaction as long as the speeds involved are non-relativistic. Thus, the particles ϕ,$\phi^\dagger$ essentially move like free particles unless they directly encounter another particle, or spontaneously give a $\phi - \phi^\dagger$ pair while propagating.

Now let's suppose the other case, when m^2 is less than 0. In this case, we find it difficult to directly identify m as the mass parameter, since a particle at rest would seemingly have imaginary mass. This has happened because the classical vacuum of the theory no longer corresponds to $|\phi(x)| = 0$, but to the non-zero $|\phi(x)| = \phi_0(m, \lambda) = \sqrt{\frac{-m^2}{\lambda}} = \sqrt{\frac{\bar{m}^2}{\lambda}}$. Instead we see that if we change complex variable $\phi(x)$ to $\phi(x) - \phi_0 = \frac{1}{\sqrt{2}} r(x) e^{i\varphi(x)}$ we are able to sensibly interpret the lagrangian. Suppressing the potential terms, we get,

$$\mathcal{L}_1 = \frac{\phi_0^2}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\partial_\mu r \partial^\mu r - \bar{m}^2 r^2 - V(r, \varphi) \tag{2.5}$$

Notice that there is no quadratic term for the $\varphi(x)$ field, and there's a proper sign mass term for the $r(x)$ field with mass $\sqrt{m}$. The potential term $V(r, \varphi)$ contains cubic and quartic interaction terms for the fields which can be found out explicitly if required. This is the case of spontaneous breaking of the $U(1)$ symmetry, to give the massless Goldstone mode φ.

Again, we have two propagating field-theoretic degrees of freedom, as before. But only one of them is massive, the other is massless. This phase of the theory, for $m^2 < 0$, is called gapless, since the energy spectrum of the theory has no 'gap' between the vacuum energy and the excited states with particles. Comparing this to the previous phase, where $m^2 > 0$, we see that there was a 'gap' of the order m in the energy spectrum of the theory, and hence would be called the 'gapped' phase of the theory. This observation shall be important in identifying the dual phases of the theories. We shall also mention that these phases can be referred to as the 'unbroken' and 'broken' phases, respectively, wrt the global $U(1)$ symmetry of the theory.

Apart from the usual particle spectrum, this theory also exhibits topological excitations. For this discussion, we shall be in the *broken* phase of the theory and we shall see why. Notice that in the broken phase, the vacuum state is degenerate under $U(1)$. This means, that the asymptotic value, that is the value of the field at spatial infinity, of $\phi(x)$ is dictated as $\phi(x) \to \phi_0$ as $|x| \to \infty$, so that the potential term in the hamiltonian is minimized. This means the phase of ϕ at spatial infinity, let's call it φ_∞, is a boundary condition we can set by hand. This forms a map $\varphi_\infty : S^1 \to S^1$ which is, in general, characterised by a winding number.

Consider a time-independent, rotationally symmetric solution $\phi(x)$ with winding number 1. This configuration shall be referred to as the vortex configuration. The vortex has a core, where $\phi(x)$ nearly vanishes, that is characterised by a size a. Far away from the core, the field $\phi(x)$ approaches its vacuum value with a phase that winds once around the core.

$$\phi(r, \theta) \to \phi_0 e^{i\theta} \quad \text{when} \quad r \to \infty \tag{2.6}$$

Similarly, configurations with winding number -1 will be called anti-vortices. Notice, that the winding number at infinity is the total number of vortices upto vortex-antivortex pairs.

We now understand why we required to be in the broken phase of the theory. The unbroken phase has no non-trivial vacua, since $|\phi(x)| = 0$ is the only vacuum state.

A field configuration with some winding number cannot be continuously deformed into one with a different winding number continuously without going through a barrier of infinite energy. As an aside, this is exactly a generalization of the fact that, in 1+1D, a kink cannot be continuously deformed to an antikink/vacuum without the hamiltonian blowing up. Evolution in time is a continuous function. And hence, at the risk of summarizing without direct proof, the vortex by itself is stable under time-evolution.

It is not difficult to imagine multi-vortex configurations starting from single vortex configurations. In fact, a two vortex solution where the two cores are sufficiently separated is almost a solution of the equations of the motion already. The solution is exact as the separation goes to infinity. The overall winding of such multi-vortex configurations is the sum of individual vortex windings. That is, the winding number at infinity is always the number of vortices minus the number of antivortices at any time. This number is topologically protected and hence is conserved over time. For example, if we were to start with the usual vacuum and increase the temperature, we would always have the number of vortices equal to the number of antivortices at any time. A pair of vortex-antivortex may be spontaneously produced, but the total winding at infinity must always remain constant - zero in this case. Remarkably, this is quite similar to the Noether's charge in Abelian Higgs. The net charge enclosed at infinity is always conserved.

The energy of the vortex can be computed by simply plugging the configuration into the classical hamiltonian functional of $\phi(x)$. It can be shown that the vortex energy is divergent as $\log(R)$, where R is the size of space. Or, with only a slight change to the derivation, the total energy of a static vortex-antivortex pair is proportional to $\log(R)$ where R now is the separation between the vortex and antivortex. The full calculation for reference is included towards the end. This log potential is an important observation and we shall use it to identify with the interaction between particles of the dual theory.

The log potential does something very non-trivial. It leads to, essentially, a confinement of vortices in the theory into pairs at low temperatures. The vortices simply cannot move individually at low enough energies. At a critical temperature, we see the Kosterless-Thouless transition occur and deconfinement is seen in the vortices. Beyond this critical temperature, we have an essentially disordered state where vortices and antivortices move freely. Such a transition is reminiscent of a transition in the theory of 2D Coulomb Gas. And indeed in the dual theory, the Abelian Higgs model which is $2+1$ dimensional scalar QED, we see exactly similar behaviour.

Let's now discuss some aspects of the RG flow of this theory. The duality is supposedly exact as we flow towards IR. Observe that in $2+1D$, the perturbations ϕ^2 and ϕ^4 are relevant. The parameters have classical mass dimension $[m^2] = 2$ and $[\lambda] = 1$. β-functions for these parameters are non-zero and they flow. The β-functions in $4-\epsilon$ euclidean dimensions to one loop in the MS scheme are [1]

$$\begin{aligned} \mu\frac{d}{d\mu}\tilde{m}_R^2 &= \Big(-2 + \frac{\lambda_R}{16\pi^2}\Big)\tilde{m}_R^2 + \mathcal{O}(\lambda_R^2) \\ \mu\frac{d}{d\mu}\lambda_R &= -\epsilon\lambda_R + \frac{3\lambda_R^2}{16\pi^2} + \mathcal{O}(\lambda_R^3) \end{aligned} \tag{2.7}$$

where we have redefined and used the dimensionless parameters $\tilde{m}^2 = \mu^2 m^2$ using the renormalization scale μ. Again, these equations are derived for $4-\epsilon$ dimensions and hold only for $\epsilon << 1$. A sketch of the flow near the Gaussian fixed point is shown.

We can see the existence of the Wilson-Fischer fixed point at a certain $(\tilde{m}^2, \lambda) = (0, \lambda_*)$ where both the β-functions vanish. To flow to the WF fixed point, however, we must be careful in tuning the mass parameter to zero throughout the flow, otherwise we see a run-away behaviour towards infinity completely missing the fixed point. That this fixed point also exists in 3D is established through various other techniques and numerical simulations on a lattice.

Let's observe some key points regarding flow away from the Wilson-Fischer fixed point. Notice that the m^2 perturbation is still relevant and seems to take the theory away towards $m^2 = \pm\infty$, $\lambda = \lambda_*$. Thus, a small m^2 perturbation from the WF fixed point, completely changes the phase of the theory. Tuning m^2 to be greater than zero, we reside in the $U(1)$ unbroken i.e. gapped phase of the theory. Tuning m^2 less than zero takes us into the $U(1)$ broken i.e. gapless phase. This phase has topological excitations.

Let us now move on to the Abelian-Higgs model and have a look at the physics of the particles and vortices of the theory. We shall see that we find similarities.

2.2 Abelian-Higgs model

$$\mathcal{L}_2 = |(\partial_\mu - iA_\mu)\Phi|^2 - M^2|\Phi|^2 - \Lambda|\Phi|^4 - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} \tag{2.8}$$

This theory is the gauged version of the $\lambda\phi^4$ model. This introduces a number of notable differences in the physics. Notice that, the gauge field A_μ has the usual maxwell term and no explicit mass term. This allows the theory to enjoy a local U(1) symmetry, where $\Phi \to e^{i\alpha(x)}\Phi$ and $A_\mu \to A_\mu + \partial_\mu\alpha(x)$ leaves the lagrangian invariant. The conserved current J_μ is given $J_\mu = i\Phi^\dagger \overleftrightarrow{D_\mu}\Phi$, where notice that the partial derivative, as in the $\lambda\phi^4$, is replaced by the covariant derivative.

As before, one might expect the $M^2 > 0$ phase to correspond to the gapped phase of the theory, but the presence of the massless gauge field actually makes this a gapless phase. In fact, the $M^2 < 0$ case leads us to the famous Higgs mechanism, and indeed that phase is then gapped since the photon acquires a mass. We expect these phases to be dual to the corresponding gapless/gapped phases of the previous theory.

Notice that in the gapless phase, the Φ particles interact via the Φ^4 vertex, or by the exchange of a single or pair of photons whose interaction strengths are given in terms of Λ, g and g^2 respectively. The photon exhange is also called as the Coulomb interaction in $2+1D$. To leading order in g, we can show that the coulomb potential behaves like $\log(R)$ where R is the distance between the pair of static $\Phi - \Phi^\dagger$ particles. This calculation is rather straight-forward, and is included at the end for reference. Notice how the $\log(R)$ behaviour matches with that of the vortices of the previous theory.

This also means, since the potential is $\log(R)$, the Φ and $\Phi^\dagger$ particles are actually confined as pairs at low energy. There occurs a transition at a critical temperature, where these particles are liberated. This transition is famously known as the deconfinement transition of the Coulomb Gas in three spacetime dimensions. Notice how remarkably similar this is to the Kosterless Thouless transition that deconfines the vortices in the previous model.

Now let's discuss the gapped or the Higgs phase of the thoery, that is when $M^2 > 0$. In this phase, we noticed spontaneous symmetry breaking in the previous theory. But since this theory is gauge invariant, a gauge field redefinition leads to a mass term for the gauge field. This is the Higgs mechanism. The naively expected Goldstone mode in the theory disappears from the lagrangian and manifests itself as a mass term for the photon. All propagating degrees of freedom in this phase thus become massive. Therefore we call this the gapped phase of the theory.

We saw that topological excitations were present in the $\lambda\phi^4$ theory when the $U(1)$ symmetry was broken. Similarly, in the Abelian-Higgs model, the broken phase admits vortex solutions. However, there are a few key differences.

The classical time-independent rotationally symmetric vortex solution of the field $\Phi(x)$ and $A_i(x)$ can be shown to behave like

$$\Phi(r, \theta) = ve^{i\theta}F(r) \tag{2.9}$$

$$A_i(r, \theta) = -\frac{1}{r^2}\epsilon_{ij}x_jA(r) \tag{2.10}$$

where $v = \Phi_0$ is the vev of the field. The functions $F(r)$ and $A(r)$ satisfy $F(r), A(r) \to 1$ as $r \to \infty$ and $F(r), A(r) \to 0$ as $r \to 0$.

With the presence of the gauge field, the classical energy density of the vortex is now localised to a finite core, and dies down rapidly asymptotically in space. In other words, individual vortex configurations are no longer divergent in energy, but have finite total energies.

This means something important. A pair of a vortex and an antivortex does not interact with each other unless their cores significantly overlap. Individual vortices behave like free extended particles that interact only when they coincide. This closely resembles the delta interaction between particles of the $\lambda\phi^4$ theory, that only interacted if they coincided. Remember however that the delta potential was only a leading order behaviour in the non-relativistic approximation. We expect the vortices too to have complicated dynamics when their speeds become relativistic and their quantum corrections are included.

This is a nice intuitive picture that can be verified to higher orders in coupling constants by doing computations in the respective theories. We shall however be satisfied with this leading order correspondence for now.

In the case of gauged vortices, it is possible to show that the magnetic flux passing through the vortex is quantized. This is a direct consequence of the winding number of the vortex. Since as $r \to \infty$, $A_i \to -\frac{1}{r^2}\epsilon_{ij}x_j = \partial_i\theta$, integral of the flux over an area S containing vortices can be rewritten as

$$\int_S d^2x\, F_{ij} = \oint_{\partial S} dx^i A_i = \oint_{\partial S} dx^i \partial_i\theta = 2n\pi \tag{2.11}$$

when S is sufficiently large. The n corresponds to the total winding number, or the total number of vortices minus antivortices inside S. We can rewrite this fact as a topological conserved current $\mathcal{J}^\mu$ that satisfies $\partial_\mu \mathcal{J}^\mu = 0$. We only require to define the following,

$$\mathcal{J}^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda \tag{2.12}$$

Notice that the conservation equation $\partial_\mu \mathcal{J}^\mu$ is nothing but the Bianchi identity for the Maxwell tensor. The conserved charge

$$\mathcal{Q} = \int_S d^2x\, \mathcal{J}^0 \tag{2.13}$$

is the winding number of the field configuration inside, and hence corresponds to the total number of vortices inside upto vortex-antivortex pairs. This is the topological $U(1)$ symmetry of our Abelian Higgs model. It is also called the 'magnetic' $U(1)$ since the current carries a conserved magnetic charge.

Let's now collect all these ideas together and build a dictionary of the dual physics.

2.3 Particle Vortex Dictionary

All of the above discussions provide us with sufficient motivation to claim the particle vortex duality. We found the following features to be identical in the phases labelled by the sign of the mass term in the respective theories.

$\lambda\phi^4$	Abelian Higgs
$m^2 < 0$ global $U(1)$ broken massless Goldstone mode vortices vortex-antivortex pairs, log potential winding number	$M^2 > 0$ 'magnetic' $U(1)$ broken massless photon charged Φ particles particle-antiparticle pairs, log coulomb potential Noether charge
$m^2 > 0$ no massless mode, gapped phase global $U(1)$ unbroken charged ϕ particles propagating ϕ particles, delta potential* Noether charge	$M^2 < 0$ higgs mechanism, gapped phase 'magnetic' $U(1)$ unbroken vortices propagating vortices, effectively free* until cores overlap winding number/magnetic flux

* *in the non-relativistic approximation*

Broadly we see how the particles interact with each other, and vortices with each other from the above analysis. The corresponding quantities have been motivated and calculated independently. However the above discussion is somewhat qualitative and we still do not have an explicit operator map for all possible (gauge invariant) operators in the two theories. An ideal way of generating such a map would be a field-theoretic change of variables that directly relate the two lagrangians. But we see now why this would be a very complicated map, since the elementary degrees of freedom in one theory are mapped to highly non-local, non-perturbative topological degrees of freedom in the other theory.

There is however a precise way to express dualities even without an operator map. This involves rewriting the dualities as an equivalence of partition functions.

Introducing Background Fields

To express the duality as an equality of partition functions, we shall couple the dual currents to a background gauge field. Let's see how this works.

Let's include a background gauge field B_μ, that couples to the dual currents in both the theories, that is $J_\mu = i\phi^\dagger \overleftrightarrow{\partial_\mu} \phi \quad \leftrightarrow \quad \mathcal{J}_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial^\nu A^\lambda$ as follows.

$$\mathcal{L}_1[\phi; B] = |\partial_\mu - iB_\mu \phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4 \tag{2.14}$$

$$\mathcal{L}_2[\Phi, A; B] = |(\partial_\mu - iA_\mu)\Phi|^2 - M^2|\Phi|^2 - \Lambda|\Phi|^4 - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}B_\mu\partial_\nu A_\lambda \tag{2.15}$$

Now we can neatly write down the duality as an equality of the above partition functions as follows:

$$Z_{\text{scalar}}[B] = \int DA \ Z_{\text{scalar}}[A] \ e^{-\frac{i}{2\pi}\int \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu B_\lambda} \tag{2.16}$$

Writing out in this fashion helps us understand the duality from the RG perspective. Notice that it is important to identify the correct dual currents to be able to couple them to the background gauge field. These currents, in general, will be the ones corresponding to the ungauged global symmetries of the theories.

An important aspect of identifying the ungauged global symmetries is the fact that global symmetries are preserved under the RG flow. Thus, once we have correctly identified the faithful global symmetries on both sides, we are free to couple the corresponding currents to a background field and write down the duality in terms of the partition function. We shall find this conceptually very important later in the non-abelian dualities section.

For now, we see that the particle vortex-duality matches the global $U(1)$ in the lambda ϕ^4 model with the 'magnetic' $U(1)$ of the vortices in the Abelian-Higgs model, owing to the quantized magnetic flux the vortices carry.

Notice that the magnetic $U(1)$ current naturally takes the form of a Chern-Simons coupling between the dynamical gauge field and the background gauge field in theory. We shall continue to encounter such mixed Chern-Simons terms as well as pure Chern-Simons terms in our discussion. It shall, thus, be helpful to review some interesting properties of these Chern-Simons theories and build our intuition around it.

Chapter 3

Chern Simons Matter Theories

Chern-Simons theories in 2+1D bring out some of most surprising physics in quantum field theory. This interesting theory shall be crucial to our discussion henceforth so let's review its novel features.

In $2+1$ dimensions, apart from the usual Maxwell lagrangian, we have another lorentz invariant, gauge invariant and local term. In fact, the term is purely topological in that the metric tensor does not appear in it.

$$\mathcal{L}_{CS} = \frac{\kappa}{4\pi} \int \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \tag{3.1}$$

This term may not appear gauge invariant at a glance. Indeed under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha$, the term shifts by a total space-time derivative

$$\delta\mathcal{L}_{CS} = \frac{\kappa}{4\pi} \int \partial_\mu (\alpha \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda)$$

and hence, if we neglect boundary terms, the full Chern Simons action is gauge invariant as expected.

In fact, it is only in 2+1D that we can write down such a term that is quadratic in the gauge field. It might seem that the theory is quite boring since its equations of motion reduce to

$$F_{\mu\nu} = 0 \tag{3.2}$$

whose solutions are trivial. In comparison, the usual Maxwell electrodynamics has plane wave solutions which seem far more interesting. Indeed, the heart of the novel physics of the Chern-Simons term is revealed after we couple it to matter with its own dynamics, Maxwell terms, non-trivial spacetimes or a combination of them. Making the gauge group non-abelian also has interesting consequences. In the following sections we shall explore these interesting physical ideas.

3.1 Flux Attachment and Anyons

Attaching a unit of magnetic flux to particles changes their statistics from bosons to fermions to any-ons. This powerful idea first arose in non-relativistic quantum mechanics from the context of the Bohm-Aharanov effect. Generalizing to relativistic particles happens naturally via the Chern Simons coupling as we shall see.

Let's first discuss the non-relativistic version to get an intuition of the physics and see how the change in statistics of particles is brought about. We will then introduce the Chern-Simons term in the relativistic generalization.

3.1.1 Non-relativistic Composite Particles

Composite particles, that is particles with attached flux, form the central idea behind changing particle statistics. The story begins with the phase of a quantum particle in a plane as it goes around a thin solenoid running through the origin. In the absence of any other potential, such a particle is described by the Schrodinger's equation which takes the form

$$\left[\frac{1}{2m}(\vec{p} - e\vec{A}(r))^2 + e\phi(r)\right]\Psi = i\hbar\frac{\partial}{\partial t}\Psi \tag{3.3}$$

where $A^\mu = (\phi, \vec{A})$ is the vector potential of the solenoid configuration and Ψ is the wave-function of the particle. Notice that the canonical conjugate momentum of the particle in a EM field is $(\vec{p} - e\vec{A})$.

The vector potential for a solenoid magnet is a famously subtle problem. Notice first of all, that for a thin infinite solenoid passing through the plane at the origin, the magnetic field is zero everywhere else except for the origin. The electric field is identically zero everywhere. Thus, our vector potential must satisfy

$$\vec{E} = \nabla\phi(r) = 0\,, \qquad B = \nabla \times \vec{A} = 0\,(r > 0) \tag{3.4}$$

However we must be careful to also impose the condition

$$\oint_C \vec{A}.\vec{dl} = \Phi_m \tag{3.5}$$

where C is a close curve that includes the origin and Φ_m is the magnetic flux passing through the solenoid. We thus have the following vector potential solution

$$\phi(r) = 0\,, \qquad \vec{A} = \frac{\Phi_m}{2\pi r}\mathbf{e}_\phi \tag{3.6}$$

which is singular at the origin. We see that even though the magnetic field is zero outside, the vector potential is non-zero.

Now, to describe the wavefunction of the charged particle, we have to solve the Schrodinger equation. There's a cute trick that allows us to see what's happening. Check that if we substitute $\Psi(\vec{r},t) = e^{ig(\vec{r})}\Psi'(\vec{r},t)$ where $g(\vec{r})$ is defined as

$$g(\vec{r}) \equiv \frac{e}{\hbar}\int_0^{\vec{r}} \vec{A}\cdot d\vec{r} \tag{3.7}$$

then Ψ' satisfies the Schrodinger equation without the vector potential

$$-\frac{\hbar^2}{2m}\Psi' = i\hbar\frac{\partial}{\partial t}\Psi' \tag{3.8}$$

We thus see how the vector potential affects the wavefunction of a particle. The wavefunction Ψ in the presence of a vector potential is the solution without it Ψ' multiplied by a phase factor $e^{ig(\vec{r})}$. One can now show that the particle that goes around the full solenoid picks up a phase of

$$e^{i\frac{e}{\hbar}\oint \vec{A}\cdot\vec{dr}} = e^{i\frac{e\Phi_m}{\hbar}} \tag{3.9}$$

Now let's arrange so that the flux is the 'unit' flux $\Phi_m = 2\pi\frac{\hbar}{e}$. And now let's imagine two identical free particles and that such a 'flux tube' is attached to each particle. A double exchange of these particles corresponds to swapping their positions twice in an 'adiabatic' way, that is slow enough so that we may neglect radiation and other effects. This double exchange, if we change reference frames and keep one particle fixed, is equivalent to the other particle moving around the first once.

Thus, the two particle wavefunction picks up the factor $e^{i2\pi}$. This means, under a single exchange, we pick up a phase factor of $e^{i\pi} = -1$. Our bosonic particles with flux tubes behave like composite fermions.

If the initial particles were fermions to begin with, the extra factor of -1 would have cancelled the one coming from the fermionic exchange, and we would have no overall -1 multiplying the exchanged wavefunction.

Notice that had we started with any intermediate unit of flux, we would not have exact cancellation and the composites would not be described as either a boson or a fermion but would be called 'anyons'.

3.1.2 Relativisitic Composite Particles via Chern Simons term

We saw how attaching flux to non-relativistic particles changed their statistics. Let's see that the Chern-Simons term in a quantum field theory allows us to exactly that. [10].

Let us introduce a matter current J^μ and couple it to the Chern Simons gauge field. In a field theory, this current will be the conserved current of some dynamical field.

$$\mathcal{L}_{CS} = \frac{\kappa}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - A_\mu J^\mu$$

The classical equations of motion are

$$\frac{\kappa}{4\pi}\epsilon^{\mu\nu\lambda}F_{\nu\lambda} = J^\mu$$

which is a straight algebraic equation of motion i.e. it does not involve any derivatives. The Chern Simons term simply sets the dual Maxwell tensor $\tilde{F}^\mu = \epsilon^{\mu\nu\lambda}F_{\nu\lambda}$ proportional to the matter current J^μ. Note that the Bianchi identity $\epsilon^{\mu\nu\lambda}\partial_\mu F_{\nu\lambda}$ is nicely compatible with the conservation of the current $\partial_\mu J^\mu$.

The upshot of this is that locally, J^0, the matter density is proportional to the magnetic field. Integrated over an area, we get that flux is tied to matter everywhere. Flux and matter are locally proportional, with the proportionality constant being the Chern Simons coupling parameter κ.

In a way, the Chern Simons coupling acts like a pure constraint. We may have the matter field follow its own dynamics. And the Chern Simons coupling ensures that flux proportional to the local matter density is introduced that follows the matter density as it evolves. For a charged point particle of charge e, the flux attached has value $\frac{2\pi e}{\kappa}$. For $\kappa = 1$ this is exactly the unit of flux that we had wanted attached.

So far, it would seem that we are free to tune the value of κ while defining the theory. However, the value of κ is restricted by gauge invariance to integer values, and we shall see that next.

3.2 Quantization of the Chern Simons coefficient

The gauge invariance of the Chern Simons term was briefly discussed in the previous section. And it was seen that we get a total spacetime derivative, upon performing the transformation. And indeed, for gauge transformations that vanish at infinity we saw that we are left with a total derivative term that vanishes. The Chern Simons action is gauge invariant under these, so called, small gauge transformations. For gauge transformations that behave non-trivially at infinity - large gauge tranformations - we shall see that we need to impose an integer quantization on the coupling κ.

Let us adopt a few normalization conventions. We are interested in space-times S where there exist non-trivial or large gauge transformations. The key idea is therefore compactifications of the usual $2+1$ dimensional spacetimes. And we shall work in the space time $S = S^2 \times S^1$. Such a compactification is necessary to have a section of the spatial part contain a nonzero total magnetic flux. When the spatial S^2 contains a 'monopole', the gauge connection A becomes a connection over a non-trivial $U(1)$ bundle over S^2 and the field strength $F = dA$ satisfies

$$\int_{S^2} \frac{F}{2\pi} \in \mathbb{Z}$$

The Chern Simons term, expressed as an integral over differential forms, becomes

$$\mathcal{L}_{CS} = \frac{\kappa}{4\pi} \int_{S^2 \times S^1} A \wedge F \tag{3.10}$$

Notice that the large gauge transformations indeed come from the fact that $\pi_1(S^1) = \mathbb{Z}$. And in fact these are the only large gauge transformations present in this theory. Performing such a gauge transformation Ω, the Chern Simons term transforms as

$$\mathcal{L}_{CS} \to \mathcal{L}_{CS} + \frac{\kappa}{4\pi} \int_{S^2 \times S^1} \Omega^{-1} d\Omega \wedge F \tag{3.11}$$

where $\Omega^{-1} d\Omega = d\omega$ with ω winding around S^1 exactly n times, i.e. $\int_{S^1} d\omega = 2n\pi$.

One is thus tempted to conclude that the Chern Simons term shifts to have an added

$$\frac{\kappa}{4\pi} \int_{S^2 \times S^1} \Omega^{-1} d\Omega \wedge F = \frac{\kappa}{4\pi} \int_{S^1} d\omega \int_{S^2} F = \frac{\kappa}{4\pi} \cdot 2n\pi \cdot 2m\pi \tag{3.12}$$

This result is incorrect. We have naively assumed that A is everywhere defined. In fact, in the presence of a monopole, A cannot be globally defined and dA belongs to the nontrivial element of the de Rham cohomology class $H^2_{DR}(S^2, \mathbb{Z})$. Thus our integral is only defined over local patches.

Such an integral over patches is usually dependent on the choice of patching, that is the open cover collection $\{U_\alpha\}$. For the integral itself to be independent of a choice of patching, correction terms need to be included which are neatly catalogued using arguments from Cech cohomology. Let's go through the argument and see what changes.

Cech Cohomology for physicists

We assume that we are able to find a cover $\mathcal{U} = \{U_\alpha\}$ of our spacetime S such that any finite intersection of them is simply connected. Using the tic-tac-toe box and Poincare's lemma

[5][6] on the intersections, we find the following relations to hold true at double, triple and quadruple intersections respectively

$$\begin{aligned} dJ_{\alpha\beta} &= \delta\{A_\alpha \wedge F\} = A_\alpha \wedge F - A_\beta \wedge F \\ dK_{\alpha\beta\gamma} &= \delta\{J_{\alpha\beta}\} = J_{\alpha\beta} + J_{\beta\gamma} + J_{\gamma\alpha} \\ dH_{\alpha\beta\gamma\rho} &= \delta\{K_{\alpha\beta\gamma}\} = K_{\alpha\beta\gamma} + K_{\beta\gamma\rho} + K_{\gamma\rho\alpha} + K_{\rho\alpha\beta} \end{aligned} \tag{3.13}$$

where δ is the Cech coboundary operator and the equations hold on the corresponding oriented intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ and $U_{\alpha\beta\gamma\rho} = U_\alpha \cap U_\beta \cap U_\gamma \cap U_\rho$ respectively.

The differential forms J, K, H are called transition functions and can be solved for explicitly. Under the nontrivial gauge transformation Ω defined above, we can show that these transition functions change to

$$\begin{aligned} \Delta J_{\alpha\beta} &= 0 \\ \Delta K_{\alpha\beta\gamma} &= c_{\alpha\beta\gamma} d\omega \\ \Delta H_{\alpha\beta\gamma\rho} &= 0 \end{aligned} \tag{3.14}$$

where $c_{\alpha\beta\gamma}$ are the elements of the integer Cech cohomology class $H^2_C(S^2, \mathbb{Z})$ that also satisfy

$$\sum_{\alpha\beta\gamma} c_{\alpha\beta\gamma} = \int_{S^2} \frac{F}{2\pi} \in \mathbb{Z} \tag{3.15}$$

where the sum is over all triple intersections.

The final prescription to get the integral that is independent of the choice of $\{U_\alpha\}$ then becomes [5]

$$\frac{\kappa}{4\pi} \int_{S^2 \times S^1} A \wedge F = \frac{\kappa}{4\pi} \left(\sum_{\alpha} \int A_\alpha \wedge F - \sum_{\alpha\beta} \int J_{\alpha\beta} + \sum_{\alpha\beta\gamma} \int K_{\alpha\beta\gamma} - \sum_{\alpha\beta\gamma\rho} H_{\alpha\beta\gamma\rho} \right) \tag{3.16}$$

which transforms under the large gauge transformation to shift by the constant

$$
\begin{aligned}
\frac{\kappa}{4\pi}\int A\wedge F \to \frac{\kappa}{4\pi}\Bigg(\int A\wedge F + \sum_{\alpha} dA_{\alpha}\int_{S^1} d\omega + \sum_{\alpha\beta\gamma} c_{\alpha\beta\gamma}\int_{S^1} d\omega\Bigg) \\
= \frac{\kappa}{4\pi}\int A\wedge F + \frac{\kappa}{4\pi}\cdot 2\cdot 2n\pi\cdot 2m\pi
\end{aligned}
\tag{3.17}
$$

Notice that a new factor of 2 has appeared. This is the correct answer. Now we may require gauge invariance, which basically means the action shift to a integer multiple of 2π.

This amounts to the condition $\frac{\kappa}{4\pi}8\pi^2 = 2\pi\mathbb{Z}$, that is indeed $\kappa \in \mathbb{Z}$. The Chern-Simons level must be quantized with the normalization 4π for the Chern-Simons action to be gauge-invariant.

3.3 Chern-Simons and Maxwell

It is interesting to look at Chern-Simons theory in the presence of the Maxwell term. The result is a surprising way of giving the gauge field a mass. Consider the lagrangian

$$
\mathcal{L}_{\text{CS+M}} = \frac{\kappa}{4\pi}\epsilon^{\mu\nu\lambda}A_{\mu}\partial_{\nu}A_{\lambda} - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} \tag{3.18}
$$

The equations of motion are

$$
\partial_{\mu}F^{\mu\nu} + \frac{\kappa g^2}{4\pi}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0 \tag{3.19}
$$

casting this in terms of the dual field $\tilde{F}^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\nu\lambda}$ we see that the equation of motion becomes

$$
\epsilon^{\nu\mu\alpha}\partial_{\mu}\tilde{F}_{\alpha} = \frac{\kappa g^2}{2\pi}\tilde{F}^{\nu} \tag{3.20}
$$

which can be inverted as

$$
\partial_{\mu}\tilde{F}_{\alpha} = \frac{\kappa g^2}{2\pi}\epsilon_{\mu\alpha\nu}\tilde{F}^{\nu} \tag{3.21}
$$

Taking the derivative ∂^{μ} on both sides and substituting $\partial^{\mu}\tilde{F}^{\nu}$ from above gives

$$
\Big(\partial^{\mu}\partial_{\mu} + (\frac{\kappa g^2}{2\pi})^2\Big)\tilde{F}_{\alpha} = 0 \tag{3.22}
$$

which is exactly the Klein-Gordon equation with mass $m_{\text{CS+M}} = \dfrac{\kappa g^2}{2\pi}$.

Note that this field $\tilde{F}^\mu$ is manifestly gauge-invariant and also satisfies $\partial_\mu \tilde{F}^\mu = 0$, which is simply the Bianchi identity for $F^{\mu\nu}$.

It is also possible to check this by looking at the poles of the gauge field propagator (after appropriate gauge fixing) and indeed the answer matches. [10]

A free Maxwell gauge field in 2+1D has one massless degree of freedom. In the presence of a Chern-Simons term along with the Maxwell term, we see the gauge field has a massive degree of freedom. This fact shall be useful in the discussion of abelian bosonization.

3.4 Induced Chern Simons terms

One of the important features of Chern-Simons terms is that they may arise at the quantum level, even if they are not explicitly included in the classical lagrangian, as quantum anomalies. Since the Chern-Simons term in 2+1D is parity and time-reversal violating, like the fermion mass term, this anomaly is referred to as the parity anomaly. Also, we shall see that the level of this anomalous term will be half-integral which means the term by itself also violates gauge-invariance of the action. [4]

Consider the Dirac fermion lagrangian in 2+1D coupled to background field

$$S_{\text{fermion}} = \int d^3x \; i\bar{\psi}\gamma^\mu(\partial_\mu + iA_\mu)\psi + m\bar{\psi}\psi \tag{3.23}$$

Switching to euclidean space-time,

$$S_{\text{fermionEu}} = \int d^3x \; i\bar{\psi}\gamma_\mu(\partial_\mu + iA_\mu)\psi - m\bar{\psi}\psi \tag{3.24}$$

we can write the effective action directly as the fermion determinant

$$S_{\text{eff}}[A; m] = \ln\det\Big(i\gamma_\mu(\partial_\mu + iA_\mu) - m\Big) \tag{3.25}$$

Expanding[9] this in orders of A, we would like the inspect the term that is quadratic in A

$$\begin{aligned}\ln\det\Big(i\gamma_\mu(\partial_\mu + iA_\mu) - m\Big) \sim & \operatorname{tr}\ln\Big(i\gamma_\mu\partial_\mu - m\Big) + \operatorname{tr}\Big(\frac{1}{i\gamma_\mu\partial_\mu - m}\gamma_\nu A_\nu\Big) \\ & - \frac{1}{2}\operatorname{tr}\Big(\frac{1}{i\gamma_\mu\partial_\mu - m}\gamma_\nu A_\nu \frac{1}{i\gamma_\rho\partial_\rho - m}\gamma_\delta A_\delta\Big) + \dots\end{aligned} \tag{3.26}$$

The parity anomaly comes from the third term in this series

$$-\frac{1}{2}\operatorname{tr}\Big(\frac{1}{i\gamma_\mu\partial_\mu - m}\gamma_\nu A_\nu \frac{1}{i\gamma_\rho\partial_\rho - m}\gamma_\delta A_\delta\Big) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\ \Big(A_\mu(p)\Gamma_{\mu\nu}(p,m)A_\nu(p)\Big) \tag{3.27}$$

$$\Gamma_{\mu\nu}(p,m) = -\int \frac{d^3k}{(2\pi)^3}\ \operatorname{tr}\Big(\gamma_\mu \frac{\gamma_\rho(p_\rho + k_\rho) + m}{(p+k)^2 + m^2}\gamma_\nu \frac{-\gamma_\delta k_\delta - m}{k^2 + m^2}\Big) \tag{3.28}$$

Using the identity

$$\operatorname{tr}(\gamma_\mu\gamma_\nu\gamma_\rho) = 2\epsilon_{\mu\nu\rho} \tag{3.29}$$

we extract the term involving $\epsilon_{\mu\nu\rho}$ from the one containing 3 gamma matrices as

$$\Gamma'_{\mu\nu} = \epsilon_{\mu\nu\rho}\Pi(p,m)p_\rho \tag{3.30}$$

where

$$\Pi(p,m) = 2m\int \frac{d^3k}{(2\pi)^3}\ \frac{1}{((p+k)^2 + m^2)(k^2 + m^2)} \tag{3.31}$$

In the low energy limit $p \to 0$,

$$\Gamma'(p,m) \to \frac{1}{4\pi}\frac{m}{|m|}\epsilon_{\mu\nu\rho}p_\rho + \dots \tag{3.32}$$

Therefore, we obtain the 2+1D parity anomaly term as

$$S^{CS}_{\text{eff}}[A] = \frac{1}{2}\frac{1}{4\pi}\frac{m}{|m|}\int d^3x\ \epsilon_{\mu\nu\rho}A_\mu\partial_\nu a_\rho \tag{3.33}$$

which is not gauge-invariant because of the factor of half sitting outside. In the case of dualities involving fermions, this shall force us to include a bare term with half-integer level to compensate. The sign(m) factor will help us match the phases of the fermion theory with the other side of the duality.

3.5 Non-Abelian Chern Simons

The Chern-Simons term can be generalized to non-abelian gauge fields as

$$\mathcal{L}_{CS}[A] = \frac{\kappa}{4\pi}\epsilon^{\mu\nu\lambda}\,\mathrm{tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right) \tag{3.34}$$

where A_μ takes values in the finite-dimensional representation of some Lie algebra $\mathcal{G}$. We shall mostly consider $\mathcal{G} = \mathfrak{su}(N)$ for some N. In the abelian case, the cubic term vanishes since all the A_μ commute. In the non-abelian case we write $A_\mu = A^a_\mu T^a$ where T^a are the generators of $\mathcal{G}$ satisfying $[T^a, T^b] = f^{abc}T^c$ and the normalization $\mathrm{tr}(T^aT^b) = -\frac{1}{2}\delta^{ab}$ with $a = 1, \ldots, dim(\mathcal{G})$

An important feature is the behaviour of this lagrangian under the non-abelian gauge transformation

$$A_\mu \to g^{-1}A_\mu g + g^{-1}\partial_\mu g \tag{3.35}$$

which takes the Chern-Simons term to

$$\mathcal{L}_{CS} \to \mathcal{L}_{CS} - \frac{\kappa}{4\pi}\epsilon^{\mu\nu\lambda}\partial_\mu\,\mathrm{tr}(\partial_\nu g g^{-1}A_\lambda) - \frac{1}{3}\frac{\kappa}{4\pi}\epsilon^{\mu\nu\lambda}\,\mathrm{tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g) \tag{3.36}$$

where the total space-time derivative vanishes like in the abelian case. But the third term, called the winding number density of the group element g,

$$w(g) = \frac{1}{24\pi^2}\epsilon^{\mu\nu\lambda}\,\mathrm{tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g) \tag{3.37}$$

is an integer (say m) with the above normalization. Thus,

$$\mathcal{L}_{CS} \to \mathcal{L}_{CS} - 8\pi^2\frac{\kappa}{4\pi}m \tag{3.38}$$

which implies $\kappa \in \mathbb{Z}$ for the Chern-Simons term to be gauge-invariant.

In a non-abelian gauge theory with fermions, these terms will be induced at the quantum level and we shall have to include appropriate bare terms to ensure that the overall partition function remains gauge invariant. If we have N_f fermion flavors, then the induced Chern-

Simons term can be shown to be [10]

$$S_{\text{eff}}^{CS}[A] = \frac{N_f}{2}\frac{1}{4\pi}\frac{m}{|m|}\int d^3x\ \epsilon_{\mu\nu\rho}A_\mu\partial_\nu a_\rho \tag{3.39}$$

We now have all the required tools to propose the idea of bosonization in 2+1 dimensions. Each of these building blocks will play a crucial role in formulating the exact statement of the bosonization duality.

Chapter 4

2+1D Bosonization

We would now like to propose a duality between a theory of bosons and a theory of fermions, which for some reason is popularly referred to as bosonization when it could very well be called fermionization. The central idea would involve observing that a boson bound to a vortex does the same job as a particle with flux attached. This bound state can be shown to behave like a fermion.

Flux attachment and the statistical transmutation that we saw before is a non-relativistic story. This was achieved by coupling the matter fields to a gauge field whose dynamics are given by a Chern-Simons term. Let us try and generalize those ideas to quantum field theory[12].

As an interesting historical note, proposals for the non-abelian bosonization dualities actually came before the abelian one. For the typical $U(N)$, $SU(N)$, $O(N)$ gauge groups with large-N, many calculations can be simplified in the $N \to \infty$ limit and then exploring the corrections order by order in the $1/N$ expansion. This gives an extra 'handle' to understanding these theories, especially their RG behaviour.

We shall see how the abelian bosonization duality arises from our building blocks. We will discuss the phases of the theories and see that they indeed match. Then, we shall be able to derive the particle vortex duality by manipulating the duality suitably. Finally, we will propose the non-abelian version of bosonization and its properties.

4.1 Bosons with Chern Simons

We start with a bosonic theory coupled to a gauge field - the Abelian Higgs model

$$S_{\text{scalar}}[\phi, a] = \int d^3x \; |(\partial_\mu - ia_\mu)\phi|^2 - V(|\phi|^2) - \frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} \tag{4.1}$$

and introduce a Chern-Simons term for the dynamical field to bring about flux attachment

$$S_{\text{scalar+flux}}[\phi, a] = S_{\text{scalar}}[\phi, a] + \frac{1}{4\pi} \int d^3x \; \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \tag{4.2}$$

The equation of motion for a_0 reads

$$\rho_{\text{scalar}} + \frac{f}{2\pi} = 0$$

where ρ_{scalar} is the density of our particles. If these were massive, non-relativistic particles we would have flux attachment and these particles would behave as fermions.

In the second quantized theory that we are considering now, it seems natural to expect a generalization of the flux attachment. We would like to attach magnetic flux to particles in the theory so that the composite particle behaves like a fermion. Vortex excitations carry quantized magnetic flux. And indeed it can be shown that if a single mode of the scalar is excited in the vortex background, the bound state carries half-integer angular momentum. The vortex+scalar bound state behaves like a fermion.

Denoting the vortex operator as $\mathcal{M}$, we can write down the topological current corresponding to the magnetic $U(1)$ as

$$\mathcal{J}^\mu_M = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda \tag{4.3}$$

We know that the ϕ current $j^\mu = i\phi^\dagger \overleftrightarrow{D^\mu} \phi$ is charged under the global part of the gauged $U(1)$. These 'electric' charges of the theory can be attributed to the term $a_\mu j^\mu$ in the lagrangian. In the lagrangian, the Chern-Simons term $\frac{1}{4\pi} a_\mu \mathcal{J}^\mu_M$ implies that the vortices also carry electric charge. It is important that we identify gauge-invariant combination of the vortex+scalar operator for the duality. The most natural operator combination that carries no electric charge under the gauged $U(1)$ would involve cancelling the electric charge of the vortex by an oppositely charged $\phi^\dagger$. Thus we propose that the vortex+scalar bound state we are considering must be of the form $\phi^\dagger\mathcal{M}$.

Let's now introduce the background gauge field by coupling it to the current

$$S_{\text{scalar+flux}}[\phi, a; A] = S_{\text{scalar}}[\phi, a] + \frac{1}{4\pi}\int \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{1}{2\pi}\int \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \tag{4.4}$$

and we get the partition function to look like

$$Z_{\text{scalar+flux}}[A] = \int D\phi Da \;\; \exp(iS_{\text{scalar+flux}}[\phi, a; A]) \tag{4.5}$$

When we flow to the infrared, we expect two phases as before depending on the sign of the mass term in the lagrangian. For one sign, the scalar is gapped and simply drops out. The gauge field dynamics are dominated by the Chern Simons term with no explicit mass. This Chern-Simons-Maxwell theory was shown to be actually gapped, with the 'topological' mass coming from the Chern-Simons dynamics. The gauge field can be integrated out to find the Hall conductivity wrt to the background gauge field.

In an effective field theory coupled to a background gauge field, Hall effect is the response of the effective current of the theory to the background electric/magnetic field. For a partition function of the form

$$Z[A] = e^{iS_{\text{eff}}[A]} = \int D(\text{fields}) \; e^{iS[\text{fields};A]} \tag{4.6}$$

where 'fields' refers to all the effective charge carriers in the experiment, the functional derivative of the effective action wrt the background gauge field gives

$$\frac{\delta S_{\text{eff}}[A]}{\delta A_\mu(x)} = \langle J^\mu(x) \rangle \tag{4.7}$$

which tells us the effective response of the current to the background electric/magnetic fields. In the linearized approximation for charged particles in the presence of small external electric field we expect this expression to look like

$$J_i = \sigma_{ij} E_j \tag{4.8}$$

where $E_i = \epsilon_{ij} A_j$ is the background electric field and σ_{ij} is the conductivity matrix. Hall effect then is the statement that the charge carriers have a motion transverse to the applied electric field and thus the off-diagonal elements σ_{xy} then physically corresponds to the Hall coefficient.

In the phase where the scalar is gapped, we have a massless gauge field whose dynamics are dominated by the Chern-Simons term (the Maxwell term drops out in the infrared). Integrating out the dynamical field gives us a Chern-Simons action for the background gauge field. From the above equation, we get

$$J_i = \frac{-1}{2\pi} \epsilon_{ij} E_j \tag{4.9}$$

which gives the Hall conductivity to be -1.

In the other phase, with the wrong sign mass term, we see Higgs mechanism take place. The dynamical gauge field becomes massive and drops out along with the scalar as we flow to the IR. This is the gapped phase of the theory.

Since the fields are all massive and drop out as we flow to the infrared, the effective action of this theory is trivial and consists of no dynamics. Thus, the effective current simply vanishes and one may call this an 'insulating' phase.

Now let's see how this structure is reproduced in the fermionic theory.

4.2 Fermions with induced Chern Simons

We start with a free theory of Dirac fermions. In 2+1D, the gamma matrices are simply the Pauli matrices upto factors of i and hence Dirac fermions are complex two-component objects. The Dirac lagrangian is similar to the usual one

$$S_{\text{fermion}}[\psi] = \int i\bar{\psi}\gamma^\mu \partial_\mu \psi \tag{4.10}$$

This theory has a global $U(1)$ symmetry with the conserved current $j^\mu = i\bar{\psi}\gamma^\mu\psi$. We would like this fermion current to be dual to the vortex current from the previous theory.

Like before, we couple it to a background gauge field via minimal coupling

$$S_{\text{fermion}}[\psi; A] = \int i\bar{\psi}\gamma^{\mu}(\partial_{\mu} - iA_{\mu})\psi \tag{4.11}$$

to get the partition function

$$Z_{\text{fermion}}[A] = \int D\psi \exp(iS_{\text{fermion}}[\psi; A]) \tag{4.12}$$

However, this partition function is not gauge invariant. This non-trivial result, as discussed in §3.4, comes from an anomalous Chern-Simons term that has a half-integer level

$$S_{\text{eff}}[A] = \text{sign}(m)\frac{1}{2}\frac{1}{4\pi}\int \epsilon^{\mu\nu\lambda}A_{\mu}\partial_{\nu}A_{\lambda} \tag{4.13}$$

where sign(m) is the sign of the fermion mass term. Since gauge invariance demands that Chern-Simons level be an integer, the half-integer level leaves the overall theory gauge non-invariant.

To fix this, we include a half-integer Chern-Simons term for the background field by hand

$$Z_{\text{fermion+CS}}[A] = \int D\psi \; e^{iS_{\text{fermion}}[\psi;A] - \frac{i}{2}S_{CS}[A]} \tag{4.14}$$

Even though classically this action seems gauge non-invariant, the quantum anomaly leads to either a cancellation or an addition, depending on the sign of the mass term, and the overall partition function continues to be gauge invariant as required.

If the fermion mass term has negative sign, then the fermion is gapped and has mass m. Thus after integrating out the fermion, we are left with a Hall conductivity zero. This corresponds to the wrong sign mass^2 term of the previous scalar+flux theory. If the fermion has a positive mass term, then we still get a gapped phase and a Hall conductivity of -1.

We may now write down the exact statement of abelian bosonization[12]

$$Z_{\text{fermion}}[A]e^{-\frac{i}{2}S_{\text{CS}}[A]} = Z_{\text{scalar+flux}}[A] \tag{4.15}$$

4.3 Abelian Bosonization

The statement above tells us how a theory of scalars with attached flux relates to a theory of fermions.

$$\text{boson+flux} \leftrightarrow \text{fermion} \tag{4.16}$$

One naturally expects another form of bosonization to hold, where instead of attaching flux to bosons, we attach flux to fermions to change their statistics. That is,

$$\text{fermion+flux} \leftrightarrow \text{boson} \tag{4.17}$$

It is interesting that the above can actually be derived from one another. In other words,

$$\text{boson+flux} \leftrightarrow \text{fermion} \qquad \Leftrightarrow \qquad \text{fermion+flux} \leftrightarrow \text{boson} \tag{4.18}$$

Let us prove this and in the process learn two techniques that will continue to help us derive new dualities from old ones.

Gauging the Background Field

Promoting the background gauge field to a dynamical one, allows us to derive further dualities from already established ones. This basically implies that we introduce a functional integral over the background field after introducing a new background field.

Consider the statement of abelian bosonization

$$Z_{\text{fermion+CS}}[A] = Z_{\text{scalar+flux}}[A] \tag{4.19}$$

and introduce a functional integral over A and couple A to a new background field B using a mixed Chern-Simons term.

$$\int DA \; Z_{\text{fermion+CS}}[A] \; e^{-\frac{i}{2\pi}\int \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu B_\lambda} = Z_{\text{scalar+flux}}[A] \tag{4.20}$$

Let's call the LHS $Z_{\text{fermion+flux}}$ to indicate the now dynamical Chern-Simons field A responsible for flux attachment.

$$Z_{\text{fermion+flux}}[B] = \int DA\ D\psi\ e^{iS_{\text{fermion}}[\psi;A] - \frac{i}{2}\frac{1}{4\pi}\int \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - \frac{i}{2\pi}\int \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu B_\lambda} \tag{4.21}$$

In the RHS,

$$Z'_{\text{scalar+flux}}[B] = \int DA\ D\phi\ e^{iS_{\text{scalar}}[\phi,a] + \frac{i}{4\pi}\int \epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{i}{2\pi}\int \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - \frac{i}{2\pi}\int \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu B_\lambda} \tag{4.22}$$

we may actually integrate out A since it appears linearly in the path integral and get

$$Z'_{\text{scalar+flux}}[B] = \int D\phi\ e^{iS_{\text{scalar}}[\phi,B] + \frac{i}{4\pi}\int \epsilon^{\mu\nu\lambda}B_\mu\partial_\nu B_\lambda} = Z_{\text{scalar}}[B]\ e^{iS_{\text{CS}}[B]} \tag{4.23}$$

Equating, we get,

$$Z_{\text{fermion+flux}}[B] = Z_{\text{scalar}}[B]\ e^{iS_{\text{CS}}[B]} \tag{4.24}$$

which is the other precise statement of abelian bosonization.

One may check that gauging B and introducing another background field takes us back to the original duality.

Finally, there is one more operation that shall be helpful in deriving dualities, namely time-reversal.

Time Reversal

For both the dualities,

$$\begin{aligned} Z_{\text{scalar+flux}}[B] &= Z_{\text{fermion}}[B]\ e^{-\frac{i}{2}S_{CS}[B]} \\ Z_{\text{fermion+flux}}[B] &= Z_{\text{scalar}}[B]\ e^{iS_{\text{CS}}[B]} \end{aligned} \tag{4.25}$$

the RHS is time-reversal symmetric (ignoring the background fields). The LHS however, because of the presence of the Chern-Simons terms, is not manifestly time reversal symmetric. Thus, the duality must imply that time-reversal arises as a quantum symmetry of the theories on the left. This is a completely non-trivial result and allows us to write down two new statements, one from each duality.

Applying 'time-reversal' on both sides,

$$\begin{aligned} \bar{Z}_{\text{scalar+flux}}[B] &= Z_{\text{fermion}}[B]\ e^{+\frac{i}{2}S_{CS}[B]} \\ \bar{Z}_{\text{fermion+flux}}[B] &= Z_{\text{scalar}}[B]\ e^{-iS_{\text{CS}}[B]} \end{aligned} \tag{4.26}$$

we see that only the background field Chern-Simons terms in RHS theories change sign. The LHS however are time-reversed counterparts of the original theories. We shall soon use these versions.

4.4 Particle Vortex Duality from Bosonization

Let's try and derive the particle-vortex duality from the bosonization dualities[12].

Start with the fermion+flux ↔ boson duality and shift the background Chern-Simons term to the other side,

$$Z_{\text{fermion+flux}}[B]\ e^{-iS_{\text{CS}}[B]} = Z_{\text{scalar}}[B] \tag{4.27}$$

Gauge the background field and couple it to a new background field C. The RHS simply becomes

$$\int DB\ Z_{\text{scalar}}[B]\ e^{\frac{i}{2\pi}\int \epsilon^{\mu\nu\lambda}B_\mu \partial_\nu C_\lambda} \tag{4.28}$$

which is already the RHS of the original particle-vortex duality.

Let's see how the LHS simplifies.

$$\int DB\ Z_{\text{fermion+flux}}[B]\ e^{-\frac{i}{4\pi}\int \epsilon^{\mu\nu\lambda}B_\mu \partial_\nu B_\lambda + \frac{i}{2\pi}\int \epsilon^{\mu\nu\lambda}B_\mu \partial_\nu C_\lambda} \tag{4.29}$$

Expanding the fermion+flux partition function,

$$\int D\phi\ DA\ DB\ e^{iS_{\text{fermion}}[\psi,A] - \int \frac{1}{2}\frac{i}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu A_\lambda - \frac{i}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu B_\lambda - \frac{i}{4\pi}\epsilon^{\mu\nu\lambda}B_\mu \partial_\nu B_\lambda + \frac{i}{2\pi}\epsilon^{\mu\nu\lambda}B_\mu \partial_\nu C_\lambda} \tag{4.30}$$

we see that B may be integrated out by substituting $B = C - A$ to give,

$$\int D\psi\ DA\ e^{iS_{\text{fermion}}[\psi,A] + \int \frac{1}{2}\frac{i}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu A_\lambda - \frac{i}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu C_\lambda + \frac{i}{4\pi}\epsilon^{\mu\nu\lambda}C_\mu \partial_\nu C_\lambda} \tag{4.31}$$

The first three terms are the time-reversed partition function $\bar{Z}_{\text{fermion+flux}}[-C]$ that we discussed earlier. This means, we can set the LHS equal to $Z_{\text{scalar}}[-C]\ e^{-iS_{\text{CS}}[-C]}$ using the duality. This simplifies the LHS to

$$\bar{Z}_{\text{fermion+flux}}[-C]\ e^{\frac{i}{4\pi}\int \epsilon^{\mu\nu\lambda}C_\mu \partial_\nu C_\lambda} = Z_{\text{scalar}}[-C]\ e^{-iS_{\text{CS}}[-C]Z_{\text{scalar}}[-C]\ e^{-iS_{\text{CS}}[-C]}} \tag{4.32}$$

where we may cancel the background Chern-Simons.

Equating LHS and RHS,

$$Z_{\text{scalar}}[-C] = \int DB\ Z_{\text{scalar}}[B]\ e^{\frac{i}{2\pi}\int \epsilon^{\mu\nu\lambda}B_\mu \partial_\nu C_\lambda} \tag{4.33}$$

sending $C \to -C$ gets us exactly to the particle-vortex duality

$$Z_{\text{scalar}}[C] = \int DB\ Z_{\text{scalar}}[B]\ e^{-\frac{i}{2\pi}\int \epsilon^{\mu\nu\lambda}B_\mu \partial_\nu C_\lambda} \tag{4.34}$$

The fact that particle-vortex duality can be derived as a consequence of bosonization is very strong evidence in its favour, and a common feature of the 2+1D web of dualities. Dualities in the web are connected to each other via such derivations and each link rests on multiple others for its validity. The fact there has been no inconsistent link in the entire web goes a long way in demonstrating its robustness.

Let's now take a look at the non-abelian generalizations of bosonization and the dualities we can derive from it.

4.5 Non-Abelian Bosonization

The statements of non-abelian bosonization, first established by Aharony[7] and then made precise by Seiberg and colaborators[8], involves relating a theory of N_f scalars with N_f Dirac fermions. Different gauge groups on the two sides and their Chern-Simons levels give rise to different statements. We shall look at the simplest duality

$$SU(N)_{-k+\frac{N_f}{2}} \text{ with } N_f\ \psi \qquad \leftrightarrow \qquad U(k)_N \text{ with } N_f\ \phi \tag{4.35}$$

The other case, with the SU/U interchanged, is

$$U(N)_{-k+\frac{N_f}{2}} \text{ with } N_f\ \psi \qquad \leftrightarrow \qquad SU(k)_N \text{ with } N_f\ \phi \tag{4.36}$$

will involve a similar discussion.

Both sides involve matter fields that transform in the fundamental representation of the respective gauge groups. The fermion side involves interactions due to the Chern-Simons gauge field, while the scalar side has a quartic potential tuned to the Wilson-Fischer fixed point. Both dualities assume $N_f \leq k$.

To write down the partition functions of the two sides, we should identify the ungauged global symmetries on both sides and the corresponding currents. We shall see that the global symmetries on the two sides match in a non-trivial way.

4.5.1 Fermion theory

The fermions transform in the fundamental of the $SU(N)$ gauge group. The kinetic term can be written as

$$i\bar{\psi}^{im}\gamma^{\mu}(\partial_{\mu}\delta^{mn} - iA_{\mu}^{mn})\psi^{in} \tag{4.37}$$

where A_μ is an $SU(N)$ gauge field and ψ carries two indices ψ_{im} where $i = 1, \ldots, N_f$ and $m = 1, \ldots, N$. Clearly we have a manifest $U(N_f)$ symmetry. However, the Z_N of the diagonal $U(1)$ subgroup is also a subgroup of the gauged $SU(N)$ group.

The ungauged global symmetry of the theory is then $U(N_f)/Z_N$. We shall see that this is indeed true on the other side.

4.5.2 Scalar theory

The scalars transform in the fundamental of the $U(k)$ gauge group and have the kinetic term

$$|(\partial_{\mu}\delta^{ba} - iB_{\mu}^{ba})\phi^{ia}|^2 \tag{4.38}$$

with quartic interactions of the form

$$|\phi^{ia}\phi^{ia}|^2 \tag{4.39}$$

that are assumed to be tuned such that the theory lies at the Wilson-Fischer fixed point.

In this case, we have a manifest $SU(N_f)$ symmetry, and a magnetic $U(1)$ symmetry that acts on the topological monopole in the theory. Together, we expect the global symmetry of the theory to be $SU(N_f) \times U(1)$. However, the Z_{N_f} subgroup of the diagonal $U(1)$ in $U(k)$ is gauged. And to quotient it out, we need to the representation of Z_{N_f} that acts on the $SU(N_f)$ and the $U(1)$. We shall see that it acts differently.

Like in abelian bosonization, the Chern-Simons term for the gauge field implies the monopole operator $\mathcal{M}$ has electric charge N. To construct a gauge invariant operator, we must cancel the magnetic charge which can be done by taking the product $\mathcal{M}(\phi^\dagger)^N$. This ensures that the combination is gauge-invariant.

This means that the Z_{N_f} acts as $e^{\frac{2\pi i}{N_f}}\,\mathbb{I}$ on the $SU(N_f)$ and as $e^{\frac{2\pi i N}{N_f}}\,\mathbb{I}$ on the magnetic $U(1)$. In other words, $Z_{N_f} = (e^{\frac{2\pi i}{N_f}}, e^{\frac{2\pi i N}{N_f}})$. Thus, the appropriate quotient is

$$\frac{SU(N_f) \times U(1)}{Z_{N_f}} = U(N_f)/Z_N$$

which matches the symmetry on the other side.

We are now free to couple these theories to the appropriate gauge fields and proceed to write the dual partition functions.

Chapter 5

Non-Abelian Quiver Dualities

Generalizations to quiver theories from the statements of non-abelian bosonization is a rather recent program by K. Jensen and collaborators[3]. It allows one to build, in principle, an infinite sequence of quiver dualities by starting from the bosonization dualities above and gauging the background fields. The result involves a quiver duality with bifundamental matter and non-abelian product gauge groups.

Let's see how this works for the simplest of the quiver dualities. We start with the non-abelian bosonization duality

$$U(k_1)_N \text{ with } k_2\ \phi \qquad \leftrightarrow \qquad SU(N)_{-k_1+\frac{k_2}{2}} \text{ with } k_2\ \psi \tag{5.1}$$

where $k_2 \leq k_1$ and couple the $SU(k_2)$ global currents to the background field A and $U(1)$ global current to $\tilde{A}$

$$\begin{aligned} Z_{\text{scalar}}[A, \tilde{A}] = \int Da\ \exp i \Bigg(S_{\text{scalar}}[\phi, a, A] + \frac{N}{4\pi} \int \text{tr}(ada + \frac{2}{3}a^3) - \frac{N}{2\pi} \int \text{tr}(a) d\tilde{A} \\ + \frac{Nk_1}{4\pi} \int \tilde{A} d\tilde{A}) \Bigg) \end{aligned} \tag{5.2}$$

$$Z_{\text{fermion}}[A,\tilde{A}] = \int Db \ \exp i\Big(S_{\text{fermion}}[\psi,b,A,\tilde{A}] + \frac{-k_1+\frac{k_2}{2}}{4\pi}\int \text{tr}(bdb+\frac{2}{3}b^3) + \frac{N}{4\pi}\int \text{tr}(AdA+\frac{2}{3}A^3) + \frac{Nk_2}{4\pi}\int \tilde{A}d\tilde{A}\Big) \tag{5.3}$$

Decompose the $U(k_1)$ gauge field a into a $SU(k_1)$ field a' and a $U(1)$ field $\tilde{a}$

$$a = a' + \tilde{a}\,\mathbb{I}_{k_1} \tag{5.4}$$

and shift the abelian part by $\tilde{A}$

$$\tilde{a} \to \tilde{a} + \tilde{A} \tag{5.5}$$

to get

$$Z_{\text{scalar}}[A,\tilde{A}] = \int Da \ \exp i\Big(S_{\text{scalar}}[\phi,a,A,\tilde{A}] + \frac{N}{4\pi}\int \text{tr}(ada+\frac{2}{3}a^3)\Big) \tag{5.6}$$

This gets rid of the Chern-Simons terms outside and introduces a minimally coupled $\tilde{A}$.

Now, on both sides of the duality, we may combine A and $\tilde{A}$ into a single $U(k_2)$ gauge field C

$$C = A + \tilde{A}\,\mathbb{I}_{k_2} \tag{5.7}$$

to get the partition functions

$$Z_{\text{scalar}}[C] = \int Da \ \exp i\Big(S_{\text{scalar}}[\phi,a,C] + \frac{N}{4\pi}\int \text{tr}(ada+\frac{2}{3}a^3)\Big) \tag{5.8}$$

$$Z_{\text{fermion}}[C] = \int Db \ \exp i\Big(S_{\text{fermion}}[\psi,b,A,\tilde{A}] + \frac{-k_1+\frac{k_2}{2}}{4\pi}\int \text{tr}(bdb+\frac{2}{3}b^3) + \frac{N}{4\pi}\int \text{tr}(CdC+\frac{2}{3}C^3) \tag{5.9}$$

To obtain the quiver duality, we simply have to gauge the $U(k_2)$ field C. We represent it as

$$U(k_1)_N \rightleftharpoons U(k_2)_0 \text{ with bifundamental } \phi \quad \leftrightarrow \quad SU(N)_{-k_1+\frac{k_2}{2}} \rightleftharpoons U(k_2)_N \text{ with bifundamental } \psi \tag{5.10}$$

We might as well multiply both sides of the duality by a Chern-Simons term for C, at some level N', to get the more general version of this quiver duality[3]

$$U(k_1)_N \rightleftharpoons U(k_2)_{N'} \text{ with bifundamental } \phi \;\; \leftrightarrow \;\; SU(N)_{-k_1+\frac{k_2}{2}} \rightleftharpoons U(k_2)_{N+N'} \text{ with bifundamental } \psi \tag{5.11}$$

Quiver theories, in general, have very interesting phase structure since their phases correspond to rich non-abelian theories. And these phases must be replicated on both sides of the duality. In fact, it can be shown that the phases are dual to each other via the level/rank dualities of Chern-Simons theories.

We may use this strategy to derive dualities between arbitrary linear quivers by bosonizing one of the nodes. Abelian versions of these dualities can be formulated by taking suitable values for N, N' etc.

Chapter 6

Conclusion

The particle vortex duality and the bosonization dualities are powerful tools to understand the behaviour of interacting quantum field theories in 2+1 dimensions. With the strategy to develop dualities involving non-abelian quiver theories we would like to conclude this work.

The central objective of this program so far has been to construct new dualities from previously established ones in novel ways. We must not forget that the program has some of the most powerful ideas waiting to find applications. We expect these duality conjectures to find home in condensed matter systems and aid in computations via dual quantities. In future, this line of thought connects best with experiment and would in turn provide fresh evidence for their validity.

Another important aspect of these dualities, for theorists, is the fact that these dualities are so surprising and rather counter-intuitive. After decades of fundamental distinction between bosonic and fermionic theories we now have statements claiming their equivalence at some level. One may be hopeful that this effort might bring in completely new ideas of describing theories in a way to directly see their underlying equivalence. A new, as yet undiscovered, description for quantum field theories that makes brings out these dualities manifestly. Such a bold expectation is probably the most exciting direction to look forward to.

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