Properties of elliptic modular forms

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Rohit Shelke

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2018

Supervisor: Debargha Banerjee

Certificate

This is to certify that this dissertation entitled Properties of elliptic modular forms towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Rohit Shelke at Indian Institute of Science Education and Research under the supervision of Debargha Banerjee, Assistant Professor, Department of Mathematics , during the academic year 2017-2018.

Debargha Banerjee

Debargha Banerjee

Committee:

Debargha Banerjee

Diganta Borah

This is dedicated to my teachers

Declaration

I hereby declare that the matter embodied in the report entitled Properties of elliptic modular forms are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Debargha Banerjee and the same has not been submitted elsewhere for any other degree.

Rohit Shelke

Abstract

The theory of modular forms is very rich. Modular forms for $SL_2(\mathbb{Z})$ and it's congruence subgroups have very interesting properties which we will explore. We will also focus on studying some comutational aspects of the modular forms along with the theory. We will see why fourier expansion of modular forms exits and why it plays an important role.

Contents

Chapter 1

Modular Forms

1.1 Fourier expansion of modular forms

Let f be a modular form of weight k. $U_{y_1} = \{z \in \boldsymbol{H} : Im(z) > y_1\}$ is mapped *via* $z \to e^{2\pi iz}$ into the punctured open disc centered at zero of radius $e^{-2\pi y_1}$.

$$A := \{z : 0 < |z| < 1\}$$

We define $F : A \to \mathbb{C}$ by $F(q) = f(z) \quad \forall\, q \in A,\;$ where $z \in \boldsymbol{H},\; q = e^{2\pi iz}$.
Here, $F(q) = f(log(q)/(2\pi i))$. F is well defined since $f(z+1) = f(z)$. f is holomorphic and logarithm function can be defined holomorphically on A. Since composition of holomorphic functions is a holomorphic function, $F(q)$ will be holomorphic on A.
Thus, F will have a Laurent expansion centered at $q = 0$,

$$F(q) = \sum_{n=-\infty}^{n=\infty} a_n q^n \;\; \forall\, q \in A,\; a_n \in \mathbb{C}$$

Now, we have $q = e^{2\pi iz}$. Let's take $z = x + iy$. By substituting this value of z, we get $q = e^{-2\pi y} \cdot e^{2\pi ix}$. So, $|q| = e^{-2\pi x}$. This gives us $q \to 0$ as $y \to \infty$. Therefore, we define f to be holomorphic at ∞ if F extends holomorphically to the point $q = 0$, which means the Laurent series sums over $n \in \mathbb{Z}$ for $n \geq 0$.

This implies that *Fourier expansion of* f will be as follows:

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi izn}$$

This expansion is called the *Fourier expansion* of the function f.

1.2 Modular forms for $SL_2(\mathbb{Z})$

Definition 1.2.1. $SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} m & t \\ h & a \end{bmatrix} : a, h, m, t \in \mathbb{Z} \ , \ ma - th = 1 \right\}$

Now we denote $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

These two matrices generate the *modular group* $SL_2(\mathbb{Z})$.

We define the upper half plane as $\boldsymbol{H} = \{z \in \mathbb{C} : Im(z) > 0\}$.
We denote the Riemann sphere by $\tilde{\mathbb{C}}$. For an element $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ and a point $z \in \mathbb{C}$, we define

$$\gamma z := \frac{az+b}{cz+d}, \ \gamma\infty := \frac{a}{c}$$

This map is also known as *fractional linear transformation* of the Riemann sphere. We can easily note that this defines a group action on $\tilde{\mathbb{C}}$. However more interestingly this also defines a group action on the upper half plane $\boldsymbol{H}$.
Firstly note that for any $\gamma \in SL_2(\mathbb{Z})$, Im $z > 0$ implies Im $\gamma z > 0$. This is because

$$Im \ \gamma z = Im \frac{az+b}{cz+d} = Im \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{Im(adz+bc\bar{z})}{|cz+d|^2} = \frac{Im \ z}{|cz+d|^2}$$

So, it makes sense to talk about the group action on $\boldsymbol{H}$.
Now take $\gamma_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $\gamma_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$(\gamma_1\gamma_2)(z) = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}(z) = \dfrac{(a_1a_2 + b_1c_2)z + a_1b_2 + b_1d_2}{(c_1a_2 + d_1c_2)z + c_1b_2 + d_1d_2}$,

$$\gamma_1(\gamma_2(z)) = \gamma_1(\frac{a_2z+b_2}{c_2z+d_2}) = \frac{a_1(\frac{a_2z+b_2}{c_2z+d_2})+b_1}{c_1(\frac{a_2z+b_2}{c_2z+d_2})+d_1} = \frac{(a_1a_2+b_1c_2)z+a_1b_2+b_1d_2}{(c_1a_2+d_1c_2)z+c_1b_2+d_1d_2}\ .$$

We can see that $I(z) = z$ and $(\gamma_1\gamma_2)(z) = \gamma_1(\gamma_2(z))\ \ \forall\ \gamma_1, \gamma_2 \in SL_2(\mathbb{Z})$.

Definition 1.2.2. *For an integer k, a meromorphic function $f : \boldsymbol{H} \to \mathbb{C}$ is weakly modular of weight k if*

$$f(\gamma(z)) = (cz+d)^k f(z) \qquad \forall\ \gamma \in SL_2(\mathbb{Z}) \text{ and } z \in \boldsymbol{H}$$

Now we will show that if this transformation law holds for S and T then it holds for all $\gamma \in SL_2(\mathbb{Z})$.

Proposition 1.2.1. *f is weakly modular of weight k if*

$$f(z+1) = f(z) \text{ and } f(-1/z) = z^k f(z)$$

Proof. First we prove that if γ_1 and γ_2 satisfy this transformation then $\gamma_1\gamma_2$ also satisfies the transformation.
Since by assumption f is weakly modular of weight k for γ_1 and γ_2, we have

$$f(\gamma_1(z)) = (c_1z+d_1)^k f(z) \text{ and } f(\gamma_2(z)) = (c_2z+d_2)^k f(z)$$

$$f(\gamma_1\gamma_2(z)) = f(\gamma_1(\gamma_2(z))) = f(\gamma_1(\tfrac{a_2z+b_2}{c_2z+d_2})) = (c_1(\tfrac{a_2z+b_2}{c_2z+d_2})+d_1)^k f(\tfrac{a_2z+b_2}{c_2z+d_2})$$

Thus,

$$f(\gamma_1\gamma_2(z)) = ((c_1a_2+d_1c_2)z+c_1b_2+d_1d_2)^k \cdot \frac{f(\gamma_2(z))}{(c_2z+d_2)^k} = ((c_1a_2+d_1c_2)z+c_1b_2+d_1d_2)^k f(z).$$

This shows that $\gamma_1\gamma_2$ satisfies the given transformation. Since $SL_2(\mathbb{Z})$ is generated by matrices S and T, any element in $SL_2(\mathbb{Z})$ can be written in the form as a combination of products of S's and T's.
Hence, it implies that if S and T follow the transformation law then f is weakly modular of weight k. □

Definition 1.2.3. *$f(z)$ is called a "Modular form of weight k" for $SL_2(\mathbb{Z})$ if*

1. $f(\gamma z) = (cz+d)^k f(z) \ \ \forall\ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$
2. *$f(z)$ is holomorphic on $\boldsymbol{H}$ and at infinity.*

As $-I \in SL_2(\mathbb{Z})$, there are no modular forms of odd weight for $SL_2(\mathbb{Z})$. Now we will look at some examples of modular forms for $SL_2(\mathbb{Z})$.

Eisenstein Series. Let k be an even integer which is greater than 2 and let $z \in \boldsymbol{H}$. The function

$$G_k(z) =: \sum_{\substack{m,n\ \in\ \mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

is a nonzero modular form of weight k for $SL_2(\mathbb{Z})$.

Proof. For k $\geq$ 4, $G_k(z)$ converges absolutely and uniformly in any compact subset of $\boldsymbol{H}$. But we know that a limit of an uniformly convergent sequence of holomorphic functions is holomorphic. Thus, $G_k(z)$ is a holomorphic function on $\boldsymbol{H}$.

Now we look at the holomorphicity of $G_k(z)$ at $i\infty$.

$$\lim_{z\to i\infty} G_k(z) = \lim_{z\to i\infty} \sum_{\substack{m,n\ \in\ \mathbb{Z} \\ (m,n)\neq(0,0) \\ m=0}} \frac{1}{(mz+n)^k} + \lim_{z\to i\infty} \sum_{\substack{m,n\ \in\ \mathbb{Z} \\ (m,n)\neq(0,0) \\ m\neq 0}} \frac{1}{(mz+n)^k} = \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{1}{n^k} = 2\zeta(k) < \infty$$

This is because,

$$\lim_{z\to i\infty} \sum_{\substack{m,n\ \in\ \mathbb{Z} \\ (m,n)\neq(0,0) \\ m\neq 0}} \frac{1}{(mz+n)^k} = 0 \ \ \textit{and} \ \sum_{n\in\mathbb{Z},\ n\geq 1} \frac{1}{n^k} = \zeta(k) < \infty$$

So, $G_k(z)$ is holomorphic at $i\infty$ as limit of $G_k(z)$ as $z \to i\infty$ exists and is finite.

Now it remains to show that $G_k(z)$ is weakly modular function, that is $G_k(z+1) = G_k(z)$ and $G_k(-1/z) = z^k G_k(z)$.

$$G_k(z+1) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m(z+1)+n)^k} = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(mz+(m+n))^k}$$

Let $m' = m,\ n' = m+n$.
We get,

$$G_k(z+1) = \sum_{\substack{m',n' \in \mathbb{Z} \\ (m',n')\neq(0,0)}} \frac{1}{(m'z+n')^k} = G_k(z)$$

$$G_k(-1/z) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m(-1/z)+n)^k} = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{z^k}{(nz+(-m))^k}$$

Let $m'' = n,\ n'' = -m$.
We get,

$$G_k(-1/z) = z^k \cdot \sum_{\substack{m'',n'' \in \mathbb{Z} \\ (m'',n'')\neq(0,0)}} \frac{1}{(m''z+n'')^k} = z^k \cdot G_k(z)$$

Note that we were able to interchange the summation because $G_k(z)$ is absolutely and uniformly convergent in compact subsets of the upper half plane. Also it is important to see that as (m,n) runs through $\mathbb{Z}^2 - \{(0,0)\}$, so do (m',n') and (m'',n'').
Thus, $G_k(z)$ is a nonzero modular form of weight k for $SL_2(\mathbb{Z})$. $\square$

Fourier expansion of Eisenstein series

First we define *Bernoulli numbers.*
The *Bernoulli numbers* B_k are defined using the following power series expansion.

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$

It is worth noting the fact that *Bernoulli numbers* are rational.

Proposition 1.2.2. *The modular form $G_k(z)$ has following q-expansion.*

$$G_k(z) = -\frac{(2\pi i)^k B_k}{k!} + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

$$\textit{where } q = e^{2\pi iz},\ \sigma_{k-1}(n) = \sum_{d|n,d>0} d^{k-1}.$$

Proof. We know,

$$sin\pi z = \pi z\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2})$$

Taking ln on both sides, we get

$$ln(sin\pi z) = ln(\pi z) + \sum_{n=1}^{\infty} ln(1-\frac{z^2}{n^2})$$

Now differentiating w.r.t. z gives,

$$\pi\frac{cos\pi z}{sin\pi z} = \frac{\pi}{\pi z} + \sum_{n=1}^{\infty}\frac{1}{1-\frac{z^2}{n^2}}\cdot\frac{-2z}{n^2} = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2} = \sum_{n\in\mathbb{Z}}\frac{1}{z+n} \tag{1.1}$$

But we also have,

$$\pi\frac{cos\pi z}{sin\pi z} = \pi i\cdot\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}} = \pi i\frac{q+1}{q-1} = \pi i - \frac{2\pi i}{1-q} = \pi i - 2\pi i\sum_{n=0}^{\infty} q^n \tag{1.2}$$

as $|q|<1$. Equating (2.1) and (2.2) and differentiating k-1 times w.r.t. z, we obtain

$$\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} n^{k-1}q^n$$

We have,

$$G_k(z) = \sum_{\substack{m,n\ \in\ \mathbb{Z}\\(m,n)\neq(0,0)\\m=0}}\frac{1}{(mz+n)^k} + \sum_{\substack{m,n\ \in\ \mathbb{Z}\\(m,n)\neq(0,0)\\m\neq 0}}\frac{1}{(mz+n)^k} = \sum_{n\in\mathbb{Z},\ n\neq 0}\frac{1}{n^k} + 2\cdot\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^k}$$

So,

$$G_k(z) = 2\zeta(k) + 2 \cdot \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} q^{mn}$$

And we can observe that the coefficient of q^x in this double series expansion is nothing but $\sigma_{k-1}(x)$. Thus, it only remains to show that $2\zeta(k) = -\frac{(2\pi i)^k B_k}{k!}$.
From (2.1), we get

$$\pi z cot \pi z = 1 - 2 \cdot \sum_{n=1}^{\infty} \frac{z^2}{n^2 - z^2} = 1 - 2 \cdot \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\frac{z}{n})^{2k} = 1 - 2 \cdot \sum_{k=2,\ even\ k}^{\infty} \zeta(k) z^k \tag{1.3}$$

From (2.2) and the definition of *Bernoulli numbers*, we get

$$\pi z cot \pi z = i\pi z + \sum_{k=0}^{\infty} B_k \frac{(2\pi i)^k}{k!} z^k \tag{1.4}$$

Equating (2.3), (2.4) and comparing the coefficient of z^k for an even k gives $2\zeta(k) = -\frac{(2\pi i)^k B_k}{k!}$. This gives us the desired Fourier expansion of Eisenstein series $G_k(z)$. □

Now we normalize this Eisenstein series by dividing it by the constant $2\zeta(k)$. Thus, $E_k(z)$ has rational coefficients.

$$E_k(z) = -\frac{k!}{B_k \cdot (2\pi i)^k} G_k(z)$$

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Now we will look at Fourier expansions of some Eisenstein series. The command used to calculate these Fourier expansions in SAGE is
`eisenstein_series_qexp(4, 10, normalization='constant')`.

$E_2(z) = 1-24*q-72*q^2-96*q^3-168*q^4-144*q^5-288*q^6-192*q^7-360*q^8-312*q^9+O(q^{10})$

$E_4(z) = 1 + 240 * q + 2160 * q^2 + 6720 * q^3 + 17520 * q^4 + 30240 * q^5 + 60480 * q^6 + 82560 * q^7 + 140400 * q^8 + 181680 * q^9 + O(q^{10})$

$E_6(z) = 1 - 504 * q - 16632 * q^2 - 122976 * q^3 - 532728 * q^4 - 1575504 * q^5 - 4058208 * q^6 - 8471232 * q^7 - 17047800 * q^8 - 29883672 * q^9 + O(q^{10})$

$E_8(z) = 1 + 480 * q + 61920 * q^2 + 1050240 * q^3 + 7926240 * q^4 + 37500480 * q^5 + 135480960 * q^6 + 395301120 * q^7 + 1014559200 * q^8 + 2296875360 * q^9 + O(q^{10})$

$E_{10}(z) = 1 - 264 * q - 135432 * q^2 - 5196576 * q^3 - 69341448 * q^4 - 515625264 * q^5 - 2665843488 * q^6 - 10653352512 * q^7 - 35502821640 * q^8 - 102284205672 * q^9 + O(q^{10})$

$E_{12}(z) = 1 + 65520/691 * q + 134250480/691 * q^2 + 11606736960/691 * q^3 + 274945048560/691 * q^4 + 3199218815520/691 * q^5 + 23782204031040/691 * q^6 + 129554448266880/691 * q^7 + 563087459516400/691 * q^8 + 2056098632318640/691 * q^9 + O(q^{10})$

$E_{14}(z) = 1 - 24 * q - 196632 * q^2 - 38263776 * q^3 - 1610809368 * q^4 - 29296875024 * q^5 - 313495116768 * q^6 - 2325336249792 * q^7 - 13195750342680 * q^8 - 61004818143672 * q^9 + O(q^{10})$

$E_{16}(z) = 1 + 16320/3617 * q + 534790080/3617 * q^2 + 234174178560/3617 * q^3 + 17524001357760/3617 * q^4 + 498046875016320/3617 * q^5 + 7673653657232640/3617 * q^6 + 77480203842286080/3617 * q^7 + 574226476491096000/3617 * q^8 + 3360143509958850240/3617 * q^9 + O(q^{10})$

Eisenstein series of weight 2 is not a modular form as it is not a weakly modular function.

Proposition 1.2.3. *For $z \in \boldsymbol{H}$,*

$$E_2(-1/z) = z^2 \cdot E_2(z) - \frac{6i}{\pi} z$$

1.3 Congruence subgroups of $SL_2(\mathbb{Z})$

For $N \in \mathbb{N}$, the principal congruence subgroup of level N is

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (modN) \right\}$$

Definition 1.3.1. *A congruence subgroup of $SL_2(\mathbb{Z})$ is a subgroup of $SL_2(\mathbb{Z})$ which contains $\Gamma(N)$.*

1.3.1 Examples and properties of congruence subgroups

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0(mod\ N) \right\}$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : a \equiv 1(mod\ N),\ c \equiv 0(mod\ N)\ and\ d \equiv 1(mod\ N) \right\}$$

Now we will state some properties of these congruence subgroups and the relations between them.

We have,

$$|SL_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{p|N} (1 - \frac{1}{p^2})$$

So,the index of congruence subgroup $\Gamma(N)$ in $SL_2(\mathbb{Z})$ is finite.

$$[SL_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p|N} (1 - \frac{1}{p^2}) < \infty$$

Since $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z}),\ \Gamma_1(N)$ and $\Gamma_0(N)$ will have finite index in $SL_2(\mathbb{Z})$. In particular,

$$[SL_2(\mathbb{Z}) : \Gamma_1(N)] = N^2 \prod_{p|N} (1 - \frac{1}{p^2})$$

and

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1 + \frac{1}{p})$$

Definition 1.3.2. *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let $k \in \mathbb{Z}$. A function $f : \boldsymbol{H} \to \mathbb{C}$ is a "Modular form of weight k" for Γ if*

1. $f(\gamma z) = (cz+d)^k f(z) \ \ \forall\, \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$
2. *$f(z)$ is holomorphic on $\boldsymbol{H}$*
3. *$f(z)$ is holomorphic at the cusps.*

We have seen that Eisenstein series of weight 2 is not a modular form for $SL_2(\mathbb{Z})$. But interestingly it can be made into a modular form for $\Gamma_0(N)$.
We define,

$$E_2^N(z) := E_2(z) - NE_2(Nz)$$

This $E_2^N(z)$ is a modular form for $\Gamma_0(N)$. We only have to check that it is a weakly modular function $\forall\, \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$.

We need to show that $E_2^N(\gamma z) = (cz+d)^2 E_2^N(z)$, that is $(cz+d)^{-2} E_2^N(\gamma z) = E_2^N(z)$.

$$(cz+d)^{-2} E_2^N(\frac{az+b}{cz+d}) = (cz+d)^{-2} E_2(\frac{az+b}{cz+d}) - N(cz+d)^{-2} E_2(N \cdot \frac{az+b}{cz+d})$$

We can write,

$$(cz+d)^{-2} E_2^N(\frac{az+b}{cz+d}) = (cz+d)^{-2} E_2(\frac{az+b}{cz+d}) - N((c/N)(Nz)+d)^{-2} E_2(\frac{a(Nz)+bN}{(c/N)(Nz)+d})$$

Using the fact that $(cz+d)^{-2} E_2(\frac{az+b}{cz+d}) = E_2(z) - \frac{1}{4\pi i} \cdot \frac{c}{cz+d}$,
we get,

$$(cz+d)^{-2} E_2^N(\frac{az+b}{cz+d}) = E_2(z) - \frac{1}{4\pi i} \cdot \frac{c}{cz+d} - N(E_2(Nz) - \frac{1}{4\pi i} \cdot \frac{c/N}{(c/N)(Nz)+d})$$

So,

$$(cz+d)^{-2} E_2^N(\frac{az+b}{cz+d}) = E_2(z) - NE_2(Nz) = E_2^N(z)$$

This concludes that $E_2^N(z)$ is a weakly modular function $\forall\, \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$.

Chapter 2

Modular Curves as Riemann Surfaces

Modular curve $Y(\Gamma)$ for a congruence subgroup Γ is defined as follows:

$$Y(\Gamma) = \{\Gamma z : z \in \boldsymbol{H}\}$$

We define local coordinates on the modular curve $Y(\Gamma)$ to make it into a Riemann surface which can be further compactified. In order to compactify the modular curve $Y(\Gamma)$, we take the extended quotient by $\boldsymbol{H}'$, where $\boldsymbol{H}' = \boldsymbol{H} \cup \mathbb{Q} \cup \{\infty\}$.
We have,

$$X(\Gamma) = \Gamma \backslash \boldsymbol{H}'$$

Theorem 2.0.1. *The modular curve $X(\Gamma)$ is connected, Housdorff and compact.*

2.1 Fundamental domain

Definition 2.1.1. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ and $\mathcal{F} \subset \boldsymbol{H}$ be a closed set which is simply connected. We say $\mathcal{F}$ is a $fundamental$ $domain$ for Γ if*

1. *every $z \in \boldsymbol{H}$ is Γ-equivalent to a point in $\mathcal{F}$,*

2. *no two distinct interior points of $\mathcal{F}$ are Γ-equivalent.*

$$\mathcal{F} =: \{z \in \boldsymbol{H} | -\frac{1}{2} \leq Re\ z \leq \frac{1}{2}\ and\ |z| \geq 1\}.$$

This defined region $\mathcal{F}$ is a *fundamental domain* for $SL_2(\mathbb{Z})$. Now we will look at some fundamental domains for congruence subgroups of $SL_2(\mathbb{Z})$. For a congruence subgroup G of $SL_2(\mathbb{Z})$, the command used to calculate these fundamental domains in SAGE is `FareySymbol(G).fundamental_domain()`

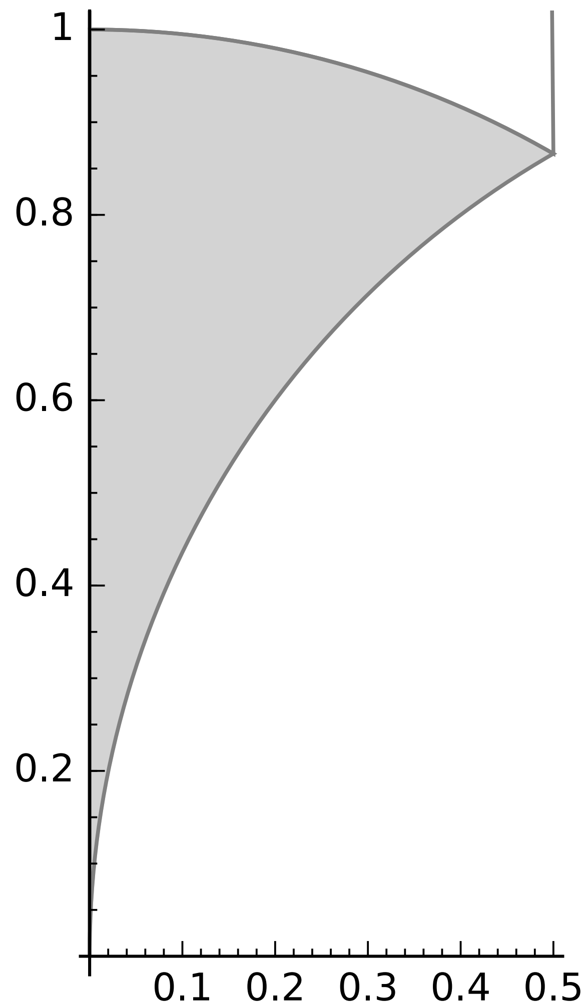

Figure 2.1: Fundamental domain for $SL_2(\mathbb{Z})$

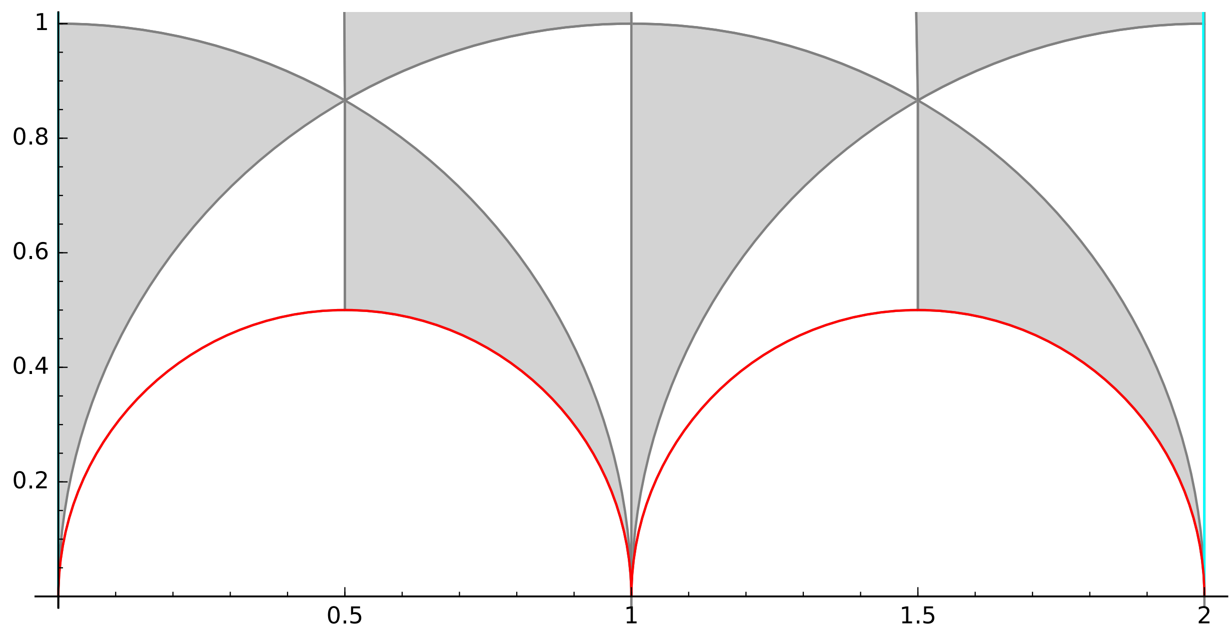

Figure 2.2: Fundamental domain for $\Gamma(2)$

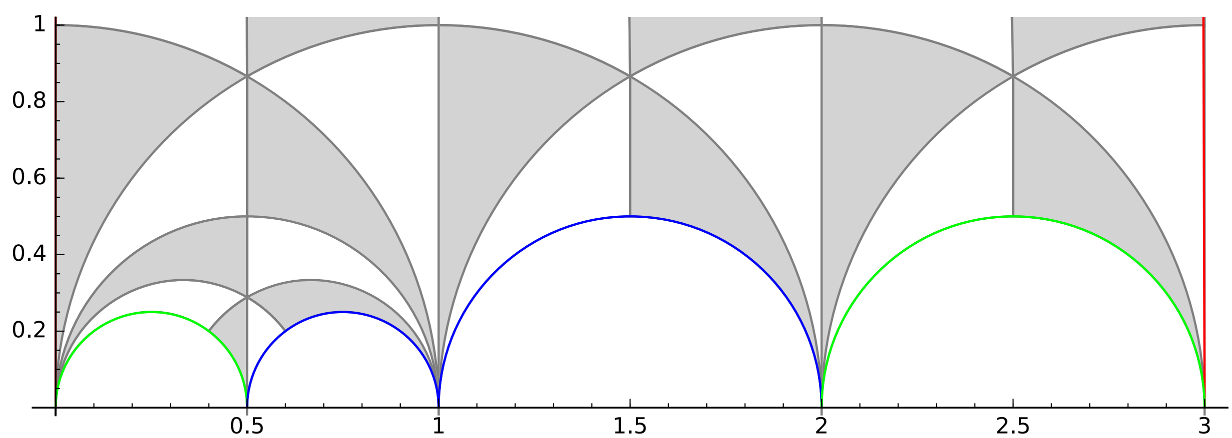

Figure 2.3: Fundamental domain for $\Gamma(3)$

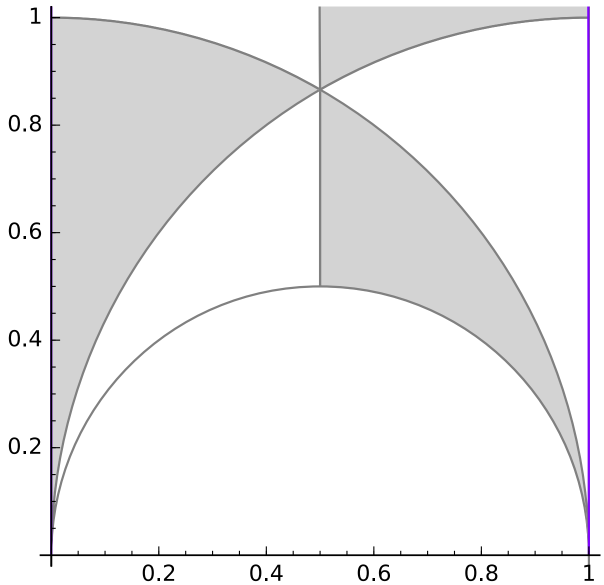

Figure 2.4: Fundamental domain for $\Gamma_0(2)$

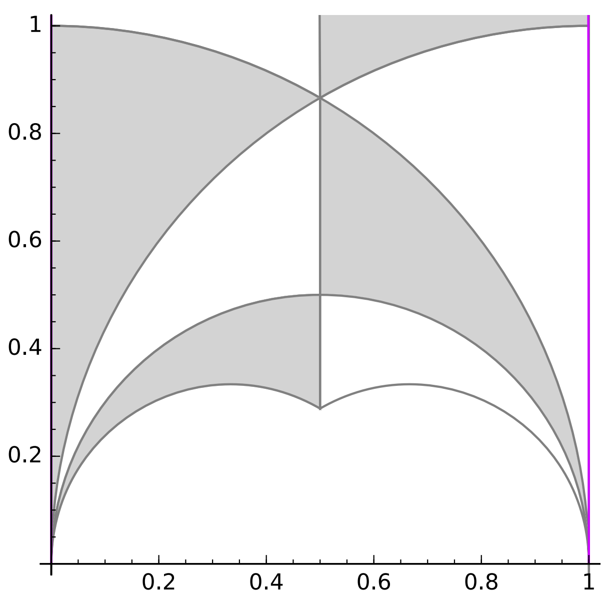

Figure 2.5: Fundamental domain for $\Gamma_0(3)$

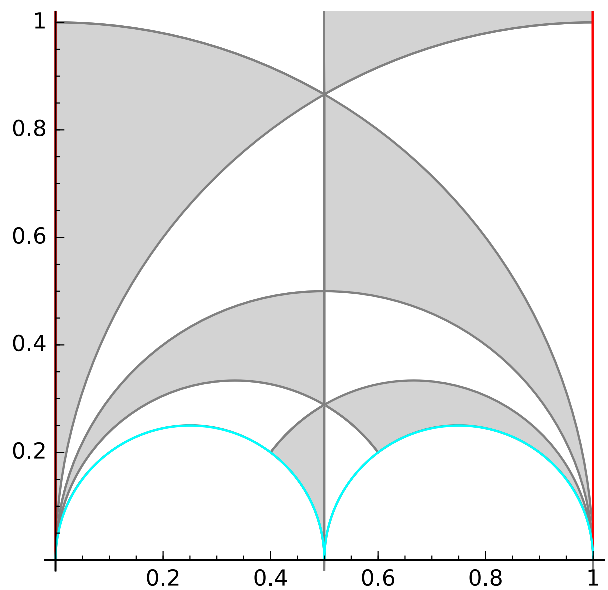

Figure 2.6: Fundamental domain for $\Gamma_0(4)$

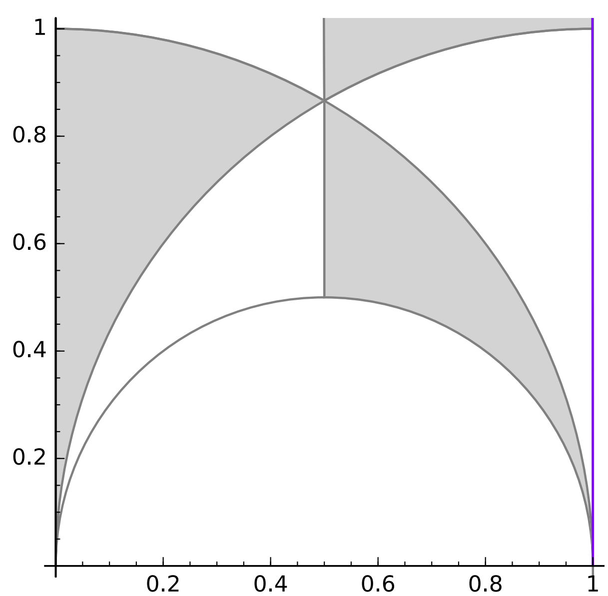

Figure 2.7: Fundamental domain for $\Gamma_1(2)$

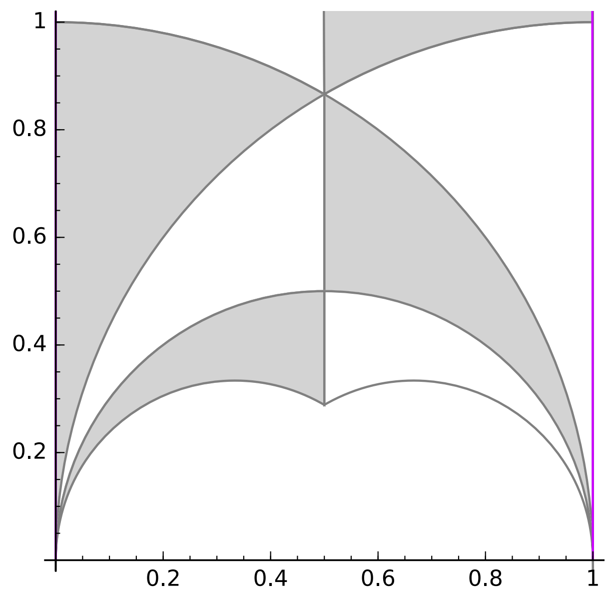

Figure 2.8: Fundamental domain for $\Gamma_1(3)$

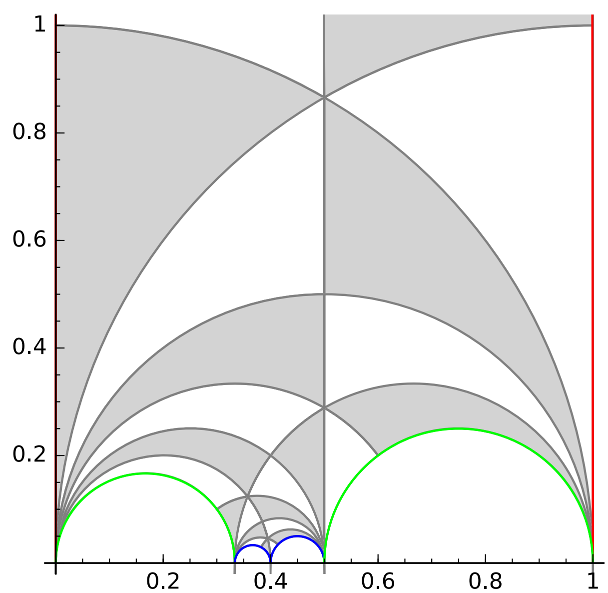

Figure 2.9: Fundamental domain for $\Gamma_1(5)$

It is important to note that if $\mathcal{F}$ is a fundamental domain then $\gamma\mathcal{F}$ is also a fundamental domain for $SL_2(\mathbb{Z})$.

Proposition 2.1.1. *Let* Γ *be a congruence subgroup of* $SL_2(\mathbb{Z})$, *written in the form of disjoint union of cosets as*

$$SL_2(\mathbb{Z}) = \bigcup_{i=1}^{n} \alpha_i \Gamma.$$

Then $\mathcal{F}' := \bigcup_{i=1}^{n} \alpha_i^{-1}\mathcal{F}$ *is a fundamental domain for* Γ*.*

2.2 Elliptic points

Definition 2.2.1.

$$\Gamma_z := \{\gamma \in \Gamma : \gamma z = z\}$$

Γ_z is known as the isotropy subgroup of z.

Definition 2.2.2. *Let* Γ *be a congruence subgroup of* $SL_2(\mathbb{Z})$*. A point* $z \in \boldsymbol{H}$ *is an elliptic point for* Γ *if the isotropy group of* z *is nontrivial as a group of transformation.*

Proposition 2.2.1. *The elliptic points for* $SL_2(\mathbb{Z})$ *are* i, $e^{\pi i/3}$ *and* $e^{2\pi i/3}$.

2.3 Cusps

Cusps of a congruence subgroup Γ are the $\Gamma - \textit{equivalence}$ classes of $\mathbb{Q} \cup \{\infty\}$.

2.3.1 Examples

The command used to calculate these cusps in SAGE is
`G.cusps()` for some congruence subgroup G of $SL_2(\mathbb{Z})$.

Cusps for $\Gamma_0(12)$ are [0, 1/6, 1/4, 1/3, 1/2, Infinity].

Cusps for $\Gamma(10)$ are [0, 1/5, 1/4, 3/10, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 1, 5/4, 4/3, 3/2, 5/3, 7/4, 2, 9/4, 7/3, 5/2, 8/3, 3, 10/3, 7/2, 4, 13/3, 9/2, 5, 16/3, 6, 19/3, 7, 8, 9, Infinity].

Cusps for $\Gamma_1(10)$ are [0, 1/5, 1/4, 3/10, 1/3, 2/5, 1/2, Infinity].

Cusps for $\Gamma_0(10)$ are [0, 1/5, 1/2, Infinity].

Cusps for $\Gamma(4)$ are [0, 1/2, 1, 2, 3, Infinity].

Cusps for $\Gamma_1(4)$ are [0, 1/2, Infinity].

Cusps for $\Gamma_0(4)$ are [0, 1/2, Infinity].

Chapter 3

Dimension formulas

3.1 Dimension formula for modular forms for $SL_2(\mathbb{Z})$

Theorem 3.1.1 (The valence formula)**.** *Let $f(z)$ be a nonzero modular function of weight k for $SL_2(\mathbb{Z})$. For $z \in \boldsymbol{H}$, let $v_z(f)$ denote the order of zero (or minus the order of pole) of $f(z)$ at the point z. Let $v_\infty(f)$ be the index of the first non vanishing term in the $q-$expansion of $f(z)$. Then*

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\omega(f) + \sum_{\substack{z \in SL_2(\mathbb{Z})\backslash \boldsymbol{H} \\ z \neq i,\omega}} v_z(f) = \frac{k}{12}. \tag{3.1}$$

Let $M_k(\Gamma)$ denote the vector space of modular forms for congruence subgroup Γ.
Using this valence formula we can deduce that $M_k(SL_2(\mathbb{Z})) = \{0\}\ \forall\ k < 2$ *and* $k = 2$. Also, we can write $M_k(SL_2(\mathbb{Z})) = \mathbb{C}E_k$ for $k = 4, 6, 8, 10$.

Proposition 3.1.2. *$M_k(SL_2(\mathbb{Z})) = \mathbb{C}E_k \oplus S_k(SL_2(\mathbb{Z}))\ \forall\ k \geq 4$.*

From this proposition it is easy to see that dim $M_k(SL_2(\mathbb{Z})) = 1+$ dim $S_k(SL_2(\mathbb{Z}))$.

Theorem 3.1.3.

$$dim\ M_k(SL_2(\mathbb{Z})) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1, & if\ k \not\equiv 2\ (mod\ 12) \\ \lfloor \frac{k}{12} \rfloor, & if\ k \equiv 2\ (mod\ 12) \end{cases} \tag{3.2}$$

$$dim\ S_k(SL_2(\mathbb{Z})) = \begin{cases} \lfloor \frac{k}{12} \rfloor, & if\ k \not\equiv 2\ (mod\ 12) \\ \lfloor \frac{k}{12} \rfloor - 1, & if\ k \equiv 2\ (mod\ 12) \end{cases} \tag{3.3}$$

Thus, the spaces $S_k(\Gamma)$ and $M_k(\Gamma)$ are finite-dimensional complex vector spaces.

3.2 Dimension formula for modular forms for congruence subgroups of $SL_2(\mathbb{Z})$

Theorem 3.2.1 (The valence formula). *Let f be a nonzero modular function of weight k for the congruence subgroup Γ of $SL_2(\mathbb{Z})$). Then*

$$\sum_{z \in \Gamma \backslash \boldsymbol{H}'} \frac{v_z(f)}{|\Gamma_z|} = \frac{k}{2} \cdot \left(\frac{\epsilon_2}{2} + \frac{2\epsilon_3}{3} + \epsilon_\infty + 2g - 2\right) \tag{3.4}$$

where, Γ_z is the isotropy subgroup of z, g is the genus of $\Gamma \backslash \boldsymbol{H}'$, ϵ_2 is the number of elliptic points of order 2, ϵ_3 is the number of elliptic points of order 3, and ϵ_∞ is the number of Γ-inequivalent cusps.

Theorem 3.2.2 (The dimension formula). *For a congruence subgroup Γ of $SL_2(\mathbb{Z})$) and an even non-negative integer k, We have,*

$$dim\ M_k(\Gamma) = \begin{cases} (k-1)(g-1) + \lfloor \frac{k}{4} \rfloor \epsilon_2 + \lfloor \frac{k}{3} \rfloor \epsilon_3 + \frac{k}{2}\epsilon_\infty\ , & if\ k \geq 2 \\ 1\ , & if\ k = 0 \end{cases} \tag{3.5}$$

and

$$dim\ S_k(\Gamma)) = \begin{cases} dim\ M_k(\Gamma) - \epsilon_\infty\ , & if\ k \geq 4 \\ g\ , & if\ k = 2 \\ 0\ , & if\ k = 0 \end{cases} \tag{3.6}$$

Bibliography

[1] Diamond,F., Shurman,J. *A First Course in Modular Forms*, Springer, 2005.

[2] Lang,S. *INTRODUCTION TO MODULAR FORMS*, Springer, 2001.

[3] Koblitz,N. *Introduction to Elliptic Curves and Modular Forms*, Springer, 1993.

[4] Murty,R., Dewar,M., Graves,H. *Problems in the Theory of Modular Forms*, HINDUSTAN BOOK AGENCY, 2015.

[5] Kilford, L.J.P. *MODULAR FORMS A Classical and Computational Introduction*, Imperial College Press, 2008.