

Emergence of controlled dynamics and role of topology in complex networks

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by

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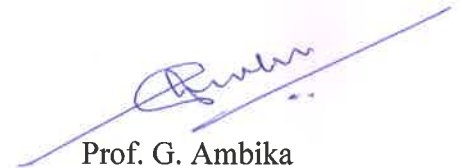
Supervisor: Prof G Ambika

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Certificate

This is to certify that this dissertation entitled “Emergence of controlled dynamics and role of topology in complex networks”, submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents the study/work carried out by Harsh Gakhare at Indian Institute of Science Education and Research under the supervision of Prof. G. Ambika, Professor, Department of Physics, during the academic year 2017-2018.



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Date: March 30, 2018.

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Declaration

I hereby declare that the matter embodied in the report entitled “Emergence of controlled dynamics and role of topology in complex networks” are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Prof. G. Ambika and the same has not been submitted elsewhere for any other degree.



Harsh Gakhare

Date: March 30, 2018.

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Abstract

The complexity of several systems is a confluence of the intricate structure of their interactions and the intrinsic dynamics of its constituent elements. Complex network framework can be used to study such systems effectively. Several recent studies indicate that understanding of the complex systems may be better achieved by using both the nodal dynamics and the interaction topology synergetically to alter the behaviour of the system.

We discuss properties of individual nonlinear systems and the dynamics displayed in collective behaviour including synchronization, phase-synchronization, amplitude and oscillation deaths, chimera state behaviour etc. We explore the emergent dynamics especially synchronization, by investigating control strategies where: i) strategic rewiring brings system toward desired synchronization dynamics and ii) the system dynamics drives changes in the interaction topology leading to an emergent stable network structure.

We observe the formation of synchronization in clusters where the clusters either remain or coalesce into larger clusters. We are also able to change the number of clusters by varying control parameters. For dynamics driven control of topology, we observe resultant networks showing characteristics of scalefree type networks. Finally, we identify possible improvements to control procedures studied and propose a method involving externally driven control to extend the study further.

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Chapter 1

Introduction

Complex systems are studied using the framework of complex networks [1] [2] [3] .

Simple networks such as lattices have found use in describing many physical systems, for example, the Ising models [4] and many other solid state models [5]. Similarly, the star and ring topology have been used extensively in modelling early physical networks of computers [6] [7]. In biology, the trophic pyramid has been used to represent a simple hierarchy of energy transport up the food chain. With the rapid increase in information available and naturally occurring systems being complex these simple models do not suffice. Ecological relations are represented as a complex web of interactions between species and their environment with every unit considered to play a key role in the network [8]. The naive modelling of the internet as a set of star networks connected to each other do not suffice and are now treated as a giant growing complex network [9]. Some examples of real-world complex systems where complex networks framework have found useful are:

- World trade networks and financial market networks [10]
- Transportation (Air travel Networks) [11]
- Disease spreading models [12]

- Ecological webs [13]
- Dependency graphs in software package distributions [14] [15]
- Social network analysis [16]

All of these have descriptions that involve the language of graph theory. A brief review of some graph theory fundamentals used in describing complex networks is given below.

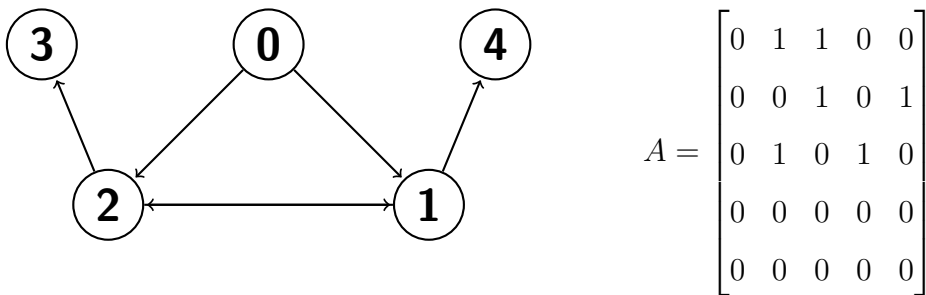
1.0.1 Complex Network Framework

Topology of a network can be represented by a graph.

Graph: A graph G is a pair (V, E) on sets satisfying $E \subseteq V^2$ where V is the set of vertices (nodes) and E is the set of edges (links).

Adjacency Matrix: An adjacency matrix $A_{i,j}$ is a representation of a graph, where (i, j) represents an edge from node i to node j such that:

$$A_{i,j} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{if } (i, j) \notin E \end{cases}$$



Connected graph: A graph G is connected if for any two vertices v_i, v_j in the set of all vertices V there exists a path joining v_i and v_j .

Degree of node: The degree of a node or vertex, $deg(i)$ is the number of edges that vertex i is incident to.

$$deg(i) = \sum_j A_{i,j}$$

Clustering coefficient: Clustering coefficient is a measure of the likelihood of a vertex's neighbours themselves being neighbours. For a node v_i it is defined as the number of triangles with vertex v_i divided by the number of possible triangles given its degree.

$\tau(i)$ - No of triangles in graph such that v_i is a vertex

$$cc(i) = \frac{\sum_i \tau_i}{\binom{deg(i)}{2}}$$

Degree Distribution: In a graph the probability function $p(k)$ of obtaining a node with degree k is called the degree distribution. It is usually defined for graph at the large N limit $N \rightarrow \infty$. For random Erdős–Rényi graphs the degree distribution is Poisson, i.e. for random graph with average degree λ :

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Another type of distribution commonly found in many real world networks is the power law distribution of scale free networks. The degree distribution for scale free graph with parameter λ is:

$$p(k) = k^{-\lambda}$$

Characteristic path length: The mean of lengths of shortest paths between every pair of nodes in a network is called the characteristic path length. For a graph on N nodes:

$$d(i, j) = \text{length of a shortest path from } i \text{ to } j$$

$$cpl = \sum_{i,j} \frac{d(i,j)}{N(N-1)}$$

1.0.2 Types of networks

Regular Network

A network in which all nodes are connected to the same number of immediate neighbours is a regular network. Such networks can have varying topologies from lattices to regular ring graphs. Regular networks represent systems with ordered components with local connections, for example, a crystal lattice, a regular road network etc. In regular network, all nodes have the same degree and the average path lengths are large.

Erdős–Rényi model (Random graphs)

Erdős–Rényi (ER) graph is a model for obtaining random graphs among all possible graphs of given node number N . These are often used as a baseline network while studying more complex graphs. A random ER graph on N nodes can be described in two ways.

- $G(n, m)$: A graph with m edges randomly chosen from set of all N^2 possible edges.
- $G(n, p)$: A graph constructed by accepting each possible edge with probability p .

Barabási – Bollobás model (for scale free networks.)

Barbasi and Bollabas came up with a model of growing networks with a preferential attachment of new nodes to higher degree nodes of the existing network. These networks have been shown to have very high clustering coefficients and small world properties which are also properties of many real-world systems. In these networks, the degree distribution shows power law. A generative procedure to create a scale-free network $G(N, m_0)$ is as follows

1. Start with connected network of m_0 nodes.
2. Add a new node j to graph and connect it to m already existing nodes with sampling probability

$$p_i = \frac{k_j}{\sum_j k_j}$$

1.0.3 Dynamical systems

The dynamics of complex systems can be modelled with dynamical systems on the nodes of the underlying network. A dynamical system is a particle or ensemble of particles whose state varies over time and thus obeys differential equations involving time derivatives. Dynamical systems are used to model a large number of real-world systems with application in physics, biology, engineering, chemistry, economics and recently growing use in social sciences. Evolution of a continuous dynamical system can be represented as a differential equation.

$$\dot{x} = f(x)$$

Simple nonlinear dynamical systems can show unpredictable behaviour. Such systems are called chaotic systems. Chaotic systems are completely deterministic systems whose dynamics after a short time of evolution cannot be predicted given any margin of error in initial conditions. While they are bounded and aperiodic, most chaotic systems are characterised by a sensitivity to initial conditions.

Chaos is often confused with stochastic behaviour, but these are completely different. Stochastic processes have some probabilistic component in them while chaos is completely deterministic. Chaotic systems which occur in natural systems like weather [17] and which involve systems as simple as the double pendulum [19] have potential application in communications, computing[18], systems engineering, understanding biological processes etc. In some complex systems including biological systems like the brain or heart also chaos is observed. Following are some of the well-studied examples of non-linear systems that show

chaotic behaviour for certain parameter ranges.

Lorenz System

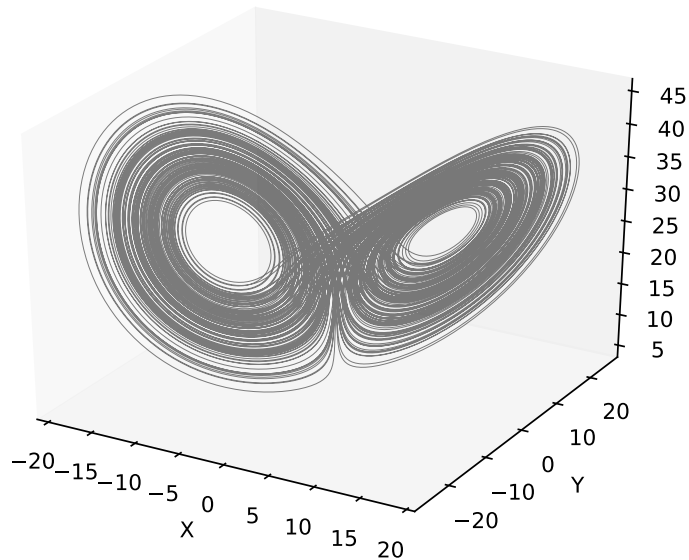


Figure 1.1: Trajectory of Lorenz system for parameter values $\sigma = 10, r = \frac{8}{3}, b = 28$ shows a chaotic butterfly pattern.

$$\begin{aligned}\dot{x} &= \sigma \cdot (y - x) \\ \dot{y} &= rx - xz - y \\ \dot{z} &= xy - bz\end{aligned}\tag{1.1}$$

The Lorenz system (Eqn. 1.1) is a set of ordinary differential equations that were first derived by Edward Lorenz in 1963 to model convection in atmospheric systems. The system exhibits non periodic behaviour and for certain range of parameters, it exhibits chaos. The system is known for its characteristic butterfly shaped chaotic attractor. One set of parameter values at which it shows chaos that are extensively studied and often used are $\sigma = 10, r = \frac{8}{3}, b = 28$. The trajectory of Lorenz system is shown in in Figure 1.1

Rössler System

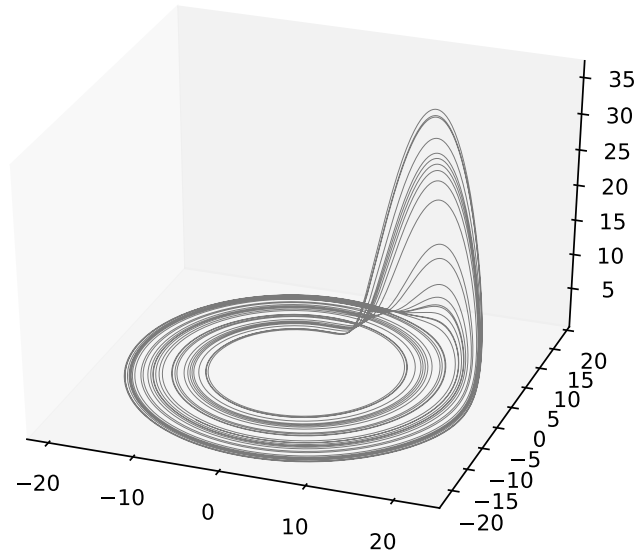


Figure 1.2: Trajectory of Rössler system for parameter values $a = 0.1, b = 0.1, c = 14$

$$\begin{aligned} \dot{x}_i &= -y - z \\ \dot{y}_i &= x + ay \\ \dot{z}_i &= b + z_i(x - c) \end{aligned} \tag{1.2}$$

The Rössler system is a chaotic system first studied by Otto Rössler. The system was intended to be a simpler version of the Lorenz attractor. The phase space trajectory of a Rössler system is shown in Figure 1.2.

In this project, we present the results of our study using standard non-linear systems connected on a complex network. We analyse various measures to identify the nature of the collective behaviour of such a system especially synchronization. We propose different control mechanisms to optimize the synchronization with low coupling and minimum con-

nections. We also study the formation of synchronization in clusters before the full network synchronizes.

Chapter 2

Emergent phenomenon on complex networks

2.1 Dynamics of interacting non linear systems

Linear systems are associated with properties of homogeneity and additivity [20] and therefore they are considered superimposable. For example, the dynamics of a particle in a medium carrying two waves can be represented as just the sum of deviation from mean position caused by both waves. Non linear systems show a diversity of behaviour in response to coupling together with other non linear systems. Often multiple interacting chaotic systems will evolve and exhibit dynamics different from intrinsic dynamics in the chaotic regime. As mentioned these can range from periodic oscillations, amplitude death, unbound destabilization etc. Some classes of behaviour displayed by interacting non-linear systems are described here:

- **Synchronization:** Non-linear chaotic systems are very sensitive to initial conditions and very hard to predict in general Two almost identical systems with small differences

will diverge very soon from each other if they are independent. Interacting systems, on the other hand, have been observed to show decreasing deviation from each other until their dynamics is synchronized.

- Complete synchronization: Complete synchronization is when the systems evolve identically over time. $x_1(t) = x_2(t)$ as $t \rightarrow \infty$

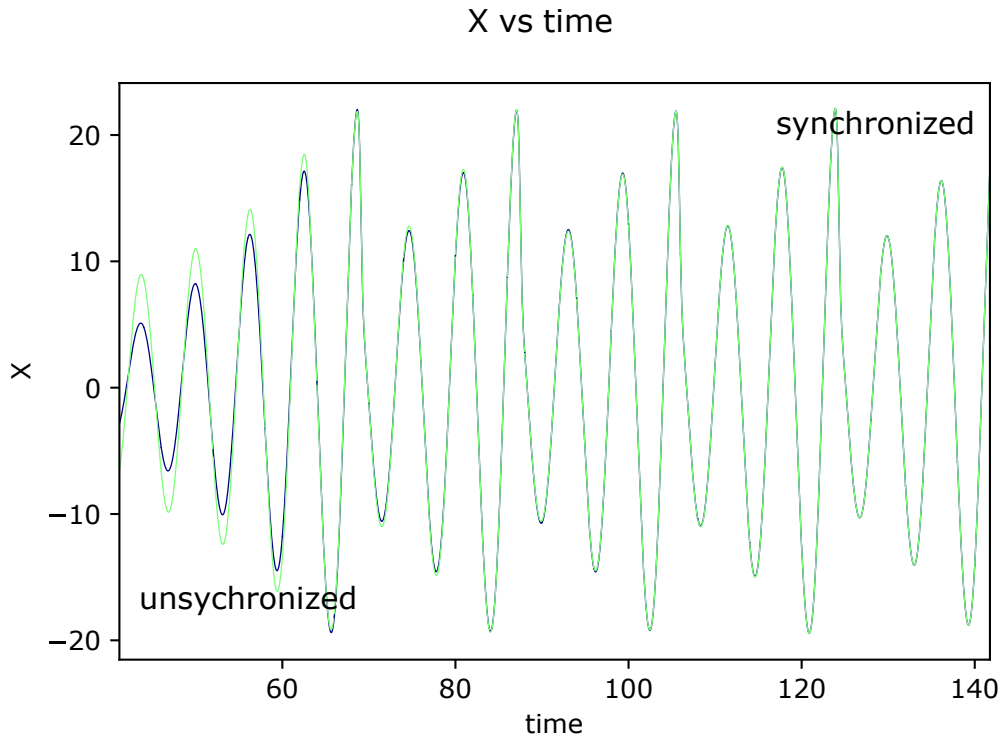


Figure 2.1: Times series of x-component of two Rössler systems exhibiting full synchronization.

- Generalized synchronization: Systems can show synchronized behaviour without having identical trajectories in time. Two heterogeneous systems when coupled often do not completely overlap in their dynamics. But when the dynamics of one system can be described as a function of the other for any given time after synchronization is said to be achieved, it is called generalised synchronization.

$$x_1(t) = \Phi(x_2(t))$$

- Phase synchronization: Given a defined phase angle for the dynamical state of

each system, two systems are said to be phase synchronized when there is a linear relation between the phase of the two systems.

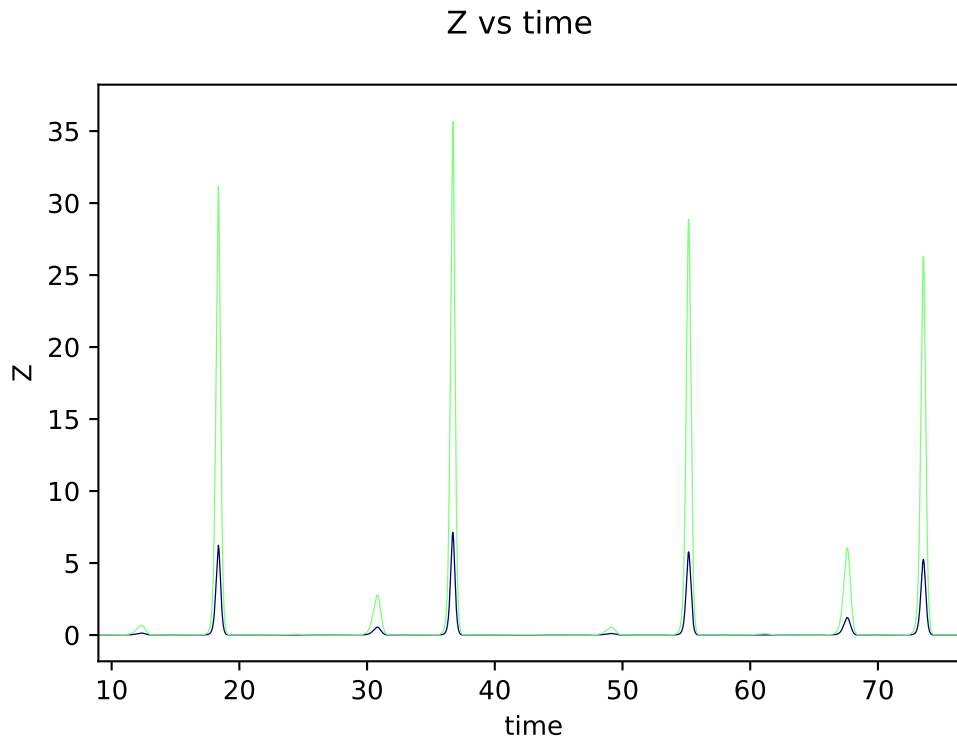


Figure 2.2: Times series of z-component of two Rössler systems showing phase synchronization.

- Lag synchronization: When one system follows the trajectory of the other with a constant time-delay the two systems are said to be lag-synchronized or delay synchronized.
- **Amplitude death:** When systems are connected by an interaction the previously unstable equilibrium points of the individual systems may be stabilized due to the interaction. When this leads to a decay of oscillations towards the newly stabilized equilibrium point, it is called amplitude death. If the systems are identical then it is very likely that the oscillations decay to the same equilibrium point for all of them. The resultant state is also called a homogeneous steady state (HSS) and is distinct from the heterogeneous fixed point behaviour called oscillation death.

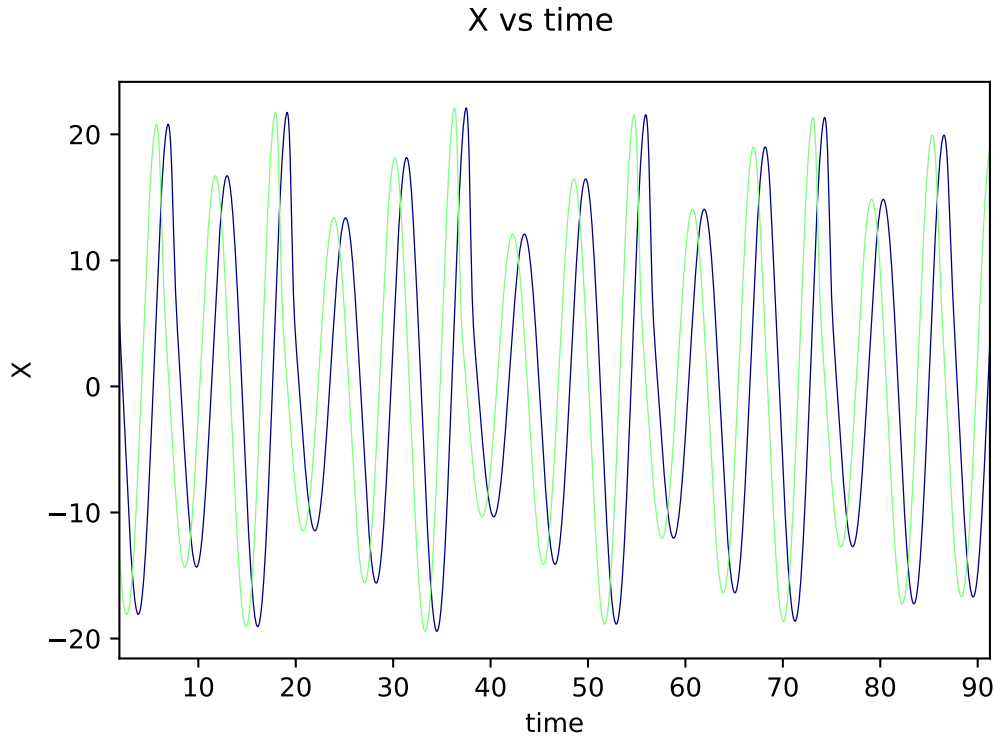


Figure 2.3: Times series of x-component of two Rössler systems showing lag synchronization. The green timeseries follows the blue timeseries exactly but with a time lag.

- **Oscillation Death:** The coupling interaction between systems may give rise to new equilibrium states that do not exist in the individual independent systems. If the individual dynamics was originally limit-cycles or non-periodic oscillations, after coupling the systems can transient to one of the new equilibrium states through a decrease in amplitude of these oscillations. This behaviour is called oscillation death. When the transients are very short it is also called explosive oscillation death.[21] [22]
- **Chimera state:** Chimeras are a peculiar state first reported in networks of non-locally coupled Kuramoto oscillators which showed dynamically stable groups of coherent and incoherent nodes existing simultaneously. Named after monstrous creatures from Greek mythology which are a composition of disparate body parts of many animals, chimeras in the context of non-linear dynamics are states of coupled systems where multiple behaviours coexist. While different behaviour is expected in complex networks

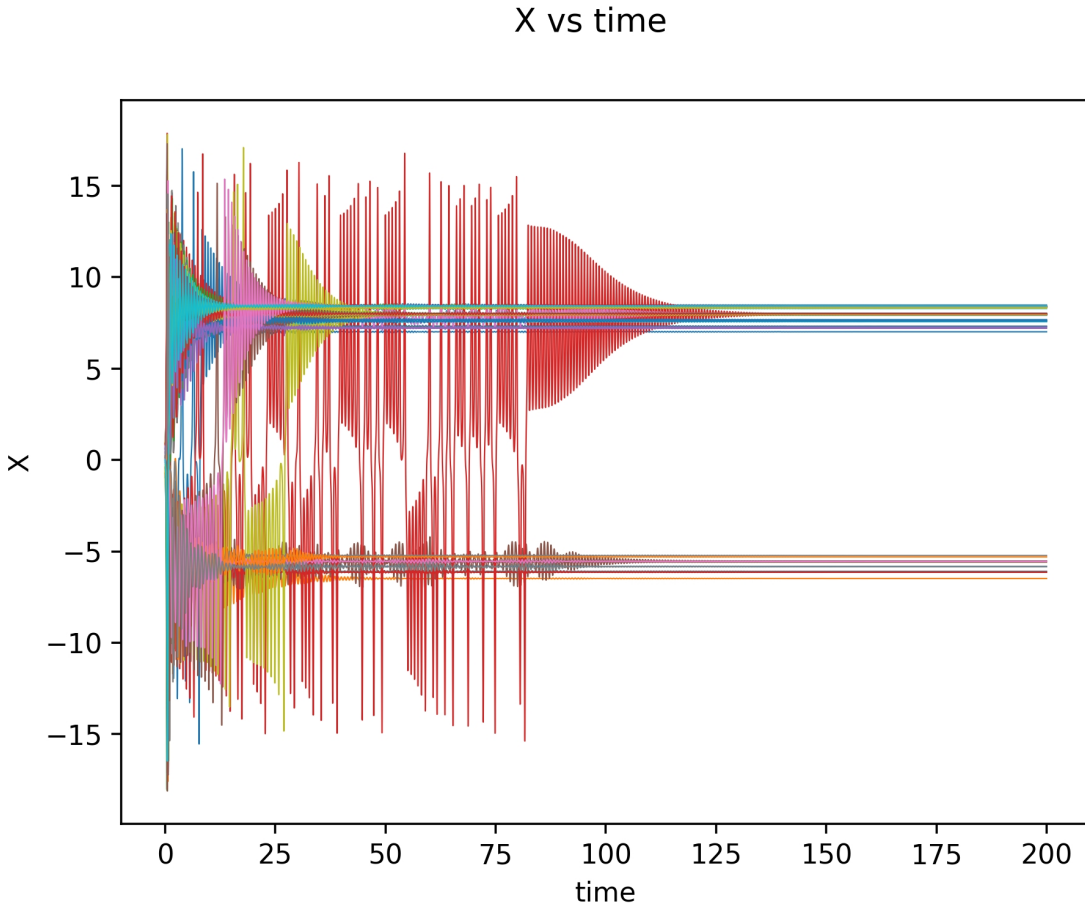


Figure 2.4: A network of 100 identical Lorenz systems exhibiting oscillation death. Note the heterogeneous positions of equilibrium states

with non-identical components, what makes chimeras interesting is the fact that they can also be found in networks of identical nodes and that they have been observed experimentally [23]. The observation of chimeras in experiments demonstrates that the chimeras are robust against noise which is intrinsic to experimental conditions.[23]

It is required in many applications to synchronize multiple systems retaining their intrinsic chaotic nature. Synchronization is known to depend not only on the dynamics of the system and type of interaction but also on the topology of the interaction network [24]. After initial observations of the role of network topology on control of systems, many methods seemed to suggest that the topology of the network alone could explain the behaviour

of interacting systems. Recently, however, it has been noted that a more holistic approach considering both structure and dynamics might be necessary for optimal control [25].

The aim of this project is to study a few control strategies that involve both topology and dynamics that can lead to desirable collective behaviour. Some classes of control that are to be investigated are:

- Dynamics driven control of topology for optimal stable structure.
- Topology driven control where strategic rewiring can result in desirable dynamics.
- Extending the study to externally driven controls.

Chapter 3

Measures of collective dynamics

3.1 Measures used to characterize synchronization

There are various metrics to measure the extent of synchronization in coupled system. Here we summarize some well-known measures generally used to characterize synchronization and define new metrics used in this study.

- **Complete-Sync Order Parameter [26]:**

Full synchronization of two connected systems $x(t), y(t)$ can be defined to be achieved at time T if $x(t) - y(t) = 0$ for all $t \geq T$. This can be extended to an N -node network considering all interacting nodes:

$$\sum_{edges(x,y)} x(t) - y(t) = 0$$
$$\equiv \sum_{i,j}^N A_{i,j} \cdot |x_i(t) - x_j(t)| = 0$$

The quantity $\sum_{i,j}^N A_{i,j} \cdot |x_i(t) - x_j(t)|$ can be used as a measure of extent of synchro-

nization.

- **Trajectory Correlation:**

Pearson Correlation Coefficient between time series of every two nodes is computed to measure correlation in dynamics in the asymptotic limit. The RMS of the values in the correlation matrix is used as a measure of the degree of synchronization.

- **Deviation from mean-trajectory:**

Standard deviation of node positions at time t from mean trajectory of all nodes. For full synchronization, this goes to zero.

- **Average inter-node distance:**

The distribution of inter-node distances is a useful indicator of the synchronization in a system. The average distance between positions of every pair of nodes at time t can be used as a metric for synchronization. For full synchronization, this goes to zero. For some purposes, this measure is found to be better in a sense that it gives smaller values in case of cluster synchronization where the deviation from mean trajectory might be still larger.

- **Number of synchronized clusters:**

While the above metrics were explored for their effectiveness in measuring the degree of synchronization the following method can be used to create a discrete measure for clear binary identification of whether synchronization was reached. It is particularly useful in systems exhibiting cluster synchronization.

Two nodes are said to be synchronized if their mutual distance remains less than a cutoff distance ϵ . Construct a synchronization network by placing an edge between every pair of synchronized nodes. The number of connected components of this network is evaluated. Each connected component is a synchronized cluster with a maximum inter-node distance of $N\epsilon$. For full synchronization, the number of synchronized clusters is 1 and for null synchronization, it is N .

- **Asymptotic cluster distribution:**

Given the number of synchronized clusters, it is also useful to know the distribution of nodes among these clusters. This allows us to differentiate for example whether the synchronization procedure has resulted in one large synchronized cluster with other clusters very small in comparison or an even distribution with multiple clusters of comparable size.

3.2 Methods of analysis

The ODEs representing the dynamics of the systems in the network were solved numerically in Python 3 using the SciPy's [27] ODEINT method which uses the LSODA (Livermore Solver for Ordinary Differential equations with Automatic method switching for stiff and non-stiff problems).

For this at each node of network the dynamical system of interest with identical parameters but different initial condition was placed. Random graphs using Erdős–Rényi model were used for network topology.

Effect of coupling strength and network density on synchronization properties were studied. Various metrics to measure extent of synchronization were explored for Rössler and Lorenz system in chaotic regime.

Different methods of synchronization were devised and applied on precomputed sets of initial conditions. Same set of initial conditions of systems were used across different methods of control and variation of parameters. This was done to avoid the choice of initial condition from having any relative affect on the results of a control procedure. For computing initial conditions of N systems, N points are chosen at random from a unit cube in the vicinity of the system and are allowed to evolve until they settle on trajectories close to the underlying chaotic attractor of the system. This provides N initial conditions homogeneously distributed

close to the chaotic attractor. For another set of initial conditions the process is repeated with different set of random points.

On application of method of control to the initial points, the time series of resulting trajectories are stored for analysis. After every rewiring event in the coupling network a timestamped snapshot of the adjacency matrix is also saved. Later the resultant synchronization of the system trajectories was quantified using suitable metrics and measures of graph topology at any given time were computed from the adjacency matrix.

3.3 Qualitative aspects and visualization

Tools for visualization of the system state for better understanding of the system were developed. As demonstrated by the Anscombe's Quartet [28] (where four completely different and identifiably distinct data-sets produce the same statistical measures) statistical measures cannot always be solely relied upon to understand the information contained in any data-set. While the computed statistical measures are useful they are limited in the insight provided about the system.

Therefore methods for visualization of the evolving network of dynamical systems were developed as a python toolkit. This allowed us for example to investigate the conditions associated with oscillation death in some random networks of Lorenz systems.

In this chapter we present the different measures that can be used in general to identify states of synchronization in any network. We now discuss the control strategies and dynamics of evolved networks of our study

Chapter 4

Methods of control

In this chapter we present cost effective and efficient methods of inducing synchronization in a network of dynamical systems.

4.1 Network rewiring based on distance threshold

The rewiring scheme was devised as follows. The scheme prohibits connection of nodes far away in the system which reduces the likelihood of them being in widely different environment in the same chaotic system. It also limits the maximum absolute value of the coupling term in the dynamics.

Nodes i and j are connected by an edge if $|x_i(t) - x_j(t)| < D$ where D is a control parameter that can be modulated. A simple heuristic to determine D is to choose a scale size that is slightly smaller than the size of topological features of the system.

4.1.1 Discrete time rewiring

The rewiring of the network can be modelled as being done continuously(i.e. at arbitrary small intervals of time) over the entire span of the synchronization procedure. This scenario, however, is ideal and unlikely to be translated into a real-world implementation. To account for this we model the re-wirings as being done after discrete intervals of time. After a rewiring event, the system is allowed to relax into the new network state until the next rewiring event. For the discrete updating method of control the time between rewiring updates or relaxation time τ is a control parameter.

This method of control is simulated for Rössler and Lorenz systems. The network consists of a collection of nodes with identical systems placed on each node but different initial conditions. The systems are initially randomly positioned close to the respective chaotic attractors at beginning of the simulation.

The results of this control procedure are compared with the same systems evolving on a static complete network topology. Above a critical value of coupling strength, the rewiring scheme successfully synchronized all Lorenz systems and the resulting synchronized system remained chaotic. For the same value of coupling strength, random (ER) networks of density up to $p = 0.12$ failed to synchronize. Rössler systems show two distinct types of behaviour on application of the control procedure, the outcome was either full synchronization or cluster synchronization. In full synchronization all nodes become part of a singleton connected network and synchronize, while in cluster synchronization they settle into multiple internally synchronized components or clusters (Figure 4.1).

All networks were allowed to evolve until $t = 200$. We document the outcome of the synchronization procedure for variations of values of the control parameters; coupling strength (k) and distance threshold (D).

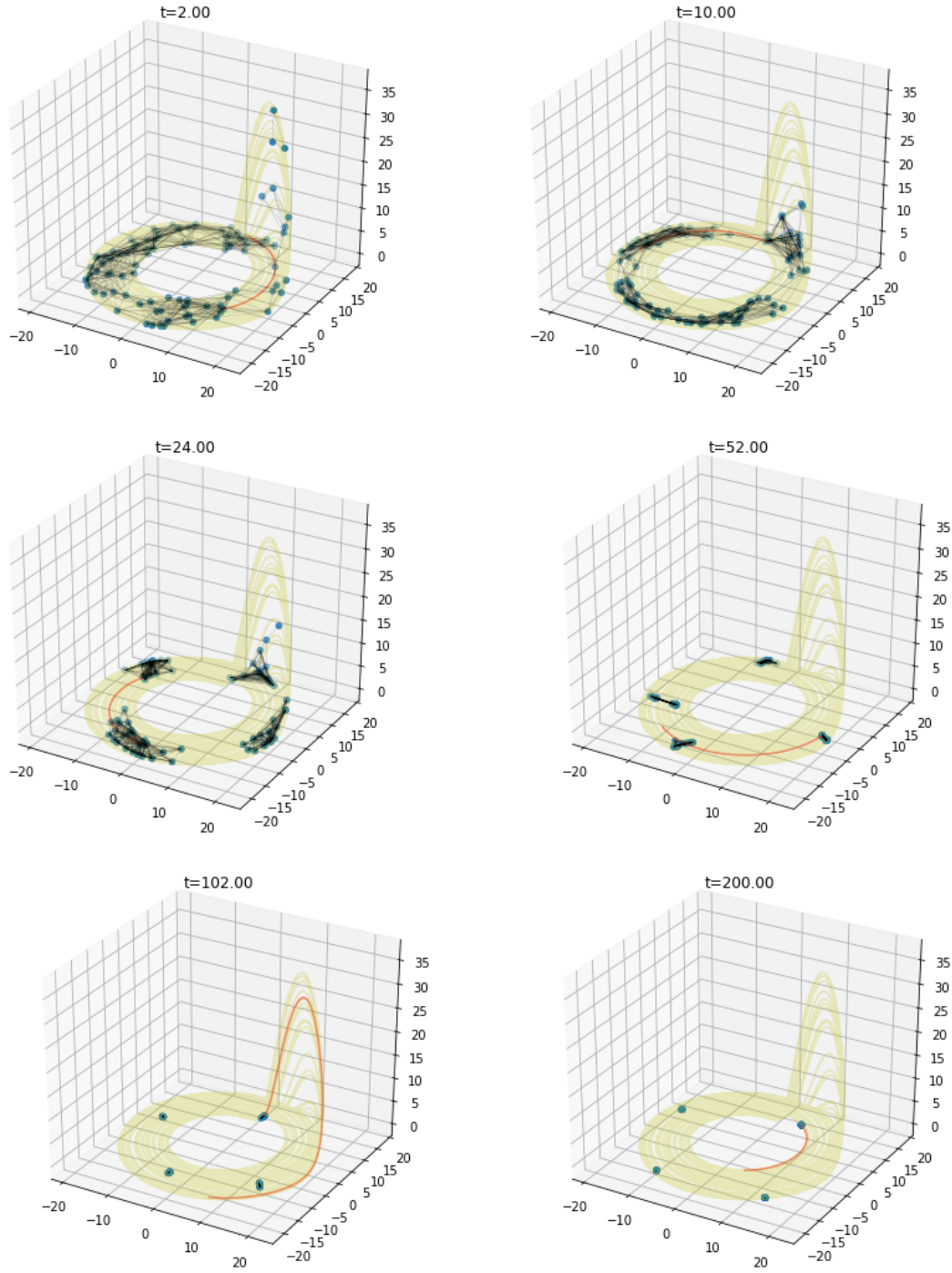


Figure 4.1: Evolution of Rössler systems under discrete-time rewiring for threshold distance $D = 10$, relaxation time $\tau = 2$ and coupling strength $k = 0.10$. The state of an individual system or node is marked as a blue circle. Edges between nodes are marked as black lines connecting nodes. A free Rössler trajectory (yellow) is plotted in the background.

Effect of control parameters on synchronization time

Full network topology leads to successful synchronization of Lorenz systems for all the coupling strengths considered ($k = 0.1$ to $k = 10$). These show a decreasing trend for synchronization time with increasing coupling strengths, levelling off near $t = 7.6$ (Figure 4.2).

The time to synchronize vs coupling strength for this control procedure on Lorenz systems showed a similar decreasing trend but with fluctuations. This behaviour for varying choice of relaxation time (τ) is shown in (Figure 4.2).

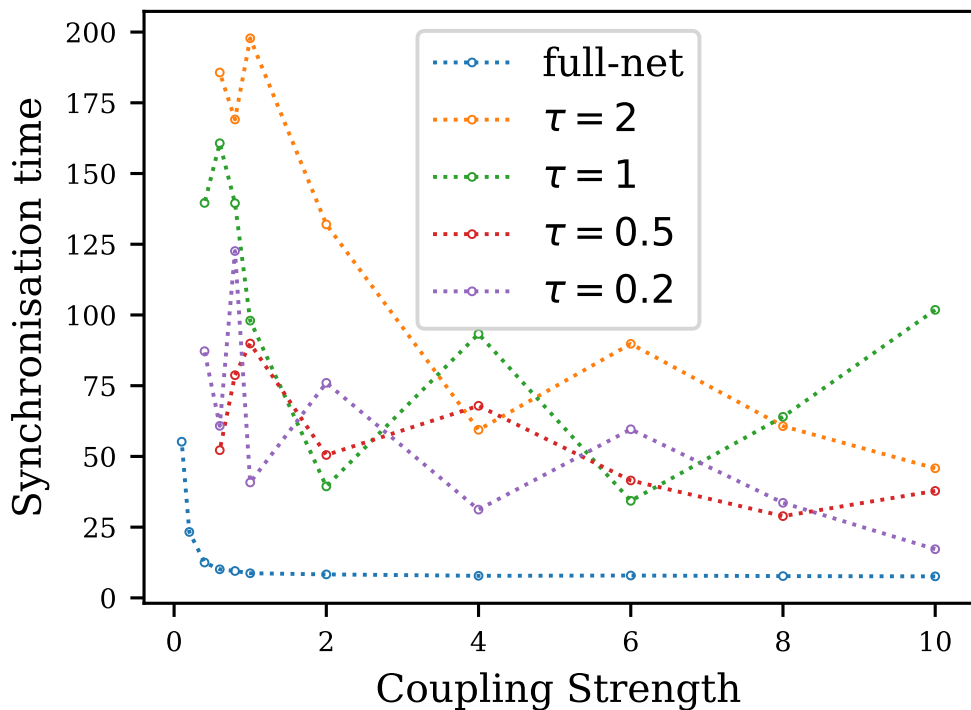


Figure 4.2: Time taken for $N = 100$ Lorenz nodes to synchronize as a function of coupling strength for varying choice of relaxation time τ . The distance threshold for connecting nodes used here is $D = 5$ Time for synchronization of fully connected network (full-net) is shown for comparison.

Rössler systems are not synchronized by fully connected networks for high values of

coupling strength. For lower values of coupling strength, synchronization time decreases with increasing coupling until an optimal coupling value is reached. For greater coupling strengths increasing time for synchronization is observed. For even higher values the dynamics of Rössler nodes show unbounded destabilization.

The behaviour for Rössler systems under the control procedure under discussion cannot be characterised appropriately by synchronization time as under varying conditions the Rössler system shows cluster synchronization with different clusters synchronizing at variable times. We instead focus on the number of synchronized clusters after a fixed length of time. Systems of Rössler networks under this control procedure do not destabilise like the fully connected network even at higher coupling strengths.

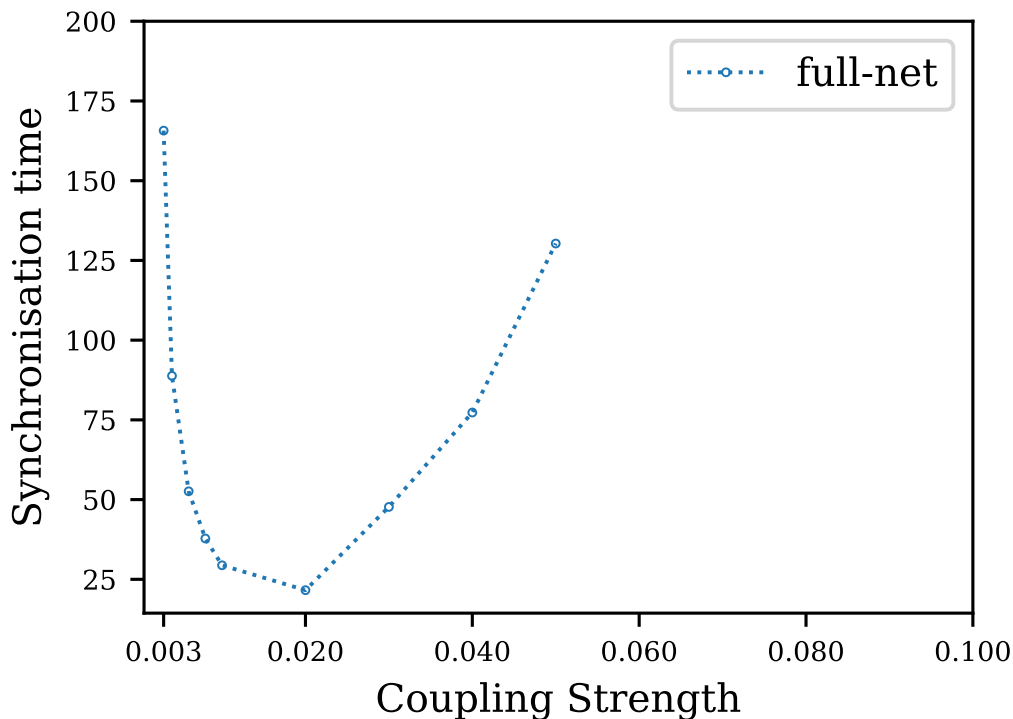


Figure 4.3: Time taken for $N = 100$ Rössler systems to synchronize as a function of coupling strength for fully connected network. For coupling strength $k > 0.5$ the systems either do not synchronize or destabilize into unbounded trajectories

Behaviour under discrete-time rewiring for varying distance threshold (D)

Rössler networks show cluster synchronization for the distance threshold based rewiring scheme. That is they settle into dynamically stable clusters of completely synchronized nodes. Once synchronized the clusters remain stable over time.

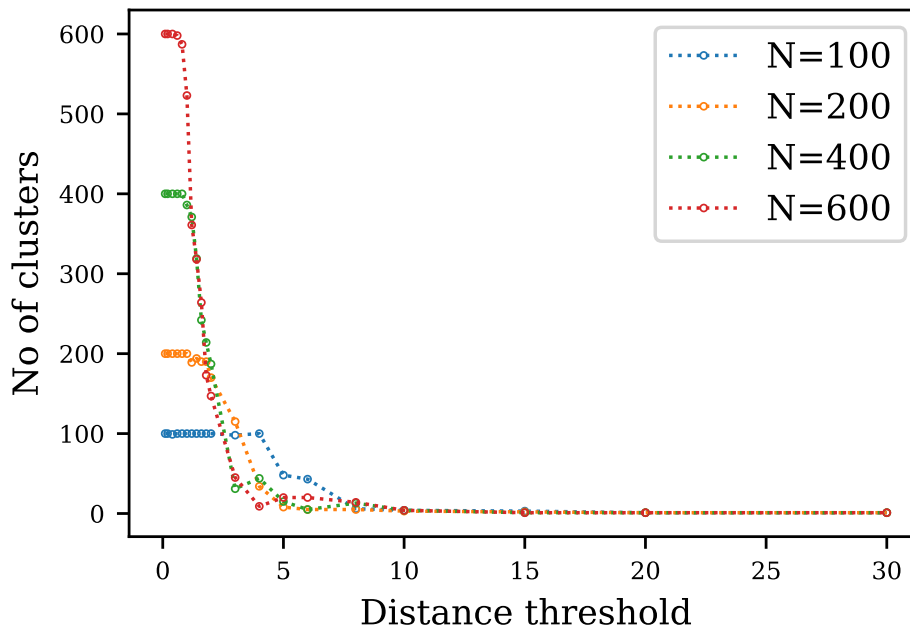


Figure 4.4: Number of clusters vs the distance threshold for various sizes of Rössler networks for coupling strength $k = 0.01$

Figure 4.4 and Figure 4.5 show number of clusters at the end of $t = 300$ for different values of distance threshold (D) for coupling strengths $k = 0.01$ and $k = 0.05$ respectively. While for lower coupling strength ($k = 0.01$) full synchronization (1 fully synchronized cluster) is seen for large (D), for the higher coupling ($k = 0.05$) systems tend not to synchronize for larger (D). For $k = 0.05$ the no of clusters tends to reduce for an initial increase in (t) and onset of reverse trend appears earlier for larger values of the number of nodes (N). The abrupt increase in the number of clusters can be explained by destabilisation of Rössler nodes

due to high coupling forces. For networks of larger sizes the density of nodes in the system position phase space is higher leading to larger node degree during rewiring, which results in larger effective coupling forces on each node which in-turn explains the earlier onset of increasing number of cluster. It must be noted however that the systems remain bounded, unlike fully connected networks.

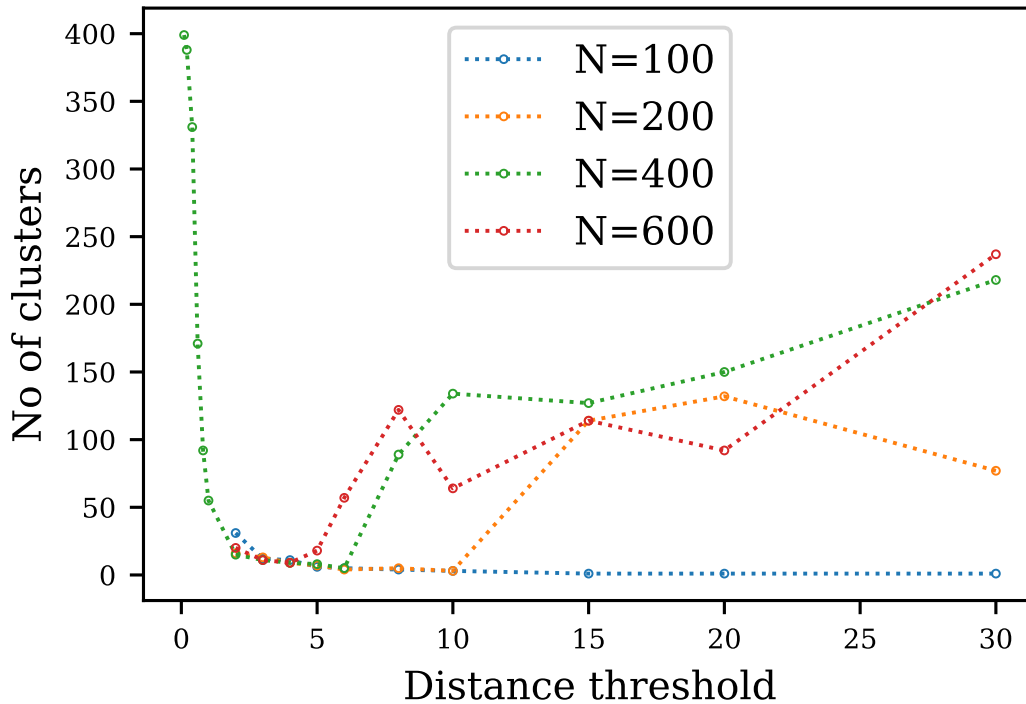


Figure 4.5: Number of clusters vs the distance threshold for various sizes of Rössler networks for coupling strength $k = 0.05$

Unlike Rössler systems Lorenz Systems do not show lasting stable cluster behaviour. In the intermediate stages before synchronization, the nodes do condense into fully synchronized sub-clusters which remain intact briefly before coalescing together into a completely synchronized system. Figure 4.6 shows the relationship between number of clusters and distance threshold for two values of coupling strength for 400 Lorenz systems.

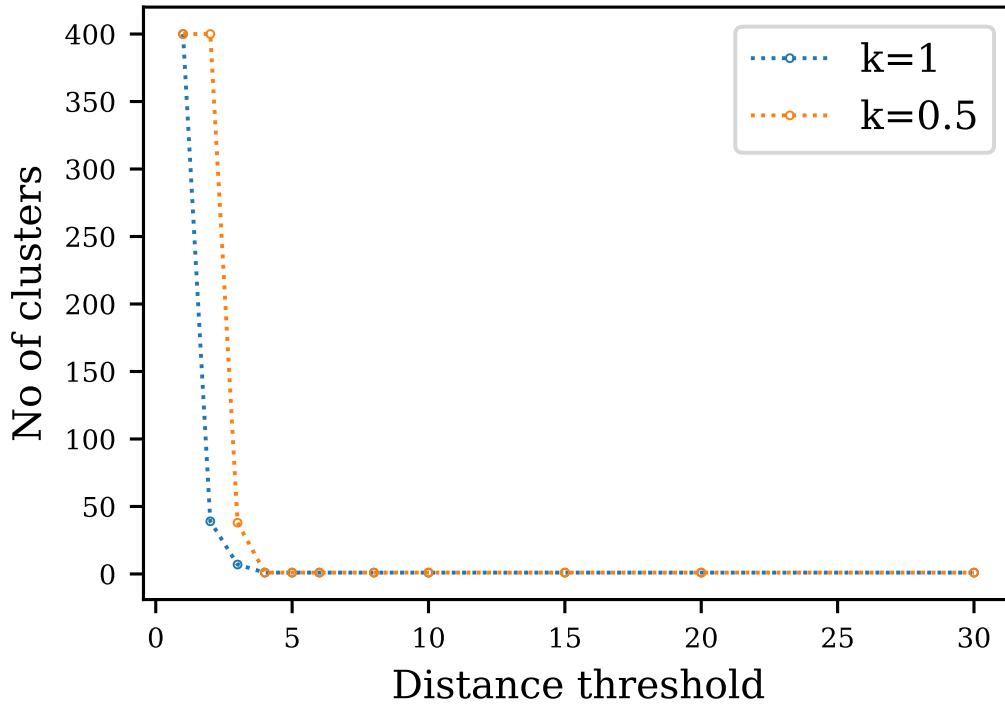


Figure 4.6: Number of clusters vs the distance threshold for Lorenz networks of size $N = 400$ for coupling strengths $k = 1, k = 0.5$ at $t=200$

Another way of characterising the effectiveness of synchronization is the distribution of node numbers across synchronized clusters. The number of nodes in the greatest component is plotted for varying D for different networks in Figure 4.7 . While Rössler shows a gradual increase in the size of giant component Lorenz systems switch from unsynchronized (N clusters) to synchronized (1 cluster) very sensitively with the change in distance threshold D .

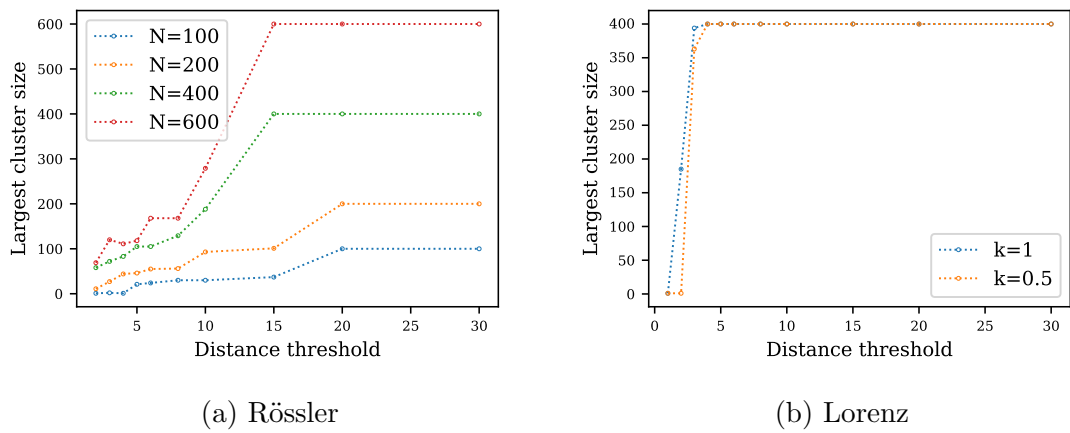


Figure 4.7: Rössler networks (a) show gradual increase in the size of largest cluster for increasing distance threshold while networks of Lorenz systems (b) show switch like behaviour.

In this section we have seen that for discrete-time rewiring based on network threshold, Lorenz and Rössler systems follow a route to synchronization through cluster formation. That is, we see separate clusters, with nodes inside a cluster wholly synchronized with each other, yet evolving independently of other clusters. In Lorenz case, the clusters formed are relatively short-lived and gradually coalesce with each other turning into larger clusters and eventually form a singleton cluster. In the case of Rössler systems, however, we see that the dynamics eventually stabilises into a fixed number of clusters synchronized internally. The number of clusters depends on the distance threshold used and the relationship between the no of clusters formed and the distance threshold is shown for different values of coupling strengths. It is also shown that while for a network of Rössler systems the size of the greatest cluster formed increases gradually with increasing D , for Lorenz systems it is more of a switch like behaviour, with the maximal cluster size jumping from zero to N across a critical value of distance threshold.

4.2 Dynamics driven control of coupling criteria

This rewiring scheme was devised as follows. For any rewiring procedure depending on a criteria, the scheme prevents certain nodes from making further connections or disconnections (so their adjacency list is frozen). For this study, we have used transient synchronization or swarming as a criterion to determine coupling behaviour of nodes. When a group of k nodes show transient synchrony (i.e. inter-node distances remain less than a threshold for synchronization for short transient interval δt) all but l nodes are deactivated. For selection of l nodes that remain active, we choose the l highest degree nodes from the k transiently synchronized nodes. At the end of synchronization procedure when equilibrium is attained a stable network structure is obtained based on the dynamics of the system.

This scheme was implemented on an array of Rössler systems undergoing distance threshold based rewiring. The transient interval used is $\delta t = 0.1$ and no of active nodes in a synchronized cluster is $l = 1$. A sample of resultant states of system for some values of distance threshold D for coupling strength $k = 0.1$ are shown in Figure 4.8. The resultant topology structure and degree-distribution is shown in (Figure 4.9 - Figure 4.20)

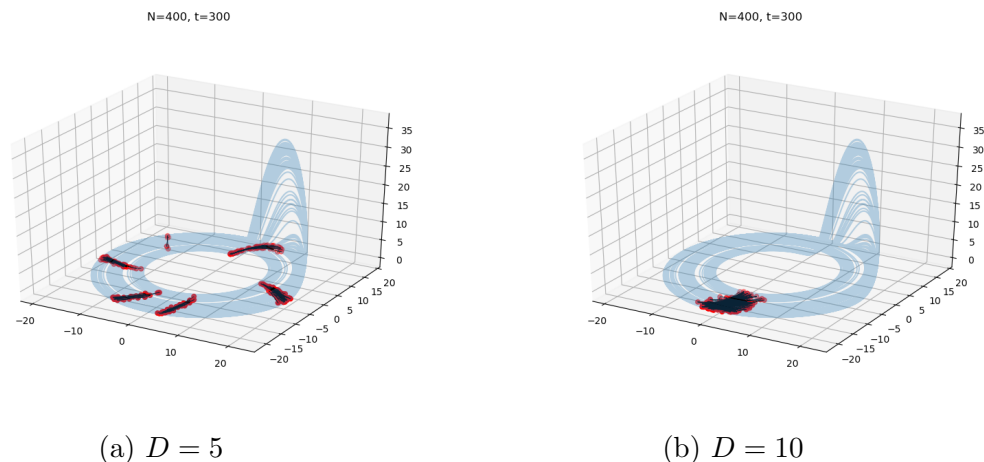


Figure 4.8: State of Rössler network of $N = 400$ nodes at $t = 300$ with red circles as nodes and black edges for $k = 0.1$. The system shows formation of clusters for (a) and single cluster for (b)

4.2.1 Evolved topology of networks

The topology evolved from this control procedure is scale free type. The degree distribution shows a fat tailed nature. The log-log plots of degree distribution fit to straight lines which is characteristic of power law distribution. Deviations are seen from straight line at low degree which is indicative of saturation of small degree nodes.

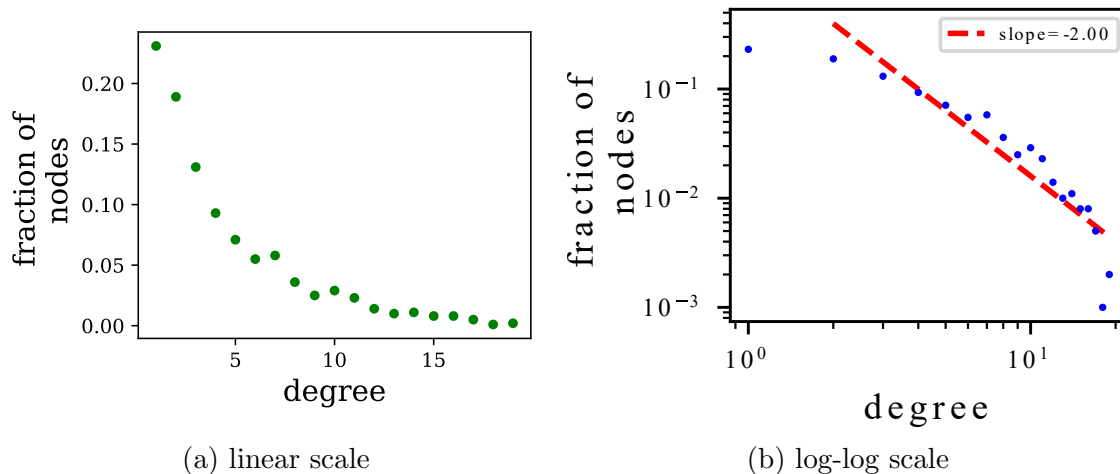


Figure 4.9: (a) Degree distribution for Rössler network of $N = 1000$ nodes at $t = 500$. (b) The log-log plot of distribution is shown. The rewiring control parameters are $D = 5, k = 0.02$

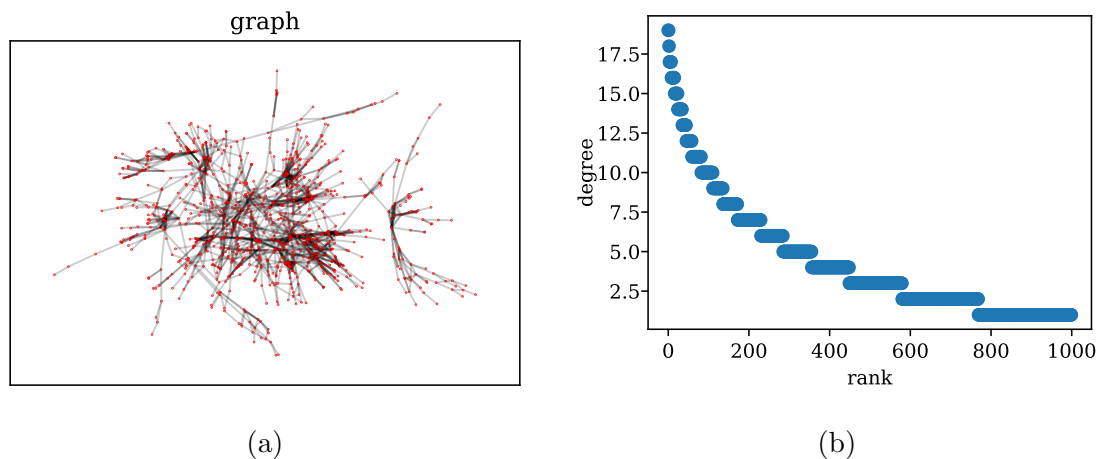


Figure 4.10: (a) 2D representation of the network-graph showing the nodes as red circles with black edges. (b) The degree-rank plot is displayed. The rewiring control parameters are $D = 5, k = 0.02$

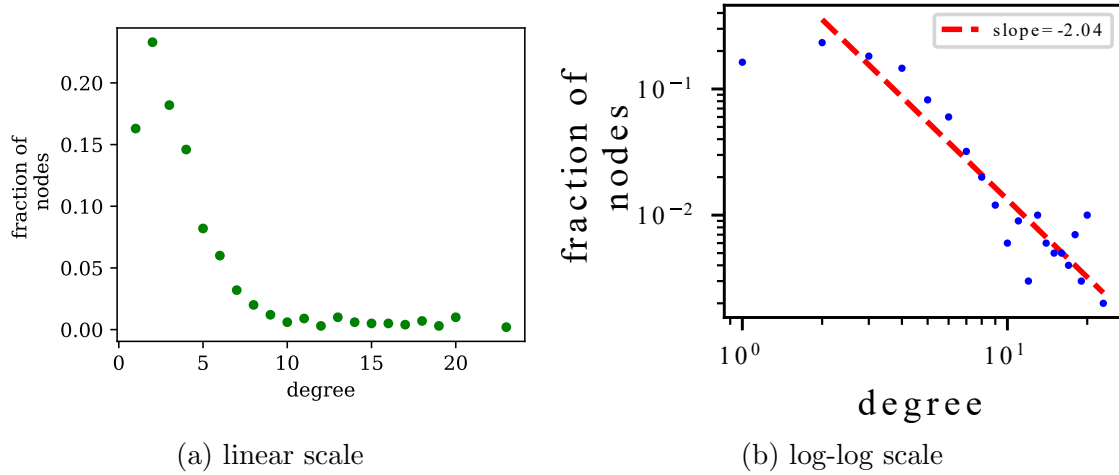


Figure 4.11: (a) Degree distribution for Rössler network of $N = 1000$ nodes at $t = 500$. (b) The log-log plot of distribution is shown. The rewiring control parameters are $d = 5, k = 0.12$

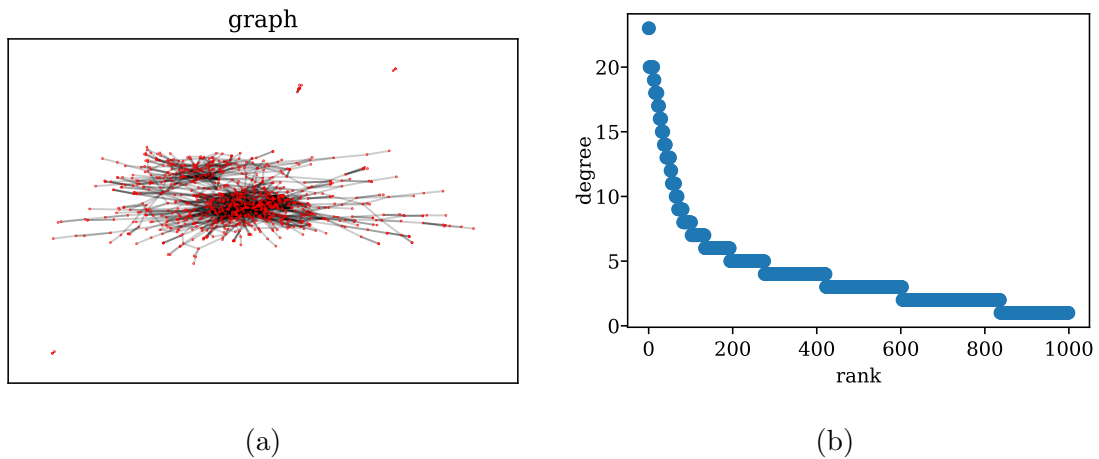


Figure 4.12: (a) 2D representation of the network-graph showing the nodes as red circles with black edges. (b) The degree-rank plot is displayed. The rewiring control parameters are $d = 5, k = 0.12$

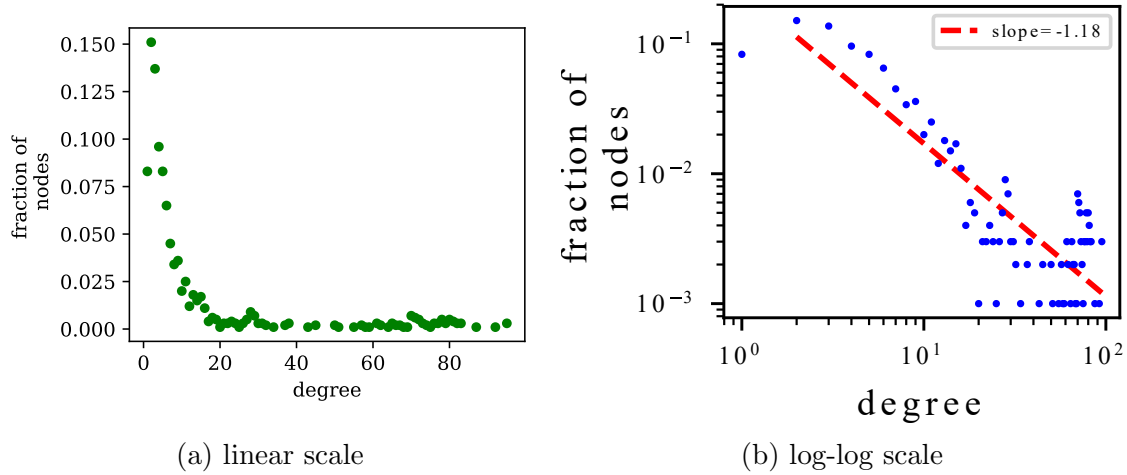


Figure 4.13: (a) Degree distribution for Rössler network of $N = 1000$ nodes at $t = 500$. (b) The log-log plot of distribution is shown. The rewiring control parameters are $d = 5, k = 0.22$

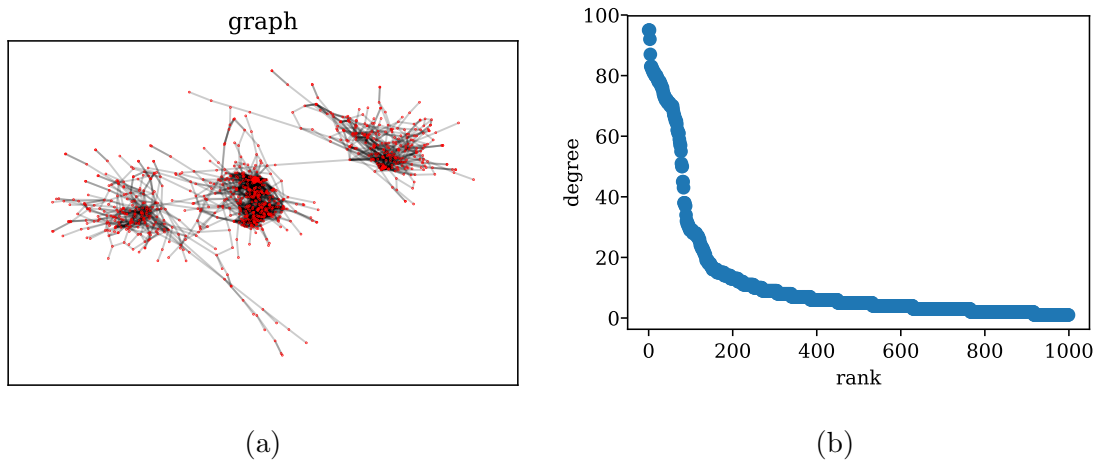


Figure 4.14: (a) 2D representation of the network-graph showing the nodes as red circles with black edges. (b) The degree-rank plot is displayed. The rewiring control parameters are $d = 5, k = 0.22$

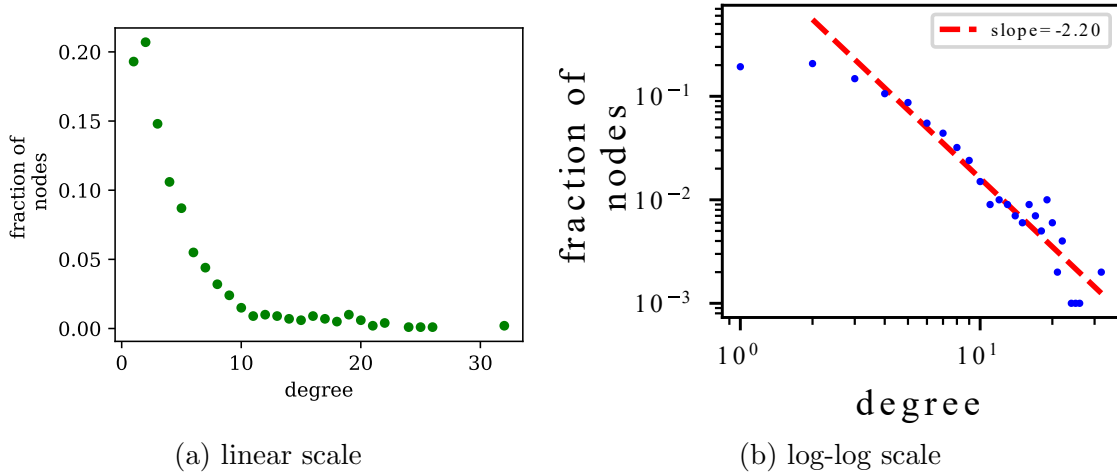


Figure 4.15: (a) Degree distribution for Rössler network of $N = 1000$ nodes at $t = 500$. (b) The log-log plot of distribution is shown. The rewiring control parameters are $d = 10, k = 0.02$

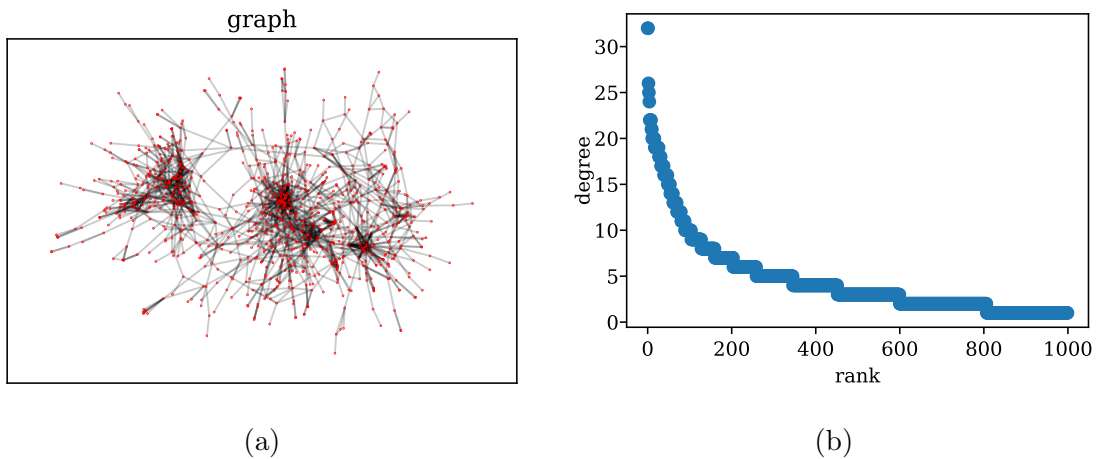


Figure 4.16: (a) 2D representation of the network-graph showing the nodes as red circles with black edges. (b) The degree-rank plot is displayed. The rewiring control parameters are $d = 10, k = 0.02$

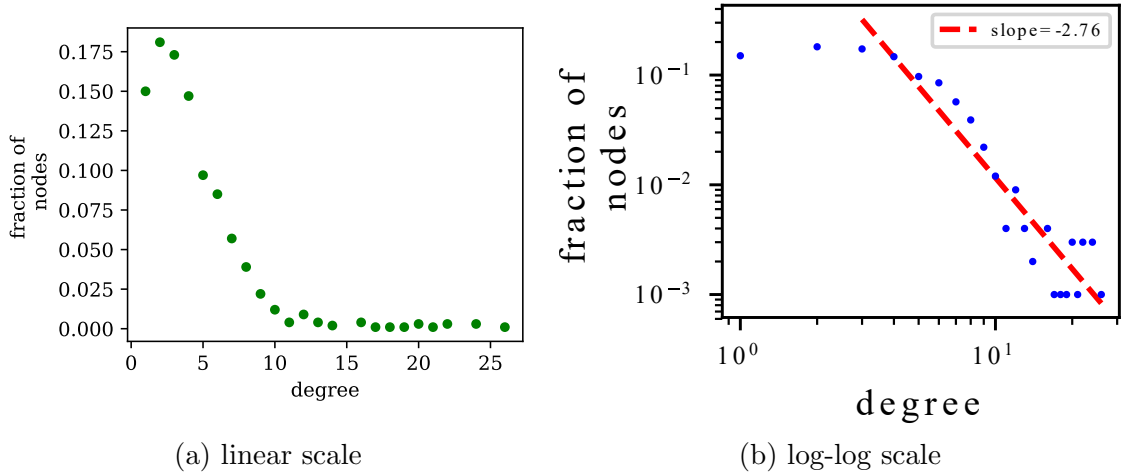


Figure 4.17: (a) Degree distribution for Rössler network of $N = 1000$ nodes at $t = 500$. (b) The log-log plot of distribution is shown. The rewiring control parameters are $d = 10, k = 0.12$

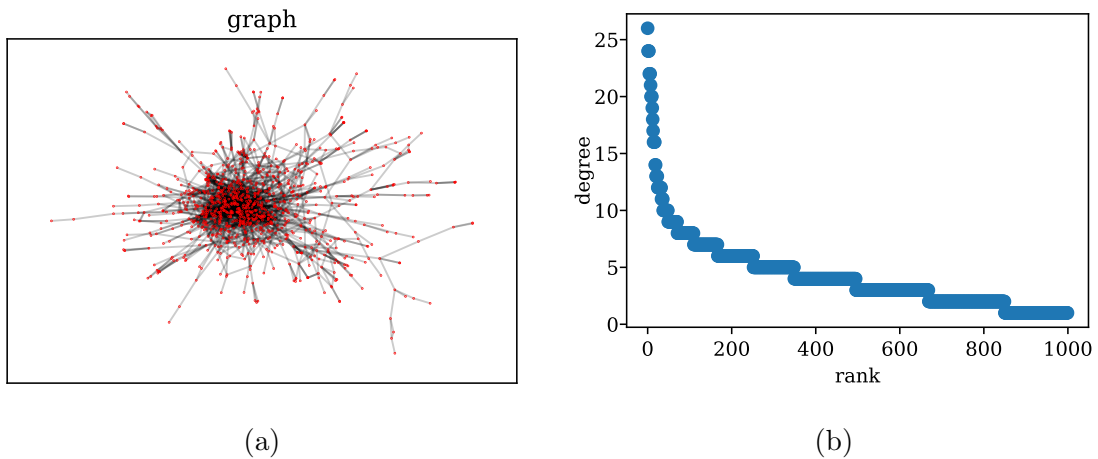


Figure 4.18: (a) 2D representation of the network-graph showing the nodes as red circles with black edges. (b) The degree-rank plot is displayed. The rewiring control parameters are $d = 10, k = 0.12$

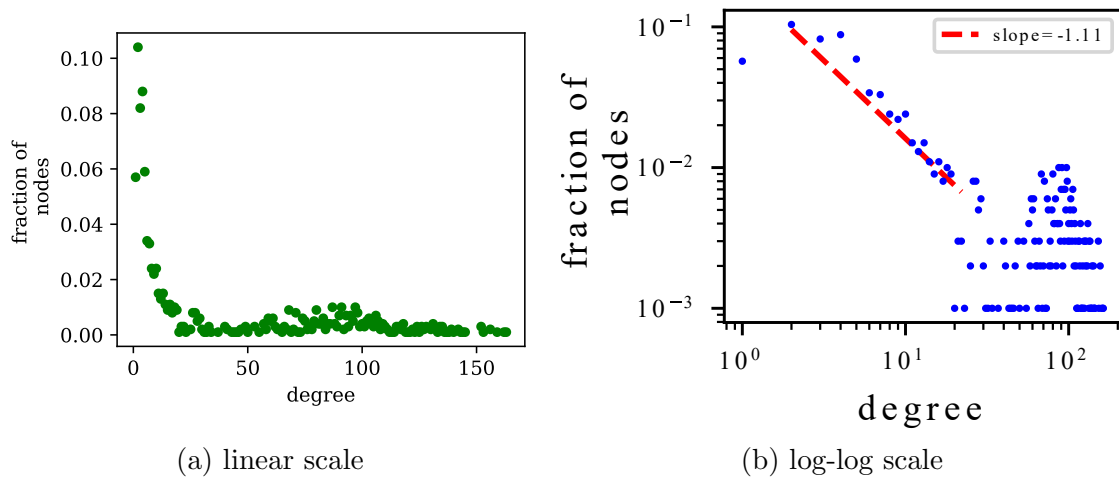


Figure 4.19: (a) Degree distribution for Rössler network of $N = 1000$ nodes at $t = 500$. (b) The log-log plot of distribution is shown. The rewiring control parameters are $d = 10, k = 0.22$

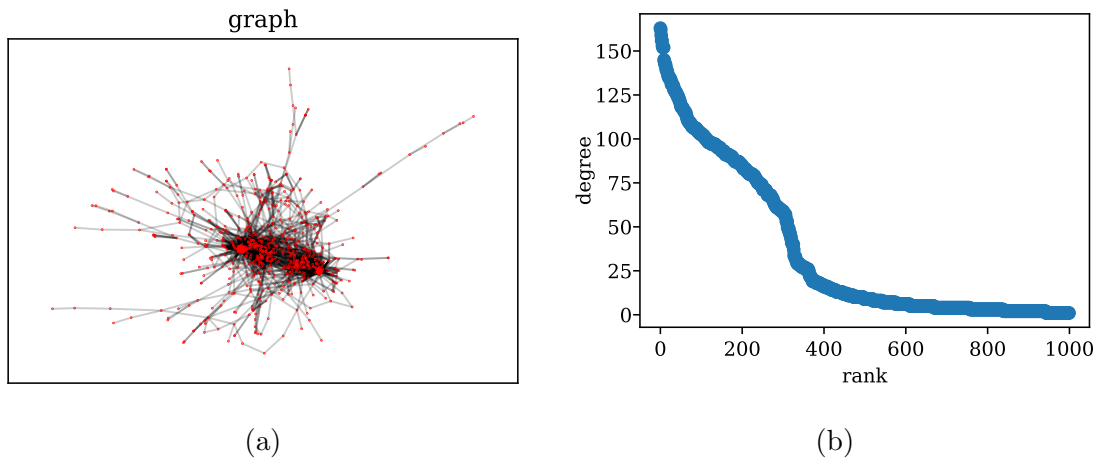


Figure 4.20: (a) 2D representation of the network-graph showing the nodes as red circles with black edges. (b) The degree-rank plot is displayed. The rewiring control parameters are $d = 10, k = 0.22$

Table 4.1: Power law exponents of the evolved network for different values of control parameters.

D	k	γ
5	0.02	2
	0.12	2.04
	0.22	1.18
10	0.02	2.2
	0.12	2.76
	0.22	1.11

We see that the control procedure results in networks with power law distribution. Therefore we conclude that the networks that are generated from this procedure are of scale free type. A straight line is fit to the log-log plot of degree distribution. The slope (γ) of the fitted line estimates of the exponent of the power law. The obtained exponents (γ) for different control parameters are summarised in Table 4.1 . It is seen that the value of γ first increases then decreases with increasing coupling strength. The decrease in γ is associated with the rise of very high degree nodes giving a more fat tailed degree-distribution.

Chapter 5

Summary and outlook

The study under this project was mainly to design control strategies that involve both topology and dynamics and the objectives were to investigate classes of control where:

- * Topology driven control where strategic rewiring can result in desirable dynamics
- * Dynamics driven control of network topology results in a stable structure
- * Extend the study to externally driven control

In the distance threshold based network rewiring scheme, strategic rewiring of network topology results in desirable dynamics while in the second method studied dynamics driven control of coupling criteria, dynamics (synchronization) of the system is used to control network topology resulting in stable scale-free type structure of the network.

The distance threshold based network rewiring scheme was successful in synchronizing networks of Rössler and Lorenz systems of varying sizes. An interesting pattern of synchronization through cluster formation was observed. The control method is also robust as it allows tuning coupling strength (k) to high values in Rössler systems without them reaching unbounded instability. Also synchronized Lorenz systems were able to retain their intrinsic chaotic behaviour. We have also shown that these properties are achieved even when the rewiring is not continuous but at discrete intervals even at the time scales of the system os-

cillations therefore allowing for less frequent re-wirings. A drawback of the system is that in the final limit when all nodes synchronize the network between synchronized nodes is always a complete network. This is because for any two nodes that are synchronized the condition $|x_i(t) - x_j(t)| < D$ automatically holds for any non-zero D and they are connected by this procedure. However, this is different from starting with complete network as the evolving network reaches full network topology only very close to final synchronized state when the coupling terms tend to vanish anyway.

Another form of control procedure involving dynamics dependent control of topology was studied. The method involved disabling some already synchronized nodes from making further connections. This was intended to allow distributing coupling edges more efficiently among unsynchronized components for an optimal resultant topology. The method results in phase synchronization with very few edges compared to pure distance threshold based network rewiring. All networks obtained have $< 4\%$ of maximum edges that can be present in the network. The resultant networks also show non-trivial topology. The networks exhibit power-law degree distribution with saturation at small-degrees which is a variation [29] of scale-free networks [3] We, therefore, conclude that the control scheme results in the generation of networks with scale-free topology with less frequent nodes of very small degrees.

Future trends

Using different distance thresholds (D) for discrete rewiring procedure on Rössler systems, we see different cluster distribution with reducing number of clusters for increasing D . It is possible to dynamically change D during the control procedure in a stepwise manner to arrive at full synchronization with only one cluster. Dynamically varying D through the control procedure or switching D based on system feedback could give an optimal route to full synchronization. However, for high coupling strengths, the number of clusters increases for larger D in network of Rössler systems. Care must be taken to detect onset of destabilisation 4.5 and stop/slow the increase of D at that stage for optimal synchronization. The

destabilisation is believed to be due to higher magnitude of effective coupling due to high degrees in larger networked components. Therefore, alternatively, the use of degree normalised coupling strength to counter the variation in effective magnitude of coupling on nodes can be studied.

In this study all networks were undirected and the effect of coupling edge between any two nodes was bi-directional. It has been shown that for directed graph improving the network structure can, in fact, hinder synchronization [30]. Using directed coupling for the above methods could be studied to observe the difference in behaviour.

The motivation for the first control procedure was to avoid non-local couplings which along with being higher in magnitude often pull the nodes away from the underlying chaotic attractors to unstable regions or fixed points. We propose a new method to localise interactions. This involves relaying coupling through a Delaunay triangulation network of a separate set of control nodes randomly dispersed and evolving on the chaotic system. A subset of the nodes to be synchronized can also be tasked as control nodes instead of using separate nodes.

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