The Graphs Associated with Ordered Structures and Algebraic Structures

A Thesis

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Samruddha Gonde

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

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Supervisor: Dr Vinayak Joshi

Certificate

This is to certify that this dissertation entitled "The Graphs Associated with Ordered Structures and Algebraic Structures" towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Samruddha Gonde at Savitribai Phule Pune University, Pune under the supervision of Dr Vinayak Joshi, Professor, Department of Mathematics, Savitribai Phule Pune University, Pune, during the academic year 2017-2018.

Dr Vinayak Joshi

Committee:

Dr Vinayak Joshi

Dr Chandrasheel Bhagwat

To the memories of Aajoba

Declaration

I hereby declare that the matter embodied in the report entitled "The Graphs Associated with Ordered Structures and Algebraic Structures" are the results of the work carried out by me at the Department of Mathematics, Savitribai Phule Pune University, Pune, under the supervision of Dr Vinayak Joshi and the same has not been submitted elsewhere for any other degree.

Samruddha Gonde

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Finally I thank to all who directly or indirectly helped for successful completion of this project.

Last but not the least is my family members without whom I stand nowhere. My heart fills with emmense gratitude for them for their moral support.

Abstract

This project focuses on the interplay between the zero-divisor graphs of semigroups and zero-divisor graphs of meet-semilattices. Mainly, we examine the weakly perfectness of the zero-divisor graphs of semigroups and annihilating-ideal graphs of semigroups. In particular, we solve DeMeyer and Schneider (L. DeMeyer and A. Schneider, *The annihilating-ideal graph of commutative semigroups*, J. Algebra 469 (2017), 402-420.) conjecture about the annihilating-ideal graphs of semigroups negatively.

In the first chapter, we provide a new proof of an analogue of Beck's Conjecture for the zero-divisor graphs of posets. Further, we study the partial order given by LaGrange and Roy [20] for reduced commutative semigroups. In fact, we prove that the minimal prime ideals of reduced commutative semigroups S are nothing but the minimal prime semi-ideals of S treated as a poset (under the partial order given in [20]). In fact, we also observe that a similar result holds for reduced commutative rings with unity. This gives a new insight about the Beck's conjecture for reduced rings via ordered sets.

It is known that the set of ideals of semigroups forms a multiplicative lattice. Hence in the last section, we deal with the annihilating-ideal graphs of semigoups and its connections with the zero-divisor graphs of multiplicative lattices.

Contents

Introduction

In last two decades, a lot of research has been done on assigning a graph to an algebraic structure and investigating algebraic properties of that structure using the associated graph. Some examples of such graphs are the commutating graphs of groups, intersection graphs of groups and rings, zero-divisor graphs of semigroups and rings, etc.

The focus of the project is to study zero-divisor graphs of algebraic structures as well as ordered structures and relations between the two. To study the coloring of commutative rings, the concept of the zero-divisor graph of a commutative ring with unity was introduced by Beck [4] in 1988. Since then it has received significant attention in the area of algebra. Following Beck [4], the **zero-divisor graph** of a commutative ring R with unity is a simple undirected graph G whose vertices are the elements of R, and vertices x, y are adjacent, if $xy = 0$. Here x, y are called **zero-divisor** and the set of zero-divisors of ring R is denoted by $Z(R)$. Note that Anderson and Livingston [2] changed the definition of zero-divisor graphs. They consider only nonzero zero-divisors of rings as vertices of the zero-divisor graph and x, y are adjacent if $xy = 0$. Also, we denote $Z^*(R) = Z(R) \setminus \{0\}$. Now, researchers are using this definition of zero-divisor graphs.

The **chromatic number** of a graph G is the minimum number of colors in a coloring of the vertices of G such that adjacent vertices receive different colors and is denoted by $\chi(G)$. If this number is not finite, we write $\chi(G) = \infty$. A subset C of G is a **clique**, if any two distinct vertices of C are adjacent. The **clique number** of a graph G, is the maximum number of vertices in a clique in G, and it is denoted by $\omega(G)$. If the sizes of the cliques are not bounded, then $\omega(G) = \infty$. For any graph G, always $\chi(G) \geq \omega(G)$. If $\chi(G) = \omega(G)$, then G is called a **weakly perfect** graph. In [4], Beck conjectured that $\chi(G) = \omega(G)$, (referred as Beck's Conjecture), that is, G is weakly perfect, if G is the zero-divisor graph of a commutative ring with unity. Moreover, Beck [4, Theorem 3.8] proved this conjecture for reduced commutative ring with unity, which is essentially given below.

Theorem 1. *Let R be a reduced commutative ring with unity. If $\chi(G(R)) < \infty$, then R has only finite number of minimal prime ideals. If n is this number then $\chi(G(R)) = \omega(G(R)) = n$.*

Now, we illustrated the concept of zero-divisor graphs by an example. Further, we observe that $G(\mathbb{Z}_{12})$ satisfies Beck's conjecture.

Example 1. *For a ring $\mathbb{Z}_{12}$, the zero-divisor graph $G(\mathbb{Z}_{12})$ as depicted in Figure is weakly perfect having $\chi(G(\mathbb{Z}_{12})) = \omega(G(\mathbb{Z}_{12})) = 2$.*

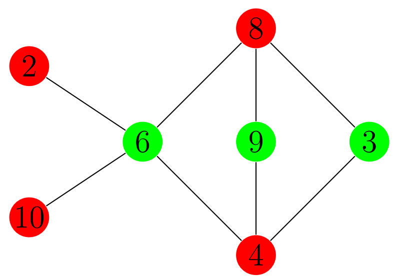


Figure 1: Zero-divisor graph of ring $\mathbb{Z}_{12}$

In 1993, Anderson and Naseer [1] solved Beck's conjecture negatively by providing an example of a commutative local ring R with 32 elements for which $\chi(G) > \omega(G)$.

This concept of zero-divisor graphs is also extended to ordered structures such as semi-lattices, poset and qosets. Nimbhorkar et al. [22] introduced it for meet-semilattices whereas Halaš and Jukl [12] extended it to posets which was further modified by Lu and Wu [21]. In the last five years, Joshi, with his collaborators, developed the theory of zero-divisor graphs through series of papers, see [13, 14, 16, 17].

From Lagrange and Roy [20, Remark 3.4] (see also Devhare, Joshi and LaGrange [9]) it can be observed that a commutative reduced ring (semigroup) can be treated as a partially ordered set (poset).

In the third chapter, we give a relation between minimal prime ideals of a reduced commutative semigroup S and minimal prime semi-ideals of S treated as a meet-semilattice. Moreover, using this result, we obtain Beck's result for reduced commutative semigroups.

In ring theory, the ideals play a crucial role. Hence Behboodi and Rakeei [5, 6][1], introduced the concept of an **annihilating-ideal graph** $\mathbb{AG}(R)$ of a commutative ring R with unity, where the vertex set $V(\mathbb{AG}(R))$ is the set of nonzero ideals with nonzero annihilator. That is, a nonzero ideal I belongs to $V(\mathbb{AG}(R))$ if and only if there exists a nonzero ideal J of R such that $IJ = (0)$, and two distinct vertices I and J are adjacent if and only if $IJ = (0)$.

In [6], Behboodi and Rakeei raised the following conjecture about annihilating-ideal graphs of commutative rings, which is still open.

Conjecture 2. *For every commutative ring R with unity, $\chi(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R))$.*

Recently, DeMeyer and Schneider [7] raised the following analogues conjecture for annihilating-ideal graphs of commutative semigroups.

Conjecture 3. *For every commutative semigroup S with unity, $\chi(\mathbb{AG}(S)) = \omega(\mathbb{AG}(S))$.*

In the second part of this project, we answer Conjecture 3 negatively by providing a counter-example.

For undefined notions and terminologies in lattices and graphs, a reader is referred to Grätzer [11] and West [27] respectively.

[1] M. Behboodi and Z. Rakeei, *The annihilating-ideal graph of commutative rings I*, J. Algebra Appl. **10 (4)** (2011), 727-739.

Chapter 1

Beck's Conjecture

In 1988, Beck [4][1] introduced the idea of coloring of a commutative ring establishing a connection between the graph theory and ring theory, specifically commutative ring theory. He also characterized some class of commutative rings with respect to their zero-divisor graphs. The zero-divisor graphs of rings are helpful in studying the algebraic properties of rings using graph theoretical tools.

In this chapter we study the premises of Beck's Conjecture and a counter-example given by Anderson and Naseer [1].

For a graph G, we have $\chi(G) \geq \omega(G)$. If the equality $\chi(G) = \omega(G)$ holds for a graph G, then we say that G is weakly perfect. A simple example of non-weakly perfect graph is a 5-cycle graph which has chromatic number 3 and has a clique number 2. In [4], Beck conjectured that for a commutative ring with unity, its zero-divisor graph $G(R)$ is a weakly perfect graph and proved it for some classes including a class of commutative reduced rings.

1.1 Preliminaries

Definitions 1.1.1. *Let R be a ring. An element $a \in R$ is a* ***nilpotent element*** *if there is a positive integer n such that $a^n = 0$. A ring R is called* ***reduced*** *if it contains no nonzero*

[1] I. Beck, *Coloring of a commutative ring*, J. Algebra **116 (1)** (1988), 208-226.

nilpotent element.

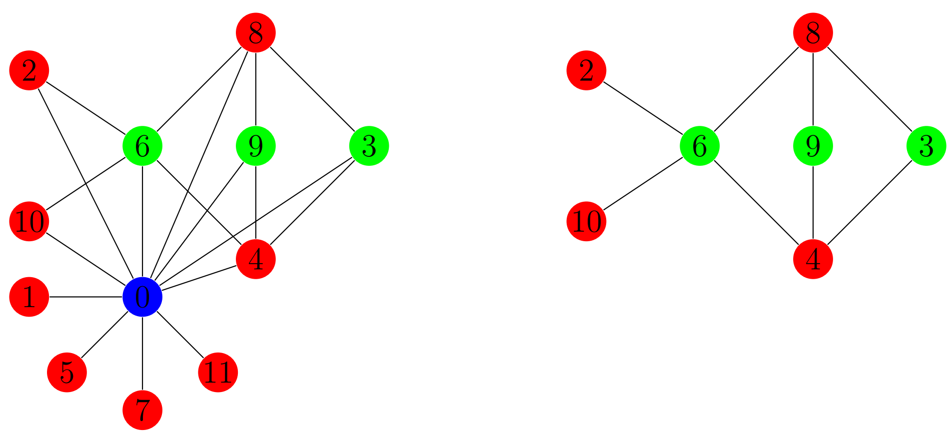
Definition 1.1.2. *(See [4]). Given subsets I and K of R, $I : K = \{r \in R | (rK \subset I)\}$. Moreover, $0 : I = ann(I)$ and if $I = \{x\}$ we simply write $0 : x = ann(x)$ and these last ideals are called* ***annihilators****.*

As mentioned earlier, we are using the modified definition of the zero-divisor graph of a ring given by Anderson and Livingston [2][2] given below.

Definition 1.1.3. *For a ring R, $Z(R) = \{x \in R \mid xy = 0$ for some nonzero $y \in R\}$ denotes the set of zero-divisors of R, then the* ***zero-divisor graph of*** *R, denoted by $G(R)$, is a simple undirected graph with vertex set $Z^*(R) = Z(R) \setminus \{0\}$ in which two distinct vertices x and y are adjacent if and only if $xy = 0$.*

The following example lays out the difference between two definitions of zero-divisor graph of rings given by Beck [4] and modified one by Anderson and Livingston [2].

Example 2. *The zero-divisor graph of the ring $\mathbb{Z}_{12}$*



$\chi(G_B(\mathbb{Z}_{12})) = \omega(G_B(\mathbb{Z}_{12})) = 3$
(a)

$\chi(G(\mathbb{Z}_{12})) = \omega(G(\mathbb{Z}_{12})) = 2$
(b)

Figure 1.1: (a) Beck Definition (b) Anderson and Livingston Definition

[2] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434-447.

1.2 Zero-divisor graphs of reduced rings

Beck [4] proved that a reduced commutative ring R has a.c.c. on it's ideals of the form $ann(x)$ whenever it's zero-divisor graph does not contain an infinite clique. The following two lemmas are due to Beck [4].

Lemma 1.2.1. *Let R be a reduced ring such that $G(R)$ does not contain an infinite clique. Then R has a.c.c. on ideals of the form $ann(x)$.*

Proof. Assume that ideals of the form $ann(x)$ do not have a.c.c., that is $ann(a_1) < ann(a_2) < \cdots$. Let $x_i \in ann(a_i) \setminus ann(a_{i-1})$ for $i = 2, 3, \cdots$. Then for $n = 2, 3, ...$ non-zero elements $y_n = x_n a_{n-1}$ form an infinite clique, where $y_i \neq y_j$ for $i \neq j$. Otherwise, we have that $y_i y_j = 0$ and the equality $y_i = y_j$ would yield $y_i^2 = y_j^2 = 0$ which implies that $y_i = y_j = 0$, a contradiction.

Lemma 1.2.2. *Let x and y be elements in R such that $ann(x)$ and $ann(y)$ are different prime ideals. Then $xy = 0$.*

Proof. On the contrary, assume that $xy \neq 0$ implies that $y \notin ann(x)$ and $x \notin ann(y)$. Since $ann(x)$ and $ann(y)$ are prime ideals we get $ann(x) : y = ann(x)$ and $ann(y) : x = ann(y)$. Since, $ann(x) : y = ann(y) : x = ann(xy)$ we get $ann(x) = ann(y)$.

Using Lemma 1.2.1 and Lemma 1.2.2, Beck [4] essentially proved the following theorem.

Theorem 1.2.3. *For a reduced ring R the following statements are equivalent :*

1. *$\chi(G(R))$ is finite.*
2. *The zero-ideal in R is a finite intersection of prime ideals.*
3. *R does not contain an infinite clique.*

Proof. As $\omega(G(R)) \leq \chi(G(R))$, (1) implies (3) is evident.
To see (2) implies (1), let $(0) = P_1 \cap P_2 \cap \cdots \cap P_n$, where $P_1, P_2, \cdots, P_n$ are prime ideals. We define a coloring f on $V(G(R))$ as $f(x) = \min\{i | x \notin P_i\}$. Note that $\chi(G(R)) \leq n$.
Now we show that (3) implies (2). So we assume that R is reduced and does not contain an

infinite clique. Then by Lemma 1.2.1 R satisfies a.c.c. on ideals of the form $ann(a)$. Let $ann(x_i)$, $i \in I$ be the maximal elements of the set $\{ann(a)\}$. Then each $ann(x_i)$ is also a prime ideal and by Lemma 1.2.2 index set I is finite implies that the number of prime ideals is also finite. Pick $y \in V(G(R))$, then $ann(y) \subset ann(x_i)$ for some $i \in I$. If $yx_i = 0$ then $x_i \in ann(y) \subset ann(x_i)$ implies that $x_i^2 = 0$, but since R is reduced we get $x_i = 0$. So we have $yx_i \neq 0$ and thus $y \notin ann(x_i)$. Therefore, $\bigcap_I ann(x_i) = (0)$.

Using Theorem 1.2.3, Beck [4] essentially proved that finitely colorable commutative reduced rings are weakly perfect. Moreover, clique and chromatic number are equal to number of minimal prime ideals of reduced ring.

Theorem 1.2.4. *Let $R \neq (0)$ be a reduced ring. If $\chi(G(R)) < \infty$ then R has only finite number of minimum prime ideals. If n is this number then $\chi(G(R)) = \omega(G(R)) = n$.*

Proof. From Theorem 1.2.3 we get $\chi(G(R)) \leq n$, where n is the number of minimal prime ideals P_i of R. Moreover, choose x_i such that $x_i \notin P_i$ and $x_i \in P_j$ for some $j \neq i$. Then $\{x_1, x_2, \cdots x_n\}$ is the clique. Hence $\omega(G(R)) \geq n$, implies that $n \geq \chi(G(R)) = \omega(G(R)) \leq n$.

1.3 A Counter-Example

Anderson and Naseer [1][3] gave some characterizations for zero-divisor graphs of commutative rings and more importantly gave an example of a ring for which zero-divisor graph is not weakly perfect disproving Beck's conjecture. We reproduce this example. (Note: We use the modified definition of zero-divisor graphs).

Theorem 1.3.1. *([1]) Let $R = \mathbb{Z}_4[X,Y,Z]/(X^2-2, Y^2-2, Z^2, 2X, 2Y, 2Z, XY, XZ, YZ-2)$ be a commutative ring with identity. Then $4 = \omega(G(R)) < \chi(G(R)) = 5$.*

Proof. $M = \{0, 2, x, x+2, y, y+2, z, z+2, x+y, x+y+2, y+z, y+z+2, x+y+z, x+y+z+2\}$ is a maximal ideal of R and (R, M) is a finite local ring such that $R/M \approx \mathbb{Z}_2$. The remaining 16 elements of $R - M$ are units $U(R)$. 0 and 2 both annihilate complete M and also, $0 : M = M^2 = \{0, 2\}$ and $M^3 = 0$.

[3] D. D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra **159 (2)** (1993), 500-514.

To show that $\omega(G(R)) = 4$ it is sufficient to show that $\omega(G(M)) = 4$. Since (apart from 0) 2 annihilates every element of M it has to be part of every maximal clique. $\{2, x, y, y+z\}$ is a maximal clique, therefore $\omega(G(R)) \geq 4$.

Next we show that no clique has more than 4 elements. From Table 1.1, we have for any element $i \in \{x, y, z, x+y, x+z, y+z, x+y+z\}$ there is an element $i+2$ such that pair of i and $i+2$ have same annihilators. Similarly, if for some $i \in M$ has annihilator $j \in M$ then $j+2$ is also an annihilator of i.

Suppose that clique contains x then other than 2 it contains one of three pairs viz; pair y and $y+2$ or pair z and $z+2$ or pair $y+z$ and $y+z+2$. The same is true for $x+2$. So largest clique with x or $x+2$ is of length 4. Similarly it is evident from Table 1.1, that a clique containing y or $y+2$, z or $z+2$, $x+y$ or $x+y+2$, $x+z$ or $x+z+2$, $y+z$ or $y+z+2$, and $x+y+z$ or $x+y+z+2$ will have at most 4 elements. Therefore, $\omega(G(M)) \leq 4$.

Since $C = \{2, x, y, y+z\}$ is the maximal clique in M we have $\chi(G(R)) \geq 4$. We color C with 4 colors (viz; c1,c2,c3, and c4). Let 2 is colored by c1, then no other element can be colored by c1 and color x with c2, y with c3, and $y+z$ with c4.

Next we claim that the subgraph $\{2, x, y, z, y+z, x+y, x+z, z+2, x+y+2\}$ can not be colored by 4 colors. Since $xz = x(z+2) = 0$ and $z(z+2) = 0$ implies that we have to color one of z and $z+2$ with c3 and then other with c4. So we color z with c3 and $z+2$ with c4. Also, $x+y$ and $x+y+2$ are annihilators of each other. In addition $y+z$ annihilates both $x+y$ and $x+y+2$. Therefore, we color $x+y$ with c2 and $x+y+2$ with c3 since $y+z$ is already colored by c4. Now since $x+y$, z, and $z+2$ all annihilates $x+z$ it cannot be colored with c1, c2, c3, c4. This implies that $\chi(G(R)) \geq 5$. In fact, the following partition of R gives coloring of $G(R)$ using five colors.

$\{2\}, \{x, x+2, x+y, x+y+z\}, \{y, y+2, z, x+y+2\}, \{y+z, y+z+2, z+2, x+y+z+2\}, \{x+z, x+z+2\}$.

<table>
<tr><th></th><th>x</th><th>x+2</th><th>y</th><th>y+2</th><th>x+y</th><th>x+y+2</th><th>z</th><th>z+2</th><th>x+z</th><th>x+z+2</th><th>y+z</th><th>y+z+2</th><th>x+y+z</th><th>x+y+z+2</th></tr>
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<tr><td>x+z+2</td></tr>
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<tr><td>y+z+2</td></tr>
<tr><td>x+y+z</td><td colspan="2" rowspan="2">2</td><td colspan="2" rowspan="2">0</td><td colspan="2" rowspan="2">2</td><td colspan="2" rowspan="2">2</td><td colspan="2" rowspan="2">0</td><td colspan="2" rowspan="2">2</td><td colspan="2" rowspan="2">0</td></tr>
<tr><td>x+y+z+2</td></tr>
</table>

Table 1.1: Multiplication Table for M

Chapter 2

Zero-divisor Graphs of Semigroups

In this chapter, we study the partial ordering relation given by LaGrange and Roy [20][1] on a reduced commutative ring. Using this relation, we study the interplay of a zero-divisor graph of a reduced commutative semigroup S and the zero-divisor graph of S treated as a meet-semilattice.

We begin with few definitions and terminologies.

2.1 Preliminaries

2.1.1 Semigroups

Throughout this chapter S is a commutative semigroup with 0 and 1. If S has no identity element, one can simply be adjoined to S.

Definitions 2.1.1. *An element $a \in S$ is a* ***nilpotent element*** *if there is a positive integer n such that $a^n = 0$. A semigroup S is called* ***reduced*** *if it contains no nonzero nilpotent elements.*

A nonempty subset I of S is called **ideal** if $SI \subseteq S$. If a is an element of a commutative

[1] J.D. LaGrange and K.A. Roy, *Poset graphs and the lattice of graph annihilators*, Discrete Math. 313 (10) (2013), 1053-1062.

semigroup S, then the smallest ideal containing a is called the **principal ideal generated by** a. Similarly as in rings, this ideal will contain $aS = \{as \mid s \in S\}$, the set of multiples of a. The zero ideal will be denoted (0).

Note that the product, union, and intersection of ideals of S will again be an ideal of S, and that each nonzero ideal must necessarily be composed of a union of principal ideals.

Definition 2.1.2. *For a subset $A \subseteq S$, the **annihilator of** A, denoted $ann(A)$ is given by $ann(A) = \{x \in S \mid xa = 0$ for all $a \in A\}$.*

We summarize some properties of annihilators of subsets of commutative semigroup S which we will be use repeatedly.
1. $ann(0) = S$.
2. $ann(A)$ is an ideal of S.
3. $A \subseteq B$ implies that $ann(A) \supseteq ann(B)$.
4. $A \subseteq ann((annA))$ and $ann(A) = ann(ann(ann(A)))$.

Definitions 2.1.3. *Let S be a semigroup. A proper ideal P of S is called a **prime** ideal if $ab \in P$ implies either $a \in P$ or $b \in P$. A prime ideal P of S is said to be a **minimal** prime ideal if there does not exist any prime ideal Q such that $Q \subsetneqq P$.*

Definition 2.1.4. *A semigroup S is said to satisfy **annihilating condition (a.c.)** if for $a, b \in S$, there exists $c \in S$ such that $ann(a) \cap ann(b) = ann(c)$.*

DeMeyer, McKenzie and Schneider [8], defined the zero-divisor graph of a commutative semigroup analogues to the definition given by Anderson and Naseer [1] for zero-divisor graph of a commutative ring.

Definition 2.1.5. *An element x of a semigroup S is a **zero-divisor** if there is a nonzero $y \in S$ such that $xy = 0$. The set of nonzero zero-divisors of a semigroup S is denoted by $Z^*(S) = \{x \in S | xy = 0$ for some nonzero $y \in S\}$, also we define $Z(S) = Z^*(S) \bigcup \{0\}$. The **zero-divisor graph** of a commutative semigroup S with unity is a simple undirected graph, denoted by $G(S)$, whose vertices are the nonzero zero-divisors of S and vertices x, y are adjacent, if $xy = 0$.*

2.1.2 Poset and Lattices

Definitions 2.1.6. *A **partially ordered set (poset)** $< P; \leq >$ consists of a non-empty set P and a binary relation $\leq$ on P such that $\leq$ satisfies the following three properties;*

$(P1)$	*Reflexivity* :	$a \leq a, \forall a \in P$
$(P2)$	*Antisymmetry* :	$a \leq b$ *and* $b \leq a$ *imply that* $a = b$
$(P3)$	*Transitivity* :	$a \leq b$ *and* $b \leq c$ *imply that* $a \leq c$.

*A poset $< L; \leq >$ is a **meet-semilattice**, if the greatest lower bound of $\{a, b\}$, i.e.,* $\inf\{a, b\}$ *exists for all $a, b \in L$. Similarly, it is a **join-semilattice**, if the least upper bound of $\{a, b\}$, i.e.,* $\sup\{a, b\}$ *exists for all $a, b \in L$. A poset $< L; \leq >$ is a **lattice**, if it is a meet-semilattice as well as a join-semilattice. Equivalently, a lattice can also be defined as an algebra $< L; \wedge, \vee >$ with two binary operations $\wedge$ (meet) and $\vee$ (join) on L that satisfy associativity, commutativity, idempotency and the absorption identities, i.e., $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$ for $a, b \in L$.*

Let P be a poset with the least element 0. Given any nonempty subset X of P, the sets $X^u = \{y \in P \mid y \geq x \text{ for every } x \in X\}$ and $X^\ell = \{y \in P \mid y \leq x \text{ for every } x \in X\}$ are called the **upper cone** and the **lower cone**, respectively. Also, if $x \in P$ then the sets $\{x\}^u$ and $\{x\}^\ell$ will be denoted by x^u and x^ℓ, respectively.

In [25], Varlet introduced and studied the concept of a 0-distributive lattice.

Definition 2.1.7. *A lattice L with 0 is said to be **0-distributive** if $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$.*

Definitions 2.1.8. *Let L be a meet-semilattice. A non-empty subset I of L is said to be a **semi-ideal** if $x \leq y \in I$ implies that $x \in I$. A proper subset I of L is said to be **prime** if $a \wedge b \in I$ implies that $a \in I$ or $b \in I$. A prime semi-ideal $P \subseteq L$ is said to be a **minimal prime semi-ideal** if there does not exists any prime semi-ideal Q such that $Q \subsetneqq P$.*

Definitions 2.1.9. *Let L be a meet-semilattice. A nonempty subset F of L is a **filter** if (i) $a, b \in F$ implies $a \wedge b \in F$ and (ii) $a \in F$ and $a \leq b$ imply $b \in F$. A filter F of L is said to be a **maximal** filter if there does not exists a filter $J \neq L$ such that $F \subsetneqq J$. If a is an element of the meet-semilattice L, then the smallest filter containing a is called the **principal filter generated by** a. This filter is denoted by $[a) = \{b \in L \mid a \leq b\}$.*

As mentioned earlier, Nimbhorkar et al. [22] introduced the concept of a zero-divisor graph for meet-semilattices with 0 which was further generalized by Halaš and Jukl [12] for poset. Lu and Wu [21] modified this definition on the lines of Anderson and Livingston [2]. The zero-divisor graph of a poset with respect to an ideal I was introduced by Joshi [13]. Note that this definition coincides with the definition of Lu and Wu, when $I = \{0\}$. It should be noted that both the papers ([13] and [21]) are submitted around same time. We quote this definition when a poset is a meet-semilattice.

Definition 2.1.10. *Let L be a meet-semilattice with 0. An element x of a meet-semilattice L is a **zero-divisor** if there is some nonzero $y \in L$ such that $x \wedge y = 0$. We associate with L, a simple undirected graph $\Gamma(L)$, called the **zero-divisor graph of** L, with the vertex set $V(\Gamma(L))$ which is the set of nonzero zero-divisors in L and distinct vertices a, b are adjacent if $a \wedge b = 0$. In this case, the vertex set of $\Gamma(L)$ is the set of nonzero zero-divisors of L , denoted by $Z^*(L)$. Further, we denote $Z(L) = Z^*(L) \cup \{0\}$.*

Example 3. *Consider the lattice L shown in Figure 2.1.2.1 (a) and its zero-divisor graph $\Gamma(L)$ is shown in Figure 2.1.2.1 (b).*

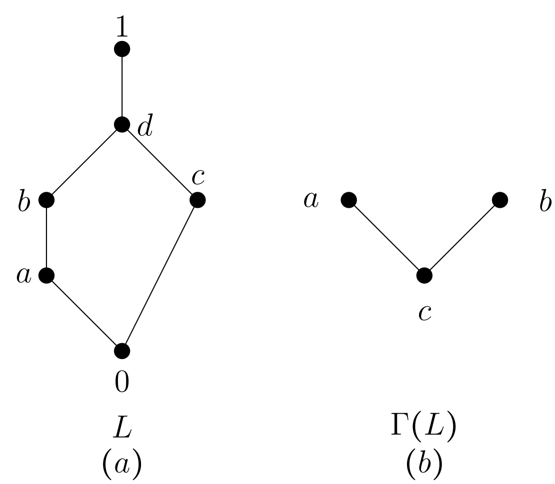


Figure 2.1: Zero-divisor graph of L

2.2 Zero-divisor graphs of posets

On the similar lines of Beck's theorem for reduced rings, Halaš and Jukl [12] essentially proved the following Theorem 2.2.1 for a poset.

Theorem 2.2.1. *Let P be a poset with 0, and assume that $\omega(\Gamma(P)) < \infty$. Then the number n of all minimal prime ideals of P is finite and $\chi(\Gamma(P)) = \omega(\Gamma(P)) = n$.*

Let L be a meet-semilattice with the least element 0. Define the set $x^{\perp} = \{y \in L \mid x \wedge y = 0\}$. Next, we provide a direct proof of [12, Theorem 2.9] for meet-semilattices with 0.

Theorem 2.2.2. *Let L be a meet-semilattice with 0 and $\omega(\Gamma(L)) < \infty$. Then $\omega(\Gamma(L)) = \chi(\Gamma(L))$.*

Proof. Let $\omega(\Gamma(L)) = n$ (say). Then there exists a clique C of $\Gamma(L)$ with cardinality n such that $C = \{x_1, x_2, \cdots, x_n\}$ where $x_i \wedge x_j = 0$ for $i \neq j$ and for all $i, j \in \{1, 2, \cdots, n\}$. Also if $i \neq j$ it is easy to observe that $x_i^{\perp}$ and $x_j^{\perp}$ are distinct. Otherwise, $x_j \in x_i^{\perp} = x_j^{\perp}$ will give $x_j = 0$, a contradiction. That is, all $x_j^{\perp}$ are distinct and total number is again n. Further observe that $\bigcap_{i=1}^{n} x_i^{\perp} = \{0\}$. Otherwise let $0 \neq t \in \bigcap_{i=1}^{n} x_i^{\perp}$. Then it can be verify that $\{t, x_1, x_2, \cdots, x_n\}$ is a clique of $n+1$ elements, a contradiction to $\omega(\Gamma(L)) = n$. We denote $x_i^{\perp} = P_i$. Define $f(x) = \min\{i \mid x \notin P_i\}$. We claim that $f(x)$ is a coloring. Let x and y be adjacent vertices of $\Gamma(L)$, i.e., $x \wedge y = 0$. Let $f(x) = k+1$. Then $x \notin P_{k+1}$ but $x \in P_i$ for all $i = 1, 2, \cdots, k$. We now claim that $y \in P_{k+1} = x_{k+1}^{\perp}$. On the contrary, assume that $y \notin P_{k+1}$, i.e., $x_{k+1} \wedge y \neq 0$. Therefore $x \wedge x_{k+1}$, $y \wedge x_{k+1}$ are vertices of $\Gamma(L)$. Now consider the set $C_1 = \{(x \wedge x_{k+1}), (y \wedge x_{k+1}), x_1, x_2, \cdots, x_k, x_{k+2}, \cdots, x_n\}$. Note that $x_{k+1} \notin C_1$. We claim that C_1 is a new clique. All x_i of C_1 are distinct, as they are the elements of the clique C. Now if $x \wedge x_{k+1} = x_i$ for some $x_i \in C_1$, then $x_i = x \wedge x_{k+1} \wedge x_i = 0$, as $x_i \wedge x_{k+1} = 0$, a contradiction. Similarly, $y \wedge x_{k+1} \neq x_i$ for every $x_i \in C_1$. Again, if $x \wedge x_{k+1} = y \wedge x_{k+1}$ then we get $x \wedge x_{k+1} = (x \wedge x_{k+1}) \wedge (y \wedge x_{k+1}) = 0$, since $x \wedge y = 0$, again a contradiction. Thus C_1 is the new clique with $|C_1| = n+1$, a contradiction to the fact that $\omega(\Gamma(L)) = n$. Therefore $y \in P_{k+1}$ and hence the function f is a coloring. Thus $n = \omega(\Gamma(L)) \leq \chi(\Gamma(L)) = n$.

Let S be a reduced commutative semigroup with $0 \neq 1$. Define a relation $\leq$ such that $r \leq s$ in S if and only if either $ann(s) \subsetneq ann(r)$ or $r \leq s$ in some predetermined linear order on the set $[r] = \{x \in S \mid ann(r) = ann(x)\}$. LaGrange and Roy [20, Remark 3.4], proved that $\leq$ is a partial order on S. In fact it follows from Remark 4.8 of [19] that S is a meet-semilattice. The last two statements of the following theorem follow from [3, Lemma 3.5 (1), (2) and (5)] and [10, Theorem 3.4]. For the sake of completeness, we provide its proof.

Theorem 2.2.3. *Let S be a reduced commutative semigroup with 0 and 1. Then the following statements are true.*

1. $< S; \leq >$ *is a meet-semilattice.*

2. For $a, b \in S$ *we have* $ab = 0$ *if and only if* $a \wedge b = 0$.
Therefore, the zero-divisor graph of S *(treated as a meet-semilattice) and zero-divisor graph of a semigroup* S *are essentially same, i.e.,* $\Gamma(S) \cong G(S)$.

3. If S *satisfies annihilating condition (a.c.), then* S *is a 0-distributive lattice.*

Proof. (1.) From the above discussion, S is a poset. Now we prove that S is a meet-semilattice. Let $a, b \in S$. If a and b are comparable, then $\inf\{a, b\}$ exists and we are through in this case. Assume that a and b are incomparable. Then $ann(a) \subsetneqq ann(ab)$ and $ann(b) \subsetneqq ann(ab)$. Hence ab is a lower bound of $\{a, b\}$. Consider the elements of $[ab]$ in predetermined linear order. Without loss of generality assume that ab is the largest, i.e., $x \leq ab$ for all $x \in [ab]$. Let t be any lower bound of $\{a, b\}$, i.e, $t \leq a, b$. Then $ann(a), ann(b) \subseteq ann(t)$. Without loss of generality, if $ann(t) = ann(a)$ and $ann(b) \subsetneqq ann(t)$, then $a \leq b$, a contradiction. Hence $ann(a), ann(b) \subsetneqq ann(t)$. We claim that $ann(ab) \subseteq ann(t)$. Let $x \in ann(ab)$. Then $xab = 0$. Hence we have $xb \in ann(a) \subsetneqq ann(t)$. This gives $xbt = 0$, i.e., $xt \in ann(b) \subsetneqq ann(t)$ which further implies that $xt^2 = (xt)^2 = 0$. Since S is reduced, we get $xt = 0$. Hence $x \in ann(t)$. This gives $ann(ab) \subseteq ann(t)$. Now if $ann(ab) \subsetneqq ann(t)$ we get $t \leq ab$ or if $ann(ab) = ann(t)$ again we get $t \leq ab$, as considered in predetermined order that the element ab is the largest, proving that $\inf\{a, b\} = ab$. Thus S is a meet-semilattice.

(2.) If $a, b \in S$ and $ab = 0$, then as S is a meet-semilattice, from the proof of (1.) we get that $a \wedge b = \inf\{a, b\} = ab = 0$. Conversely, if $a \wedge b = 0$ then $ab \leq a \wedge b = 0$ implies that $ab = 0$.

(3.) Assume that S satisfies the annihilating condition (a.c.). Let $a, b \in S$. Then by (a.c.), we have $c \in S$ such that $ann(c) = ann(a) \cap ann(b)$. Consider the elements of $[c]$ in predetermined linear order. Without loss of generality, assume that c is the least of them, i.e., $c \leq x$ for all $x \in [c]$. Arguing similarly as above we get $\sup\{a, b\}$ is a or b, if $ann(c) = ann(b)$ or $ann(c) = ann(a)$. Now, if $ann(c) \subsetneqq ann(a)$ and $ann(c) \subsetneqq ann(b)$ then $a, b \leq c$. Thus c is a upper bound of $\{a, b\}$. Let u be any upper bound of $\{a, b\}$. Then $u \geq a, b$ which implies $ann(a), ann(b) \subseteq ann(u)$. Again we have similar conditions and therefore by similar arguments we get the existence of $\sup\{a, b\}$. Now if $ann(u) \subsetneqq ann(a), ann(b)$ then we claim that $ann(u) \subseteq ann(c)$. Let $x \in ann(u) \subsetneqq ann(a), ann(b)$. Then $xa = xb = 0$. This gives $x \in ann(a) \cap ann(b) = ann(c)$. Hence $ann(u) \subseteq ann(c)$. Now if $ann(u) \subsetneqq ann(c)$ we get $c \leq u$ or if $ann(u) = ann(c)$ again we get $c \leq u$ as considered in predetermined order, proving that

$\sup\{a,b\} = c$. Thus S is a lattice.

Now, suppose that $x, y, z \in S$ such that $x \wedge y = x \wedge z = 0$. Since $\Gamma(S) = G(S)$, it follows that $xy = xz = 0$. Let $u = x \wedge (y \vee z)$. Since $y, z \in S$, by (a.c.) there exists $a \in S$ such that $ann(a) = ann(y) \cap ann(z)$. Hence $ann(a) \subseteq ann(y)$ and $ann(a) \subseteq ann(z)$, so either $y, z \leq a$ or $ann(a) \in \{ann(y), ann(z)\}$. But if $ann(a) = ann(y)$ then replace a by any element of $\{s \in S | ann(s) = ann(y)\}$ that is greater than or equal to y, i.e., $y \leq a$. Similarly, it can be assumed that $z \leq a$. Hence it follows that a can be chosen such that $ann(a) = ann(y) \cap ann(z)$ and $y \vee z \leq a$. Therefore, $u \leq a$, and since $x \in ann(y) \cap ann(z) = ann(a) \subseteq ann(u)$, we have $xu = 0$, i.e., $x \wedge u = 0$. But $u \leq x$ implies $u = 0$. Thus, $x \wedge (y \vee z) = 0$.

The following well-known result is due to Kist [18].

Lemma 2.2.4. *Let S be a reduced semigroup and P be a prime ideal of S. Then P is a minimal prime ideal if and only if it satisfies the condition (§).*

(§) : For any $x \in P$, there exists $y \notin P$ such that $xy = 0$.

The following result is a modified version of [23, Theorem 4] by Pawar and Thakare, which is an analogue of the above result. For the sake of completeness, we provide its proof.

Lemma 2.2.5. *Let L be a meet-semilattice with 0 and P be a prime semi-ideal. Then P is a minimal prime semi-ideal if and only if it satisfies the condition ($\star$).*

($\star$) : For any $x \in P$, there exists $y \notin P$ such that $x \wedge y = 0$.

Proof. Let P be a prime semi-deal satisfying the condition ($\star$). Suppose on the contrary that P is not minimal. Hence there exists a minimal prime semi-ideal, say Q such that $Q \subsetneqq P$ Then there exists $x \in P$ such that $x \notin Q$. By the condition ($\star$), there exist $y \notin P$ such that $x \wedge y = 0 \in Q$. But $x, y \notin Q$ a contradiction to primeness of Q.

Conversely, assume that P is minimal prime semi-ideal. Then $L \setminus P$ is a filter. Moreover, $L \setminus P$ is a maximal filter. Let $x \in P$. Consider the filter $K = [x) \vee L \setminus P$. By maximality of $L \setminus P$, we have $K = L$. Since, $0 \in L = K = [x) \vee L \setminus P$ implies that $x \wedge y = 0$ for some $y \in L \setminus P$. This proves that $x \wedge y = 0$ for some $y \notin P$.

With this preparation, we are ready to prove our main result which relates minimal prime ideals of a reduced semigroup S and minimal prime semi-ideals of S treated as a meet-semilattice.

Theorem 2.2.6. *Let S be a reduced semigroup with 0 and 1 and $P \subseteq R$. Then P is a minimal prime ideal of S (treated as a semigroup) if and only if P is a minimal prime semi-ideal of S (treated as a meet-semilattice).*

Proof. Let P be a minimal prime ideal of S (treated as a semigroup). First, we prove that it is a semi-ideal of S as a meet-semilattice. Let $x \leq y \in P$. We claim that $x \in P$. Since $y \in P$, by Lemma 2.2.4, there exists $z \notin P$ such that $yz = 0$. By Theorem 2.2.3, we have $y \wedge z = 0$. Therefore $x \wedge z = 0$. Again by Theorem 2.2.3 $xz = 0$. This gives $x \in P$. Thus P is a semi-ideal.

Next we prove that P is a prime semi-ideal. Let $x \wedge y \in P$. From proof of Lemma 2.2.3, we observe that xy is a lower bound of x, y. Hence $xy \leq x \wedge y \in P$. Since P is a semi-ideal, we have $xy \in P$. Therefore, either $x \in P$ or $y \in P$, as P is a prime ideal (treated as a semigroup), proving that P is a prime semi-ideal of S (treated as a meet-semilattice). Minimality of a prime semi-ideal P follows from the condition $(\star)$ of Lemma 2.2.5 and Theorem 2.2.3.

Conversely, let P be a minimal prime semi-ideal of S (treated as a meet-semilattice). First, we prove that P is an ideal of a semigroup S. Let $a \in P$ and $r \in R$. By the condition $(\star)$ of Lemma 2.2.5 there exists $c_1 \notin P$ such that $a \wedge c_1 = 0$. This further gives that $ac_1 = 0$ (by Theorem 2.2.3). Hence $(ar)c_1 = 0$ which yields $(ar) \wedge c_1 = 0 \in P$. This gives $ar \in P$. Hence P is an ideal of S (treated as a semigroup).

Next we claim that P is a prime ideal of a semigroup S. Let $ab \in P$. Then by the condition $(\star)$ of Lemma 2.2.5 there exists $c \notin P$ such that $ab \wedge c = 0$. Hence $abc = 0$. Note that if $abc = 0$ then $a \wedge b \wedge c = 0$. For this let $t \in \{a, b, c\}^{\ell}$. Then $ann(a) \subseteq ann(t)$, $ann(b) \subseteq ann(t)$ and $ann(c) \subseteq ann(t)$. Clearly, $bc \in ann(a) \subseteq ann(t)$. Hence $bct = 0$ which further implies that $ct \in ann(b) \subseteq ann(t)$. Since S is reduced, we have $ct = 0$. Therefore, $t \in ann(c) \subseteq ann(t)$ implies that $t^2 = 0$. Thus we have $t = 0$. Thus $a \wedge b \wedge c = 0 \in P$ implies that $a \wedge b \in P$, since $c \notin P$. As P is a prime semi-ideal, we have either $a \in P$ or $b \in P$. This proves that P is a prime ideal of a semigroup S. Minimality of a prime ideal P follows from the condition (§) of Lemma 2.2.4 and Theorem 2.2.3.

Let $Min(S)$ denotes the set of all minimal prime ideals of S treated as a semigroup and $Min_s^p(S)$ denotes the set of all minimal prime semi-ideals of S treated as a meet-semilattice.

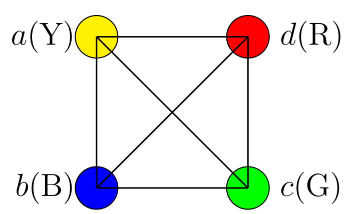
By Theorem 2.2.3, we have $\Gamma(S) \cong G(S)$. Using Theorem 2.2.2 and Theorem 2.2.6, we have the following result.

Theorem 2.2.7. *Let S be a reduced commutative semigroup with 0 and 1. Let $\omega(\Gamma(S)) < \infty$. Then $\omega(\Gamma(S)) = \chi(\Gamma(S)) = \omega(G(S)) = \chi(G(S)) = |Min(S)| = |Min_s^p(S)|$.*

To illustrate the idea of zero-divisor graph of reduced semigroup and verification of Theorem 2.2.7, we provide the following simple example.

Example 4. *Let semigroup $S = \{0, a, b, c, d, 1\}$ such that $xy = 0$ for $x \neq y$, where $x \neq 1$, $y \neq 1$ also $x^2 = x$ for all $x \in X$, and $x \cdot 1 = x$ for all $x \in X$. Note that S is commutative reduced semigroup with 0 and 1.*

For a semigroup S, we have $Z^*(S) = \{a, b, c, d\}$, such that $xy = 0$ for all $x \neq y$. Then the zero-divisor graph $G(S)$ is weakly perfect with $\omega(G(S)) = \chi(G(S)) = |Min(S)| = 4$, where $Min(S) = \{(a), (b), (c), (d)\}$, (see Figure 2.2).



$G(S) \cong \Gamma(S)$

(R)-red, (G)-green, (Y)-yellow, (B)-blue

Figure 2.2: Zero-divisor graph of S

Now, if we treat S as a meet-semilattice with the partial order given by LaGrange and Roy [20]. Then we have $ann(x) \neq ann(y)$ for all $x, y \in Z^*(S)$. Therefore we have $x \wedge y = \inf\{x, y\} = xy = 0$ for all $x, y \in Z^*(S)$. Then again all $x, y \in Z^*(S)$ are adjacent, that is, $\Gamma(S)$ is a complete graph. Hence, the zero-divisor graph $\Gamma(S)$ is also weakly perfect with $\omega(\Gamma(S)) = \chi(\Gamma(S)) = |Min_s^p(S)| = 4$, where $Min_s^p(S) = \{\{0, a\}, \{0, b\}, \{0, c\}, \{0, d\}\}$. Moreover, we get that $\Gamma(S) \cong G(S)$ which is in accordance with second statement of Theorem 2.2.1.

Therefore, for S in Example 4, we get $\omega(\Gamma(S)) = \chi(\Gamma(S)) = \omega(G(S)) = \chi(G(S)) = |Min(S)| = |Min_s^p(S)| = 4$ which verifies the Theorem 2.2.7.

Though results are written for reduced semigroups, it can be easily observed that these are also true for reduced rings. Therefore we can extend [4, Theorem 3.8] and rewrite above theorem for rings as follow.

Corollary 2.2.8. *Let R be a reduced commutative ring with unity. Let $\omega(\Gamma(R)) < \infty$. Then $\omega(\Gamma(R)) = \chi(\Gamma(R)) = \omega(G(R)) = \chi(G(R)) = |Min(R)| = |Min_s^p(R)|$.*

Chapter 3

Annihilating-ideal Graphs of Semigroups

3.1 Introduction

Behboodi and Rakeei [5] introduced the concept of annihilating-ideal graph of a ring associating a simple graph to set ideals of a ring. They established the weakly perfectness of annihilating-ideal graph for reduced rings, and also in particular to counterexample for Becks conjecture given by Anderson and Naseer [1]. In addition to this, due to the lack of counterexample to weakly perfect annihilating-ideal graph they conjectured (see Conjecture 2) that annihilating-ideal graph for every commutative ring R is weakly perfect. Therefore analogously DeMeyer and Schneider [7][1], introduced the annihilating-ideal graph of a commutative semigroup. Similarly they showed that equality also holds for reduced semigroups as well as semigroups S with $ann(S) = S$ and raised the following conjecture analogues to conjecture raised by Behboodi and Rakeei [5] for rings.

Conjecture 3.1.1. *For every commutative semigroup S with unity,* $\chi(\mathbb{AG}(S)) = \omega(\mathbb{AG}(S))$.

In this chapter, we study the theory of annihilating-ideal graph of semigroups, theory of zero-divisor graph of multiplicative lattice and compressed graphs. With help of this, we

[1]L. DeMeyer and A. Schneider, *The annihilating-ideal graph of commutative semigroups*, J. Algebra 469 (2017), 402-420.

provide an example of commutative semigroup S such that the annihilating-ideal graph of S is not weakly perfect. This solves the Conjecture 3.1.1 negatively.

3.2 Preliminaries

Inspired from the concept of annihilating-ideal graph of a ring given by Behboodi and Rakeei [5], Demeyer and Schneider [7] introduced the idea of annihilating-ideal graph of a semigroup as follows.

Definition 3.2.1. *Let $\mathbb{I}(S)$ be the set of ideals of a commutative semigroup S with 0 and let $\mathbb{I}^*(S) = \mathbb{I}(S) \setminus \{(0)\}$. A proper ideal $I \in \mathbb{I}(S)$ is an* ***annihilating-ideal*** *of S if there exists a nonzero ideal $J \in \mathbb{I}(S)$ such that $IJ = (0)$. Let $\mathbb{A}(S)$ be the set of all annihilating-ideals of S, and $\mathbb{A}^*(S) = \mathbb{A}(S) \setminus \{(0)\}$.*

Let S be a commutative semigroup with 0 and 1. We say that $x \sim y$ if and only if $ann(x) = ann(y)$. Clearly, $\sim$ is an equivalence relation and let $[x] = \{y \in L | \, ann(x) = ann(y)\}$ be an equivalence class corresponding to $\sim$. Furthermore, if $x_1 \sim x_2$ and $x_1 y = 0$, then $y \in ann(x_1) = ann(x_2)$ and hence $x_2 y = 0$. It follows that the multiplication is well-defined on the equivalence classes of $\sim$, that is, $[x][y] = [xy]$. Note that $[0] = \{0\}$ and $[1] = S \setminus Z(S)$.

We denote the set of all equivalence classes of the relation $\sim$ on S by $S_E = \{[x] \mid x \in L\}$. Note that S_E is a commutative monoid. Also, S_E forms a poset under the partial order $\leq$ as follows: $[a] \leq [b]$ if and only if $ann(b) \subseteq ann(a)$.

Consider the following Definition 3.2.2 given by LaGrange [19] of the compressed zero-divisor graph of a commutative semigroup S with 0.

Definition 3.2.2. *Let S be a commutative semigroup with 0. We associate an simple undirected graph, called the* ***compressed zero-divisor graph*** *of S and denoted by $C(G(S))$, in which the set of vertices are the equivalence classes of vertices in $G(S)$, and two nonzero distinct vertices $[x], [y]$ are adjacent if and only if $[x][y] = [0]$.*

The following definition of the annihilating-ideal graph of a semigroup is given by DeMeyer and Schneider [7] which is an analogue of definition of the annihilating-ideal graph of a ring.

Definitions 3.2.3. *The **annihilating-ideal graph** $\mathbb{AG}(S)$ of a commutative semigroup S with $1 \neq 0$ is the graph where the vertex set $V(\mathbb{AG}(S))$ is the collection of nonzero ideals with nonzero annihilator. That is, a nonzero ideal I belongs to $V(\mathbb{AG}(S))$ if and only if there exists a nonzero ideal J of S such that $IJ = (0)$, and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. Also, we associate an simple undirected graph, called the **compressed annihilating-ideal graph** of S and denoted by $C(\mathbb{AG}(S))$, in which the set of vertices $V(\mathbb{AG}(S)) = S_E$ and two distinct vertices $[x], [y] \in S_E$ are adjacent if and only if $[x][y] = [0]$.*

The set of ideals of semigroup S, denoted by $\mathbb{I}(S)$ forms a multiplicative lattice. Moreover, $\mathbb{I}_E(S)$ is again a multiplicative lattice, in fact $\mathbb{I}_E(S)$ is a compressed multiplicative lattice (see Definition 3.2.7).

The concept of multiplicative lattices was introduced by Ward and Dilworth [26][2] to study abstract commutative ideal theory of commutative rings.

Definition 3.2.4. *A complete lattice (see Definition 2.1.6) L is a **multiplicative lattice**, if there is a binary operation "$\cdot$" called the multiplication on L satisfying the following conditions:*

1. *$a \cdot b = b \cdot a$, for all $a, b \in L$.*
2. *$a \cdot (b \cdot c) = (a \cdot b) \cdot c$, for all $a, b, c \in L$.*
3. *$a \cdot (\bigvee_\alpha b_\alpha) = \bigvee_\alpha (a \cdot b_\alpha)$, for all $a, b_\alpha \in L$, $\alpha \in \Lambda$(an index set).*
4. *$a \cdot 1 = a$, for all $a \in L$.*

Definitions 3.2.5. *(See [11]). An element c of a complete lattice L is a **compact element**, if $c \leq \bigvee_\alpha a_\alpha$, $\alpha \in \Lambda$ (Λ is an index set) implies $c \leq \bigvee_{i=1}^n a_{\alpha_i}$, where $n \in \mathbb{Z}^+$.*

*The set of all compact elements of a multiplicative lattice L is denoted by L_*. A lattice L is **compactly generated** or **algebraic**, if for every $x \in L$, there exist $x_\alpha \in L_*$ for $\alpha \in \Lambda$(an index set) such that $x = \bigvee_\alpha x_\alpha$, that is, every element is a join of compact elements. Equivalently, if L is a compactly generated lattice and if $a \nleq b$ for $a, b \in L$, then there is a nonzero compact element $c \in L_*$ such that $c \leq a$ and $c \nleq b$.*

[2] M. Ward and R. P. Dilworth, *Residuated lattices*, Trans. Amer. Math. Soc. **45** (1939), 335-354.

*A multiplicative lattice L is 1-**compact**, if 1 is a compact element of L. A multiplicative lattice L is **compact**, if every element of L is a compact element.*

The following concept of m-SSC lattice is introduced and studied by Sarode and Joshi [24].

Definition 3.2.6. *A multiplicative lattice L is a m-**section semi-complemented** (in brief m-SSC), if for $a, b \in L$ with $b \nleq a$, there exists $c \in L$ such that $b \cdot c \neq 0$ and $a \cdot c = 0$.*

Remark 3.2.1. *It is easy to observe that a multiplicative lattice L is m-SSC if and only if the equivalence class (treated L as a multiplicative semigroup) $[a] = \{a\}$ for every $a \in L$.*

Let L be a multiplicative lattice and $x, y \in L$. We say that $x \sim y$ if and only if $ann(x) = ann(y)$. Clearly, $\sim$ is an equivalence relation and let $[x]_m = \{y \in L \mid ann(x) = ann(y)\}$ be an equivalence class corresponding to $\sim$. Furthermore, if $x_1 \sim x_2$ and $x_1 \cdot y = 0$. Then $y \in ann(x_1) = ann(x_2)$ and hence $x_2 \cdot y = 0$. It follows that the multiplication is well defined on the equivalence classes of $\sim$, that is, $[x]_m \cdot [y]_m = [x \cdot y]_m$. Note that $[0]_m = \{0\}$ and $[1]_m = L \smallsetminus Z(L)$.

Definition 3.2.7. *(See [15]) We denote the set of all equivalence classes of the relation $\sim$ on a multiplicative lattice L by $L_E = \{[x]_m \mid x \in L\}$ and called **compressed multiplicative lattice**.*

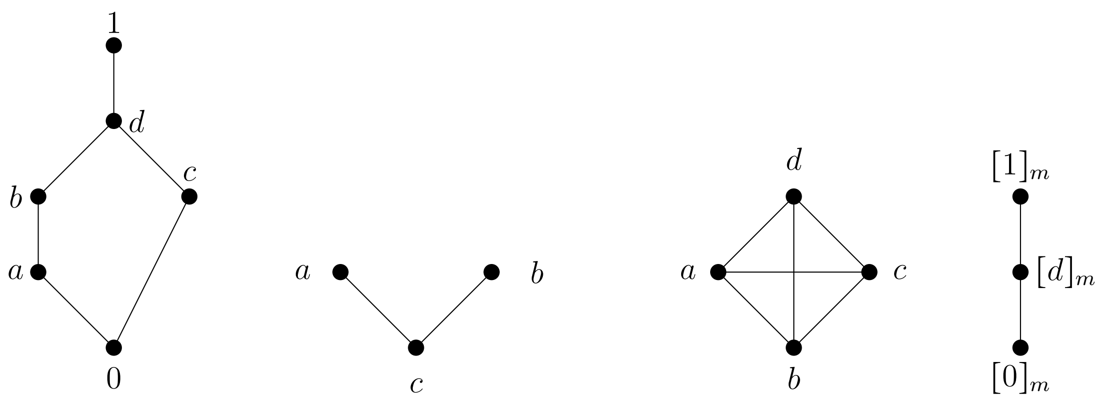
Note that L_E is a commutative monoid. Also, L_E forms a poset under the partial order $\leq$ as follows: $[a]_m \leq [b]_m$ if and only if $ann(b) \subseteq ann(a)$. In Figure 3.1 (d) we have depicted compressed multiplicative lattice L_E for lattice in Example 3.

Next, we introduce the concept of a multiplicative zero-divisor graph $\widetilde{\Gamma}(L)$ of a multiplicative lattice L given by Sarode and Joshi [24].

Definition 3.2.8. *Let L be a multiplicative lattice. An undirected simple graph is said to be the **multiplicative zero-divisor graph of** L and denoted by $\widetilde{\Gamma}(L)$, with the set of vertices is set of nonzero zero-divisors of L and it is denoted by $(Z^*(L))$, and two distinct vertices a, b are adjacent if and only if $a \cdot b = 0$. Further, we denote by $Z(L) = (Z^*(L)) \cup \{0\}$.*

Example 5. *Consider the lattice L shown in Figure 3.1 (a) with the trivial multiplication $x \cdot y = 0 = y \cdot x$ for $x,\ y \in L \smallsetminus \{1\}$ and $x \cdot 1 = x = 1 \cdot x$ for every $x \in L$. It is easy to see that L*

is a multiplicative lattice. Then, its zero-divisor graph $\Gamma(L)$ *and multiplicative zero-divisor graph* $\widetilde{\Gamma}(L)$ *are shown in Figure 3.1* (b) *and* (c) *respectively.*



L
(a)

$\Gamma(L)$
(b)

$\widetilde{\Gamma}(L)$
(c)

L_E
(d)

Figure 3.1: For a multiplicative lattice L (a) Hasse Diagram, (b) Zero-divisor graph, (c) Multiplicative zero-divisor graph and (d) Hasse Diagram of compressed multiplicative lattice L_E

Definition 3.2.9. *Let* G *be a graph.* ***Duplicating a vertex*** x *of* G *produces a new graph* $G \circ x$ *by adding a new vertex* x' *with* $N(x') = N(x)$ *(West [27, page 320]). Thus a vertex* y *is adjacent to* x' *in* $G \circ x$ *if and only if* y *is adjacent to* x *in* G*.*

3.3 Results

In [15], Joshi and Sarode gave an example of multiplicative lattice for which Beck's conjecture is not true, that is zero-divisor graph of multiplicative lattice of L (Example 6) is not weakly perfect.

Example 6. *A multiplicative lattice* L *with the multiplication given in Table 3.1.*

$\bullet$	0	a	b	c	d	e	f	$(a \vee c)$	$(a \vee d)$	$(b \vee e)$	$(c \vee e)$	$(b \vee d)$	t	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	f	0	f	f	0	0	f	f	0	f	f	f	a
b	0	0	f	0	f	f	0	0	f	f	f	f	f	b
c	0	f	0	f	0	f	0	f	f	f	f	0	f	c
d	0	f	f	0	f	0	0	f	f	f	0	f	f	d
e	0	0	f	f	0	f	0	f	0	f	f	f	f	e
f	0	0	0	0	0	0	0	0	0	0	0	0	0	f
$(a \vee c)$	0	f	0	f	f	f	0	f	f	f	f	f	f	$(a \vee c)$
$(a \vee d)$	0	f	f	f	f	0	0	f	f	f	f	f	f	$(a \vee d)$
$(b \vee e)$	0	0	f	f	f	f	0	f	f	f	f	f	f	$(b \vee e)$
$(c \vee e)$	0	f	f	f	0	f	0	f	f	f	f	f	f	$(c \vee e)$
$(b \vee d)$	0	f	f	0	f	f	0	f	f	f	f	f	f	$(b \vee d)$
t	0	f	f	f	f	f	0	f	f	f	f	f	f	t
1	0	a	b	c	d	e	f	$(a \vee c)$	$(a \vee d)$	$(b \vee e)$	$(c \vee e)$	$(b \vee d)$	t	1

Table 3.1: Multiplication on L

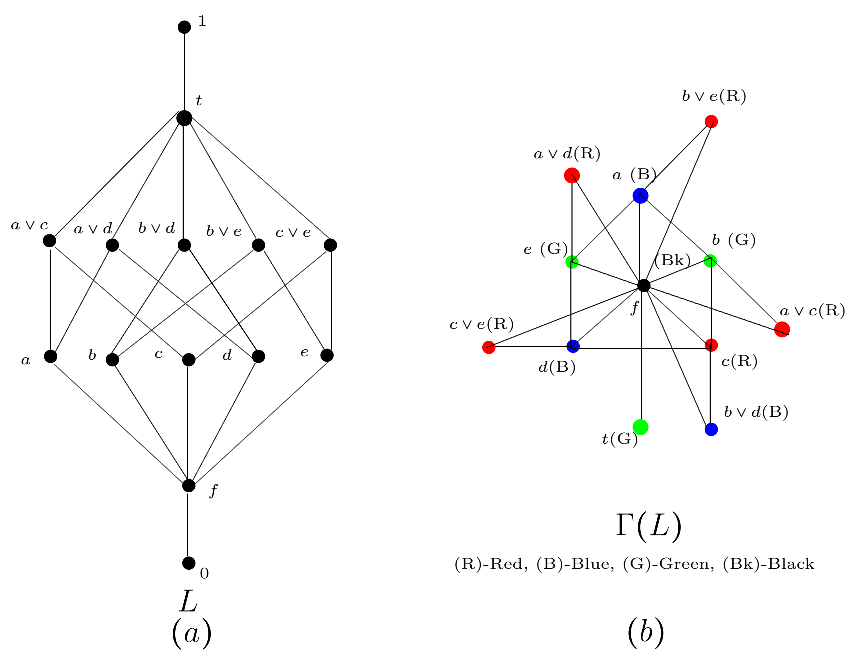


Figure 3.2: Multiplicative lattice L for which the Beck's Conjecture does not hold (a) Hasse Diagram of L (b) Multiplicative zero-divisor graph of L

Now we consider the multiplicative lattice L given above as a semigroup. Then the nonzero annihilating ideals of L are as follows.

(a)	$(b)\cup(e)$	$(b\vee e)$	$(b)\cup(b\vee e)$ $(e)\cup(b\vee e)$	$(b)\cup(e)\cup(b\vee e)$
(b)	$(a)\cup(c)$	$(a\vee c)$	$(a)\cup(a\vee c)$ $(c)\cup(a\vee c)$	$(a)\cup(c)\cup(a\vee c)$
(c)	$(b)\cup(d)$	$(b\vee d)$	$(b)\cup(b\vee d)$ $(d)\cup(b\vee d)$	$(b)\cup(d)\cup(b\vee d)$
(d)	$(c)\cup(e)$	$(c\vee e)$	$(c)\cup(c\vee e)$ $(e)\cup(c\vee e)$	$(c)\cup(e)\cup(c\vee e)$
(e)	$(a)\cup(d)$	$(a\vee d)$	$(a)\cup(a\vee d)$ $(d)\cup(a\vee d)$	$(a)\cup(d)\cup(a\vee d)$
(f)			(t)	

Table 3.2: Nonzero annihilators of L (treated as semigroup)

Lemma 3.3.1. *Ideals which are the union of more than three distinct principal ideals, except* (0) *and* (f)*, are not annihilating ideals.*

Proof. It is evident from the multiplication table that any one element (except 0 and f) annihilates only three elements (other than 0 and f) thus, any combination of four distinct non-trivial principal ideals do not annihilate single element and vice-versa.

Moreover, for each principal ideal only annihilating ideals are unions of combination of principal ideal generated by its three annihilators.

Lemma 3.3.2. *Let L be an m-SSC lattice and $I, J \subseteq L$. If $ann(I) = ann(J)$ (treated L as a multiplicative semigroup) then $\bigvee I = \bigvee J$.*

Proof. Without loss of generality, assume that $\bigvee I \nleq \bigvee J$. Since L is m-SSC, there exists $c \in L$ such that $c \cdot (\bigvee I) \neq 0$ and $c \cdot (\bigvee J) = 0$. But then $c \in ann(J) = ann(I)$. This gives $c \cdot x = 0$ for every $x \in I$. As L is a multiplicative lattice, the multiplication distributes over an infinite join, we have $c \cdot (\bigvee I) = 0$, a contradiction.

With this preparation, we are ready to prove a crucial lemma which will help disprove the Conjecture 3.1.1.

Lemma 3.3.3. *Let L be a compactly generated, m-SSC multiplicative lattice and $\mathbb{AG}(L)$ be its annihilating-ideal graph (treated L as a multiplicative semigroup). Then the (semigroup theoretic) zero-divisor graph $G(L)$ and the compressed annihilating-ideal graph of L (treated as a multiplicative semigroup) are isomorphic. That is, $G(L) \cong C(\mathbb{AG}(L))$.*

Proof. Let $\Phi : C(\mathbb{AG}(L)) \longrightarrow G(L)$ be a map such that

$$\Phi([I]) = \bigvee I.$$

First, we prove that Φ is well-defined. Let $[I] = [J]$. Then by Lemma 3.3.2, $\bigvee I = \bigvee J$. Thus Φ is well-defined.

Now, we prove that Φ is one-to-one. For this, let $\bigvee I = \bigvee J$, where $I, J \in \mathbb{AG}(L)$. We claim that $[I] = [J]$, i.e., $ann(I) = ann(J)$. Let $I = \bigcup(x_i)$ and $J = \bigcup(y_j)$ be two ideals of L (treated as a multiplicative semigroup). Assume that $t \in ann(I) = ann(\bigcup(x_i))$. Then $t \cdot x_i = 0$ for every i. Since L is a multiplicative lattice, we have $t \cdot (\bigvee x_i) = t \cdot (\bigvee I) = t \cdot (\bigvee J) = 0$. Hence $t \cdot y_j = 0$ for every j. This proves that $ann(I) \subseteq ann(J)$. On similar lines, we can prove the reverse inclusion. Thus Φ is one-one.

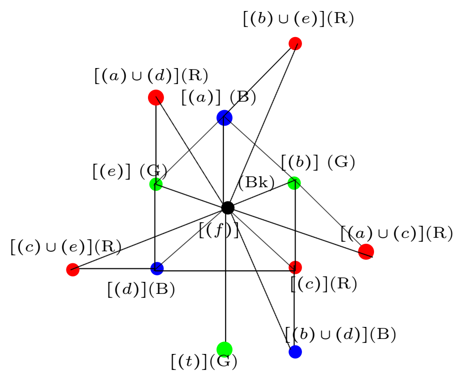
To prove that Φ is onto, assume that $x \in G(L)$. Since L is compactly generated, we have $x = \bigvee c_i$, where c_i's are compact elements of L. We put $J = \bigcup(c_i)$. Then $\Phi(J) = \bigvee J = \bigvee c_i = x$. Thus Φ is onto.

Lastly, we prove that Φ is a graph isomorphism. Let $[I]$ and $[J]$ be two adjacent vertices of $C(\mathbb{AG}(L))$. Then $[I] \cdot [J] = [(0)]$. This further gives $I \cdot J = (0)$. If $I = \bigcup(x_i)$ and $J = \bigcup(y_j)$, we have $x_i \cdot y_j = 0$, $\forall i$, $\forall j$.

For fixed i, we have $x_i \cdot (\bigvee y_j) = x_i \cdot (\bigvee J) = 0$, as the multiplication distributes over infinite join. Thus we have $x_i \cdot (\bigvee J) = 0$ for every i. Again using the fact that the multiplication distributes over infinite join, we have $(\bigvee I) \cdot (\bigvee J) = 0$. This proves that $\bigvee I$ and $\bigvee J$ are adjacent in $G(L)$.

Conversely, assume that $[I]$ and $[J]$ are not adjacent, where $I = \bigcup(x_i)$ and $J = \bigcup(y_j)$. If possible, $(\bigvee I) \cdot (\bigvee J) = 0$, then $x_i \cdot y_j = 0$, $\forall i$, $\forall j$. Therefore, we have $I \cdot J = (0)$ which will yield $[I] \cdot [J] = [(0)]$, a contradiction. Thus Φ is a graph isomorphism, i.e., $C(\mathbb{AG}(L)) \cong G(L)$.

Therefore, from Lemma 3.3.3 and the fact that for a lattice L given in Table 3.1, $3 = \omega(G(L)) < \chi(G(L)) = 4$ (see Figure 3.2 (b)) we have $C(\mathbb{AG}(L))$ (see Figure 3.3) which is not weakly perfect.



$C(\mathbb{AG}(L)$

(R)-Red, (B)-Blue, (G)-Green, (Bk)-Black

Figure 3.3: $C(\mathbb{AG}(L))$

$[(a)]$	$[(b)\cup(e)] = \{\ (b)\cup(e), (b\vee e)\ , (b)\cup(b\vee e)\ , (b)\cup(e)\cup(b\vee e)\ , (e)\cup(b\vee e)\ \}$
$[(b)]$	$[(a)\cup(c)] = \{\ (a)\cup(c), (a\vee c)\ , (a)\cup(a\vee c)\ , (a)\cup(c)\cup(a\vee c)\ , (c)\cup(a\vee c)\ \}$
$[(c)]$	$[(b)\cup(d)] = \{\ (b)\cup(d), (b\vee d)\ , (b)\cup(b\vee d)\ , (b)\cup(d)\cup(b\vee d)\ , (d)\cup(b\vee d)\ \}$
$[(d)]$	$[(c)\cup(e)] = \{\ (c)\cup(e), (c\vee e)\ , (c)\cup(c\vee e)\ , (c)\cup(e)\cup(c\vee e)\ , (e)\cup(c\vee e)\ \}$
$[(e)]$	$[(a)\cup(d)] = \{\ (a)\cup(d), (a\vee d)\ , (a)\cup(a\vee d)\ , (a)\cup(d)\cup(a\vee d)\ , (d)\cup(a\vee d)\ \}$
$[(t)]$	$[(f)]$

Table 3.3: Equivalence classes of $\mathbb{A}^*(L)$

Remark 3.3.1. *The duplication of a vertex does not enlarge any clique. Also, it is easy to extend proper coloring of G to a proper coloring of $G\circ x$ by giving the color of x to x'. Hence, we get the following result.*

The following lemma is well-known in the literature.

Lemma 3.3.4. *Let G' be a graph obtained by duplicating a vertex v of a graph G, i.e., $G' = G\circ v$. Then $\chi(G') = \chi(G)$ and $\omega(G') = \omega(G)$.*

Now annihilating-ideal graph $\mathbb{AG}(L)$ can be obtained by successive duplication of vertices of $C(\mathbb{AG}(L))$. Observe from Table 3.3 that $\mathbb{AG}(L)$ will have 4 duplicate vertices (ideals from equivalence class) for each vertex of type $[(x)\cup(y)]$ in $C(\mathbb{AG}(L))$. Hence Lemma 3.3.4 proves that $\mathbb{AG}(L)$ is not weakly perfect which is a counter-example for Conjecture 3.1.1.

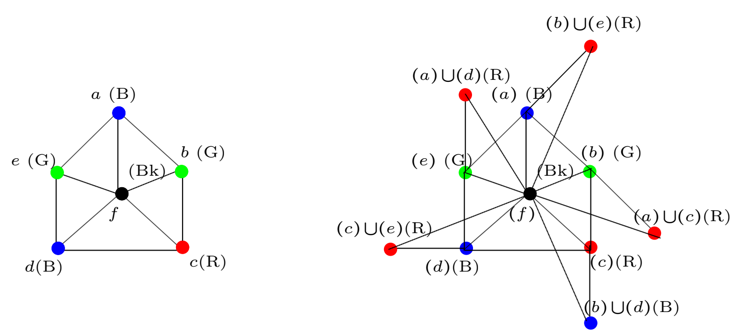
We close the thesis by providing a smaller counter-example to Conjecture 3.1.1.

Example 7. *Let $S = \{0, a, b, c, d, e, f, 1\}$ with $fx = 0$ $\forall x \in S$ and $x^2 = f$ $\forall x$ (except $f, 1$) in S. Also $ab = bc = cd = de = ae = 0$ and $ac = ad = bd = be = ce = f$. Note that, since $f^2 = 0$, S is a non-reduced semigroup. It is interesting to note that $G(S)$ as well as $\mathbb{AG}(S)$ both are not weakly perfect. In fact, $4 = \chi(\mathbb{AG}(S)) = \chi(G(S)) \neq \omega(G(S))\omega(\mathbb{AG}(S)) = 3$.*

In Table 3.4 given below the nonzero annihilating ideals of S are given.

(f)	(a)	(b)	(c)	(d)	(e)
	$(b)\cup(e)$	$(a)\cup(c)$	$(b)\cup(d)$	$(c)\cup(e)$	$(a)\cup(d)$

Table 3.4: Nonzero annihilating ideals of S



(R)-Red, (B)-Blue, (G)-Green, (Bk)-Black

$G(S)$ (a) $\qquad$ $\mathbb{AG}(S)$ (b)

Figure 3.4: (a) Zero-divisor graph of S (b) Annihilating-ideal graph of S

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