2D CONFORMAL FIELD THEORY AND LIOUVILLE THEORY

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this dissertation entitled '2D Conformal Field Theory and Liouville Theory' towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents study/work carried out by Lakshya Agarwal at Indian Institute of Science Education and Research, Pune, under the supervision of Sunil Mukhi, Professor, department of Physics, during the academic year 2017-2018

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Declaration

I hereby declare that the matter embodied in the report entitled '2D Conformal field theory and Liouville Theory' are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Prof. Sunil Mukhi and the same has not been submitted elsewhere for any other degree.

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Acknowledgements

I would like to thank my supervisor, Prof. Sunil Mukhi, for his guidance and for the numerous insights and valuable inputs he provided. I would also like to thank Dr. Sachin Jain for accepting to be on my Thesis Advisory Committee.

I thank the Indian Institute of Science Education and Research, Pune, for providing its students with all the opportunities and a conducive research environment. I thank Rugved Pund for participating in intellectually stimulating discussions over the last five years which have helped to build the groundwork for a scientific aptitude.

I would like to thank my family for the truly endless love and support they have provided.

I have been continuously supported, including the current thesis work, with INSPIRE grant from Department of Science and Technology, Government of India. Finally, I thank the citizens of India for their continued support for Basic Science.

Abstract

Conformal field theory is a formalism encountered in many branches of physics, such as String Theory and condensed matter physics. Given the wide range of its applicability it has become a subject of extensive study and research. In such field theoretic descriptions we are usually interested in computing observables called correlators. Two dimensional CFTs are important not only because they are simplified by the presence of an infinite dimensional symmetry algebra, thereby making it easier to compute correlators, but also because they play a very important role in the Polyakov String action.

Liouville theory emerges when one couples a conformally invariant field to a two dimensional quantized gravitational background. The gravity sector of Liouville theory matches that of non-critical string theory, hence assigning it more importance.

In this project we first try to understand the conceptual and computational aspects of two dimensional conformal field theories. Thereafter, the discussion will move onto Liouville theory, which is an example of an irrational conformal field theory. This will include the study of how Liouville theory emerges from two-dimensional quantum gravity plus a conformal field theory, and studying the DOZZ proposal: The conjectural formula for the three point structure constant in Liouville theory.

Contents

Chapter 1

Conformal invariance and Correlation functions

Conformal invariance is a powerful tool that gives us a handle on seemingly complicated formalisms. The quantities of interest in a conformal field theory are the spectrum of the theory, the central charge and the structure constant of the three-point function. Given these data, one has essentially solved the theory and can determine any correlation function, which form the observables of the theory. This chapter will focus on detailing some of the most important properties of a conformal field theory in general d dimensions.

1.1 Conformal invariance in d dimensions

A coordinate transformation in d dimensions $x \to x^{'}$ is considered to be a conformal transformation if it scales the metric by a scalar function [6]:

$$g^{'}_{\mu\nu}(x^{'}) = \Lambda(x) g_{\mu\nu}(x) \tag{1.1}$$

The set of conformal transformations form a group structure which contains the Poincare group as a subgroup. In a d dimensional spacetime, the conformal group is $SO(d+1,1)$ if the spacetime is Euclidean and $SO(d,2)$ if it is Minkowski. The conformal group has the following generators in a d dimensional Euclidean spacetime :

Translation	$P_\mu = -i\partial_\mu$
Dilation	$D = -ix^\mu \partial_\mu$
Roatation	$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
Special Conformal Transformation	$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

A theory is said to be conformally invariant if its action is invariant under conformal transformations. A spinless, primary conformal field transforms as :

$$\phi^{'}(x^{'}) = \left|\frac{\partial x^{'}}{\partial x}\right|^{-\frac{\Delta}{d}} \phi(x) \tag{1.2}$$

[Where Δ is the eigenvalue of the Dilation operator]

In scale invariant descriptions such as a system undergoing a second-order phase transition, we can check for conformal invariance by confirming the tracelessness of the energy-momentum tensor. For a conformal transformation $x^\mu \to x^\mu + \epsilon^\mu$, the action transforms as :

$$\delta S = \frac{1}{d}\int d^d x T^\mu_\mu \partial_\rho \epsilon^\rho \tag{1.3}$$

This makes it apparent the tracelessness of $T_{\mu\nu}$ implies conformal invariance, however the vice-versa is not true because $\partial_\rho \epsilon^\rho$ is not an arbitrary function. The Energy momentum tensor can always be made symmetric by adding the Belinfante term.

1.2 Correlation functions

Correlation functions are of utmost importance in field theory. An n-point correlator is defined as :

$$\langle \phi_1(x_1)\phi_2(x_2)...\phi_n(x_n)\rangle = \frac{1}{Z}\int [d\phi]\phi_1(x_1)\phi_2(x_2)...\phi_n(x_n)\exp\{-S[\phi]\} \tag{1.4}$$

where Z is the partition function given by : $Z = \int [d\phi]\exp\{-S[\phi]\}$

Under a conformal transformation, primary fields transform in a coordinate dependant manner and this dependence can be extracted from the integral because the measure $[d\phi]$ is coordinate invariant. This technique gives us the general rule for how correlation functions transform under a conformal transformation [Here we consider the two-point function of a spinless primary field for simplicity] :

$$\langle \phi_1^{'}(x_1^{'})\phi_2^{'}(x_2^{'})\rangle = \left|\frac{\partial x^{'}}{\partial x}\right|_{x=x_1}^{-\frac{\Delta_1}{d}} \left|\frac{\partial x^{'}}{\partial x}\right|_{x=x_2}^{-\frac{\Delta_2}{d}} \langle \phi_1(x_1)\phi_2(x_2)\rangle \tag{1.5}$$

Using the conformal covariance of the correlator, one can completely fix the functional form of the two and three point correlation functions as follows:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}}\left(\delta_{\Delta_1,\Delta_2}\right) \tag{1.6}$$

This signifies that the two-point correlator is 0 if $\Delta_1 \neq \Delta_2$

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_1+\Delta_3-\Delta_2}} \tag{1.7}$$

$$\Big[\text{ Where } x_{ij} = |x_i - x_j| \Big]$$

The four and higher point functions cannot be completely fixed by conformal invariance. However one can express the higher point functions in terms of conformally invariant cross ratios. For eg: In the case of four point functions the cross ratios are $\frac{x_{12}x_{34}}{x_{13}x_{24}}$ and $\frac{x_{12}x_{34}}{x_{23}x_{14}}$. For a general N-point function there are $\frac{N(N-3)}{2}$ independent cross-ratios.

Chapter 2

CFT in 2 dimensions

2 dimensional conformal field theory has emerged as the most solvable description of a CFT after the publication of the seminal work done by BPZ [1]. In this chapter we will outline the properties which make 2-D CFTs so favourably transparent.

2.1 Conformal transformation in 2 dimensions

In flat space, i.e. $ds^2 = dzd\bar{z}$, CFT in 2 dimensions is described by two independent coordiantes z (holomorphic) and $\bar{z}$ (anti-holomorphic) where conformal transformations are of the form $(z, \bar{z}) \to (w(z), \bar{w}(\bar{z}))$. This results in an infinite dimensional conformal group in the space of each coordinate (let's call them Γ and $\bar{\Gamma}$), and the overall conformal group is given by $\Gamma \otimes \bar{\Gamma}$.

The generators of the group have commutation relations given by: $[l_n, l_m] = (n-m)l_{n+m}$, which defines the Witt algebra, where the l_{-1}, l_0 and l_1 generators along with their anti-holomorphic counterparts generate the global subgroup SL(2,C) of conformal transformations. The global transformations are of the form:

$$z \to w = \frac{az+b}{cz+d} \text{ given } ad - bc = 1 \tag{2.1}$$

The basic objects in conformal field theory are correlators, and as covered in the last chapter, they are defined as the average expectation value of operators in a given theory. The Operator Algebra is a generalised version of the operator product expansion in conformal field theory and is described by the equation :

$$\phi_n(z,\bar{z})\phi_m(0,0) = \sum_{p,k,\bar{k}} C_{nm}^{p,k,\bar{k}} \left(z^{h_p-h_n-h_m+\sum_i k_i}\right)\left(\bar{z}^{\bar{h}_p-\bar{h}_n-\bar{h}_m+\sum_i \bar{k}_i}\right)\phi_p^{k,\bar{k}}(0,0) \tag{2.2}$$

We get this equation by requiring that both sides of it transform in the same way under a transformation. The operator algebra is equivalent to the statement that once we know the central charge of a theory along with the conformal dimensions of the primaries involved, and all the three-point structure constants, we have essentially solved the theory since all correlators and the spectrum of the theory can be computed directly from the Operator Algebra.

Using the fact that the correlators have coordinate co-variance, one can write down equations known as Ward identities :

$$\langle \delta X \rangle = \langle X \delta S \rangle \tag{2.3}$$

Writing the change in action in terms of the current generated by conformal transformations, one can find the ward identity in terms of the current and the generator of the transformation :

$$\frac{\partial}{\partial x^\mu}\langle j^\mu(x)\phi(x_1)\phi(x_2)...\phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\langle\phi(x_1)...G\phi(x_i)...\phi(x_n)\rangle \tag{2.4}$$

Here j is the current and G is the generator of the transformations. Integrating this equation in a thin pill-box results in a very fundamental equation in all of quantum field theory :

$$[Q,\phi] = -iG\phi \qquad Q = \int dx j^0(x) \tag{2.5}$$

This is equivalent to the quantum version of Noether's theorem and it states that the conserved charge of the current is the generator of transformations in the operator formalism.

In a theory we will have a set of fields called primary fields. They have the unique property of transforming as : $\delta_\epsilon \phi = -\epsilon\partial\phi - h\phi\partial\epsilon$ under any local conformal transformation (h is the conformal dimension of ϕ). In the Hilbert space we can construct asymptotic states corresponding to these primaries which are eigenfuctions of the Hamiltonian : $\hat{H}\,|\phi\rangle = (h+\bar{h})\,|\phi\rangle$.

But they are not the only eigenfunctions of the Hamiltonian in the theory, we can construct a whole "tower" of states by acting operators L_{-n}s and $\bar{L}_{-n}$s, which are the modes of the EM Tensor, on the state to increase it's eigenvalues in the left and right direction respectively:

$$\hat{H}\big(\bar{L}_{-k_1}...\bar{L}_{-k_n}L_{-k_1}...L_{-k_m}\left|\phi\right\rangle\big) = \big[\big(\bar{h}+\bar{k}_1+...+\bar{k}_n\big)+\big(h+k_1+...+k_m\big)\big]\left|\phi\right\rangle \tag{2.6}$$

These states are known as descendants of the primary. Corresponding to each eigenvalue $h+N$ are independent states equal to the number $p(N)$ of partitions of the integer N.

2.2 The Energy Momentum tensor

Using the conformal ward identity :

$$\delta_\epsilon\langle X\rangle = -\frac{1}{2\pi i}\int dz\,\epsilon(z)\,\langle T(z)X\rangle \tag{2.7}$$

one can show that the energy momentum tensor acts as a generator of conformal transformations for conformal fields :

$$\delta_\epsilon\phi(w) = -[Q_\epsilon,\phi(w)] \quad \text{where} \quad Q_\epsilon = \frac{1}{2\pi i}\int dz\,\epsilon(z)\,T(z) \tag{2.8}$$

If we expand the Energy momentum tensor into its constituent modes :

$$T(z) = \sum_{n\in Z} z^{-n-2}L_n \tag{2.9}$$

The operators L_n and $\bar{L}_n$ are the generators of local conformal transformations on the Hilbert space. The Hamiltonian of the system is given by $L_0+\bar{L}_0$. These operators form the the Virasoro algebra which is the central extension of the Witt Algebra :

$$[L_n,L_m] = L_{n+m}+\frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{2.10}$$

The conformal ward identity yet again proves to be useful when showing that the components of the energy momentum tensor don't transform as a true tensor because it is a quasi-primary field (i.e. it behaves as a primary only under global conformal transformations):

$$T(z) = \left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^{-2}\left[T(w)-\frac{c}{12}(z;w)\right] \tag{2.11}$$

(z;w) is known as the Schwarzian derivative] and is defined by

$$(z;w) = \frac{d^3z/dw^3}{dz/dw} - \frac{3}{2}\left(\frac{d^2z/dw^2}{dz/dw}\right)^2 \tag{2.12}$$

Using the ward identity, one can show that the OPE of the Energy Momentum Tensor with a primary field takes the form :

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{(z-w)}\partial_w\phi(w,\bar{w}) \tag{2.13}$$

And using the way tensors transform under a local transformation, it can be shown that the OPE of T with itself is given by the general structure :

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{2.14}$$

Where c is the central charge of the theory

2.3 Correlation functions

Using the covariance of correlation functions, it can be shown that the general two and three point correlators in 2 dimensional CFTs take the form :

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}}$$

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle = \frac{C_{123}}{(z_{12}^{h_1+h_2-h_3}z_{23}^{h_3+h_2-h_1}z_{13}^{h_1+h_3-h_2})(\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\bar{z}_{23}^{\bar{h}_3+\bar{h}_2-\bar{h}_1}\bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_1})}$$

Here also, the two-point function is zero if the fields do not have equal conformal dimensions. An easy way to compute correlation functions is to use the equations of motion of correlators, which is derived by taking the variation of Ward identities with respect to a conformal field :

$$\left\langle\frac{\delta X}{\delta\phi(y)}\right\rangle = \left\langle X\frac{\delta S}{\delta\phi(y)}\right\rangle \tag{2.15}$$

For example, in case of the Free Boson action :

$$S = \frac{1}{8\pi} \int d^2x \, \partial^\mu \phi \partial_\mu \phi \tag{2.16}$$

Putting $X = \phi(x)$ in the equation of motion of correlation functions, we get the result :

$$\langle \phi(x)\phi(y) \rangle = -ln(x-y)^2 \tag{2.17}$$

2.4 Vertex Operators

Putting the free boson on the cylinder with coordinates x, t and the property $\phi(x,t) = \phi(x+L,t)$, allows us to fourier expand ϕ as follows :

$$\phi(x,t) = \sum_n e^{2\pi i n x/L} \, \phi_n(t) \tag{2.18}$$

The Hamiltonian in this picture is given by :

$$H = \frac{2\pi}{L} \sum_n \{\pi_n \pi_{-n} + (2\pi n)^2 \phi_n \phi_{-n}\} \tag{2.19}$$

Where π_ns are the conjugate momenta to the ϕ_ns. The mode expansion of the field $\phi(x)$ at t=0 is :

$$\phi(x) = \phi_0 + i \sum_{n \neq 0} \frac{1}{n}(a_n - \bar{a}_{-n}) e^{2\pi i n x/L} \tag{2.20}$$

ϕ_0 is the zero mode of the field. Using the Hamiltonian, we can get the time evolution of the operator in the Heisenberg picture. We can then move to the coordinates $z = e^{2\pi(\tau - ix)/L}$ and $\bar{z} = e^{2\pi(\tau+ix)/L}$ (where $\tau = it$), to get the general dependence of the operator in terms of the modes :

$$\phi(z, \bar{z}) = \phi_0 - i\pi_0 ln(z\bar{z}) + i \sum_{n \neq 0} \frac{1}{n}(a_n z^{-n} + \bar{a}_n \bar{z}^{-n}) \tag{2.21}$$

The fact that we can extract the z and $\bar{z}$ dependence of ϕ separately is due to its periodic nature.

Vertex Operators are defined by $\mathcal{V}_\alpha(z,\bar{z}) =: e^{i\alpha\phi(z,\bar{z})} :$. The normal ordering here means that :

$$\mathcal{V}_\alpha(z,\bar{z}) = \exp\left\{ i\alpha\phi_0 + \alpha \sum_{n>0} \frac{1}{n}(a_{-n}z^n + \bar{a}_{-n}\bar{z}^n) \right\} \exp\left\{ \alpha\pi_0 - \alpha \sum_{n>0} \frac{1}{n}(a_n z^{-n} + \bar{a}_n \bar{z}^{-n}) \right\} \tag{2.22}$$

Given this expression, one can find the OPE of $T(z)$ with $\mathcal{V}_\alpha(z,\bar{z})$ to get :

$$T(z)\mathcal{V}_\alpha(w,\bar{w}) \sim \frac{\alpha^2}{2}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{\partial_w \mathcal{V}_\alpha(w,\bar{w})}{z-w} \tag{2.23}$$

Thus we know that $h_\alpha = \frac{\alpha^2}{2}$. The OPE of operators $\mathcal{V}_{\alpha_1}(z_1,\bar{z}_1)...\mathcal{V}_{\alpha_i}(z_i,\bar{z}_i)...\mathcal{V}_{\alpha_n}(z_n,\bar{z}_n)$ vanishes unless $\sum_i \alpha_i = 0$. This is known as the Neutrality Condition and can be obtained by imposing translation invariance on the correlator.

Chapter 3

Rational Conformal Field Theory

RCFTs contain a finite number of primary fields. Although the techniques detailed in this chapter are for rational conformal fields, in a few cases they can be applied to even irrational conformal field theories. An example of this is Liouville theory which we will be detailing in chapter 5.

3.1 Minimal Models

In certain theories, the representation of the Virasoro Algebra (The Verma module) is not irreducible as they contain Null Vectors with the following properties: $L_n |\chi\rangle = 0$ for $n > 0$ and $L_0 |\chi\rangle = (h + K) |\chi\rangle$. As one can immediately notice, these are properties of the primary state, however, the Null vector is a secondary of another primary in the theory.

One can check that since a null vector is a secondary corresponding to another primary, the $\langle\chi|$ will be acted on by $n > 0$ modes of the operators L_n and if we take the inner product with some state $|\phi\rangle$ on the right that is annihilated by positive modes (Like Primaries and Null Vectors), the inner product will vanish. But if $\langle\psi|\chi\rangle = 0$ and $\langle\chi|\chi\rangle = 0$, one can self-consistently set the null field to zero. In such a case the conformal family contains "less" fields than usual and it is known as a degenerate conformal family and we call the field ψ a degenerate primary field. The appearance of such null vectors happens at special values of h and all such values have been listed by Kac [1] through the formula:

$$h_{(n,m)} = h_0 + \left(\frac{\alpha_+ n + \alpha_- m}{2}\right)^2 : \text{ where } h_0 = \frac{1}{24}\big(c-1\big) \text{ and } \alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \tag{3.1}$$

If $h = h_{(n,m)}$, the corresponding null vector has the dimension $h_{(n,m)} + nm$. The differential equations satisfied by these degenerate fields impose hard constraints on the operator algebra and give rise to the **Fusion Rules**:

$$\chi_{(n,m)}\phi_\alpha = \sum_{l=1-m}^{1+m}\sum_{k=1-n}^{1+n}\left[\phi_{(\alpha+l\alpha_-+k\alpha_+)}\right] \text{ where } \left(\alpha = \alpha_+ n' + \alpha_- m'\right) \tag{3.2}$$

These differential equations are generated by expressing correlators involving secondary fields in terms of correlators involving their corresponding primaries. Since null vectors are secondary with respect to some primary field, we can obtain a differential equation for the primary field. As an example, we will look at the $\psi_{1,2}$ field of the BPZ paper which has the null vector :

$$|\chi\rangle = \left[L_{-2} + \frac{3}{2(2\Delta+1)}L_{-1}^2\right]|\Delta\rangle \tag{3.3}$$

The formula to compute correlators of secondaries is given by :

$$\langle\phi_n^{(-k_1\ldots-k_m)}(z)\phi_1(z_1)\ldots\phi_N(z_N)\rangle = \hat{\mathcal{L}}_{-k_m}(z,z_i)\ldots\hat{\mathcal{L}}_{-k_1}(z,z_i)\langle\phi_n(z)\phi_1(z_1)\ldots\phi_N(z_N)\rangle \tag{3.4}$$

where

$$\hat{\mathcal{L}}_{-k}(z,z_i) = \sum_{i=1}^{N}\left[\frac{(1-k)\Delta_i}{(z-z_k)^k} - \frac{1}{(z-z_i)^{k-1}}\frac{\partial}{\partial z_i}\right] \tag{3.5}$$

Using this equation, we can find the correlator for the state $|\chi\rangle$ but since $|\chi\rangle = 0$, this correlator is zero too which gives us the differential equation :

$$\left\{\frac{3}{2(2\Delta+1)}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{N}\frac{\Delta_i}{(z-z_i)^2} - \sum_{i=1}^{N}\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right\}\langle\psi_{1,2}(z)\phi_1(z_1)\ldots\phi_n(z_n)\rangle = 0 \tag{3.6}$$

Let the OPE of ϕ_Δ with $\psi_{1,2}$ be :

$$\psi_{1,2}(z)\phi_\Delta(z_1) = (z-z_1)^\kappa[\phi_{\Delta'}(z_1) + \ldots] \quad \text{where} \quad \kappa = \Delta' - \Delta - \delta \tag{3.7}$$

δ is the dimension of the field $\psi_{1,2}$. Substituting this equation into the differential equation for the correlator of $\psi_{1,2}$ and comparing the most singular terms in $(z - z_1)$ gives us the allowed values of Δ' :

$$\frac{3\kappa(\kappa-1)}{2(2\delta+1)} - \Delta + \kappa = 0 \tag{3.8}$$

Once we know about degenerate conformal families, it is natural to ask the question: How many such null vectors are present in a given conformal family? Let's examine the case where $0 < c \leq 1$ because it is forbidden for a quantum field theory to have degenerate fields beyond these values of the central charge. It can be seen from the Kac formula that if $\alpha_+ p + \alpha_- q = 0$, where p and q are integers, then there will be infinitely many null vectors in the conformal family. For eg: if $h = h_{(n,m)}$ then $h = h_{(n+p,m+q)}$ but the null vectors corresponding to both lie at different levels. Therefore, the correlation functions in minimal theories satisfy infinitely many differential equations and as a result the operator algebra is truncated from both above and below. This means that the operator algebra is closed for the conformal families $\left[\psi_{(n,m)}\right]$ with $0 < n < p$ and $0 < m < q$.

To each Verma module $V(c,h)$ associated with a highest-weight state $|h\rangle$, we associate a generating function $\chi_{(c,h)}(\tau)$ defined by:

$$\chi_{(c,h)}(\tau) = \sum_{n=0}^{\infty} dim(h+n) q^{n+h-\frac{c}{24}} \ \left(\text{where } q = e^{2\pi i\tau}\right) \tag{3.9}$$

Since the Verma module associated with minimally degenerate highest weight vectors is infinitely reducible, the formula has to account for all the null states that will be removed from the module. For any theory described by integers $(p,p^{'})$ where $c = 1 - 6\frac{(p-p^{'})^2}{pp^{'}}$, the degenerate primary will have conformal dimension $h_{r,s} = \frac{(pr-p^{'}s)^2-(p-p^{'})^2}{4pp^{'}}$ [This is done to take care of unitarity]. The irreducible character will be equal to :

$$\chi_{(r,s)}(q) = K_{r,s}^{(p,p^{'})}(q) \ - \ K_{r,-s}^{(p,p^{'})}(q) \ \left(\text{where } K_{r,s}^{(p,p^{'})}(q) = \frac{q^{\frac{-1}{24}}}{\phi(q)} \sum_{n \in Z} q^{\frac{(2pp^{'}n+pr-p^{'}s)^2}{4pp^{'}}}\right) \tag{3.10}$$

Below is a table that lists characters upto order q^6 for certain well-known minimal theories:

Characters of specific minimal models		
(p,p')	$h_{r,s}$	$q^{-h_{r,s}+c/24}\chi_{r,s}(q)$
(5,2) Yang-Lee	$h_{1,1}=0$	$1+q^2+q^3+q^4+q^5+2q^6....$
	$h_{1,2}=-2/5$	$1+q+q^2+q^3+2q^4+2q^5+3q^6....$
(4,3) Ising	$h_{1,1}=0$	$1+q^2+q^3+2q^4+2q^5+3q^6...$
	$h_{2,1}=1/16$	$1+q+q^2+q^3+2q^4+2q^5+3q^6...$
	$h_{1,2}=1/2$	$1+q+q^2+2q^3+2q^4+3q^5+4q^6...$
(5,4) Tri-crit. Ising	$h_{1,1}=0$	$1+q^2+q^3+2q^4+2q^5+4q^6...$
	$h_{2,1}=7/16$	$1+q+q^2+2q^3+3q^4+4q^5+6q^6...$
	$h_{1,2}=1/10$	$1+q+q^2+2q^3+3q^4+4q^5+6q^6...$
	$h_{1,3}=3/5$	$1+q+2q^2+2q^3+4q^4+5q^5+7q^6...$
	$h_{2,2}=3/80$	$1+q+2q^2+3q^3+4q^4+6q^5+8q^6...$
	$h_{3,1}=3/2$	$1+q+q^2+2q^3+3q^4+4q^5+6q^6...$
(6,5) 3-state Potts	$h_{1,1}=0$	$1+q^2+q^3+2q^4+2q^5+4q^6...$
	$h_{2,1}=2/5$	$1+q+q^2+2q^3+3q^4+4q^5+6q^6...$
	$h_{3,1}=7/5$	$1+q+2q^2+2q^3+4q^4+5q^5+8q^6...$
	$h_{1,3}=2/3$	$1+q+2q^2+2q^3+4q^4+5q^5+8q^6...$
	$h_{4,1}=3$	$1+q+2q^2+3q^3+4q^4+5q^5+8q^6...$
	$h_{2,3}=1/15$	$1+q+2q^2+3q^3+5q^4+7q^5+10q^6...$

3.2 Conformal Bootstrap

The conformal bootstrap is an associativity based consistency check for correlators in Conformal Field Theory. Let's consider the four point function $\langle\phi_k(x_1)\phi_l(x_2)\phi_m(x_3)\phi_n(x_4)\rangle$. Due to conformal invariance we can make the correlator depend on conformal cross-ratios $x=\frac{z_{12}z_{34}}{z_{13}z_{24}}$ and $\bar{x}=\frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}$. Sending $z_1=\bar{z}_1=\infty; z_2=\bar{z}_2=1; z_3=x; \bar{z}_3=\bar{x}; z_4=\bar{z}_4=0$, we can define the functions :

$$G^{lk}_{nm}(x,\bar{x})=\langle k|\,\phi_l(1,1)\phi_n(x,\bar{x})\,|m\rangle \tag{3.11}$$

The crossing symmetry condition then reads :

$$G^{lk}_{nm}(x,\bar{x})=G^{mk}_{nl}(1-x,1-\bar{x})=x^{-2\Delta_n}\bar{x}^{-2\bar{\Delta}_n}G^{lm}_{nk}(1/x,1/\bar{x}) \tag{3.12}$$

These functions can be written in terms of conformal blocks :

$$G^{lk}_{nm}(x,\bar{x})=\sum_p C^p_{nm}C_{klp}\mathcal{F}^{lk}_{nm}(p|x)\bar{\mathcal{F}}^{lk}_{nm}(p|\bar{x}) \tag{3.13}$$

Where each conformal block can be expressed as a power series expansion :

$$\mathcal{F}^{lk}_{nm}(p|x) = x^{\Delta_p - \Delta_n - \Delta_m} \sum_{\{k\}} \beta^{\{k\}}_{nm} x^{\sum k_i} \frac{\langle k| \, \phi_l(1,1) L_{-k_1} ... L_{-k_N} \, |p\rangle}{\langle k| \, \phi_l(1,1) \, |p\rangle} \tag{3.14}$$

Analytically, the crossing symmetry relation is expressed as :

$$\sum_p C^p_{nm} C_{klp} \mathcal{F}^{lk}_{nm}(p|x) \bar{\mathcal{F}}^{lk}_{nm}(p|\bar{x}) = \sum_q C^q_{nl} C_{mkq} \mathcal{F}^{mk}_{nl}(q|1-x) \bar{\mathcal{F}}^{mk}_{nl}(q|1-\bar{x}) \tag{3.15}$$

Diagrammatically, this relation can be expressed as :

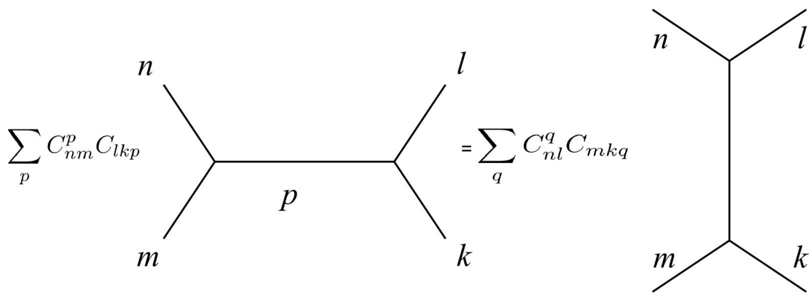

Figure 3.1: Conformal Bootstrap

Chapter 4

2-D Quantum Gravity and CFT in curved spacetime

Conformal field theory in a curved spacetime coupled to quantum gravity in the conformal gauge gives birth to the Liouville field theory, whose classical equation of motion is the generalization of the Liouville differential equation. In this chapter we will study in great detail, how ghost fields arise from quantum gravity in two dimensions and how they generate the critical dimension in Bosonic String Theory.

4.1 CFT in Curved Spacetime

A field theory in curved spacetime is considered conformally invariant if its EM Tensor obeys the conformal anomaly equation $T^{\mu}_{\mu} = -\frac{c}{12}R$, where c is the central charge of the theory and R is the scalar curvature of the metric [14]. A general subtlety here is that usually the conformal anomaly equation is written as $\langle T^{\mu}_{\mu} \rangle = -\frac{c}{12}R$ but here we will assume that we have 'absorbed' the curvature dependence of the measure $[d\phi]$ into the field T^{μ}_{μ}. A general reparametrisation of the co-ordinates $x^{\mu} \to x^{\mu} + \epsilon^{\mu}$ induces the variation of the metric according to : $\delta g_{\mu\nu} = 2\nabla_{(\mu}\epsilon_{\nu)}$. This, combined with the definition of the Energy Momentum tensor :

$$\delta S = -\frac{1}{4\pi} \int \sqrt{g}\, T_{\mu\nu}\, \delta g^{\mu\nu}\, d^2x \tag{4.1}$$

directly implies that the continuity equation in curved spacetime takes the form $\nabla_{\mu} T^{\mu\nu} = 0$.

It is wise to extract the traceless part of the EM Tensor by writing it as

$T_{\mu\nu} = T^0_{\mu\nu} - \frac{c}{24} t_{\mu\nu}$, where $t^\mu_\mu = 2R$ and therefore T^0 is traceless. Using the continuity equation once can then verify that $\partial_{\bar{z}} T^0_{zz} = 0$, and the same goes for its anti-holomorphic counterpart. The tensor T^0 acts as the generator for conformal transformations in curved spacetime and its modes follow the Virasoro commutation relations.

4.2 Weyl Response of the Partition function and the Liouville action

This section will deal with the mathematics of the emergence of the Liouville action. The partition function of a conformal field in a fixed background metric is given by :

$$Z[g] = \int e^{-S[g,\phi]} [D\phi] \tag{4.2}$$

Since ϕ is a conformal field, its EM tensor will follow $T^\mu_\mu = -\frac{c}{12} R$. If we make an infinitesimal Weyl transformation to the metric, i.e. $g(x) \to (1 + \delta\sigma(x))g(x)$, the action changes by :

$$\delta S = \frac{1}{4\pi} \int \sqrt{g}\, \delta\sigma\, T^\mu_\mu \, d^2x \tag{4.3}$$

This variation can be plugged into the partition function to obtain the differential equation governing the Weyl response :

$$\delta \log Z[e^\sigma g] = \frac{c}{48\pi} \int \sqrt{g}\, R(x)\, \delta\sigma(x)\, d^2x \tag{4.4}$$

Integrating with respect to σ, we obtain the famous Liouville action :

$$\frac{Z[e^\sigma g]}{Z[g]} = \exp\left\{ \frac{c}{48\pi} \int \sqrt{g}\, \left[R(x)\, \sigma(x) + \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \right] d^2x \right\} \tag{4.5}$$

This equation shows us how the physics of a conformal field theory changes at different length scales.

4.3 Quantum gravity in the conformal gauge

In 2-D quantum gravity, the Einstein-Hilbert action does not generate any dynamics because it is proportional to the Euler-Characteristic of the space due to the Gauss-Bonnet theorem. Hence all dynamics is generated strictly by the underlying topology, given that there is no cosmological term. The partition function in quantum gravity is given by :

$$Z[g] = \int e^{-S[g,\phi]}[Dg_{\mu\nu}][D\phi] \tag{4.6}$$

This involves integrating over all possible configurations of the metric (Here ϕ is any scalar field). The measure $[Dg_{\mu\nu}]$ poses a problem, however, because metrics related by a coordinate reparametrization represent the same geometry. This can be interpreted as redundancies in the description of gravity and therefore the transformations $x^\mu \to x^\mu + \epsilon^\mu$ are to be regarded as gauge transformations. In the conformal gauge, we can always choose a geometry to be conformally flat within a given open set, and this description is uniquely labelled by the conformal factor e^σ. Therefore, a good gauge slice consists of the conformal class $[\delta_{\mu\nu}]$ of metrics, all conformal to the Euclidean metric [Friedan reference].

This problem of unnecessary infinities arising in the partition function, due to overcounting of descriptions related by gauge transformations, is overcome by introducing the Faddeev-Popov determinant in the following procedural manner: A metric in 2 dimensions has three degrees of freedom and therefore we can split the measure $[Dg_{\mu\nu}]$ in terms of three independent components $[Dg_{\mu\nu}] = [Dg_{zz}][Dg_{\bar{z}z}][Dg_{\bar{z}\bar{z}}]$ [7]. Looking at reparametrization as a vector field $(\epsilon^z, \epsilon^{\bar{z}})$, one can then make the change of variables : $(g_{zz}, g_{\bar{z}z}, g_{\bar{z}\bar{z}}) \to (\epsilon^z, \epsilon^{\bar{z}}, \sigma)$ which are related in the following way :

$$\begin{aligned}
\delta g_{z\bar{z}} &= \nabla_z \epsilon_{\bar{z}} + \nabla_{\bar{z}} \epsilon_z \\
\delta g_{zz} &= 2\nabla_z \epsilon_z \\
\delta g_{z\bar{z}} &= e^\sigma g_{z\bar{z}}
\end{aligned}$$

Since we are making a change of variables, this has to be accompanied by the corresponding Jacobian :

$$[Dg_{\mu\nu}] = [D\epsilon^z][D\epsilon^{\bar{z}}][D\sigma]\frac{\partial(g_{zz}, g_{\bar{z}z}, g_{\bar{z}\bar{z}})}{\partial(\epsilon^z, \epsilon^{\bar{z}}, \sigma)} \tag{4.7}$$

The Jacobian $\frac{\partial(g_{zz}, g_{\bar{z}z}, g_{\bar{z}\bar{z}})}{\partial(\epsilon^z, \epsilon^{\bar{z}}, \sigma)}$ is computed to be equal to $\det(\nabla^z)\det(\nabla^{\bar{z}})$. Usually, we would throw out the measures corresponding to reparametrizations because they would be perpendicular to the gauge slice. However, here they are not strictly perpendicular as their projection along the gauge slice gives us conformal transformations. We will assume that these dependencies will be absorbed into the remaining path integral implicitly [Friedan reference]. It is a standard mathematical trick to rewrite determinants of differential operators in terms of a path integral over fermionic [anti-commuting] fields. Here we will employ the same method and write :

$$\det(\nabla^z)\det(\nabla^{\bar{z}}) = \int [Dc][Db]\exp\left\{-\int \frac{d^2x}{2\pi}\sqrt{g}\,(b_{zz}\nabla^z c^z + b_{\bar{z}\bar{z}}\nabla^{\bar{z}}c^{\bar{z}})\right\} \tag{4.8}$$

4.4 Ghost fields in quantum gravity

The ghost action, as encountered in the the previous section, can be written as :

$$S_{gh} = \int \frac{d^2x}{2\pi}\sqrt{g}\,(b_{\mu\nu}\nabla^\mu c^\nu) \quad \text{[Where b is a symmetric, traceless tensor]} \tag{4.9}$$

Computing the Energy momentum tensor of this theory, we find:

$$T_{\mu\nu} = \frac{2\pi}{\sqrt{g}}\frac{\delta S_{gh}}{\delta g^{\mu\nu}} = \frac{1}{2}\nabla_\rho(b_{\mu\nu}c^\rho) + b_{\rho(\mu}(\nabla_{\nu)}c^\rho) - \frac{1}{2}g_{\mu\nu}b_{\rho\sigma}(\nabla^\sigma c^\rho) \tag{4.10}$$

Immediately, one can check that $g^{\mu\nu}T_{\mu\nu} = 0$ which implies that the ghost field theory is conformally invariant! Hence we can use techniques similar to section 4.2 to extract the Weyl dependence of the theory, if we can compute the central charge of the theory. The standard technique to do this is of course to compute the $T_{zz}T_{zz}$ OPE. To do this, we will put the theory on a flat background and find its equations of motion. The action on a flat background is given by :

$$S_{gh} = \int \frac{d^2x}{2\pi}\,(b\partial c + \bar{b}\bar{\partial}\bar{c}) \tag{4.11}$$

The equations of motion are given by :

$$\bar{\partial} c = 0 \qquad \bar{\partial} b = 0$$
$$\partial \bar{c} = 0 \qquad \partial \bar{b} = 0$$
$$\partial c + \bar{\partial} \bar{c} = 0$$

The OPE of the b and c fields is special in the sense that it is off-diagonal even though they are primary fields :

$$b(z)c(w) \sim \frac{1}{z-w} \qquad b(z)b(w) \sim 0 \qquad c(z)c(z) \sim 0 \tag{4.12}$$

Incorporating the classical equations of motion, the energy momentum tensor becomes :

$$T(z) = :(2(\partial c)b + c(\partial b)): \tag{4.13}$$

Here the order of the operators matters because they are fermionic. The OPE of T with the primary fields and itself is given as follows :

$$T(z)c(w) \sim -\frac{c(w)}{(z-w)^2} + \frac{\partial_w c(w)}{z-w}$$
$$T(z)b(w) \sim 2\frac{b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{z-w}$$
$$T(z)T(w) \sim \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

This result makes it evident that the central charge of ghost fields is **c=-26**

4.5 Bosonic String Theory

In Polyakov's description of String Theory [9], two-dimensional Strings trace out world-sheets that are embedded in the spacetime. This is similar to how a particle traces out its world-line. Polyakov wanted to place surfaces with equal 'areas' on equal footing in terms of the probability with which they occur. The way to do this was to define the Polyakov action :

$$S_P = \frac{1}{2}\int \frac{d^2x}{2\pi}\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu \tag{4.14}$$

Here $X^\mu(x)$ are the coordinates on the world sheet and the free field term measures the area of the world sheet. The full partition function also includes summing over different metrics with the weight of the cosmological constant μ_0 :

$$Z = \int_{metrics} [Dg] e^{-\mu_0 \int d^2x \sqrt{g}} \int_{surfaces} [DX] e^{-S_P} \tag{4.15}$$

The free field term forms a conformal field theory with central charge equal to d (dimension of the spacetime) and the sum over metrics is done exactly as in section 4.3, thereby giving the central charge -26. This generates a Weyl dependence proportional to $d-26$ and hence $d=26$ is the dimension at which the scale dependence of the partition function disappears. This is the critical dimension in Bosonic String Theory, which is required to build a consistent String Theory.

Chapter 5

Liouville Theory

As we have studied in the previous chapter, the critical dimension in Bosonic String theory is $d = 26$. Therefore, an important area to explore is what happens when $d \neq 26$. In Liouville Theory, as we will see, we can describe and study the theory for $d < 1$ [Here d will lose its interpretation as the dimension of spacetime in a String Theory and instead it will be the central charge of some underlying conformal matter field]. In a particular limit of the correlator, Liouville theory becomes easier to solve, and we have a strong contender for the 3-point structure constant conjectured by DOZZ [3,13]. In this chapter we will study the properties of classical and quantum Liouville theory, including the DOZZ proposal.

5.1 Classical Liouville theory

In previous chapter we have shown how to extract the Weyl dependence of the gravity sector and external conformal field. The resulting partition function is :

$$Z = \int_{moduli} d\mu(\hat{g})\, Z_{matter}(\hat{g})\, Z_{ghost}(\hat{g}) \int e^{-S_L[\sigma,\hat{g}]}[D\sigma] \tag{5.1}$$

Here $\hat{g}$ is a fixed background metric and we have extracted the explicit metric dependence in terms of the Liouville action :

$$S_L[\sigma,\hat{g}] = \frac{26-c}{48\pi} \int \sqrt{\hat{g}} \left[\hat{R}\sigma + \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma + \Lambda e^\sigma \right] d^2x \tag{5.2}$$

Here c is the central charge of the underlying conformal field and the Liouville term Λe^σ has been added by hand as a cosmological term. The prefactor proportional to $(26-c)$ acts like a $\frac{1}{\hbar}$ term and therefore $\hbar \to 0$ corresponds to $c \to -\infty$. In this limit, classical configurations of the field dominate : $\sigma(x) \to \sigma_{cl}(x)$.

The Euler Lagrange equation for the field σ is :

$$\hat{R}(x) + \Lambda e^{\sigma} - \Delta_{\hat{g}}\sigma(x) = 0 \tag{5.3}$$

Using the equation $\sqrt{\hat{g}}\big(\hat{R} - \Delta_{\hat{g}}\sigma\big) = \sqrt{g}R$, given $g = e^{\sigma}\hat{g}$, we can write the above equation as :

$$R(x) + \Lambda = 0 \tag{5.4}$$

Hence classical configurations of the field describe geometries where the scalar curvature is negative of the cosmological constant. Since we are free to choose any background metric $\hat{g}$ we like, it is wise to put the theory on a flat background and look at its equation of motion. Using $R = -4e^{-\sigma}\partial_z\partial_{\bar{z}}\sigma$, we get :

$$-4\partial_z\partial_{\bar{z}}\sigma + \Lambda e^{\sigma} = 0 \tag{5.5}$$

This is the famous Liouville equation. The next task will be to compute the Energy momentum tensor of the theory. Using the definition :

$$\delta S = -\frac{1}{4\pi}\int \sqrt{\hat{g}}\,\delta\hat{g}^{\mu\nu}\,t_{\mu\nu}\,d^2x \tag{5.6}$$

we obtain the tensor $t_{\mu\nu}$ as :

$$t_{\mu\nu} = -\partial_\mu\sigma\partial_\nu\sigma + \hat{g}_{\mu\nu}\left(\frac{1}{2}(\partial\sigma)^2 + \Lambda e^{\sigma}\right) + 2\big(\partial_\mu\partial_\nu\sigma - \hat{g}_{\mu\nu}\partial^2\sigma\big) \tag{5.7}$$

This tensor has the trace $t^{\mu}_{\mu} = 2(\Lambda e^{\sigma} - \partial^2\sigma)$, which vanishes because of the classical equations of motion. Moreover, one can check that $\partial_z t_{\bar{z}\bar{z}} = 0$ and $\partial_{\bar{z}} t_{zz} = 0$, therefore they can be written as $t(\bar{z})$ and $t(z)$ respectively. Because of the tracelessness of the EM tensor, one can check that the classical equations of motion are invariant under conformal transformations provided one transforms the field σ in the following way :

$$\sigma(w,\bar{w}) = \sigma(z,\bar{z}) - \log\left(\frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}\right) \tag{5.8}$$

Having checked certain nice properties of the classical Liouville equation, let's concern ourselves with finding the solution to the Liouville equation. We will construct the solution to the classical equation using solutions to a set of different differential equations [14] :

$$-4\partial_z^2\psi(z) = t(z)\psi(z)$$
$$-4\partial_{\bar{z}}^2\bar{\psi}(\bar{z}) = t(\bar{z})\psi(\bar{z})$$

One known solution to these equations is $\psi = e^{-\frac{\sigma}{2}}$. Since these are both second order differential equations, each will have two independent solutions. Let

$$\bar{\psi}(\bar{z}) = [\bar{\psi}_1(\bar{z}), \bar{\psi}_2(\bar{z})] \qquad \text{and} \qquad \psi(z) = \begin{bmatrix}\psi_1(z)\\ \psi_2(z)\end{bmatrix}$$

Where $\psi_1(z)$ and $\psi_2(z)$ are the two independent solutions to the holomorphic differential equation and $\bar{\psi}_1(\bar{z})$ and $\bar{\psi}_2(\bar{z})$ are solutions to its anti-holomorphic counterpart. Since the equations lack a first order derivative term, they have a constant Wronskian and we can choose the basis of solutions to have Wronskian equal to one :

$$W(z) = \psi_1(z)\partial_z\psi_2(z) - \psi_2(z)\partial_z\psi_1(z) = 1 \tag{5.9}$$

Similarly one can choose the anti-holomorphic Wronskian to be equal to one : $\bar{W}(\bar{z}) = 1$. Defining a new matrix $\tilde{\Lambda}$, it is straightforward to verify that the field :

$$\sigma(z, \bar{z}) = -2\log\left(\bar{\psi}(\bar{z})\tilde{\Lambda}\psi(z)\right) + \log 8 \tag{5.10}$$

solves the Liouville equation upto Monodromy and sign issues, if we set $\det(\tilde{\Lambda}) = \Lambda$. Now we will move on to the quantum regime in Liouville theory.

5.2 Quantum Liouville theory

We will start with the Liouville action :

$$S_L[\sigma, \hat{g}] = \frac{26 - c_M}{48\pi}\int\sqrt{\hat{g}}\left[\hat{R}\sigma + \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma + \Lambda e^\sigma\right]d^2x \tag{5.11}$$

c_M is the central charge of the matter field. The measure associated with the Liouville action $[D\sigma]$ poses a subtle problem as it is not linear. This can be seen from how the inner product is defined on the space of σ fields :

$$||\delta\sigma||^2 = \int \sqrt{\hat{g}}\, e^{\sigma} (\delta\sigma)^2 \, d^2x \tag{5.12}$$

Here the exponential term makes the measure non-linear. Following the approach of Distler and Kawai [2], we will make an intelligent guess based on general coordinate invariance and write a renormalised action in which the non-linearity of the measure is absorbed into the action :

$$S_L^{(r)} = \frac{1}{8\pi b^2} \int \sqrt{\hat{g}} \left[\frac{1}{2}(\partial\sigma)^2 + q\hat{R}\sigma + \tilde{\Lambda} e^{\sigma} \right] d^2x \tag{5.13}$$

The parameters b and q have to be fixed and $\tilde{\Lambda}$ is a free parameter. Since classical saddle points are not affected by the measure, we should expect the same classical limits to hold in the renomarlised action :

$$\frac{1}{b^2} \to -\frac{c_M}{6} \quad \text{and} \quad q \to 1 \quad \text{as} \quad c \to -\infty \tag{5.14}$$

In standard literature, the convention is to use the field $\sigma = 2b\phi$, in terms of which the action becomes :

$$S_L^{(r)} = \frac{1}{4\pi} \int \sqrt{\hat{g}} \left[(\partial\phi)^2 + Q\hat{R}\phi + 4\pi\mu e^{2b\phi} \right] d^2x \tag{5.15}$$

After absorbing the non-linearity in the action, the measure $[D\phi]$ now follows : $D[\phi(x) + C(x)] = D[\phi(x)]$. Using the linearity of the measure we will now aim to establish background independence which a key feature in quantum gravity. Substituting $\hat{g}_{\mu\nu} = e^{\tilde{\sigma}} g_{\mu\nu}$, one should expect the $\tilde{\sigma}$ dependence to drop out of the total partition function. We will first address the problem in the absence of the cosmological term, i.e. $\mu = 0$. Given that $\sqrt{\hat{g}}\hat{R} = \sqrt{g}(R - \Delta\tilde{\sigma})$, after integrating by parts we can write :

$$S_L^{(r)} = \frac{1}{4\pi} \int \sqrt{g} \left[(\partial\phi)^2 + QR\phi + Q\partial^{\mu}\tilde{\sigma}\partial_{\mu}\phi \right] d^2x \tag{5.16}$$

After making a shift of the field $\phi = \tilde{\phi} - \frac{Q}{2}\tilde{\sigma}$, the expression can be rewritten as :

$$S_L^{(r)} = \frac{1}{4\pi} \int \sqrt{g} \left[(\partial\tilde{\phi})^2 + QR\tilde{\phi} - \frac{Q^2}{2}\left(R\tilde{\sigma} + \frac{1}{2}(\partial\tilde{\sigma})^2 \right) \right] d^2x \tag{5.17}$$

Here we must keep in mind that the partition function of the theory is given by :

$$Z = \int_{moduli} d\mu(\hat{g})\, Z_{matter}(\hat{g})\, Z_{ghost}(\hat{g}) \int e^{-S_L^{(r)}[\tilde{\phi},\hat{g}]}[D\tilde{\phi}] \tag{5.18}$$

Since the measure is linear, $[D\phi] = [D\tilde{\phi}]$. Also as we can see, the first half of the action above is exactly the term we want to keep after we have factored out the Weyl dependence. Since we are dealing with the full partition function, we have to deal with the Weyl dependence of the Ghost and matter fields partition functions :

$$Z_{matter}[\hat{g}]Z_{ghosts}[\hat{g}] = \exp\left\{\frac{c_M - 26}{48\pi}\int \sqrt{g}\left[R\tilde{\sigma} + \frac{1}{2}(\partial\tilde{\sigma})^2\right]d^2x\right\}Z_{matter}[g]Z_{ghosts}[g] \tag{5.19}$$

From the calculation of the Weyl anomaly, it can be checked that the curvature term does not affect the anomaly. Therefore, the measure $[D\tilde{\phi}]$ contributes a free scalar field term of $\frac{1}{48\pi}$. Collecting all the prefactors, we will have a term like :

$$\frac{25 - c_M - 6Q^2}{48\pi}\int \sqrt{g}\left[R\tilde{\sigma} + \frac{1}{2}(\partial\tilde{\sigma})^2\right]d^2x \tag{5.20}$$

But we want background independence, hence this term has to vanish. This implies that :

$$Q^2 = \frac{25 - c_M}{6} \tag{5.21}$$

Now we will turn our attention to the exponential term : $\mu\int\sqrt{\hat{g}}\, e^{2b\phi}\, d^2x$. This term has 3 contributions coming from the Weyl factor. One is obviously the $\sqrt{\hat{g}}$ term, the second is due to the shifted Liouville field $\tilde{\phi}$ and the third is quantum corrections to the exponential vertex operator, resulting in it transforming like : $\left[e^{2a\phi}\right]_{e^{\sigma}g} = e^{a^2\sigma}\left[e^{2a\phi}\right]_g$. Collecting all these factors, we get the result :

$$e^{(\tilde{\sigma}+b^2\tilde{\sigma}-bQ\tilde{\sigma})}\,\mu\int\sqrt{g}\, e^{2b\tilde{\phi}}\, d^2x \tag{5.22}$$

Using this, we can express Q in terms of b

$$Q = b + \frac{1}{b} \tag{5.23}$$

Here we can find two solutions for b, but we choose the one that results in the correct classical limit. In this form, it is obvious that $Q \geq 2$ and in order to have real and positive b, $c_M < 1$. b is chosen to be real and positive since we want a Weyl factor interpretation for $e^{2b\phi}$ and we want large ϕ to correspond to a large rescaling of the metric. Since c_M is not an integer we have also lost the spacetime interpretation of the theory. The EM tensor of the theory without the cosmological term is given by :

$$T = -(\partial_z \phi)^2 + Q\partial_z^2 \phi$$
$$\bar{T} = -(\partial_{\bar{z}} \phi)^2 + Q\partial_{\bar{z}}^2 \phi$$

The term proportional to Q is known as the improvement term. Finding the OPE of the TT operator gives us the central charge of the theory which comes out to be equal to $1 + 6Q^2$. The total central charge of the theory, i.e. $c_M + c_{ghosts} + c_L$ is equal to 0 [One should note that this is essentially the same equation which gives us Q^2 is terms of c_M], which tells us we have successfully built a quantum theory of gravity. Since the Weyl response of any partition function is proportional to the central charge, the partition function in quantum gravity has no Weyl response. And this is quite a sane expectation, because integrating over metrics should eliminate all metric dependence.

5.3 Physical operators

The Liouville field ϕ transforms as :

$$\phi(w, \bar{w}) = \phi(z, \bar{z}) - \frac{Q}{2} \log \left| \frac{dw}{dz} \right|^2 \tag{5.24}$$

This suggests that the Liouville field $e^{2\alpha Q}$ has holomorphic conformal dimension αQ. However, we know from the theory of vertex operators that normal ordering will introduce a $-\alpha^2$ term to each conformal dimension. Therefore the total left conformal dimension of the field $e^{2\alpha Q}$ is : $\Delta_\alpha = \alpha(Q - \alpha)$

While computing correlators, one can interpret the $QR\phi$ term in the Liouville action as the curvature singularities in the topology of the sphere. This is done by putting a flat metric on the sphere except at two points,the North and South pole : $\sqrt{\hat{g}}\hat{R} = 4\pi\delta(x - x_s) + 4\pi\delta(x - x_n)$. Therefore it is easy to check that :

$$\frac{1}{4\pi} \int \sqrt{g}\, Q\, \hat{R}\, \phi\, d^2x = Q\phi(x_s) + Q\phi(x_n) \tag{5.25}$$

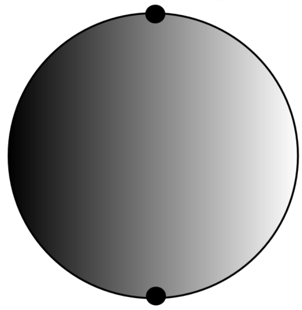

Figure 5.1: The topology on the sphere

While computing correlators, we can insert the operators $e^{Q\phi(x_s)}e^{Q\phi(x_n)}$ into the correlator instead of writing the curvature term of the Liouville action.

With this knowledge, let us put the theory on the cylinder via the transformation law $z = e^{iu}$ and $\bar{z} = e^{-i\bar{u}}$, where $u = \sigma + i\tau$ [Here σ is just the spatial coordinate on the cylinder is not to be confused with the Weyl scaling factor used above in the text]. This means that the coordinate σ is identified with itself after a 2π translation. The modes of the EM Tensor on the sphere :

$$T(z) = \sum_{-\infty}^{+\infty} \frac{L_n}{z^{n+2}} \qquad \bar{T}(\bar{z}) = \sum_{-\infty}^{+\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}} \tag{5.26}$$

Are represented in the following way when put on the cylinder :

$$T(u) = \frac{c_L}{24} - \sum_{-\infty}^{+\infty} e^{-inu} L_n \qquad \bar{T}(\bar{u}) = \frac{c_L}{24} - \sum_{-\infty}^{+\infty} e^{in\bar{u}} \bar{L}_n \tag{5.27}$$

The Hamiltonian, by definition, is given by :

$$H = \frac{1}{2\pi} \int T_{\tau\tau} d\sigma = -\frac{1}{2\pi} \int (T + \bar{T})\, d\sigma \tag{5.28}$$

This becomes $H = -\frac{c_L}{12} + L_0 + \bar{L}_0$. To get a better idea of the space of states in the theory, we will move onto the approximation where the zero mode of the field ϕ, i.e. ϕ_0 is taken to $-\infty$. In this limit, the theory reduces to a free field and therefore ϕ admits the following mode expansion :

$$\phi(\sigma,\tau) = \phi_0 + 2\hat{P}\tau + \sum_{n\neq 0}\big(\frac{ia_n}{n}e^{-inu} + \frac{i\bar{a}_n}{n}e^{in\bar{u}}\big) \tag{5.29}$$

Where $\hat{P} = -\frac{i}{2}\frac{\partial}{\partial\phi_0}$ and the modes of the field follow the commutation relations : $[a_n, a_m] = \frac{m}{2}\delta_{n+m,0}$ and $[\bar{a}_n, \bar{a}_m] = \frac{m}{2}\delta_{n+m,0}$. The energy momentum tensor on the cylinder is : $T(u) = -(\partial_u\phi)^2 + Q\partial_u^2\phi + \frac{Q^2}{4}$. Using this expression, one can express the modes of the EM tensor in terms of the modes of the field ϕ :

$$L_n = \sum_{k\neq 0,n} a_k a_{n-k} + \big(2\hat{P} + inQ\big)a_n$$
$$L_0 = \frac{Q^2}{4} + \hat{P}^2 + 2\sum_{k>0} a_{-k}a_k$$

and the same expression for the barred modes. The oscillator Fock Vacuum is defined by $a_k\,|vac\rangle = 0$ for k>0. The oscillator Fock space is obtained by applying negative modes to the vacuum like : $\mathcal{F}_{osc} = \text{Span}\{a_{-k_1}...a_{-k_n}\bar{a}_{-\bar{k}_1}...\bar{a}_{-\bar{k}_m}\,|vac\rangle\}$. The full Hilbert space will be the product of the oscillator space of states with the space of states of the zero mode.

As we can check, the eigenstates and eigenvalues of the theory are of the form :

$$\psi_{(P,k_i,\bar{k}_j)} = e^{\pm 2iP\phi_0}a_{-k_1}...a_{-k_n}\bar{a}_{-\bar{k}_1}...\bar{a}_{-\bar{k}_m}\,|vac\rangle$$

$$h = \frac{Q^2}{4} + \hat{P}^2 + \sum_i k_i$$
$$\bar{h} = \frac{Q^2}{4} + \hat{P}^2 + \sum_i \bar{k}_i$$

These are the eigenstates in the region where the zero mode of the field is extremely small. These modes are modified on their journey to the interaction barrier and then reflected back to the free field region. The full state in the very small ϕ region is then :

$$|\Psi_P\rangle = \left(e^{2iP\phi_0} + \hat{S}(P)e^{-2iP\phi_0}\right)|s\rangle$$
$$\text{where } |s\rangle = a_{-k_1}...a_{-k_n}\bar{a}_{-\bar{k}_1}...\bar{a}_{-\bar{k}_m}\,|vac\rangle$$

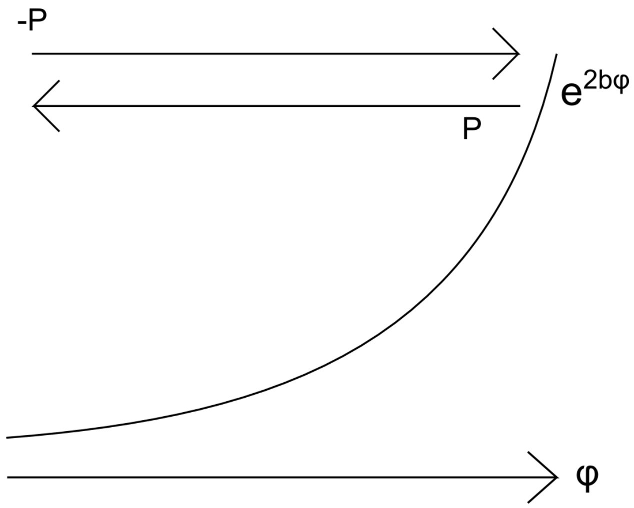

Figure 5.2: Reflection of the Liouville field

Here $\hat{S}(P)$ is the Liouville reflection operator. This reflection amplitude can be physically interpreted if we imagine the universe to start with a metric $e^{2b\phi}$ along with a few bosonic excitations and then imagine that the universe grows in size, only to be reflected back to its small size after a period of time. The operator $\hat{S}(P)$ is easy to compute for small excitations :

$$\hat{S}(P)a_{-1}\left|vac\right\rangle = \frac{Q-2iP}{Q+2iP}S(P)a_{-1}\left|vac\right\rangle \tag{5.30}$$

Here S(P) is a phase factor known as the Liouville reflection amplitude. The above calculation is performed by expressing the mode L_{-1} in terms of the modes a_k, acting it on the state $|\Psi_P\rangle$ and then comparing the result with $S(P)a_{-1}\left|vac\right\rangle$.

5.4 Two and Three point correlation functions

The two-point correlator of two physical states is normalised in the following way :

$$\langle V_{Q/2+iP}(0)V_{Q/2-iP'}(\infty)\rangle = \pi\delta(P-P') \tag{5.31}$$

Here $V_a(x) = e^{2a\phi(x)}$ and $P, P' > 0$. States with real P always have $\Delta \geq \frac{Q^2}{4}$

as shown in the previous section. However, states for which $\Delta \leq \frac{Q^2}{4}$ have imaginary values of P or $a \in [0, \frac{Q}{2}]$, and were shown to be non-normalisable by Seiberg [That is, they do not correspond to delta function integrable functions] [5, 10]. This is known as the *Seiberg bound.*

As one can compute, the states $V_a(x)$ and $V_{Q-a}(x)$ have the same conformal dimension. In the modern interpretation these states are the same upto a factor in the quantum regime [8] :

$$V_a = R(a)V_{Q-a} \tag{5.32}$$

This will be justified using the DOZZ proposal and we will later compute the value of $R(a)$ but for now we will turn our attention to three point-functions. Following the approach of BPZ [1], we can find the null vectors at level 2 in Liouville theory. The conformal dimension of the degenerate fields $\psi_{1,2}$ and $\psi_{2,1}$ in Liouville theory can be obtained from the formula :

$$\Delta = \frac{1}{16}[5 - c \pm \sqrt{(c-1)(c-25)}] \tag{5.33}$$

Plugging $c_L = 1 + 6Q^2$ in the above expression gives us the following two fields whose conformal dimensions satisfy the equation :

$$V_{-b/2} : \Delta = -\frac{1}{2} - \frac{3b^2}{4}$$
$$V_{-1/2b} : \Delta = -\frac{1}{2} - \frac{3}{4b^2}$$

As BPZ have shown, degenerate fields satisfy differential equations that limit the number of fields that can appear in the Fusion Rules. For eg : The OPE of the degenerate field $V_{-b/2}$ with another field V_a can only result in two fields :

$$V_{-b/2}V_a = (C_-)V_{a-b/2} + (C_+)V_{a+b/2} \tag{5.34}$$

Where C_- and C_+ are three point structure constants. Any field which is not present in the Fusion Rules will have a 0 correlation with the two other fields.

5.5 Conformal Bootstrap and the DOZZ proposal

Let's consider the four-point function $\mathcal{G}(x_1, x_2, x_3, x_4)$ defined by :

$$\mathcal{G}(x_1, x_2, x_3, x_4) = \langle 0| V_{\alpha_4}(z_4, \bar{z}_4) V_{\alpha_3}(z_3, \bar{z}_3) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_1}(z_1, \bar{z}_1) |0\rangle \quad (5.35)$$

where $\alpha_2 = -b/2$. We know that due to conformal invariance, we can reduce this expression to a function of cross ratios in the following way :

$$\mathcal{G}(x_1, x_2, x_3, x_4) = |z_{42}|^{-4\Delta_2} |z_{41}|^{2(\Delta_3+\Delta_2-\Delta_4-\Delta_1} |z_{43}|^{2(\Delta_1+\Delta_2-\Delta_4-\Delta_3} |z_{31}|^{2(\Delta_4-\Delta_2-\Delta_1-\Delta_3} G(z, \bar{z}) \quad (5.36)$$

Where $z = \frac{z_{21}z_{43}}{z_{31}z_{42}}$. Since V_{α_2} is a degenerate field, the correlator will follow the null-vector differential equation (Conformal invariance has been used to convert the partial differential equation to an ordinary differential equation) :

$$\left(-\frac{1}{b^2}\frac{d^2}{dz^2} + \left(\frac{1}{z-1} + \frac{1}{z}\right)\frac{d}{dz} - \frac{\Delta_3}{(z-1)^2} - \frac{\Delta_1}{z^2} + \frac{\Delta_3 + \Delta_2 - \Delta_4 + \Delta_1}{z(z-1)}\right) G(z, \bar{z}) = 0 \quad (5.37)$$

The equation given above corresponds to the S-channel. We will choose to combine fields 1 and 2 to split the four point correlator into two conformal blocks :

$$G(z, \bar{z}) = \sum_{s=+,-} C(\alpha_4, \alpha_3, \alpha_1 + sb/2) C(s) |\mathcal{F}_s(z)|^2 \quad (5.38)$$

In this form, the solution to the differential equation is given in terms of Hypergeometric functions : $\mathcal{F}_s(z) = z^{a_s}(1-z)^{b'} F(A_s, B_s, C_s; z)$ where

$$\begin{aligned}
a_s &= \Delta_{\alpha_1+sb/2} - \Delta_2 - \Delta 1 \\
b' &= \Delta_{\alpha_3-b/2} - \Delta_3 - \Delta 2 \\
A_s &= -sb(\alpha_1 - Q/2) + b(\alpha_3 + \alpha_4 - b) - 1/2 \\
B_s &= -sb(\alpha_1 - Q/2) + b(\alpha_3 - \alpha_4) + 1/2 \\
C_s &= 1 - sb(2\alpha_1 - Q)
\end{aligned}$$

The Gamma function identity :

$$\begin{aligned}
F(A, B, C; z) = &\frac{\Gamma(C)\Gamma(B-A)}{\Gamma(B)\Gamma(C-A)}(-z)^A F(A, 1-C+A, 1-B+A, 1/z) \quad + \\
&\frac{\Gamma(C)\Gamma(A-B)}{\Gamma(A)\Gamma(C-B)}(-z)^B F(B, 1-C+B, 1-A+B, 1/z)
\end{aligned}$$

can be used to yield a relation of the form [11]:

$$\mathcal{F}_s(z) = z^{-2\Delta_2} \sum_{t=+,-} B_{st} \mathcal{F}_t\left(\frac{1}{z}\right) \tag{5.39}$$

Given the crossing relations $G(z, \bar{z}) = |z|^{-4\Delta_2} G(1/z, 1/\bar{z})$, one can then determine the ratio of the three point functions :

$$\frac{C(\alpha_3, \alpha_2, \alpha_1 + b)}{C(\alpha_3, \alpha_2, \alpha_1)} = -\frac{\gamma(-b^2)}{\pi\mu} \frac{\gamma(b(2\alpha_1 + b))\gamma(2b\alpha_1)\gamma(b(\alpha_2 + \alpha_3 - \alpha_1 - b))}{\gamma(b(\alpha_2 + \alpha_3 + \alpha_1 - Q))\gamma(b(\alpha_2 + \alpha_1 - \alpha_3))\gamma(b(\alpha_1 + \alpha_3 - \alpha_2))} \tag{5.40}$$

where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$ and we have used the integral representation of $\frac{C(-)}{C(+)}$ given in Dotsenko and Fateev [4]:

$$\frac{C(-)}{C(+)} = \frac{\gamma(-b^2)}{\pi\mu} \gamma(2b\alpha_1)\gamma(2 - b(2\alpha_1 - b)) \tag{5.41}$$

This ratio of the three-point functions differing by b is exactly what one gets from the **DOZZ proposal** [13] :

$$C(\alpha_1, \alpha_2, \alpha_3) = \frac{\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(Q-\sum\alpha_i)/b} \Upsilon'(0)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1 + \alpha_2 + \alpha_3 - Q)\Upsilon(\alpha_1 + \alpha_2 - \alpha_3)\Upsilon(\alpha_3 + \alpha_2 - \alpha_1)\Upsilon(\alpha_1 + \alpha_3 - \alpha_2)} \tag{5.42}$$

Here

$$\Upsilon(x|b, 1/b) = \Upsilon_b(x) = \frac{\Gamma_2^2(Q/2|b, 1/b)}{\Gamma_2(x|b, 1/b)\Gamma_2(Q - x|b, 1/b)} \tag{5.43}$$

And $\Gamma_2(x)$ is the Barnes double Gamma function defined by :

$$\log \Gamma_2(x|b, 1/b) = \frac{d}{dz} \sum_{m,n=0}^{\infty} (x + mb + n/b)^{-z}|_{z=0} \tag{5.44}$$

The DOZZ proposal can be used to calculate the value of the reflection amplitude $R(a)$:

$$R(a) = \frac{\pi\mu\gamma(b^2))^{(Q-2a)/b}}{b^2} \frac{\gamma(2ab - b^2)}{\gamma(2 - 2ab^{-1} + b^{-2})} \tag{5.45}$$

Because of the DOZZ proposal, Liouville theory is *self-dual*, i.e. it in invariant under $b \to 1/b$ and $\mu \to \tilde{\mu}$ where $\pi\tilde{\mu}\gamma(1/b^2) = (\pi\mu\gamma(b^2))^{1/b^2}$. Apart from giving the correct ratio of three-point functions, the DOZZ proposal has passed tests such as produce the correct thermodynamic Bethe ansatz for the Sinh-Gordon model which is related to Liouville theory in the ultraviolet limit and generate the right quantum version of Liouville equation of motion [12].

Summary

In this project, we first studied conformal transformations in general d-dimensions and then specialized to two dimensions. This allowed us to delve into a special class of conformal field theories known as minimal models. After that, we discussed quantum gravity in two dimensions and derived the Liouville action as the Weyl response of the partition function of a conformal field theory. Coupling conformal matter to quantum gravity in the conformal gauge in two dimensions allows us to explore the full system in terms of the Weyl mode of the gravity sector. We studied the properties of the action governing this Weyl sector, both classical and quantum, and ran a consistency check for the DOZZ proposal using the conformal bootstrap. This approach was valid only if the central charge of the external conformal matter field was less than one and it will be interesting to see what happens when $c > 1$ since in this limit, the uniqueness of Teschner's single-valued recursion relation for the three point function breaks down. In this context, the $c = 1$ barrier is an interesting limit to explore.

Appendix A

Mathematical identities

A.1 Properties of the Gamma function

The Gamma function, for $\alpha > 0$ is given by :

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$$

It follows certain properties which make computation easier in the context of Liouville theory :

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

Given that $\gamma(x) = \Gamma(x)/\Gamma(1-x)$, we have the following property :

$$\frac{\Gamma^2(2-z)}{\Gamma^2(z)} = -\frac{\gamma(2-z)}{\gamma(z)}$$

A.2 Properties of the Upsilon function

The Upsilon function, as defined in the text, has the following identities :

$$\Upsilon(x) = \Upsilon(Q - x)$$

It follows the shift-recursion relations :

$$\Upsilon(x + b) = \gamma(bx) b^{1-2bx} \Upsilon(x)$$

$$\Upsilon(x+1/b) = \gamma(x/b) b^{2x/b-1} \Upsilon(x)$$

Combining these, we obtain :

$$\Upsilon(x+Q) = b^{-2+2x(\frac{1}{b}-b)} \gamma(bx+1) \gamma(x/b) \Upsilon(x)$$

References

[1] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Physics B*, 241(2):333–380, 1984.

[2] J. Distler and H. Kawai. Conformal field theory and 2d quantum gravity. *Nuclear physics B*, 321(2):509–527, 1989.

[3] H. Dorn and H.-J. Otto. Two-and three-point functions in liouville theory. *Nuclear Physics B*, 429(2):375–388, 1994.

[4] V. S. Dotsenko and V. A. Fateev. Conformal algebra and multipoint correlation functions in 2d statistical models. *Nuclear Physics B*, 240(3):312–348, 1984.

[5] H. Erbin. Notes on 2d quantum gravity and liouville theory, 2015.

[6] P. Francesco, P. Mathieu, and D. Sénéchal. *Conformal field theory*. Springer Science & Business Media, 2012.

[7] D. Friedan. Introduction to polyakov's string theory. Technical report, 1982.

[8] D. Harlow, J. Maltz, and E. Witten. Analytic continuation of liouville theory. *Journal of High Energy Physics*, 2011(12):71, 2011.

[9] A. M. Polyakov. Quantum geometry of bosonic strings. *Physics Letters B*, 103(3):207–210, 1981.

[10] N. Seiberg. Notes on quantum liouville theory and quantum gravity. *Progress of Theoretical Physics Supplement*, 102:319–349, 1990.

[11] J. Teschner. On the liouville three-point function. *Physics Letters B*, 363(1-2):65–70, 1995.

[12] J. Teschner. Liouville theory revisited. *Classical and Quantum Gravity*, 18(23):R153, 2001.

[13] A. Zamolodchikov and A. Zamolodchikov. Conformal bootstrap in liouville field theory. *Nuclear Physics B*, 477(2):577–605, 1996.

[14] A. Zamolodchikov and A. Zamolodchikov. Lectures on liouville theory and matrix models. *Google Scholar*, 2007.