Chaos in Field Theory and Gravity

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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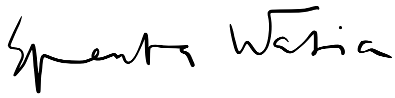
Certificate

This is to certify that this thesis entitled *Chaos in Field Theory and Gravity* submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune contains the summary of a project on quantum chaos, the Sachdev-Ye-Kitaev (SYK) model and related ideas and includes a new derivation of the effective action of the charged SYK model done by Anup Anand Singh at the International Centre for Theoretical Sciences - Tata Institute of Fundamental Research, Bengaluru, under the supervision of Prof. Spenta R. Wadia during the academic year 2017-2018.

Student
ANUP ANAND SINGH



Supervisor
PROF. SPENTA R. WADIA

Declaration

I hereby declare that the matter embodied in the report entitled *Chaos in Field Theory and Gravity* are the results of the investigations carried out by me at the International Centre for Theoretical Sciences - Tata Institute of Fundamental Research, Bengaluru, under the supervision of Prof. Spenta R. Wadia and the same has not been submitted elsewhere for any other degree.

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Dedicated
to the lives and work of
Stephen Hawking
and
Joseph Polchinski

Acknowledgements

I'd like to begin by thanking my supervisor Prof. Spenta R. Wadia for being both a sweet teacher and a harsh critic, and equally importantly, for keeping me on my toes for the better part of the year. I'd also like to thank Prof. Suneeta Vardarajan for being on my thesis advisory committee and for providing precious feedback at different points of time during the project.

Next in line, but deserving an equal share in all the good (and the bad) that ever happened, are the members of my family: Kaku, Papa and Leo; none of this would have been possible without them around.

Sincere thanks are due to Ramesh Ammanamanchi, Krishnendu Ray and Divya Singh for many useful conversations throughout the year and for not making any fuss about being *forced* to proofread parts of this thesis. It is also imperative that I thank my *math-buddies* here: Nagarjuna Chary, Arpith Shanbhag and Shashank Pratap Singh; thank you, guys, for providing the periodic stimulants my brain needed.

This work would not have taken its present shape without the discussions with Prof. Pallab Basu, Prof. Avinash Dhar, Prof. R. Loganayagam, Dr. Alexandre Serantes and Dr. Junggi Yoon. I also thank Prof. Varun Bhalerao, Prof. Sunil Mukhi, Prof. Raghav Rajan and Prof. MS Santhanam for their continuous guidance in the past.

I'm really indebted to the International Centre for Theoretical Sciences for hosting me as a Long Term Visiting Student for this entire year and to the Department of Science and Technology, Government of India for the continuous financial support over the last five years.

In the end, my heartfelt gratitude to the thousand and nine cups of coffee that served a very important role throughout the course of this work.

Abstract

The fate of information contained in an object when it falls into a black hole has been a matter of debate since Hawking's discovery that all black holes emit radiation and can eventually evaporate away. Though there has been no complete resolution to the black hole information puzzle, it is now widely accepted that black holes do not destroy information and that one can, at least in principle, recover this information from the radiation.

As it turns out, the relevant time scale for this recovery is the time required by the black hole to scramble the information over its degrees of freedom. This result has led to an ongoing fruitful exchange of ideas between quantum chaos, quantum information, quantum matter and black hole physics, which we discuss and attempt to summarise in this thesis.

Field theories, especially the ones that are solvable, will be another focus of this thesis. In particular, we discuss two such models, the Sachdev-Ye-Kitaev (SYK) model and one of its variants, the charged SYK model, a generalised version of the SYK model with Dirac fermions, with a global $U(1)$ symmetry. We also provide a new derivation for the effective action of the charged SYK model.

Contents

Chapter 1

Introduction

1.1 Chaos, Scrambling and Black Holes

Trying to understand nature's ways is an ambitious task that is made further challenging by our biases and the limitations of human intuition. It has been partly because of such a bias, having its roots in the difficulty to analyse chaotic systems, that despite its ubiquity, very little was understood about chaos till roughly the middle of the previous century. This changed significantly in the sixties when computers made studying complicated and strongly-interacting classical systems really easy; these numerical studies led to some very useful and interesting insights into our understanding of chaos.

A typical manifestation of chaos is the growth of small initial perturbations to significantly change the configuration of the system at later times – a marked sensitive dependence on initial conditions. In a classical set-up, this notion of chaos can be characterised as the exponential divergence of nearby phase space trajectories. For an observable q of a chaotic classical system evolving in time t, one expects

$$\frac{\partial q(t)}{\partial q(0)} \sim \exp \lambda_L t, \tag{1.1}$$

where the *Lyapunov exponent* λ_L dictates how chaotic the system is. In contrast, the Lyapunov exponent for a non-chaotic system is zero – the distance between two phase space trajectories may increase, say, as a power function in time, but never exponentially.

Interpreting classical chaos as the classical limit of an underlying quantum mechanical phenomenon is rather tricky, and for a very basic reason: a complete understanding of quantum chaos itself remains elusive [1]. It is worthwhile pursuing this goal of decoding the meaning of chaos in quantum systems – it turns out to be a prerequisite to answering many important questions, a large fraction of which come from quantum information, quantum matter and black hole physics.

However, any attempts to directly extend our classical definition to quantum chaos are rendered useless by the meaninglessness of the notion of trajectories in quantum mechanics. And because memory and processing requirements for simulating quantum systems increases exponentially with the number of components, using computers to study quantum systems that are complicated enough to exhibit chaos remains particularly challenging as well.

Let us make a modest attempt to answer an *apparently* straightforward question, though: *what happens to small perturbations made to a quantum chaotic system?* It is intuitive to expect this perturbation to grow in time and spread throughout the system. The spreading of the initial perturbation over the entire system can be equivalently described as the delocalisation or *scrambling* of the information about the initial state over the degrees of freedom of the system [2].

This interpretation of the growth of perturbation in quantum systems as the scrambling of information is central to many quantum information problems and to much of what follows here. In the recent years, these ideas have also helped us gain insights into understanding the fate of information when something falls into a black hole [3, 4] – a problem which in turn promises to teach us valuable lessons about the features of a quantum theory of gravity. More on this in the next chapter, we will first focus on the arsenal needed to tackle the challenges that lie ahead.

1.2 Out-of-Time-Ordered Correlators

The formulation of a quantitative description of quantum chaos requires us to go back to our classical definition where the dependence of the final position on small changes in the initial position is essentially the Poisson bracket between $q(t)$ and $p(0)$:

$$\frac{\partial q(t)}{\partial q(0)} = \{q(t), p(0)\} \tag{1.2}$$

which corresponds to the quantity $\frac{1}{i\hbar}[q(t), p(0)]$, the commutator between $q(t)$ and $p(0)$ [5]. We can use this relation in our semi-classical description where after substituting q and p with two generic Hermitian operators V and W defined on the system, we can treat the growth of the quantity $[W(t), V(0)]$ with time as a signature of chaos. This quantity measures the effect of perturbations by V on W at later times t and vice versa.

However, to diagnose chaos in generic quantum systems, it makes sense using a positive definite quantity instead. This motivates the introduction of the thermal expectation value of the square of the commutator at temperature $T = 1/(k_B\beta)$:

$$C(t) = -\langle [W(t), V(0)]^2 \rangle_\beta, \tag{1.3}$$

the growth of which can treated as a signature of chaos in the system.[1]

What does it mean for a quantity like $C(t)$ to be a diagnostic of chaos? Chaos results in a non-trivial growth of the operator $W(t)$ that appears in 1.3, which is manifested in the growth of $C(t)$ at large times [5]. This allows us to use $C(t)$ as a measure of the strength of chaos, and equivalently, of the rate at which the system scrambles information. *Scrambling time*, the time scale at which $C(t)$ grows to become significant, is essentially the minimum time the system takes to delocalise the initial information in a manner that measurements, unless done over a large fraction of the degrees of freedom of the system [2], cannot distinguish distinct initial states.

A physical quantity of particular interest for discussions on quantum chaos is

$$F(t) = \langle W(t)^\dagger V(0)^\dagger W(t) V(0) \rangle_\beta, \tag{1.4}$$

the out-of-time-ordered (OTO) correlator – the ordering of the operators lending the quantity its name. This particular correlator is related to the commutator in 1.3 through the relation:

$$C(t) = 2(1 - Re[F(t)]), \tag{1.5}$$

where $Re[F(t)]$ denotes the real part of the quantity $F(t)$. It can be viewed as an inner product of two states: one given by first applying $V(0)$ and then $W(t)$, and the other given by applying $W(t)$ followed by $V(0)$. It is an artefact of the chaotic dynamics of the system, that at sufficiently large times, these states are quite different from each other and the overlap between them is small [6]. This is what distinguishes the behaviour of $F(t)$ from that of a time-ordered correlator of the form $\langle W(t)^\dagger W(t)^\dagger V(0) V(0) \rangle_\beta$ at large times, making it a suitable diagnostic of chaos.

1.3 Systems with Strongest Chaos

Early attempts to study chaos almost completely relied on numerical simulations and random matrix analysis of the spectral statistics of chaotic systems [1, 7]. The more recent introduction of the out-of-time-ordered correlators has been particularly fruitful

[1] Here $\langle \cdot \rangle_\beta = Z^{-1} \mathrm{tr}[e^{-\beta H} \cdot]$, with Z and H as the partition function and the Hamiltonian of the system respectively, denotes the thermal expectation value.

– apart from helping characterise chaos, it has also resulted in improved understanding of the connections between the physics of black holes and that of quantum many-body systems [8, 9, 10].

For black holes, time-ordered correlators of physical observables do not change. However, at early times, but after the decay of all two-point correlation functions, the OTO correlators for black holes take a particular form that depends on time t in the following manner [10, 11]:

$$\langle D(t)C(0)B(t)A(0)\rangle - \langle DB\rangle\langle CA\rangle \propto e^{\lambda_L t}. \tag{1.6}$$

The Lyapunov exponent λ_L equals $2\pi T$ for a black hole at temperature T, the Hawking temperature of the black hole.[2]

Maldacena, Shenker and Stanford showed that this rate of exponential growth is, in fact, the fastest possible in quantum mechanics [5]. Nature appears to limit how chaotic a system can be – it bounds the value of the Lyapunov exponent such that:

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar} \tag{1.7}$$

for any quantum mechanical system at a temperature T.

As we have come to realise, this notion of maximal chaos has a very central role to play in understanding black holes. Quantum systems which also exhibit the saturation of this bound are potential candidates for providing dual description to black holes [8]. In particular, models that are solvable can possibly can help us gain useful insights into the quantum aspects of gravity.

One such model that has attracted a lot of attention over the last few years, and is also the primary focus of this thesis, is the Sachdev-Ye-Kitaev (SYK) model [8, 9], a one-dimensional quantum mechanical model of N Majorana fermions.

The Hamiltonian for the SYK model is

$$H = \sum_{iklm} j_{iklm}\chi_i\chi_k\chi_l\chi_m \tag{1.8}$$

and the couplings j_{iklm} come from a random Gaussian distribution. What makes this model hugely fascinating is that it is solvable in the large N limit and that the growth of the OTO correlators saturates the bound on chaos [8, 9].

[2] We will formally define and derive the Hawking temperature in the next chapter.

Since the SYK model and its variants (one of which we will discuss in Chapter 4) are *near* conformal field theories which exhibit maximal chaos, it has been suggested [8, 5, 12, 13] that they are dual to *near* extremal black holes in two-dimensional anti-de Sitter space, making them interesting and possibly useful for understanding black holes better.

Chapter 2

Black Holes as Scramblers

"Alice: How long is forever?
White Rabbit: Sometimes, just one second."
Lewis Carroll, *Alice in Wonderland*

2.1 The Information Puzzle

Alice has a secret diary, but unlike the traditional quantum information set-ups, where Alice and Bob function by cooperation, this is a story of competition. Naively thinking that a black hole will keep the contents of her diary safely hidden behind its event horizon *forever*, Alice throws it into a black hole. What she doesn't know (but Bob does) is that black holes radiate everything that they swallow; albeit in a thermal fashion. *But can Bob really recover and reconstruct the information contained in Alice's diary from the thermal radiation emitted by the black hole? And if he indeed can, how long does it take for the black hole to spew out information for Bob to extract it?*

2.1.1 Early Lessons

In a classical description of gravity, every black hole is completely described by its mass, charge and angular momentum. Any two black holes with the same mass, charge and angular momentum should be completely identical (irrespective of how they would have formed or what lies behind their horizons). A black hole can represent no other information, which also means that it carries no entropy. However, as Bekenstein pointed out in the early seventies [14, 15], this results in a conundrum. One can take an object with some entropy and throw it into a black hole. The entropy of the object disappears once the object is swallowed by the black hole. This is a clear violation of the very *sacred* second law of thermodynamics: *the entropy of a closed system can never decrease.*

The issue was resolved after the realisation that a black hole must possess both entropy and temperature [14, 15, 16]. If A is the surface area of the event horizon of a black hole, then the black hole entropy, in its dimensionless form, is given by

$$S_{\mathrm{BH}} = \frac{c^3 A}{4G\hbar} \tag{2.1}$$

where c, G and $\hbar$ denote the speed of light, the gravitational constant and the reduced Planck constant respectively.

Let us get down to deriving the entropy relation in (2.1) to justify our treatment of black holes as thermodynamic objects.[1] The Schwarzschild metric is given as:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + sin^2\theta \mathrm{d}\phi^2) \tag{2.2}$$

where the Schwarzschild time t is the time recorded by a standard clock at rest at spatial infinity and the Schwarzschild radius r is defined such that the area of a 2-sphere at r is $4\pi r^2$. Setting $r_s = 2GM$, we get:

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + sin^2\theta \mathrm{d}\phi^2). \tag{2.3}$$

We will look at the region close to the horizon which is where much of the dynamics of interest occurs as a consequence of gravitational redshift. Expanding the metric in this region by substituting $r = r_s + \delta$ for small δ gives

$$\mathrm{d}s^2 \cong -\frac{\delta}{r_s}\mathrm{d}t^2 + \frac{r_s}{\delta}\mathrm{d}\delta^2 + {r_s}^2(\mathrm{d}\theta^2 + sin^2\theta \mathrm{d}\phi^2). \tag{2.4}$$

Let us use the redefinition $\delta = \rho^2/4r_s$ to obtain

$$\mathrm{d}s^2 \cong -\frac{\rho^2}{4{r_s}^2}\mathrm{d}t^2 + \mathrm{d}\rho^2 + {r_s}^2(\mathrm{d}\theta^2 + sin^2\theta \mathrm{d}\phi^2) \tag{2.5}$$

and then switch to Euclidean time $\tau = it$:

$$\mathrm{d}s^2 \cong \frac{\rho^2}{4{r_s}^2}\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + {r_s}^2(\mathrm{d}\theta^2 + sin^2\theta \mathrm{d}\phi^2). \tag{2.6}$$

The metric can be rewritten as:

$$\mathrm{d}s^2 \cong \mathrm{d}X^2 + \mathrm{d}Y^2 + {r_s}^2(\mathrm{d}\theta^2 + sin^2\theta \mathrm{d}\phi^2) \tag{2.7}$$

[1] We will primarily follow [17] for this derivation.

using

$$X = \rho cos(\tau/2r_s) \quad \text{and} \quad X = \rho sin(\tau/2r_s). \tag{2.8}$$

We can similarly write down the metric for the Lorentzian case

$$\begin{aligned} ds^2 &\cong -dT^2 + dX^2 + {r_s}^2(d\theta^2 + sin^2\theta d\phi^2) \\ &= -dUdV + {r_s}^2(d\theta^2 + sin^2\theta d\phi^2), \end{aligned} \tag{2.9}$$

where

$$X = \rho cos(t/2r_s), \quad T = \rho sin(\tau/2r_s) \tag{2.10}$$

and

$$U = T - X = -\rho e^{-t/2r_s}, \quad V = T + X = \rho e^{t/2r_s}. \tag{2.11}$$

Understanding the causal properties of black hole geometry becomes easy in the (U,V) coordinates (called the Kruskal-Szekeres coordinates), because here light rays and time-like trajectories always lie within a two-dimensional light cone.

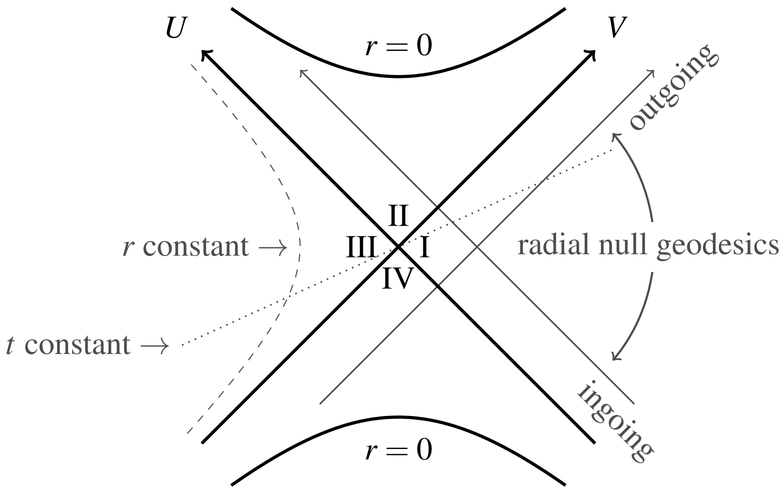

Figure 2.1: The maximal extension of the Schwarzschild geometry can be described in the Kruskal-Szekeres coordinates (U,V).

The original geometry described by Schwarzschild coordinates (r,t) which covers only the region with $U > |T|$ can be analytically continued and maximally extended to contain four quadrants in total as shown in the figure.

A path integral with periodic Euclidean time $\tau \sim \tau + \beta$ generates the thermal partition function $\mathrm{Tr}e^{-\beta H}$. Since the smoothness of the Euclidean metric requires τ to be periodic, with period $4\pi r_s$ (as we can see from the coordinates in (2.8)), the path integral for quantum fields in the Euclidean black hole geometry describes a gas at temperature

$$T_{\mathrm{H}} = \frac{1}{4\pi r_s} \tag{2.12}$$

in equilibrium with the black hole. T_{H} is essentially the Hawking temperature of the black hole.

Having derived the temperature of a black hole, we can compute the corresponding entropy using the standard thermodynamic relation:

$$\mathrm{d}S = \frac{\mathrm{d}M}{T_{\mathrm{H}}} = -\frac{\mathrm{d}T_{\mathrm{H}}}{8\pi G T_{\mathrm{H}}} \tag{2.13}$$

from which we get the expression for the black hole entropy we saw in (2.1):

$$S = S_{\mathrm{BH}} = \frac{c^3 \pi r_s^2}{G\hbar} = \frac{c^3 A}{4G\hbar} \tag{2.14}$$

where A is the area of the black hole event horizon.

For usual thermodynamic systems, the exponential of the entropy is a measure of the number of states available to a system – the entropy of a system acts as a point of contact to the second law of thermodynamics. We expect that black hole entropy measures something similar, and understanding what the black hole microstates correspond to is a rather important question for which we do not have a final, conclusive answer yet. However, this has also been an aspect that string theory has found itself making significant contributions to over the last few decades [18, 19].

2.1.2 Black, but Not Quite So

Within a few years of the proposal that a black hole is a thermodynamic object with temperature and entropy, Hawking did a calculation [16] that led to a further surprising result: *all black holes emit radiation*. All that falls into a black hole possibly comes out as the radiation. But what is even more profound and perplexing is that this radiation appears to be thermal.

The spectrum of a black hole at Hawking temperature T_{H} looks exactly like the spectrum of a blackbody at temperature T_{H} – a black hole radiates away its energy in the form of blackbody radiation.[2] Matter that falls into a black hole does not stay hidden behind the horizon, it leaks out as the black hole evaporates away. This leaking out of

[2] We formally show this in Appendix A.

information seemingly results in the violation of a basic quantum mechanical principle: *the principle of information conservation*. While in classical physics, this principle is manifested in the conservation of phase space volume, it is the unitarity of the S-matrix that encapsulates information conservation in quantum mechanics. The evolution

$$|\psi^{\text{final}}\rangle = S|\psi^{\text{initial}}\rangle \tag{2.15}$$

denotes a process that takes pure states to pure states, consistent with Schrödinger-like evolution

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle. \tag{2.16}$$

One can trace back the initial states from the final states by reversing the sign of the Hamiltonian H – in a sense – by running the dynamics backward. But a black hole appears to do something different, in fact, quite contrary to this basic tenet. Irrespective of what a star is made of when it collapses into a black hole, or what all a black hole engulfs during its lifetime, it evaporates in a thermal manner, the nature of which seems to depend only the temperature of the black hole [16]. More precisely, what a black hole appears to be doing is taking an initial pure density matrix

$$\rho^{\text{initial}} = |\psi^{\text{initial}}\rangle\langle\psi^{\text{initial}}| \tag{2.17}$$

to a final mixed density matrix

$$\rho^{\text{final}} = \sum_i p_i|\psi_i^{\text{final}}\rangle\langle\psi_i^{\text{final}}|. \tag{2.18}$$

To resolve this apparent conflict, one can argue that the radiation *actually* carries the information in subtle correlations between the photons that constitute the Hawking radiation and information can, in principle, be recovered from the radiation [18].

Though the whole issue of the fate of information in black hole environments is far from settled, evidences from the gauge/gravity duality and string theory strongly suggest that black holes do not destroy information when they evaporate [18, 19]. We will not concern ourselves with the possible resolutions to the conundrum, though. In the rest of this chapter, we will look at the time scales of the dynamics involved in the process – the delocalisation of information by a black hole over its degrees of freedom and its possible recovery from the radiation.

2.2 The Hayden-Preskill Protocol

Black holes are not completely black, and one should be able to recover information about initial states of matter that falls into a black hole from the emitted Hawking radiation, at least in principle.

But before we jump into believing that Bob, our antagonist (or protagonist, depending on whose side you are), can really recover and reconstruct from the radiation the information that Alice's diary contained, we should try to answer if he can do so in some reasonable time.

Early estimates [20, 21], in fact, showed that Bob would need to wait for the entire of the black hole to evaporate before he can completely recover the information. But for most astrophysical black holes we know about this time is multiple orders of magnitudes longer than the present age of the universe. However, as we have come to realise over the last decade [3, 4], this is not really so – Bob need not wait that long.

In their relatively recent work [3], Hayden and Preskill came up with a remarkable result using insights from quantum information theory: *the time required for information to be recovered and reconstructed from the black hole radiation is essentially the time the black hole takes to scramble the information over its degrees of freedom*, a time scale much shorter than the black hole evaporation time scale. We will now look into the details of this proposal.

The Hayden-Preskill analysis requires the assumption that Alice's diary is absorbed by black hole instantly. Since this assumption holds no danger of affecting the rigour of our proof, we will accept it without any fuss. Let's say we have a diary D which is maximally entangled with a reference system S before it is thrown into a black hole B that starts in a pure state. What we do next is to model the action of the black hole on the joint system BD of the black hole and the diary with a random unitary operator U, including the process of radiation in this unitary operator as well. The BD system can be therefore reinterpreted as a tensor product of Hawking radiation R and the black hole that remains afterwards, which we will denote by B'.

The question we had set out to answer is now essentially reduced to the following: *How large does the Hawking radiation R have to be before Bob can recover the diary?* To be able to answer this question, it needs a further (more precise) reformulation: How long does Bob have to wait for the following to be true:

$$\int \mathrm{d}U \|\rho_{SB'} - \rho_S \otimes \rho_{SB'}\|_1 \ll 1 \tag{2.19}$$

where the operator trace norm $\|M\|_1$ equals $\mathrm{tr}\sqrt{M^\dagger M}$ for an operator M and denotes the closeness of the states. The integral is over the group-invariant Haar measure.

Until the Page time – the time scale at which the entropy of a black hole is reduced to half of its initial value – the Hawking radiation carries no information as it is maximally mixed. Since the black hole starts in a pure state, we can treat the diary as something

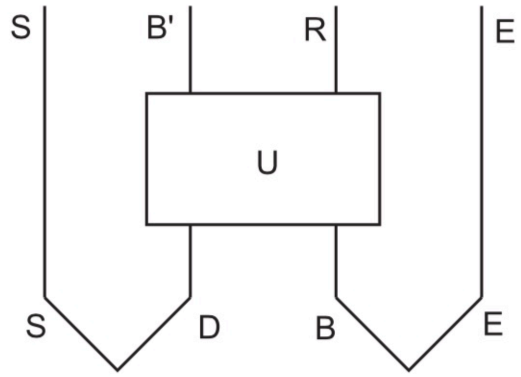

Figure 2.2: The Hayden-Preskill thought experiment follows the fate of information contained in a diary D maximally entangled with a reference system S when it is thrown into a black hole B. B is already maximally entangled with the Hawking radiation E emitted before the diary was thrown into the black hole.

that creates the black hole. What is a subtle point to realise that if the diary is thrown into the black hole after the Page time, that is, after half of the original black hole has evaporated away, the black hole is already in a state of maximal entanglement with the Hawking radiation from the earlier stage, denoted here by E. Hayden and Preskill went on to show that

$$\int \mathrm{d}U \|\rho_{SB'} - \rho_S \otimes \rho_{SB'}\|_1 \leq \sqrt{\frac{(|D|^2-1)(|B'|^2-1)}{|D|^2|E|^2-1}}. \tag{2.20}$$

This can further be approximated to

$$\int \mathrm{d}U \|\rho_{SB'} - \rho_S \otimes \rho_{SB'}\|_1 \leq \frac{|D|}{|R|} \tag{2.21}$$

for large enough systems for which $|B'||R| = |E||D|$ holds true. If the number of bits that were radiated after the diary was thrown into the black hole is b bits more than what the diary contained, then the right hand side is 2^{-b} which becomes increasing small at a rapid rate.

This analysis does multiple things – one, it shows that recovery of information takes much less time than the evaporation time for black holes, and is, therefore, possible in principle. And since a black hole scrambles information this fast (which turns out to be the fastest allowed rate in nature [2, 4, 5]), other systems which scramble equally fast can be analysed to understand aspects of black hole physics. This is a theme that has been carried forward quite significantly over the last decade and has resulted in numerous fruitful exchanges between the physics of quantum information, quantum matter and black holes [2, 8, 9, 11].

Chapter 3

Chaos, Scrambling and Holography

The objective of this chapter is threefold. In the first section, we will formally prove the bound on the Lyapunov exponent for quantum systems, as worked out by Maldacena, Shenker and Stanford in [5], which leads to the notion of maximal chaos we have been referring to while talking about connections between certain quantum mechanical models and black holes. We will then move on to discuss the fast scrambling conjecture [2, 4] and its relationship with holography. And finally, in the third section, we will continue our discussion of the Sachdev-Ye-Kitaev model [8, 9] which we had briefly talked about in the first chapter.

3.1 A Bound on Chaos

In the first chapter, we introduced the OTO correlator

$$F(t) = \langle W(t)^{\dagger} V(0)^{\dagger} W(t) V(0) \rangle_{\beta}, \tag{3.1}$$

the ordering of the operators making it a suitable diagnosis of chaos in quantum systems. We also saw that this quantity is related to thermal expectation value of the square of the commutator at temperature $T = 1/(k_B \beta)$:

$$C(t) = -\langle [W(t), V(0)]^2 \rangle_{\beta} \tag{3.2}$$

through the relation

$$C(t) = 2(1 - Re[F(t)]), \tag{3.3}$$

where $Re[F(t)]$ denotes the real part of the quantity $F(t)$. We will now move on to using these definitions to prove that there exists a bound on how chaotic a quantum system can be.[1] We discuss the implications of the bound on classical systems in Appendix B.

[1] We follow [5] for this entire derivation.

Let us represent $F(t)$ on a thermal circle and rotate a pair of operators by an angle $2\pi\tau/\beta$ to obtain a generalised function of complex times $F(t+i\tau)$.

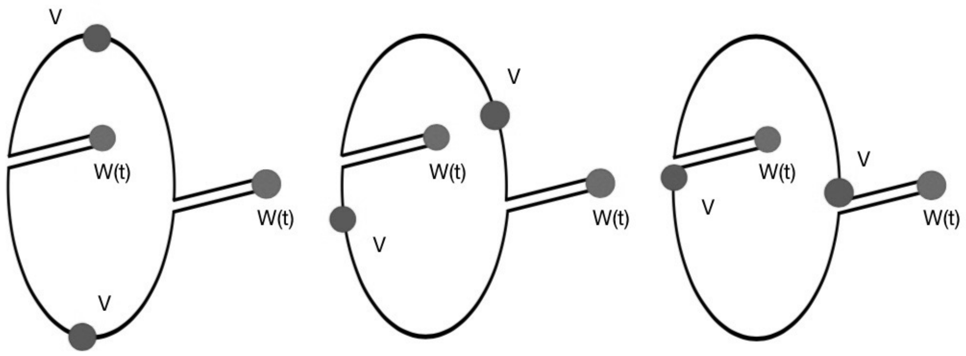

Figure 3.1: The thermal circle representation of $F(t+i\tau)$ involves a particular configuration of the operators that depends on t and τ. Evolution in the Lorentzian time t to produce $W(t)$ are indicated as folds in the circle. The configuration of operators in the second circle corresponds to $|\tau| < \beta/4$ and the one in the third circle corresponds to $\tau = \beta/4$.

$F(t+i\tau)$ is analytic in a strip of width $\beta/2$ in the complex plane (given by $|\tau| \leq \beta/4$):

$$F(t+i\tau) = \mathrm{tr}\left[y^{1-4\tau/\beta}Vy^{1+4\tau/\beta}W(t)y^{1-4\tau/\beta}Vy^{1+4\tau/\beta}W(t)\right], \tag{3.4}$$

where y is defined as

$$y^4 = \frac{1}{Z}e^{-\beta H}. \tag{3.5}$$

In the first chapter, we had defined the scrambling time (t_*) as the time scale at which $C(t)$ becomes significant. Another time scale that is relevant to quantum systems and one that we will need in our proof here is the dissipation time (t_d) which we define as the exponential decay time for two point expectation values like $\langle V(0)V(t)\rangle$ and is expected to be of the order $1/\lambda_L$. On the other hand, since $C(t) \sim \hbar^2 e^{2\lambda_L t}$, $t_* \sim (1/\lambda_L)\log 1/\hbar$. This provides strong motivation to the idea that there exists a parametrically large hierarchy between the scrambling and dissipation time scales for strongly interacting quantum systems.

We have discussed earlier how one expects the OTO correlators to decay at large times – scrambling causes $F(t)$ to decrease with time. Let us focus our attention to times after

t_d but well before t_*. At some point of time in this interval, $F(t)$ takes a factorised value

$$F_d \equiv \mathrm{tr}\left[y^2Vy^2V\right]\mathrm{tr}\left[y^2W(t)y^2W(t)\right], \tag{3.6}$$

a product of disconnected correlators which is independent of t because of time translational invariance. Let us now introduce a conjecture that states that rate of this decrease is bounded as

$$\frac{\mathrm{d}}{\mathrm{d}t}(F_d - F(t)) \leq \frac{2\pi}{\beta}(F_d - F(t)). \tag{3.7}$$

Our primary expectation from a chaotic system is that the difference $(F_d - F(t))$ grows exponentially[2]:

$$(F_d - F(t)) = \varepsilon \exp \lambda_t T + \ldots \tag{3.8}$$

Combining our conjecture in (3.6) with the above gives us

$$\lambda_L \leq \frac{2\pi}{\beta} = 2\pi T \tag{3.9}$$

which is the bound on the Lyapunov exponent for quantum mechanical systems. Thus, all we need to do now is to prove the conjecture we stated above.

We will first make and prove another claim: For a function $f(t)$

$$\frac{1}{1-f}\left|\frac{\mathrm{d}f}{\mathrm{d}t}\right| \leq \frac{2\pi}{\beta} + \mathscr{O}(e^{-4\pi T/\beta}) \tag{3.10}$$

if it satisfies the following properties:

1. $f(t+i\tau)$ is analytic in the half strip $t > 0$ and $-\beta/4 \leq \tau \leq \beta/4$ and $f(t+i\tau) \in \mathbb{R}$ for $\tau = 0$.

2. $|f(t+i\tau)| \leq 1$ in the entire half strip.

To prove this claim, we first map the half strip to the unit circle in the complex plane using the transformation:

$$z = \frac{1 - \sinh\left[\frac{2\pi}{\beta}(t+i\tau)\right]}{1 + \sinh\left[\frac{2\pi}{\beta}(t+i\tau)\right]}. \tag{3.11}$$

For a function f as defined above, $f(z)$ is an analytic function from the unit disk to a unit disk, as $|f(t+i\tau)| \leq 1$.

[2] This was also motivated in [22].

We will now use an important result from complex geometry, the *Schwarz-Pick theorem*: *An analytic map of the disk* $\mathbb{D}$ *into* $\mathbb{D}$ *that preserves the hyperbolic distance between any two points is a disk map and preserves all distances.* This theorem essentially states that functions from unit disk into unit disk cannot increase distances in the hyperbolic metric:

$$\mathrm{d}s^2 = \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{\left(1-|z|^2\right)^2} \tag{3.12}$$

Therefore,

$$\frac{|\mathrm{d}f|}{1-|f|^2} \leq \frac{|\mathrm{d}z|}{1-|z|^2} \tag{3.13}$$

$$\frac{1}{(1-f)}\left|\frac{\mathrm{d}f}{\mathrm{d}t}\right| \leq \left|\frac{\mathrm{d}z}{\mathrm{d}t}\right| \frac{(1+f)}{(1-|z|^2)} = \frac{2\pi}{\beta}\coth\left(\frac{2\pi t}{\beta}\right)\frac{(1+f)}{2} \tag{3.14}$$

which essentially proves our claim

$$\frac{1}{(1-f)}\left|\frac{\mathrm{d}f}{\mathrm{d}t}\right| \leq \frac{2\pi}{\beta} + \mathscr{O}(e^{-4\pi T/\beta}). \tag{3.15}$$

Let us see how this statement is related to the original conjecture in (3.6). Let us define $f = F(t)/F_d$ (we will justify this afterwards). Plugging this definition of f in the expression we proved above gives

$$\frac{1}{1-\frac{F(t)}{F_d}}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(1-\frac{F(t)}{F_d}\right)\right| \leq \frac{2\pi}{\beta} + \mathscr{O}(e^{-4\pi T/\beta}) \tag{3.16}$$

which reduces to our original conjecture which we had used to prove the bound on Lyapunov exponent. So, all we need to show now is that our definition $f = F(t)/F_d$ is valid.

It is rather straightforward to show that $F(t)/F_d$ satisfies the first of the required properties. For general t and τ, we can write $F(t+i\tau)$ as

$$F(t+i\tau) = \frac{1}{Z}\mathrm{tr}\left[e^{-(\beta/4-\tau)H}Ve^{-(\beta/4+\tau)H}W(t)e^{-(\beta/4-\tau)H}Ve^{-(\beta/4+\tau)H}W(t)\right] \tag{3.17}$$

which is clearly analytic in the strip $|\tau| \leq \beta/4$. Also, for $\tau = 0$, $f(t+i\tau) = f(t) \in \mathbb{R}$, establishing the first property.

We now need to show that $|f(t+i\tau)| \leq 1$ for $f(t+i\tau) = F(t+i\tau)/F_d$. However, we will show this for $f(t) = F(t+t_0)/F_d + \varepsilon$ instead. Here, ε is some small error up to the relation will be shown to hold true for times greater than t_0. To do so, we will use the *Phragmén-Lindelöf principle* which states that the modulus of a holomorphic function in the interior of a region is bounded by its modulus on the boundary of the region. All we need to show now is that $|f(t+i\tau)| \leq 1$ holds true on the three boundaries.

For $\tau = \pm\beta/4$, we have

$$F\left(t - \frac{i\beta}{4}\right) = \mathrm{tr}\left[y^2 V W(t) y^2 V W(t)\right] \tag{3.18}$$

which is, in fact, the inner product of two matrices $[yVW(t)y]_{ij}$ and $[yW(t)Vy]_{ij}$. Therefore, using the *Cauchy-Schwarz inequality* ($\|\bar{a}\cdot\bar{b}\| \leq \|a\|\|b\|$), we can write

$$F\left(t - \frac{i\beta}{4}\right) \leq \mathrm{tr}\left[y^2 W(t) V y^2 V W(t)\right]. \tag{3.19}$$

For times large compared to t_d, we expect the dynamics of the system to result in the factorisation of $\mathrm{tr}\left[y^2 W(t) V y^2 V W(t)\right]$ to $F_d = \mathrm{tr}\left[y^2 V y^2 V\right]\left[y^2 W(t) y^2 W(t)\right]$.

To accommodate for all possible errors, we assert that this factorisation holds true for $t \geq t_0$ with some error ε, the value of ε depending on the choice of t_0. To get a good approximation, we restrict $t_0 \leq t_*$. This allow us to write

$$\mathrm{tr}\left[y^2 W(t) V y^2 V W(t)\right] = \mathrm{tr}\left[y^2 V y^2 V\right]\mathrm{tr}\left[y^2 W(t) y^2 W(t)\right] + \varepsilon \tag{3.20}$$

which gives

$$\left|\frac{F(t+t_0)}{F_d + \varepsilon}\right| \leq 1 \tag{3.21}$$

This establishes the bound for the $|\tau| = \beta/4$ boundaries. The same argument holds true for the $t = 0$ boundary where one would obtain the same factorisation as well. The only subtlety is to ensure that the t_0 we choose is well less than t_* to keep the error smaller than ε.

To complete the proof, we need to show that f is bounded by some constant in the interior of the half strip as well. We proceed as follows:

$$\begin{aligned}
|F(t+i\tau)| &\leq \mathrm{tr}\left[y^{1+4\tau/\beta} V y^{1-4\tau/\beta} W y^{1+4\tau/\beta} W y^{1-4\tau/\beta} V\right] \\
&\sim \mathrm{tr}\left[y^{1+4\tau/\beta} V y^{3-4\tau/\beta} V\right]\mathrm{tr}\left[y^{1+4\tau/\beta} W y^{3-4\tau/\beta} W\right] \\
&\leq \mathrm{tr}\left[y V y^2 V\right]\mathrm{tr}\left[y W y^2 W\right]
\end{aligned} \tag{3.22}$$

which is some finite quantity. This establishes that f is bounded in the interior of the half strip as well. Hence, the function $F(t+t_0)/F_d + \varepsilon$ satisfies the two required properties.

Tracing it back to where we began, we have proved our original conjecture in (3.6) which we had used to show the existence[3] of the bound on Lyapunov exponent:

$$\lambda_L \leq \frac{2\pi}{\beta}. \tag{3.23}$$

3.2 Scrambling and Holography

We discussed in the previous chapter how ideas and techniques from quantum information theory suggest that black holes scramble information quickly, in fact, at the quickest rate possible. Here, we discuss this idea further in connection to the fast scrambling conjecture and talk about how holographic considerations have led to improved understanding of both black holes and quantum systems that scramble quantum information rapidly.

3.2.1 The Fast Scrambling Conjecture

Building up on the motivation that came in the form of the analysis by Hayden and Preskill [3], Sekino and Susskind came up with the fast scrambling conjecture [2, 4], which primarily asserts that the fastest scramblers in nature take a time logarithmic in the number of degrees of freedom and that black holes are the fastest scramblers of quantum information.

Solvable models which saturate the bound on the rate of scrambling appear to have a great use in providing dual descriptions to black holes, a connection that requires a formal correspondence between the two sides of the gauge/gravity duality[4] [8, 9, 11, 13, 25]. One can attempt to understand sensitivity to initial conditions in the context of holography by studying the effect of perturbations on the boundary field theory. It

[3]The proof depends on the assumption that OTO correlators factorise at times after t_d with the added restriction that it stays valid only till times much smaller than t_*. To tackle any possible failures, we need to impose an additional boundary on the half-strip to ensure $t \ll t_*$. We will require $|F| \leq F_d + \varepsilon$ to be true at this boundary which is in fact the case as F becomes exceedingly smaller with time. The only other modification one would need to do after adding a fourth boundary (and essentially changing the region from a half strip to a full closed strip) is to choose a different and rather complicated transformation instead of the transformation in (3.11).

[4]The gauge/string duality, first proposed formally by Juan Maldacena in 1997 [23, 24], conjectures that all the physics in the bulk of a region of spacetime with a theory of gravity can be described by a quantum field theory that lives on the boundary. The simplest examples of the duality exist between theories of gravity defined on a $(d+1)$-dimensional anti-de Sitter spacetime (AdS) and conformal field theories (CFTs) defined in d dimensions.

turns out, as Shenker and Stanford showed in [11], that the scrambling time[5] in such considerations (with N being the number of degrees of freedom) is

$$t_* \sim \frac{\beta}{2\pi} \log N^2 \tag{3.24}$$

which substantiates the fast scrambling conjecture [4]. In the next section, we will discuss a particular example – that of the thermofield double state and the eternal black hole – to better understand this idea.

3.2.2 The Thermofield Double State

The thermofield double formalism is a clever technique to treat a thermal, mixed state as a pure state in a bigger system. Consider a conformal field theory (CFT) with Hamiltonian H and a complete set of eigenvalues $|n\rangle$. Taking two copies of the CFT, one can construct a thermofield double state:

$$|TFD\rangle = \frac{1}{Z^{1/2}} \sum_n e^{-\beta E_n/2} |n\rangle_L |n\rangle_R. \tag{3.25}$$

One can compute the total density matrix corresponding to the prescribed state:

$$\rho_{total} = |TFD\rangle \langle TFD| \tag{3.26}$$

and show that the thermofield double state describes a particular pure state in the doubled system. Also, the reduced density matrix for each of the subsystems can be computed as:

$$\rho_L = tr_{\mathscr{H}_R} |TFD\rangle \langle TFD| \tag{3.27}$$

One can look at the effect of scrambling in the thermofield double state by perturbing one of the CFTs at some time t_w in the past and then analysing how quantities like the correlations and the mutual information between subsystems on the two sides decay in time due to the growth of the perturbation [11].

The perturbed state can be represented as

$$\left|\widetilde{TFD}\right\rangle = e^{-iH_L t_w} \mathscr{O}_L e^{iH_L t_w} |TFD\rangle \tag{3.28}$$

where $\mathscr{O}_L$ is an operator defined on the left CFT [11]. The thermofield double has been used to a good extent both in understanding effect of scrambling in quantum systems and also because it is dual to an eternal black hole, a two-boundary black hole in AdS, with each of the copies of the CFT living on each of the two boundaries.

[5] Since the effect of the perturbation is to drive the system from an atypical state to a state of typicality, the time taken for the entanglement in the system to be disrupted is essentially the scrambling time of the system.

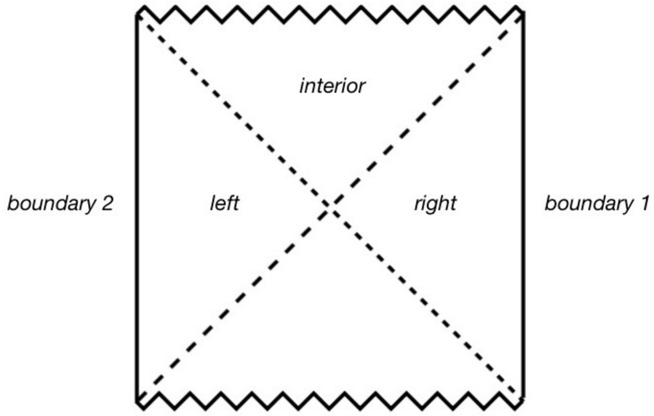

Figure 3.2: An eternal black hole is a black hole with the full, two-sided Penrose diagram. It has a past singularity, a future singularity, two asymptotic regions, and is dual to the thermofield double state.

It was for one such set-up that Shenker and Stanford studied the effect of scrambling in [11], where using holography, they probed the sensitive dependence on initial conditions for both the thermofield double state and the corresponding bulk geometry.

3.3 The Sachdev-Ye-Kitaev Model

A maximally chaotic model is naturally a strong candidate for a black hole dual. Kitaev's proposal of a variant of the Sachdev-Ye model has held particular promises in this regard [8, 9]. The Sachdev-Ye-Kitaev (SYK) model is a quantum mechanical model of N Majorana fermions with a Hamiltonian given by:

$$H = \sum_{ijkl} j_{ijkl}\chi_i\chi_j\chi_k\chi_l \tag{3.29}$$

where the couplings j_{ijkl} come from a zero-mean random Gaussian distribution with a width of order $\mathcal{J}/N^{3/2}$. The random interactions, which represent a disorder in the system, lend the model a number of interesting features. One of them of particular significance from the point of view of chaotic dynamics is the saturation of the chaos bound by the SYK model in the large N, low energy limit.

In the large N limit, the IR theory has an emergent time reparametrisation symmetry [8, 9]. As we will derive in the next chapter, the first correction to the effective action is the *Schwarzian action*:

$$I_{\mathrm{Sch}} = -N\frac{a_0}{\mathcal{J}}\int \mathrm{d}\tau\{f,\tau\}, \tag{3.30}$$

where the Schwarzian derivative

$$\{f,\tau\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \tag{3.31}$$

is invariant under $SL(2)$ symmetry $f \to \frac{af+b}{cf+d}$ and a_0 is a numerically determined constant.

We will also derive the effective action for a variant of the SYK model in the next chapter – the charged SYK model with a global $U(1)$ symmetry [13, 26]. In particular, we will see how the added symmetry gives rise to a further correction to the Schwarzian action.

Chapter 4

Deriving the Effective Action

We discussed in the previous chapter the properties of the Sachdev-Ye-Kitaev (SYK) model that make it interesting and and possibly useful in gaining insights into the physics of black holes. In this chapter, we will derive the effective action for the SYK model using a method similar to one used by Kitaev and Suh in [8]. We will also use the method to provide a new derivation for the effective action for a generalised SYK model with Dirac fermions, with a global $U(1)$ symmetry, introduced in [26] where the effective action was originally written down using symmetry arguments.

4.1 The SYK Model

The SYK model is a quantum mechanical model with N Majorana fermions with random interactions. The Lagrangian is

$$\mathscr{L} = \sum_i \chi_i \partial_t \chi_i + \sum_{ijkl} j_{ijkl} \chi_i \chi_j \chi_k \chi_l \tag{4.1}$$

with $\langle j_{ijkl}^2 \rangle = J^2 3!/N^3$. The couplings come from a random Gaussian distribution and the model can be analysed in both its quenched and annealed versions.[1]

We will use the annealed version of the SYK model and sum over the fermions and

[1] One can show, as in [27], that the quenched and the annealed versions of the model are identical at leading order in $1/N$.

average over all the random couplings to obtain the partition function:

$$
\begin{aligned}
Z &= \int \mathscr{D}\chi \mathscr{D} j_{ijkl} \exp\left(- j_{ijkl}^2 \frac{N^3}{3!J^2} \right) \exp\Big(- \int \mathrm{d}\tau \mathscr{L} \Big) \\
&= \int \mathscr{D}\chi \mathscr{D} j_{ijkl} \exp\left(- j_{ijkl}^2 \frac{N^3}{3!J^2} \right) \exp\Big(- \int \mathrm{d}\tau \sum \chi_i \partial_t \chi_i + \sum j_{ijkl} \chi_i \chi_j \chi_k \chi_l \Big) \\
&= \int \mathscr{D}\chi \exp\Big(J^2 \Big(\sum \int \mathrm{d}\tau \chi_i(\tau) \chi_j(\tau) \chi_k(\tau) \chi_l(\tau) \Big)^2 \Big) \\
&= \int \mathscr{D}\chi \exp\Big(J^2 \int \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \Big(\frac{1}{N} \sum_i \chi_i(\tau_1) \chi_i(\tau_2) \Big)^4 \Big)
\end{aligned} \tag{4.2}
$$

We use the $O(N)$ symmetry of the expression to introduce bilocal fields $\tilde{\Sigma}(\tau_1, \tau_2)$ and $\tilde{G}(\tau_1, \tau_2)$ with

$$
\tilde{G}(\tau_1, \tau_2) = \frac{1}{N} \sum_{i=1}^{N} \chi_i(\tau_1) \chi_i(\tau_2) \tag{4.3}
$$

and

$$
\int \mathscr{D}\tilde{\Sigma} \mathscr{D}\tilde{G} \exp\Big(- \tilde{\Sigma}(\tau_1, \tau_2) \Big(\tilde{G}(\tau_1, \tau_2) - \frac{1}{N} \sum_{i=1}^{N} \chi_i(\tau_1) \chi_i(\tau_2) \Big) \Big) = 1 \tag{4.4}
$$

We can now write down the effective action:

$$
I_{\text{eff}} = -\frac{N}{2} \log\det\big(\partial_\tau - \tilde{\Sigma} \big) + \frac{N}{2} \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \Big(\tilde{\Sigma}(\tau_1, \tau_2) \tilde{G}(\tau_1, \tau_2) - \frac{J^2}{4} \tilde{G}(\tau_1, \tau_2)^2 \Big) \tag{4.5}
$$

where the ∂_t term features because of the kinetic term in the Lagrangian.

At the leading order in $1/N$, we have the Schwinger-Dyson equations:

$$
\Sigma(\tau_1, \tau_2) = J^2 G(\tau_1, \tau_2)^3 \tag{4.6}
$$

$$
G^{-1}(\omega) = -i\omega - \Sigma(\omega) \tag{4.7}
$$

which we can write down after summing up the melonic diagrams.

For free fermions, we have

$$
G_0(\tau) = \frac{1}{2} \text{sgn}(\tau) \tag{4.8}
$$

$$
G_0(\omega) = \int \mathrm{d}\tau e^{i\omega\tau} G_0(\tau) = -(i\omega)^{-1}. \tag{4.9}
$$

On including interactions, we obtain the following:

$$G = G_0 + G_0 \Sigma G \tag{4.10}$$

$$\implies \Sigma = {G_0}^{-1} - G^{-1} \tag{4.11}$$

where G_0 is the bare propagator and Σ is the self-energy.

What we have achieved from the above is a reduction of the original theory of Majorana fermions to a theory of bilocal fields. We will proceed with this and compute the first correction to the IR action. We expand perturbatively around the saddle to get

$$\langle \tilde{G}(\tau_1, \tau_2) \tilde{G}(\tau_1, \tau_2) \rangle \sim \frac{1}{N^2} \sum_{ij} \langle \chi_i(\tau_1) \chi_j(\tau_2) \chi_i(\tau_3) \chi_j(\tau_4) \rangle \tag{4.12}$$

where the right hand side has a manifest $O(N)$ symmetry.

In the infrared limit of the theory, we can drop ∂_t in the action. This results in an emergent *time-reparametrisation* invariance which can be used to write

$$G(\tau_1, \tau_2) \to (f'(\tau_1) f'(\tau_2))^{\Delta} G(f(\tau_1), f(\tau_2)). \tag{4.13}$$

The infrared solution at zero temperature is

$$G(\tau_1, \tau_2) = b \frac{\operatorname{sgn}(\tau_{12})}{|\tau_{12}|^{1/2}} \tag{4.14}$$

where $\tau_{12} = \tau_1 - \tau_2$. To obtain the solution at non-zero temperature, we choose $f(\tau) = \tan(\tau\pi/\beta)$ and use it along with (4.13). This gives

$$G(\tau_1, \tau_2) = b \cdot \operatorname{sgn}(\tau_{12}) \left[\frac{\pi}{\beta \sin\left(\frac{\pi \tau_{12}}{\beta}\right)} \right]^{1/2}. \tag{4.15}$$

We will use this along with a change of variable $\tilde{\Sigma} \to \tilde{\Sigma} + \sigma$ to compute the first correction to the action. Here

$$\sigma(\tau_1, \tau_2) = \partial_\tau \delta(\tau_1 - \tau_2). \tag{4.16}$$

We now write down the effective action after implementing the above:

$$\begin{aligned} I_{\text{eff}} &= -\frac{N}{2} \log\det\big(-\tilde{\Sigma}\big) + \frac{N}{2} \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \Big(\big(\tilde{\Sigma}(\tau_1, \tau_2) + \sigma(\tau_1, \tau_2)\big) \tilde{G}(\tau_1, \tau_2) - \frac{J^2}{4} \tilde{G}(\tau_1, \tau_2)^2 \Big) \\ &= -\frac{N}{2} \log\det\big(\partial_t - \tilde{\Sigma}\big) + \frac{N}{2} \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \Big(\tilde{\Sigma}(\tau_1, \tau_2) \tilde{G}(\tau_1, \tau_2) - \frac{J^2}{4} \tilde{G}(\tau_1, \tau_2)^2 \Big) \\ &+ \frac{N}{2} \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \Big(\sigma(\tau_1, \tau_2) \tilde{G}(\tau_1, \tau_2) \Big). \end{aligned} \tag{4.17}$$

The first two terms in the expression are time-reparametrisation invariant. The third term essentially becomes a perturbation to the IR action to get the first correction term.

Let us take $\tilde{G}$ to be the IR solution at non-zero temperature:

$$\tilde{G}(\tau_1,\tau_2) \to b\cdot \mathrm{sgn}(\tau_{12})(f'(\tau_1)f'(\tau_2))^{\Delta}\left[\frac{\pi}{\beta\sin\left(\frac{\pi(f(\tau_1)-f(\tau_2))}{\beta}\right)}\right]^{2\Delta}. \tag{4.18}$$

Since $\sigma(\tau_1,\tau_2)=\partial_t\delta(\tau_1-\tau_2)$ picks up $\tilde{G}(\tau_1,\tau_2)$ in (4.17) for which $\tau_1\approx\tau_2$, we expand $\tilde{G}(\tau_1,\tau_2)$ about $\tau_{12}\approx 0$. We define

$$\tau_+ = \frac{\tau_1+\tau_2}{2} \tag{4.19}$$

and Taylor expand around τ_+ to obtain

$$f(\tau_1) = f(\tau_+) + \frac{\tau_{12}}{2}f'(\tau_+) + \frac{\tau_{12}^2}{8}f''(\tau_+) + \frac{\tau_{12}^3}{48}f'''(\tau_+) + \ldots \tag{4.20}$$

$$f(\tau_2) = f(\tau_+) - \frac{\tau_{12}}{2}f'(\tau_+) + \frac{\tau_{12}^2}{8}f''(\tau_+) - \frac{\tau_{12}^3}{48}f'''(\tau_+) + \ldots \tag{4.21}$$

We have, up to third order:

$$f(\tau_1)-f(\tau_2) = \tau_{12}f'(\tau_+) + \frac{\tau_{12}^3}{24}f'''(\tau_+). \tag{4.22}$$

Also,

$$f'(\tau_1) = f'(\tau_+) + \frac{\tau_{12}}{2}f''(\tau_+) + \frac{\tau_{12}^2}{8}f'''(\tau_+) + \ldots \tag{4.23}$$

$$f'(\tau_2) = f'(\tau_+) - \frac{\tau_{12}}{2}f''(\tau_+) + \frac{\tau_{12}^2}{8}f'''(\tau_+) + \ldots \tag{4.24}$$

from which we get

$$f'(\tau_1)f'(\tau_2) = (f'(\tau_+))^2 + \frac{\tau_{12}^2}{4}f'(\tau_+)f'''(\tau_+) - \frac{\tau_{12}^2}{4}(f''(\tau_+))^2. \tag{4.25}$$

Putting these in

$$\tilde{G}(\tau_1,\tau_2) \to b\cdot \mathrm{sgn}(\tau_{12})(f'(\tau_1)f'(\tau_2))^{\Delta}\left[\frac{\pi}{\beta\sin\left(\frac{\pi(f(\tau_1)-f(\tau_2))}{\beta}\right)}\right]^{2\Delta} \quad (\Delta = 1/q = 1/4), \tag{4.26}$$

we get, for small τ_{12},

$$\tilde{G}(\tau_1,\tau_2)=\mathrm{sgn}(\tau_{12})\frac{\left[(f'(\tau_+))^2+\frac{\tau_{12}^2}{4}f'(\tau_+)f'''(\tau_+)-\frac{\tau_{12}^2}{4}(f''(\tau_+))^2\right]^{1/4}}{\left[\frac{\beta}{\pi}\right]^{1/2}\left[\frac{\pi}{\beta}\left(\tau_{12}f'(\tau_+)+\frac{\tau_{12}^3}{24}f'''(\tau_+)\right)\right]^{1/2}} \tag{4.27}$$

$$\tilde{G}(\tau_1,\tau_2)=\mathrm{sgn}(\tau_{12})\frac{\left[(f'(\tau_+))^2+\frac{\tau_{12}^2}{4}f'(\tau_+)f'''(\tau_+)-\frac{\tau_{12}^2}{4}(f''(\tau_+))^2\right]^{1/4}}{\left[\tau_{12}f'(\tau_+)+\frac{\tau_{12}^3}{24}f'''(\tau_+)\right]^{1/2}}. \tag{4.28}$$

Simplifying further we get

$$\begin{aligned}\tilde{G}(\tau_1,\tau_2)&\approx\frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{1/2}}\frac{\left[1+\frac{\tau_{12}^2}{4}\frac{f'''(\tau_+)}{f'(\tau_+)}-\frac{\tau_{12}^2}{4}\left(\frac{f''(\tau_+)}{f'(\tau_+)}\right)^2\right]^{1/4}}{\left[1+\frac{\tau_{12}^2}{24}\frac{f'''(\tau_+)}{f'(\tau_+)}\right]^{1/2}}\\&\approx\frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{1/2}}\left(1+\frac{\tau_{12}^2}{16}\frac{f'''(\tau_+)}{f'(\tau_+)}-\frac{\tau_{12}^2}{16}\left(\frac{f''(\tau_+)}{f'(\tau_+)}\right)^2\right)\left(1-\frac{\tau_{12}^2}{48}\frac{f'''(\tau_+)}{f'(\tau_+)}\right)\\&=\frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{1/2}}\left(1+\frac{\tau_{12}^2}{24}\left(\frac{f'''(\tau_+)}{f'(\tau_+)}-\frac{3}{2}\left(\frac{f''(\tau_+)}{f'(\tau_+)}\right)^2\right)\right).\end{aligned} \tag{4.29}$$

We will now use this to derive the first correction term to the action where we integrate the above expression after subtracting out the ground state energy:

$$I_{\mathrm{Sch}}=\frac{N}{2}\int \mathrm{d}\tau_1\mathrm{d}\tau_2\sigma(\tau_1,\tau_2)\frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{1/2}}\{f(\tau_+),\tau_+\} \tag{4.30}$$

where

$$\{f(\tau_+),\tau_+\}=\frac{f'''(\tau_+)}{f'(\tau_+)}-\frac{3}{2}\left(\frac{f''(\tau_+)}{f'(\tau_+)}\right)^2 \tag{4.31}$$

is the Schwarzian derivative. Evaluating the integral gives us the first correction to the action, the Schwarzian action

$$I_{\mathrm{Sch}}=\alpha_s N\int_0^{2\pi}\mathrm{d}\tau\{f(\tau),\tau\} \tag{4.32}$$

where the coefficient α_S can be computed up to a constant. This can be done by choosing an appropriate representation of $\sigma(\tau_1,\tau_2)=\partial_\tau\delta(\tau_1-\tau_2)$ with a cutoff proportional to J. The sharp peak of the delta function represents its UV characteristic. Since the formalism depends on implementing $\tilde{\Sigma}\to\tilde{\Sigma}+\sigma$, where $\tilde{\Sigma}$ itself is a bilocal field defined in IR, σ should also be defined appropriately in IR as a function that peaks when $\tau_1 \approx \tau_2$ as well (as a Gaussian, for example). The kernel σ has a sharp, delta function-like behaviour in UV but is more appropriately represented in IR as a function with a wider base but one that peaks at the same value. This modification is also necessitated by the fact that the scaling dimension of $\tilde{\Sigma}$ in UV does not match that of σ in IR, the quantity we are adding it to – the kernel σ needs to be changed from UV to IR in a way that fixes the issue of the scaling dimension.

4.2 The Charged SYK Model

We will now derive the effective action for the charged SYK model, a generalised version of the SYK model with Dirac fermions, with a global $U(1)$ symmetry, for which we have the Hamiltonian

$$\mathscr{H} = \sum j_{i_1\dots i_q} \psi^\dagger_{i_1} \cdots \psi^\dagger_{i_q/2} \psi_{i_{q/2}+1} \cdots \psi^\dagger_{i_q}, \quad \langle |j_{i_1\dots i_q}|^2 \rangle = \frac{J^2\Big((q/2)!\Big)^2}{N^{q-1}}, \tag{4.33}$$

$$j_{i_1,\dots,i_{q/2},i_{q/2+1},\dots,i_q} = j^*_{i_{q/2+1},\dots,i_q,i_1,\dots,i_{q/2}}. \tag{4.34}$$

We will again work with $q=4$, for which $\langle |j_{i_1\dots i_q}|^2\rangle = \frac{J^2 4}{N^3}$.

Following the derivation for the standard SYK model in the previous section, we will introduce bilocal fields $\tilde{\Sigma}(\tau_1,\tau_2)$ and $\tilde{G}(\tau_1,\tau_2)$ with

$$\tilde{G}(\tau_1,\tau_2) = \frac{1}{N}\sum_{i=1}^{N} \psi_i^\dagger(\tau_1)\psi_1(\tau_2) \tag{4.35}$$

and

$$\int \mathscr{D}\tilde{\Sigma}\mathscr{D}\tilde{G}\exp\Big(-\tilde{\Sigma}(\tau_1,\tau_2)\Big(\tilde{G}(\tau_1,\tau_2) - \frac{1}{N}\sum_{i=1}^{N}\psi_i^\dagger(\tau_1)\psi_i(\tau_2)\Big)\Big) = 1. \tag{4.36}$$

The effective action in terms of the bilocal fields can then be written as:

$$\begin{aligned} I_{\text{eff}} = -\frac{N}{2}\log\det\big((\partial_t - \mu) - \tilde{\Sigma}\big) - \frac{N}{2}\int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \Big(&\tilde{\Sigma}(\tau_1,\tau_2)\tilde{G}(\tau_1,\tau_2) \\ &- \frac{J^2}{4}(\tilde{G}(\tau_1,\tau_2))^2(\tilde{G}(\tau_2,\tau_1))^2\Big) \end{aligned} \tag{4.37}$$

for which we derive the Schwinger-Dyson equations:

$$\Sigma(\tau) = -4J^2(G(\tau))^2(G(-\tau))^2 \tag{4.38}$$

$$G(i\omega_n) = ((i\omega_n + \mu) - \Sigma(i\omega_n))^{-1}. \tag{4.39}$$

In the IR limit, we can drop the $(i\omega_n + \mu)$ term from the equations above. As a result of this, we obtain an emergent *time-reparametrisation* and $U(1)$ symmetry. This allows us to write

$$G(\tau_1,\tau_2) \to (f'(\tau_1)f'(\tau_2))^{\Delta} G(f(\tau_1), f(\tau_2)) \exp(i\phi(\tau_1) - i\phi(\tau_2)). \tag{4.40}$$

We will use this along with a change of variable $\tilde{\Sigma} \to \tilde{\Sigma} + \sigma$ to compute the first correction to the action. Here

$$\sigma(\tau_1,\tau_2) = \partial_\tau \delta(\tau_1 - \tau_2). \tag{4.41}$$

Implementing the above, we write down the effective action:

$$\begin{aligned} I_{\text{eff}} = -\frac{N}{2}\log\det\big(-\tilde{\Sigma}\big) - \frac{N}{2}\int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \Big(\Big(\tilde{\Sigma}(\tau_1,\tau_2) + \sigma(\tau_1,\tau_2)\Big)\tilde{G}(\tau_1,\tau_2) \\ - \frac{J^2}{4}\tilde{G}(\tau_1,\tau_2)^2\tilde{G}(\tau_2,\tau_1)^2\Big) \\ = -\frac{N}{2}\log\det\big(\partial_t - \tilde{\Sigma}\big) + \frac{N}{2}\int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \Big(\tilde{\Sigma}(\tau_1,\tau_2)\tilde{G}(\tau_1,\tau_2) - \frac{J^2}{4}\tilde{G}(\tau_1,\tau_2)^2\tilde{G}(\tau_2,\tau_1)^2\Big) \\ + \frac{N}{2}\int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \Big(\sigma(\tau_1,\tau_2)\tilde{G}(\tau_2,\tau_1)\Big) \end{aligned} \tag{4.42}$$

where the first two terms in the expression are invariant under time-reparametrisation and $U(1)$ transformations. The third term essentially becomes a perturbation to the IR action to get the first correction term. Since $\sigma(\tau_1,\tau_2) = \partial_t \delta(\tau_1 - \tau_2)$ picks up $\tilde{G}(\tau_2,\tau_1)$ in (4.42) for which $\tau_1 \approx \tau_2$, we expand $\tilde{G}(\tau_2,\tau_1)$ about $\tau_{12} \approx 0$. Similar to the derivation for the standard SYK model, we define

$$\tau_+ = \frac{\tau_1 + \tau_2}{2} \tag{4.43}$$

and Taylor expand around τ_+ to obtain, up to third order:

$$f(\tau_1) - f(\tau_2) = \tau_{12} f'(\tau_+) + \frac{\tau_{12}^3}{24} f'''(\tau_+) \tag{4.44}$$

$$\phi(\tau_1) - \phi(\tau_2) = \tau_{12}\phi'(\tau_+) + \frac{\tau_{12}^3}{24}\phi'''(\tau_+) \tag{4.45}$$

$$f'(\tau_1)f'(\tau_2) = (f'(\tau_+))^2 + \frac{\tau_{12}^2}{4} f'(\tau_+) f'''(\tau_+) - \frac{\tau_{12}^2}{4}(f''(\tau_+))^2. \tag{4.46}$$

We take the IR solution

$$G(\tau) = \exp(-2\pi\varepsilon T\tau)(\sin\pi T\tau)^{-1/2} \tag{4.47}$$

and use it in

$$G(\tau_1,\tau_2) = (f'(\tau_1)f'(\tau_2))^{1/4} G(f(\tau_1), f(\tau_2)) \exp(i\phi(\tau_1) - i\phi(\tau_2)) \tag{4.48}$$

where

$$
\begin{aligned}
G(f(\tau_2), f(\tau_1)) &= \exp(-2\pi\varepsilon T(f(\tau_2)-f(\tau_1)))(\sin\pi T(f(\tau_2)-f(\tau_1)))^{-1/2} \\
&\approx \exp(-2\pi\varepsilon T(f(\tau_2)-f(\tau_1)))\cdot \\
&\left(\pi T(f(\tau_2)-f(\tau_1)) - \frac{\pi^3 T^3}{6}(f(\tau_2)-f(\tau_1))^3\right)^{-1/2}.
\end{aligned} \tag{4.49}
$$

Simplifying the above expression, we get

$$
\begin{aligned}
\tilde{G}(\tau_2,\tau_1) &= \frac{(f'(\tau_1)f'(\tau_2))^{1/4}}{(f(\tau_1)-f(\tau_2))^{1/2}} \cdot \frac{exp(i(\phi(\tau_1)-\phi(\tau_2)) - 2\pi\varepsilon T(f(\tau_1)-f(\tau_2))}{(1-\frac{\pi^2T^2}{6}(f(\tau_2)-f(\tau_1))^2))^{1/2}} \\
&= \mathscr{S}\cdot\frac{exp(i(\phi(\tau_1)-\phi(\tau_2)) - 2\pi\varepsilon T(f(\tau_1)-f(\tau_2))}{(1-\frac{\pi^2T^2}{6}(f(\tau_2)-f(\tau_1))^2))^{1/2}}
\end{aligned} \tag{4.50}
$$

where the first factor

$$
\mathscr{S} = \frac{1}{|\tau_{12}|^{1/2}}\left(1+\frac{\tau_{12}^2}{24}\{f(\tau_+),\tau_+\}\right) \tag{4.51}
$$

as we had evaluated in the derivation for the standard SYK model. We can now write

$$
\begin{aligned}
\tilde{G}(\tau_2,\tau_1) = \frac{1}{|\tau_{12}|^{1/2}}&\left[1+\frac{\tau_{12}^2}{24}\{f(\tau_+),\tau_+\}\right]\cdot \\
\left[\frac{exp(i(\phi(\tau_1)-\phi(\tau_2)) - 2\pi\varepsilon T(f(\tau_1)-f(\tau_2))}{(1-\frac{\pi^2T^2}{6}(f(\tau_2)-f(\tau_1))^2))^{1/2}}\right]&.
\end{aligned} \tag{4.52}
$$

We will put the expressions for $f(\tau_1)-f(\tau_2)$ and $\phi(\tau_1)-\phi(\tau_2)$ in the above to get

$$
\begin{aligned}
\tilde{G}(\tau_2,\tau_1) = \frac{1}{|\tau_{12}|^{1/2}}&\left[1+\frac{\tau_{12}^2}{24}\{f(\tau_+),\tau_+\}\right]\cdot \\
\left[\frac{exp(i(\tau_{12}\phi'(\tau_+)+\frac{\tau_{12}^3}{24}\phi'''(\tau_+)) - 2\pi\varepsilon T(\tau_{12}f'(\tau_+)+\frac{\tau_{12}^3}{24}f'''(\tau_+))}{(1-\frac{\pi^2T^2}{6}(\tau_{12}f'(\tau_+)+\frac{\tau_{12}^3}{24}f'''(\tau_+))^2))^{1/2}}\right]&.
\end{aligned} \tag{4.53}
$$

Expanding the above and retaining the term up to third order gives

$$
\begin{aligned}
\tilde{G}(\tau_2,\tau_1) \approx \frac{1}{|\tau_{12}|^{1/2}} \bigg[1 + \frac{\tau_{12}^2}{24}\{f(\tau_+),\tau_+\} + \frac{\tau_{12}^2}{2}\Big((\phi'(\tau_+))^2 \\
+ (f'(\tau_+))^2 - 2f'(\tau_+) + 4i\pi\varepsilon T \phi'(\tau_+) f'(\tau_+) - 4i\pi\varepsilon T \phi'(\tau_+) - 2\pi\varepsilon T \Big) \bigg] \\
= \frac{1}{|\tau_{12}|^{1/2}} \bigg[1 + \frac{\tau_{12}^2}{24}\{f(\tau_+),\tau_+\} + \frac{\tau_{12}^2}{2}\Big[(\phi' + 2\pi i\varepsilon T(f'(\tau)-1))^2 \Big] \bigg]
\end{aligned}
\tag{4.54}
$$

which we use to compute the correction to the effective action of the standard SYK model due to added $U(1)$ symmetry. The second term in the above expression gives the Schwarzian part of the effective action.

After subtracting the ground state energy, similar to how we evaluated the Schwarzian action for the standard SYK model, we use the above expression to obtain the effective action:

$$
I_{\text{eff}} = \int d\tau_1 d\tau_2 \Big(\sigma(\tau_1,\tau_2)\tilde{G}(\tau_2,\tau_1) \Big) \tag{4.55}
$$

which on evaluating gives the effective action for the charged SYK model:

$$
I_{\text{eff}} = I_{\text{Sch}} + \alpha' N \int_0^{2\pi} \left[\phi' + 2\pi i\varepsilon T(f'(\tau)-1) \right]^2 \tag{4.56}
$$

where the first term

$$
I_{\text{Sch}} = \alpha_s N \int_0^{2\pi} d\tau \{f(\tau),\tau\} \tag{4.57}
$$

is the Schwarzian action that we had derived for the uncharged SYK model at finite temperature. The second term in the effective action describes the correction to the Schwarzian part of the action due to the added global $U(1)$ symmetry. The coefficients can be fixed up to constants – in a manner one would do it for the SYK model – by choosing an appropriate representation of σ to describe it in IR.

Chapter 5

Discussion

“Exactly!" said Deep Thought.
“So once you do know what the question actually is,
you’ll know what the answer means."
Douglas Adams, *The Hitchhiker’s Guide to the Galaxy*

First discovered as a solution of the Einstein field equations by Karl Schwarzschild in 1915,[1] black holes have held us in awe for over a century now. Every time we have discovered something new about them, they have appeared to pose even further challenges.[2] The field of black hole physics which has largely been theoretical till recently has begun getting new insights from astrophysical observations, primarily from the more recent gravitational-wave detections of binary black hole mergers [28]. Though these discoveries will stand as one of the finest achievements of human endeavours, we will possibly have to wait for quite a while before we can probe the quantum aspects of black holes using our observatories.

Since theoretical set-ups are all we have at the moment to work our way through the difficulties posed by the at-times-counter-intuitive physics of black holes, we can expect to make some steady progress by harnessing what we have. The Sachdev-Ye-Kitaev model and its variants are a few such models that can act as useful toy models to study black holes. At the same time, given how closely questions from quantum chaos, quantum information and black holes are to each other, there is a great deal of physics to be learnt from working on the solutions to these problems.

We began this thesis discussing the difficulties in developing a complete understanding of quantum chaos. Recent advances in the study of chaos have led to the belief

[1]Interestingly, Schwarzschild worked out the solution while serving the German army on the Russian front during World War I and published it after a few months in early 1916.

[2]Such is the nature of almost all of scientific research.

that understanding chaotic dynamics has an important role to play in solving the issues that have been hindrances to completely resolving the black hole information puzzle [3, 4, 17].

Models like the Sachdev-Ye-Kitaev model and its variants have possibly crucial roles to play in solving such issues. That these models are solvable makes them particularly important. In the previous chapter, we looked at the derivation for the effective action for the SYK and the charged SYK models and tried to see how the low energy physics of these models is encoded in the Schwarzian action and the correction to the Schwarzian action in the case of the charged SYK model. It will be really interesting in the coming years to see how much we are able to understand in terms of the bulk dual to these theories, instances of which have been explored in [13, 22, 25]. A slightly more ambitious problem would be to learn how to use these models to explore the region behind the black hole horizon. This can, in turn, help us more conclusively understand what transpires in black hole environments.

More generally, it will also be interesting to see how the confluence of ideas from quantum chaos, quantum information and quantum matter, that has helped us discover surprising new results about black holes, leads us further into better understanding the quantum aspects of gravity.

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Appendix A

Deriving the Hawking Radiation

Imagine a futuristic space-traveller, Ringo, who attempts to jump into a black hole just out of sheer curiosity. As he falls toward the black hole, his speed keeps on increasing, and in his time frame, which we will denote here as t_{R}, he takes a finite time to fall through the future horizon of the black hole. Ringo's mate, George, who is slightly less adventurous, chooses to observe Ringo from far, far away, far enough from the black hole to essentially make him an asymptotic observer. But George never really sees Ringo reaching the horizon till an infinite time t_{R}, indicating that there is a nonlinear relation between the two times. In the proper time t_{R} of the infalling observer, U goes through zero linearly, while V is approximately constant. Therefore,

$$\mathrm{d}t_{\mathrm{R}} \propto e^{-t_{\mathrm{G}}/r_s} \mathrm{d}t_{\mathrm{G}}. \tag{A.1}$$

The two time coordinates are characteristically different. While Ringo, with the t_{R} coordinate crosses the horizon freely, George with the t_{G} coordinate interprets space ending at the horizon. We will use the difference between the coordinates used by George and Ringo that to see how the black hole emits radiation.[1]

Ringo, in his t_{R} coordinate, expands the quantum fields in modes of t_{R}-frequency ν, while George expands them in modes of t_{G}-frequency ω. One should make a note here that these are two different expansions with positive and negative frequencies mixed. Here, for simplicity, we treat our set-up as a $(1+1)$-dimensional system[2] with ϕ, a

[1] We have followed [17] for this derivation.
[2] We have just ignored the angular directions.

massless scalar. The metric can therefore be written as

$$\begin{aligned} \mathrm{d}s^2 &= -\left(1-\frac{r_s}{r}\right)\mathrm{d}t^2+\left(1-\frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 \\ &= -\left(1-\frac{r_s}{r}\right)\mathrm{d}u\mathrm{d}v \\ &= -\frac{4r_s^2}{r}e^{r/r_s}\mathrm{d}U\mathrm{d}V \end{aligned} \tag{A.2}$$

where we have used the following definition:

$$u = t-(r+r_s\ln(r-r_s)) = -2r_s\ln(-U/r_s), \quad v = t+(r+r_s\ln(r-r_s) = 2r_s\ln(V/r_s). \tag{A.3}$$

The coordinates above $(u, v)$are defined only in the first quadrant but are conformally related to the KruskalâĂŞSzekeres (U, V) coordinates. One can simply write the Klein-Gordon equation as

$$\partial_u\partial_v\phi = \partial_U\partial_V\phi = 0 \tag{A.4}$$

which can be used to obtain both the modes.

We expand the right-moving part in modes:

$$\begin{aligned} \phi_R &= \int_0^\infty \frac{\mathrm{d}\nu}{2\pi(2\nu)^{1/2}}(a_\nu e^{-i\nu U}+a_\nu^\dagger e^{i\nu U}) \\ &= \int_0^\infty \frac{\mathrm{d}\omega}{2\pi(2\omega)^{1/2}}(b_\omega e^{-i\omega U}+b_\omega^\dagger e^{i\omega U}). \end{aligned} \tag{A.5}$$

Here a_ν and b_ω are the modes for Ringo and George respectively. Using mode expansions, we can show that the two are related as:

$$b_\omega = \int_0^\infty \frac{\mathrm{d}\nu}{2\pi}(\alpha_{\omega\nu}a_\nu+\beta_{\omega\nu}a_\nu^\dagger) \tag{A.6}$$

where

$$\begin{aligned} \alpha_{\omega\nu} &= 2r_s(\omega/\nu)^{1/2}(2r_s\nu)^{2ir_s\omega}e^{\pi r_s\omega}\Gamma(-2ir_s\omega), \\ \beta_{\omega\nu} &= 2r_s(\omega/\nu)^{1/2}(2r_s\nu)^{2ir_s\omega}e^{-\pi r_s\omega}\Gamma(-2ir_s\omega). \end{aligned} \tag{A.7}$$

Since $a_\nu|\psi\rangle = 0$ holds true because of the adiabatic principle, the eternal modes have the following behaviour:

$$\begin{aligned}
\langle\psi|b_{\omega}^{\dagger}b_{\omega'}|\psi\rangle &= 2(\omega\omega')^{1/2}\int_0^{\infty}\frac{\mathrm{d}\nu}{2\pi(2\nu)^{1/2}}\frac{\mathrm{d}\nu'}{2\pi(2\nu')^{1/2}}\beta^*_{\omega\nu}\beta_{\omega'\nu'}\langle\psi|a_\nu a^{\dagger}_{\nu'}|\psi\rangle \\
&= 2(\omega\omega')^{1/2}\int_0^{\infty}\frac{\mathrm{d}\nu}{4\pi\nu}\beta^*_{\omega\nu}\beta_{\omega'\nu'} \\
&= \frac{2\pi\delta(\omega-\omega')}{e^{4\pi r_s\omega}-1} \\
&= \frac{2\pi\delta(\omega-\omega')}{e^{\omega/T_{\mathrm{H}}}-1}.
\end{aligned} \tag{A.8}$$

This is exactly the spectrum of a blackbody at temperature T_{H} – a black hole radiates away its energy in the form of Hawking radiation which is thermal.

Appendix B

The Bound on Classical Chaos

While discussing the bound on the Lyapunov exponent for quantum systems, we did not address the issue of the validity of our proof for classical mechanics.

It may appear that Lyapunov exponents for classical systems which can take any value violate the Maldacena-Shenker-Stanford bound [5]. However, the expression for the bound on the Lyapunov exponent with the dimensionful factors is

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar}. \tag{B.1}$$

Since in the strict classical limit, we take $\hbar \to 0$, the Lyapunov exponent in the classical case is allowed to take any possible value. Hence, there is no possible violation of the bound that classical chaos can exhibit.