Congruences between the ratios of Rankin-Selberg L-functions

विद्या वाचस्पति की

उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

A thesis submitted in partial fulfillment of the requirements

for the degree of Doctor of Philosophy

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भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्थान पुणे

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH

PUNE

2026

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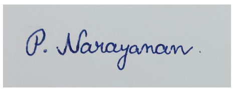

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Acknowledgments

First and foremost, I thank my advisors, Prof. A. Raghuram and Dr. Baskar Balasubramanyam for guiding me and for being patient with me. I thank my parents Panchapakesan and Chitra for their encouragement. I thank my brother Avinash. Also, a special thanks to my dear বন্ধু Saikat for his unwavering support. Without them this thesis would not have been possible.

I thank IISER Pune and the department of mathematics at IISER Pune. I thank the computational facility, the Bhaskara Lab at IISER Pune, where some of the computations in this thesis were carried out. I thank the Institute for Advanced Study (IAS), Princeton, where a part of the research regarding this thesis was carried out through their summer collaborators program. My gratitude to the Harish-Chandra Research Institute (HRI), Allahabad for their hospitality. I also thank the CSIR fellowship which supported me during my Ph.D.

I thank my Ph.D. committee members, Dr. Chandrasheel Bhagwat and Dr. Dipramit Majumdar for their invaluable comments. I would like to thank Prof. Jacques Tilouine for some of his invaluable remarks. I also thank Prof. Amit Hogadi for some informative discussions. I thank Tristan for helpful discussions concerning congruences. I thank, the math office staffs Mrs. Suvarna Bharadwaj, Mr. Yogesh Kolap and, the academic staff Mrs. Sayalee Damlee.

I would like to thank my friends, Sajid, Sandra, Darshan, Shubham, Sujeet, Dhruv Khatri, Digvijay, Daniel, Dilip, Firdousi, Visakh, Namrata, Ramya and many others who had helped me. I also thank Mr. Arjun, Mr. Deepak and Mr. Subba Rao and many other inspiring teachers from my school. I would like to thank my uncle Balaji for his help. I thank Mr. Ravichandran and Mr. Ganesh for letting me use their office spaces in Mayiladuthurai for my (online) comprehensive examinations. I also thank Rajaraman from Thirukodikaval. I apologize to anyone whom I may have missed and wish to express my gratitude to them as well.

This thesis is dedicated to William Stein and many others for developing the SAGE software.

Abstract

Let $h' \in S_{k'}(N', \chi')$ and $h \in S_k(N, \chi)$ be normalized newforms in the respective spaces with $k', k \geq 2$ and $k' - k \geq 2$. Let $L(s, h \times h')$ denote the completed Rankin-Selberg L-function attached to (h, h'). It is well-known that for m an integer and $\frac{k'+k}{2} - 1 < m < k' - 1$ then

$$\frac{L(m, h \times h')}{L(m+1, h \times h')} \in \overline{\mathbb{Q}}.$$

Let $h'' \in S_{k'}(N', \chi')$ be another newform and $\mathfrak{l} \subset \bar{\mathbb{Q}}$ be a prime ideal. For all $n \in \mathbb{N}$ assume $a(n, h') \equiv a(n, h'') \pmod{\mathfrak{l}}$. This thesis is concerned with the question whether the ratios of L-values are congruent modulo $\mathfrak{l}$, i.e.,

$$a(n, h') \equiv a(n, h'') \pmod{\mathfrak{l}} \implies \frac{L(m, h \times h')}{L(m+1, h \times h')} \equiv \frac{L(m, h \times h'')}{L(m+1, h \times h'')} \pmod{\mathfrak{l}}?$$

First, we develop some algorithms to compute the special values of Rankin-Selberg L-functions from well known results. Using them we verify in many instances that the ratios are congruent. Then, under some hypothesis on the prime $\mathfrak{l}$, the levels N and N' and the weights k, k' we show that the ratios are indeed congruent modulo $\mathfrak{l}$.

The ArXiv links for the preprints:

- Congruences for the ratios of Rankin-Selberg L-functions. (with A. Raghuram.) (https://arxiv.org/pdf/2512.02919)
- Eisenstein cohomology and congruences for the ratios of Rankin-Selberg L-functions. (with A. Raghuram.) (https://arxiv.org/pdf/2512.02927)

Contents

Chapter 1

Introduction

1.1 Why ratios?

The ratios of successive L-values show up naturally in a calculation of Langlands which will be explained now. Let σ, σ' are two cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ attached to the newforms $h \in S_k(N, \chi)$ and $h' \in S_{k'}(N', \chi')$. Let σ_p, σ'_p are the local components of σ, σ' respectively. Assume both σ_p and σ'_p are unramified, i.e., $p \nmid NN'$. For $\mathrm{Re}(s) \gg 0$ let $I(s, \sigma_p \otimes \sigma'_p)$ and $I(-s, \sigma'_p \otimes \sigma_p)$ denote the normalized induction from $P(\mathbb{Q}_p)$ to $\mathrm{GL}_4(\mathbb{Q}_p)$. Here $P(\mathbb{Q}_p)$ is the $(2, 2)$ parabolic subgroup of $\mathrm{GL}_4(\mathbb{Q}_p)$. The induced representations are again unramified. Fix the normalized spherical vectors f_p^0 and $\tilde{f}_p^0$ in the respective induced spaces. They span the $\mathrm{GL}_4(\mathbb{Z}_p)$ invariant subspaces which are one dimensional because of the assumption that σ_p and σ'_p are unramified and generic.

Let $T_{\mathrm{st}}(s, \sigma_p \otimes \sigma'_p) : I(s, \sigma_p \otimes \sigma'_p) \to I(-s, \sigma'_p \otimes \sigma_p)$ denote the standard intertwining operator. Since the operator commutes with the action of $\mathrm{GL}_4(\mathbb{Q}_p)$, it preserves the $\mathrm{GL}_4(\mathbb{Z}_p)$ invariant subspaces. Hence, under the standard intertwining operator, $f_p^0 \mapsto c_p \tilde{f}_p^0$ for some complex number $c_p \in \mathbb{C}$. The constant turns out to be

$$c_p = \frac{L(s, \sigma_p \times \sigma'^{\vee}_p)}{L(s+1, \sigma_p \times \sigma'^{\vee}_p)}.$$

See Langlands [13] where it is called as the Gindikin-Karpalevic formula for p-adic groups. It should be mentioned that the result of Langlands is much more general than what has been described above.

1.2 Some properties of global L-functions

It is then natural to ask about the properties of the ratios of the global L-functions by including the Euler factors at *all* the places. Here some of its properties are mentioned which will be reviewed in Chapter 2.

Let $h' \in S_{k'}(N', \chi')$ and $h \in S_k(N, \chi)$ be normalized newforms in the respective spaces with, $k', k \geq 2$. It is well-known that the Fourier coefficients are algebraic integers. For $s \in \mathbb{C}$

let $L(s, h \times h')$ denote the Rankin-Selberg L-function attached to the pair (h, h'); it is the completed L-function with Gamma factors at infinity and the local factors at the unramified places are of degree 4. It has a meromorphic continuation and satisfies a functional equation with the line of symmetry being $\mathrm{Re}(s) = \frac{k'+k-1}{2}$. The critical set is $\{m \in \mathbb{Z} \mid k \leq m < k'\}$. To have at least two successive elements in the critical set assume $k' - k \geq 2$. From Shimura [23, Theorem 4] it follows that

$$\frac{L(m, h \times h')}{L(m+1, h \times h')} \in \overline{\mathbb{Q}} \text{ for } m = k, k+1 \cdots, k'-2, \tag{1.1}$$

Congruences and ratios

A well-known principle with origins in Iwasawa theory states that *a congruence between objects* should translate to a *congruence between the special values of L-functions* attached to these objects. The reader is referred to Vatsal [27] for an instance of this principle.

Hence, it is natural to ask if $h'' \in S_{k'}(N', \chi')$ be another newform and $\mathfrak{l} \subset \bar{\mathbb{Q}}$ is a prime ideal such that for all $n \in \mathbb{N}$ suppose $a(n, h') \equiv a(n, h'') \pmod{\mathfrak{l}}$ then is it whether the ratios of L-values are congruent, i.e.,

$$a(n, h') \equiv a(n, h'') \pmod{\mathfrak{l}} \implies \frac{L(m, h \times h')}{L(m+1, h \times h')} \equiv \frac{L(m, h \times h'')}{L(m+1, h \times h'')} \pmod{\mathfrak{l}}?$$

The thesis is concerned with this question. First computationally it will be verified in many examples that indeed it is true. Then it will be proved, under some hypothesis of the prime $\mathfrak{l}$ and on the levels N, N', that congruent morphisms give rise to congruent ratios of L-values.

Chapter 2

Algorithms for the special values of L-functions

In this article, we verify that the expected congruences hold in the following cases:

1. $h', h'' \in S_{24}(\mathrm{SL}_2(\mathbb{Z}))$, $\mathfrak{l} = 144169$, $h \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.
2. $h', h'' \in S_{30}(\mathrm{SL}_2(\mathbb{Z}))$, $\mathfrak{l} = 51349$, $h \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.
3. $h', h'' \in S_{13}(\Gamma_0(3), \chi)$, $\mathfrak{l} = 13$, $g \in S_6(\Gamma_0(3))$.
4. $h \in S_{26}(\mathrm{SL}_2(\mathbb{Z}))$, and $h', h'' \in S_{13}(\Gamma_0(3), \chi)$, $\mathfrak{l} = 13$.
5. $h \in S_{24}(\mathrm{SL}_2(\mathbb{Z}))$, $h' \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$, h'' a weight 12 Eisenstein series, $\mathfrak{l} = 691$. (This is the Ramanujan congruence.)

In (1) and (2), h' is a Galois conjugate of h, but in (3) h' is not a Galois conjugate of h.

The computations rely on two algorithms. The first algorithm, reviewed in 2.2.1, computes the special values of Rankin-Selberg L-functions, and is essentially due to Shimura and Hida. Roughly speaking, it relies on the interpretation of the L-value as a Petersson inner product of a cusp form with a holomorphic projection which is due to Shimura [24], and the computation of the holomorphic projection recursively from the given data and an appropriate Eisenstein series due to Hida [8]. The second well-known algorithm, reviewed in 2.2.2, computes the L-values of modular forms from the Hecke-equivariant pairing between a space of cusp forms and a suitable space of modular symbols. It should be mentioned that there is also an algorithm by Tim Dokchitser [4] to compute the L-values of modular forms numerically. All the computations in this thesis are done on SAGE [20].

Based on these examples, Conjecture 3.3.1 is formulated. In Chapter 6, a variation of the conjecture will be proved.

2.1 Review of some classical results

In this section, we begin by reviewing some preliminaries from Shimura [24] on modular forms and the special values of Rankin-Selberg L-functions and some results of Hida [8] on the holomorphic projection. We present two algorithms which will be used to verify the congruences mentioned in the introduction.

2.1.1 Some results of Shimura and Hida

Let $\Gamma_1(N)$ be the standard Hecke congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$; the volume of its fundamental domain $\Phi_1(N)$ with respect to the measure $y^{-2}dxdy$. will be denoted $\mu(\Phi_1(N))$. Let $M_k(\Gamma_1(N))$ be the space of holomorphic modular forms of weight k for $\Gamma_1(N)$. For $f_1,\ f_2 \in M_k(\Gamma_1(N))$ such that $f_1 f_2$ is a cusp form, define the Petersson inner product $\langle f_1, f_2\rangle$ by:

$$\langle f_1, f_2\rangle \ :=\ \mu(\Phi_1(N))^{-1}\int_{\Phi_1(N)} \overline{f_1(z)} f_2(z) y^{k-2}\, dxdy.$$

Let χ be a Dirichlet character modulo N. Let $S_{k'}(N,\chi')$ (resp., $S_k(N,\chi)$) denote the space of holomorphic cusp forms of weight k' (resp., k) and for $\Gamma_0(N)$ with nebentypus character χ' (resp., χ). Assume $k < k'$. Suppose $h' = \sum_{n=1}^{\infty} a(n,h')q^n \in S_{k'}(N',\chi')$ and $h = \sum_{n=1}^{\infty} a(n,h)q^n \in S_k(N,\chi)$, where $q = e^{2\pi i z}$, then, for $s \in \mathbb{C}$ and $\mathrm{Re}(s) \gg 0$ the Rankin–Selberg convolution is defined as the Dirichlet series:

$$D(s,h,h') \ :=\ \sum_{n=1}^{\infty} a(n,h)a(n,h')n^{-s}.$$

Clearly $D(s,h,h') = D(s,h',h)$. Let $h_\rho(z) = \overline{h(-\overline{z})} = \sum_{n=1}^{\infty} \overline{a(n,h)} e^{2\pi i n z}$ and $h_\rho \in S_k(N,\chi^{-1})$. Observe that

$$\int_0^1 \overline{h_\rho(z)} h'(z) dx = \sum_{n=1}^{\infty} a(n,h)a(n,h')e^{-4\pi n y}$$

and so

$$\int_0^{\infty} y^{s-1}\int_0^1 \overline{h_\rho(z)} h'(z) dxdy = (4\pi)^{-s}\Gamma(s)D(s,h,h') = (4\pi)^{-s}\Gamma(s)D(s,h',h)$$

For an integer $\lambda \geq 0$ and a Dirichlet character ω modulo N such that $\omega(-1) = (-1)^\lambda$, define an Eisenstein series:

$$E^*_{\lambda,N}(z,s,\omega) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \omega(d)(cz+d)^{-\lambda}|cz+d|^{-2s}, \quad (\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix}),$$

where $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} | m \in \mathbb{Z}\}$. For the same λ and ω, define:

$$E_{\lambda,N}(z,s,\omega) = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \omega(n)(mNz+n)^{-\lambda}|mNz+n|^{-2s}.$$

The following relation holds between $E^*_{\lambda,N}$ and $E_{\lambda,N}$:

$$E_{\lambda,N}(z,s,\omega) = 2L_N(2s+\lambda,\omega)E^*_{\lambda,N}(z,s,\omega),$$

where $L_N(s,\omega)$ denotes the Dirichlet L-series $\sum_{n=1}^{\infty}\omega(n)/n^s$ and the subscript N means that with $\omega(n)=0$ if $\gcd(n,N)\neq 1$. If ω is primitive modulo N then this condition is automatic and the subscript may be dropped. One has the following integral representation:

$$\begin{aligned} 2(4\pi)^{-s}\Gamma(s)D(s,h,h') &= \int_{\Phi_0(N)} \overline{h_\rho}h' \cdot E^*_{k'-k,N}(z,s+1-k,\chi\chi')\, y^{s-1}\, dxdy \\ &= \int_{\Phi_0(N)} \overline{h'_\rho}h \cdot E^*_{k'-k,N}(z,s+1-k,\chi\chi')\, y^{s-1}\, dxdy. \end{aligned}$$

Here $\Phi_0(N)$ is the fundamental domain for $\Gamma_0(N)\backslash\mathbb{H}$. Define $E^*_{\lambda,N}(z,\omega) = E^*_{\lambda,N}(z,0,\omega)$.

The Maaß–Shimura differential operators on the space of smooth functions on the Poincare upper half plane $\mathbb{H}$ by

$$\delta_\lambda = \frac{1}{2\pi i}\left(\frac{\lambda}{2iy}+\frac{\partial}{\partial z}\right),\quad 0<\lambda\in\mathbb{Z},\quad \text{and}\quad \delta_\lambda^{(r)} = \delta_{\lambda+2r-2}\cdots\delta_{\lambda+2}\delta_\lambda,\quad 0\le r\in\mathbb{Z}.$$

It is understood that $\delta_\lambda^{(0)}$ is the identity operator. Define $d = \frac{1}{2\pi i}\frac{\partial}{\partial z} = q\frac{d}{dq}$. Then

$$\delta_\lambda^{(r)} \;=\; \sum_{0\le t\le r}\binom{r}{t}\frac{\Gamma(\lambda+r)}{\Gamma(\lambda+t)}(-4\pi y)^{t-r}d^t, \tag{2.1}$$

where the symbol d^0 means the identity operator; see [24]. The following theorem is one of the main results from *loc.cit.*

Theorem 2.1.1. *Suppose $h'\in S_{k'}(N,\chi')$, $h\in M_k(N,\chi)$, and $k+2r<k'$ with a non-negative integer r. Then*

$$D(k'-1-r,h,h') \;=\; c_r\pi^{k'}\langle h'_\rho,\; h.\delta_\lambda^{(r)}E^*_{\lambda,N}(z,\chi\chi')\rangle,$$

where $\lambda = k'-l-2r$, and c_r is the rational number explicitly given by:

$$c_r \;=\; \frac{\Gamma(k'-k-2r)}{\Gamma(k'-1-r)\Gamma(k'-k-r)}.\frac{(-1)^r4^{k'-1}N}{3}\prod_{p|N}(1+p^{-1}),$$

where the product is taken over all prime factors p of N.

The action of the mass Shimura operator operators on $1/4\pi y$ is given by the following

Lemma 2.1.1.

$$d^n\left(\frac{1}{4\pi y}\right) = \frac{n!}{(4\pi y)^{n+1}}, \quad n = 0, 1, 2, \dots,$$

$$\delta_\lambda^{(r)}\left(\frac{1}{4\pi y}\right) = (-4\pi y)^{-(r+1)} \cdot \sum_{t=0}^{r} \binom{r}{t}(-1)^{t+1}\frac{\Gamma(\lambda+r)\Gamma(t+1)}{\Gamma(\lambda+t)}.$$

Proof. The first is proved by induction on n; the second from the first and (2.1). □

Let A_r denote the set of all functions of the form $H(z) = \sum_{\nu=0}^{r} y^{-nu} g_\nu(z)$ with holomorphic functions g_ν on $\mathbb{H}$ which have Fourier expansions $g_\nu(z) = \sum_{n=0}^{\infty} b_{\nu n} q^n$. Elements of A_r may be called *nearly holomorphic modular forms.* The following lemma is [24, Lemma 7].

Lemma 2.1.2. *For a positive integer $k' > 2r$ and a Dirichlet χ' modulo N, suppose that an element h of A_r satisfies*

$$\begin{aligned} H(\gamma(z))(cz+d)^{-k'} &\in A_r \textit{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \\ H(\gamma(z))(cz+d)^{-k'} &= \chi'(d)H(z) \textit{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \end{aligned} \tag{2.2}$$

Then $H(z) = \sum_{\nu=0}^{r} \delta_{k'-2\nu}^{(\nu)} h_\nu$, with h_ν of $M_{k'-2\nu}(N, \chi)$ being uniquely determined by H.

The following orthogonality relation is [24, Lemma 6].

Lemma 2.1.3. *Suppose $h' \in S_{k'}(N, \chi')$, $h \in M_k(N, \chi^{-1})$, and $k' = k + 2r$ with a positive integer r. Then $\langle h'_\rho, \delta_l^{(r)} g\rangle = 0$.*

As a consequence of the above two lemmas one has:

Lemma 2.1.4. $\langle h'_\rho, \delta_\lambda^{(r)} E_{\lambda,N}\rangle \;=\; \langle h'_\rho, h_0\rangle.$

The following result of Shimura gives the algebraicity of $D(s, h, h')$ at certain integers.

Theorem 2.1.2. *Let f be a normalized eigenform of $S_{k'}(\Gamma_1(N'))$, g an element of $M_k(\Gamma_1(N))$, and m a positive integer. Suppose $k < k'$ and $\frac{1}{2}(k' + k - 2) < m < k'$. Then, the quantity $\pi^{-k'}\langle h', h'\rangle^{-1} D(m, h, h')$ belongs to $\mathbb{Q}(h)\mathbb{Q}(h')$, the compositum of $\mathbb{Q}(h)$ and $\mathbb{Q}(h')$. Moreover, for every automorphism σ of $\mathbb{C}$, we have*

$$\sigma\left(\pi^{-k'}\langle h', h'\rangle^{-1} D(m, h, h')\right) \;=\; \pi^{-k'}\langle h'^\sigma, h'^\sigma\rangle^{-1} D(m, h^\sigma, h'^\sigma).$$

Proof. See Shimura [24, Lemma 4, Lemma 5] □

Recursive relation satisfied the holomorphic projection

When $\lambda = 2$ and $\omega = \mathbb{1}$ is the trivial character modulo N, then $E^*_{\lambda,N}(z,\omega)$ is not holomorphic at ∞; its Fourier expansion at ∞ is given by

$$E^*_{2,N}(z,\mathbb{1}) \;=\; \frac{c}{4\pi y} + \sum_{n=0}^{\infty} c_n q^n,$$

where c and all the c_n are in $\mathbb{Q}$. This is due to Hecke; see, for example, Miyake [15, p. 288].

Let $h' \in S_{k'}(N,\chi')$ and $h \in S_k(N,\chi)$. Then as noted above

$$E^*_{\lambda,N}(z,\chi\chi') \;=\; \frac{c}{4\pi y} + \sum_{n=0}^{\infty} c_n q^n \;=:\; \frac{c}{4\pi y} + E,$$

with $c \neq 0$ only when $\lambda = 2$ and $\chi\chi' = \mathbb{1}$. Then, by Lemma 2.1.1 and Lemma 2.1.2

$$\begin{aligned} h.\delta_\lambda^{(r)} E^*_{\lambda,N} &= g\cdot\delta_\lambda^{(r)}\left(\frac{c}{4\pi y} + E\right) \\ &= h\cdot\frac{c\cdot c'}{(-4\pi y)^{r+1}} + h\cdot\delta_\lambda^{(r)}(E) = \begin{cases} \sum_{\nu=0}^{r+1}\delta_{k'-2\nu}^{(\nu)} h_\nu & \text{if } \chi\chi' = \mathbb{1}, \lambda = 2, \\ \sum_{\nu=0}^{r}\delta_{k'-2\nu}^{(\nu)} h_\nu & \text{otherwise,}\end{cases} \end{aligned} \tag{2.3}$$

where $h_\nu \in M_{k'-2\nu}(N,\chi\chi')$ and $c' \in \mathbb{Q}$ is obtained from Lemma 2.1.1.

Lemma 2.1.5. *The h_ν, as in equation* (2.3)*, satisfy the following recursive relations:*

(i) When $\lambda = 2$ and $\chi\psi =$ trivial then

$$\frac{\Gamma(k-(r+1))}{\Gamma(k-2(r+1))} h_{r+1} = ch\sum_{t=0}^{r}\binom{r}{t}(-1)^{t+1}\frac{\Gamma(\lambda+r)\Gamma(t+1)}{\Gamma(\lambda+t)}$$

and

$$\begin{aligned}\frac{\Gamma(k-\nu)}{\Gamma(k-2\nu)} h_\nu = &\binom{r}{r-\nu}\frac{\Gamma(\lambda+r)}{\Gamma(\lambda+r-\nu)} h d^{r-\nu} E^*_{\lambda,N}(z,\chi\psi) \\ &- \sum_{n=\nu+1}^{r+1}\binom{n}{n-\nu}\frac{\Gamma(k-n)}{\Gamma(k-n-\nu)} d^{n-\nu} h_n \qquad \textit{for } \nu = 0,1,\ldots,r.\end{aligned}$$

(ii) When $\lambda \neq 2$ or $\chi\psi \neq$ trivial then

$$\begin{aligned}\frac{\Gamma(k-\nu)}{\Gamma(k-2\nu)} h_\nu = &\binom{r}{r-\nu}\frac{\Gamma(\lambda+r)}{\Gamma(\lambda+r-\nu)} h d^{r-\nu} E^*_{\lambda,N}(z,\chi\psi) \\ &- \sum_{n=\nu+1}^{r}\binom{n}{n-\nu}\frac{\Gamma(k-n)}{\Gamma(k-n-\nu)} d^{n-\nu} h_n \qquad \textit{for } \nu = 0,1,\ldots,r.\end{aligned}$$

Proof. Statement (ii) is proved in Hida [8, p.177]. For (i), when $\lambda = 2$ and $\chi\psi =$ trivial,

$$h.\delta_\lambda^{(r)} E^*_{\lambda,N} = h \cdot \delta_\lambda^{(r)} \left(\frac{c}{4\pi y} + E \right) = h \cdot \delta_\lambda^{(r)} \left(\frac{c}{4\pi y} \right) + h \cdot \delta_\lambda^{(r)} (E) = \sum_{n=0}^{r+1} \delta_{k-2n}^{(n)} h_n.$$

Use (2.1) and expand both sides of the above equation to get:

$$h \cdot \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(\lambda + r)}{\Gamma(\lambda + t)} (-4\pi y)^{t-r} d^t \left(\frac{c}{4\pi y} + E \right) = \sum_{n=0}^{r+1} \sum_{t=0}^{n} \binom{n}{t} \frac{\Gamma(k-n)}{\Gamma(k-2n+t)} (-4\pi y)^{t-n} (d^t h_n).$$

The left hand side of the equation becomes

$$ch \cdot (-4\pi y)^{-(r+1)} \cdot \sum_{t=0}^{r} \binom{r}{t} (-1)^{t+1} \frac{\Gamma(\lambda + r)\Gamma(t+1)}{\Gamma(\lambda + t)} + h \cdot \delta_\lambda^{(r)} E.$$

By comparing the coefficients of $(-4\pi y)^t$ for $t = 0, 1, \ldots, r+1$, one gets the desired equality. □

Completed Rankin-Selberg L-functions

For normalized newforms, $h' \in S_{k'}(N', \chi')$, and $h \in S_k(N, \chi)$, define the completed Rankin–Selberg L-function (with a 'classical' normalization) as:

$$L(s, h \times h') = L_\infty(s, h \times h') L_M(2s + 2 - k - k', \chi\chi') D(s, h, h'), \tag{2.4}$$

where $L_\infty(s, h \times h') = (2\pi)^{-2s}\Gamma(s)\Gamma(s+1-k)$ is the archimedean factor and for a Dirichlet character ω modulo M put $L_M(s, \omega) = \sum_{n=1}^{\infty} \omega(n) n^{-s}$ with $\omega(n) = 0$ if $(n, M) = 1$. Here M is the least common multiple of N, N'. It satisfies a functional equation

$$L(s, h \times h') \approx L(k + k' - 1 - s, h^\rho, h'^\rho). \tag{2.5}$$

See Shimura [24, Section 3] and Hida [8, Section 9]. The line of symmetry for this functional equation is $\mathrm{Re}(s) = (k + k' - 1)/2$. An integer m is *critical* for $L(s, h \times h')$, if the Gamma factors on both sides of the functional equation are finite at $s = m$, i.e., if $\Gamma(m)\Gamma(m + 1 - k)$ and $\Gamma(k + k' - 1 - m)\Gamma(k - m)$ are finite. Therefore the critical set is:

$$\{m \in \mathbb{Z} \,|\, k \le m \le k' - 1\}.$$

The parameters determining the abelian part and the infinite part of $L(s, h \times h')$ in (2.4) depend only on the weights and the nebentype characters involved. So for two $h', h'' \in S_{k'}(N', \chi')$ and $h \in S_k(N, \chi)$ one has

$$L_\infty(s, h \times h') L_N(2s + 2 - k' - k, \chi\chi') = L_\infty(s, h \times h'') L_N(2s + 2 - k' - k, \chi\chi'). \tag{2.6}$$

For brevity, let $\omega = \chi\chi'$ modulo M. Suppose ω is a Dirichlet character modulo M which is primitive. Then $L_M(s, \omega) = L(s, \omega)$. Let $\epsilon \in \{0, 1\}$ such that $\omega(-1) = (-1)^\epsilon$. For $m \in \mathbb{N}$ and $m \equiv \epsilon \pmod 2$ it is well known that

$$L(m, \omega) = (-1)^{1+(m-\epsilon)/2} \frac{\mathfrak{g}(\omega)}{2\sqrt{-1}^\epsilon} \left(\frac{2\pi}{M}\right)^m \frac{B_{m,\bar{\omega}}}{m!},$$

where $\mathfrak{g}(\omega)$ is the Gauss sum associated to ω and $B_{m,\bar{\omega}}$ is the generalized Bernoulli number for the character $\bar{\omega} = \omega^{-1}$. See Neukirch [18]. From this it follows that

$$\frac{L(m, \omega)}{L(m+2, \omega)} = (m+2)(m+1)\left(\frac{M}{2\pi}\right)^2 \frac{B_{m,\bar{\omega}}}{B_{m+2,\bar{\omega}}}. \tag{2.7}$$

If ω modulo N is not primitive take ω^{prim} to be the primitive Dirichlet character defined by ω. Then $L_N(s, \omega) = \prod_{p|N} \left(1 - \omega^{\text{prim}}(p)p^{-s}\right) L(s, \omega^{\text{prim}})$. Hence, from (2.7) and Theorem 2.1.2, for any integer m with $\frac{k+l-2}{2} < m < m+1 < k$, it follows that:

$$\frac{L(m, h \times h')}{L(m+1, h \times h')} \in \mathbb{Q}(h)\mathbb{Q}(h'). \tag{2.8}$$

2.2 Algorithms

2.2.1 An algorithm for the algebraic part of $D(m, f \times g)$

If $f \in S_{k'}(N, \chi')$ is a newform then $f_\rho = \overline{f(-\overline{z})} = f^\rho$ is again a newform of $S_{k'}(N, \chi'^{-1})$, where, recall that for any automorphism σ of $\mathbb{C}$, one defines $f^\sigma := \sum_{n=1}^\infty a(n, h)^\sigma q^n$.

Input: $f \in S_{k'}(N, \chi)$ a newform of level N, and $g \in S_k(N, \chi)$ an arbitrary cusp form with algebraic Fourier coefficients.

Output: The algebraic number $\frac{D(m, f \times g)}{\pi^k \langle f, f \rangle}$ for $\frac{k'+k-2}{2} < m < k'$.

Step 1. Extend f_ρ to a basis consisting of normalized eigenforms $S_{k'}(\Gamma_1(N))$ which we denote by $\mathcal{B} = \{f_0 := f_\rho, f_1, \ldots, f_n\}$. (The dimension is $n+1$.)

Step 2. Fix an m in the critical set. Put $r = k' - 1 - m$. Pick the correct Eisenstein series $E^*_{\lambda,N}(z, \chi\chi')$ where $\lambda = k' - k - 2r$.

Step 3. Find the holomorphic projection h_0 of $g\delta_\lambda^{(r)} E_{\lambda,N}(z, \chi\chi')$ using the recursive relations in Lemma 2.1.5.

Step 4. Write the modular form h_0 in terms of $\mathcal{B}$, i.e., $h_0 = \alpha_0 f_0 + \alpha_1 f_1 + \cdots + \alpha_n f_n$.

Step 5. **Return:** $c_r \alpha_0$.

This is the desired value due to Lemma 2.1.4, Lemma 2.1.3, Theorem 2.1.1, and $\langle f_0, f_i\rangle = 0$ for $i = 1, \cdots, n$ and $\langle f^\rho, f^\rho\rangle = \langle f, f\rangle$. The constant c_r appears in Theorem 2.1.1.

This algorithm can be extended to the case when f is only assumed to be an eigenform instead of newform. See Section 3.1.4 for an example.

2.2.2 An algorithm for the algebraic part of $D(m, f)$

We describe a well-known algorithm to calculate the algebraic part of the special value of $D(s, f)$ for the standard L-function of a new form f at $s = m$ a critical value. The term algebraic is explained below. We need some preliminaries on computing L-values by modular symbols.

Special values of L-functions via modular symbols

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. Define $\mathbb{M}_2$ to be the free abelian group on the symbols $\{\alpha, \beta\} \in \mathbb{P}^1(\mathbb{Q})$ modulo the relations $\{\alpha, \beta\}+\{\beta, \gamma\}+\{\gamma, \alpha\} = 0$ for all $\alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q})$ and modulo all torsion. Let $\mathbb{Z}[X, Y]_n$ be the abelian group of homogeneous polynomials of degree n in two variables X and Y. Make $\mathbb{Z}[X, Y]_n$ into a Γ-module: if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $P \in \mathbb{Z}[X, Y]_n$ then define $(\gamma P)(X, Y) = P(dX - bY, -cX + aY)$. The abelian group $\mathbb{M}_2$ can also be made into a Γ-module by the action $g\{\alpha, \beta\} := \{g\alpha, g\beta\}$ for all $g \in \Gamma$ and $\{\alpha, \beta\} \in \mathbb{M}_2$. Define the Γ-module $\mathbb{M}_k = \mathbb{Z}[X, Y]_{k-2} \otimes_{\mathbb{Z}} \mathbb{M}_2$ with the Γ acting diagonally.

Definition 2.2.1. *(Modular Symbols) For an integer $k \geq 2$ and a congruence subgroup Γ, the space $\mathbb{M}_k(\Gamma)$ of weight k modular symbols for Γ is the quotient of $\mathbb{M}_k$ by all the relations $\gamma x - x$ for $x \in \mathbb{M}_k$, and $\gamma \in G$ and by any torsion.*

For $P \in \mathbb{Z}[X, Y]_{k-2}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ the associated *Manin symbol* is

$$[P, \gamma] := \gamma(P \otimes \{0, \infty\}) \in \mathbb{M}_k(\Gamma)$$

When $\Gamma = \Gamma_1(N)$ and if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Gamma_1(N)$ are such that $(c, d) \equiv (c', d')$ (mod N), then $[P, \gamma] = [P, \gamma']$. Hence, the Manin symbol $[P, \gamma]$ is determined by P and the lower two entries c, d of the matrix γ. So we take $[P, (c, d)]$ itself to be a Manin symbol. For the following theorem, see, for example, Stein [25, Prop. 8.3].

Theorem 2.2.1. *The Manin Symbols generate $\mathbb{M}_k(\Gamma)$.*

One can define Hecke operators on the space of modular symbols; see Stein [25, Section 8.3]. There is also an action of $\left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right)$ on the space of modular symbols. This action commutes with the Hecke action. Its isotypic component will be denoted by $\mathbb{M}_k(\Gamma)^{\pm}$.

Theorem 2.2.2. *Let $S_k(\Gamma), \bar{S}_k(\Gamma)$ denote the space of holomorphic and anti-holomorphic cusp forms respectively. Then the pairing $(\cdot, \cdot) : (S_k(\Gamma) \oplus \bar{S}_k(\Gamma)) \times \mathbb{M}_k(\Gamma) \to \mathbb{C}$*

$$((f_1, f_2), P \otimes \{\alpha, \beta\}) \mapsto \int_\alpha^\beta f_1(z)P(z, 1)dz + \int_\alpha^\beta f_2(z)P(\bar{z}, 1)d\bar{z}$$

is Hecke equivariant, i.e., for $x \in \mathbb{M}_k(\Gamma)$ *we have* $(T_n(f_1, f_2), x) = (f_1, f_2, T_n(x))$.

For a cusp form $f \in S_k(N)$, the associated L-function is

$$D(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \text{and} \quad \int_0^{\infty} f(it) t^s \frac{dt}{t} = (2\pi)^{-s}\Gamma(s) D(s, f), \quad \Re(s) \gg 0. \tag{2.9}$$

Consider the modular symbols $X^j Y^{k-2-j} \otimes \{0, \infty\}$ for $j = 0, 1, 2, \ldots, k-2$. Then the non-degenerate and Hecke equivariant bilinear pairing $(\cdot, \cdot)$ of these modular symbols against the cusp form f is equal to

$$(f, X^j Y^{k-2-j} \otimes \{0, \infty\}) = \int_0^{i\infty} f(z) z^j dz = i^{j+1} \int_0^{\infty} f(it) t^j dt.$$

Hence, by (2.9), we get the following relation between the bilinear pairing and the special value of L-function. For $m = 1, 2 \ldots, k-1$,

$$D(m, f) = \frac{(-2\pi\sqrt{-1})^m}{(m-1)!} \cdot (f, X^{m-1} Y^{k-2-(m-1)} \otimes \{0, \infty\}). \tag{2.10}$$

For $m = 1, 2 \ldots, k-1$, define algebraic part of $L(m, f)$ by:

$$\frac{D(m, f)}{(-2\pi\sqrt{-1})^{m-1} D(1, f)} = \frac{1}{(m-1)!} \cdot \frac{(f, X^{m-1} Y^{k-2-(m-1)} \otimes \{0, \infty\})}{(f, Y^{k-2} \otimes \{0, \infty\})} \quad \text{for } m = 1, 3, \ldots, \tag{2.11}$$

$$\frac{D(m, f)}{(-2\pi\sqrt{-1})^{m-2} D(2, f)} = \frac{1}{(m-1)!} \cdot \frac{(f, X^{m-1} Y^{k-2-(m-1)} \otimes \{0, \infty\})}{(f, X^1 Y^{k-3} \otimes \{0, \infty\})} \quad \text{for } m = 2, 4, \ldots. \tag{2.12}$$

Here we present the algorithm to calculate the algebraic part of the special values $D(m, f)$.

Input: The first d Fourier coefficients $a(1, f) = 1, a(2, f), \ldots, a(d, f)$ of a normalized eigenform $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ which are assumed to be real. (Here d is the dimension of $S_k(\mathrm{SL}_2(\mathbb{Z}))$.)

Output: $\frac{D(m,f)}{(-2\pi\sqrt{-1})^{m-2} D(2,f)}$ for $m = 2, 4, \ldots, k-2$.

Step 1. Find a basis $b_1^-, b_2^-, \ldots, b_d^-$ for the space $\mathbb{M}_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q})^-$.

Step 2. Express the Manin symbols $(i, 0, 1)$ for i odd and $0 \le i \le k-3$ in terms of the basis $b_1^-, b_2^-, \ldots, b_d^-$. Let $A^- = [a_{ij}^-]$ be the matrix with rational entries such that $(i, 0, 1) = \sum_{j=1}^{d} a_{ji}^- b_j^-$.

Step 3. Compute the matrices $M_2^-, \ldots, M_d^-$ for the action of the Hecke operators $T_2^-, \ldots, T_d^-$ on $\mathbb{M}_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q})^-$ with respect to the basis $b_1^-, b_2^-, \ldots, b_d^-$.

Step 4. Form the matrix

$$\mathcal{L}^- = \begin{pmatrix} {}^tM_2^- - a(2,f)I_d \\ \vdots \\ {}^tM_d^- - a(d,f)I_d \end{pmatrix}.$$

Step 5. Row-reduce the matrix $\mathcal{L}^-$ and pick any non-zero vector $w^- := {}^t(w_1^-, w_2^-, w_3^-, \ldots, w_d^-)$ from the null space of $\mathcal{L}^-$.

Step 6. Compute the $\frac{k-2}{2} \times 1$ column matrix $E := {}^tA^- \cdot w^-$. Let $c_1, c_3, \ldots, c_{k-3}$ be the entries of E.

Step 7. **Return:** $\frac{1}{(i-1)!} \cdot \frac{c_i}{c_1}$ for $i = 1, 3, \ldots, k-3$.

This is the desired output by (2.12) and

$$\frac{(f, X^{i-1}Y^{k-2-(i-1)} \otimes \{0, \infty\})}{(f, Y^{k-2} \otimes \{0, \infty\})} = \frac{c_i}{c_1} \quad \text{for } i = 1, 3, \ldots k-3.$$

See [17, Section 2.3] for a proof of the algorithm and an algorithm for odd critical values. They are originally due to Manin [14]. See Section 3.2.1 for a computational example when the level is not necessarily 1.

Chapter 3

Explicit computations of some L-values

The aim of this chapter is to compute the ratios of Rankin-Selberg L-values for congruent cusp forms and verify if the ratios are congruent. Also, an example is given which uses the second algorithm in the previous chapter. Towards the end a precise conjecture is stated based on the examples.

3.0.1 Sturm's criterion for congruence between cusp forms

To check congruence between cusp forms we will use the following criterion of Sturm [26]; see also Stein [25, p. 173].

Theorem 3.0.1 (Sturm)**.** *Let $\mathfrak{l}$ be a prime ideal of a number field E. Let $f, g \in M_k(\Gamma, \mathcal{O}_E)$ be modular forms of weight k for a congruence subgroup Γ of level N with Fourier coefficients in the ring of integers $\mathcal{O}_E$ of E. Let $m = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. Suppose*

$$a(n, f) \equiv a(n, g) \pmod{\mathfrak{l}}$$

for all

$$n \leq \begin{cases} \frac{km}{12} - \frac{m-1}{N} & \text{if } f - g \in S_k(\Gamma, \mathcal{O}), \\ \frac{km}{12} & \text{otherwise.} \end{cases}$$

Then $f \equiv g \pmod{\mathfrak{l}}$.

The bound for n in the above theorem is called the Sturm bound for Γ and weight k.

3.1 Examples verifying congruence

3.1.1 $S_{24}(\mathrm{SL}_2(\mathbb{Z})) \times S_{12}(\mathrm{SL}_2(\mathbb{Z}))$

The dimension of $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ is 2. There are no old forms as the level is 1. The two distinct newforms, say, h' and h'' have Fourier expansions:

$$h' = q + (\beta_0)q^2 + \cdots ,$$
$$h'' = q + (-\beta_0 + 1080)q^2 + \cdots ,$$

where $\beta_0 = 12\sqrt{144169}$. The coefficients are in the number field $\mathbb{Q}(\beta_0)$. The prime number 144169 is ramified in the number field $\mathbb{Q}(\beta_0)$. Let the prime ideal lying above 144169 be $\mathfrak{l}_{144169}$. Also, $h'_\rho = h'^\rho = h'$ and $h''_\rho = h''^\rho = h''$ as the coefficients are totally real. Their difference is

$$h' - h'' = (2\beta_0 - 1080)q^2 + \cdots .$$

The ideal factorization of $(2\beta_0 - 1080)$ in the number field $\mathbb{Q}(\beta_0)$ is

$$(\left(2, \frac{1}{24}\beta_0 - 23\right))^3 \cdot (\left(2, \frac{1}{24}\beta_0 - 22\right))^3 \cdot (\left(3, \frac{1}{24}\beta_0 - 23\right))$$
$$\cdot (\left(3, \frac{1}{24}\beta_0 - 22\right)) \cdot (\left(144169, \frac{1}{24}\beta_0 + 72062\right)).$$

Therefore, since the Sturm bound is 2 for $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$, by Theorem 3.0.1 one has:

$$h' \equiv h'' \pmod{\mathfrak{l}_{144169}},$$

The space $S_l(\Gamma_0(N)) = S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is of dimension 1 and generated by the unique Ramanujan cusp form Δ. Put $h = \Delta$. The Fourier expansion of h is $q - 24q^2 + \cdots$.

Theorem 3.1.1. *For $h', h'' \in S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ and $h \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ as above, and any integer m with $18 \leq m \leq 22$, one has:*

$$\frac{L(m, h \times h')}{L(m+1, h \times h')} \equiv \frac{L(m, h \times h'')}{L(m+1, h \times h'')} \pmod{\mathfrak{l}_{144169}}.$$

This is verified using the algorithm given in Section 2.2.1.

Values of $D(23, h' \times h)$ and $D(23, h'' \times h)$

As *Step 1*, fix the basis $\mathcal{B} = \{h', h''\}$ of $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ consisting of newforms. Proceed to *Step 2*. The critical point is 23. So put $r = 0$ and $\lambda = k' - k - 2r = 12$. The Eisenstein series to be considered is $E^*_{\lambda,N}(z,\mathbb{1}) = E^*_{12,1}(z,\mathbb{1})$. The space of Eisenstein series of weight 12 and trivial character is of dimension 1. The q-expansion of $E^*_{12,1}$ is

$$E^*_{12,1}(z,\mathbb{1}) = 1 + \frac{65520}{691}q + \frac{134250480}{691}q^2 + \frac{11606736960}{691}q^3 + O(q^4).$$

Furthermore,

$$h.\delta^{(0)}_{12}E^*_{12,1}(z,\mathbb{1}) = h.E^*_{12,1}(z,\mathbb{1}) = q + \frac{48936}{691}q^2 + \frac{132852132}{691}q^3 + O(q^4).$$

Also, $h.E^*_{12,1}(z,\mathbb{1})$ is a modular form (in fact, a cusp form) of weight 24. So there is no need to take the holomorphic projection. Proceed to *Step 4*. In terms of the basis $\mathcal{B} = \{h', h''\}$

$$hg.E^*_{12,1}(z,\mathbb{1}) = \left(-\frac{27017}{2390898696}\beta_0 + \frac{50418272}{99620779}\right)h' + \left(\frac{27017}{2390898696}\beta_0 + \frac{49202507}{99620779}\right)h''.$$

For *Step 5* one gets

$$\frac{D(23, h'\times h)}{\pi^{24}\langle h', h'\rangle} = c_0\frac{\langle h'^{\rho}, g\delta^{(r)}_{\lambda}E^*_{\lambda,N}\rangle}{\pi^{24}\langle h', h'\rangle} = c_0\left(-\frac{27017}{2390898696}\beta_0 + \frac{50418272}{99620779}\right).$$

Similarly, for the pair (h'', h) one gets

$$\frac{D(23, h''\times h)}{\pi^{24}\langle h'', h''\rangle} = c_0\frac{\langle h''^{\rho}, h\delta^{(r)}_{\lambda}E^*_{\lambda,N}\rangle}{\pi^{24}\langle h'', h''\rangle} = c_0\left(\frac{27017}{2390898696}\beta_0 + \frac{49202507}{99620779}\right).$$

Values of $D(22, h'\times h)$ and $D(22, h''\times h)$

As the basis is already fixed proceed to *Step 2*. For the critical point 22 put $r = 1$ and so $\lambda = k' - k - 2r = 10$. So the Eisenstein Series is $E^*_{\lambda,N}(z,\mathbb{1}) = E^*_{10,1}(z,\mathbb{1})$ and its q-expansion is

$$E^*_{10,1}(z,\mathbb{1}) = 1 - 264q - 135432q^2 - 5196576q^3 - 69341448q^4 - 515625264q^5 + O(q^6).$$

Proceeding to *Step 4*, by Lemma 2.1.2, there are unique h_0 and h_1 satisfying

$$g\delta^{(1)}_{10}E^*_{10,1}(z,\mathbb{1}) = h_0 + \delta^{(1)}_{22}h_1.$$

Using the recursive relations in Lemma 2.1.5 calculate the holomorphic projection h_0 :

$$\begin{aligned} h_0 &= -\frac{5}{11}q - \frac{24}{11}q^2 - \frac{977148}{11}q^3 + O(q^4) \\ &= \left(\frac{223}{38060616}\beta_0 - \frac{365440}{1585859}\right)h' + \left(-\frac{223}{38060616}\beta_0 - \frac{355405}{1585859}\right)h''. \end{aligned}$$

Proceeding to the last *Step 6*,

$$\frac{D(22, h' \times h)}{\pi^{24}\langle h', h'\rangle} = \frac{\langle h'^{\rho}, \delta_{10}^{(1)} E^*_{10,1}\rangle}{\pi^{24}\langle h', h'\rangle} = \frac{\langle h', h_0\rangle}{\pi^{24}\langle h', h'\rangle} = c_1\left(\frac{223}{38060616}\beta_0 - \frac{365440}{1585859}\right).$$

Similarly,

$$\frac{D(22, h'' \times h)}{\pi^{24}\langle h'', h''\rangle} = c_1\left(-\frac{223}{38060616}\beta_0 - \frac{355405}{1585859}\right).$$

Ratios of successive critical values

Since $N = 1$ and χ and χ' are trivial, we have $L_N(s, \chi\chi') = \zeta(s)$, the Riemann Zeta function. It is well known that:

$$\zeta(2\cdot 23 - k' - k) = \zeta(10) = \frac{691}{638512875}\,\pi^{12}, \quad \zeta(2\cdot 22 + 2 - k' - k) = \zeta(8) = \frac{1}{93555}\,\pi^{10}.$$

Therefore, for $i = 1, 2$

$$\frac{L_\infty(22, h \times h')\zeta(8)}{L_\infty(23, h \times h')\zeta(10)} = \frac{13650}{83611}.$$

The ratios of the special values of the completed L-functions are

$$\frac{L(22, h \times h')}{L(23, h \times h')} = -\frac{1}{1905750}\beta_0 + \frac{51866}{317625} \quad \text{and} \quad \frac{L(22, h \times h'')}{L(23, h \times h'')} = \frac{1}{1905750}\beta_0 + \frac{51686}{317625}.$$

The ideal factorization of the quantity $\frac{L(22,h'\times h)}{L(23,h\times h')} - \frac{L(22,h\times h'')}{L(23,h\times h'')}$ in the number field $\mathbb{Q}(\beta_0)$ is

$$\left(\left(2, \frac{1}{24}\beta_0 - 23\right)\right)^2 \cdot \left(\left(2, \frac{1}{24}\beta_0 - 22\right)\right)^2 \cdot \left(\left(3, \frac{1}{24}\beta_0 - 23\right)\right)^{-1} \cdot \left(\left(3, \frac{1}{24}\beta_0 - 22\right)\right)^{-1} \cdot$$
$$\left(\left(5, \frac{1}{24}\beta_0 - 21\right)\right)^{-3} \cdot \left(\left(5, \frac{1}{24}\beta_0 - 19\right)\right)^{-3} \cdot \left(\left(7, \frac{1}{24}\beta_0 - 20\right)\right)^{-1} \cdot \left(\left(7, \frac{1}{24}\beta_0 - 18\right)\right)^{-1} \cdot$$
$$\left(\left(11, \frac{1}{24}\beta_0 - 20\right)\right)^{-2} \cdot \left(\left(11, \frac{1}{24}\beta_0 - 14\right)\right)^{-2} \cdot \left(\left(144169, \frac{1}{24}\beta_0 + 72062\right)\right).$$

Hence

$$\frac{L(22, h \times h')}{L(23, h \times h')} \equiv \frac{L(22, h \times h'')}{L(23, h \times h'')} \pmod{\mathfrak{l}_{144169}}.$$

Other ratios

For $m = 21, 20, 19$, the ideal factorization of

$$\frac{L(m, h \times h')}{L(m+1, h \times h')} - \frac{L(m, h \times h'')}{L(m+1, h \times h'')}$$

in the number field $\mathbb{Q}(\beta_0)$ are stated below. The method of computation is along the same lines as in the previous sub-sections.

$$\frac{L(21, h \times h')}{L(22, h \times h')} - \frac{L(21, h \times h'')}{L(22, h \times h'')}$$
$$= (\left(2, \frac{1}{24}\beta_0 - 23\right))^{-1} \cdot (\left(2, \frac{1}{24}\beta_0 - 22\right))^{-1} \cdot (\left(3, \frac{1}{24}\beta_0 - 23\right))^{-3} \cdot (\left(3, \frac{1}{24}\beta_0 - 22\right))^{-3} \cdot$$
$$(\left(5, \frac{1}{24}\beta_0 - 21\right))^{-1} \cdot (\left(5, \frac{1}{24}\beta_0 - 19\right))^{-1} \cdot (\left(7, \frac{1}{24}\beta_0 - 20\right))^{-1} \cdot (\left(7, \frac{1}{24}\beta_0 - 18\right))^{-1} \cdot$$
$$(\left(17, \frac{1}{24}\beta_0 - 21\right))^{-1} \cdot (\left(17, \frac{1}{24}\beta_0 - 7\right))^{-1} \cdot (\left(144169, \frac{1}{24}\beta_0 + 72062\right)),$$

$$\frac{L(20, h \times h')}{L(21, h \times h')} - \frac{L(20, h \times h'')}{L(21, h \times h'')}$$
$$= (\left(2, \frac{1}{24}\beta_0 - 23\right))^{-3} \cdot (\left(2, \frac{1}{24}\beta_0 - 22\right))^{-3} \cdot (\left(3, \frac{1}{24}\beta_0 - 23\right))^{-1} \cdot (\left(3, \frac{1}{24}\beta_0 - 22\right))^{-1} \cdot$$
$$(\left(5, \frac{1}{24}\beta_0 - 21\right))^{-1} \cdot (\left(5, \frac{1}{24}\beta_0 - 19\right))^{-1} \cdot (\left(103, \frac{1}{24}\beta_0 + 18\right))^{-1} \cdot (\left(103, \frac{1}{24}\beta_0 + 40\right))^{-1} \cdot$$
$$(\left(144169, \frac{1}{24}\beta_0 + 72062\right)),$$

and,

$$\frac{L(19, h \times h')}{L(20, h \times h')} - \frac{L(19, h \times h'')}{L(20, h \times h'')}$$
$$= (\left(2, \frac{1}{24}\beta_0 - 23\right))^{-1} \cdot (\left(2, \frac{1}{24}\beta_0 - 22\right))^{-1} \cdot (\left(7, \frac{1}{24}\beta_0 - 20\right))^{-1} \cdot (\left(7, \frac{1}{24}\beta_0 - 18\right))^{-1} \cdot$$
$$(\left(17, \frac{1}{24}\beta_0 - 21\right))^{-1} \cdot (\left(17, \frac{1}{24}\beta_0 - 7\right))^{-1} \cdot (\left(19, \frac{1}{24}\beta_0 - 15\right))^{-1} \cdot (\left(19, \frac{1}{24}\beta_0 - 11\right))^{-1} \cdot$$
$$(\left(144169, \frac{1}{24}\beta_0 + 72062\right)).$$

The critical point $m = 18$

When the critical point under consideration is $m = 18$, the Eisenstein series is no longer holomorphic. So $r = 5$ and $\lambda = 24 - 12 - 2r = 2$; the characters of f, f' and g are all trivial. The correct Eisenstein series is a non-holomorphic one:

$$E^*_{2,1}(z, \mathbb{1}) = \frac{-12}{4\pi y} + 1 - 24q - 72q^2 - \cdots$$

See Miyake [15, Chapter 7]. By using Lemma 2.1.5 the holomorphic projection h_0 to be

$$\begin{aligned} h_0 &= -\frac{5}{24871}q + \frac{10536}{24871}q^2 - \frac{212004}{3553}q^3 + O(q^4) \\ &= \left(\frac{1103}{86055052776}\beta_0 - \frac{385240}{3585627199}\right)h' + \left(-\frac{1103}{86055052776}\beta_0 - \frac{335605}{3585627199}\right)h''. \end{aligned}$$

A similar calculation like in the previous sub-sections yields the ideal factorization of

$$\begin{aligned} &\frac{L(18, h\times h')}{L(19, h\times h')} - \frac{L(18, h\times h'')}{L(19, h\times h'')} \\ &= (\left(3, \frac{1}{24}\beta_0 - 23\right))^{-1} \cdot (\left(3, \frac{1}{24}\beta_0 - 22\right))^{-1} \cdot (\left(5, \frac{1}{24}\beta_0 - 21\right))^{-1} \cdot (\left(5, \frac{1}{24}\beta_0 - 19\right))^{-1} \cdot \\ &(\left(7, \frac{1}{24}\beta_0 - 20\right))^{-1} \cdot (\left(7, \frac{1}{24}\beta_0 - 18\right))^{-1} \cdot (\left(17, \frac{1}{24}\beta_0 - 21\right))^{-1} \cdot (\left(17, \frac{1}{24}\beta_0 - 7\right))^{-1} \cdot \\ &\qquad (\left(144169, \frac{1}{24}\beta_0 + 72062\right)). \end{aligned}$$

This completes the verification of Thm. 3.1.1 for all successive ratios of critical points to the right of the line of symmetry.

3.1.2 $S_{30}(\mathrm{SL}_2(\mathbb{Z})) \times S_{12}(\mathrm{SL}_2(\mathbb{Z}))$

The space $S_{30}(\mathrm{SL}_2(\mathbb{Z}))$ is again of dimension 2. Call the newforms h' and h''. The form h' is a Galois conjugate of h'. The Fourier coefficients of h' and h'' lie in the number field $\mathbb{Q}(\beta_1)$ where β_1 satisfies the polynomial $x^2 - 8640x - 454569984$. The prime 51349 is ramified in $\mathbb{Q}(\beta_1)$. Let $\mathfrak{l}_{51349}$ be the prime ideal lying above 51349. Like in 3.1.1, one verifies that $h' \equiv h''$ (mod $\mathfrak{l}_{51349}$). Take $h = \Delta$ to be Ramanujan cusp form which is of weight 12 and level 1. Then exactly like in the 3.1.1 we get:

Theorem 3.1.2. *For $h', h'' \in S_{30}(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ as above, and for an integer m with $21 \le m \le 28$, one has:*

$$\frac{L(m, h\times h')}{L(m+1, h\times h')} \equiv \frac{L(m, h\times h'')}{L(m+1, h\times h'')} \pmod{\mathfrak{l}_{51349}}.$$

3.1.3 $S_{13}(\Gamma_0(3), \chi) \times S_6(\Gamma_0(3))$

Let χ be the unique quadratic character modulo 3. All the eigenforms in $S_{13}(\Gamma_1(3)) = S_{13}(\Gamma_0(3), \chi')$ are newforms. The q-expansions of the newforms are

$$\begin{aligned} h' &= q + 729q^3 + (4096)q^4 + O(q^5), \\ h'' &= q + \nu_1 q^2 + (-3\nu_1 - 675)q^3 + (-4328)q^4 + O(q^5), \\ h''' &= q + -\nu_1 q^2 + (3\nu_1 - 675)q^3 + (-4328)q^4 + O(q^5), \end{aligned}$$

where $\nu_1 = \sqrt{-8424}$. It can be checked that h'' and h''' are Galois conjugates of each other, and in fact, $h''_\rho = h''^\rho = h'''$ and $h'''_\rho = h'''^\rho = h''$ as the coefficients are in the imaginary quadratic field $\mathbb{Q}(\nu_1)$. However, h' and h'' are *not* Galois conjugate of each other, since the coefficients of h' are in $\mathbb{Q}$; also, $h'_\rho = h'^\rho = h'$, Using the Sturm bound (Thm. 3.0.1) one verifies

$$h' \equiv h'' \pmod{\mathfrak{l}_{13}},$$

where $\mathfrak{l}_{13} = (\left(13, \frac{1}{18}\nu_1\right))$ is the prime ideal lying above $(13) \subset \mathbb{Z}$ in the number field $\mathbb{Q}(\nu_1)$.

The space $S_6(\Gamma_0(3))$ is one-dimensional, spanned by g whose Fourier expansion is

$$h = q - 6q^2 + 9q^3 + 4q^4 + O(q^5).$$

Ratios of successive critical values

Theorem 3.1.3. *For $h', h'' \in S_{13}(\Gamma_0(3), \chi')$ and $h \in S_6(\Gamma_0(3))$ as above, and for an integer m with $9 \leq m \leq 11$, one has:*

$$\frac{L(m, h \times h')}{L(m+1, h \times h')} \equiv \frac{L(m, h \times h'')}{L(m+1, h \times h'')} \pmod{\mathfrak{l}_{13}}.$$

The ratios of the Rankin-Selberg L-values are calculated exactly as in the previous subsections. The reader should keep in mind that $h''^\rho = h'''$, $h'''^\rho = h''$, $\langle h''', h''' \rangle = \langle h'''^\rho, h'''^\rho \rangle = \langle h'', h'' \rangle$, and $L_N(s, \chi'\chi) = L(s, \chi')$ is the Dirichlet L-function for the unique quadratic character modulo 3. The ideal factorization of the difference of the ratios $\frac{L(m,h\times h')}{L(m+1,h\times h')} - \frac{L(m,h\times h'')}{L(m+1,h\times h'')}$, for $m = 11$, in the number field $\mathbb{Q}(\nu_1)$ is given by:

$$\frac{L(11, h \times h')}{L(12, h \times h')} - \frac{L(11, h \times h'')}{L(12, h \times h'')} = (\left(2, \frac{1}{18}\nu_1\right))^2 \cdot (\left(3, \frac{1}{18}\nu_1 + 1\right))^2 \cdot (\left(3, \frac{1}{18}\nu_1 + 2\right))$$
$$\cdot (\left(5, \frac{1}{18}\nu_1 + 2\right))^{-1} \cdot (\left(5, \frac{1}{18}\nu_1 + 3\right))^{-1} \cdot (11)^{-1} \cdot (\left(13, \frac{1}{18}\nu_1\right)) \cdot (\left(71, \frac{1}{18}\nu_1 + 20\right))^{-1}.$$

So they are congruent modulo $\mathfrak{l}_{13}$; the same conclusion holds for the other ratios as well.

3.1.4 $S_{26}(\mathrm{SL}_2(\mathbb{Z})) \times S_{13}(\Gamma_0(3), \chi')$

Unlike the previous examples, here $k > k'$ and the cusp form of higher weight k is fixed, and the modular forms of lower weight k' vary in a congruence class.

For the weight $k = 26$ cusp form, take $h \in S_{26}(\mathrm{SL}_2(\mathbb{Z}))$ with q-expansion

$$h = q - 48q^2 - 195804q^3 - 33552128q^4 - 741989850q^5 + 9398592q^6 + O(q^7).$$

and view it as an element of $S_{26}(\Gamma_1(3))$.

We take two cusp forms of weight 13 and level 3. In order to have consistency of notations we call the nebentypus character, which is the non-trivial primitive character modulo 3, to be χ'. It is already established that $h' \equiv h'' \pmod{\mathfrak{l}_{13}}$, where $\mathfrak{l}_{13}$ is the prime ideal lying above $(13) \subset \mathbb{Z}$ in $\mathbb{Q}(\nu_1)$, where $\nu_1^2 = -8424$.

As a Hecke module, the 7-dimensional space $S_{26}(\Gamma_1(3))$ decomposes into a sum of:

1. a 2-dimensional subspace with a basis of the old forms h and $\hat{h} := \langle 3 \rangle \cdot f = f(3z)$,
2. a 2-dimensional subspace with a basis consisting of a newform and its nontrivial Galois conjugate,
3. a 3-dimensional subspace with a basis consisting of a newform and its two distinct Galois conjugates.

This data is obtained from the L-functions and modular forms database LMFDB [**?**]. As there are in total 5 newforms, finding an extension of $\mathbb{Q}(\nu_1)$ containing all the Fourier coefficients of such newforms is computationally taxing. Instead, as a work around, we compute a basis consisting of Fourier coefficients in $\mathbb{Q}$, for the space in (2) and for the space in (3). This can be achieved in SAGE by applying the `decomposition()` command on any Hecke operator away from the level.

If h_0 is the holomorphic projection and $h_0 = c_0 f + c_1 \hat{f} + \cdots \in S_{26}(\Gamma_1(3))$ then it is *no longer true* that $\langle h, h_0 \rangle = c_0 \langle h, h \rangle$, since $\langle h, \hat{h} \rangle \neq 0$. Hence $c_r \alpha_0$ is *not* the correct L-value. So while returning the L-value this has to be taken into account. The relation between $\langle h, h \rangle$ and $\langle h, \hat{h} \rangle$ is as follows: Suppose $h \in S_k(\Gamma_1(N))$ be a newform. Let $a(p, h)$ be its coefficient at p. In $S_k(\Gamma_0(Np))$, taking $\hat{h}(z) = h(pz)$ one has:

$$\langle \hat{h}, h \rangle_{\Gamma_0(Np)} = \langle h(pz), h(z) \rangle_{\Gamma_0(Np)} = \langle h(z), h(pz) \rangle_{\Gamma_0(Np)} = \frac{p^{-k+1} a(p, h)}{1 + p} \langle h, h \rangle_{\Gamma_0(Np)}.$$

See Bellaïche [2, p. 284].

Ratios of special values

In this example, we verify the congruence in all cases except one recorded in the following theorem:

Theorem 3.1.4. *For $h \in S_{26}(\mathrm{SL}_2(\mathbb{Z}))$ and $h', h'' \in S_{13}(\Gamma_0(3), \chi')$ as above, and an integer m with $19 \leq m \leq 23$, one has:*

$$\frac{L(m, h \times h')}{L(m+1, h \times h')} \equiv \frac{L(m, h \times h'')}{L(m+1, h \times h'')} \pmod{\mathfrak{l}_{13}}.$$

For the ratios of the right extreme critical values, one has for the Rankin–Selberg convolutions, the congruence:

$$\frac{D(24, h \times h')}{D(25, h \times h')} \equiv \frac{D(24, h \times h'')}{D(25, h \times h'')} \pmod{\mathfrak{l}_{13}},$$

however, for the ratios of completed L-values, one has:

$$\frac{L(24, h \times h')}{L(25, h \times h')} \not\equiv \frac{L(24, h \times h'')}{L(25, h \times h'')} \pmod{\mathfrak{l}_{13}}.$$

For $19 \le m \le 23$, the congruence is verified as in the previous sections. In the exceptional case $m = 24$, the non-congruence for the ratios of extreme critical values is explained like this. The character of $f \in S_k(\Gamma_1(3))$ is trivial modulo 3 and the character of $g_1, g_2 \in S_k(\Gamma_0(3), \chi')$ is the non-trivial quadratic character modulo 3. So

$$\frac{L_\infty(24, h \times h')L_N(2 \cdot 24 + 2 - 26 - 13, \chi\chi')}{L_\infty(25, h \times h')L_N(2 \cdot 25 + 2 - 26 - 13, \chi\chi')} = \frac{L_\infty(24, h \times h'')L(11, \chi')}{L_\infty(25, h \times h'')L(13, \chi')} = \frac{60951}{444808}.$$

The first few prime ideals (written in the increasing order of their norms) appearing in the ideal factorization of the quantity $\frac{L(24,h\times h')}{L(25,h\times h')} - \frac{L(24,h\times h'')}{L(25,h\times h'')}$ in the number field $\mathbb{Q}(\nu_1)$ is

$$(\left(2, \frac{1}{18}\nu_1\right))^{-9} \cdot (\left(3, \frac{1}{18}\nu_1 + 1\right))^{2} \cdot (\left(5, \frac{1}{18}\nu_1 + 2\right))^{-1} \cdot (\left(5, \frac{1}{18}\nu_1 + 3\right))^{-1}$$
$$\cdot(\left(7, \frac{1}{18}\nu_1 + 3\right))^{-2}\cdot(\left(7, \frac{1}{18}\nu_1 + 4\right))^{-2}\cdot(\left(13, \frac{1}{18}\nu_1\right))^{-2}\cdot(19)^{-1}\cdot(\left(31, \frac{1}{18}\nu_1 + 6\right))^{-1}\cdots.$$

Hence

$$\frac{L(24, h \times h')}{L(25, h \times h')} \not\equiv \frac{L(24, h \times h'')}{L(25, h \times h'')} \pmod{\mathfrak{l}_{13}}. \tag{3.1}$$

In fact,

$$v_{\mathfrak{l}_{13}}\left(\frac{D(24, h \times h')}{D(25, h \times h')} - \frac{D(24, h \times h'')}{D(25, h \times h'')}\right) = 1. \tag{3.2}$$

The reason for non-congruence is due to the ratio $\frac{L(11,\chi')}{L(13,\chi')}$ of the abelian L-functions. Since $L(13, \chi)$ has in it the generalized Bernoulli number $B_{13,\chi'} = -1445626/3 = -1 \cdot 2 \cdot 3^{-1} \cdot 7 \cdot 13^3 \cdot 47$, the congruence modulo $\mathfrak{P}_{13}$ in (3.2) gets 'cancelled' by the 13^3 in the denominator of $\frac{L(11,\chi')}{L(13,\chi')}$ explaining the non-congruence in (3.1); see also Section 6.3.6.

3.1.5 $S_{24}(\mathrm{SL}_2(\mathbb{Z})) \times M_{12}(\mathrm{SL}_2(\mathbb{Z}))$

We may also vary the modular forms in a congruence class with one being cuspidal and the other an Eisenstein series. In this section, we vary the lower weight modular forms of weight $k' = 12$, with one form being the Ramanujan Δ-function, and the other form the weight 12 Eisenstein series. More precisely, let h be the newform of weight $k' = 24$ for $\mathrm{SL}_2(\mathbb{Z})$ with q-expansion $h = q + \beta_0 q^2 + (-48\beta_0 + 195660)\, q^3 + O(q^4)$, where β_0 satisfies the polynomial $x^2 - 1080x - 20468736 \in \mathbb{Q}[x]$. This is one of the newforms from Section 3.1.1. For the smaller weight $l = 12$, take $h' := \Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ and $h'' := E_{12}(z) = 691/65520 + q + \cdots \in M_{12}(\mathrm{SL}_2(\mathbb{Z}))$, the Eisenstein series of weight 12 and level 1. The well-known Ramanujan congruence asserts that $h' \equiv h'' \pmod{691}$.

Theorem 3.1.5. *For* $f \in S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ *and* $g_1, g_2 \in M_{12}(\mathrm{SL}_2(\mathbb{Z}))$ *as above*

$$\frac{L(m, h \times h')}{L(m+1, h \times h')} \equiv \frac{L(m, h \times h'')}{L(m+1, h \times h'')} \pmod{\mathfrak{l}_{691}} \text{ for } m = 22, 21, 20, 19, 18.$$

The p-th Fourier coefficient of E_{12} being $p^{11} + 1$, one has $L(s, h \times E_{12}) = L(s, h)L(s + 11, h)$, it is probably more enlightening to write the above congruence as:

$$\frac{L(m, h \times \Delta)}{L(m+1, h \times \Delta)} \equiv \frac{L(m, h)L(m-11, h)}{L(m+1, h)L(m-10, h)} \pmod{\mathfrak{l}_{691}}$$

In a similar vein, by Shimura [**?**, Lem. 1], one has:

$$D(s, h \times h'') = \frac{D(s, h) \cdot D(s-11, h)}{\zeta(2s-34)},$$

where $\zeta(s)$ is the Riemann zeta function and $D(s, f) = \sum_{n=1}^{\infty} a(n, f)n^{-s}$. Therefore,

$$\begin{aligned} \frac{D(m, h, h'')}{D(m+1, h, h'')} &= \frac{\zeta(2m+2-34)}{\zeta(2m-34)} \frac{D(m, h)}{D(m-10, h)} \frac{D(m-11, h)}{D(m+1, h)} \\ &= \left(\frac{1}{(2\pi\sqrt{-1})^2} \frac{\zeta(2m+2-34)}{\zeta(2m-34)} \right) \cdot \left(\frac{D(m, h)}{(2\pi\sqrt{-1})^{-1} D(m-10, h)} \right) \\ &\quad \cdot \left(\frac{D(m-11, h)}{(2\pi\sqrt{-1})^{-1} D(m+1, h)} \right). \end{aligned}$$

The first term inside $(\cdots)$ is rational due to well known result on the special values of the Riemann zeta function. The second and the third term are algebraic due to equations (2.11) and (2.12). Hence we use the algorithm described in Section 2.2.2 to calculate the ratios of the special values $\frac{L(m, h \times h'')}{L(m+1, h \times h'')}$. The ratios $\frac{D(m,f)}{(-2\pi\sqrt{-1})^{m-1} D(1,f)}$ and $\frac{D(m,h)}{(-2\pi\sqrt{-1})^{m-2} D(2,h)}$ depending on the parity of m are summarized in the table.

m	$\frac{D(m,h)}{(-2\pi\sqrt{-1})^{m-1}D(1,h)}$
1	1
3	$-\frac{569}{18825760800}\beta_0-\frac{16757416}{196101675}$
5	$\frac{3403}{218993544000}\beta_0+\frac{40511069}{4562365500}$
7	$-\frac{147089}{17869873190400}\beta_0-\frac{854671739}{744578049600}$
9	$\frac{6538127}{1250891123328000}\beta_0+\frac{10060850717}{52120463472000}$
11	$-\frac{320477}{77436117158400}\beta_0-\frac{169463087}{3226504881600}$
13	$\frac{320477}{77436117158400}\beta_0+\frac{169463087}{3226504881600}$
15	$-\frac{6538127}{1250891123328000}\beta_0-\frac{10060850717}{52120463472000}$
17	$\frac{147089}{17869873190400}\beta_0+\frac{854671739}{744578049600}$
19	$-\frac{3403}{218993544000}\beta_0-\frac{40511069}{4562365500}$
21	$\frac{569}{18825760800}\beta_0+\frac{16757416}{196101675}$
23	-1

m	$\frac{D(m,h)}{(-2\pi\sqrt{-1})^{m-2}D(2,h)}$
2	1
4	$-\frac{1}{15120000}\beta_0-\frac{59221}{630000}$
6	$\frac{1}{26593920}\beta_0+\frac{12031}{1108080}$
8	$-\frac{1783}{80372736000}\beta_0-\frac{5312293}{3348864000}$
10	$\frac{37}{2344204800}\beta_0+\frac{31327}{97675200}$
12	$-\frac{847}{60279552000}\beta_0-\frac{371437}{2511648000}$
14	$\frac{37}{2344204800}\beta_0+\frac{31327}{97675200}$
16	$-\frac{1783}{80372736000}\beta_0-\frac{5312293}{3348864000}$
18	$\frac{1}{26593920}\beta_0+\frac{12031}{1108080}$
20	$-\frac{1}{15120000}\beta_0-\frac{59221}{630000}$
22	1

From the table above one concludes

$$\frac{L(22,h\times h'')}{L(23,h\times h'')}=-\frac{19153}{3052249200}\beta_0+\frac{23359724}{63588525}.$$

Similarly, for the other ratios in the right hand side of Thm. 3.1.5 are determined. For the left hand side of Thm. 3.1.5, note that $\frac{D(m,h\times h')}{\pi^{24}\langle h\times h\rangle}$ and the ratios of successive critical values $\frac{L(m,h\times h')}{L(m+1,h\times h')}$ have already been calculated in Sect. 3.1.1. The prime ideal factorization of the difference of the ratios has the following shape:

$$\frac{L(22,h\times h')}{L(23,h\times h')}-\frac{L(22,h\times h'')}{L(23,h\times h'')}=\cdots\left(\left(13,\frac{1}{24}\beta_0-20\right)\right)^{-1}\cdot\left(\left(73,\frac{1}{24}\beta_0-6\right)\right)$$
$$\cdot\left(\left(691,\frac{1}{24}\beta_0-12\right)\right)\cdot\left(\left(691,\frac{1}{24}\beta_0+658\right)\right)\cdot\left(\left(23003,\frac{1}{24}\beta_0+2325\right)\right).$$

The only prime that ramifies in $\mathbb{Q}(\beta_0)$ is 144169. The ideal $\mathfrak{l}_{691}:=(691)\subset\mathbb{Q}(\beta_0)$ is split and has the factorization $\left(\left(691,\frac{1}{24}\beta_0-12\right)\right)\cdot\left(\left(691,\frac{1}{24}\beta_0+658\right)\right)$. Hence

$$\frac{L(22,h\times h')}{L(23,h\times h')}\equiv\frac{L(22,h\times h'')}{L(23,h\times h'')}\pmod{\mathfrak{l}_{691}}.$$

Same conclusion holds for other ratios as well.

3.2 An example for the algebraic part of $D(m, f)$

3.2.1 $S_5(\Gamma_1(4))$

In this section we shall see the second algorithm in action when the level is not necessarily 1.

Theorem 3.2.1. *Let $f \in S_5(\Gamma_1(4))$ be a newform with q-expansion $f(q) = q - 4q^2 + 16q^4 + \cdots$. Then*

$$\frac{D(4,f)}{D(2,f)} = \frac{\pi^2}{5} \quad \textit{and} \quad \frac{D(3,f)}{D(1,f)} = \frac{5\pi^2}{12}.$$

Proof. The space of modular symbols is generated by the following Manin basis

$$e_1 := [X^3, (0,3)],\ e_2 := [X^3, (1,2)],\ e_3 := [X^3, (2,3)],\ e_4 := [X^3, (3,3)].$$

Express the modular symbols $b_1 := [Y^3, (0,1)],\ b_2 := [XY^2, (0,1)],\ b_3 := [X^2Y, (0,1)],\ b_4 := [X^3, (0,1)]$ in terms of the Manin symbols.

$$b_1 = 0e_1 + \frac{1}{3}e_2 + \frac{8}{3}e_3 + \frac{4}{3}e_4,\ \ b_2 = 0e_1 + 0e_2 - \frac{2}{3}e_3 + 0e_4,$$
$$b_3 = 0e_1 - \frac{1}{9}e_2 - \frac{2}{9}e_3 - \frac{1}{9}e_4,\ \ b_4 = -e_1 + 0e_2 + 0e_3 + 0e_4.$$

Note that we write the modular symbols $X^{j-1}Y^{k-2-(j-1)} \otimes \{0, \infty\}$ equivalently as the Manin symbol $[X^{j-1}Y^{k-2-(j-1)}, (0,1)]$.

So the matrix A in algorithm in Section 2.2.2 in our case is

$$A = \begin{pmatrix} 0 & 1/3 & 8/3 & 4/3 \\ 0 & 0 & -2/3 & 0 \\ 0 & -1/9 & -2/9 & -1/9 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

From the previous equations one obtains

$$\begin{aligned} (f, b_1) &= 0(f, e_1) + \frac{1}{3}(f, e_2) + \frac{8}{3}(f, e_3) + \frac{4}{3}(f, e_4), \\ (f, b_2) &= 0(f, e_1) + 0(f, e_2) - \frac{2}{3}(f, e_3) + 0(f, e_4), \\ (f, b_3) &= 0(f, e_1) - \frac{1}{9}(f, e_2) - \frac{2}{9}(f, e_3) - \frac{1}{9}(f, e_4), \\ (f, b_4) &= -(f, e_1) + 0(f, e_2) + 0(f, e_3) + 0(f, e_4). \end{aligned} \tag{3.4}$$

The matrix of the Hecke operator T_2 in terms of the Manin basis is

$$B = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & -3 & 0 & -1 \\ 4 & -8 & -4 & 8 \\ 0 & -4 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

It is known that $T_2(f) = a(p, f)f = -4f$. Invoking the Hecke equivariance of the pairing $(\cdot, \cdot)$ in Theorem 2.2.2 and using the first equation in (3.4) we get

$$\begin{aligned} &-4(f, e_1) = (-4f, e_1) = (T_2(f), e_1) = (f, T_2(e_1)) = (f, 16e_1 + 4e_3) \\ &\qquad\qquad \Longrightarrow\ 20(f, e_1) + 4(f, e_3) = 0 \ \text{ (Due to the bilinearity of the pairing } (\cdot,\cdot)) \end{aligned}$$

The relation $T_2(e_1) = 16e_1 + 4e_3$ is due to the first column in (3.5). Similarly,

$$\begin{pmatrix} 20 & 0 & 4 & 0 \\ 0 & 1 & -8 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 8 & 4 \end{pmatrix} \begin{pmatrix} (f, e_1) \\ (f, e_2) \\ (f, e_3) \\ (f, e_4) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.6}$$

The matrix $B - a(2, f)I_4$ is nothing but the one in (3.6).

$$B - a(2, f)I_4 = \begin{pmatrix} 20 & 0 & 4 & 0 \\ 0 & 1 & -8 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 8 & 4 \end{pmatrix}.$$

Combine the equations (3.4) and (3.6) to get

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{-1}{3} & \frac{-8}{3} & \frac{-4}{3} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{1}{9} \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 8 & 4 \end{pmatrix} \begin{pmatrix} (f, b_1) \\ (f, b_2) \\ (f, b_3) \\ (f, b_4) \\ (f, e_1) \\ (f, e_2) \\ (f, e_3) \\ (f, e_4) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The above system of equations after row reduction is equivalent to solving the following system of equations.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{-16}{3} & \frac{-8}{3} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{10}{9} & \frac{5}{9} \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (f,b_1) \\ (f,b_2) \\ (f,b_3) \\ (f,b_4) \\ (f,e_1) \\ (f,e_2) \\ (f,e_3) \\ (f,e_4) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} (f,b_1) - \frac{16}{3}(f,e_3) - \frac{8}{3}(f,e_4) &= 0, \\ (f,b_2) + \frac{2}{3}(f,e_3) &= 0, \\ (f,b_3) + \frac{10}{9}(f,e_3) + \frac{5}{9}(f,e_4) &= 0, \\ (f,b_4) - \frac{1}{5}(f,e_3) &= 0. \end{aligned}$$

Assuming $(f,e_3) \neq 0$ and $2(f,e_3) + (f,e_4) \neq 0$ we get

$$\frac{(f,b_4)}{(f,b_2)} = \frac{-3}{10} \text{ and } \frac{(f,b_3)}{(f,b_1)} = \frac{-5}{24}.$$

The claim follows from the equations (2.11) and (2.12). □

If the algorithm is implemented in SAGE one might get a different Manin basis but the ratios will be the same.

3.3 A conjecture on the Rankin-Selberg L-functions

Recall, we wanted to investigate whether

$$h' \equiv h'' \pmod{\mathfrak{l}} \implies \frac{L(m, h \times h')}{L(m+1, h \times h')} \equiv \frac{L(m, h \times h'')}{L(m+1, h \times h'')} \pmod{\mathfrak{l}}?$$

Keeping the above examples in mind which lend credence to this principle, while accounting for the exceptional situation as in Thm. 3.1.4, we formulate the following conjecture:

Conjecture 3.3.1. *Let $h', h'' \in S_{k'}(N, \chi')$ and $h \in S_k(N, \chi)$ be normalized newforms with $k' - k > 2$ (resp., $k - k' > 2$) depending on whether $k' > k$ (resp., $k > k'$.) Let $\mathfrak{l} \subset \mathbb{Q}(h)\mathbb{Q}(h')\mathbb{Q}(h'')$ be a prime ideal such that*

$$a(n, h') \equiv a(n, h'') \pmod{\mathfrak{l}} \quad \textit{for all } n \in \mathbb{N}.$$

For an integer m with $\frac{k'+k}{2} - 1 < m \leq k' - 2$ (resp. $\frac{k'+k}{2} - 1 < m \leq k - 2$), assume:

$$v_{\mathfrak{P}}\left(\frac{L_\infty(m, h \times h')L_M(2m + 2 - k' - k, \chi\chi')}{L_\infty(m + 1, h \times h')L_M(m + 4 - k' - k, \chi\chi')}\right) \geq 0. \tag{3.7}$$

Then for all such m one has the congruence:

$$\frac{L(m, h \times h')}{L(m + 1, h \times h')} \equiv \frac{L(m, h \times h'')}{L(m + 1, h \times h'')} \pmod{\mathfrak{l}}. \tag{3.8}$$

In the preceding section this was verified in many examples. Especially, note that the hypothesis (3.7) has been introduced to take care of the exceptional situation in Theorem 3.1.4; see also Section 6.3.6 for another explanation. In the upcoming chapters notations will be set-up and some classical results will be recalled. In the final chapter, a variation of the conjecture will be proved using the machinery of Eisenstein Cohomology developed by Harder and Raghuram [7].

Chapter 4

Cohomology groups and automorphic L-functions

In this chapter we set-up some notations and recall some classical results.

4.1 Preliminaries

Throughout $h', h'' \in S_{k'}(N', \chi')^{\text{new}}$ and $h \in S_k(N, \chi)^{\text{new}}$ are newforms. Fix a large number field E containing the Fourier coefficients of h, h', h''. The ideal $\mathfrak{l}$ in the ring of integers $\mathcal{O}_E$ of E such that $h' \equiv h'' \pmod{\mathfrak{l}}$. We also make the assumption that $l \geq 5$, where $l = \mathfrak{l} \cap \mathbb{Z}$.

Let $E_{\mathfrak{l}}$ denote the $\mathfrak{l}$-adic completion of E and $\mathcal{O}_{\mathfrak{l}}$ the ring of integers of $E_{\mathfrak{l}}$. Fix $\iota : \hat{\bar{E}}_{\mathfrak{l}} \cong \mathbb{C}$ an embedding throughout. Suppose V is any $\mathcal{O}_E, \mathcal{O}_{\mathfrak{l}}, E, E_{\mathfrak{l}}$ module then $V \otimes \mathbb{C}$ will mean $V \otimes_\iota \mathbb{C}$. We suppress the notation ι when context is clear.

4.1.1 Basic notations

Let G_n be the algebraic group $\mathrm{GL}_n/\mathbb{Q}$ with the chain of subgroups

$$G_n \supset B_n = T_n U_n \supset T_n \supset Z_n,$$

where B_n is the upper triangular Borel subgroup, T_n the torus consisting of all the diagonal matrices, U_n the unipotent radical of the Borel, and Z_n the center of G_n. Let $X^*(T_n)$ denote the group of characters of T_n which is free abelian group on the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n$, where $\mathbf{e}_i : \mathrm{diag}(t_1, t_2, \ldots, t_n) \mapsto t_i$ for $i = 1, 2, \ldots, n$. Then $\boldsymbol{\delta}_n = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n$ is the determinant character. Let $\gamma_1, \gamma_2, \ldots, \gamma_{n-1} \in X^*_{\mathbb{Q}}(T_n) = X^*(T_n) \otimes \mathbb{Q}$ be the fundamental dominant weights with respect to the usual pairing. Let Δ_n denote the set of roots and Δ_n^+ is the subset of positive roots with respect to B_n. Let Π_n denote the set of simple roots $\mathbf{e}_i - \mathbf{e}_{i+1}$ for $i = 1, \cdots, n-1$. Let W_n denote the Weyl group of G_n which will be identified with the set of permutation matrices.

We will be using rank-one Eisenstein cohomology when the ambient group is $G_4 = \mathrm{GL}_4/\mathbb{Q}$. In that context, we will let P denote the standard parabolic subgroup of G_4 corresponding to the deletion of the simple root $\mathbf{e}_2 - \mathbf{e}_3$ of Π_4. Then P is the $(2,2)$ block upper matrices in G_4. Let U_P denote the unipotent radical of P and $\kappa : P \to P/U_P \cong M_P$ be the projection onto the Levi quotient $M_P \cong G_2 \times G_2$. (The notation G_2 standing for $\mathrm{GL}_2/\mathbb{Q}$ will cause absolutely no confusion as we won't use any exceptional group.) The simple roots of M_P are $\Pi_{M_P} = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3 - \mathbf{e}_4\}$. The Weyl group of the Levi is denoted by W^{M_P} which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and realized as a subgroup of the Weyl group W_4 of G_4.

The set $W^P = \{w \in W_4 \mid w^{-1}\alpha > 0, \forall \alpha \in \Pi_{M_P}\}$ is the set of Kostant representatives which is in bijection with the right cosets $W_M \backslash W_4$; there are six elements in W^P. They are enumerated in Section 5.1.

4.1.2 Sheaves and cohomology

Locally symmetric spaces

Let $\mathbb{A}$ denote the ring of adeles and $\mathbb{A}_f \subset \mathbb{A}$ denote the finite adeles. If $\mathrm{SO}(n)$ denotes the standard special orthogonal group as a subgroup of $G_n(\mathbb{R})$, then let $K_{n,\infty} = \mathrm{SO}(n) \times Z_n(\mathbb{R})^0$, where $(\cdot)^0$ denotes the connected component of the group. Let $K_f \subset G_n(\mathbb{A}_f)$ be an open compact subgroup. The adelic locally symmetric space is

$$S^{(n)}_{K_f} := G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}) / K_{n,\infty} \cdot K_f.$$

Highest weight representations

A weight $\mu = b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n$ is integral if $b_1, \ldots, b_n \in \mathbb{Z}$, and dominant if $b_1 \geq \cdots \geq b_n$. For such a dominant integral weight, let $\mathcal{M}_\mu$ denote the finite dimensional irreducible representation of G_n with highest weight μ which is defined over $\mathbb{Q}$. For $A = E, E_\mathfrak{l}, \mathbb{C}$, define $\mathcal{M}_{\mu,A} := \mathcal{M}_{\mu,E} \otimes A$. For an integer $m \in \mathbb{Z}$ put $\mu(m) = \mu + m\boldsymbol{\delta}_n$; then $\mathcal{M}_{\mu(m)} = \mathcal{M}_\mu \otimes \det^m$.

Sheaves and their cohomology

Let $\pi : G_n(\mathbb{A})/K_{n,\infty} \times K_f \to S^{(n)}_{K_f}$ be the projection. For $A = E, E_\mathfrak{l}, \mathbb{C}$, define a sheaf $\widetilde{\mathcal{M}}_{\mu,A}$ whose sections over an open $U \subset S^{(n)}_{K_f}$ is given by:

$$\widetilde{\mathcal{M}}_{\mu,A}(U) = \{s : \pi^{-1}(U) \to \mathcal{M}_{\mu,A} \mid s(\gamma.g) = \gamma s(g)\ \forall \gamma \in G_2(\mathbb{Q})\}.$$

Let $H^\bullet(S^{(n)}_{K_f}, \widetilde{\mathcal{M}}_{\mu,E})$ denote the sheaf cohomology groups. For $A = E, E_\mathfrak{l}, \mathbb{C}$, one has isomorphisms for changing the base: $H^\bullet(S^{(2)}_{K_f}, \widetilde{\mathcal{M}}_{\mu,A}) \cong H^\bullet(S^{(2)}_{K_f}, \widetilde{\mathcal{M}}_{\mu,E}) \otimes A$. If $K_{1,f} \subset K_{2,f}$ then there is a natural map $H^\bullet(S^{(n)}_{K_{1,f}}, \widetilde{\mathcal{M}}_{\mu,E}) \to H^\bullet(S^{(n)}_{K_{2,f}}, \widetilde{\mathcal{M}}_{\mu,E})$, letting us define

$$H^\bullet(S^{(n)}, \widetilde{\mathcal{M}}_{\mu,E}) = \mathrm{colim}_{K_f} H^\bullet(S^{(n)}_{K_f}, \widetilde{\mathcal{M}}_{\mu,E}).$$

The mirahoric congruence subgroups in G_2 and G_4

For a prime p and an integer $n_p \geq 0$, let

$$K_p(n_p) \ := \ \{g \in \mathrm{GL}_2(\mathbb{Z}_p) \mid g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{p^{n_p}}\}.$$

For a positive integer N, put $n_p = v_p(N)$ and let $K_1(N)$ be the subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ defined as $K_1(N) = \prod_{p<\infty} K_p(n_p)$. Similarly, in G_4, define

$$K_p^{n_p} \ := \ \left\{ g \in \mathrm{GL}_4(\mathbb{Z}_p) \mid g \equiv \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod{p^{n_p}} \right\}.$$

and define K_f^N to be the subgroup of $\mathrm{GL}_4(\hat{\mathbb{Z}})$ by $K_f^N = \prod_{p<\infty} K_p^{n_p}$.

Inner cohomology of $\mathrm{GL}_2/\mathbb{Q}$

For the level structure $K_1(N)$ in G_2, abbreviate $S_1^{(2)}(N) := S^{(2)}_{K_1(N)}$. For $A = E, E_{\mathfrak{l}}, \mathbb{C}$, let $H_!^1(S_1^{(2)}(N), \widetilde{\mathcal{M}}_{\mu,E})$ denote the inner cohomology group, by which one means the image of cohomology with compact supports inside full cohomology. If S denotes the set all the finite places $p|N$, then the action of the Hecke algebra $\mathcal{H}_2^{\mathsf{S}} = \bigotimes'_{p\nmid N} C_c^\infty(G_2(\mathbb{Q}_p)/\!\!/G_2(\mathbb{Z}_p))$ on $H_!^1(S_1^{(2)}(N), \widetilde{\mathcal{M}}_{\mu,E})$ is semi-simple. Let $\mathrm{Coh}_!(G_2, \mu, K_1(N))$ denote the set of all isotypic components of $H_!^1(S_1^{(2)}(N), \widetilde{\mathcal{M}}_{\mu,E})$.

Representation at infinity

We will assume the weight μ is regular. Given an absolutely simple Hecke module $\sigma_f \in \mathrm{Coh}_!(G_2, \mu, K_1(N))$, under the embedding $\hat{\bar{E}}_{\mathfrak{l}} \cong \mathbb{C}$, the module σ_f is the $K_1(N)$-invariants of the finite part of a cuspidal automorphic representation which (up to a minor abuse of notation) will be denote σ; its representation at infinity σ_∞ is an essentially discrete series representation $\mathbb{D}_\mu$ of $\mathrm{GL}_2(\mathbb{R})$ such that such that the relative Lie algebra cohomology $H^1(\mathfrak{g}_2, K_{2,\infty}, \mathbb{D}_\mu \otimes \mathcal{M}_{\mu,\mathbb{C}})$ is nonzero. The notations are as in [7, Section 3.1].

4.1.3 Integral structures on cohomology groups for GL_2

Highest weights for modular forms

For an integer $k \geq 2$, define:

$$\mu_k = (k-2)\boldsymbol{\gamma}_1 + (k/2-1)\boldsymbol{\delta}_2 \ = \ (k-2)\mathbf{e}_1 + 0\mathbf{e}_2.$$

The underlying $\mathbb{Q}$-vector space of $\mathcal{M}_{\mu_k}$ consists of homogeneous polynomials of degree $k-2$ in two variables X and Y with coefficients in $\mathbb{Q}$. If the k is clear from context then $\mu := \mu_k$. Also, $k > 2$ if and only if μ_k is a regular weight. Similarly, an integer $k' \geq 2$ determines a $\mu' = \mu_{k'}$. Hereafter the weights μ and μ' will be assumed to be regular.

Integral sheaves

Assume $N \geq 3$. One can also re-define the sheaf $\widetilde{\mathcal{M}}_{\mu,E}$ with respect to the projection:

$$\pi_1 : G_2(\mathbb{Q})\backslash\Big(G_2(\mathbb{R})/K_{2,\infty} \times G(\mathbb{A}_f)\Big) \to S_1^{(2)}(N)$$

In this case, sections over an open set U is given by:

$$\begin{aligned}\widetilde{\mathcal{M}}_{\mu,E} = \{\tilde{s} : \pi_1^{-1}(U) \to \mathcal{M}_{\mu,E} \otimes \mathbb{A}_E^{(\infty)} |\ \ \tilde{s} \text{ is locally constant,}\\ g_f \cdot \tilde{s}(x_\infty, g_f) \in \mathcal{M}_{\mu,E} \ \text{ and } \ \tilde{s}(x_\infty, g_f k_f) = k_f^{-1} \cdot s(x_\infty, g_f) \ \ \forall \ \ k_f \in K_1(N)\}.\end{aligned}$$

Take $\mathcal{M}_{\mu,\mathcal{O}_E}$ to be the $\mathcal{O}_E$–lattice generated by X^jY^{k-2-j}. It is clear that for $A = \mathcal{O}_E, E, \mathcal{O}_\mathfrak{l}, E_\mathfrak{l}, \mathbb{C}$, one has $\mathcal{M}_{\mu,\mathcal{O}_E} \otimes A = \mathcal{M}_{\mu,A}$. It is also clear that $\mathcal{M}_{\mu,\mathcal{O}_E} \otimes \hat{\mathcal{O}}_E$ is stable under the action of $K_1(N)$. For $A = \mathcal{O}_E, E, \mathcal{O}_\mathfrak{l}, E_\mathfrak{l}, \mathbb{C}$, define:

$$\begin{aligned}\widetilde{\mathcal{M}}_{\mu,A} = \{\tilde{s} : \pi_1^{-1}(U) \to \mathcal{M}_{\mu,\mathcal{O}_E} \otimes \hat{\mathcal{O}}_E |\ \ \tilde{s} \text{ is locally constant,}\\ g_f \cdot \tilde{s}(x_\infty, g_f) \in M_{\mu,\mathcal{O}_E} \otimes A \ \text{ and } \ \tilde{s}(x_\infty, g_f k_f) = k_f^{-1} \cdot s(x_\infty, g_f) \ \ \forall \ \ k_f \in K_1(N)\}.\end{aligned}$$

If $N = 1, 2$ we do usual modifications; see Hida [9, Section 5.3.2].

Classical cohomology groups

Let $\mathbb{H}$ denote the complex upper half space which is acted upon by $\mathrm{SL}_2(\mathbb{R})$ in the usual way. The group $\Gamma_1(N) = \mathrm{GL}_2^+(\mathbb{Q}) \cap K_1(N)$ is the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of matrices which are congruent to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ modulo N. Put $X_1(N) = \Gamma_1(N)\backslash\mathbb{H}$. Let $\pi_\mathbb{H} : \mathbb{H} \to X_1(N)$ be the canonical projection. For $A = \mathcal{O}_E, E, \mathcal{O}_\mathfrak{l}, E_\mathfrak{l}, \mathbb{C}$, and an open set $U \subset X_1(N)$, define

$$\begin{aligned}\underline{\mathcal{M}}_{\mu,A}(U) = \{s : \pi_\mathbb{H}^{-1}(U) \to \mathcal{M}_{\mu,A}\ |s \text{ is locally constant;}\\ s(\gamma(z)) = \gamma \cdot s(z),\ \forall \gamma \in \Gamma_1(N),\ z \in \mathbb{H}\}.\end{aligned}$$

Given a $g_f \in G_2(\mathbb{A}_f)$, one can express $g_f = \gamma k_f$ for some $\gamma \in G_2(\mathbb{Q})$ and $k_f \in K_1(N)$. The map $G(\mathbb{Q})(g_\infty, g_f)K_{2,\infty}K_1(N) \mapsto \Gamma_1(N)\gamma^{-1}g_\infty \cdot \sqrt{-1}$ is a homeomorphism between $S_1^{(2)}(N) \xrightarrow{\sim} X_1(N)$, giving then an isomorphism of sheaves $\widetilde{\mathcal{M}}_{\mu,A} \xrightarrow{\sim} \underline{\mathcal{M}}_{\mu,A}$, from which one has:

$$H_!^1(S_1^{(2)}(N), \widetilde{\mathcal{M}}_{\mu,A}) \ \cong \ H_!^1(X_1(N), \underline{\mathcal{M}}_{\mu,A}), \quad A = \mathcal{O}_E, E, \mathcal{O}_\mathfrak{l}, E_\mathfrak{l}, \mathbb{C}$$

Furthermore, there is also a canonical isomorphism

$$H_!^1(X_1(N), \underline{\mathcal{M}}_{\mu,A}) \cong H_!^1(\Gamma_1(N), \mathcal{M}_{\mu,A}),$$

where the latter are the parabolic cohomology groups defined by Shimura [22, Chapter 8]; see also Hida [10, Appendix]. Since $\mathcal{O}_{\mathfrak{l}}, E, E_{\mathfrak{l}}, \mathbb{C}$ are all flat $\mathcal{O}_E$–modules one gets

$$H^1_!(\Gamma_1(N), \mathcal{M}_{\mu,\mathcal{O}_E}) \otimes A \cong H^1_!(\Gamma_1(N), \mathcal{M}_{\mu,A}) \quad \text{for } A = \mathcal{O}_{\mathfrak{l}}, E, E_{\mathfrak{l}}, \mathbb{C}.$$

See Hida [10, p. 168] which in turn implies

$$H^1_!(S_1^{(2)}(N), \widetilde{\mathcal{M}}_{\mu,\mathcal{O}_E}) \otimes A \cong H^1_!(S_1^{(2)}(N), \widetilde{\mathcal{M}}_{\mu,A}) \quad \text{for } A = \mathcal{O}_{\mathfrak{l}}, E, E_{\mathfrak{l}}, \mathbb{C}. \tag{4.1}$$

When the

Avoiding torsion in integral cohomology

For an integer $N \geq 1$, define a finite set of prime ideals:

$$\mathsf{S}_N := \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal of } \mathcal{O}_E \text{ which divides } N\}. \tag{4.2}$$

Similarly, for an integer $k \geq 2$, define another finite set of prime ideals:

$$\mathsf{S}_k = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal of } \mathcal{O}_E \text{ over any prime } p \leq k\}. \tag{4.3}$$

If $\mathfrak{l} \notin \mathsf{S}_N \cup \mathsf{S}_k$ then by Hida [9, (1.14_b)], the cohomology group $H^1_!(S_1^{(2)}(N), \widetilde{\mathcal{M}}_{\mu,\mathcal{O}_E})$ has no $\mathfrak{l}$-torsion.

Tate twists

For $A = \mathcal{O}_E, E, \mathcal{O}_{\mathfrak{l}}, E_{\mathfrak{l}}, \mathbb{C}$ it is clear that $\underline{\mathcal{M}}_{\mu,A} \cong \underline{\mathcal{M}}_{\mu(m),A}$ due to sheaves being defined by the action of $\gamma \in \mathrm{SL}_2(\mathbb{R})$. Hence we fix *one* integral structure, namely the image of

$$H^1_!(\Gamma_1(N), \mathcal{M}_{\mu,\mathcal{O}_E}) \text{ (resp. } H^1_!(\Gamma_1(N), \mathcal{M}_{\mu,\mathcal{O}_{\mathfrak{l}}})) \tag{4.4}$$

in all of the cohomology groups $H^1_!(S_1^{(2)}(N), \widetilde{\mathcal{M}}_{\mu(m),E})$ (resp. $H^1_!(S_1^{(2)}(N), \widetilde{\mathcal{M}}_{\mu(m),E_{\mathfrak{l}}})$) with $m \in \mathbb{Z}$. It should be kept in mind that the twists appear when one considers the action of an integral Hecke algebra on integral cohomology.

For any of the cohomology groups we will be considering, the notation $\tilde{H}^{\bullet}_{?}(...)$ will be used to denote the images of the *integral* cohomology group in the *rational* cohomology group. The is done to ensure there are no torsion cohomology classes. As noted above, for $G_2 = \mathrm{GL}_2/\mathbb{Q}$, if we avoid a suitable finite set of primes as in 4.1.3 then there is no $\mathfrak{l}$-torsion and we may simplify the notation $\tilde{H}^1_!(...)$ to $H^1_!(...)$.

4.1.4 Cohomology of the Levi quotient $M_P = G_2 \times G_2$

Künneth isomorphisms

The weights μ and μ' give a highest weight $\mu+\mu'$ for M_P. For $A = E, E_{\mathfrak{l}}, \mathbb{C}$, denote the cohomology group at degree 2 of the locally symmetric space associated to the Levi quotient M_P by $H^2(S^{M_P}_{N\times N'}, \widetilde{\mathcal{M}}_{\mu+\mu',A})$. Here the level structure $K_1(N) \times K_1(N')$, and coefficient system is $\mathcal{M}_{\mu+\mu',A}$. If the weights μ and μ' are regular, then the notion of inner and strongly inner cohomology in [7] coincide. Moreover, we have a Künneth isomorphism:

$$H^2_!(S^{M_P}_{N\times N'}, \widetilde{\mathcal{M}}_{\mu+\mu',A}) \cong H^1_!(S^{(2)}_1(N), \widetilde{\mathcal{M}}_{\mu,A}) \otimes_A H^1_!(S^{(2)}_1(N'), \widetilde{\mathcal{M}}_{\mu',A}).$$

Integral structures

Via the Künneth isomorphism above, the image of

$$\tilde{H}^1_!(S^{(2)}_1(N'), \widetilde{\mathcal{M}}_{\mu,\mathcal{O}_E}) \otimes_{\mathcal{O}_E} \tilde{H}^1_!(S^{(2)}_1(N'), \widetilde{\mathcal{M}}_{\mu',\mathcal{O}_E}) \hookrightarrow H^1_!(S^{M_P}_{N\times N'}, \widetilde{\mathcal{M}}_{\mu+\mu',E})$$

gives an $\mathcal{O}_E$–lattice which will be denoted $\tilde{H}^2_!(S^{M_P}_{N'\times N}, \widetilde{\mathcal{M}}_{\mu'+\mu,\mathcal{O}_E})$. Similarly, an $\mathcal{O}_{\mathfrak{l}}$ lattice:

$$\tilde{H}^2_!(S^{M_P}_{N\times N'}, \widetilde{\mathcal{M}}_{\mu+\mu',\mathcal{O}_{E_{\mathfrak{l}}}}) \subset \tilde{H}^2_!(S^{M_P}_{N\times N'}, \widetilde{\mathcal{M}}_{\mu+\mu',E_{\mathfrak{l}}}).$$

Suppose for the moment, R and S are commutative rings with 1 and $R \to S$ is a ring homomorphism, and if M and N are R-modules then $(M \otimes_R N) \otimes_R S \cong (M \otimes_R S) \otimes_S (N \otimes_R S)$. Applying this for $\mathcal{O}_E \hookrightarrow A$, for $A = \mathcal{O}_{\mathfrak{l}}, E, E_{\mathfrak{l}}, \mathbb{C}$, we get

$$\tilde{H}^2_!(S^{M_P}_{N\times N'}, \widetilde{\mathcal{M}}_{\mu+\mu',\mathcal{O}_E}) \otimes_{\mathcal{O}_E} A \cong \tilde{H}^2_!(S^{M_P}_{N\times N'}, \widetilde{\mathcal{M}}_{\mu+\mu',A}). \tag{4.5}$$

There will be variations on the cohomology of the Levi (as when we look at both sides of an intertwining operator), but the same recipe as above will be adopted for all variations.

4.1.5 Rankin–Selberg L-functions

Relation between classical and automorphic L-functions

Given newforms $h \in S_k(N,\chi)^{\text{new}}$ and $h' \in S_{k'}(N',\chi')^{\text{new}}$ as above, consider highest weights $\mu = \mu_k = (k-2,0)$ and $\mu' = \mu_{k'} = (k'-2,0)$, and Hecke modules in inner cohomology $\sigma \in \text{Coh}_!(G_2,\mu)$ and $\sigma' \in \text{Coh}_!(G_2,\mu')$ such that

$$\sigma \cong \Pi(\mathbf{h})|\cdot|^{-k/2+1}, \quad \sigma' \cong \Pi(\mathbf{h}'^{\rho})|\cdot|^{-k'/2+1}. \tag{4.6}$$

where $\mathbf{h}$ and $\mathbf{h}'^{\rho}$ are the $\mathbb{C}$-valued automorphic forms on $\text{GL}_2(\mathbb{A})$ attached to h and h'^{ρ} and $\Pi(\mathbf{h})$ and $\Pi(\mathbf{h}'^{\rho})$ are the cuspidal automorphic representations generated by $\mathbf{h}$ and $\mathbf{h}'^{\rho}$ respectively. The reason for taking h'^{ρ} for σ' (instead of h' itself) will become clear in (4.7) below. The reader is referred to Raghuram and Tanabe [19] for details of the dictionary between the modular forms and cohomological cuspidal representations. In particular, one has the following relations:

$$L(s,\sigma) \;=\; L(s+\frac{1}{2},h), \quad L(s,\sigma^{\mathsf{v}}) \;=\; L(s+k-\frac{3}{2},h),$$

and similarly for σ', h' and k'. Furthermore, for an integer m, the Tate-twist $\sigma(-m)$ has cohomology with respect to $\mu(m)$, then we have the following relation between the automorphic-representation theoretic and the classical Rankin–Selberg L-functions:

$$L(s,\sigma(-m)\times\sigma'^{\mathsf{v}}) \;=\; L(s+k'-m-1,h\times h'). \tag{4.7}$$

All this applies just the same to the pair $h \in S_k(N,\chi)^{\text{new}}$ and $h'' \in S_{k'}(N',\chi')^{\text{new}}$.

Setting-up the context of Eisenstein cohomology

To apply the machinery of [7], we will be looking at the intertwining operator:

$$T_{\text{st}}(s)|_{s=-2} : {}^{\text{a}}\text{Ind}_P^G\,(\sigma(-m)\times\sigma')) \;\longrightarrow\; {}^{\text{a}}\text{Ind}_P^G(\sigma'(2)\times\sigma(-m-2)),$$

which in *loc. cit.*, gives a rationality result for the ratio:

$$\frac{L(-2,\sigma(-m)\times\sigma'^{\mathsf{v}})}{L(-1,\sigma(-m)\times\sigma'^{\mathsf{v}})} \;=\; \frac{L(-2-m,\sigma\times\sigma'^{\mathsf{v}})}{L(-1-m,\sigma\times\sigma'^{\mathsf{v}})} \;=\; \frac{L(k'-m-3,h\times h')}{L(k'-m-2,h\times h')},$$

provided m satisfies the constraints imposed by combinatorial lemma which is exactly equivalent to the above L-values being critical; from the fact that $\{m\in\mathbb{Z}\mid k\le m<k'-1\}$ being the critical set of $L(s,h'\times h)$, imposes the following bounds on permissible Tate-twists m:

$$-1 \;\le\; m \;\le\; k'-k-3.$$

Furthermore, to carry out [7], the data $(\mu_k(m),\mu_{k'})$ needs to be on the right of the unitary axis (required for a certain Eisenstein series to to be holomorphic), which is the condition:

$$-2+\frac{k'-2}{2}-\left(\frac{k-2}{2}+m\right)\ge 0 \quad\iff\quad m\le\frac{k'-k}{2}-2.$$

Hence, as m varies from -1 to $\frac{k'-k}{2}-2$, we are looking at the string of ratios of L-values from the rightmost up to a little more than the central value:

$$\frac{L(k'-2,h\times h')}{L(k'-1,h\times h')},\;\frac{L(k'-3,h\times h')}{L(k'-2,h\times h')},\;\dots,\;\frac{L(\lfloor\frac{k+k'-1}{2}\rfloor,h\times h')}{L(\lfloor\frac{k+k'+1}{2}\rfloor,h\times h')}.$$

If we are on the left of the unitary axis, then reversing the direction of the intertwining operator and using the functional equation offers the possibility of a result for all successive ratios critical values exactly as in [7]; see the discussion in Sect. 6.3.4.

4.2 Hecke algebras and Gorenstein property

4.2.1 Classical Hecke algebras

For $A = \mathcal{O}_E, E, \mathcal{O}_{\mathfrak{l}}, E_{\mathfrak{l}}$, define A sub-modules of the space of classical cusp forms $S_k(N) := S_k(\Gamma_1(N))$ to be $S_k(N, A) = \{f = \sum_{n=1}^{\infty} a(n, f)q^n \in S_k(N) \mid a(n, f) \in A \ \forall n \in \mathbb{N}\}$. Recall a theorem of Shimura, Deligne, Rapoport and Katz; see Hida [11, Chapter 3].

Theorem 4.2.1. *For $A = \mathcal{O}_E, E, \mathcal{O}_{\mathfrak{l}}, E_{\mathfrak{l}}$ the space $S_k(N, A)$ is an A-module of full rank in $S_k(N)$, i.e.,*

$$S_k(N, A) \otimes_\iota \mathbb{C} = S_k(N). \tag{4.8}$$

For $A = \mathcal{O}_E, E, \mathcal{O}_{\mathfrak{l}}, E_{\mathfrak{l}}$ define $h_k(N, A) \subset \mathrm{End}_A(S_k(N, A))$ to be the usual Hecke algebra generated by the operators $T(p)$ for all primes p and $T(p, p)$ for $p \nmid N$. Due to the perfect pairing $(\cdot, \cdot) : S_k(N, A) \times h_k(N, A) \to A, \ (f, T) \mapsto a(1, f|T)$, Hida [11, Theorem 3.17], one gets $h_k(N, \mathcal{O}_E) \otimes A = h_k(N, A)$ for $A = E, \mathcal{O}_{\mathfrak{l}}, E_{\mathfrak{l}}, \mathbb{C}$ and

$$S_k(N, A) \cong \mathrm{Hom}_A(h_k(N, A), A) \ \text{ (as } h_k(N, A)\text{–modules)}\ . \tag{4.9}$$

4.2.2 Formalism of a Gorenstein datum

Suppose R is the ring of integers of a local field of characteristic 0 and $\mathfrak{l} \subset R$ will be the unique maximal principal ideal of R. Let $\mathbb{T}$ be a commutative R-algebra with 1 which is also finite and free as an R-module. Since R is complete $\mathbb{T}$ is complete as well. It is well known that $\mathbb{T}$ has only finitely many maximal ideals and each such ideal $\mathfrak{L}$ defines an idempotent $e_{\mathfrak{L}} \in \mathbb{T}$. Furthermore, $e_{\mathfrak{L}}\mathbb{T} \cong \mathbb{T}_{\mathfrak{L}}$ and $\mathbb{T} = \sum_{\mathfrak{L}} e_{\mathfrak{L}}\mathbb{T} \cong \oplus_{\mathfrak{L}}\mathbb{T}_{\mathfrak{L}}$, where the sum is over the finitely many maximal ideals. Let H be a *fixed* free R–module and also a $\mathbb{T}$–module (not necessarily free) and $\mathfrak{L} \subset \mathbb{T}$ a *fixed* maximal ideal. Observe $H_{\mathfrak{L}} \cong e_{\mathfrak{L}}H$ and so $H_{\mathfrak{L}} \hookrightarrow H$. In applications we will be under the following assumptions:

1. There is a $\mathbb{T}_{\mathfrak{L}}$ equivariant isomorphism $\Phi_{\mathfrak{L}} : \mathbb{T}_{\mathfrak{L}} \xrightarrow{\sim} H_{\mathfrak{L}}$,
2. $\mathrm{Hom}_{R-\mathrm{mod}}(\mathbb{T}_{\mathfrak{L}}, R) \cong \mathbb{T}_{\mathfrak{L}}$ as $\mathbb{T}_{\mathfrak{L}}$ modules.

The second assumption is the definition of a ring (here $\mathbb{T}_{\mathfrak{L}}$) being Gorenstein. Hereafter call the tuple $(R, \mathbb{T}, H, \mathfrak{L})$ which satisfies the the above assumptions to be a *freely Gorenstein datum*.

4.2.3 Presence of two congruent morphisms

Assume now there are two *distinct* R-algebra morphisms $\Theta', \Theta'' : \mathbb{T} \to R$ such that their compositions with the map $R \to R/\mathfrak{l}$ are the same, i.e.,

$$\overline{\Theta}' = \overline{\Theta}'',$$

where, $\overline{\Theta}' := \Theta' \pmod{\mathfrak{l}}$ and $\overline{\Theta}'' := \Theta'' \pmod{\mathfrak{l}}$; the kernels of $\overline{\Theta}'$ and $\overline{\Theta}''$ are the same; put $\mathfrak{L} := \ker \overline{\Theta}' = \ker \overline{\Theta}''$. Hence the morphisms Θ' and Θ'' factors through $\mathbb{T}_{\mathfrak{L}}$ which will be denoted by the same symbols. It will be *assumed* that $(R, \mathbb{T}, H, \mathfrak{L})$ is a freely-Gorenstein datum.

Lemma 4.2.1. *Under the $\mathbb{T}_{\mathfrak{L}}$-equivariant isomorphism, say $\Psi_{\mathfrak{L}} : Hom_{R-mod}(\mathbb{T}_{\mathfrak{L}}, R) \cong \mathbb{T}_{\mathfrak{L}}$, one has*

$$\Psi_{\mathfrak{L}}(\Theta'), \Psi_{\mathfrak{L}}(\Theta'') \notin \mathfrak{l}\mathbb{T}_{\mathfrak{L}}, \quad \Psi_{\mathfrak{L}}(\Theta') - \Psi_{\mathfrak{L}}(\Theta'') \in \mathfrak{l}\mathbb{T}_{\mathfrak{L}}.$$

The algebra $\mathbb{T}_{\mathfrak{L}}$ acts on $\Psi_{\mathfrak{L}}(\Theta')$ and $\Psi_{\mathfrak{L}}(\Theta'')$ by the characters Θ' and Θ'' respectively.

Proof. The natural map $\mathrm{Hom}_{R-\mathrm{mod}}(\mathbb{T}_{\mathfrak{L}}, R) \otimes_R R/\mathfrak{l} \longrightarrow \mathrm{Hom}_{R-\mathrm{mod}}(\mathbb{T}_{\mathfrak{L}}, R/\mathfrak{l})$ is an isomorphism. Since $\mathbb{T}$ is a free R-module, fix an R basis $\{e_i\}_{i=1}^n$ of $\mathbb{T}_{\mathfrak{L}}$ to get a dual basis $\{e_i^\vee\}_{i=1}^n$ of $\mathrm{Hom}_{R-\mathrm{mod}}(\mathbb{T}_{\mathfrak{L}}, R)$. Suppose $\sum_i r_i e_i^\vee$ maps to the 0 element of $\mathrm{Hom}_{R-\mathrm{mod}}(\mathbb{T}_{\mathfrak{L}}, R/\mathfrak{l})$. Then evaluating at e_i we see that $r_i \in \mathfrak{l}$ for all $i = 1, \cdots, n$. Therefore, the element $(\sum_i r_i e_i^\vee) \otimes 1 = 0$ in $\mathrm{Hom}_{R-\mathrm{mod}}(\mathbb{T}_{\mathfrak{L}}, R) \otimes_R R/\mathfrak{l}$ which proves injectivity. Using the dual basis $e_i^\vee$ one can show it is surjective as well.

Due to the isomorphism $\Psi_{\mathfrak{L}}$ we get the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{R-\mathrm{mod}}(\mathbb{T}_{\mathfrak{L}}, R) & \xrightarrow{\sim \text{ via } \Psi_{\mathfrak{L}}} & \mathbb{T}_{\mathfrak{L}} \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{R-\mathrm{mod}}(\mathbb{T}_{\mathfrak{L}}, R) \otimes_R R/\mathfrak{l} & \xrightarrow{\sim} & \mathbb{T}_{\mathfrak{L}} \otimes_R R/\mathfrak{l} \\
\sim\downarrow & & \downarrow\sim \\
\mathrm{Hom}_{R-\mathrm{mod}}(\mathbb{T}_{\mathfrak{L}}, R/\mathfrak{l}) & \xrightarrow{\sim \text{ via } \overline{\Psi}_{\mathfrak{L}}} & \mathbb{T}_{\mathfrak{L}}/\mathfrak{l}\mathbb{T}_{\mathfrak{L}}.
\end{array}$$

Since $\overline{\Theta}' = \overline{\Theta}''$ in $\mathrm{Hom}_{R-\mathrm{mod}}(\mathrm{T}_{\mathfrak{L}}, R/\mathfrak{l})$ we have that $\overline{\Psi}_{\mathfrak{L}}(\overline{\Theta}') - \overline{\Psi}_{\mathfrak{L}}(\overline{\Theta}'') = \overline{\Psi}_{\mathfrak{L}}(\overline{\Theta}' - \overline{\Theta}'') = 0$ in $\mathbb{T}_{\mathfrak{L}}/\mathfrak{l}\mathbb{T}_{\mathfrak{L}}$. Due to the commutativity of the above diagram the first assertion is true as $\Theta'(1) = \Theta''(1) = 1 \notin \mathfrak{l}$.

Now, for the second claim. For all $t \in \mathbb{T}_{\mathfrak{L}}$ one has $t \cdot \Theta' = \Theta'(t)\Theta'$ since for all $x \in \mathbb{T}_{\mathfrak{L}}$ we have $(t \cdot \Theta')(x) = \Theta'(xt) = \Theta'(tx) = \Theta'(t)\Theta'(x)$. Since $\Psi_{\mathfrak{L}}$ is $\mathbb{T}_{\mathfrak{L}}$ equivariant it is R–linear as well and so $t \cdot \Psi_{\mathfrak{L}}(\Theta') = \Psi_{\mathfrak{L}}(t \cdot \Theta') = \Psi_{\mathfrak{L}}(\Theta'(t)\Theta') = \Theta'(t)\Psi_{\mathfrak{L}}(\Theta')$. ☐

Lemma 4.2.2. *Set $v(\Theta') := (\Phi_{\mathfrak{L}} \circ \Psi_{\mathfrak{L}})(\Theta')$ and $v(\Theta'') := (\Phi_{\mathfrak{L}} \circ \Psi_{\mathfrak{L}})(\Theta'')$. On the vectors $v(\Theta')$ and $v(\Theta'')$, the algebra $\mathrm{T}_{\mathfrak{L}}$ acts by Θ' and Θ'', respectively. Moreover,*

$$v(\Theta'),\ v(\Theta'') \ \notin\ \mathfrak{l}e_{\mathfrak{L}}H, \quad \textit{and} \quad v(\Theta') - v(\Theta'') \ \in\ \mathfrak{l}e_{\mathfrak{L}}H.$$

Proof. Follows from Lemma 4.2.1 and that both $\Psi_{\mathfrak{L}}$ and $\Phi_{\mathfrak{L}}$ are $\mathbb{T}_{\mathfrak{L}}$ equivariant. ☐

4.2.4 Specializing to a particular $\mathbb{T}$ and H

Take $\mathbb{T}$ to be the Hecke algebra $h_{k'}(N', \mathcal{O}_{\mathfrak{l}})$ and H the cohomology group $H^1_!(X_1(N'), \widetilde{\mathcal{M}}_{\mu',\mathcal{O}_{\mathfrak{l}}})$. Let $\Theta' : h_{k'}(N', \mathcal{O}_{\mathfrak{l}}) \to \mathcal{O}_{\mathfrak{l}}$ (resp., $\Theta'' : h_{k'}(N', \mathcal{O}_{\mathfrak{l}}) \to \mathcal{O}_{\mathfrak{l}}$) be the morphisms $T(p) \mapsto a(p, h')$ (resp., $T(p) \mapsto a(p, h'')$). Let $\mathfrak{L} \subset h_{k'}(N', \mathcal{O}_{\mathfrak{l}})$ be the maximal ideal determined by the hypothesis $\Theta' \equiv \Theta'' \pmod{\mathfrak{l}}$. Assume E is large so that $\mathbb{T}/\mathfrak{L} \cong \mathcal{O}_{\mathfrak{l}}/\mathfrak{l}$.

Let $\varrho_{\Theta'} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}/\mathfrak{L})$ be the Galois representation modulo l attached to the Hecke algebra morphism Θ'. It is a semi-simple representation determined by

$$\text{Trace } \varrho_{\Theta'}(\mathrm{Frob}_p) = a(p, h') \pmod{\mathfrak{l}} \quad \text{and} \quad \det \varrho_{\Theta'}(\mathrm{Frob}_p) = \chi'(p)p^{k'-1} \pmod{\mathfrak{l}}. \tag{4.10}$$

for $p \nmid N'l$. This was constructed first by Deligne. See Gross [6, p. 483, Proposition 11.1]. Consider the twist $\varrho_{\Theta'}\chi'^{-1}$ of the representation of $\varrho_{\Theta'}$ by the character χ'^{-1} which satisfies:

$$\text{Trace } \varrho_{\Theta'}\chi'^{-1}(\mathrm{Frob}_p) = a(p, h')\chi'^{-1}(p) \pmod{\mathfrak{l}} \quad \text{and}$$
$$\det \varrho_{\Theta'}\chi'^{-1}(\mathrm{Frob}_p) = \chi'^{-1}(p)p^{k'-1} \pmod{\mathfrak{l}}. \tag{4.11}$$

for $p \nmid N'l$. This is the mod l representation attached to the newform $h'^{\rho} = \sum_{n=1}^{\infty} \overline{a(n, h')} q^n$ or equivalently the Hecke algebra morphism $\Theta'^{\rho} : h_{k'}(N', \mathfrak{l}) \to \mathcal{O}_{\mathfrak{l}},\ T(p) \mapsto \chi'^{-1}(p)\ a(p, h)$, due to $\overline{a(p, h)} = \chi'^{-1}(p)\ a(p, h)$ for $p \nmid Nl'$. Hence $\varrho_{\Theta'}$ is irreducible if and only if $\varrho_{\Theta'}\chi'^{-1}$ is irreducible. Also, the maximal ideal of $h_{k'}(N', \mathcal{O}_{\mathfrak{l}})$ determined by Θ' and Θ'^{ρ} are the same.

Theorem 4.2.2. *Assume $l > k'$ and the Galois representation $\varrho_{\Theta'}$ is irreducible. Then*

$$(\mathcal{O}_{\mathfrak{l}},\ h_{k'}(N', \mathcal{O}_{\mathfrak{l}}),\ H^1_!(X_1(N'), \underline{\mathcal{M}}_{\mu',\mathcal{O}_{\mathfrak{l}}})(\epsilon'),\ \mathfrak{L}) \tag{4.12}$$

is a freely Gorenstein datum.

Proof. The $\mathcal{O}_{\mathfrak{l}}$-freeness of $H^1_!(X_1(N'), \underline{\mathcal{M}}_{\mu',\mathcal{O}_{\mathfrak{l}}})(\epsilon')$ is due to the fact $l > k'$. It is proved in Faltings and Jordan [5, Theorem 2.1] that $h_{k'}(N', \mathcal{O}_{\mathfrak{l}})_{\mathfrak{L}}$ is Gorenstein. The freeness condition in the definition of a freely Gorenstein datum follows from the perfect pairing between Hecke algebras and cusp forms as in (4.9) and the Eichler-Shimura isomorphism. □

As a historical remark, the Gorenstein property of l-adic weight 2 Hecke algebras of prime level were first studied by Mazur. In the ordinary case, this property is necessary and sufficient for $\rho_{\Theta'}$ to appear with multiplicity-one in the l-torsion points of the Jacobian of modular curve. Later for weights $k' \geq 2$ it was shown that the corresponding Hecke algebras were Gorenstein. See Hida [11, Chapter 3 and Chapter 5] and the references therein for an alternative proof of Gorensteiness using Taylor-Wiles system.

4.2.5 Explicit congruent cohomology classes

Theorem 4.2.3. *There are cohomology classes $^{\circ}v^{\epsilon'}_{\mu'}(h'^{\rho})$ and $^{\circ}v^{\epsilon'}_{\mu'}(h''^{\rho})$ inside the cohomology group $\tilde{H}^1_!(S^{(2)}_1(N'), \widetilde{\mathcal{M}}_{\mu',\mathcal{O}_{\mathfrak{l}}})(\epsilon')$ on which $h_{k'}(N', \mathcal{O}_{\mathfrak{l}})$ acts by Θ'^{ρ} and Θ''^{ρ}, respectively. They* *are non-zero modulo $\mathfrak{l}$ but their difference is* 0 *modulo* $\mathfrak{l}$*, i.e.,*

$$\begin{aligned} {}^{\circ}v^{\epsilon'}_{\mu'}(h'^{\rho}),\ {}^{\circ}v^{\epsilon'}_{\mu'}(h''^{\rho}) &\notin \mathfrak{l}\tilde{H}^1_!(S^{(2)}_1(N'),\widetilde{\mathcal{M}}_{\mu',\mathcal{O}_{\mathfrak{l}}})(\epsilon') \\ {}^{\circ}v^{\epsilon'}_{\mu'}(h'^{\rho}) - {}^{\circ}v^{\epsilon'}_{\mu'}(h''^{\rho}) &\in \mathfrak{l}\tilde{H}^1_!(S^{(2)}_1(N'),\widetilde{\mathcal{M}}_{\mu',\mathcal{O}_{\mathfrak{l}}})(\epsilon'). \end{aligned} \tag{4.13}$$

Also, fix a non-zero modulo $\mathfrak{l}$ cohomology class for the cusp form h:

$${}^{\circ}v^{\epsilon'}_{\mu}(h) \in \tilde{H}^1_!(S^{(2)}_1(N),\widetilde{\mathcal{M}}_{\mu,\mathcal{O}_{\mathfrak{l}}})(\epsilon'\times\sigma_f) \setminus \mathfrak{l}\tilde{H}^1_!(S^{(2)}_1(N),\widetilde{\mathcal{M}}_{\mu,\mathcal{O}_{\mathfrak{l}}})(\epsilon'\times\sigma_f), \tag{4.14}$$

where $\tilde{H}^1_!(S^{(2)}_1(N),\widetilde{\mathcal{M}}_{\mu,\mathcal{O}_{\mathfrak{l}}})(\epsilon'\times\sigma_f)$ is $\tilde{H}^1_!(S^{(2)}_1(N),\widetilde{\mathcal{M}}_{\mu,\mathcal{O}_{\mathfrak{l}}})(\epsilon') \cap \tilde{H}^1_!(S^{(2)}_1(N),\widetilde{\mathcal{M}}_{\mu,E_{\mathfrak{l}}})(\epsilon'\times\sigma_f)$.

It is helpful to keep in mind that ${}^{\circ}v^{\epsilon'}_{\mu'}(h'^{\rho}) = {}^{\circ}v^{\epsilon'}_{\mu'(-n)}(h'^{\rho})$ and ${}^{\circ}v^{\epsilon'}_{\mu}(h) = {}^{\circ}v^{\epsilon'}_{\mu(-m)}(h)$ for any $m,n \in \mathbb{Z}$. Only the action of the Hecke algebra is different on these vectors. Fix the notations

$$\begin{aligned} {}^{\circ}v^{\epsilon'}_{\mu+\mu'}(h,h'^{\rho}) &:= {}^{\circ}v^{\epsilon'}_{\mu}(h)\otimes{}^{\circ}v^{\epsilon'}_{\mu'}(h'^{\rho}), & {}^{\circ}v^{\epsilon'}_{\mu'(-2)+\mu(2)}(h'^{\rho},h) &:= {}^{\circ}v^{\epsilon'}_{\mu'(-2)}(h'^{\rho})\otimes{}^{\circ}v^{\epsilon'}_{\mu(2)}(h), \\ {}^{\circ}v^{\epsilon'}_{\mu+\mu'}(h,h''^{\rho}) &:= {}^{\circ}v^{\epsilon'}_{\mu}(h)\otimes{}^{\circ}v^{\epsilon'}_{\mu'}(h''^{\rho}), & {}^{\circ}v^{\epsilon'}_{\mu'(-2)+\mu(2)}(h''^{\rho},h) &:= {}^{\circ}v^{\epsilon'}_{\mu'(-2)}(h''^{\rho})\otimes{}^{\circ}v^{\epsilon'}_{\mu(2)}(h). \end{aligned} \tag{4.15}$$

The vectors ${}^{\circ}v^{\epsilon'}_{\mu}(h)\otimes{}^{\circ}v^{\epsilon}_{\mu^?}(h^{?\rho})$ are in the isotypic components $H^2_!(S^{M_P}_{N\times N'},\widetilde{\mathcal{M}}_{\mu+\mu',E})(\tilde{\epsilon}'\times\sigma_f\otimes\sigma^?_f)$ for $^? \in \{','' \}$. Similar comments apply to the other vectors.

Corollary 4.2.1. *The vectors ${}^{\circ}v^{\epsilon'}_{\mu+\mu'}(h,h'^{\rho}), {}^{\circ}v^{\epsilon'}_{\mu+\mu'}(h,h''^{\rho})$ are not zero modulo $\mathfrak{l}$, but are congruent modulo $\mathfrak{l}$, i.e.,*

$$\begin{aligned} {}^{\circ}v^{\epsilon'}_{\mu+\mu'}(h,h'^{\rho}),\ {}^{\circ}v^{\epsilon'}_{\mu+\mu'}(h,h''^{\rho}) &\notin \mathfrak{l}H^2_!(S^{M_P}_{N\times N'},\widetilde{\mathcal{M}}_{\mu+\mu',\mathcal{O}_{\mathfrak{l}}})(\tilde{\epsilon}'), \\ {}^{\circ}v^{\epsilon'}_{\mu+\mu'}(h,h'^{\rho}) - {}^{\circ}v^{\epsilon'}_{\mu+\mu'}(h,h''^{\rho}) &\in \mathfrak{l}H^2_!(S^{M_P}_{N\times N'},\widetilde{\mathcal{M}}_{\mu+\mu',\mathcal{O}_{\mathfrak{l}}})(\tilde{\epsilon}'). \end{aligned}$$

Similarly,

$$\begin{aligned} {}^{\circ}v^{\epsilon'}_{\mu'(-2)+\mu(2)}(h'^{\rho},h),\ {}^{\circ}v^{\epsilon'}_{\mu'(-2)+\mu(2)}(h''^{\rho},h) &\notin \mathfrak{l}H^2_!(S^{M_P}_{N'\times N},\widetilde{\mathcal{M}}_{\mu'(-2)+\mu(2),\mathcal{O}_{\mathfrak{l}}})(\tilde{\epsilon}'), \\ {}^{\circ}v^{\epsilon'}_{\mu'(-2)+\mu(2)}(h'^{\rho},h) - {}^{\circ}v^{\epsilon'}_{\mu'(-2)+\mu(2)}(h''^{\rho},h) &\in \mathfrak{l}H^2_!(S^{M_P}_{N'\times N},\widetilde{\mathcal{M}}_{\mu'(-2)+\mu(2),\mathcal{O}_{\mathfrak{l}}})(\tilde{\epsilon}'). \end{aligned}$$

The reader should note that the auxiliary cusp form h has no bearing on the choice of the prime $\mathfrak{l}$. So we don't impose the hypothesis that $l \notin \mathsf{S}_k$, where k is the weight of the form h.

Chapter 5

Integral structures on induced spaces

Here, a formalism of defining an integral structure on the invariant subspace of an induced space through Mackey isomorphism is given. The following, combined with the double coset representatives calculation in the next section, will enable us to give integral structure on induced space considered in [7] in the next chapter.

Suppose V is an admissible $M_P(\mathbb{A}_f)$–module over E (resp., $E_{\mathfrak{l}}$). After inflating it to $P(\mathbb{A}_f)$, let ${}^a\mathrm{Ind}(V)$ denote the algebraically induced space from $P(\mathbb{A}_f)$ to $\mathrm{GL}_4(\mathbb{A}_f)$. If K_f is an open compact subgroup of $\mathrm{GL}_4(\mathbb{A}_f)$, then one has the Mackey isomorphism:

$$ {}^a\mathrm{Ind}(V)^{K_f} \xrightarrow{\sim} \bigoplus_{\xi_f \in P(\mathbb{A}_f)\backslash \mathrm{GL}_4(\mathbb{A}_f)/K_f} V^{K_f^{M_P}(\xi_f)}, \qquad \phi_f \mapsto \sum_{\xi_f} \phi_f(\xi_f), \tag{5.1} $$

where $K_f^{M_P}(\xi_f) = \kappa_P(P(\mathbb{A}_f) \cap \xi_f\, K_f\, \xi_f^{-1})$ is a subgroup of $M_P(\mathbb{A}_f)$ for every ξ_f. Suppose now each $V^{K_f^{M_P}(\xi_f)}$ has an $\mathcal{O}_E$ (resp. $\mathcal{O}_{\mathfrak{l}}$)–lattice, say ${}^\circ V^{K_f^{M_P}(\xi_f)}$, then an $\mathcal{O}_E$ (resp. $\mathcal{O}_{\mathfrak{l}}$)–lattice in ${}^a\mathrm{Ind}(V)^{K_f}$ is defined to be all the vectors ϕ_f in the algebraically induced space such that $\phi_f(\xi_f) \in {}^\circ V^{K_f^{M_P}(\xi_f)}$.

In this chapter the double coset representatives will be enumerated when $K_f = K_f^{N+N'}$. From Corollary 5.2.1 of this chapter one obtains

Theorem 5.0.1. *Let V be an admissible $M_P(\mathbb{A}_f)$-module, $K_f^{N+N'}$ be the mirahoric subgroup of $\mathrm{GL}_4(\mathbb{A}_f)$ of level $N + N'$ and $K_1(M)$ is the congruence subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ of level M then*

$$ {}^a\mathrm{Ind}(V)^{K_f^{N+N'}} \cong \bigoplus_{\substack{M,M' \\ MM'=NN'}} V^{K_1(M)\times K_1(M')} $$

With the choice of the integral structure on ${}^a\mathrm{Ind}(V)^{K_f^{N+N'}}$, say ${}^\circ\, {}^a\mathrm{Ind}(V)^{K_f^{N+N'}}$, as mentioned in the beginning, it is clear that if $\phi'_f, \phi''_f \in {}^a\mathrm{Ind}(V)^{K_f^{N+N'}}$ then

$$\phi'_f - \phi''_f \in \mathfrak{l}\left({}^{\circ\, a}\mathrm{Ind}(V)^{K_f^{N+N'}}\right) \iff \sum_M \left(\phi'_f(\xi_f^{(M)}) - \phi''_f(\xi_f^{(M)})\right) \in \mathfrak{l}\left(\bigoplus_M {}^{\circ}V^{K_1(NN'/M)\times K_1(M)}\right).$$

(The notation $\xi_f^{(N)}$ is explained at the end of this chapter.)

This observation will enable us to construct congruent vectors ${}^{\circ}\phi'_f$ and ${}^{\circ}\phi''_f$ in an induced space defined in Section 6.2.3 with the help of Corollary 4.2.1.

5.1 Double coset representatives

In the next chapter, we define certain specific vectors in induced representations built from the vectors in Section 4.2.5. Towards this, while using the Mackey isomorphism Eq. (5.1), we need to understand double cosets: first locally and then globally at all the finite places. Specifically, in this section, we calculate an explicit set of representatives Ξ_p of the double cosets $P(\mathbb{Q}_p)\backslash \mathrm{GL}_4(\mathbb{Q}_p)/K_p^{n_p+n'_p}$, where $K_p^{n_p+n'_p}$ is the Mirahoric subgroup of $\mathrm{GL}_4(\mathbb{Z}_p)$ of level $n_p + n'_p$ and P is the $(2,2)$ parabolic subgroup.

5.1.1 Calculation for the Borel and principal congruence subgroups

Theorem 5.1.1 and Corollary 5.1.1 below are essentially due to Januszewski [12]. We follow closely the notation therein and reproduce the proof with a minor modification. Recall the Iwahori decomposition of $\mathbb{Q}_p$ points of $G_n = \mathrm{GL}_n/\mathbb{Q}$.

$$G_n(\mathbb{Q}_p) = \coprod_{w\in W_n} B_n(\mathbb{Q}_p)wI_n,$$

where B_n is the set of upper triangular matrices, W_n is the Weyl group identified with the set of permutation matrices and I_n is the Iwahori subgroup of $G_n(\mathbb{Z}_p)$. Let J_m denote the set of principal congruence subgroup of level m of $G_n(\mathbb{Z}_p)$, i.e., the set of matrices of $g \in G_n(\mathbb{Z}_p)$ such that $g \equiv \mathbf{1}_N \pmod{p^m}$. Let $R_m \subset \mathbb{Z}_p$ denote the complete set of coset representatives $\{0, p, p^2, \cdots, p^{m-1}\}$ of $\mathbb{Z}_p/p^m\mathbb{Z}_p$. Then the set

$$\mathcal{R}_m = \{r = (r_{ij})_{ij} \in I_N | r_{ij} \in R_m\}$$

forms a complete set of (left) coset representatives for I_n/J_m. For $w \in W_n$ we have the obvious inclusion: $B_n(\mathbb{Q}_p)wsJ_m \subset B_n(\mathbb{Q}_p)wI_n$, for $s \in \mathcal{R}_m$. We have

$$B_n(\mathbb{Q}_p)wI_n = \bigcup_{s\in\mathcal{R}_m} B_n(\mathbb{Q}_p)wsJ_m.$$

The cosets on the right need not be distinct. The following theorem shows that we can take the union over a smaller set of representatives and still get $B(\mathbb{Q}_p)wI_N$. Before stating it we need some more notations. Let U_B denote the unipotent radical of B_n consisting of strictly upper triangular matrices and U_B^- denote the unipotent radical of the opposite Borel subgroup B^- of lower triangular matrices of G_n. For a fixed $w \in W_n$ define $B^w := B_n(\mathbb{Q}_p) \cap wI_nw^{-1}$ and $\mathcal{R}_m^w := \mathcal{R}_{m,B} \cap w^{-1}U_B^-(\mathbb{Q}_p)w$.

Theorem 5.1.1. *For a fixed $w \in W_n$ the set $\mathcal{R}_{m,B}^w$ forms a complete system of coset representatives for $B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)wI_n/J_m$, i.e.,*

$$B_n(\mathbb{Q}_p)wI_n = \bigcup_{s \in \mathcal{R}_{m,B}^w} B_n(\mathbb{Q}_p)wsJ_m.$$

Proof. The map sending the coset $B^wwsJ_m \mapsto B_n(\mathbb{Q}_p)wsJ_m$, for $s \in \mathcal{R}_m$, is injective. Clearly, it is surjective as well. So it is enough to show that $w\mathcal{R}_{m,B}^w$ forms a system of representatives for $B^w\backslash B^wwI_n/J_m$. Consider the following

Assumption 5.1.1. *One can find elements $u^{(0)} := \mathbf{1}_n, u^{(1)}, u^{(2)}, \cdots, u^{(n)} \in B^w$, and recursively define $r^{(0)}, r^{(1)}, \cdots, r^{(n)} \in I_n$ where $r^{(0)} = 1$ and $r^{(v+1)} = w^{-1}u^{(v)}w \cdot r^{(v)}$ for $v > 1$, such that $r^{(n)} \in w^{-1}U_B^-(\mathbb{Z}_p)w$.*

From this assumption it follows if we define $u := u^{(n-1)} \cdots u^{(1)} \in B^w$, then

$$u \cdot wrw^{-1} \;=\; w \cdot w^{-1}uwrw^{-1} \;=\; ww^{-1}u^{n-1} \cdots u^{(1)}wrw^{-1} \;=\; wr^{(n)}w^{-1} \in U_B^-(\mathbb{Z}_p).$$

Suppose $s \in \mathcal{R}_m$ is a representative of the left coset $w^{-1}uwrJ_m \in I_n/J_m$ then it follows

$$wsJ_m = uwrJ_m \implies B^wwsJ_m = B^wwrJ_m.$$

So ws represents the same double coset as wr in $B^w\backslash B^wwI_n/J_m$. Since $s \in w^{-1}uwrJ_m$, this implies $wsw^{-1} \in uwrw^{-1}J_m \subset U_B^-(\mathbb{Z}_p)J_m$, hence $s \in w^{-1}U_B^-(\mathbb{Z}_p)wJ_m$, whence $s \in \mathcal{R}_m \cap w^{-1}U_B^-(\mathbb{Z}_p)w = \mathcal{R}_{m,B}^w$. □

Assumption 5.1.1 can be shown by changing the definition of the $(n-v, n-v)$ entry of $u^{(v)}$ as in Januszewski [12, Prop. 2.2]. This is possible because here $u^{(v)} \in B_n(\mathbb{Q}_p)$ and not just in $U_B(\mathbb{Q}_p)$ as in *loc.cit.*

Corollary 5.1.1. *With the notations as in the previous theorem we have*

$$B_n(\mathbb{Q}_p)wI_n = \coprod_{s \in \mathcal{R}_{m,B}^w} B_n(\mathbb{Q}_p)wsJ_m.$$

In other words, the double cosets in the previous theorem are all disjoint.

Proof. Assume two of the cosets, $B_n(\mathbb{Q}_p)wsJ_m = B_n(\mathbb{Q}_p)ws'J_m$ for $s, s' \in \mathcal{R}_{m,B}^w$ are the same. This means $B^wwsJ_m = B^wws'J_m$. Since $B^w = B_n(\mathbb{Q}_p) \cap wI_nw^{-1} = B_n(\mathbb{Z}_p) \cap wI_nw^{-1}$, we see that there exists $u \in B^w \subset B_n(\mathbb{Z}_p)$ and $j \in J_m$ such that $uws = ws'j$.

Observing that J_n is normal in $G_n(\mathbb{Z}_p)$ we get that $u = ws'w^{-1} \cdot ws^{-1}w^{-1}j'$ for some $j' \in J_m$. But both the elements $ws'w^{-1}$ and $ws^{-1}w^{-1}$ are in $U_B^-(\mathbb{Z}_p)$ from the previous theorem. Combined with the fact that $u \in B_n(\mathbb{Z}_p)$ and $u \in U_B^-(\mathbb{Z}_p)J_m$ we get $u \equiv \mathbf{1}_n \pmod{J_m}$ if and only if $u \in J_m$. Then for some $j'' \in J_m$ depending on u we have $uws = wsj'' = ws'j$ because J_m is normal in $G_n(\mathbb{Z}_p)$), hence $s \equiv s' \pmod{J_m}$, whence $s = s'$. □

5.1.2 Calculation for the parabolic and the mirahoric subgroups

Now we focus on $G_4 = \mathrm{GL}_4/\mathbb{Q}$, and P the $(2,2)$ parabolic subgroup of with the Levi decomposition $P = M_P U_P$. Let U_P^- be the opposite unipotent radical contained in B_4^-. The Iwahori decomposition gives

$$G_4(\mathbb{Q}_p) = \coprod_{w \in W^P} P(\mathbb{Q}_p) w I_4,$$

where W^P is the set of Kostant representatives. This is due to the fact that the Weyl group of the Levi $W_{M_P} \subset P(\mathbb{Q}_p)$ and there is a bijection between $W_{M_P}\backslash W_4 \cong W^P$. For a fixed $w \in W_4$ define $P^w := P(\mathbb{Q}_p) \cap wI_4w^{-1}$ and $\mathcal{R}_m^w := \mathcal{R}_m \cap w^{-1}U_P^-(\mathbb{Q}_p)w$. (This $\mathcal{R}_m^w$ is different from the one in the previous subsection; this abuse of notation will not cause any confusion.)

Theorem 5.1.2. *For a fixed $w \in W^P$ the set $\mathcal{R}_m^w$ forms a complete system of coset representatives for $P(\mathbb{Q}_p)\backslash P(\mathbb{Q}_p)wI_4/J_m$, i.e.,*

$$P(\mathbb{Q}_p) w I_4 = \coprod_{s \in \mathcal{R}_m^w} P(\mathbb{Q}_p) w s J_m.$$

Moreover, we have from the Iwahori decomposition

$$G_4(\mathbb{Q}_p) = \coprod_{w \in W^P} \coprod_{s \in \mathcal{R}_m^w} P(\mathbb{Q}_p) w s J_m.$$

Proof. (The proof is essentially the same as Theorem 5.1.1.) The map $P^w w s J_m \mapsto P(\mathbb{Q}_p) w s J_m$, for $s \in \mathcal{R}_m$, is injective. Clearly, it is surjective as well. So it is enough to show that $w\mathcal{R}_m^w$ forms a system of representatives for $P^w\backslash P^w w I_N/J_m$. Suppose Assumption 5.1.1 holds for $u \in U_B^-(\mathbb{Q}_p)$, giving $u^{(0)}, u^{(1)}, u^{(2)}, u^{(3)} \in B^w$ and $r^{(0)}, r^{(1)}, r^{(2)}, r^{(3)}, r^{(4)} \in I_4$. Define $u^{(4)} \in P(\mathbb{Z}_p)$ and $r^{(5)} \in I_4$ by

$$u^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -r^{(4)}_{\sigma_w(2)\sigma_w(1)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -r^{(4)}_{\sigma_w(4)\sigma_w(3)} & 1 \end{pmatrix}, \qquad r^{(5)} = w^{-1}u^{(4)}w \cdot r^{(4)}.$$

Here σ_w is the image of w under the usual (group) isomorphism $W_4 \cong S_4$, where S_4 is the permutation group of the set $\{1,2,3,4\}$. It is clear that $u^{(4)} \in P^w$ and $wr^{(5)}w^{-1} \in U_P^-(\mathbb{Z}_p)$. Therefore if $u = u^{(4)}u^{(3)}u^{(2)}u^{(1)}$ then $u \cdot wrw^{-1} \in wI_4w^{-1} \cap U_P^-(\mathbb{Z}_p)$. The rest of the arguments are essentially the same as in Theorem 5.1.1 and Corollary 5.1.1. □

Recall the notations $n_p = v_p(N)$ and $n'_p = v_p(N')$, and $K_p^{n_p+n'_p}$ is the mirahoric subgroup of $\mathrm{GL}_4(\mathbb{Z}_p)$ of level $n_p + n'_p$. Since $J_{n_p+n'_p} \subset K_p^{n_p+n'_p}$, from Theorem 5.1.2 one gets

Corollary 5.1.2. *The set* $\bigcup_{w\in W^P}\{ws|s \in \mathcal{R}^w_{n_p+n'_p}\}$ *contains a complete set of coset representatives for* $P(\mathbb{Q}_p)\backslash G_4(\mathbb{Q}_p)/K_p^{n_p+n'_p}$.

The representatives in the above corollary up to left action of $P(\mathbb{Q}_p)$ and right action of $K_p^{n_p+n'_p}$ need not be distinct.

5.2 Explicit representatives

For $\xi_p \in P(\mathbb{Q}_p)\backslash\mathrm{GL}_4(\mathbb{Q}_p)/K_p^{n_p+n'_p}$ define $K_p^P(\xi_p) = P(\mathbb{Q}_p)\cap\xi_p K_p^{n_p+n'_p}\xi_p^{-1}$ and $K_p^{M_P}(\xi_p) = \kappa(K_p^P(\xi_p))$ denote its projection onto the Levi quotient via the canonical map $\kappa_P : P \to P/U_P \cong M_P$. Since we need a explicit set of representatives, first, the set W^P will be determined.

Lemma 5.2.1. *The following matrices are the explicit set of Kostant representatives* W^P *for the* $(2,2)$*-Parabolic subgroup* P *of* $\mathrm{GL}_4/\mathbb{Q}$.

$$w_1 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ w_2 = \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix},\ w_3 = \begin{pmatrix} 0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1 \end{pmatrix},$$
$$w_4 = \begin{pmatrix} 1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0 \end{pmatrix},\ w_5 = \begin{pmatrix} 0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0 \end{pmatrix},\ w_6 = \begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix}.$$

Proof. Recall, the definition $W^P = \{w \in W_4 \mid w^{-1}\alpha > 0, \forall\alpha \in \Pi_{M_P}\}$. Since $\Pi_{M_P} = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3 - \mathbf{e}_4\}$, a simple check shows that the enumerated elements indeed belong to W^P. Since they are distinct and a total of *six*, they are *all* the elements of W^P due to the fact $W^P \overset{\sim}{\leftrightarrow} W^{M_P}\backslash W_4$. ☐

So Corollary 5.1.2 may be restated as that $\cup_{I=1}^6\{w_i s|s \in \mathcal{R}^{w_i}_{n_p+n'_p}\}$ contains a complete set of coset representatives for $P(\mathbb{Q}_P)\backslash G_4(\mathbb{Q}_p)/K_p^{n_p+n'_p}$. For $u = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\x_1&x_2&1&0\\x_3&x_4&0&1 \end{pmatrix} \in U_P^-(\mathbb{Z}_p)$, and $i = 4,5,6$, the matrices $u_i := w_i^{-1}uw_i$, explicitly given by:

$$u_4 = \begin{pmatrix} 1&0&0&0\\x_1&1&0&x_2\\x_3&0&1&x_4\\0&0&0&1 \end{pmatrix},\quad u_5 = \begin{pmatrix} 1&x_1&0&x_2\\0&1&0&0\\0&x_3&1&x_4\\0&0&0&1 \end{pmatrix},\quad \text{and}\quad u_6 = \begin{pmatrix} 1&0&x_1&x_2\\0&1&x_3&x_4\\0&0&1&0\\0&0&0&1 \end{pmatrix},$$

are clearly in $K_p^{n_p+n'_p}$. Therefore $P(\mathbb{Q}_p)w_i sK_p^{n_p+n'_p} = P(\mathbb{Q}_p)w_iK_p^{n_p+n'_p}$ for $i = 4,5,6$, and $s \in \mathcal{R}^{w_i}_{n_p+n'_p}$. Define $\xi_p^{(0)} := \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&0&1 \end{pmatrix}$.

Lemma 5.2.2.

(i) $P(\mathbb{Q}_p)w_4K_p^{n_p+n'_p} = P(\mathbb{Q}_p)w_5K_p^{n_p+n'_p} = P(\mathbb{Q}_p)w_6K_p^{n_p+n'_p} = P(\mathbb{Q}_p)\xi_p^{(0)}K_p^{n_p+n'_p}$.

(ii) $K_p^M(\xi_p^{(0)}) = \kappa_P(P(\mathbb{Q}_p) \cap \xi_p^{(0)}K_p^{n_p+n'_p}\xi_p^{(0)^{-1}}) = K_p(n_p+n'_p) \times \mathrm{GL}_2(\mathbb{Z}_p)$.

Proof. For (i), observe that

$$w_4 = w_6 \begin{pmatrix} 0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1 \end{pmatrix} \in K_p^{n_p+n'_p}$$

and, similarly,

$$w_5 = w_6 \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix} \in K_p^{n_p+n'_p}.$$

Hence $P(\mathbb{Q}_p)w_4K_p^{n_p+n'_p} = P(\mathbb{Q}_p)w_5K_p^{n_p+n'_p} = P(\mathbb{Q}_p)w_6K_p^{n_p+n'_p}$. To get the last equality of (i), further observe that

$$w_6 = \begin{pmatrix} 1&0&0&0\\0&-1&0&1\\0&0&1&0\\0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&0&0&1 \end{pmatrix} \begin{pmatrix} 0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1 \end{pmatrix}$$

with $\begin{pmatrix} 1&0&0&0\\0&-1&0&1\\0&0&1&0\\0&0&0&1 \end{pmatrix} \in P(\mathbb{Q}_p)$ and $\begin{pmatrix} 1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1 \end{pmatrix} \in K_p^{n_p+n'_p}$. This completes the proof of (i).

For (ii), since $P(\mathbb{Q}_p) \cap \xi_p^{(0)}K_p^{n_p+n'_p}\xi_p^{(0)^{-1}} = P(\mathbb{Q}_p) \cap w_6K_p^{n'_p+n_p}w_6^{-1}$, and for an element $k = \begin{pmatrix} k_{11}&k_{12}&k_{13}&k_{14}\\k_{21}&k_{22}&k_{23}&k_{24}\\k_{31}&k_{32}&k_{33}&k_{34}\\k_{41}&k_{42}&k_{43}&k_{44} \end{pmatrix}$ in $K_p^{n_p+n'_p}$, one has $w_6kw_6^{-1} = \begin{pmatrix} k_{33}&k_{34}&k_{31}&k_{32}\\k_{43}&k_{44}&k_{41}&k_{42}\\k_{13}&k_{14}&k_{11}&k_{12}\\k_{23}&k_{24}&k_{21}&k_{22} \end{pmatrix}$, (ii) follows. □

From Lemma 5.2.2, and the discussion preceding it, one has:

$$\bigcup_{i=4,5,6} \bigcup_{s\in\mathcal{R}^{w_i}_{n_p+n'_p}} P(\mathbb{Q}_p)w_isK_p^{n'_p+n_p} = P(\mathbb{Q}_p)\xi_p^{(0)}K_p^{n'_p+n_p}.$$

Now, we will consider the other double cosets represented by w_is with $i = 1, 2, 3$, and s as before. For an element $u \in U_P^-(\mathbb{Z}_p)$ as before, and for $i = 1, 2, 3$, the matrices $u_i := w_i^{-1}uw_i$ are

$$u_1 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\x_1&x_2&1&0\\x_3&x_4&0&1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1&0&0&0\\x_1&1&x_2&0\\0&0&1&0\\x_3&0&x_4&1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1&x_1&x_2&0\\0&1&0&0\\0&0&1&0\\0&x_3&x_4&1 \end{pmatrix}.$$

For $i = 1, 2, 3$, if $w_i^{-1}uw_i \in \mathcal{R}^{w_i}_{n_p+n'_p}$ then since $\mathcal{R}^{w_i}_{n_p+n'_p} \subset I_4$, it is necessary that either $x_1, x_2, x_3, x_4 \in p\mathbb{Z}_p$, or $x_1, x_3, x_4 \in p\mathbb{Z}_p$, or $x_3, x_4 \in p\mathbb{Z}_p$ depending on whether $i = 1$ or 2 or 3. Henceforth, assume $v_p(x_3), v_p(x_4) > 0$. Moreover, $P(\mathbb{Q}_p)w_iw_i^{-1}uw_iK_p^{n_p+n'_p} = P(\mathbb{Q}_p)uK_p^{n_p+n'_p}$ since $w_i \in K_p^{n_p+n'_p}$.

Lemma 5.2.3. *For $u \in U_P^-(\mathbb{Z}_p)$ such that $u \in \mathcal{R}^{w_i}_{n_p+n'_p}$ for $i = 1, 2, 3$, The double coset $P(\mathbb{Q}_p)uK_p^{n_p+n'_p}$ is also represented by one of the elements:*

$$\xi_p^{(j)} := \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&p^j&0&1 \end{pmatrix},$$

for some $0 < j \leq n_p + n'_p$.

Proof. Let $u = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ x&y&1&0\\ z&w&0&1 \end{pmatrix} \in U_P^-(\mathbb{Z}_p)$, with $v_p(z), v_p(w) > 0$. If both $w, z = 0$ then $u \in K_p^{n_p+n'_p}$. So assume that's not the case. If necessary, conjugating u by the matrix $\begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$ which belongs to both $P(\mathbb{Q}_p)$ and $K_p^{n_p+n'_p}$, assume $v_p(z) \geq v_p(w) > 0$ with $v_p(w) \neq \infty$ or in other words $v_p(z) \geq v_p(w) > 0$ and $w \neq 0$. If $z = 0$ then we skip to the next step. If $z \neq 0$ then since

$$\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ x&y&1&0\\ z&w&0&1 \end{pmatrix} = \begin{pmatrix} z^{-1}wp^{v_p(z)-v_p(w)} & 0&0&0\\ -p^{v_p(z)-v_p(w)} & 1&0&0\\ (xz^{-1}w-y)p^{v_p(z)-v_p(w)} & y&1&0\\ 0 & w&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\ p^{v_p(z)-v_p(w)}&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} w^{-1}zp^{-(v_p(z)-v_p(w))}&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}.$$

and the observation that the last two matrices are in $K_p^{n'_p+n_p}$, one is reduced to the case that $z = 0$. Define $x' := (xz^{-1}w - y)p^{v_p(z)-v_p(w)}$. It is clear that $x' \in \mathbb{Z}_p$. Again, if $y = 0$ then skip to the next step. If $y \neq 0$ then since

$$\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ x'&y&1&0\\ 0&w&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\ 0&y^{-1}p^{v_p(y)}&0&0\\ x'&0&1&0\\ 0&wy^{-1}v_p(y)&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&p^{v_p(y)}&1&0\\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\ 0&p^{-v_p(y)}y&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$$

and the observation that the last two matrices are in $K_p^{n'_p+n_p}$ and so one can assume $y = 0$. Note that $v_p(wy^{-1}p^{v_p(y)}) = v_p(w)$. Finally, due to

$$\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ b&0&1&0\\ 0&c&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&c&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ b&0&1&0\\ 0&0&0&1 \end{pmatrix}$$

it can be assumed $x' = 0$. Finally, if necessary, conjugate by a diagonal matrix which is in both $P(\mathbb{Q}_p)$ and $K_p^{n_p+n'_p}$ and what remains is one of the $\xi_p^{(j)}$. □

Theorem 5.2.1. *For $0 \leq i \leq n_p + n'_p$, we have $K_p^{M_P}(\xi_p^{(i)}) = K_p(n_p + n'_p - i) \times K_p(i)$. In particular,*

$$K_p^{M_P}(\xi_p^{(n_p)}) = K_p(n'_p) \times K_p(n_p), \quad \textit{and} \quad K_p^{M_P}(\xi_p^{(n'_p)}) = K_p(n_p) \times K_p(n'_p).$$

Proof. For $k = \begin{pmatrix} k_{11}&k_{12}&k_{13}&k_{14}\\ k_{21}&k_{22}&k_{23}&k_{24}\\ k_{31}&k_{32}&k_{33}&k_{34}\\ k_{41}&k_{42}&k_{43}&k_{44} \end{pmatrix} \in K_p^{n'_p+n_p}$ note that

$$\xi_p^{(i)} k \xi_p^{(i)^{-1}} = \begin{pmatrix} k_{11} & -k_{14}p^i+k_{12} & k_{13} & k_{14}\\ k_{21} & -k_{24}p^i+k_{22} & k_{23} & k_{24}\\ k_{31} & -k_{34}p^i+k_{32} & k_{33} & k_{34}\\ k_{21}p^i+k_{41} & -\big(k_{24}p^i+k_{44}\big)p^i+k_{22}p^i+k_{42} & k_{23}p^i+k_{43} & k_{24}p^i+k_{44} \end{pmatrix}. \tag{5.2}$$

The case $i = 0$ has already been proved. For $i = n_p + n'_p$, since $\xi_p^{(n_p+n'_p)} \in K_p^{n_p+n'_p}$, observe that

$$\begin{aligned} K_p^M(\xi_p^{(n_p+n'_p)}) &= \kappa_P(P(\mathbb{Q}_p) \cap \xi_p^{(n_p+n'_p)} K_p^{n_p+n'_p} \xi_p^{(n_p+n'_p)^{-1}}) \\ &= \kappa_P(P(\mathbb{Q}_p) \cap K_p^{n_p+n'_p}) = \mathrm{GL}_2(\mathbb{Z}_p) \times K_p(n_p + n'_p). \end{aligned}$$

Assume now that $0 < i < n_p + n'_p$. Since $k_{44} \equiv 1 \pmod{p^{n_p+n'_p}}$, one has $k_{24}p^{n_p} + k_{44} \equiv 1 \pmod{p^i}$. Also, since $v_p(k_{42}) \geq n'_p + n_p$ one has $v_p(k_{42}p^{-i}) \geq n_p + n'_p - i$. The $(4,2)$-entry of the matrix in (5.2) is 0 (because we are looking at a situation when $\xi_p^{(i)} k \xi_p^{(i)^{-1}}$ is in $P(\mathbb{Q}_p)$), which is

$$- (k_{24}p^i + k_{44})p^i + k_{22}p^i + k_{42} = (-k_{24}p^i + k_{22})p^i - k_{44}p^i + k_{42} = 0$$
$$\implies (-k_{24}p^i + k_{22}) = k_{44} - k_{42}p^{-i} \equiv 1 \pmod{p^{n_p+n'_p-i}}.$$

In other words, under the assumption $\xi_p^{(i)} k \xi_p^{(i)^{-1}} \in P(\mathbb{Q}_p)$, the $(2,4)$ and the $(4,4)$ entry of the matrix in (5.2) are congruent to $1 \pmod{p^{n_p+n'_p-i}}$ and $1 \pmod{p^i}$, respectively. Since $k_{21}p^i + k_{41} = 0$, we get that $v_p(k_{21}) \geq n_p + n'_p - i$. Similarly, $v_p(k_{23}p^i + k_{43}) \geq i$ as $v(k_{43}) \geq n'_p + n_p$. Also, note that $-k_{34}p^i + k_{32} = 0 \implies v_p(k_{32}) \geq i > 0$ and $k_{31} = 0$. Now, calculating the determinant by expanding the last row we get

$$\begin{aligned}
\mathbb{Z}_p^\times \ni \det k &= -k_{44}[k_{31}(\cdots) - k_{32}(\cdots) + k_{33}(k_{11}k_{22} - k_{12}k_{21})] \\
&\quad + k_{43}[\cdots] - k_{42}[\cdots] + k_{41}[\cdots] \\
&= -k_{44}[0 - p^i(\cdots) + k_{33}(k_{11}k_{22} - p^{n_p+n'_p-i}(\cdots))] \\
&\quad + p^{n_p+n'_p}[\cdots] - p^{n_p+n'_p}[\cdots] + p^{n_p+n'_p}[\cdots] \\
&= -k_{44}k_{33}k_{11}k_{22} + p^i(\cdots) + p^{n_p+n'_p-i}(\cdots) \quad \text{(after re-grouping)}.
\end{aligned}$$

This shows that $k_{11}k_{22}$ and $k_{33}k_{44}$ are units in $\mathbb{Z}_p$ as $0 < i < n_p + n'_p$. Therefore,

$$\det \begin{pmatrix} k_{11} & -k_{14}p^i + k_{12} \\ k_{21} & -k_{24}p^i + k_{22} \end{pmatrix} = -\big(k_{24}p^i - k_{22}\big)k_{11} + \big(k_{14}p^i - k_{12}\big)k_{21} = k_{11}k_{22} + p^i(\cdots) \in \mathbb{Z}_p^\times.$$

Similarly,

$$\det \begin{pmatrix} k_{33} & k_{34} \\ k_{23}p^i + k_{43} & k_{24}p^i + k_{44} \end{pmatrix} \in \mathbb{Z}_p^\times.$$

Combined with the previous observations $v_p(k_{21}) \geq n'_p + n_p - i$ and $v_p(k_{23}p^{n_p} + k_{43}) \geq i$, and $(-k_{24}p^i + k_{22}) \equiv 1 \pmod{p^{n_p+n'_p-i}}$ and $k_{24}p^{n_p} + k_{44} \equiv 1 \pmod{p^i}$ shows that

$$K_p^{M_P}(\xi_p^{(i)}) \subset K_p(n'_p + n_p - i) \times K_p(i).$$

For the reverse containment take an arbitrary $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)\right) \in K_p(n_p + n'_p - i) \times K_p(i)$. Then, one checks that

$$\tilde{k} = \begin{pmatrix} a & b & 0 & 0 \\ c & d+d'-1 & c'p^{-i} & (d'-1)p^{-i} \\ 0 & b'p^i & a' & b' \\ -cp^i & (1-d)p^i & 0 & 1 \end{pmatrix}$$

is in $K_p^{n_p+n'_p}$, $\xi_p^{(i)} \tilde{k} \xi_p^{(i)^{-1}} \in P(\mathbb{Q}_p)$ and $\kappa_P(\xi_p^{(i)} \tilde{k} \xi_p^{(i)^{-1}}) = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)\right)$. □

5.2.1 A corollary for global level structures

Let $\underline{i} = (i_p)_{p|NN'}$ with $i_p \in \{0, \cdots, n_p + n'_p\}$. Set

$$\xi_f^{(\underline{i})} = \prod_{p|NN'} \{\xi_p^{(i_p)}\} \times \prod_{p\nmid NN'} \{\mathbf{1}_4\} \in \mathrm{GL}_4(\mathbb{A}_f), \quad N_{\underline{i}} = \prod_{p|NN'} p^{i_p}, \quad N^{(\underline{i})} = NN'/N_{\underline{i}}.$$

Corollary 5.2.1. *For $K_f = K_f^{N+N'}$*

$$K_f^{M_P}(\xi_f^{(\underline{i})}) = K_1(N^{(\underline{i})}) \times K_1(N_{\underline{i}}).$$

*In particular, if $\underline{i} = (n_p)_{p|NN'}$ (*resp., *$\underline{i} = (n'_p)_{p|NN'}$) we have*

$$K_f^{M_P}(\xi_f^{(\underline{i})}) = K_1(N') \times K_1(N) \quad (\text{resp., } K_f^{M_P}(\xi_f^{(\underline{i})}) = K_1(N) \times K_1(N').)$$

As a shorthand for the notation $\xi_f^{(\underline{i})}$ when $\underline{i} = (m_p)_{p|NN'}$ will be denoted as $\xi_f^{(M)}$ where $M = \prod_{p|NN'} p^{m_p}$. Hence $K_f^{M_P}(\xi_f^{(M)}) = K_1(NN'/M) \times K_1(M)$. Note that in particular $K_f^{M_P}(\xi_f^{(N)}) = K_1(N') \times K_1(N)$ and $K_f^{M_P}(\xi_f^{(N')} = K_1(N) \times K_1(N')$.

Chapter 6

The main theorems on the ratios of L-values

In this chapter we use the formalism and the calculation of double coset representatives from the previous chapter to give integral structures on algebraically induced representations given in Harder and Raghuram [7].

6.1 Integral structures on the induced space

Specialize by considering the mirahoric subgroups; let $K_f = K_f^{N+N'}$. For Hecke modules in inner cohomology $\sigma' \in \mathrm{Coh}_!(G_2, \mu')$ and $\sigma \in \mathrm{Coh}_!(G_2, \mu)$, and for $A = E, E_{\mathfrak{l}}, \mathbb{C}$, define $I_4^{\mathsf{S}}(\sigma_f, \sigma'_f, \epsilon', A)$ to be the isotypic component

$$ {}^a\mathrm{Ind}\left(H^2_!(S^{M_P}, \widetilde{\mathcal{M}}_{\mu+\mu',A})(\tilde{\epsilon}' \otimes \sigma_f \otimes \sigma'_f)\right)^{K_f^{N+N'}}, $$

where $\tilde{\epsilon'} = \epsilon' \times \epsilon'$. Similarly, one can define the isotypic spaces

$$ I_4^{\mathsf{S}}(\sigma_f, \sigma''_f, \epsilon', A), \quad I_4^{\mathsf{S}}(\sigma'_f(2), \sigma_f(-2), \epsilon', A), \quad I_4^{\mathsf{S}}(\sigma''_f(2), \sigma_f(-2), \epsilon', A). $$

The latter two are algebraically induced representations of the isotypic components of cohomology of the Levi with coefficients $\mathcal{M}_{\mu'(-2)+\mu(2),A}$. The subscript 4 in $I_4^{\mathsf{S}}(...)$ is because these induced spaces appear in the cohomology of the boundary strata $\partial_P S^{(4)}$ with certain coefficients at degree 4; see Section 6.2.1 for some details.

Now, collect all these induced spaces (for $A = E, E_{\mathfrak{l}}, \mathbb{C}$):

$$ I_4^{\mathsf{S}}(\mu+\mu', A) := \bigoplus_{\epsilon'} \bigoplus_{\sigma \in \mathrm{Coh}_!(G_2,\mu)} \bigoplus_{\sigma' \in \mathrm{Coh}_!(G_2,\mu')} I_4^{\mathsf{S}}(\sigma_f, \sigma'_f, \epsilon', A), $$

$$ I_4^{\mathsf{S}}(\mu'(-2)+\mu(2), A) := \bigoplus_{\epsilon'} \bigoplus_{\sigma \in \mathrm{Coh}_!(G_2,\mu)} \bigoplus_{\sigma' \in \mathrm{Coh}_!(G_2,\mu')} I_4^{\mathsf{S}}(\sigma'_f(2), \sigma'_f(-2), \epsilon', A). $$

The notations $I_4^{\mathsf{S}}(\mu+\mu',\epsilon',A)$ and $I_4^{\mathsf{S}}(\mu'(-2)+\mu(2),\epsilon',A)$ will mean the $\tilde{\epsilon}' = \epsilon' \times \epsilon'$ isotypic component. There are only finitely many summands because we have taken $K_f^{N+N'}$ invariants.

Applying the isomorphism in Eq. (5.1) and Corollary 5.2.1 one gets for $A = E, E_{\mathfrak{l}}, \mathbb{C}$

$$I_4^{\mathsf{S}}(\mu+\mu',A) \cong \bigoplus_{\epsilon'} \bigoplus_{\substack{M,M' \\ MM'=NN'}} H_!^2(S^{M_P}_{M\times M'}, \widetilde{\mathcal{M}}_{\mu+\mu',A})(\tilde{\epsilon}'),$$

$$I_4^{\mathsf{S}}(\mu'(-2)+\mu(2),A) \cong \bigoplus_{\epsilon'} \bigoplus_{\substack{M,M' \\ MM'=NN'}} H_!^2(S^{M_P}_{M'\times M}, \widetilde{\mathcal{M}}_{\mu'(-2)+\mu(2),A})(\tilde{\epsilon}').$$

Now we can appeal to the discussion in the beginning of Chapter 5; the $\mathcal{O}_E$ or $\mathcal{O}_{\mathfrak{l}}$–lattice are clear; define for $A^\circ = \mathcal{O}_E, \mathcal{O}_{\mathfrak{l}}$:

$$I_4^{\mathsf{S}}(\mu+\mu',A^\circ) := \bigoplus_{\epsilon'} \bigoplus_{\substack{M,M' \\ MM'=NN'}} \tilde{H}_!^2(S^{M_P}_{M\times M'}, \widetilde{\mathcal{M}}_{\mu+\mu',A^\circ}),$$

$$I_4^{\mathsf{S}}(\mu'(-2)+\mu(2),A^\circ) := \bigoplus_{\epsilon'} \bigoplus_{\substack{M,M' \\ MM'=NN'}} \tilde{H}_!^2(S^{M_P}_{M'\times M}, \widetilde{\mathcal{M}}_{\mu'(-2)+\mu(2),A^\circ}).$$

These are lattices in $I_4^{\mathsf{S}}(\mu+\mu',A)$ and $I_4^{\mathsf{S}}(\mu'(-2)+\mu(2),A)$ respectively when $A = E, E_{\mathfrak{l}}$. It follows from the definitions that for $A = \mathcal{O}_{\mathfrak{l}}, E, E_{\mathfrak{l}}, \mathbb{C}$:

$$I_4^{\mathsf{S}}(\mu+\mu',\mathcal{O}_E)\otimes_{\mathcal{O}_E} A \cong I_4^{\mathsf{S}}(\mu+\mu',A), \text{ and}$$
$$I_4^{\mathsf{S}}(\mu'(-2)+\mu(2),\mathcal{O}_E)\otimes_{\mathcal{O}_E} A \cong I_4^{\mathsf{S}}(\mu'(-2)+\mu(2),A).$$

6.2 Congruence of the Eisenstein operator

6.2.1 Review of the work Harder and Raghuram

We briefly summarize the technical results of [7]; especially, Section 5.3.7, Theorem 5.12, and the proof of Theorem 6.2 in Section 6.3.7.

Here we assume that the pair of weights (μ,μ') satisfies the conditions of the combinatorial lemma (see [7, Lem. 7.14]); this then gives us λ a weight on $\mathrm{GL}_4/\mathbb{Q}$ which is of the form $\lambda = w^{-1}\cdot(\mu+\mu')$ for a Kostant representative $w \in W^P$ with $l(w) = 2 = \dim(U_P)/2$. The Eisenstein operator comes about as follows: assume the pair of weights (μ,μ') is on the right of the unitary axis. For $\tau' \in \mathrm{Coh}_!(G_2,\mu')$ and $\tau \in \mathrm{Coh}_!(G_2,\mu)$ the image of the composition of maps:

$$H^4(S^{(4)},\widetilde{\mathcal{M}}_{\lambda,E})^{K_f^{N+N'}} \xrightarrow{\mathfrak{r}^*} H^4(\partial S^{(4)},\widetilde{\mathcal{M}}_{\lambda,E})^{K_f^{N+N'}}$$
$$\xrightarrow{\mathfrak{R}^4_{\tau_f,\tau'_f,\epsilon'}} I_4^{\mathsf{S}}(\tau_f,\tau'_f,\epsilon',E)\oplus I_4^{\mathsf{S}}(\tau'_f(2),\tau_f(-2),\epsilon',E)$$

is a k_τ-dimensional subspace of $I_4^{\mathsf{S}}(\tau_f, \tau'_f, \epsilon', E) \oplus I_4^{\mathsf{S}}(\tau'_f(2), \tau_f(-2), \epsilon', E)$, where k_τ is the common dimension of the two summands $I_4^{\mathsf{S}}(\tau_f, \tau'_f, \epsilon', E)$ and $I_4^{\mathsf{S}}(\tau'_f(2), \tau_f(-2), \epsilon', E)$. The image, denoted by $\mathfrak{I}^4(\tau_f, \tau'_f, \epsilon', E)$, is of the form

$$\mathfrak{I}^4(\tau_f, \tau'_f, \epsilon', E) = \{(\phi_f, \phi_f + T_{\text{Eis}}(\tau, \tau', \epsilon', E)\phi_f) \mid \phi_f \in I_4^{\mathsf{S}}(\tau_f, \tau'_f, \epsilon', E)\}$$

where the Eisenstein operator T_{Eis} is such that

$$T_{\text{Eis}}(\tau, \tau', \epsilon', E) \otimes_\iota \mathbb{C} = T_{\text{st}}(-2, \tau \otimes \tau')^\bullet.$$

The Eisenstein operator is a rational 'avatar' of the map induced in cohomology by the standard intertwining operator which is only defined at the transcendental level.

Now collect all the summands by running over all the Hecke modules in inner cohomology. Define:

$$\mathfrak{R}^4_{\mu,\mu'} := \sum_{\epsilon'} \sum_{\tau'_f} \sum_{\tau_f} \mathfrak{R}^4_{\tau_f, \tau'_f, \epsilon'}.$$

Applying the above discussion on the Eisenstein operator, the image

$$\begin{aligned} H^4(S^{(4)}, \widetilde{\mathcal{M}}_{\lambda,E})^{K_f^{N+N'}} &\xrightarrow{\mathfrak{r}^*} H^4(\partial S^{(4)}, \widetilde{\mathcal{M}}_{\lambda,E})^{K_f^{N+N'}} \\ &\xrightarrow{\mathfrak{R}^4_{\mu,\mu'}} I_4^{\mathsf{S}}(\mu + \mu', E) \oplus I_4^{\mathsf{S}}(\mu'(-2) + \mu(2), E) \end{aligned} \quad (6.1)$$

is a k-dimensional subspace of $I_4^{\mathsf{S}}(\mu + \mu', E) \oplus I_4^{\mathsf{S}}(\mu'(-2) + \mu(2), E)$, where k now is the common dimension of $I_4^{\mathsf{S}}(\mu+\mu', E)$ and $I_4^{\mathsf{S}}(\mu'(-2)+\mu(2), E)$. The image of the composition of maps, denoted by $\mathfrak{I}^4(\mu, \mu', E)$, is of the form:

$$\mathfrak{I}^4(\mu, \mu', E) = \bigoplus_{\epsilon'} \bigoplus_{\tau'} \bigoplus_{\tau} \{(\phi_f, \phi_f + T_{\text{Eis}}(\tau, \tau', \epsilon', E)\phi_f) \mid \phi_f \in I_4^{\mathsf{S}}(\tau_f, \tau'_f, \epsilon', E)\}.$$

Define an E–linear morphism from the sum of induced representations to this image as follows:

$$\pi^{\mathfrak{I}}_{\mu+\mu'} : I_4^{\mathsf{S}}(\mu + \mu', E) \to \mathfrak{I}^4(\mu, \mu', E),$$

given by

$$\sum_{\tau, \tau', \epsilon'} \phi_{\tau_f, \tau'_f, \epsilon'} \mapsto \sum_{\tau, \tau', \epsilon'} (\phi_{\tau_f, \tau'_f, \epsilon'} \, , \, \phi_{\tau_f, \tau'_f, \epsilon'} + T_{\text{Eis}}(\tau, \tau', \epsilon', E)\phi_{\tau_f, \tau'_f, \epsilon'}),$$

and, similarly, define an E–linear morphism from the image to the sum of induced representations:

$$\pi_{\mathfrak{I}}^{\mu'(-2)+\mu(2)} : \mathfrak{I}^4(\mu, \mu', E) \to I_4^{\mathsf{S}}(\mu'(-2) + \mu(2), E)$$

given by

$$\sum_{\tau, \tau', \epsilon'} (\phi_{\tau_f, \tau'_f, \epsilon'} \, , \, \phi_{\tau_f, \tau'_f, \epsilon'} + T_{\text{Eis}}(\tau, \tau', \epsilon', E)\phi_{\tau_f, \tau'_f, \epsilon'}) \mapsto \sum_{\tau', \tau, \epsilon'} T_{\text{Eis}}(\tau, \tau', \epsilon', E)\phi_{\tau_f, \tau'_f, \epsilon'},$$

where $\phi_{\tau_f, \tau'_f, \epsilon'} \in I_4^{\mathsf{S}}(\tau_f, \tau'_f, \epsilon', E)$.

For $A = E, E_{\mathfrak{l}}, \mathbb{C}$ define

$$T_{\text{Eis}}(\tau, \tau', \epsilon', A) := T_{\text{Eis}}(\tau, \tau', \epsilon', E) \otimes A. \quad (6.2)$$

6.2.2 Another integral structure on the induced space

Using Eq. (6.1) define an $\mathcal{O}_E$–lattice of full rank in $\mathfrak{I}^4(\mu,\mu',E)$ as follows:

$$\mathfrak{I}^4(\mu,\mu',\mathcal{O}_E) := \mathrm{Im}\left(\tilde{H}^4(\partial S^{(4)},\widetilde{\mathcal{M}}_{\lambda,\mathcal{O}_E})^{K_f^{N+N'}} \xrightarrow{\mathfrak{R}^4_{\mu,\mu'}} \mathfrak{I}^4(\mu,\mu',E)\right),$$
$$\mathfrak{I}^4(\mu,\mu',A) := \mathfrak{I}^4(\mu,\mu',\mathcal{O}_E)\otimes_{\mathcal{O}_E} A \qquad \text{for } A=\mathcal{O}_{\mathfrak{l}}, E_{\mathfrak{l}}, \mathbb{C}.$$

The extension of $\pi^{\mathfrak{I}}_{\mu+\mu'}$ and $\pi_{\mathfrak{I}}^{\mu'(-2)+\mu(2)}$ to the $\mathfrak{l}$–adic completions will again be denoted by the same symbols. The E–linear isomorphisms $\pi^{\mathfrak{I}}_{\mu+\mu'}$ and $\pi_{\mathfrak{I}}^{\mu'(-2)+\mu(2)}$ need not preserve the $\mathcal{O}_E$–lattices in either of the co-domains. Also, to obtain the main result for other critical values, one would also like to consider Tate twists. For an integer m with $-1\le m<\frac{k'-k}{2}-1$ consider the ideals

$$\begin{gathered}\{x\in\mathcal{O}_E \mid x\cdot\pi^{\mathfrak{I}}_{\mu(m)+\mu'}\left(I_4^{\mathsf{S}}(\mu(m)+\mu,\mathcal{O}_E)\right)\subset\mathfrak{I}^4(\mu(m),\mu,\mathcal{O}_E)\},\\ \{x\in\mathcal{O}_E \mid x\cdot\pi_{\mathfrak{I}}^{\mu'(-2)+\mu(2+m)}\left(\mathfrak{I}^4(\mu(m),\mu',\mathcal{O}_E)\right)\subset I_4^{\mathsf{S}}(\mu'(-2)+\mu(2+m),\mathcal{O}_E)\},\end{gathered}\tag{6.3}$$

and define the union of their supports to be the set S_{Eis} of primes which we would like to avoid in Eisenstein cohomology. By definition it follows that $\mathsf{S}_{\text{Eis}}=\mathsf{S}_{\text{Eis}}(\mu,\mu',m)$ depends only on the weights μ' and μ and the finitely many Tate twists and not on any of the isotypic components of the cohomology group of the Levi. The following lemma follows from the definition of S_{Eis}:

Lemma 6.2.1. *If $\mathfrak{l}\notin S_{\text{Eis}}$ then*

$$\begin{gathered}\pi^{\mathfrak{I}}_{\mu'+\mu}\left(I_4^{\mathsf{S}}(\mu(m)+\mu',\mathcal{O}_{\mathfrak{l}})\right)\subseteq\mathfrak{I}^4(\mu(m),\mu',\mathcal{O}_{\mathfrak{l}}),\\ \pi_{\mathfrak{I}}^{\mu'(-2)+\mu(2+m)}\left(\mathfrak{I}^4(\mu(m),\mu',\mathcal{O}_{\mathfrak{l}})\right)\subseteq I_4^{\mathsf{S}}(\mu'(-2)+\mu(2+m),\mathcal{O}_{\mathfrak{l}}).\end{gathered}\tag{6.4}$$

6.2.3 Congruence of the Eisenstein operator

Theorem 6.2.1. *Let $\phi'_f\in I_4^{\mathsf{S}}(\sigma_f,\sigma'_f,\epsilon',E_{\mathfrak{l}})$ and $\phi''_f\in I_4^{\mathsf{S}}(\sigma_f,\sigma''_f,\epsilon',E_{\mathfrak{l}})$. Assume that $\mathfrak{l}\notin\mathsf{S}_{Eis}$. Also, assume ϕ'_f and ϕ''_f are integral in that they are in $I_4^{\mathsf{S}}(\mu+\mu',\epsilon',\mathcal{O}_{\mathfrak{l}})$. We have:*

$$\phi'_f\equiv\phi''_f \pmod{\mathfrak{l}} \implies T_{\text{Eis}}(\sigma,\sigma',\epsilon',E_{\mathfrak{l}})\phi'_f\equiv T_{\text{Eis}}(\sigma,\sigma'',\epsilon',E_{\mathfrak{l}})\phi''_f \pmod{\mathfrak{l}}.$$

Proof. Suppose $B\subset V$ and $B'\subset V'$ are $\mathcal{O}_{\mathfrak{l}}$ lattices inside $E_{\mathfrak{l}}$ vector spaces. Let $T:V\to V'$ be a morphism of vector spaces such that $T(B)\subset B'$ then for $x,y\in V$ one has $x\equiv y \pmod{\mathfrak{l}}$, which, by definition means $x-y\in\mathfrak{l}B$, implies $T(x)\equiv T(y)\pmod{\mathfrak{l}}$, which, by definition, means $T(x)-T(y)\in\mathfrak{l}B'$. Since $\mathfrak{l}\notin S_{\text{Eis}}$ one has $\phi'_f-\phi''_f\in\mathfrak{l}\left(I_4^{\mathsf{S}}(\mu+\mu',\epsilon',\mathcal{O}_{\mathfrak{l}})\right)$. From which one gets $\pi^{\mathfrak{I}}_{\mu+\mu'}\left(\phi'_f\right)-\pi^{\mathfrak{I}}_{\mu+\mu'}\left(\phi''_f\right)\in\mathfrak{l}\,(\mathfrak{I}^4(\mu,\mu',\epsilon',\mathcal{O}_{\mathfrak{l}}))$. Hence,

$$\pi_{\mathfrak{I}}^{\mu'(-2)+\mu(2)}\left(\pi^{\mathfrak{I}}_{\mu+\mu'}\left(\phi'_f\right)\right)-\pi_{\mathfrak{I}}^{\mu'(-2)+\mu(2)}\left(\pi^{\mathfrak{I}}_{\mu+\mu'}\left(\phi''_f\right)\right)\in\mathfrak{l}\left(I_4^{\mathsf{S}}(\mu'(-2)+\mu(2),\epsilon',\mathcal{O}_{\mathfrak{l}})\right),$$

whence,

$$T_{\mathrm{Eis}}(\sigma,\sigma',\epsilon',E_{\mathfrak{l}})\phi'_f - T_{\mathrm{Eis}}(\sigma,\sigma'',\epsilon',E_{\mathfrak{l}})\phi''_f \in \mathfrak{l}\left(I_4^{\mathsf{S}}(\mu'(-2)+\mu(2),\epsilon',\mathcal{O}_{\mathfrak{l}})\right).$$

□

Define vectors in the induced space $I_4^{\mathsf{S}}(\mu+\mu',\epsilon',\mathcal{O}_{\mathfrak{l}})$:

$$^\circ\phi'_f(\xi_f) = \begin{cases} ^\circ v^{\epsilon'}_{\mu+\mu'}(h,h'^\rho) & \xi_f = \xi_f^{(N')} \\ 0 & \xi_f \neq \xi_f^{(N')} \end{cases} \text{ and } ^\circ\tilde{\phi}'_f(\xi_f) = \begin{cases} ^\circ v^{\epsilon'}_{\mu'(-2)+\mu(2)}(h'^\rho,h) & \xi_f = \xi_f^{(N)} \\ 0 & \xi_f \neq \xi_f^{(N)}. \end{cases}$$

Similarly, define vectors for the pair (h,h''). Then the vectors are in the induced spaces $^\circ\phi'_f \in I_4^{\mathsf{S}}(\sigma_f,\sigma'_f,\epsilon',E_{\mathfrak{l}})$ and $^\circ\phi''_f \in I_4^{\mathsf{S}}(\sigma_f,\sigma''_f,\epsilon',E_{\mathfrak{l}})$. From the definition of $I_4^{\mathsf{S}}(\mu+\mu',\epsilon',\mathcal{O}_{\mathfrak{l}})$ it follows that $^\circ\phi'_f \equiv {}^\circ\phi''_f \pmod{\mathfrak{l}}$. We get the following

Corollary 6.2.1. $T_{\mathrm{Eis}}(\sigma,\sigma',\epsilon',E_{\mathfrak{l}})^\circ\phi'_f \equiv T_{\mathrm{Eis}}(\sigma,\sigma'',\epsilon',E_{\mathfrak{l}})^\circ\phi''_f \pmod{\mathfrak{l}}$.

6.3 Computing the Eisenstein operator on the special vectors

In this section the effect of $T_{\mathrm{Eis}}(\sigma,\sigma',\epsilon',E_{\mathfrak{l}})\otimes\mathbb{C}$ on $^\circ\phi'_f$ will be determined. To do so we shall introduce periods attached to cohomology classes by comparing them with certain canonically defined vectors at the transcendental level.

6.3.1 Periods attached to the cohomology classes

For $\mathrm{GL}_2/\mathbb{Q}$

For a dominant integral weight μ for $\mathrm{GL}_2/\mathbb{Q}$, recall from 4.1.2, the relative Lie algebra cohomology $H^1(\mathfrak{g}_2,K_{2,\infty},\mathbb{D}_\mu\otimes\mathcal{M}_{\mu,\mathbb{C}})$ is nonzero. It is a two-dimensional space, on which for the action of $\mathrm{O}(2)/\mathrm{SO}(2)$, one has both the trivial and sign character appearing; for each character ϵ' of $\mathrm{O}(2)/\mathrm{SO}(2)$, the ϵ'-isotypic component $H^1(\mathfrak{g}_2,K_{2,\infty},\mathbb{D}_\mu\otimes\mathcal{M}_{\mu,\mathbb{C}})(\epsilon')$ is one-dimensional. Fix a basis $w^{\epsilon'}_\infty(\mu)$ for this one-dimensional space as in [7, Section 5.2.1].

Now, in our situation of $h'\in S_{k'}(N',\chi')^{\mathrm{new}}$ and $h\in S_k(N,\chi)^{\mathrm{new}}$, let $\mathbf{h}$ and $\mathbf{h}'^\rho$ be $\mathbb{C}$-valued automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ attached to h and h'^ρ, respectively. Let $\mathbf{h}_f$ and $\mathbf{h}'^\rho_f$ denote their restrictions to $\mathrm{GL}_2(\mathbb{A}_f)$ respectively. We have isomorphisms

$$\Phi_{(2)}: H^1(\mathfrak{g}_2,K_{2,\infty},\mathbb{D}_\mu\otimes\mathcal{M}_{\mu,\mathbb{C}})(\epsilon')\otimes\mathbb{C}\mathbf{h}_f \xrightarrow{\sim} H^1_!(S_1^{(2)}(N),\mathcal{M}_{\mu,\mathbb{C}})(\epsilon'\otimes\sigma_f),$$
$$\Phi'_{(2)}: H^1(\mathfrak{g}_2,K_{2,\infty},\mathbb{D}_{\mu'}\otimes\mathcal{M}_{\mu',\mathbb{C}})(\epsilon')\otimes\mathbb{C}\mathbf{h}'^\rho_f \xrightarrow{\sim} H^1_!(S_1^{(2)}(N'),\mathcal{M}_{\mu',\mathbb{C}})(\epsilon'\otimes\sigma'_f)$$

between one-dimensional spaces. But there are vectors which already the span one dimensional target-spaces, namely the base change of ${}^{\circ}v_{\mu}^{\epsilon'}(h)$ and ${}^{\circ}v_{\mu'}^{\epsilon'}(h'^{\rho})$ to $\mathbb{C}$ via the embedding ι. Hence, there are two complex numbers $\Omega^{\epsilon'}(\Phi'_{(2)}, \mu', \sigma')$ and $\Omega^{\epsilon'}(\Phi_{(2)}, \mu, \sigma)$ such that

$$\Phi'_{(2)}(w_{\infty}^{\epsilon'}(\mu') \otimes \mathbf{h}_f'^{\rho}) \;=\; \Omega^{\epsilon'}(\Phi'_{(2)}, \mu', \sigma') \; {}^{\circ}v_{\mu'}^{\epsilon'}(h'^{\rho})$$

and

$$\Phi_{(2)}(w_{\infty}^{\epsilon'}(\mu) \otimes \mathbf{h}_f) \;=\; \Omega^{\epsilon'}(\Phi_{(2)}, \mu, \sigma) \; {}^{\circ}v_{\mu}^{\epsilon'}(h).$$

Exactly as in [7, Section 5.2.4], there is an invariance with respect to even Tate twists; for the generators of the relative Lie algebra cohomology, one has $w_{\infty}^{\epsilon'}(\mu(2m)) = w_{\infty}^{\epsilon'}(\mu)$, and hence for the periods:

$$\Omega^{\epsilon'}(\Phi^{?}_{(2)}, \mu^{?}(2m), \sigma^{?}(-2m)) = \Omega^{\epsilon'}(\Phi^{?}_{(2)}, \mu^{?}, \sigma^{?}). \tag{6.5}$$

For the Levi quotient M_P

The discussion above on periods for GL_2 naturally boot-straps via Künneth theorem for periods for the cohomology classes for the Levi $M_P = \mathrm{GL}_2 \times \mathrm{GL}_2$. Begin by fixing the basis element $w_{\infty}^{\epsilon'}(\mu{+}\mu')$ for $H^1(\mathfrak{g}_2, K_{2,\infty}, \mathbb{D}_{\mu}\otimes\mathcal{M}_{\mu,\mathbb{C}})(\epsilon')\otimes H^1(\mathfrak{g}_2, K_{2,\infty}, \mathbb{D}_{\mu'}\otimes\mathcal{M}_{\mu',\mathbb{C}})(\epsilon')$ defined as $w_{\infty}^{\epsilon'}(\mu{+}\mu') := w_{\infty}^{\epsilon'}(\mu)\otimes w_{\infty}^{\epsilon'}(\mu')$. One has the isomorphism Φ_{M_P} from the one-dimensional

$$\Big(H^1(\mathfrak{g}_2, K_{2,\infty}, \mathbb{D}_{\mu} \otimes \mathcal{M}_{\mu,\mathbb{C}})(\epsilon') \otimes H^1(\mathfrak{g}_2, K_{2,\infty}, \mathbb{D}_{\mu'} \otimes \mathcal{M}_{\mu',\mathbb{C}})(\epsilon')\Big) \otimes (\mathbb{C}\mathbf{h}_f \otimes \mathbb{C}\mathbf{h}_f'^{\rho})$$

to the one-dimensional space

$$H^2_!(S^{M_P}_{N\times N'}, \widetilde{\mathcal{M}}_{\mu+\mu',\mathbb{C}})(\tilde{\epsilon}' \otimes {}^{\iota}\sigma_f \otimes {}^{\iota}\sigma'_f).$$

Using the the base change of the element ${}^{\circ}v_{\mu+\mu'}^{\epsilon'}(h, h'^{\rho})$ generating the target space, gives us a period $\Omega^{\tilde{\epsilon}'}(\Phi_{M_P}, w_{\infty}^{\epsilon'}(\mu+\mu'), {}^{\iota}\sigma \otimes {}^{\iota}\sigma')$. Analogously, there is a map $\tilde{\Phi}_{M_P}$, a basis element $w_{\infty}^{\epsilon'}(\mu'(-2)+\mu(2)) = w_{\infty}^{\epsilon'}(\mu'(-2)) \otimes w_{\infty}^{\epsilon'}(\mu(2))$ for the weight $\mu'(-2)+\mu(2)$ and the representation $\sigma'(2)\otimes\sigma(-2)$ spanned by the vector ${}^{\circ}v_{\mu'(-2)+\mu(2)}^{\epsilon'}(h'^{\rho}, h)$. Using (6.5) one has the following period relation:

Theorem 6.3.1.

$$\Omega^{\tilde{\epsilon}'}(\Phi_{M_P}, w_{\infty}^{\epsilon'}(\mu+\mu'), {}^{\iota}\sigma \otimes {}^{\iota}\sigma') \;=\; \Omega^{\tilde{\epsilon}'}(\tilde{\Phi}_{M_P}, w_{\infty}^{\epsilon'}(\mu'(-2)+\mu(2)), {}^{\iota}\sigma'(2) \otimes {}^{\iota}\sigma(-2)).$$

For the ambient group $\mathrm{GL}_4/\mathbb{Q}$

The discussion above on periods for cohomology classes for the Levi M_P naturally boot-straps via Delorme's lemma for periods for the cohomology classes for the ambient group $\mathrm{GL}_4/\mathbb{Q}$. By Delorme's lemma one has the isomorphism between the relative Lie algebra cohomology of a parabolically induced representation with that of the inducing representation:

$$H^1(\mathfrak{g}_2, K_{2,\infty}, \mathbb{D}_\mu \otimes \mathcal{M}_{\mu,\mathbb{C}})(\epsilon') \otimes H^1(\mathfrak{g}_2, K_{2,\infty}, \mathbb{D}_{\mu'} \otimes \mathcal{M}_{\mu',\mathbb{C}})(\epsilon') \\ \cong H^4(\mathfrak{g}_4, K_{4,\infty}, {}^{\mathrm{a}}\mathrm{Ind}(\mathbb{D}_\mu \otimes \mathbb{D}_{\mu'}) \otimes \mathcal{M}_{\lambda,\mathbb{C}})(\epsilon' \times \epsilon'),$$

Recall that we had assumed the pair of weights (μ, μ') satisfies the conditions of the combinatorial lemma which gave a weight λ on GL_4 which is of the form $\lambda = w^{-1} \cdot (\mu + \mu')$ for a Kostant representative w with $l(w) = 2 = \dim(U_P)/2$. The vector $w_\infty^{\epsilon'}(\mu + \mu')$ is now also to be thought of as a generator for the cohomology group on GL_4 via Delorme's lemma. Similarly, for $w_\infty^{\epsilon'}(\mu'(-2) + \mu(2))$.

At the finite places fix vectors in the one-dimensional space of invariants under $K_f^{N+N'}$ of induced representations which are supported only on one double-coset:

$$\psi'_f \in {}^{\mathrm{a}}\mathrm{Ind}_{P(\mathbb{A}_f)}^{G_4(\mathbb{A}_f)}(\sigma_f \otimes \sigma'_f)^{K_f^{N+N'}}, \qquad \tilde{\psi}'_f \in {}^{\mathrm{a}}\mathrm{Ind}_{P(\mathbb{A}_f)}^{G_4(\mathbb{A}_f)}(\sigma'_f(2) \otimes \sigma_f(-2))^{K_f^{N+N'}}$$

such that

$$\psi'_f(\xi_f) = \begin{cases} \mathbf{h}_f \otimes \mathbf{h}'^{\rho}_f & \xi_f = \xi_f^{(N')} \\ 0 & \xi_f \neq \xi_f^{(N')}, \end{cases} \quad \text{and} \quad \tilde{\psi}'_f(\xi_f) = \begin{cases} \mathbf{h}'^{\rho}_f(2) \otimes \mathbf{h}_f(-2) & \xi_f = \xi_f^{(N)} \\ 0 & \xi_f \neq \xi_f^{(N)}. \end{cases}$$

We have an isomorphism $\Phi_{(4)}$ between the one-dimensional space

$$H^4(\mathfrak{g}_4, K_{4,\infty}, {}^{\mathrm{a}}\mathrm{Ind}(\mathbb{D}_\mu \otimes \mathbb{D}_{\mu'}) \otimes \mathcal{M}_{\lambda,\mathbb{C}})(\epsilon' \times \epsilon') \otimes {}^{\mathrm{a}}\mathrm{Ind}_{P(\mathbb{A}_f)}^{G_4(\mathbb{A}_f)}(\sigma_f \otimes \sigma'_f)^{K_f^{N+N'}}$$

and the space $I_4^{\mathsf{S}}(\sigma_f, \sigma'_f, \epsilon', \mathbb{C})$ (see Chapter 5) giving a period construction via comparison of chosen basis elements:

$$\Phi_{(4)}(w_\infty^{\epsilon'}(\mu + \mu') \otimes \psi'_f) \;=\; \Omega^{\tilde{\epsilon}'}(\Phi_{M_P},\, w_\infty^{\epsilon'}(\mu + \mu'),\, \sigma \otimes \sigma') \,{}^\circ\phi'_f. \tag{6.6}$$

Similarly, we have a map $\tilde{\Phi}_{(4)}$ such that

$$\tilde{\Phi}_{(4)}(w_\infty^{\epsilon'}(\mu'(-2) + \mu(2)) \otimes \tilde{\psi}'_f) = \Omega^{\tilde{\epsilon}'}(\Phi_{M_P}, w_\infty^{\epsilon'}(\mu'(-2) + \mu(2)), \sigma'(2) \otimes \sigma(-2))$$

6.3.2 The standard intertwining operator on the special vectors

The reader is referred to [7, 6.3.3] for the definition and notations for the standard intertwining operator.

At the infinite place

Recall that we have assumed (μ, μ') satisfies the conditions of [7, Lem. 7.14]; in particular, the values of $L(s, \sigma_\infty \times \sigma'^{\mathsf{v}}_\infty)$ are finite at $s = -1$ and $s = -2$. Define an operator between induced representations:

$$T_{\rm loc}(\sigma_\infty \otimes \sigma'_\infty) : {}^{\rm a}{\rm Ind}_{P_\infty}^{G_{4,\infty}}(\mathbb{D}_\mu \otimes \mathbb{D}_{\mu'}) \ \to \ {}^{\rm a}{\rm Ind}_{P_\infty}^{G_{4,\infty}}(\mathbb{D}_{\mu'}(2) \otimes \mathbb{D}_\mu(-2))$$

such that the map it induces at the level of the relative Lie algebra cohomology is pinned down by:

$$T_{\rm loc}(\sigma_\infty \otimes \sigma'_\infty)^\bullet(w_\infty^{\epsilon'}(\mu + \mu')) \ = \ w_\infty^{\epsilon'}(\mu'(-2) + \mu(2)).$$

On the other hand, there is the standard intertwining operator

$$T_{\rm st}(-2, \sigma_\infty \otimes \sigma'_\infty) : {}^{\rm a}{\rm Ind}_{P_\infty}^{G_\infty}(\mathbb{D}_\mu \otimes \mathbb{D}_{\mu'}) \ \to \ {}^{\rm a}{\rm Ind}_{P_\infty}^{G_\infty}(\mathbb{D}_{\mu'}(2) \otimes \mathbb{D}_\mu(-2)).$$

The operator $T_{\rm loc}(\sigma_\infty \otimes \sigma'_\infty)^\bullet$ and the map induced at the level of cohomology by the standard intertwining operator are equal up to a scalar multiple. From [7, Thm. 7.25], there exists a $c'_\infty \in \mathbb{Q}^\times$ such that

$$T_{\rm st}(-2, \sigma_\infty \otimes \sigma'_\infty)^\bullet(\epsilon') \ = \ c'_\infty \frac{L(-2, \sigma_\infty \times \sigma'^{\mathsf{v}}_\infty)}{L(-1, \sigma_\infty \times \sigma'^{\mathsf{v}}_\infty)} T_{\rm loc}(\sigma_\infty \otimes \sigma'_\infty)^\bullet.$$

Hence

$$T_{\rm st}(-2, \sigma_\infty \otimes \sigma'_\infty)^\bullet(w_\infty^{\epsilon'}(\mu + \mu')) \ = \ c'_\infty \frac{L(-2, \sigma_\infty \times \sigma'^{\mathsf{v}}_\infty)}{L(-1, \sigma_\infty \times \sigma'^{\mathsf{v}}_\infty)} w_\infty^{\epsilon'}(\mu'(-2) + \mu(2)). \qquad (6.7)$$

At the finite places

Let S_f denote the set of all finite places where either σ_f or σ'_f is ramified; it is the support of the integer NN'. Let S denote S_f together with the archimedean place. We will now compute the effect of the standard intertwining operator at all finite places:

$$T_{\rm st}(-2, \sigma_f \otimes \sigma_f) : {}^{\rm a}{\rm Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\sigma_f \otimes \sigma'_f) \ \to \ {}^{\rm a}{\rm Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\sigma_f(2) \otimes \sigma'_f(-2))$$

on the vector ψ'_f. By one-dimensionality of invariants under the mirahoric $K_f^{N+N'}$, the operator maps ψ'_f to a multiple of $\tilde{\psi}'_f$. This is stated in the following

Theorem 6.3.2. *There exists a nonzero constant $c'_{\mathsf{S}_f} \in E^\times$, such that*

$$T_{st}(-2, \sigma_f \otimes \sigma'_f)\psi'_f = c'_{\mathsf{S}_f} \frac{L^{\mathsf{S}}(-2, \sigma \times \sigma'^{\mathsf{v}})}{L^{\mathsf{S}}(-1, \sigma \times \sigma'^{\mathsf{v}})} \tilde{\psi}'_f.$$

Proof. It is enough to compute the value $\big(T_{\rm st}(2,\sigma'_f\otimes\sigma_f)\psi'{}_f\big)\,(\xi_f^{(N')})$. Going through the definitions, there is a $c\in\mathbb{C}^\times$ such that $T_{\rm st}(-2,\sigma_f\otimes\sigma'_f)(\psi'{}_f(\xi_f^{(N')}))=c\ \mathbf{h}'^{\rho}_f(2)\otimes\mathbf{h}_f(-2)$. The scalar c can be determined by evaluating at $\underline{\mathbf{1}}\in M_P(\mathbb{A}_f)$ as $(\mathbf{h}'^{\rho}_f(2)\otimes\mathbf{h}_f(-2))(\underline{\mathbf{1}})=1\in\mathbb{C}$. At the unramified places $p\notin \mathsf{S}_f$ this is exactly the calculation of Langlands (known as the Gindikin-Karpalevic formula) that the constant is the ratio of local L-values. The convergence is guaranteed here and $c'_{\mathsf{S}_f}=\prod_{p\in\mathsf{S}_f}c'_p$ are local constants which are in E follows from the main result in [**?**]. □

When the levels N' and N of h' and h are square-free and coprime to each other, the local constants are calculated in Section 6.4, where it is shown that c'_{S_f} is exactly the product of ratios of the correct local L-values.

At the global level

Recall once again that we have assumed (μ,μ') satisfies the conditions of [7, Lemma 7.14]; in particular, $s=-1$ and $s=-2$ are critical points for $L(s,\sigma\times\sigma'^{\mathsf{v}})$. Furthermore, we now assume that the pair (μ,μ') is on the right of the unitary axis guaranteeing holomorphy of an Eisenstein series; see Thm. 6.4 in [7, 6.3.6]. The consequence of these conditions for the classical Rankin–Selberg L-functions were discussed in Section 4.1.5.

Theorem 6.3.3. *Under* $T_{\rm Eis}(\sigma,\sigma',\epsilon',E_{\mathfrak{l}})\otimes_\iota\mathbb{C}$ *the image of* ${}^\circ\phi'_f$ *is*

$$(T_{\rm Eis}(\sigma,\sigma',\epsilon',E_{\mathfrak{l}})\otimes_\iota\mathbb{C})\ {}^\circ\phi'_f\ =\ c'_\infty c'_{\mathsf{S}_f}\frac{L^{\mathsf{S}}(-2,\sigma\times\sigma'^{\mathsf{v}})}{L^{\mathsf{S}}(-1,\sigma\times\sigma'^{\mathsf{v}})}{}^\circ\tilde{\phi}'_f.$$

Proof. The map $T_{\rm Eis}(\sigma,\sigma',\epsilon')\otimes_E\mathbb{C}:I_4^{\mathsf{S}}(\sigma_f,\sigma'_f,\epsilon',\mathbb{C})\ \to\ I_4^{\mathsf{S}}(\sigma'_f(2),\sigma_f(-2),\epsilon',\mathbb{C})$ after using the isomorphisms $\Phi_{(4)}$ and $\tilde{\Phi}_{(4)}$ is the same as the map $T_{\rm st}(-2,\sigma_\infty\otimes\sigma_\infty)^\bullet\otimes T_{\rm st}(-2,\sigma_f\otimes\sigma'_f)$. The theorem follows from (6.7) and Theorem 6.3.2. And the result follows from the fact that the periods in Theorem 6.3.1 are equal. □

6.3.3 The main results on the right of the unitary axis

Recall our notations: the cusp forms $h\in S_k(N,\chi)^{\rm new}$ and $h',h''\in S_{k'}(N',\chi')^{\rm new}$; the weights $\mu=(k-2,0)$ and $\mu'=(k'-2,0)$ which we assume are regular, i.e., $k',k>2$; a number field E containing all the Fourier coefficients of h,h',h''; the Hecke modules $\sigma\in{\rm Coh}_!(G_2,\mu)$ and $\sigma',\sigma''\in{\rm Coh}_!(G_2,\mu')$ such that for an embedding $\iota:\hat{\hat{E}}_{\mathfrak{l}}\cong\mathbb{C}$ one has: $\sigma\cong\Pi(\mathbf{h})|\cdot|^{-k/2+1}$, $\sigma'\cong\Pi(\mathbf{h}'^{\rho})|\cdot|^{-k'/2+1}$, and $\sigma''\cong\Pi(\mathbf{h}''^{\rho})|\cdot|^{-k'/2+1}$, where $\Pi(h)$ (resp., $\Pi(\mathbf{h}'^{\rho})$, $\Pi(\mathbf{h}'^{\rho})$) is the unitary cuspidal automorphic representation attached to h (resp., h'^{ρ}, h''^{ρ}). The pair (μ,μ') is such that $s=-1$ and $s=-2$ are critical points for $L(s,\sigma\times\sigma'^{\mathsf{v}})$ ([7, Lemma 7.14]) and is on the right of the unitary axis ([7, 6.3.6]).

Theorem 6.3.4. *For notations and assumptions as above, suppose $\mathfrak{l}$ is a prime ideal in E outside of $\mathsf{S}_{k'} \cup \mathsf{S}_{N'} \cup \mathsf{S}_{\text{Eis}} \cup \mathsf{S}_{c'_\infty}$, and such that for all $n \in \mathbb{N}$, we have $a(n,h') \equiv a(n,h'')$ $(\text{mod } \mathfrak{l})$ then*

$$\left(c'_{\mathsf{S}_f} \frac{L^{\mathsf{S}_f}(-2, \sigma \times \sigma'^{\mathsf{v}})}{L^{\mathsf{S}_f}(-1, \sigma \times \sigma'^{\mathsf{v}})} - c''_{\mathsf{S}_f} \frac{L^{\mathsf{S}_f}(-2, \sigma \times \sigma''^{\mathsf{v}})}{L^{\mathsf{S}_f}(-1, \sigma \times \sigma''^{\mathsf{v}})} \right) \in \mathfrak{l}. \tag{6.8}$$

Proof. From Theorem 6.3.3 one has

$$(T_{\text{Eis}}(\sigma, \sigma', \epsilon', E_{\mathfrak{l}}) \otimes_\iota \mathbb{C})^\circ \phi'_f = c'_\infty c'_{\mathsf{S}_f} \frac{L^{\mathsf{S}_f}(-2, \sigma \times \sigma'^{\mathsf{v}})}{L^{\mathsf{S}_f}(-1, \sigma \times \sigma'^{\mathsf{v}})} {}^\circ\tilde{\phi}'_f.$$

Similarly, from Theorem 6.3.3 for the pair (σ, σ'') one has

$$(T_{\text{Eis}}(\sigma, \sigma'', \epsilon', E_{\mathfrak{l}}) \otimes_\iota \mathbb{C})^\circ \phi''_f = c''_\infty c''_{\mathsf{S}_f} \frac{L^{\mathsf{S}_f}(-2, \sigma \times \sigma''^{\mathsf{v}})}{L^{\mathsf{S}_f}(-1, \sigma \times \sigma''^{\mathsf{v}})} {}^\circ\tilde{\phi}''_f.$$

Note that $c'_\infty = c''_\infty$ because they depend only on the representations at infinity and $\sigma'_\infty = \sigma''_\infty$. Applying Theorem 6.2.1 for the vectors ${}^\circ\phi'_f$ and ${}^\circ\phi''_f$ and then base-changing to $\mathbb{C}$ one gets

$$(T_{\text{Eis}}(\sigma, \sigma', \epsilon', E_{\mathfrak{l}}) \otimes_\iota \mathbb{C}\ {}^\circ\phi'_f)(\xi_f^{(N)}) \equiv (T_{\text{Eis}}(\sigma, \sigma'', \epsilon', E_{\mathfrak{l}}) \otimes_\iota \mathbb{C}\ {}^\circ\phi''_f)(\xi_f^{(N)}) \pmod{\mathfrak{l}}$$

where $x \equiv y \pmod{\mathfrak{l}}$ means that $x - y \in \mathfrak{l} H^2_!(S^{M_P}_{N' \times N}, \widetilde{\mathcal{M}}_{\mu'(-2)+\mu(2), \mathcal{O}_{\mathfrak{l}}}) \otimes_\iota \mathbb{C}$. By Theorem 6.3.3 one has

$$c'_\infty c'_{\mathsf{S}_f} \frac{L^{\mathsf{S}_f}(-2, \sigma \times \sigma'^{\mathsf{v}})}{L^{\mathsf{S}_f}(-1, \sigma \times \sigma'^{\mathsf{v}})} {}^\circ\tilde{\phi}'_f(\xi_f^{(N)}) \equiv c'_\infty c''_{\mathsf{S}_f} \frac{L^{\mathsf{S}_f}(-2, \sigma \times \sigma''^{\mathsf{v}})}{L^{\mathsf{S}_f}(-1, \sigma \times \sigma''^{\mathsf{v}})} {}^\circ\tilde{\phi}''_f(\xi_f^{(N)}) \pmod{\mathfrak{l}}.$$

Since ${}^\circ\tilde{\phi}'_f(\xi_f^{(N)}) \equiv {}^\circ\tilde{\phi}''_f(\xi_f^{(N)}) \pmod{\mathfrak{l}}$), one has

$$\left(c'_\infty c'_{\mathsf{S}_f} \frac{L^{\mathsf{S}_f}(-2, \sigma \times \sigma'^{\mathsf{v}})}{L^{\mathsf{S}_f}(-1, \sigma \times \sigma'^{\mathsf{v}})} - c'_\infty c''_{\mathsf{S}_f} \frac{L^{\mathsf{S}_f}(-2, \sigma \times \sigma''^{\mathsf{v}})}{L^{\mathsf{S}_f}(-1, \sigma \times \sigma''^{\mathsf{v}})} \right) {}^\circ\tilde{\phi}'_f(\xi_f^{(N)}) \equiv 0 \pmod{\mathfrak{l}}$$

But ${}^\circ\tilde{\phi}'_f(\xi_f^{(N)}) = {}^\circ v^{\epsilon'}_{\mu'(-2)}(h'^\rho) \otimes {}^\circ v^{\epsilon'}_{\mu(2)}(h) \not\equiv 0 \pmod{\mathfrak{l}}$. Hence (6.3.4) follows since $\mathfrak{l} \notin \mathsf{S}_{c'_\infty}$. □

Theorem 6.3.5. *Let $h', h'' \in S_{k'}(N', \chi')$ and $h \in S_k(N, \chi)$ with $k', k > 2$ and $k' - k > 2$. Assume $\mathfrak{l} \notin \mathsf{S}_{\mu'} \cup \mathsf{S}_{\text{level}} \cup \mathsf{S}_{\text{Eis}} \cup \mathsf{S}_{c_\infty}$. If for all $n \in \mathbb{N}$, we have $a(n,h') \equiv a(n,h'') \pmod{\mathfrak{l}}$, and the Galois representations attached to h' and h'' modulo $\mathfrak{l}$ are both absolutely irreducible then, for integer m and $-1 \le m < \frac{k'-k}{2} - 1$,*

$$c'_{\mathsf{S}_f}(m) \frac{L^{\mathsf{S}_f}(k' - m - 3, h \times h')}{L^{\mathsf{S}_f}(k' - m - 2, h \times h')} - c''_{\mathsf{S}_f}(m) \frac{L^{\mathsf{S}_f}(k' - m - 3, h \times h'')}{L^{\mathsf{S}_f}(k' - m - 2, h \times h'')} \in \mathfrak{l}, \tag{6.9}$$

where $c'_{\mathsf{S}_f}(m) = \prod_{p|NN'} c'_p(m)$ and $c''_{\mathsf{S}_f}(m) = \prod_{p|NN'} c''_p(m)$ with $c'_p(m), c''_p(m) \in E$.

Proof. Recall the identification $H^1_!(S_1^{(2)}(N), \widetilde{\mathcal{M}}_{\mu,\mathcal{O}_{\mathfrak{l}}}) \cong H^1_!(S_1^{(2)}(N), \widetilde{\mathcal{M}}_{\mu(m),\mathcal{O}_{\mathfrak{l}}})$. Now apply Theorem 6.3.4 to the pair $(\mu(m), \mu')$ and then use the dictionary between classical and automorphic L-functions. □

Theorem 6.3.6. *Let the notations and assumptions be as in Theorem 6.3.5. Further assume that the levels N and N' are square-free and relatively prime. Then for $-1 \leq m < \frac{k'-k}{2} - 1$*

$$\frac{L(k'-m-3, h\times h')}{L(k'-m-3, h\times h')} - \frac{L(k'-m-2, h\times h'')}{L(k'-m-2, h\times h'')} \in \mathfrak{l}. \tag{6.10}$$

Proof. Under the hypothesis on the levels N and N', for $p|NN'$, it is proved in Section 6.4 that

$$c'_p(m) = \frac{L_p(k'-3-m, h\times h')}{L_p(k'-2-m, h\times h')}.$$

Similarly for $c''_p(m)$. Then the congruence in Theorem 6.3.5 becomes exactly (6.10). □

6.3.4 The left of the unitary axis

If the pair (μ, μ') is on the left of the unitary axis, then we reverse the direction of the intertwining operator and consider the intertwining operator

$$T_{\mathrm{st}}(s)|_{s=2} : {}^{\mathrm{a}}\mathrm{Ind}_P^G({}^{\iota}\sigma'(2) \times {}^{\iota}\sigma(-m-2)) \longrightarrow {}^{\mathrm{a}}\mathrm{Ind}_P^G({}^{\iota}\sigma(-m) \times {}^{\iota}\sigma')\,.$$

Now we are in the right of the unitary axis for the pair $(\mu'(-2), \mu(m+2))$. One can now define the E-linear isomorphisms analogous to $\pi^{\mathfrak{J}}_{\mu+\mu'}$ and $\pi_{\mathfrak{J}}^{\mu'(-2)+\mu(2+m)}$, say $\pi^{\mathfrak{J}}_{\mu'(-2)+\mu(m+2)}$ and $\pi_{\mathfrak{J}}^{\mu(m)+\mu'}$. Enlarge the set $\mathsf{S}_{\mathrm{Eis}}$ and S_{c_∞} if necessary, which we shall denote again by $\mathsf{S}_{\mathrm{Eis}}$ and S_{c_∞} respectively.

Assuming $\mathfrak{l} \notin \mathsf{S}_{\mathrm{weight}} \cup \mathsf{S}_{\mathrm{level}} \cup \mathsf{S}_{\mathrm{Eis}} \cup \mathsf{S}_{c_\infty}$, for an integer m and $\frac{k'-k}{2} - 1 \leq m < k'-k-2$ one gets

$$\tilde{c}'_{\mathsf{S}_f}(m)\frac{L^{\mathsf{S}_f}(2, \sigma' \times \sigma(-m)^{\mathsf{v}})}{L^{\mathsf{S}_f}(3, \sigma' \times \sigma(-m)^{\mathsf{v}})} - \tilde{c}''_{\mathsf{S}_f}(m)\frac{L^{\mathsf{S}_f}(2, \sigma'' \times \sigma(-m)^{\mathsf{v}})}{L^{\mathsf{S}_f}(3, \sigma'' \times \sigma(-m)^{\mathsf{v}})} \in \mathfrak{l}$$

which is equivalent to

$$\tilde{c}'_{\mathsf{S}_f}(m)\frac{L^{\mathsf{S}_f}(m+k+1, h'^{\rho} \times h^{\rho})}{L^{\mathsf{S}_f}(m+k+2, h'^{\rho} \times h^{\rho})} - \tilde{c}''_{\mathsf{S}_f}(m)\frac{L^{\mathsf{S}_f}(m+k+1, h'^{\rho} \times h^{\rho})}{L^{\mathsf{S}_f}(m+k+2, h'^{\rho} \times h^{\rho})} \in \mathfrak{l}.$$

Yet we cannot conclude, using the functional equation, that the ratios to the left of the line of the symmetry are congruent modulo $\mathfrak{l}$, because $\mathfrak{l} \notin \mathsf{S}_{\mathrm{Eis}} \cup \mathsf{S}_{c_\infty}$ ensures that

$$\tilde{c}'_{\mathsf{S}_f}(m)\frac{L^{\mathsf{S}_f}(m+k+1, h'^{\rho} \times h^{\rho})}{L^{\mathsf{S}_f}(m+k+2, h'^{\rho} \times h^{\rho})}, \quad \tilde{c}''_{\mathsf{S}_f}(m)\frac{L^{\mathsf{S}_f}(m+k+1, h''^{\rho} \times h^{\rho})}{L^{\mathsf{S}_f}(m+k+2, h''^{\rho} \times h^{\rho})} \tag{6.11}$$

are in $\mathcal{O}_{\mathfrak{l}}$ but they need not be in $\mathcal{O}_{\mathfrak{l}}^{\times}$. Therefore, if one further assumes that the quantities in (6.11) are $\mathfrak{l}$-adic units then one gets, using the functional equation, for $k \leq m < k' - 1$:

$$c'_{\mathsf{S}_f}(m)\frac{L^{\mathsf{S}_f}(m, h \times h')}{L^{\mathsf{S}_f}(m+1, h \times h')} - c'_{\mathsf{S}_f}(m)\frac{L^{\mathsf{S}_f}(m, h \times h'')}{L^{\mathsf{S}_f}(m+1, h \times h'')} \in \mathfrak{l}.$$

As before, if the levels N and N' are square-free and relatively prime then for $k \leq m < k' - 1$

$$\frac{L(m, h \times h')}{L(m+1, h \times h')} - \frac{L(m, h \times h'')}{L(m+1, h \times h'')} \in \mathfrak{l}.$$

6.3.5 Varying the modular forms of lower weight

For convenience take twists $\mu(-m)$ instead of $\mu(m)$. Only for the integers $3 \leq m \leq k - k' + 1$ two successive L-values are critical. The pair $(\mu(-m), \mu')$ is on the right of the unitary axis only for integers m and $\frac{k-k'}{2} + 1 < m \leq k - k' + 1$. Then going through the above proof with a newly defined $\mathsf{S}_{\mathrm{Eis}}$ and S_{c_∞} one gets

Theorem 6.3.7. *Let $h \in S_k(N, \chi)$ and $h', h'' \in S_{k'}(N', \chi')$ with $k', k > 2$ and $k - k' \geq 2$. Assume $\mathfrak{l} \notin \mathsf{S}_{k'} \cup \mathsf{S}_{\mathrm{level}} \cup \mathsf{S}_{\mathrm{Eis}} \cup \mathsf{S}_{c_\infty}$. If $a(n, h') \equiv a(n, h'') \pmod{\mathfrak{l}}$ for all $n \geq 1$, and the Galois representations attached to h' and h'' modulo $\mathfrak{l}$ are both absolutely irreducible, then for an integer m such that $\frac{k-k'}{2} + 1 < m \leq k - k' + 1$, one has*

$$c'_{\mathsf{S}_f}(m)\frac{L^{\mathsf{S}_f}(k' + m - 3, h \times h')}{L^{\mathsf{S}_f}(k' + m - 2, h \times h')} - c''_{\mathsf{S}_f}(m)\frac{L^{\mathsf{S}_f}(k' + m - 3, h \times h'')}{L^{\mathsf{S}_f}(k + m - 2, h \times h'')} \in \mathfrak{l},$$

where $c'_{\mathsf{S}_f}(m) = \prod_{p|NN'} c'_p(m)$ and $c''_{\mathsf{S}_f}(m) = \prod_{p|NN'} c''_p(m)$ with $c'_p(m), c''_p(m) \in E$. If the levels N and N' are square-free and relatively prime then

$$\frac{L(k' + m - 3, h \times h')}{L(k' + m - 2, h \times h')} - \frac{L(k' + m - 3, h \times h'')}{L(k' + m - 2, h \times h'')} \in \mathfrak{l}.$$

The reader should bear in mind that $c'_{\mathsf{S}_f}(m), c''_{\mathsf{S}_f}(m)$ in Theorem 6.3.5 and Theorem 6.3.7 are different. When one is on the left of the unitary axis same remarks as in Section 6.3.4 applies here.

6.3.6 A non-example

Recall the non-example from Theorem 3.1.4 where the ratios of the Rankin–Selberg L-values at certain critical values are not congruent. We had $h \in S_{26}(\mathrm{SL}_2(\mathbb{Z}))$ and $h', h'' \in S_{13}(\Gamma_1(3))$. The newform h' had rational Fourier coefficients and h'' had coefficients in $K := \mathbb{Q}(\sqrt{-8424})$, an imaginary quadratic extension and $\mathfrak{l}$ was a prime ideal of K lying above 13. It happened that for all $n \in \mathbb{N}$

$$a(n, h') \equiv a(n, h'') \pmod{\mathfrak{l}} \quad \text{but} \quad \frac{L(24, h \times h')}{L(25, h' \times h)} \not\equiv \frac{L(24, h \times h'')}{L(25, h \times h'')} \pmod{\mathfrak{l}}. \tag{6.12}$$

The levels are square-free and coprime to each other yet the congruence for the ratios of this particular L-values failed. There are two reasons our main theorem does not hold here. First the hypothesis $l > k'$ is violated as $l = k' = 13$. The second being the hypothesis that the mod l Galois representation $\varrho_{\Theta'}$ obtained from h' is irreducible is *not* satisfied here. To see this observe that there is an Eisenstein series $E_{13} \in M_{13}(\Gamma_1(3))$ with q-expansion $E_{13} = \frac{55601}{3} + q - 4095q^2 + q^3 + 16773121q^4 \cdots$ and that

$$E_{13} \equiv h' \equiv h'' \pmod{\mathfrak{l}},$$

which can be seen from Sturm's bound. It should be observed however that the ratios of L-values at other critical points are still congruent modulo $\mathfrak{l}$ even though it follows outside the purview of the Theorem 6.3.4

6.4 Local calculation

Let σ_p and σ'_p denote the local representations of $\Pi(\mathbf{h})|\cdot|^{-k/2+1}$ at p and $\Pi(\mathbf{h}'^{\rho})|\cdot|^{-k'/2+1}$ respectively at a prime $p|NN'$. From the Lemma on double coset representatives, both the spaces $I_P^{G_4}(s, \sigma_p \otimes \sigma'_p)^{K_p^{n_p+n'_p}}$ and $I_P^{G_4}(-s, \sigma'_p \otimes \sigma_p)^{K_p^{n_p+n'_p}}$ are one-dimensional. Let ϕ'_p and $\tilde{\phi}'_p$ respectively span those one-dimensional spaces. Again from the same lemma ϕ'_p is supported only on the double coset $P(\mathbb{Q}_p)\xi_p^{(n'_p)}K_p^{N'+N}$ and $\tilde{\phi}'_p$ is supported on $P(\mathbb{Q}_p)\xi_p^{(n_p)}K_p^{N'+N}$.

Consider the standard intertwining operator $T_{\text{st}}(s, \sigma_p \otimes \sigma'_p) : I_{P(\mathbb{Q}_p)}^{G_4(\mathbb{Q}_p)}(s, \sigma_p \otimes \sigma'_p) \to I_{P(\mathbb{Q}_p)}^{G_4(\mathbb{Q}_p)}(-s, \sigma'_p \otimes \sigma_p)$ given by the integral

$$(T_{\text{st}}(s, \sigma_p \otimes \sigma'_p)\phi'_p)(g) = \int_{U_P(\mathbb{Q}_p)} \phi'_p(w_0^{-1}ug)du,$$

where $w_0 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix}$. We know from [7] that the integral converges when $s = -2$. Since it is a map between two one-dimensional spaces there exists $c'_p \in \mathbb{C}$, such that $(T_{\text{st}}(-2, \sigma_p \otimes \sigma'_p)\phi'_p)(g) = c'_p\tilde{\phi}'_p$.

Hereafter we *assume* N and N' are square-free and coprime to each other. Due to which there is a constraint on the local representations σ_p and σ'_p. Either $\sigma_p \cong \text{St} \otimes \chi_p$ is a twist of the Steinberg representation by an unramified character and $\sigma'_p \cong \pi(\chi'_{1,p}, \chi'_{2,p})$ is an unramified principal series representation or $\sigma_p \cong \pi(\chi_{1,p}, \chi_{2,p})$ is an unramified principal series representation and $\sigma'_p \cong \text{St} \otimes \chi'_p$ is a twist of the Steinberg representation by an unramified character.

The main result of this section, Theorem 6.4.1, is to evaluate the constant c'_p when $\sigma_p \cong \mathrm{St} \otimes \chi_p$ is a twist of the Steinberg representation by an unramified character and $\sigma'_p \cong \pi(\chi'_{1,p}, \chi'_{2,p})$ is an unramified principal series representation. It will turn out to be the ratio of the correct local L-values. The same proof goes through for other possible twists accounting for the other critical L-values. The other case when $\sigma_p \cong \pi(\chi_{1,p}, \chi_{2,p})$ and $\sigma'_p \cong \mathrm{St} \otimes \chi'_p$ as well.

6.4.1 Digression on 'correct' local Euler factors

For $s \in \mathbb{C}$ and $\mathrm{Re}(s) \gg 0$ the local Euler factor $L_p(s, h \times h')$ when $p \nmid NN'$ is as follows: Suppose $1 - a(p,h)X + \chi(p)p^{k-1}X^2 = (1-\alpha_p X)(1-\beta_p X)$ and $1 - a(p,h')X + \chi'(p)p^{k'-1}X^2 = (1-\alpha'_p X)(1-\beta'_p X)$ then

$$L_p(s, h \times h')^{-1} = (1 - \alpha_p\beta_p p^{-s})(1 - \alpha'_p\beta_p p^{-s})(1 - \alpha_p\beta'_p p^{-s})(1 - \alpha'_p\beta'_p p^{-s}). \qquad (6.13)$$

Note that $1 - a(p,h)p^{-s} + \chi(p)p^{k-1}p^{-2s}$ and $1 - a(p,h')p^{-s} + \chi'(p)p^{k'-1}p^{-2s}$ are precisely $L_p(s,h)^{-1}$ and $L_p(s,h')^{-1}$ respectively. Here $L(s,h)$ and $L(s,h')$ are the completed L-functions attached to the modular forms h and h'.

Now, if $p|N$ and $p \nmid N'$ then $L_p(s,h)^{-1} = 1 - a(p,h)p^{-s}$ and $L_p(s,h')^{-1} = 1 - a(p,h')p^{-s} + \chi'(p)p^{k'-1}p^{-2s}$. With the calculation of Langlands mentioned in Section 1.1 as a cue, it would be expected that the effect of the standard intertwining operator at the place p will give the ratio

$$\frac{L_p(s, h \times h')}{L_p(s+1, h \times h')} = \frac{(1 - a(p,h)\beta_p p^{-s})(1 - a(p,h)\beta'_p p^{-s})}{(1 - a(p,h)\beta_p p^{-s-1})(1 - a(p,h)\beta'_p p^{-s-1})}.$$

A similar discussion holds in the cases when $p \nmid N$ and $p|N'$ or when p divides both N and N'.

In the sequel we shall show that this is indeed true for all $p|NN'$ when N and N' are square-free and coprime to each other.

6.4.2 Fixing canonical new vectors

Let $\sigma_p \cong \mathrm{St} \otimes \chi_p$ where χ_p is unramified, hence $n_p = 1$. Fix the new vector $v_p \in \mathrm{St} \otimes \chi_p \subsetneq I_{B_2(\mathbb{Q}_p)}^{G_2(\mathbb{Q}_p)}(|\cdot|^{1/2}\chi_p, |\cdot|^{-1/2}\chi_p)$ as follows: It is a map $v_p : G_2(\mathbb{Q}_p) \to \mathbb{C}$ such that

$$\left(\sigma_p\left(\begin{smallmatrix} t_1 & * \\ & t_2 \end{smallmatrix}\right)v_p\right)(\mathbf{1}_2) = \chi_p(t_1)\chi_p(t_2), \quad v_p(\mathbf{1}_2) = 1 \quad \text{and } v_p(\mathbf{w}) = -1/p, \qquad (6.14)$$

where $\mathbf{w} := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. See Schmidt [21, Sect. 2.1]. This normalization is a little different from the one given in *loc.cit.* This is done so that there is a *canonical* isomorphism between $\sigma_p \cong \mathrm{St} \otimes \chi_p$ and the $G_2(\mathbb{Q}_p)$ representation generated by the restriction $\mathbf{h}|_{G_2(\mathbb{Q}_p)}$ which is obtained by mapping the new vector of one space to the new vector of the other. This mapping determines χ_p.

For $\sigma'_p \cong \pi(\chi'_{1,p}, \chi'_{2,p})$, an unramified principal series representation, one has n_p is 0. The characters $\chi'_{1,p}$ and $\chi'_{2,p}$ are unramified. The normalized spherical vector of σ'_p is a function $v'_p : G_2(\mathbb{Q}_p) \to \mathbb{C}$ such that

$$\left(\sigma'_p \left(\begin{smallmatrix} t_1 & * \\ & t_2 \end{smallmatrix}\right)\right) v'_p)(1) = |t_1 t_2^{-1}|^{1/2} \chi'_{1,p}(t_1)\chi'_{2,p}(t_2). \tag{6.15}$$

Again there is a *canonical* isomorphism between $\sigma'_p \cong \pi(\chi'_{1,p}, \chi'_{2,p})$ and the $G_2(\mathbb{Q}_p)$ representation generated by the restriction $\mathbf{h}'^{\rho}|_{G_2(\mathbb{Q}_p)}$ and $\pi(\chi'_{1,p}, \chi'_{2,p})$.

One has the following relations with Fourier coefficients:

$$\chi_p(p) = p^{-1/2} a(p,h), \quad \chi'_{1,p}(p) + \chi'_{2,p}(p) = p^{-1/2} a(p, h'^{\rho}), \quad \chi'_{1,p}(p)\chi'_{2,p}(p) = p^{k'-2}\chi'^{-1}(p). \tag{6.16}$$

6.4.3 Fixing vectors in the induced space

Given $f_p \in I_P^{G_4}(s, \sigma_p \otimes \sigma'_p)$, since $f_p(g) \in V_{\sigma_p} \otimes V_{\sigma'_p}$, and the local representations being sub-representations of induced representations, one can evaluate $f_p(g)$ at an element of $\underline{m} \in M_P(\mathbb{Q}_p)$ to get: $f_p(g)(\underline{m}) = (\underline{m} \cdot f_p(g))(\underline{\mathbf{1}}) = f_p(\underline{m} \cdot g)(\underline{\mathbf{1}})$. So one can identify the induced vector $f_p(g)$ with the complex number $f_p(g)(\underline{\mathbf{1}})$. Next, since $n_p = 1, n'_p = 0$ and so $n_p + n'_p = 1$, the coset representative $\xi_p^{(n'_p)}$ in Section 5.1 is an element of $K_p^{n'_p+n_p}$. Hence $P(\mathbb{Q}_p)\xi_p^{(n_p)} K_p^{n'_p+n_p} = P(\mathbb{Q}_p)\mathbf{1}_4 K_p^{n'_p+n_p}$. To make evaluations less cumbersome we take $\xi_p^{(n_p)} = \mathbf{1}_4$. Fix the vectors in $I_P^{G_4}(-2, \sigma_p \otimes \sigma'_p)$ and $I_P^{G_4}(2, \sigma'_p \otimes \sigma_p)$ by:

$$\phi'_p(\xi_p^{(n'_p)}) = v_p \otimes v'_p, \quad \text{and} \quad \tilde{\phi}'_p(\xi_p^{(n_p)}) = v'_p \otimes v_p, \tag{6.17}$$

respectively. So, $\phi'_p(\xi_p^{(n'_p)})(\underline{\mathbf{1}}) = (v_p \otimes v'_p)(\underline{\mathbf{1}}) = 1$ and $\phi'_p(\xi_p^{(n_p)})(\mathbf{w}, \mathbf{1}) = v'_p(\mathbf{w}) \otimes v_p(\mathbf{1}) = -1/p$, where $(\mathbf{w}, \mathbf{1}) \in M_P(\mathbb{Q}_p)$. Since $\tilde{\phi}'_p$ is normalized as $\tilde{\phi}'_p(\xi_p^{(n_p)})(\underline{\mathbf{1}}) = 1$, to determine the scalar c'_p, it is enough to evaluate the integral at $\underline{\mathbf{1}}$, i.e.,

$$c'_p = (T_{\mathrm{st}}(-2, \sigma_p \otimes \sigma'_p)\phi'_p)(\xi_p^{(n_p)})(\underline{\mathbf{1}}) = (T_{\mathrm{st}}(-2, \sigma_p \otimes \sigma'_p)\phi'_p)(\mathbf{1}_4)(\underline{\mathbf{1}}).$$

This constant c'_p will be shown to be exactly the ratio of the local L-values:

Theorem 6.4.1. *Assume the levels N and N' are square-free and coprime to each other. With the notations as above*

$$c'_p = \frac{L_p(k'-3, h \times h')}{L_p(k'-2, h \times h')}.$$

In the sequel only the constant will be evaluated in the case $\sigma_p \cong \mathrm{St} \otimes \chi_p$ and $\sigma'_p \cong \pi(\chi'_{1,p}, \chi'_{2,p})$. The other calculation can be done analogously. Same comment goes for the twists of the representation.

6.4.4 Certain formal integrals

Fix a measure on $\mathbb{Q}_p$ by $\int_{\mathbb{Z}_p} dg = 1$. Fix the product measure $dx_1dx_2dx_3dx_4$ on $U_P(\mathbb{Q}_p)$ normalized by $\mathrm{vol}(U_P(\mathbb{Z}_p)) = 1$. Suppose $\underline{m} = (m_1, m_2) \in M_P(\mathbb{Q}_p)$ then for $f_p \in C_c^\infty(G_4(\mathbb{Q}_p))$

$$\int f_p\left(\underline{m}\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ x_1 & x_2 & 1 & \\ x_3 & x_4 & 0 & 1\end{pmatrix}\underline{m}^{-1}\right) dx_1dx_2dx_3dx_4 = \delta_P(\underline{m})^{1/2}\int f_p\left(\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ x_1 & x_2 & 1 & \\ x_3 & x_4 & 0 & 1\end{pmatrix}\right) dx_1dx_2dx_3dx_4, \tag{6.18}$$

where $\delta_P(\underline{m}) = |\det(m_1)|^2|\det(m_2)|^{-2}$.

Let $x_1, x_2, x_3, x_4 \in \mathbb{Q}_p^\times$. Define

$$t(x_1) = \begin{pmatrix} 1/x_1 & & & \\ & 1 & & \\ & & x_1 & \\ & & & 1\end{pmatrix},\ t(x_2) = \begin{pmatrix} 1 & & & \\ & 1/x_2 & & \\ & & x_2 & \\ & & & 1\end{pmatrix},\ t(x_3) = \begin{pmatrix} 1/x_3 & & & \\ & 1 & & \\ & & 1 & \\ & & & x_3\end{pmatrix},\ t(x_4) = \begin{pmatrix} 1 & & & \\ & 1/x_4 & & \\ & & 1 & \\ & & & x_4\end{pmatrix}.$$

For any x_i's as above and $M \in \mathbb{Z}$ formally define the operators

$$T_{<M}(x_i) := \int\limits_{\substack{x_i\in\mathbb{Q}_p\\ v_p(x_4)<M}} \delta(t(x_i))^{1/2}\ \underline{\sigma}'_p(t(x_i))dx_i, \qquad T_{\geq M}(x_i) := \int\limits_{\substack{x_i\in\mathbb{Q}_p\\ v_p(x_i)\geq M}} dx_i, \tag{6.19}$$

where $\underline{\sigma}'_p(t(x_i)) = (\sigma_p \otimes \sigma'_p)(t(x_i))$.

Lemma 6.4.1. *One has*

$$(T_{\geq 0}(x_1)\phi'_p)(\xi_p^{(n_p)})(\underline{\mathbf{1}}) + \left(T_{<0}(x_1)\phi'_p\right)(\xi_p^{(n_p)})(\underline{\mathbf{1}}) = \frac{1 - p\,\chi_p(p)\chi'_{1,p}(p^{-1})}{1 - p^2\,\chi_p(p)\chi'_{1,p}(p^{-1})}. \tag{6.20}$$

Similarly,

$$(T_{\geq 0}(x_2)\phi'_p)(\xi_p^{(n_p)})(\underline{\mathbf{1}}) + \left(T_{<0}(x_2)\phi'_p\right)(\xi_p^{(n_p)})(\underline{\mathbf{1}}) = \frac{1 - p\,\chi_p(p)\chi'_{1,p}(p^{-1})}{1 - p^2\,\chi_p(p)\chi'_{1,p}(p^{-1})},$$
$$(T_{\geq 0}(x_3)\phi'_p)(\xi_p^{(n_p)})(\underline{\mathbf{1}}) + \left(T_{<0}(x_3)\phi'_p\right)(\xi_p^{(n_p)})(\underline{\mathbf{1}}) = \frac{1 - p\,\chi_p(p)\chi'_{2,p}(p^{-1})}{1 - p^2\,\chi_p(p)\chi'_{2,p}(p^{-1})},$$
$$(T_{\geq 0}(x_4)\phi'_p)(\xi_p^{(n_p)})(\underline{\mathbf{1}}) + \left(T_{<0}(x_4)\phi'_p\right)(\xi_p^{(n_p)})(\underline{\mathbf{1}}) = \frac{1 - p\,\chi_p(p)\chi'_{2,p}(p^{-1})}{1 - p^2\,\chi_p(p)\chi'_{2,p}(p^{-1})}.$$

Proof. We prove it only when $i = 1$. Other cases are similar.

$$\phi'_p(\xi_p^{n_p})(\underline{\mathbf{1}}) + \left(T_{<0}(x_1)\phi'_p)(\xi_p^{(n_p)})\right)(\underline{\mathbf{1}}) = 1 + \int\limits_{\substack{x_i\in\mathbb{Q}_p\\ v_p(x_1)<0}} \delta_P(t(x_i))^{1/2}\ \left(\underline{\sigma}'_p(t(x_1))\phi'_p\right)(\xi_p^{(n_p)})(\underline{\mathbf{1}})dx_1,$$

the right hand side evaluates to

$$1+\sum_{M=1}^{\infty}\int_{p^{-M}\mathbb{Z}_p^{\times}} \delta_P(t(x_i))^{1/2}\ \left(\underline{\sigma}'_p(t(x_1))\phi'_p\right)(\xi_p^{(n_p)})(\underline{\mathbf{1}})dx_1$$

$$= 1+\sum_{M=1}^{\infty}\left(\frac{p-1}{p}\right)p^{2M}\chi_p(p^M)\chi'_{1,p}(p^{-M})\phi'_p(\xi_p^{(n_p)})(\underline{\mathbf{1}}),$$

$$= 1+\sum_{M=1}^{\infty}\left(\frac{p-1}{p}\right)p^{2M}\chi_p(p^M)\chi'_{1,p}(p^{-M}).$$

further simplifying as

$$1+\left(\frac{p-1}{p}\right)\sum_{M=1}^{\infty}(p^2\ \chi'_p(p)\chi_{1,p}(p^{-1}))^M$$

$$= 1+\left(\frac{p-1}{p}\right)\left(\frac{p^2\ \chi_p(p)\chi'_{1,p}(p^{-1})}{1-p^2\ \chi'_p(p)\ \chi_{1,p}(p^{-1})}\right) = \frac{1-p\,\chi_p(p)\chi'_{1,p}(p^{-1})}{1-p^2\,\chi_p(p)\ \chi'_{1,p}(p^{-1})}.$$

□

The convergence is guaranteed here because we will be in the context of [7].

Some matrix identities

Let us record some matrix identities in $\mathrm{GL}_4(\mathbb{Q}_p)$ which will be useful in the computation.

$$\begin{pmatrix}\frac{1}{c}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&c\end{pmatrix}^{-1}\begin{pmatrix}1&0&0&\frac{1}{c}\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0&0\\0&1&0&0\\a&b&1&0\\0&d&0&1\end{pmatrix}\begin{pmatrix}1&0&0&\frac{1}{c}\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\frac{1}{c}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&c\end{pmatrix}=\begin{pmatrix}1&-d&0&0\\0&1&0&0\\\frac{a}{c}&b&1&a\\0&\frac{d}{c}&0&1\end{pmatrix}.$$

$$\begin{pmatrix}1&-d&0&0\\0&1&0&0\\\frac{a}{c}&b&1&a\\0&\frac{d}{c}&0&1\end{pmatrix}=\begin{pmatrix}1&-d&0&0\\0&1&0&0\\0&0&1&a\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\\frac{a}{c}&b-\frac{ad}{c}&1&0\\0&\frac{d}{c}&0&1\end{pmatrix}.$$

$$\begin{pmatrix}1&0&0&0\\0&\frac{1}{d}&0&0\\0&0&1&0\\0&0&0&d\end{pmatrix}^{-1}\begin{pmatrix}1&0&0&0\\0&1&0&\frac{1}{d}\\0&0&1&0\\0&0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0&0\\0&1&0&0\\a&b&1&0\\c&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&\frac{1}{d}\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\frac{1}{d}&0&0\\0&0&1&0\\0&0&0&d\end{pmatrix}=\begin{pmatrix}1&0&0&0\\-c&1&0&0\\a&\frac{b}{d}&1&b\\\frac{c}{d}&0&0&1\end{pmatrix}.$$

$$\begin{pmatrix}1&0&0&0\\-c&1&0&0\\a&\frac{b}{d}&1&b\\\frac{c}{d}&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\-c&1&0&0\\0&0&1&b\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\a-\frac{bc}{d}&\frac{b}{d}&1&0\\\frac{c}{d}&0&0&1\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{a}&0&0&0\\0&1&0&0\\0&0&a&0\\0&0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&\frac{1}{a}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&b&1&0\\c&d&0&1\end{pmatrix}\begin{pmatrix}1&0&\frac{1}{a}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\frac{1}{a}&0&0&0\\0&1&0&0\\0&0&a&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&-b&0&0\\0&1&0&0\\0&\frac{b}{a}&1&0\\\frac{c}{a}&d&c&1\end{pmatrix}$$

$$\begin{pmatrix}1&-b&0&0\\0&1&0&0\\0&\frac{b}{a}&1&0\\\frac{c}{a}&d&c&1\end{pmatrix}=\begin{pmatrix}1&-b&0&0\\0&1&0&0\\0&0&1&0\\0&0&c&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&\frac{b}{a}&1&0\\\frac{c}{a}&-\frac{bc}{a}+d&0&1\end{pmatrix}.$$

Some integral identities.

Define the following matrices in $U_4(\mathbb{Q}_p)$, where U_4 is the unipotent radical of the upper triangular $B_4 \subset G_4 = \mathrm{GL}_4/\mathbb{Q}$.

$$u(x_1) = \begin{pmatrix} 1 & 0 & 1/x_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; u(x_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/x_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; u(x_3) = \begin{pmatrix} 1 & 0 & 0 & 1/x_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; u(x_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x_1, x_2, x_3, x_4 \in \mathbb{Q}_p^\times$.

Lemma 6.4.2. *If $f_p \in C^\infty(G_4(\mathbb{Q}_p))$ with $f_p(ug) = f(g)$ for all $u \in U_4(\mathbb{Q}_p)$ we have the following three identities:*

$$\int f_p\left(t(x_3)^{-1}u(x_3)^{-1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_1 & x_2 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{pmatrix}u(x_3)t(x_3)\right)dx_1dx_2dx_4$$
$$= \delta_P(t(x_3))^{1/2}\int f_p\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_1 & x_2 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{pmatrix}\right)dx_1dx_2dx_4, \quad (6.21)$$

$$\int f_p\left(t(x_4)^{-1}u(x_4)^{-1}\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ x_1 & x_2 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}u(x_4)t(x_4)\right)dx_1dx_2$$
$$= \delta_P(t(x_4))^{1/2}\int f_p\left(\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ x_1 & x_2 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)dx_1dx_2, \quad (6.22)$$

$$\int f_p\left(t(x_1)^{-1}u(x_1)^{-1}\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & x_2 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}u(x_1)t(x_1)\right)dx_2$$
$$= \delta_P(t(x_4))^{1/2}\int f_p\left(\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & x_2 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)dx_2. \quad (6.23)$$

Proof. To begin, the integral in (6.21) simplifies as

$$\int f_p\left(\begin{pmatrix} 1 & -x_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{x_1}{x_3} & x_2 & 1 & x_1 \\ 0 & \frac{x_4}{x_3} & 0 & 1 \end{pmatrix}\right)dx_1dx_2dx_4 = \int f_p\left(\begin{pmatrix} 1 & -x_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{x_1}{x_3} & x_2-\frac{x_1x_4}{x_3} & 1 & 0 \\ 0 & \frac{x_4}{x_3} & 0 & 1 \end{pmatrix}\right)dx_1dx_2dx_4.$$

The integral on the right is the same as

$$\int f_p\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{x_1}{x_3} & x_2-\frac{x_1x_4}{x_3} & 1 & 0 \\ 0 & \frac{x_4}{x_3} & 0 & 1 \end{pmatrix}\right)dx_1dx_2dx_4 = \int f_p\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{x_1}{x_3} & x_2 & 1 & 0 \\ 0 & \frac{x_4}{x_3} & 0 & 1 \end{pmatrix}\right)dx_1dx_2dx_4$$

which evaluates to

$$\int f_p\left(\begin{pmatrix} \frac{1}{x_3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x_3 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_1 & x_2 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{x_3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x_3 \end{pmatrix}\right)dx_1dx_2dx_4$$
$$= \delta_P\left(t(x_3)\right)^{1/2}\int f_p\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_1 & x_2 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{pmatrix}\right)dx_1dx_2dx_4,$$

the last equality is due to (6.18). This completes the verification of (6.21). The other integrals are similar. □

6.4.5 The evaluation

The purpose of this section is to evaluate the constant

$$I := \int \phi_p'\left(w_0^{-1}\begin{pmatrix}1&0&x_1&x_2\\0&1&x_3&x_4\\0&0&1&0\\0&0&0&1\end{pmatrix}\xi_p^{(n_p)}\right)(\underline{\mathbf{1}})dx_3dx_1dx_4dx_2$$
$$= \int \phi_p'\left(w_0^{-1}\begin{pmatrix}1&0&x_1&x_2\\0&1&x_3&x_4\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)(\underline{\mathbf{1}})dx_3dx_1dx_4dx_2. \quad (6.24)$$

This is due to our choice $\xi_p^{(n_p)} = \mathbf{1}_4$. Writing out the evaluation at $\underline{\mathbf{1}}$ makes the notation cumbersome so we shall drop it and assume implicitly.

Eliminating the variable x_3

Split the innermost integral as

$$\int_{x_3\in\mathbb{Q}_p} = \int_{x_3\in\mathbb{Q}_p:\ v_p(x_3)\geq 0} + \int_{x_3\in\mathbb{Q}_p:\ v_p(x_3)<0}.$$

If $v_p(x_3)\geq 0$ then $\begin{pmatrix}1&0&0&0\\0&1&x_3&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\in K_p^{n_p'+n_p}$, hence the first integral is

$$\int_{x_3\in\mathbb{Q}_p:\ v_p(x_3)\geq 0} \phi_p'\left(w_0^{-1}\begin{pmatrix}1&0&x_1&x_2\\0&1&x_3&x_4\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)dx_3dx_1dx_4dx_2$$
$$= \int_{x_3\in\mathbb{Q}_p:\ v_p(x_3)\geq 0} \phi_p'\left(w_0^{-1}\begin{pmatrix}1&0&x_1&x_2\\0&1&0&x_4\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)dx_3dx_1dx_4dx_2$$
$$= T_{\geq 0}(x_3)\left[\int \phi_p'\left(w_0^{-1}\begin{pmatrix}1&0&x_1&x_2\\0&1&0&x_4\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)dx_1dx_4dx_2\right]. \quad (6.25)$$

If $x_3\in\mathbb{Q}_p^\times$ and $v_p(x_3)<0$, then second integral is

$$\int \phi_p'\left(w_0^{-1}\begin{pmatrix}1&0&x_1&x_2\\0&1&x_3&x_4\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)dx_3dx_1dx_4dx_2 = \int \phi_p'\left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\x_1&x_2&1&0\\0&x_4&0&1\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&x_3&0\end{pmatrix}\right)dx_3dx_1dx_4dx_2.$$

which can be written as

$$\int \phi_p'\left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\x_1&x_2&1&0\\0&x_4&0&1\end{pmatrix}\begin{pmatrix}1&0&0&\frac{1}{x_3}\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\frac{1}{x_3}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&x_3\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0&-1&0&0\\0&\frac{1}{x_3}&1&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\right)dx_3dx_4dx_1dx_2$$

When $x_3 \in \mathbb{Q}_p^\times$ and $v_p(x_3) < 0$, $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & \frac{1}{x_3} & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in K_p^{n_p+n_p'}$. Hence it simplifies as

$$\int\limits_{x_3\in\mathbb{Q}_p^\times,\ v_p(x_3)<0} \phi_p'\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_1 & x_2 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{pmatrix} u(x_3)t(x_3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right) dx_3dx_1dx_4dx_2,$$

which can be written as

$$\int\limits_{x_3\in\mathbb{Q}_p^\times,\ v_p(x_3)<0} \underline{\sigma}_p'(t(x_3))\cdot\phi_p'\left(t(x_3)^{-1}u(x_3^{-1})\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_1 & x_2 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{pmatrix} u(x_3)t(x_3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right) dx_3dx_4dx_1dx_2,$$

which in turn becomes

$$\int\limits_{x_3\in\mathbb{Q}_p^\times,\ v_p(x_3)<0} \delta_P(t(x_3))^{1/2}\underline{\sigma}_p'(t(x_3))\cdot\phi_p'\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_1 & x_2 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right) dx_3dx_4dx_1dx_2.$$

Using the notation in (6.19) we can write the above integral as

$$\begin{aligned} T_{<0}(t(x_3))&\left[\int \phi_p'\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_1 & x_2 & 1 & 0 \\ 0 & x_4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right) dx_1dx_4dx_2\right] \\ &= T_{<0}(t(x_3))\left[\int \phi_p'\left(w_0^{-1}\begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 0 & 1 & 0 & x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} w_0\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right) dx_1dx_4dx_2\right] \\ &\qquad = T_{<0}(t(x_3))\left[\int \phi_p'\left(w_0^{-1}\begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 0 & 1 & 0 & x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) dx_1dx_4dx_2\right]. \end{aligned}$$

The last integral equals

$$T_{<0}(t(x_3))\left[\int \phi_p'\left(w_0^{-1}\begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 0 & 1 & 0 & x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) dx_1dx_4dx_2\right]. \tag{6.26}$$

Combine (6.25) and (6.26) to get

$$I = (T_{\geq 0}(x_3) + T_{<0}(x_3))\left[\int \phi_p'\left(w_0^{-1}\begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 0 & 1 & 0 & x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) dx_1dx_4dx_2\right]. \tag{6.27}$$

Eliminating the variable x_1

Carrying out a very similar computation with the the inner integral in (6.27), by splitting it up as $\int\limits_{x_1\in\mathbb{Q}_p} = \int\limits_{x_1\in\mathbb{Q}_p:\ v_p(x_1)\geq 0} + \int\limits_{x_1\in\mathbb{Q}_p^\times;\ v_p(x_1)<0}$, the integral I equals

$$I = (T_{\geq 0}(x_3) + T_{<0}(x_3))(T_{\geq 0}(x_1) + T_{<0}(x_1))\left[\int \phi_p'\left(w_0^{-1}\begin{pmatrix} 1 & 0 & 0 & x_2 \\ 0 & 1 & 0 & x_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) dx_4dx_2\right]. \tag{6.28}$$

Eliminating the variable x_4

Once again, carrying out a very similar computation with the the inner integral in (6.28), we now get

$$I = (T_{\geq 0}(x_3) + T_{<0}(x_3))(T_{\geq 0}(x_1) + T_{<0}(x_1))\big[\\ T_{\geq 0}(x_4)\Bigg[\int \phi'_p\Bigg(w_0^{-1}\begin{pmatrix} 1&0&0&x_2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\Bigg)dx_2\Bigg] + T_{<0}(x_4)\Bigg[\int \phi'_p\Bigg(w_0^{-1}\begin{pmatrix} 1&0&0&x_2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}w_0\Bigg)dx_2\Bigg]\big]. \tag{6.29}$$

Eliminating the variable x_2

There are two integrals in (6.29) to be evaluated now. The second integral in (6.29) will turn out to be 0, i.e.,

$$\int \phi_p\Bigg(w_0^{-1}\begin{pmatrix} 1&0&0&x_2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}w_0\Bigg)dx_2 = 0. \tag{6.30}$$

To see this, split the integral $\int\limits_{x_2\in\mathbb{Q}_p} = \int\limits_{x_2\in\mathbb{Q}_p:\ v_p(x_2)\geq 0} + \int\limits_{x_2\in\mathbb{Q}_p:\ v_p(x_2)<0}$, and observe that

$$\int\limits_{x_2\in\mathbb{Q}_p:\ v_p(x_2)\geq 0} \phi'_p\left(w_0^{-1}\begin{pmatrix} 1&0&0&x_2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}w_0\right)dx_2 = \int \phi'_p\left(\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&x_2&1&0\\0&0&0&1\end{pmatrix}\right)dx_2 = 0$$

because $\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&x_2&1&0\\0&0&0&1\end{pmatrix} \in K_p^{n'_p+n_p} \subset P(\mathbb{Q}_p)\mathbf{1}_4K_p^{n'_p+n_p}$ and ϕ'_p is *not* supported in the coset $P(\mathbb{Q}_p)\mathbf{1}_4K_p^{n'_p+n_p}$. Similarly,

$$\int\limits_{v(x_2)<0} \phi'_p\left(\begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&x_2\\0&1&0&0\end{pmatrix}w_0\right)dx_2 = \int \phi'_p\left(u(x_2)t(x_2)\begin{pmatrix} 1&0&0&0\\0&0&-1&0\\0&1&\frac{1}{x_2}&0\\0&0&0&1\end{pmatrix}\right)dx_2 = 0,$$

since $\begin{pmatrix} 1&0&0&0\\0&0&-1&0\\0&1&\frac{1}{x_2}&0\\0&0&0&1\end{pmatrix} \in K_p^{n_p+n'_p}$. This proves (6.30). Next, we evaluate the first integral in (6.29)

$$\int \phi'_p\Bigg(w_0^{-1}\begin{pmatrix} 1&0&0&x_2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\Bigg)dx_2,$$

By once again splitting the integral as $\int\limits_{x_2\in\mathbb{Q}_p} = \int\limits_{x_2\in\mathbb{Q}_p:\ v_p(x_2)\geq 0} + \int\limits_{x_2\in\mathbb{Q}_p:\ v_p(x_2)<0}$; the first of which equals

$$\int\limits_{x_2\in\mathbb{Q}_p:\ v_p(x_2)\geq 0} \phi'_p\Bigg(w_0^{-1}\begin{pmatrix} 1&0&0&x_2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\Bigg)dx_2 = \int\limits_{x_2\in\mathbb{Q}_p:\ v_p(x_2)\geq 0} \phi'_p(w_0^{-1})dx_2 = (v_p\otimes v'_p)(\underline{\mathbf{1}});$$

(the last equality is due to the facts, that $w_0^{-1} = Q\xi_p^{(n_p)}K_1K_2$ for some $Q \in P(\mathbb{Q}_p)$ and $K_1, K_2 \in K_p^{n'_p+n_p}$ and that ϕ'_p is supported in $P(\mathbb{Q}_p)\xi_p^{(n_p)}K_p^{n'_p+n_p}$ and on the double coset it takes the value $v'_p \otimes v_p$. Recall that in the beginning of the evaluation the evaluation at $\underline{\mathbf{1}}$ was assumed implicitly.) The latter integral over $v_p(x_2) < 0$ is

$$\int\limits_{x_2\in\mathbb{Q}_p:\, v_p(x_2)<0} \phi'_p\left(w_0^{-1}\left(\begin{smallmatrix}1&0&0&x_2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\right)dx_2$$

$$= \int \phi'_p\left(\left(\begin{smallmatrix}1&0&0&0\\0&1&\frac{1}{x_2}&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0&0&0\\0&\frac{1}{x_2}&0&0\\0&0&x_2&0\\0&0&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}-1&0&0&0\\0&0&1&0\\0&1&0&0\\\frac{1}{x_2}&0&0&1\end{smallmatrix}\right)\right)dx_2 = 0.$$

Again due to the fact ϕ'_p is not supported in $P(\mathbb{Q}_p)\mathbf{1}_4K_p^{n'_p+n_p}$. Therefore,

$$\int \phi'_p\left(w_0^{-1}\left(\begin{smallmatrix}1&0&0&x_2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\right)dx_2 = T_{\geq 0}(x_2)\phi'_p(\xi_p^{(n_p)}).$$

Final evaluation

Using the above calculations, (6.29) reduces to

$$I = (T_{\geq 0}(x_3) + T_{<0}(x_3))(T_{\geq 0}(x_1) + T_{<0}(x_1))$$
$$\left(T_{\geq 0}(x_4)\left[\int \phi_p\left(w_0^{-1}\left(\begin{smallmatrix}1&0&0&x_2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\right)dx_2\right] + T_{<0}(x_4)\left[\int \phi'_p\left(w_0^{-1}\left(\begin{smallmatrix}1&0&0&x_2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)w_0\right)dx_2\right]\right)$$

which reduces to

$$(T_{\geq 0}(x_3) + T_{<0}(x_3))(T_{\geq 0}(x_1) + T_{<0}(x_1))\left(T_{\geq 0}(x_4)\left[\int \phi'_p\left(w_0^{-1}\left(\begin{smallmatrix}1&0&0&x_2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\right)dx_2 + 0\right]\right)$$

which in turn simplifies as

$$(T_{\geq 0}(x_3) + T_{<0}(x_3))(T_{\geq 0}(x_1) + T_{<0}(x_1))\left(T_{\geq 0}(x_4)\left[T_{\geq 0}(x_2)\phi'_p(\xi_p^{(n_p)})(\underline{\mathbf{1}})\right]\right).$$

Using Lemma 6.4.1 for x_2, x_1 and x_3, we get:

$$I = \left(\frac{1 - p\,\chi_p(p)\chi'_{1,p}(p^{-1})}{1 - p^2\,\chi_p(p)\;\chi'_{1,p}(p^{-1})}\right)\left(\frac{1 - p\,\chi_p(p)\;\chi'_{2,p}(p^{-1})}{1 - p^2\,\chi_p(p)\;\chi'_{2,p}(p^{-1})}\right) = \frac{L_p(k'-3, h\times h')}{L_p(k'-2, h\times h')}.$$

The last equality is due to (6.16).

Similar computation yields the results for the other possible twists as well.

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