Regularity theory and fractionally convex functions

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

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May, 2026

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Certificate

This is to certify that this dissertation entitled Regularity theory and fractionally convex functionstowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Vishnu Vaidya at Indian Institute of Science Education and Research under the supervision of Dr. Anup Biswas, Associate Professor, Department of Mathematics , during the academic year 2025-2026.

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This thesis is dedicated to my parents

Declaration

I hereby declare that the matter embodied in the report entitled Regularity theory and fractionally convex functions are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Anup Biswas and the same has not been submitted elsewhere for any other degree. I do not claim originality for any of the content presented in Chapters 1 to 5, and the material used therein has been duly sourced from the works cited. Chapter 6 contains some original contributions; however, any material derived from other works has been properly cited and acknowledged.

Vishnu

Vishnu Vaidya

Acknowledgments

I would first like to thank my family for supporting my curiosities throughout my life, without which I would not have been able to pursue this thesis.

I first asked for advice from Dr. Bhakta on possible thesis advisors based on my interest. I was lucky that Dr. Biswas had a problem in mind and was willing to supervise me. I started the thesis with very naive knowledge of partial differential equations and discussion with Dr. Biswas helped me gain intuition behind the concepts in this wonderful field. He also let me learn at my own pace, and freedom to read topics which I thought were relevant. I could not have asked for more, and I am really grateful to Dr. Biswas for this opportunity and Dr. Bhakta for supporting me throughout my master thesis.

Finally I would like to thank all my friends who have provided company during this crucial time.

Abstract

In this thesis, we study the regularity theory developed for uniformly elliptic operators and adapt some of the techniques to the fractional convex operator, as defined in [8],. The degeneracy and nonlocal nature of the fractional convex operator posses significant challenges, which limits the extent to which the theory of the uniform elliptic regularity can be adapted to this operator. We first study the regularity theory developed for local and nonlocal operators, and showcase the importance of fractional convexity. Then, we describe the properties of the fractionally convex functions.

Contents

Introduction

In recent years, there has been significant interest in the regularity theory of nonlocal operators due to their wide range of applications. For local operators, regularity questions for second-order elliptic equations in divergence form were resolved by the so-called De Giorgi–Nash–Moser theory, and for nondivergence form operators by the Krylov–Safonov theory. Caffarelli adapted the theory of Krylov and Safonov for viscosity solutions of fully nonlinear operators, in which the properties of the convex envelope of a function are crucially exploited. For the regularity theory of nonlocal operators, there is no well-developed theory that plays the role of convex analysis in the study of local operators. In this thesis, we explore fractional convexity as defined in [8], and present material that provides context for why it may be important to study this new entity.

In the first chapter, we discuss viscosity solutions and their properties, such as the comparison principle. In the second chapter, we study the local ABP estimate for viscosity solutions and how it leads to the Harnack inequality and Hölder regularity. The third chapter is dedicated to understanding the properties of the convex envelope and the application of PDE techniques to study its regularity. In the fourth chapter, we define the notion of nonlocal operators, viscosity solutions for nonlocal operators, and state theorems that are analogous to those for local operators. We choose not to present the proofs, as they are very technical; moreover, the key ideas are similar to those in the local case. Fractional convexity and the nonlocal operator associated with the fractional convex envelope are discussed in the fifth chapter. Finally, in the sixth chapter, we present some new results obtained by adapting techniques from elliptic regularity theory.

Chapter 1

Viscosity Solutions

1.1 Introduction

The term viscosity solutions first appear in the work of Michael G. Crandall and Pierre-Louis Lions [7] in 1983 regarding the Hamilton–Jacobi equation. In fluid dynamics, viscosity has 'smoothening' effect and existence of solutions is easier in a viscous flow. In Hamilton-Jacobi equations, existence of solutions was justified by adding a viscosity term and letting that term go to zero. Hence, these solutions became known as viscosity solutions.

For a while, it was not known if viscosity solutions could be adapted to second order operators because uniqueness had not been proved. Then, in 1988, Jensen proved the comparison principle for the viscosity solutions using an approximation scheme [12].

The concept of viscosity solutions is very popular in modern approaches to non-linear PDE, because it is formulated in such a way that the comparison principle holds for a large class of operators. Once comparison principle is established, existence and uniqueness are proved using Perron's method, see [6]. Viscosity solutions also provide a rigorous foundation for many numerical schemes used to approximate PDE solutions. In particular, Barles and Souganidis [2] proved the convergence of certain finite difference schemes to viscosity solutions. Furthermore, the work of Caffarelli [3], established improved regularity results, showing that under natural assumptions viscosity solutions can, in some cases, enjoy the regularity of classical solutions. As a result, the viscosity framework has been crucial in demonstrating that numerical approximations may converge to classical solutions in appropriate settings.

For all these reasons, the theory of viscosity solutions has become ubiquitous in contemporary PDE research.

1.2 Ellipticity

For the next two sections we will be following [11]. The generalized statements for degenerate elliptic operators can be found in [6]. Before we move onto defining viscosity solutions and explore its properties, it is essential that we understand what ellipticity means in a non-linear context. We will consider the equations of the type

$$F(D^2u(x)) = 0, \qquad x \in \Omega$$

where $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ and for now, $u \in C^2(\Omega)$ and Ω is a domain of $\mathbb{R}^n$. First we give a definition to motivate the idea of ellipticity.

Definition 1.2.1. *We call $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda$, if for any two symmetric matrices M, N such that $N \geq 0$ we have,*

$$\lambda\|N\| \leq F(M+N) - F(N) \leq \Lambda\|N\|,$$

where $\|N\| = \mathrm{tr}(N)$.

What is the intuition behind this definition? Let us consider the standard equation

$$\mathrm{tr}(D^2u) = \Delta u = 0,$$

which is the harmonic equation. It is well known that harmonic functions are very well-behaved that is, these functions are smooth. The reason is that all the eigenvalues (which are directional derivatives) of D^2u can neither be too positive nor negative. This implies the solution cannot oscillate too much in any direction, and therefore is regular. This is what ellipticity captures, it enforces that the eigenvalues of the Hessian of the solution is neither too negative nor too positive. Now we define the notion of extremal Pucci operators. These operators first appeared in the work [16]. It turns out that studying the 'end' points of an ellipticity class is enough to understand every operator in that class. The reason for this will become apparent in the coming sections.

Definition 1.2.2. *Let $0 < \lambda \leq \Lambda$. $\mathcal{M}^+$, $\mathcal{M}^-$ are called extremal Pucci operators with ellipticity constants λ, Λ, where*

$$\mathcal{M}^+(M) = \sup_{\lambda Id \leq A \leq \Lambda Id} \operatorname{tr}(AM),$$
$$\mathcal{M}^-(M) = \inf_{\lambda Id \leq A \leq \Lambda Id} \operatorname{tr}(AM),$$

where A, M are symmetric matrices.

It is easy to that, as shown in [11],

$$\mathcal{M}^+(M) = \Lambda \sum_{k_i^+ > 0} k_i^+ + \lambda \sum_{k_i^- < 0} k_i^-,$$
$$\mathcal{M}^-(M) = \lambda \sum_{k_i^+ > 0} k_i^+ + \Lambda \sum_{k_i^- < 0} k_i^-,$$

where k_i^+ (k_i^-) are positive (negative) eigenvalues of M. By the definition of uniform ellipticity, it is clear that $\mathcal{M}^-(N) \leq F(M+N) - F(M) \leq \mathcal{M}^+(N)$

1.3 Viscosity solutions and existence

Recall that f is said to be lower semi-continuous or $f \in \mathrm{LSC}(\bar{\Omega})$ in $\bar{\Omega}$ if

$$\liminf_{x \to y} f(x) \geq f(y),$$

and upper semi-continuous or $f \in \mathrm{USC}(\bar{\Omega})$ if

$$\limsup_{x \to y} f(x) \leq f(y).$$

Definition 1.3.1. *Let $F : \mathbb{R}^{n \times n} \to \mathbb{R}$ be uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda$. Consider the PDE*

$$F(D^2 u) = 0$$

in Ω.

- *We call $u \in USC(\bar{\Omega})$ a subsolution in viscosity sense to $F(D^2 u) = 0$ at y if for all $\phi \in C^2(\bar{\Omega})$ such that $\phi \geq u$ in Ω and $\phi(y) = u(y)$, we have $F(D^2 \phi) \geq 0$.*

- *We call $u \in LSC(\bar{\Omega})$ a supersolution in viscosity sense to $F(D^2u) = 0$ at y if for all $\phi \in C^2(\bar{\Omega})$ such that $\phi \leq u$ in Ω and $\phi(y) = u(y)$, we have $F(D^2\phi) \leq 0$.*

$\tilde{u} \in C(\bar{\Omega})$ is called the viscosity solution if $\tilde{u}$ is both a subsolution and supersolution to $F(D^2u) = 0$ at all points $y \in \Omega$

One way to think about viscosity subsolution is: suppose that P is a parabola which is greater than u in a neighbourhood of y. Then $F(D^2P) \geq 0$. This notion is actually equivalent to our previous definition and can be shown rigorously. A similar notion holds for supersolutions. Therefore, heuristically what we are doing is replacing the function u in a neighbourhood of y with a parabola so that we can evaluate what $F(D^2u)$ is. We state the comparison principle, which has been proved with great generality in [6].

Theorem 1.3.2. Let Ω be a bounded domain $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ be uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda$. Suppose u, v are subsolution and supersolution to $F(D^2\tilde{u}) = 0$ and $v \geq u$ on $\partial\Omega$. Then $v \geq u$ in Ω.

Before we prove this lemma, observe that if u, v were twice differentiable, the proof would follow from the fact that $D^2u \leq 0$ at x if u has a local maximum at the point x. Assuming, we show how to proceed to prove the existence of solutions using Perron's method.

1.4 Perron's Method

We will follow the proof given in [11]. First let us define upper and lower semi-continuous envelopes.

Definition 1.4.1. *Let u be a bounded function on $\bar{\Omega}$. Then we call u^* the upper semi-continuous envelope of u, where*

$$u^*(x) = \lim_{r\to 0} \sup\{u(z) : z \in B_r(x)\}$$

.

u^* is the smallest upper continuous function such that $u^* \geq u$. Lower semi-continuous envelope is defined similarly.

Theorem 1.4.2. Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ be uniformly elliptic and u_i, $i \in$ indexing set $\mathcal{I}$, be a family of subsolutions to $F(D^2u) = 0$. Then u^* is also a subsolution of $F(D^2u) = 0$, where

$$u(x) = \sup_{i\in\mathcal{I}}\{u_i(x)\}.$$

Proof. We start by proving a technical result we will need. Let u^* have a local maximum at x_0 with $u^*(x_0) > u^*(y)$ for all $y \in \overline{B_r(x_0)} \setminus \{x_0\}$. By definition, it is clear that there exists a countable subsequence i_k and y_k such that

$$u_{i_k}(y_k) \to u^*(x_0).$$

Now we want to extract a subsequence such that $u_{i_k}(y_k)$ has a local maximum at y_k.

We can choose any $0 \leq \delta \leq \delta_0$ and $\rho < r$ such that

$$\sup_{K_\rho} u^* \leq u^*(x_0) - \delta,$$

where $K_\rho = \overline{B_r(x_0) \setminus B_\rho(x_0)}$. Indeed, $u^*(x) - u^*(x_0)$ is a negative function on the compact set K_ρ, so it attains its maximum, which is strictly less than 0. Now for large enough i_k, $\sup_{K_\rho} u^*_{i_k} \leq u^*(x_0) - \frac{\delta}{2}$. If not, we can find i_k and $z_k \in K_\rho$ such that $u^*_{i_k}(z_k) \geq u^*(x_0) - \frac{\delta}{2}$. As K_ρ is compact we can find a convergent subsequence from z_k which converges to z, and $u^*(z) \geq u^*(x_0) - \frac{\delta}{2}$, which is a contradiction.
Let x_k be where u_{i_k} attains its maximum. As $u_{i_k}(x_k) \geq u_{i_k}(y_k) \to u^*(x_0)$. Therefore, for large enough k, $u_{i_k}(x_k) \geq u^*(x_0) - \frac{\delta}{4}$ which implies that $x_k \in B(\rho, x_0)$. As we can do this for all ρ, $x_k \to x_0$ and by taking $\delta \to 0$, we can conclude $u_{i_k}(x_k) \to u^*(x_0)$ (as $u^*(x_0) \geq \limsup u_{i_k}(x_k)$).
With this technical result, we can apply the definition of viscosity solution to get the desired result. Let $\tilde{\phi} \in C^2$ such that $\tilde{\phi}(x_0) = u^*(x_0)$ and $\tilde{\phi} > u^*$ in a neighbourhood of x_0. To make sure that we have a strict maximum, we consider $\phi = \tilde{\phi} + |x - x_0|^4$, which also is equal to u^* at x_0 and $\phi \geq u^*$. Now using the result proved in the last paragraph, we obtain a sequence of indices $\{a_k\}$ and points $x_k \to x_0$ such that $u_{a_k} - \phi$ has local maximum at x_k. But as u_{a_k} are subsolutions

$$F(\phi(x_k)) \geq 0, \implies F(\phi(x_0)) \geq 0$$

by continuity of F and we are done.

To show existence of solutions to the Dirichlet problem, we will assume the existence of barrier functions on our domain. This is necessary to show that the boundary datum is attained.
If for the domain Ω, for each $x_0 \in \partial\Omega$, there exists $\psi_+ \in C^2(\bar{\Omega})$ such that $\psi_+(x_0) = 0$, $\psi|_{\partial\Omega/\{x_0\}} > 0$ and $\mathcal{M}^+(D^2\psi_+) \leq 0$ in Ω, we call Ω a *nice domain*. Note that if we let

$\psi_- = -\psi_+$ then $\psi_- \in C^2(\bar{\Omega})$ such that $\psi_-(x_0) = 0$, $\psi|_{\partial\Omega/\{x_0\}} < 0$ and $\mathcal{M}^+(D^2\psi_-) \geq 0$ in Ω.

Theorem 1.4.3. Let Ω be a nice domain, $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ be uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda$, and $g \in C(\Omega)$. Then, the Dirichlet problem

$$\begin{cases} F(D^2u(x)) & = 0 \quad x \in \Omega, \\ u(x) & = g(x) \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

has a unique solution in the viscosity sense.

Proof. We assume for now that $F(C) = 0$ where C is the constant function. The uniqueness of the solutions follows from the comparison principle stated in the last section. The key step in this proof is to show that supremum of all subsolutions for (1.1) is both a subsolution and a supersolution. It follows from previous lemma that it is a subsolution. We show that if it is not a supersolution at any point, we can find a subsolution greater than the supremum, which is a contradiction. Before we do this, we will show that boundary datum is attained due to Ω being a nice domain.

We define

$$\mathcal{A} := \{v \in USC(\overline{\Omega});\ F(D^2u) \geq 0 \,\text{in}\, \Omega, \quad u \leq g \,\text{on}\, \partial\Omega\}.$$

We note that $-||g||_{L^\infty(\partial\Omega)}$ belongs to $\mathcal{A}$ and all elements of $\mathcal{A}$ are bounded by $||g||_{L^\infty(\partial\Omega)}$ due to comparison principle . Let

$$u(x) = \sup_{v\in\mathcal{A}} v(x).$$

We first show that boundary datum is attained by u. For $\epsilon > 0$, define

$$w_\epsilon^- = g(x_0) - \epsilon + k_\epsilon\psi_- = g(x_0) - \epsilon - k_\epsilon\psi_+$$

for $x_0 \in \partial\Omega$ and $k_\epsilon > 0$ chosen such that $w_\epsilon^- \leq g$ on $\partial\Omega$. Similarly, define

$$w_\epsilon^- = g(x_0) + \epsilon + k_\epsilon\psi_+,$$

and choose k_ϵ big enough, so that $w_\epsilon^- \leq g$, $w_\epsilon^+ \geq g$. Now, by the properties of $\psi_\pm$, $\mathcal{M}^-(D^2w_\epsilon^-) = k_\epsilon\mathcal{M}^-(D^2\psi_-) \geq 0$, $\mathcal{M}^+(D^2w_\epsilon^+) = k_\epsilon\mathcal{M}^+(D^2\psi_+) \leq 0$. This implies that

$$F(D^2w_\epsilon^-) \geq 0 \text{ and } F(D^2w_\epsilon^+) \leq 0 \text{ in } \Omega.$$

By continuity of ψ_-, for each $\epsilon > 0$, there exists a $\delta > 0$ such that we have $w_\epsilon^- \geq g(x_0) - 2\epsilon$ in $B_\delta(x_0) \cap \overline{\Omega}$. As $w_\epsilon^- \in \mathcal{A}$, we have $u^* \geq w_\epsilon^- \geq g(x_0) - 2\epsilon$ (recall that u^* is the upper semicontinous envelope of u) that if $x_k \to x_0$,

$$\limsup_{k\to\infty} u(x_k) \geq g(x_0) - 2\epsilon.$$

We note that all elements of $\mathcal{A}$, and u^* are lesser than w_ϵ^+ by the comparison principle. Similarly, by continuity of w_ϵ^+, we have that $w_\epsilon^+ \geq g(x_0) + 2\epsilon$ in $B_\delta(x_0) \cap \overline{\Omega}$ and $u^* \leq w_\epsilon^+ \leq g(x_0) + 2\epsilon$, so that if $x_k \to x_0$,

$$\liminf_{k\to\infty} u(x_k) \leq g(x_0) + 2\epsilon.$$

By letting $\epsilon \to 0$, we have that

$$u^* = g \ \text{ on } \ \partial\Omega.$$

Observe that since $u^* \in \mathcal{A}$ and $u^* \geq u$, $u = u^*$ by the definition. This implies u is upper semicontinuous and $F(D^2 u) \geq 0$. Now we show that u_* (u_* is the lower semi-continuous envelope of u) is a supersolution for (1.1). Suppose that it is not. Then there exists a point $x_0 \in \Omega$ and $\phi \in C^2(\overline{\Omega})$ such that $\phi(x_0) = u_*(x_0)$, $\phi \leq u_*$ and $F(D^2\phi(x_0)) > 0$. Without loss of generality, we can assume $\phi < u_*$. Indeed, otherwise, we can replace ϕ by $\phi - |x - x_0|^4$. By continuity of F we have that $F(D^2\phi) \geq 0$ in $B_\delta(x_0)$ for some $\delta > 0$.

Now, consider $u_\epsilon = \sup\{u, \phi + \epsilon\}$. As $\phi < u_* \leq u$ in $\Omega \setminus B_\delta(x_0)$, for small enough $\epsilon > 0$, $\phi + \delta \leq u$ in $\overline{\Omega}/B_\delta(x_0)$. u_ϵ is a subsolution as supremum of subsolutions is a subsolution. As $\phi + \delta \leq u$ in $\overline{\Omega}/B_\delta(x_0)$, $\phi + \delta \leq g$ on $\partial\Omega$, therefore $u_\delta \in \mathcal{A}$. By definition of u, $\phi + \delta \leq u$, therefore $\phi + \delta \leq u_*$, which is a contradiction. Therefore, u_* is supersolution and $u = u_*$. Therefore, u is both a subsolution and supersolution, with $u = g$ on $\partial\Omega$, which concludes the proof. □

1.5 Degenerate Ellipticity

Till now, we have discussed results for uniformly elliptic operators. However, both the comparison principle and existence for the solutions are true for degenerate operators with mild assumptions. In this section, we define degenerate ellipticity and related notions. Unlike uniform elliptic operators, degenerate elliptic operators are allowed to have $\lambda = 0$

Definition 1.5.1. *We call $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ degenerate elliptic or just elliptic if $F(X) \leq F(Y)$ when $X \leq Y$, where X, Y are semi-positive definite symmetric matrices.*

Suppose u is C^2 and $F(D^2u) \geq 0$. Let ϕ be a C^2 function such that $u-\phi$ has a maximum at x_0. Then, by calculus, $Du = D\phi$ and $D^2u \leq D^2\phi$. By ellipticity of F, we have that

$$F(D^2\phi) \geq 0.$$

Note that by Taylor approximation,

$$u(x) \leq u(x_0) + \langle p, x-x_0\rangle + \langle X(x-x_0), x-x_0\rangle + o(|x-x_0|^2), \tag{1.2}$$

where $p = D\phi$, $X = D^2\phi$. It is also true that if u is C^2 at x_0 and (p, X) satisfy (1.2), then $p = Du$, $D^2u = X$. With this as motivation, we define jets which will serve as the pair '(Du,D^2u)' for non-smooth u. This will help us define an alternative (but equivalent, see [6]) definition of viscosity solutions.

Definition 1.5.2. *We call $\mathcal{J}^{2,+}u(x_0)$ the super-jet of u at x_0, where*

$$\mathcal{J}^{2,+}u(x_0) := \{(p, X)|(p, X) \textit{ satisfy } (1.2) \textit{ in some neighbourhood of } x_0\}.$$

We will also define the 'closure' of $\mathcal{J}^{2,+}u(x_0)$:

$$\bar{\mathcal{J}}^{2,+}u(x_0) =: \{(p, X)| \text{ there exists a sequence } (p_n, X_n) \in \mathcal{J}^{2,+}u(x_n), (p_n, X_n, x_n) \to (p, X, x_0)\}$$

Similarly we define $\mathcal{J}^{2,-}u(x_0)$ and $\bar{\mathcal{J}}^{2,-}u(x_0)$ but require $(p, X) \in \mathcal{J}^{2,-}u(x_0)$ to satisfy a reverse inequality to that of (1.2).

For simplicity of the exposition, we consider F which only depend on D^2u but results stated in this section are true for more general operators, see [6].

Definition 1.5.3. *Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ be elliptic. $u \in USC(\bar{\Omega})$ is said to be the subsolution of $F(D^2v) = 0$ at x_0 in Ω (in viscosity sense) if for all X*

$$F(X) \geq 0 \textit{ where } (p, X) \in \mathcal{J}^{2,+}u(x_0).$$

$u \in LSC(\bar{\Omega})$ is said to be the supersolution of $F(D^2v) = 0$ at x_0 in Ω (in viscosity sense) if for all X

$$F(X) \geq 0 \textit{ where } (p, X) \in \mathcal{J}^{2,+}u(x_0).$$

If u is a subsolution and a supersolution for all $x_0 \in \Omega$, u is called the solution (in the viscosity sense).

We will state the main theorem which showcases the usefulness of jets. From this theorem existence and uniqueness for degenerate elliptic equations can be shown similarly to that of uniformly elliptic equations.

Theorem 1.5.4. Let u_i be USC($\bar{\Omega}_i$) where Ω_i is a domain in $\mathbb{R}^n$. Let

$$\Omega := \Omega_1 \times \Omega_2 \times \cdots \times \Omega_k.$$

Let ϕ be a twice differentiable function in $\bar{\Omega}$. Define

$$v(x) = \sum_1^k u_i(x).$$

Suppose $v - \phi$ has a local maximum at $\hat{x} = (x_1, x_2....x_k)$. Then for each ε there exists $X_i \in \mathcal{S}(n)$ where $\mathcal{S}(n)$ stands for symmetric matrices of size $n \times n$ such that

$$(D_{x_i}\phi(\hat{x}), X_i) \in \bar{\mathcal{J}}^{2,+}u(x_i)$$

for all i, and the block diagonal X matrix with entries X_i satisfies

$$-\left(\frac{1}{\varepsilon} + \|A\|\right) Id \leq X \leq A + \varepsilon A^2,$$

where $X = \begin{pmatrix} X_1 & ... & 0 \\ . & ... & . \\ 0 & ... & X_k \end{pmatrix}$ and $A = D^2\phi$.

This result tells us that while approximating, jets are well-behaved. Therefore, we can prove comparison principle and the existence through Perron's method for degenerate elliptic equations. The proof of this result is involved and uses techniques from convex analysis and functional analysis, see [6].

Chapter 2

ABP estimate and Harnack Inequality

2.1 Introduction

In this chapter we will prove Alexandroff-Bakelman-Pucci estimate for viscosity solutions of uniformly elliptic operators, that is, viscosity solutions of $F(D^2u(x)) = 0$, where F is uniformly elliptic. Regularity of a function at a point quantifies how much a function oscillates very close to that point. ABP estimate tells us how large $\sup u^-$ can be depending only on the size of the domain and L^n norm of f or the R.H.S. Because the estimate is scale invariant, we can quantify oscillations at every scale, which leads to higher regularity for the solutions. The two main steps involved is to prove the convex envelope of the supersolution u is $C^{1,1}$ and re-adapt the classical proof once this is done.

2.2 Convex envelope

In this section, we follow [3],. By Hahn-Banach theorem, it is easy to show that for a convex function v defined on some set A, at each point x of A there exists a supporting hyperplane, that is , an affine function L such that $L(x) = v(x)$ and $L \leq v$ in A. We give the following definitions from [3],

Definition 2.2.1. *"Let v be a continuous function in an open convex set. The convex envelope of v in A is defined by*

$$\begin{aligned}\Gamma_v(x) &= \sup_w \{w(x) : w \leq v \textit{ in } A \textit{ and } w \textit{ is convex}\} \\ &= \sup_w \{w(x) : w \leq v \textit{ in } A \textit{ and } w \textit{ is affine}\}.'' \qquad (2.1)\end{aligned}$$

2.3 ABP inequality

We now prove the maximal curvature of the convex envelope is bounded by f. Before we proceed with the fully non-linear case, we will give the idea behind the linear case.
Consider the divergence form equation $\sum a_{ij}u_{ij} = f$. A classical supersolution to this equation satisfies $\sum a_{ij}(x_0)u_{ij}(x_0) \leq f^+(x_0)$. At points where $u(x_0) = \Gamma_u(x_0)$, $u_{ij}(x_0) \geq 0$,and by ellipticity $\lambda < a_{ij} \leq \Lambda$. Combining these two, $\lambda \sum u_{ii}(x_0) \leq f^+(x_0)$. As trace is invariant under co-ordinate change, $\lambda \sum k_{ii} \leq f^+(x_0)$ where k_{ii} are eigenvalues of $D^2u(x_0)$. By A.M-G.M inequality, $\lambda(\Pi(k_{ii}))^{\frac{1}{n}} = \lambda(\det D^2u(x_0))^{\frac{1}{n}} \leq Cf^+(x_0)$. As $D^2\Gamma_u \leq D^2u$ wherever $u = \Gamma_u$, this combined with (2.3.2) gives us the result.

Theorem 2.3.1. Let $v \geq 0$ be a supersolution to $F(D^2v) = f$ in B_d, where $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ is uniformly elliptic with ellipticity constants λ, Λ and f is bounded in B_d. Let ϕ be a convex function such that $0 \leq \phi \leq v$ in B_d and $0 = \phi(0) = v(0)$, then for some $\nu < 1$

$$\phi(x) \leq C(\sup_{B_d} f^+)|x|^2 \quad \text{for all } x \in B_{\nu d}$$

for some constant C.

Proof. For $0 < r \leq \frac{d}{4}$, define

$$C_r = \frac{1}{r^2}\sup_{B_r} \phi.$$

As ϕ, is convex, the maximum should be attained on the boundary ∂B_r, and for some $z \in \partial B_r$,

$$C_r r^2 = \phi(z)$$

The set $A := \{x \in B_d \mid \phi(x) \leq C_r r^2\}$ is convex due to the convexity of ϕ and it contains B_r. By Hahn-Banach theorem, we know that there exists a hyperplane to A at z. But a hyperplane to A should also be a hyperplane to B_r, and let H be the only hyperplane to B_r at z (it is the tangent plane at z). Then H is the only hyperplane to A, and therefore,

$$\phi \geq C_r r^2 \quad \text{in } H \cap B_d. \tag{2.2}$$

Without loss of generality we can assume that $z = (0, 0, \ldots r)$ and $H = \{x \mid x_n = r\}$, where $x := (x_1, x_2 \ldots x_n)$ and $x' = (x_1, \ldots, x_{n-1})$.

Now we obtain an upper bound on C_r by constructing a paraboloid whose curvature is dependent on the ellipticity constants. Consider the set $A := B_{\frac{d}{2}} \cap \{-r < x_n < r\}$. We break ∂A into A_1, A_2, A_3, where

$$\begin{aligned}
A_1 &= \bar{B}_{\frac{d}{2}} \cap \{x_n = r\},\\
A_2 &= \bar{B}_{\frac{d}{2}} \cap \{x_n = -r\},\\
A_3 &= \{|x| = \frac{d}{2}\} \cap \{-r < x_n < r\}.
\end{aligned}$$

By previous discussion, $\phi \geq C_r r^2$ in A_1 and $\phi \geq 0$ in $A_1 \cup A_2$. Let P be the paraboloid,

$$P(x) = \frac{C_r}{8}(x_n + r)^2 - 4C_r \frac{r^2}{d^2}|x'|^2.$$

By simple computations, it is easy to check that $P \leq C_r r^2$ in A_1 and $P \leq 0$ in $A_2 \cup A_3$. This implies $P \leq \phi \leq v$ on ∂A but $P(0) > 0 = \phi(0) = v(0)$. Therefore, by a vertical translation of P we can touch the graph of v from below at some point $y \in A$. By definition of viscosity solution and assumptions on u, we have

$$F(D^2 P) \leq f(y) \leq (\sup_{B_d} f^+). \tag{2.3}$$

$\partial_{ii} P = -8C_r \frac{r^2}{d^2}$ for $1 \leq i \leq n-1$ and $\partial_{nn} P = \frac{C_r}{4}$. By ellipticity of F,

$$\frac{\lambda C_r}{4} - \Lambda(n-1)8C_r \frac{r^2}{d^2} \leq F(D^2 P).$$

By choosing appropriate value of $r > 0$, call it r_0, we can make sure

$$F(D^2 P) \geq \lambda \frac{C_r}{8}.$$

This choice of r_0 can be made such that $\frac{r_0}{d}$ depends only on λ, Λ, n (see [3],). Using (2.3) we can conclude

$$C_r \leq \frac{8}{\lambda}(\sup_{B_d} f^+).$$

Hence, $\sup_{B_{r_0}} \phi \leq C(\sup_{B_d} f^+) r_0^2$, which concludes the proof. □.

The following lemma tells us how much a function can grow if we know its convex envelope, and the proof uses a very elegant geometric argument.

Lemma 2.3.2. Let u be a continuous function in B_d such that $u \geq 0$ on ∂B_d and let Γ_u, defined by (2.1), be $C^{1,1}(B_d)$, where Γ_u is the convex envelope in B_{2d} (we have extended u by zero outside B_d). Then there exists a set $A \subset B_d$ such that $|B_d \setminus A| = 0$, Γ_u is double differentiable at any $x \in A$ and

$$\sup_{B_d} u^- \leq C(n) d \left(\int_A \det D^2 \Gamma_u \right)^{\frac{1}{n}},$$

where $C(n)$ is a constant depending only on n and u^- is continuous in B_{2d} by our choice of extension.

Note: We will use the following two theorems in coming sections:

Theorem 2.3.3. Let $H : \bar{B}_1 \subset \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz map. Then, H is differentiable at every point in $A \subset B_1$ such that $|B_1 \setminus A| = 0$. Furthermore,

$$|H(B_1)| \leq \int_A |\text{det} DH|,$$

where DH is the differential of H.

This was first proved by Radamacher. Proof of this theorem can be found in [10].

Theorem 2.3.4. Let u be a convex function B_d. The u is second differentiable almost everywhere in B_d.

This is a famous result of Alexandrov, see [1].

Proof of Lemma (2.3.2) We assume u^- is not zero everywhere. Since $u^- \equiv 0$ on ∂B_d, we have that $M := \sup u^- = -u(x_0) > 0$ for some $x_0 \in B_d$.

Consider a function whose graph is a cone in $\mathbb{R}^n \times \mathbb{R}$ with vertex $(x_0, -M) = (x_0, -u^-(x_0))$ and base $\partial B_{3d} \times \{0\}$. For any $|\xi| \leq \frac{M}{(3d)}$,

$$H =: \{x \mid x_{n+1} = -M + \xi \cdot (x - x_0) \text{ in } \mathbb{R}^n\}$$

is a hyperplane to the cone at x_0. As $|\xi| \leq \frac{M}{(3d)} \leq \frac{\sup u^-}{3d}$ and $u^- \equiv 0$ on $\bar{B_{2d}}/B_d$, there exists a $x^* \in B_d$ such that a translation of H, lets call it H', is a hyperplane to $-u^-$. See figures (2.1)(2.2).

By the definition of Γ_u, H' is also a hyperplane to Γ_u at x^*. Since, Γ_u is differentiable, it

follows that $\nabla\Gamma_u(x^*) = \xi$.
We have proved that

$$B_{\frac{M}{3d}}(0) \subset \nabla\Gamma_u(B_d),$$

and therefore, by comparing the measures

$$C(n)\frac{M^n}{d^n} \leq |\nabla\Gamma_u(B_d)|. \tag{2.4}$$

By Theorem (2.3.3) and the fact that $D^2\Gamma_u \geq 0$, we have

$$|\nabla\Gamma_u(B_d)| \leq \int_A \det D^2\Gamma_u.$$

Hence, by (2.4), we arrive at

$$\sup_{B_d} u^- \leq C(n)d\left(\int_A \det D^2\Gamma_u\right)^{\frac{1}{n}}.$$

□

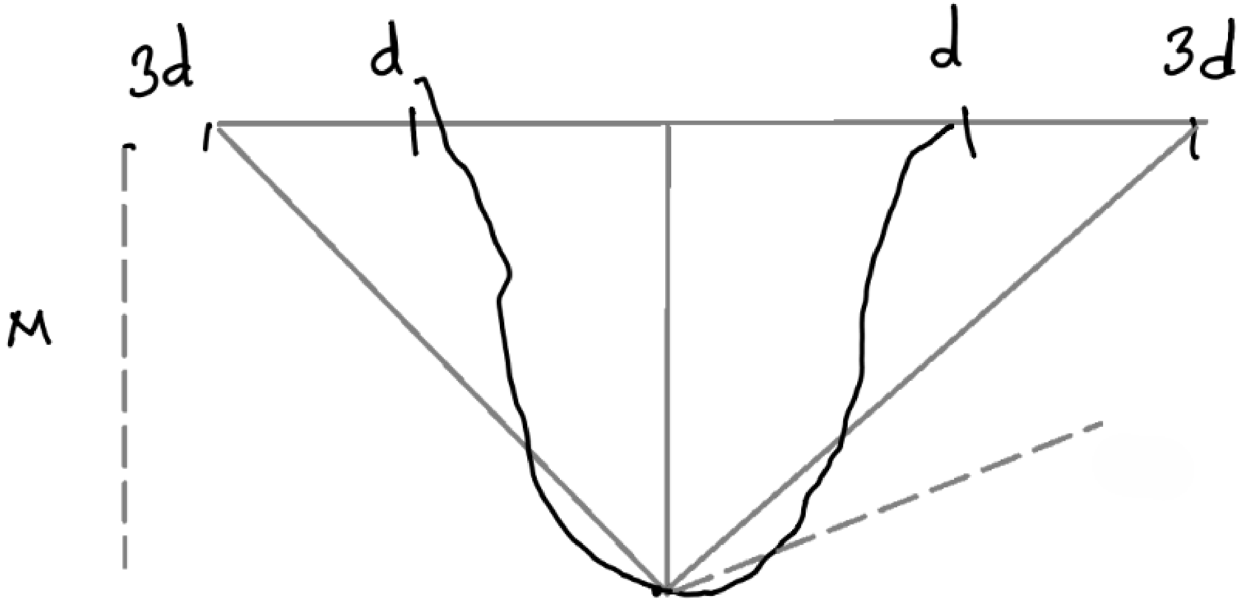

Figure 2.1: Hyperplane with slope ξ

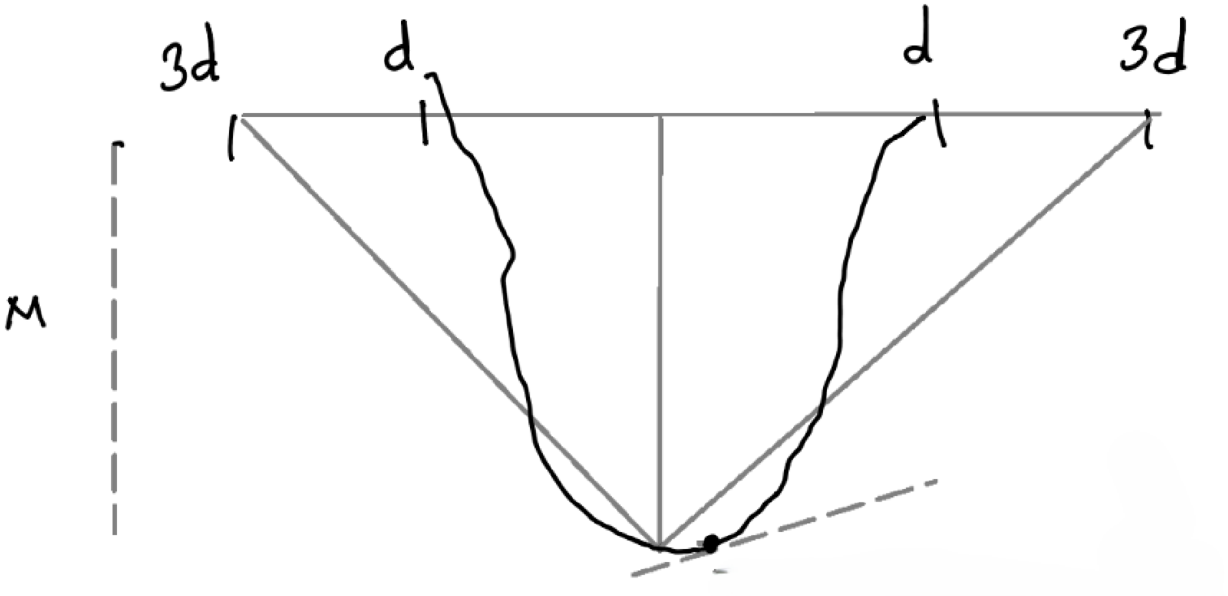

Figure 2.2: By translating, we get a supporting hyperplane

We define a few useful notions before we proceed. Let P be the paraboloid of **opening** M

$$P(x) = l_0 + l(x) + \frac{M}{2}|x|^2,$$

where l_0 is a constant, $l(x)$ is a linear function, M is a constant.

Let us quote few definitions from [3],. "Let u be a continuous function defined in an open set Ω. For $x_0 \in \Omega$, we define

$$\overline{\Theta}(u, A)(x_0) \tag{2.5}$$

to be the infimum of all positive constants M for which there is a convex paraboloid of opening M that touches u by above at x_0 in A. We define (2.5) to be ∞ if no such constant M exists. Using concave paraboloids that touch u by below, we similarly define

$$\underline{\Theta}(u, A)(x_0) \in [0, \infty].$$

We finally consider

$$\Theta(u, A)(x_0) = \sup\left\{\overline{\Theta}(u, A)(x_0), \underline{\Theta}(u, A)(x_0)\right\} \leq \infty.$$

If $\underline{\Theta}(u, A)(x_0)$ is finite we say u is $C^{1,1}$ from below and if $\overline{\Theta}(u, A)(x_0)$is finite we say it is $C^{1,1}$ from above."

Theorem 2.3.5. Let u be a continuous function in $\overline{B_d}$ such that $u \geq 0$ on ∂B_d and let Γ_u be the convex envelope of u on B_{2d} as defined by (2.1). Let $K > 0$ and $0 < \epsilon \leq d$ be constants. Assume that

$$\overline{\Theta}(\Gamma_u, B_\epsilon(x_0))(x_0) \leq K \qquad \forall x_0 \in \overline{B_d} \cap \{u = \Gamma_u\}. \tag{2.6}$$

Then $\Gamma_u \in C^{1,1}(\overline{B_d})$. Moreover, we have that

$$\sup_{B_d} u^- \leq C(n)\, d \left(\int_{A \cap \{u = \Gamma_u\}} \det D^2\Gamma_u \right)^{1/n},$$

where $C(n)$ is a constant depending only on n.

Proof By Lemma (2.3.2), we only need to prove that

$$\Gamma_u \in C^{1,1}(\overline{B}_d).$$

For any $y_0 \in \overline{B}_d \cap \{u = \Gamma_u\}$, this follows from the fact that there exists a hyperplane which touches Γ_u from below at each point and a paraboloid of opening K touches Γ_u from above almost everywhere. We also need to show

$$\det D^2\Gamma_u(x) = 0 \qquad \text{a.e. } x \in B_d \setminus \{u = \Gamma_u\}. \tag{2.7}$$

The intuition behind this fact is simple. Suppose $u : \mathbb{R} \to \mathbb{R}$ is touched by Γ_u at x and y. Consider the line between $(x, u(x))$ and $(y, u(y))$. Any convex function below u must have its graph below this line, and as Γ_u is the supremum of all the convex functions lesser than u, graph of Γ_u should coincide with this line between $(x, u(x))$ and $(y, u(y))$. As affine functions have second derivative equal to 0,

$$\det D^2\Gamma_u(x) = 0 \qquad \text{a.e. } x \in B_d \setminus \{u = \Gamma_u\}.$$

See, Figure 2.3.

Now we will rigorously prove this in two steps. As we pointed out above, we know that for any $y_0 \in \overline{B}_d \cap \{u = \Gamma_u\}$, there is a hyperplane L_0 that touches Γ_u by below at y_0; by (2.6), there is a paraboloid that touches Γ_u by above at y_0. It follows that Γ_u is $C^{1,1}$ at y_0,

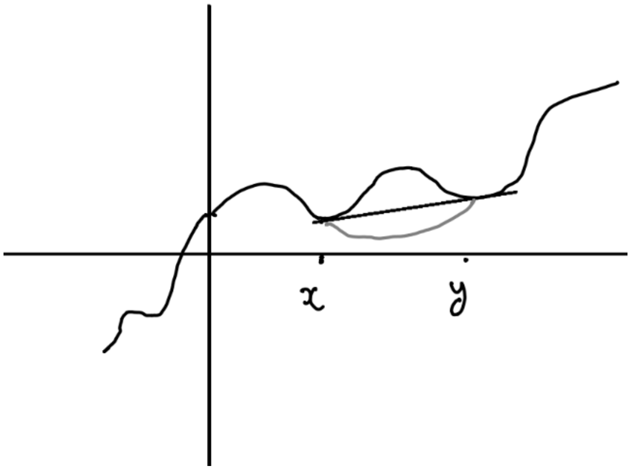

Figure 2.3: The line between x, y is the maximal convex function

that

$$L_0(y) = \Gamma_u(y_0) + D\Gamma_u(y_0)(y - y_0)$$

and

$$\Theta(\Gamma_u, B_\epsilon(y_0))(y_0) \leq K \qquad \forall y_0 \in \overline{B_d} \cap \{u = \Gamma_u\}. \tag{2.8}$$

Step 1. Let $x_0 \in B_d \setminus \{u = \Gamma_u\}$ and let L be a supporting hyperplane for Γ_u at x_0 in $\overline{B}_{2d}$. We claim the following:

(a) x_0 belongs to a simplex S with vertices $x_1, \ldots, x_{n+1}$ (i.e., S is the convex hull of the set $\{x_1, \ldots, x_{n+1}\}$), and $L = \Gamma_u$ in this simplex S. The points $x_1, \ldots, x_{n+1}$ need not be all distinct. Moreover, all vertices x_i are in $\overline{B}_d \cap \{u = \Gamma_u\}$, except for possibly $x_{n+1} \in \partial B_{2d}$.

(b) If we write

$$x_0 = \sum_{i=1}^{n+1} \lambda_i x_i \quad (\lambda_i \geq 0,\ \sum_{i=1}^{n+1} \lambda_i = 1),$$

then $\lambda_i \geq \frac{1}{3n}$ for at least one index i for which $x_i \in \overline{B}_d \cap \{u = \Gamma_u\}$.

The proof of this claim is as follows. By definition, $\Gamma_u(x) = \sup\{L(x) : L \leq -u \text{ in } \overline{B}_{2d}\}$. If $x_0 \in B_{2d}$ and L is the supporting hyperplane for Γ_u at x_0, then L realizes the supremum at x_0. It follows that L must have at least one contact point in $\overline{B}_{2d}$ with $-u$. Thus, the closed convex hull

$$C = \{x \in \overline{B}_{2d} : L(x) = -u(x)\}$$

is nonempty.

Assume that $x_0 \notin C$. Then by Hahn-Banach theorem, there is an affine function l such that $l(x_0) > 0$ and $l(C) < 0$. Therefore, $L + \delta l \leq -u$ in $\overline{B}_{2d}$ for $\delta > 0$ small, since l is negative in a neighbourhood of C and $-u - L$ is positive outside this open neighbourhood of C. In particular,

$$\Gamma_u(x_0) \geq (L + \delta l)(x_0) > L(x_0),$$

which contradicts the maximality of L at x_0, i.e. $\Gamma_u(x_0) = L(x_0)$.

We conclude that $x_0 \in C$. Now we state Cartheodory's theorem.

Theorem 2.3.6 (Carathéodory's Theorem)**.** Let $S \subset \mathbb{R}^n$ and let $x \in \operatorname{conv}(S)$. Then there exist points $x_1, \ldots, x_{n+1} \in S$ and numbers $\lambda_1, \ldots, \lambda_{n+1} \geq 0$ with

$$\sum_{i=1}^{n+1} \lambda_i = 1$$

such that

$$x = \sum_{i=1}^{n+1} \lambda_i x_i.$$

Therefore, from previous discussion, we have that x_0 is a convex combination of $n+1$ points $x_1, \ldots, x_{n+1} \in \overline{B}_{2d}$. Note there is at most one x_i with $|x_i| = 2d$, since if this is not true then $\Gamma_u = 0$ (a contradiction). Note also that if one $x_i \in \overline{B}_{2d} \setminus B_d$ then $\Gamma_u \equiv 0$. We have proved part (a) of the claim.

Intuition for Caratheodory's theorem : Even if x is expressed as a convex combination of many points in S, one never needs more than $n+1$ of them in $\mathbb{R}^n$. Indeed, in $\mathbb{R}^n$ any set of more than $n+1$ points is affinely dependent. This affine dependence allows one to eliminate a point from the convex combination without changing the value of the sum. Repeating this procedure reduces the representation to a convex combination of at most $n+1$ points.

We now prove (b). If all x_i belong to $\overline{B}_d$, then

$$\lambda_i \geq \frac{1}{n(n+1)} \geq \frac{1}{3n}$$

for at least one index i. If $x_{n+1} \in \partial B_{2d}$ and $\lambda_i < \frac{1}{3n}$ for $i \leq n$, then $\lambda_{n+1} > \frac{2}{3}$ and

$$|x_0| \geq \frac{2}{3}|x_{n+1}| - \sum_{i=1}^{n} \frac{1}{3n}|x_i| \geq \frac{4}{3}d - \frac{1}{3}d = d,$$

which is a contradiction.

Step 2. Let x_0 and L be as in Step 1, and take any $h \in \mathbb{R}^n$ with $|h| < d$. Using Step 1 we can relabel x_i such that $x_1 \in B_d \cap \{u = \Gamma_u\}$ and $\lambda_1 \geq \frac{1}{3n}$, and write

$$x_0 + h = \lambda_1 \left(x_1 + \frac{h}{\lambda_1} \right) + \lambda_2 x_2 + \cdots + \lambda_{n+1} x_{n+1}.$$

We have that

$$L(x_0+h) \leq \Gamma_u(x_0+h) \leq \lambda_1 \Gamma_u \left(x_1 + \frac{h}{\lambda_1} \right) + \lambda_2 \Gamma_u(x_2) + \cdots + \lambda_{n+1} \Gamma_u(x_{n+1}).$$

Suppose that

$$|h| < \frac{\varepsilon}{3n}.$$

We then have that $|h|/\lambda_1 < \varepsilon$. Recall that L is supporting hyperplane for Γ_u at $x_1 \in B_d \cap \{u = \Gamma_u\}$ and

$$Lx = \Gamma_u(x_1) + D\Gamma_u(x_1)(x - x_1);$$

this together with (2.8) applied with y_0 replaced by $x_1 \in B_d \cap \{u = \Gamma_u\}$ gives

$$L(x_0+h) \leq \Gamma_u(x_0+h) \leq \lambda_1 \left(L\left(x_1 + \frac{h}{\lambda_1} \right) + \frac{K}{2} \left| \frac{h}{\lambda_1} \right|^2 \right) + \lambda_2 \Gamma_u(x_2) + \cdots + \lambda_{n+1} \Gamma_u(x_{n+1}).$$

Therefore, since L is a supporting hyperplane for Γ_u at x_i for any $1 \leq i \leq n+1$ (recall that $L = \Gamma_u$ in the simplex generated by $x_1, \ldots, x_{n+1}$), we have

$$L(x_0+h) \leq \Gamma_u(x_0+h) \leq \lambda_1 \left(L\left(x_1 + \frac{h}{\lambda_1} \right) + \frac{K}{2} \left| \frac{h}{\lambda_1} \right|^2 \right) + \lambda_2 L(x_2) + \cdots + \lambda_{n+1} L(x_{n+1})$$

$$= L(x_0+h) + \frac{K}{2\lambda_1}|h|^2 \leq L(x_0+h) + \frac{3nK}{2}|h|^2.$$

We have proved that Γ_u is $C^{1,1}$ at any point $x_0 \in \overline{B}_d \setminus \{u = \Gamma_u\}$ and

$$\Theta(\Gamma_u, B_{\varepsilon/(3n)}(x_0))(x_0)) \leq 3nK \qquad \forall x_0 \in \overline{B}_d \setminus \{u = \Gamma_u\}.$$

This and previous discussion imply that $\Gamma_u \in C^{1,1}(\overline{B_d})$. (a) In Step 1 also implies that for any $x_0 \in B_d \setminus \{u = \Gamma_u\}$, there exists an open segment (i.e., an open interval of a line through x_0) on which Γ_u is affine. Therefore, we get (2.7), and we are done. $\square$

Now, we state the Alexandroff-Bekelmann-Pucci (ABP) estimate

Theorem 2.3.7. Let u be a continuous super-solution as in Theorem 2.3.1 in $\bar{B}_d$ such that $u \geq 0$ on ∂B_d. Then,

$$\sup_{B_d} u^- \leq Cd \left(\int_{B_d \cap \{u = \Gamma_u\}} (f^+)^n \right)^{\frac{1}{n}} \tag{2.9}$$

Here we have extended u by zero outside B_d, and hence $-u^-$ is continuous in B_{2d}; Γ_u is the convex envelope in B_{2d} of $-u^-$ and C is a universal constant.

Proof We claim that

$$\overline{\Theta}(\Gamma_u, B_{\nu d}(x_0))(x_0) \leq C \sup_{B_d} f^+ \qquad \forall x_0 \in \overline{B}_d \cap \{u = \Gamma_u\} \tag{2.10}$$

and

$$\det D^2\Gamma_u(x_0) \leq C f^+(x_0)^n \quad \text{a.e. } x_0 \in B_d \cap \{u = \Gamma_u\}, \tag{2.11}$$

where C and $\nu < 1$ are positive universal constants.

(2.9) follows immediately from this claim using Theorems 2.3.1 and 2.3.5 with $\varepsilon = \nu d$ and $K = C \sup_{B_d} f^+$. The claim is proved as follows. Take $x_0 \in \overline{B}_d \cap \{u = \Gamma_u\}$ and let L be a supporting hyperplane for Γ_u at x_0. We claim that

$$\tilde{F}(D^2(-u^-)) \leq f^+ \chi_{B_d},$$

where $\tilde{F}$ is uniformly elliptic with the same elliptic constants. Proof is given in [3],. Since L is affine, $\Gamma_u - L$ is convex and $\tilde{F}(D^2(-u^- - L)) \leq f^+ \chi_{B_d}$ in B_{2d}. We also have (for any $0 < \delta \leq d$) that $x_0 \in B_\delta(x_0) \subset B_{2d}$,

$$0 \leq \Gamma_u - L \leq -u^- - L \quad \text{in } B_\delta(x_0),$$

and equalities hold at x_0. We apply Lemma 2.3.2 and get

$$L(x) \leq \Gamma_u(x) \leq L(x) + C\Big(\sup_{B_\delta(x_0)} f^+ \chi_{B_d}\Big)|x - x_0|^2 \qquad \forall x \in B_{\nu\delta}(x_0).$$

From this, we immediately get (2.10); letting $\delta \to 0$ and using that f is continuous, we also conclude (2.11). $\square$

2.4 Harnack Inequality

Harnack Inequality tells us that for a positive solution of $F(D^2u) = f$, where F is uniformly elliptic as before, we can bound the supremum of u in the domain by the infimum of u, and appropriate norm of f in the domain. This is a very fundamental result as it gives us direct information about how much the function can oscillate. Indeed, using this result we can prove Hölder regularity of u in a straightforward fashion.

Before we proceed, we prove two results which will be required. One of them is a construction of an appropriate barrier function, which will help us modify the contact set $u = \Gamma_u$. The other result is known as Calderon-Zygmund cube decomposition, which will help us prove the estimates at every scale.

Theorem 2.4.1. Given $0 < \lambda \leq \Lambda$ ellipticity constants of F, there exists a smooth function φ in $\mathbb{R}^n$ and universal positive constants C and $M > 1$ such that

$$\varphi \geq 0 \quad \text{in } \mathbb{R}^n \setminus B_{2\sqrt{n}}, \tag{2.12}$$

$$\varphi \leq -2 \quad \text{in } Q_3, \tag{2.13}$$

$$F(D^2\phi) \leq C\xi \quad \text{in } \mathbb{R}^n, \tag{2.14}$$

where Q_d is the cube of length d and $0 \leq \xi \leq 1$ is a continuous function in $\mathbb{R}^n$ with $\operatorname{supp}\xi \subset \overline{Q}_1$. Moreover, $\varphi \geq -M$ in $\mathbb{R}^n$.

Proof. Consider the universal constant

$$\alpha = \max\{1, (n-1)\Lambda/\lambda - 1\}.$$

We have that

$$B_{1/4} \subset B_{1/2} \subset Q_1 \subset Q_3 \subset B_{3\sqrt{n}/2} \subset B_{2\sqrt{n}}.$$

Define

$$\varphi(x) = M_1 - M_2|x|^{-\alpha} \qquad \text{in } \mathbb{R}^n \setminus B_{1/4};$$

we choose M_1 and M_2 (positive) such that

$$\varphi\big|_{\partial B_{2\sqrt{n}}} = 0 \qquad \text{and} \qquad \varphi\big|_{\partial B_{3\sqrt{n}/2}} = -2.$$

(2.12) therefore holds; by extending ϕ inside $B_{\frac{1}{4}}$ with a radially decreasing smooth function,(2.13) will hold. This extension depends only on n, λ, Λ.

It is easy to see that (2.14) holds. If $r \geq 1/4$ then, at the point $(r, 0, \ldots, 0)$,

$$\begin{aligned} \partial_{ij}\varphi &= 0 && \text{if } i \neq j, \\ \partial_{11}\varphi &= -M_2\alpha(1+\alpha)r^{-\alpha-2}, \\ \partial_{ii}\varphi &= M_2\alpha r^{-\alpha-2} && \text{for } i > 1. \end{aligned}$$

Rotational symmetry and the definition of α give that for $|x| \geq 1/4$

$$\begin{aligned} F(D^2\phi) &\leq M_2\Big[\Lambda(n-1)\alpha|x|^{-\alpha-2} - \lambda\alpha(1+\alpha)|x|^{-\alpha-2}\Big] \\ &= M_2\Big[\alpha|x|^{-\alpha-2}\big(\Lambda(n-1) - \lambda(1+\alpha)\big)\Big] \leq 0. \end{aligned}$$

Inside $B_{\frac{1}{4}}$, as $F(D^2\varphi)$ is a continuous function,

$$F(D^2\varphi) \leq C = C(n, \lambda, \Lambda) \qquad \text{for } |x| \leq 1/4.$$

We can now take $0 \leq \xi \leq 1$ smooth such that $\xi \equiv 1$ in $\overline{B}_{1/4}$, $\xi \equiv 0$ outside $B_{1/2}$, so that (2.13) holds. □

Let Q_1 be the unit cube. We cut it into 2^n equal sized cubes. Furthermore, we continue to cut the cubes into 2^n many smaller cubes, i.e, reducing the side by half. We will call these cubes *dyadic cubes*.

If Q is a dyadic cube different from Q_1, we say that $\widetilde{Q}$ is the predecessor of Q if Q is one of the 2^n cubes obtained from dividing $\widetilde{Q}$. The proof given below is due to Calderon and Zygmund.

Lemma 2.4.2. Let $A \subset B \subset Q_1$ be measurable sets and $0 < \delta < 1$ such that

(a) $|A| \leq \delta$, and

(b) If Q is a dyadic cube such that $|A \cap Q| > \delta|Q|$, then $\widetilde{Q} \subset B$.

Then $|A| \leq \delta|B|$.

Proof. We have that

$$\frac{|Q_1 \cap A|}{|Q_1|} = |A| \leq \delta.$$

We subdivide Q_1 into 2^n dyadic cubes. If Q is a dyadic cube of Q_1 of length $\frac{1}{2}$ and satisfies $|Q \cap A|/|Q| \leq \delta$, we then split Q into dyadic cubes. We iterate this process. We will obtain $Q^1, Q^2, \dots$ dyadic cubes such that

$$\frac{|Q^i \cap A|}{|Q^i|} > \delta, \qquad \forall i.$$

Suppose $x \notin \bigcup Q^i$ then $x \in Q^i$ for $|Q^i| \to 0$, such that $|Q \cap A|/|Q| \leq \delta < 1$. Applying the Lebesgue differentiation theorem to χ_A, we get that $\chi_A(x) \leq \delta < 1$ for a.e. $x \notin \bigcup Q^i$. Hence, $A \subset \bigcup Q^i$ except for a set of measure zero. Let $\{\widetilde{Q}^i\}_{i\geq 1}$ be the predecessors of the cubes $\{Q^i\}_i$, and relabel them so that they are pairwise disjoint. We clearly have that

$$A \subset \bigcup_{i\geq 1} Q^i \subset \bigcup_{i\geq 1} \widetilde{Q}^i$$

and, from the way we chose the cubes Q^i,

$$\frac{|\widetilde{Q}^i \cap A|}{|\widetilde{Q}^i|} \leq \delta, \qquad \forall i.$$

Since $|Q^i \cap A|/|Q^i| > \delta$ and (b) holds, we have that $\widetilde{Q}^i \subset B$, for any $i \geq 1$. Hence

$$A \subset \bigcup_{i\geq 1} \widetilde{Q}^i \subset B.$$

We conclude that

$$|A| \leq \sum |\widetilde{Q}^i \cap A| \leq \delta \sum |\widetilde{Q}^i| = \delta \left|\bigcup \widetilde{Q}^i\right| \leq \delta |B|,$$

finishing the proof. □

2.4.1 Reduction of oscillations

Our next lemma roughly says that if a supersolution u takes the value 1 at some point x_0, then the set where $u < M$ has some positive measure μ, where M and μ depend only on the ellipticity constants. This implies that if u is small at some point, then it has to be small in some positive measure set. Recall that a supersolution is supposed to *curve downwards*, therefore this is to be expected.

Lemma 2.4.3. There exist universal constants $\varepsilon_0 > 0$, $0 < \mu < 1$ and $M > 1$, such that if

$F(D^2u) \leq |f|$ in $Q_{4\sqrt{n}}$, $u \in C(\overline{Q}_{4\sqrt{n}})$ and f satisfy

$$u \geq 0 \quad \text{in } Q_{4\sqrt{n}}, \tag{2.15}$$

$$\inf_{Q_3} u \leq 1, \quad \text{and} \tag{2.16}$$

$$\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0, \tag{2.17}$$

then

$$|\{u \leq M\} \cap Q_1| > \mu. \tag{2.18}$$

Proof. Take φ as in Theorem 2.4.1 and define $w = u + \varphi$. Recall that $B_{2\sqrt{n}} \subset Q_{4\sqrt{n}}$, φ is smooth and $F(D^2\varphi) \leq C\xi$. We claim that

$$F(D^2w) \leq |f| + C\xi \quad \text{in } B_{2\sqrt{n}}.$$

We have that $w \in C(\overline{B}_{2\sqrt{n}})$, $w \geq 0$ on $\partial B_{2\sqrt{n}}$ by (2.15) and (2.12). We also have that $\inf_{Q_3} w \leq -1$, by (2.16) and (2.13). We therefore, can apply the ABP estimate to w in $B_{2\sqrt{n}}$ and get

$$\begin{aligned} 1 &\leq C\left(\int_{\{w=\Gamma_w\}\cap B_{2\sqrt{n}}} (|f| + C\xi)^n\right)^{1/n} \\ &\leq C\|f\|_{L^n(Q_{4\sqrt{n}})} + C|\{w = \Gamma_w\} \cap Q_1|^{1/n}. \end{aligned}$$

We have used that $0 \leq \xi \leq 1$ and $\operatorname{supp}\xi \subset Q_1$. Taking ε_0 small enough, previous inequality and (2.17) imply

$$\frac{1}{2} \leq C|\{w = \Gamma_w\} \cap Q_1|^{1/n} \leq C|\{u \leq M\} \cap Q_1|^{1/n}.$$

$w(x) = \Gamma_w(x)$ implies $w(x) \leq 0$, because in the proof of ABP estimate, we extend w by 0 outside the domain. As convex functions obtain their maximum on the boundary, we have that $\Gamma_w \leq 0$. Therefore $u(x) \leq -\varphi(x) \leq M$ and the proof is done. □

In the previous proof, we could exploit the properties of convex functions to infer about contact points with the convex envelope. In general, the set $\{x \mid w(x) = \Gamma_w(x)\}$ contains important information about the function. The next result is an application of the previous lemma at all scales $u > M^k$.

Lemma 2.4.4. Let u be as in Lemma 2.4.3. Then

$$|\{u > M^k\} \cap Q_1| \leq (1-\mu)^k \tag{2.19}$$

for $k = 1, 2, 3, \ldots$, where M and μ are as in Lemma 2.4.3.

As a consequence, we have

$$|\{u \geq t\} \cap Q_1| \leq d\, t^{-\varepsilon} \qquad \forall t > 0, \tag{2.20}$$

where d and ε are positive universal constants.

Proof. For $k = 1$ (2.19) is just (2.18). We proceed by induction, let (2.19) hold for $k - 1$, and let

$$A = \{u > M^k\} \cap Q_1, \qquad B = \{u > M^{k-1}\} \cap Q_1.$$

(2.19) will be proved if we show that

$$|A| \leq (1-\mu)|B|.$$

We apply Lemma 2.4.2. Clearly, $A \subset B \subset Q_1$ and

$$|A| \leq |\{u > M\} \cap Q_1| \leq 1 - \mu,$$

by (2.4.3). To apply Lemma 2.4.2, we need to show that if $Q = Q_{1/2^i}(x_0)$ is a dyadic cube such that

$$|A \cap Q| > (1-\mu)|Q| \tag{2.21}$$

then $\widetilde{Q} \subset B$. We will prove this by method of contradiction. Let $\widetilde{Q} \not\subset B$ and take

$$\tilde{x} \in \widetilde{Q} \quad \text{such that} \quad u(\tilde{x}) \leq M^{k-1}. \tag{2.22}$$

Consider the transformation

$$x = x_0 + \frac{1}{2^i} y, \qquad y \in Q_1, \qquad x \in Q = Q_{1/2^i}(x_0),$$

and the function

$$\tilde{u}(y) = \frac{u(x)}{M^{k-1}}.$$

We claim that $\tilde{u}$ satisfies the hypothesis of Lemma 2.4.3. Before proceeding, let us note that, if Q is a dyadic cube, then $Q = Q_{\frac{1}{2^i}}(p)$ for some $p \in Q_1$, and $\widetilde{Q} \subset Q_{\frac{3}{2^i}}(p)$. From definition of $\tilde{u}$,

$$F(D^2\tilde{u}(y)) \leq \left(\frac{f(x)}{2^{2i}M^{k-1}}\right) =: \tilde{f}(y) \quad \text{in } Q_{4\sqrt{n}},$$

and by previous discussion

$$x \in \widetilde{Q} \Rightarrow y \in Q_3.$$

Hence $\tilde{u} \geq 0$ and $\inf_{Q_3} \tilde{u} \leq \tilde{u}(\tilde{x})/M^{k-1} \leq 1$, by (2.22). Finally,

$$\|\tilde{f}\|_{L^n(Q_{4\sqrt{n}})} = \frac{2^i}{2^{2i}M^{k-1}}\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \|f\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0.$$

By (2.17), it follows that

$$\mu < |\{\tilde{u}(y) \leq M\} \cap Q_1| = 2^{in}|\{u(x) \leq M^k\} \cap Q|.$$

Hence $|Q \setminus A| > \mu|Q|$, which contradicts (2.21).

(2.20) follows from (2.19) taking $d = (1-\mu)^{-1}$ and choosing ε such that $1-\mu = M^{-\varepsilon}$. □

The next lemma says if we have a subsolution u which is greater than some $\nu^{j-1}M$ at some point, then in a near-by cube it has to have a point x such that $u(x) \geq \nu^j M$. This heuristically means that the subsolution is curving upwards until the boundary of the domain, which is in line with our previous discussions. The proof will use the previous lemmas by exploiting the fact that if u is a subsolution, $-u$ is a supersolution.

Lemma 2.4.5. Let $F(D^2u) \geq -|f|$, where F is uniformly elliptic with λ, Λ as ellipticity constants, in $Q_{4\sqrt{n}}$. Assume that f satisfies (2.17) and u satisfies (2.20). Then there exist universal constants $M_0 > 1$ and $\sigma > 0$ such that, for ε as in (2.20) and $\nu = \dfrac{M_0}{M_0 - \frac{1}{2}} > 1$, the following holds:

If $j \geq 1$ is an integer and x_0 satisfies

$$||x_0|| \leq \frac{1}{4}$$

and

$$u(x_0) \geq \nu^{j-1}M_0, \tag{2.23}$$

then

$$Q'_j := Q_{l_j}(x_0) \subset Q_1 \qquad \text{and} \qquad \sup_{Q'_j} u \geq \nu^j M_0,$$

where

$$l_j = \sigma M_0^{-\varepsilon/n} \nu^{-\varepsilon j/n}.$$

Proof. Take $\sigma > 0$ and $M_0 > 1$ such that

$$\frac{1}{2}\sigma^n > d\, 2^\varepsilon (4\sqrt{n})^n \tag{2.24}$$

and

$$\sigma M_0^{-\varepsilon/n} + d M_0^{-\varepsilon} \leq \frac{1}{2}, \tag{2.25}$$

with d and ε as in (2.20). From the above we obtain

$$Q_{l_j/(4\sqrt{n})}(x_0) \subset Q_{l_j}(x_0) = Q'_j \subset Q_1.$$

We will proceed by the method of contradiction and suppose $\sup_{Q'_j} u < \nu^j M_0$. From the previous inclusion and (2.20) by comparing measures, we have

$$\left| \left\{ u \geq \nu^j \frac{M_0}{2} \right\} \cap Q_{l_j/(4\sqrt{n})}(x_0) \right| \leq \left| \left\{ u \geq \nu^j \frac{M_0}{2} \right\} \cap Q_1 \right| \leq d\nu^{-j\varepsilon} \left(\frac{M_0}{2} \right)^{-\varepsilon}. \tag{2.26}$$

We perform a coordinate change

$$x = x_0 + \frac{l_j}{4\sqrt{n}}\, y, \qquad y \in Q_{4\sqrt{n}}, \qquad x \in Q'_j = Q_{l_j}(x_0),$$

and define the function

$$v(y) = \frac{\nu M_0 - \frac{1}{\mu^{j-1}} u(x)}{(\nu - 1) M_0}.$$

The previous transformation defines bijections between the following sets

$$x \in Q_{l_j}(x_0) \quad \Longleftrightarrow \quad y \in Q_{4\sqrt{n}} \text{ [resp. } Q_3,\, Q_1].$$

and also [resp. $Q_{3l_j/(4\sqrt{n})}(x_0)$, $Q_{l_j/(4\sqrt{n})}(x_0)$] $\Longleftrightarrow$ [resp. Q_3, Q_1]. We claim that v satisfies the hypothesis of (2.4.4) and hence, By ellipticity of F we have,

$$F(D^2v(y)) \leq \left(\frac{l_j^2}{(4\sqrt{n})^2(\nu^j-(\nu-1)M_0)^{-1}}f(y)\right) =: \tilde{f}(y) \quad \text{in } Q_{4\sqrt{n}}.$$

By the assumption $\sup_{Q_j'} u < \nu^j M_0$, we have that $v > 0$ in $Q_{4\sqrt{n}}$. (2.23) implies that $\inf_{Q_3} v \leq 1$.

Finally,

$$\begin{aligned}\|\tilde{f}(y)\|_{L^n(Q_{4\sqrt{n}})} &= \frac{\sigma M_0^{-\varepsilon/n}\nu^{-\varepsilon j/n}}{4\sqrt{n}\nu^{j-1}(\nu-1)M_0}\|f(x)\|_{L^n(Q_{l_j}(x_0))} \\ &\leq \frac{\sigma M_0^{-\varepsilon/n}\nu^{-\varepsilon j/n}}{4\sqrt{n}\nu^{j-1}(\nu-1)M_0}\,\varepsilon_0.\end{aligned}$$

By the choice made in (2.25), $\nu > 1$ and $\nu = 2(\nu-1)M_0$, we get

$$\|\tilde{f}(y)\|_{L^n(Q_{4\sqrt{n}})} \leq \frac{\nu^{-\varepsilon j/n}}{8\sqrt{n}\nu^{j-1}(\nu-1)M_0}\,\varepsilon_0 \leq \frac{\nu^{-\varepsilon j/n}}{4\sqrt{n}\nu^j}\varepsilon_0 \leq \varepsilon_0.$$

Since v satisfies the hypothesis of (2.4.4), we have

$$|\{v(y) > M_0\} \cap Q_1| \leq dM_0^{-\varepsilon}.$$

This implies, since $u(x) < \nu^j \dfrac{M_0}{2} \Rightarrow v(y) > M_0$, that

$$\left|\left\{u(x) < \nu^j\frac{M_0}{2}\right\} \cap Q_{l_j/(4\sqrt{n})}(x_0)\right| \leq \left(\frac{l_j}{4\sqrt{n}}\right)^n dM_0^{-\varepsilon}.$$

This inequality and (2.26) give

$$\left(\frac{l_j}{4\sqrt{n}}\right)^n \leq \left(\frac{l_j}{4\sqrt{n}}\right)^n dM_0^{-\varepsilon} + d\nu^{-j\varepsilon}\left(\frac{M_0}{2}\right)^{-\varepsilon}.$$

Hence, by (2.25),

$$\frac{1}{2}\left(\frac{l_j}{4\sqrt{n}}\right)^n \leq d\nu^{-j\varepsilon}\left(\frac{M_0}{2}\right)^{-\varepsilon}.$$

Using the definition of l_j, we obtain

$$\frac{1}{2}\sigma^n \leq d2^\varepsilon(4\sqrt{n})^n,$$

which is a contradiction with (2.24) and we are done with the proof. □

If u is a solution, then it is both a subsolution and a supersolution. As we have been pointing out, subsolution tends to curve upwards and supersolution tends to curve downwards. Due to the opposing tendencies, a solution cannot curve 'too' much in either way which let us obtain higher regularity. Next result is a rigorous justification of this observation.

Lemma 2.4.6. Let $F(D^2u) = f$ where F is uniformly elliptic with the ellipticity constants λ, Λ, in $Q_{4\sqrt{n}}$, $u \in C(\overline{Q_{4\sqrt{n}}})$ satisfy $u \geq 0$ in $Q_{4\sqrt{n}}$, where f is continuous and bounded in $Q_{4\sqrt{n}}$. Assume that $\inf_{Q_{1/4}} u \leq 1$ and

$$\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0.$$

Then

$$\sup_{Q_{1/4}} u \leq C,$$

where ε_0 and C are constants dependent only on λ, Λ, n.

Proof u satisfies the hypothesis of (2.4.4), and therefore, being a subsolution too, it satisfies the hypothesis of (2.4.5). Recall that

$$l_j = \sigma M_0^{-\ell/n} \nu^{-j/n}, \qquad j = 1, 2, 3, \ldots.$$

Therefore, there exists a universal integer, that depends only on dimension and ellipticity constants, $j_0 \geq 1$, such that

$$\sum_{j \geq j_0} l_j \leq \frac{1}{4}. \tag{2.27}$$

We claim that $\sup_{Q_{1/4}} u \leq \nu^{j_0 - 1} M_0$ to finish the proof of (2.4.6). Again, we argue by contradiction and suppose not.Then there exists x_{j_0} such that

$$||x_{j_0}|| \leq \frac{1}{8} \quad \text{and} \quad u(x_{j_0}) \geq \nu^{j_0-1} M_0.$$

We can apply (2.4.5) to get the existence of a point x_{j_0+1} such that

$$||x_{j_0+1} - x_{j_0}|| \leq \frac{l_{j_0}}{2} \quad \text{and} \quad u(x_{j_0+1}) \geq \nu^{j_0} M_0.$$

By an inductive argument, we get a sequence of points x_j $(j \geq j_0)$ such that

$$||x_{j+1} - x_j|| \leq \frac{l_j}{2} \quad \text{and} \quad u(x_{j+1}) \geq \nu^j M_0 \quad \forall j \geq j_0. \tag{2.28}$$

but we will also have

$$||x_j|| \leq ||x_{j_0}|| + \sum_{k=j_0}^{j-1} ||x_{k+1} - x_k|| \leq \frac{1}{8} + \sum_{k \geq j_0} \frac{l_k}{2} \leq \frac{1}{4},$$

by (2.27). But this implies that x_j stays inside $\bar{Q}_{\frac{1}{2}}$, and $\sup_{Q_{\frac{1}{2}}} u = \infty$ as $\nu > 1$. This is a contradiction as u is continuous on $\bar{Q}_{\frac{1}{2}}$. □

Theorem 2.4.7. Let u be such that $F(D^2u) = f$ in $Q_{4\sqrt{n}}$ and $u \geq 0$ in $Q_{4\sqrt{n}}$, where f is continuous and bounded in $Q_{4\sqrt{n}}$ and F is uniformly elliptic with λ, Λ as ellipticity constants. Then

$$\sup_{Q_1} u \leq C \left(\inf_{Q_1} u + \|f\|_{L^n(Q_{4\sqrt{n}})} \right),$$

where C is a universal constant.

Proof. Consider, for any $\delta > 0$,

$$u_\delta = \frac{u}{\inf_{Q_{1/4}} u + \delta + \|f\|_{L^n}/\varepsilon_0}$$

We apply (2.4.6) to u_δ and get, after letting $\delta \to 0$,

$$\sup_{Q_{1/4}(x)} u \leq C \left(\inf_{Q_{1/4}(x)} u + F \right), \qquad F := \|f\|_{L^n(Q_{4\sqrt{n}})}. \tag{2.29}$$

for every cube $Q_{1/4}(x) \subset Q_{3/2}$. Let $x^*, y^* \in Q_1$ be such that

$$u(x^*) = \sup_{Q_1} u, \qquad u(y^*) = \inf_{Q_1} u.$$

Since Q_1 is bounded, we can connect x^* to y^* by a chain of points

$$x_0, x_1, \ldots, x_k$$

such that

$$x_0 = x^*, \qquad x_k = y^*,$$

and the cubes $Q_{1/4}(x_i)$ and $Q_{1/4}(x_{i+1})$ intersect for each i. The number k depends only on the dimension.

Applying (2.29) in the cube $Q_{1/4}(x_i)$ we obtain

$$\sup_{Q_{1/4}(x_i)} u \le C \left(\inf_{Q_{1/4}(x_i)} u + F \right).$$

Since the cubes overlap, we have

$$\inf_{Q_{1/4}(x_i)} u \le \sup_{Q_{1/4}(x_{i+1})} u.$$

Therefore

$$\sup_{Q_{1/4}(x_i)} u \le C \left(\sup_{Q_{1/4}(x_{i+1})} u + F \right).$$

Iterating this inequality along the chain yields

$$\sup_{Q_{1/4}(x_0)} u \le C^k \left(\sup_{Q_{1/4}(x_k)} u + F \right).$$

Since $x_0 = x^*$ and $x_k = y^*$, and $x^* \in Q_{1/4}(x_0)$, we obtain

$$u(x^*) \le C^k \left(\sup_{Q_{1/4}(y^*)} u + F \right).$$

Applying (2.29) once more at y^* gives

$$\sup_{Q_{1/4}(y^*)} u \le C \left(\inf_{Q_{1/4}(y^*)} u + F \right) \le C \left(u(y^*) + F \right).$$

Combining the inequalities we find

$$u(x^*) \le C^{k+1} \left(u(y^*) + F \right).$$

Since $u(x^*) = \sup_{Q_1} u$ and $u(y^*) = \inf_{Q_1} u$, we conclude

$$\sup_{Q_1} u \le C \left(\inf_{Q_1} u + \|f\|_{L^n(Q_{4\sqrt{n}})} \right),$$

where C is a universal constant.

Theorem 2.4.8 (Hölder continuity)**.** Let u be such that $F(D^2u) = f$ in Q_1, where f is continuous and bounded in Q_1 and F is uniformly elliptic with ellipticity constants λ, Λ.Then there exist universal constants $\alpha \in (0,1)$ and $C > 0$ such that

$$|u(x) - u(y)| \le C \left(\|u\|_{L^\infty(Q_2)} + \|f\|_{L^n(Q_{4\sqrt{n}})} \right) |x-y|^\alpha$$

for all $x, y \in Q_1$. In particular, $u \in C^\alpha(Q_1)$.

Proof. We prove decay of the oscillation of u in smaller cubes.

Let

$$M_r = \sup_{Q_r} u, \qquad m_r = \inf_{Q_r} u, \qquad \omega(r) = M_r - m_r.$$

We claim that there exists $\theta \in (0,1)$ such that

$$\omega(1/2) \le \theta\, \omega(1) + C\|f\|_{L^n(Q_{4\sqrt{n}})}.$$

Define

$$v = u - m_1.$$

Then $v \ge 0$ in Q_1 and

$$\sup_{Q_1} v = \omega(1).$$

Applying the Harnack inequality to v in Q_1 gives

$$\sup_{Q_{1/2}} v \le C \left(\inf_{Q_{1/2}} v + \|f\|_{L^n(Q_{4\sqrt{n}})} \right).$$

Since

$$\inf_{Q_{1/2}} v = m_{1/2} - m_1,$$

we obtain

$$M_{1/2} - m_1 \le C \left((m_{1/2} - m_1) + \|f\|_{L^n} \right).$$

Rearranging the terms yields

$$M_{1/2} - m_{1/2} \leq \theta(M_1 - m_1) + C\|f\|_{L^n},$$

for some $\theta \in (0,1)$ depending only on the ellipticity constants.

Thus

$$\omega(1/2) \leq \theta\omega(1) + C\|f\|_{L^n}.$$

Iterating this estimate gives

$$\omega(2^{-k}) \leq \theta^k\omega(1) + C\|f\|_{L^n}.$$

Since $\theta^k = (2^{-k})^\alpha$ for $\alpha = -\log_2\theta$, we obtain

$$\omega(r) \leq Cr^\alpha \left(\|u\|_{L^\infty(Q_2)} + \|f\|_{L^n(Q_{4\sqrt{n}})}\right)$$

for $0 < r < 1$.

Finally, if $x, y \in Q_1$ and $r = |x-y|$, then

$$|u(x) - u(y)| \leq \omega(r)$$

which yields

$$|u(x) - u(y)| \leq C\left(\|u\|_{L^\infty(Q_2)} + \|f\|_{L^n(Q_{4\sqrt{n}})}\right)|x-y|^\alpha.$$

Hence $u \in C^\alpha(Q_1)$. $\square$

Chapter 3

Convex Envelope

3.1 Introduction

In the last chapter we utilized the regularity of the convex envelope of a supersolution to get the ABP estimate. In this chapter, we will study in general what can be the expected regularity for the convex envelope. Here, we will only study the convex envelope of functions defined on the boundary of a domain. The convex envelope ϕ_g of a function g defined on the boundary of a convex set $\Omega \subset \mathbb{R}^n$ is defined as

$$\phi_g(x) = \sup\{\phi(x) \mid \phi \text{ is convex in } \Omega \text{ and}, \phi \leq g \text{ on } \partial\Omega\}. \tag{3.1}$$

In this chapter, we will pose the problem of finding the convex envelope as solving a PDE and then derive regularity properties of it, following the works [14] and [15].

3.2 PDE associated with Convex Envelope

In this section we will show that ϕ_g as defined in 3.1, is the viscosity solution to the PDE

$$\begin{cases} \lambda_1(u(x)) = 0 & x \in \Omega, \\ u(x) = g(x) & x \in \partial\Omega, \end{cases} \tag{3.2}$$

where $\lambda_1(u(x))$ is the first or smallest eigenvalue of the Hessian (in viscosity sense). This was first done in the paper [14], and we will be following this paper for the rest of this section.

Theorem 3.2.1. The continuous function $u : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if u satisfies $\lambda_1(u) \geq 0$ in viscosity sense.

Suppose u is C^2, then the previous theorem is obvious because in one dimension, a twice differentiable function is convex if and only if $\frac{d^2u}{dt^2} \geq 0$, and if $\lambda_1(u) \geq 0$ then $\frac{\partial^2 u}{\partial z^2} \geq 0$ for all directions z, therefore u will be convex. The opposite direction can also be shown to be true arguing similarly. Now, with this in mind we justify the arguments we made in viscosity sense. Let us define the notion of cylindrical neighbourhoods. Let $x = (x^1, x')$ for $x \in \mathbb{R}^n$. For a set $B \subset \mathbb{R}^n$ we define the cylindrical neighbourhood, $C_\epsilon(B)$,

Definition 3.2.2. $C_\epsilon(B) := \{\ y \in \mathbb{R}^n \ :\ |y^1 - x^1| \leq \epsilon$ *and* $\|y' - x'\| \leq \epsilon$ *for some* $(x^1, x') \in B\}$.

Note that convex functions on finite dimensional spaces are continuous, therefore they are upper-semi continuous. Hence, we can apply our notions of viscosity solutions on them.

Proof 1. Let u be convex. Suppose x_0 is a local maximum of $u - \phi$ for some C^2 function ϕ (we can assume $u(x_0) = \phi(x_0)$). We will call the supporting hyperplane $p(x)$ for u at x_0. As x_0 is the local maximum,

$$D^2\phi(x_0) \geq D^2 p(x_0) = 0,$$

which gives one way implication.

2. If $u(x)$ is not convex, then there are points $w, y, z \in \mathbb{R}^n$ with

$$w = \theta y + (1-\theta) z, \qquad 0 < \theta < 1,$$

such that

$$u(w) > \theta u(y) + (1-\theta) u(z).$$

Change the co-ordinate system so that w, y, z lie on the x^1 axis, and $y^1 < z^1$. Let

$$q(x) = \frac{a}{2}(x^1 - w^1)^2 + b(x^1 - w^1) + u(w)$$

be the parabola whose graph passes through $(w, u(w)), (y, u(y))$ and $(z, u(z))$. By our assumption of non-convexity , we have $a < 0$.

(i) Let $n = 1$. If w is a local maximum of $u - q$, then we are done. If not, let

$$x_* \in \operatorname{argmax}\{u(x) - q(x) \mid x \in [y, z]\}.$$

Then $x_* \in (y,z)$ because $u-q=0$ at y,z,w, and if maximum is not achieved at w it cannot be achieved at y or z. Therefore, x_* is a local maximum of $u-q$, and we have obtained a contradiction, since $\lambda_1[q]=a<0$.

(ii) Now suppose $n>1$. Our goal is to find a local maximum which is not on the boundary. To ensure this we will first increase the curvature of our parabola and then add another parabola which is curving in a different direction (this wont change the value of $\lambda_1(q)$ as it depends only on one dimensional sets). This will be enough to ensure existence of local maximum inside the domain. We redefine

$$q(x)=\frac{a+\delta}{2}(x^1-w^1)^2+b(x^1-w^1)+u(w),$$

for $0<\delta<-a$. Then $q(w)=u(w)$, but $q>u$ at y,z. Thus $q>u$ in a neighbourhoodd of each of the points y,z. Choose $\varepsilon>0$ so that

$$q(x)>u(x), \qquad x\in C_\varepsilon(y)\cup C_\varepsilon(z).$$

Define

$$\phi(x)=q(x)+\frac{M\|x'\|^2}{\varepsilon^2},$$

where

$$M=\max\{u(x)\mid x\in C_\varepsilon(I)\}$$

and I is the line segment $[y^1-\varepsilon,\, z^1+\varepsilon]$. Then $\lambda_1[\phi]=a-\delta<0$, $\phi(w)=u(w)$. Finally, $\phi\geq u$ on $\partial C_\varepsilon(I)$ since $q\geq u$ when $\|x'\|<\varepsilon$ and $M\geq u$ when $\|x'\|=\varepsilon$. If w is a local maximum of $u-\phi$, then we are done. If not, choose

$$x_*\in \operatorname{argmax}\{u(x)-\phi(x)\mid x\in C_\varepsilon(I)\}.$$

Then x_* is in the interior of $C_\varepsilon(I)$, so x_* is a local maximum of $u-\phi$ and we have obtained a contradiction as $\lambda_1(\phi)\leq 0$. □

Theorem 3.2.3. Let Ω be a convex set. Let $g:\partial\Omega\to\mathbb{R}$ be a function and ϕ_g be its convex envelope. Then ϕ_g solves (3.2) in viscosity sense.

Proof. We define

$$\mathcal{A}:=\{v\in USC(\overline{\Omega});\ \lambda_1(v)\geq 0 \text{ in } \Omega, \quad u\leq g \text{ on } \partial\Omega\}.$$

We note that $-||g||_{L^\infty(\partial\Omega)}$ belongs to $\mathcal{A}$ and all elements of $\mathcal{A}$ are bounded by $||g||_{L^\infty(\partial\Omega)}$ due to comparison principle. Let

$$u(x) = \sup_{v\in\mathcal{A}} v(x).$$

By Perron's method, $u(x)$ is the solution for (3.2). But by previous lemma (as we are dealing with local operators, $\mathbb{R}^n$ can be replaced with open convex set Ω), $\lambda_1(v) \geq 0$ iff v is convex. Therefore,

$$u(x) = \sup\{v \mid v \text{ is convex in } \Omega, \quad u \leq g \text{ on } \partial\Omega\}$$

That is $u(x) = \phi_g(x)$, and we are done. □

3.3 Regularity of the convex envelope

In the previous section, we posed the problem of finding the convex envelope as solving a PDE. This approach will help us prove optimal regularity of convex envelope because we can use PDE techniques, which will be the comparison principle here. In this section we will follow [15].

Theorem 3.3.1. Consider the PDE (3.2) for $\Omega = B_1 \subset \mathbb{R}^n$. If the boundary data g is $C^{1,\alpha}$ on $\partial\Omega = S_1$ for some $\alpha \in (0,1)$, then the convex envelope u is $C^{1,\alpha}$ in $B_{1/2}$. Moreover, the following estimate holds

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\|g\|_{C^{1,\alpha}(\partial B_1)},$$

where the constant C depends only on the dimension n and on α.

Proof. Without loss of generality, we can assume (as we can prove the general case by rescaling)

$$\|g\|_{C^{1,\alpha}(\partial B_1)} = 1$$

We will be done proving this if we can prove for any two points $x_1, x_2 \in B_{1/2}$ the following estimate holds

$$|\nabla u(x_1) - \nabla u(x_2)| \leq C|x_1 - x_2|^\alpha$$

where C depends only on n and α. We have the advantage of dealing with convex functions. *So we have a natural choice for the gradient, which is the supporting hyperplane(s).* Remember that if the supporting plane is unique at the point x_0, then the convex function is differentiable at x_0 with supporting hyperplane as the derivative. Let

$$L_i(x) = A_i \cdot (x - x_i) + b_i, \qquad i = 1, 2 \tag{3.3}$$

be the supporting hyperplanes to u at x_i, respectively. If we prove $|A_1 - A_2| \le \|x_1 - x_2\|$, uniqueness will follow trivially. Let us define

$$M(x) = \max(L_1(x), L_2(x)).$$

We can find an appropriate affine function B such that if we B subtract from u and g we will have

$$A_1 + A_2 = 0$$

and

$$\min_{x \in B_1} M = 0. \tag{3.4}$$

Note that this can be done because convex envelop of $g - B$ is $u - B$. Therefore, the $(n-1)$ dimensional hyperplane

$$P = \{x \mid L_1(x) = L_2(x)\}$$

is the same as

$$P = \{x \mid M(x) = 0\}.$$

The following equation will help us relate the value of the function and the distance between points

$$M(x) = \frac{|A_1 - A_2|}{2} \operatorname{dist}(x, P).$$

As noted before, our goal is to show that

$$|A_1 - A_2| \le C|x_1 - x_2|^\alpha \tag{3.5}$$

for some constant C depending only on the dimension n and α.

The proof will involve an estimate on $\min g$ from below in terms of $|A_1 - A_2|$, which will lead to an estimate on $|x_1 - x_2|$ from below by using the comparison principle.

Relationship between Euclidean and geodesic distances. We can consider two types of distances from points on the boundary $S_1 = \partial B_1$ to the intersection of P with S_1, one in the ambient space $\mathbb{R}^n$ and the other along the boundary, using the geodesic distance. We use the notation $\mathrm{dist}(y, x)$ for the standard Euclidean distance, and $y \in S_1$,

$$\mathrm{dist}_{S_1}(y, P \cap S_1)$$

for the geodesic distance between y and $P \cap S_1$ on S_1.

Since x_1 and x_2 lie on different sides of P, P passes through $B_{1/2}$ and intersects S_1 nontangentially (if it intersects tangentially then whole of B_1 lies on one side of P, but P is separating x_1 and x_2. Therefore, P intersects S_1 at some minimum angle $\theta_0 > 0$ (which depends only on the dimension n). We claim that there is a constant $C_2 > 1$ such that for any $y \in S_1$

$$\mathrm{dist}(y, P) \leq \mathrm{dist}_{S_1}(y, P \cap S_1) \leq C_2 \,\mathrm{dist}(y, P), \tag{3.6}$$

where

$$1 \leq C_2.$$

Indeed, fix a point $y_0 \in S_1$, then there exists a $C_2(y_0)$ such that $\mathrm{dist}_{S_1}(y_0, P \cap S_1) - C_2(y_0)\,\mathrm{dist}(y_0, P) < 0$, and because of continuity of the function

$$\mathrm{dist}_{S_1}(y, P \cap S_1) - C_2(y_0)\,\mathrm{dist}(y, P)$$

, $\mathrm{dist}_{S_1}(y, P \cap S_1) - C_2(y_0)\,\mathrm{dist}(y, P) < 0$ in a neighbourhood of y_0. As S^1 is compact, we can cover it finitely many such neighbourhoods of points $\{y_i\}$ and take $C_2 = \sup_i C_2(y_i)$ to prove the claim.

Estimates. Let $x_0 \in S_1$ be such that the minimum of g is achieved at x_0,

$$\min_{x \in S_1} g(x) = g(x_0) \geq 0.$$

and moreover, this minimum is nonnegative, by (3.4).

Now select a great circle on S_1 which is perpendicular to P and which passes through x_0. We parametrize (arc length parametrization) the circle which passes through P and x_0, by the function $\gamma(t)$.

$$\gamma(0) \in P, \qquad \gamma(t_1) = x_0.$$

Using (3.6)(and the fact that we are parameterizing using arch length), we obtain

$$\mathrm{dist}(\gamma(t), P) \leq t \leq C_2 \mathrm{dist}(\gamma(t), P), \qquad 0 < t < \pi/2. \tag{3.7}$$

where we have chosen $\gamma(0)$ in the same hemisphere as x_0. By convexity of u and the supporting plane conditions,

$$g(x) \geq u(x) \geq M(x), \qquad \text{for } x \in \overline{B}_1.$$

Let us define

$$G(t) = g(\gamma(t))$$

and by the previous result

$$G(t) \geq M(\gamma(t)), \qquad 0 < t < \pi/2. \tag{3.8}$$

Apply estimates at two points. To obtain lower bound on g we will use (3.7) and (3.8) at t_1 and at a second larger value of t to obtain the desired result.

Combining (3.8) with the expression for $M(x)$, we obtain

$$\operatorname{dist}(x_0, P) \leq \frac{2g(x_0)}{|A_1 - A_2|}.$$

Applying (3.7) at t_1 we obtain

$$t_1 \leq C_2 \operatorname{dist}(x_0, P).$$

From the previous two inequalities we obtain

$$0 \leq t_1 \leq C_2 \frac{2g(x_0)}{|A_1 - A_2|}.$$

Define

$$t_2 = 2C_2 \frac{2g(x_0)}{|A_1 - A_2|}.$$

then

$$C_2 \frac{2g(x_0)}{|A_1 - A_2|} \leq t_2 - t_1 \leq 2C_2 \frac{2g(x_0)}{|A_1 - A_2|}.$$

Then compute

$$\begin{aligned} G(t_2) &\geq M(\gamma(t_2)) \\ &= \frac{|A_1 - A_2|}{2} \operatorname{dist}(\gamma(t_2), P) \\ &\geq \frac{|A_1 - A_2|}{2} \frac{1}{C_2} t_2 \\ &= \frac{|A_1 - A_2|}{2} \frac{1}{C_2} 2C_2 \frac{2g(x_0)}{|A_1 - A_2|} \\ &= 2g(x_0) \end{aligned}$$

Therefore,

$$G(t_2) \geq 2g(x_0). \tag{3.9}$$

Use the Holder regularity of ∇g. By (3.9) and the definition $G(t_1) = g(x_0)$ we obtain

$$\int_{t_1}^{t_2} G'(t)\, dt = G(t_2) - G(t_1) \geq g(x_0).$$

By integral version of intermediate value lemma

$$\int_{t_1}^{t_2} f(t)\, dt \geq \beta$$

implies there exists $t^* \in [t_1, t_2]$ such that

$$f(t^*) \geq \beta (t_2 - t_1)^{-1}.$$

From this fact and the previous inequality, there exists $t^* \in [t_1, t_2]$ such that

$$G'(t^*) \geq g(x_0)(t_2 - t_1)^{-1} \geq \frac{1}{2C_2} \frac{|A_1 - A_2|}{2}. \tag{3.10}$$

Since $g(x_0)$ is the minimum of g,

$$G'(t_1) = 0.$$

Since $\|g\|_{C^{1,\alpha}(\partial B_1)} = 1$ then

$$|G'(s_1) - G'(s_2)| \leq |s_1 - s_2|^\alpha, \qquad \text{for all } s_1, s_2.$$

Therefore, by (3.10)

$$G'(t^*) \leq |t^* - t_1|^\alpha.$$

Combining these inequalities we get,

$$\frac{1}{2C_2}\frac{|A_1 - A_2|}{2} \leq |t^* - t_1|^\alpha \leq |t_2 - t_1|^\alpha \leq \left(2C_2\frac{2g(x_0)}{|A_1 - A_2|}\right)^\alpha.$$

This implies

$$g(x_0) \geq (4C_2)^{-1-1/\alpha}|A_1 - A_2|^{1+1/\alpha}.$$

Thus, there is a constant c such that

$$\min g = g(x_0) \geq c|A_1 - A_2|^{1+1/\alpha}.$$

Use comparison principle From the previous step, the constant function

$$L(x) = c|A_1 - A_2|^{1+1/\alpha}$$

is below g on S_1, and then, by comparison,

$$u(x) \geq c|A_1 - A_2|^{1+1/\alpha} \qquad \text{for } x \in B_1.$$

Evaluating u at x_1 and x_2, and noting the fact that M touches u from below at x_1 and x_2, we get

$$u(x_i) = M(x_i) = \frac{|A_1 - A_2|}{2}\,\mathrm{dist}(x_i, P), \qquad i = 1, 2.$$

then

$$\mathrm{dist}(x_i, P) \geq c|A_1 - A_2|^{1/\alpha},$$

which clearly implies

$$|A_1 - A_2| \leq C|x_1 - x_2|^\alpha,$$

and we are done. □

In this section we have used the supporting hyperplane property of convex functions repeatedly to derive the properties of the PDE (3.2). This geometric property is not available to us in the fractional convex case, which we will define in the fifth chapter.

Chapter 4

Nonlocal Operators

4.1 Introduction to Nonlocal Operators

Nonlocal operators arise in many areas of mathematics, physics, biology and applied sciences when the behaviour of a function at a point depends not only on nearby values but on values of the function over an entire region. Unlike classical differential operators, which involve only derivatives at a point, nonlocal operators incorporate interactions across spatial scales through integral expressions.

A classical example of a local operator is the Laplacian

$$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x),$$

which depends only on the behaviour of u arbitrarily close to the point x. In contrast, a typical nonlocal operator takes the form

$$Lu(x) = \int_{\mathbb{R}^n} (u(x+y) - u(x))K(y)\, dy,$$

where K is a kernel describing the strength of interaction between the point x and other points in space. Because the integral ranges over the entire domain, the value of $Lu(x)$ depends on the global behaviour of u.

Nonlocal operators naturally appear in the study of stochastic processes with jumps, such as Lévy processes, where the generator of the process is a nonlocal operator. They also arise in models of anomalous diffusion, material science, image processing, and mathematical finance. In many of these applications, long-range interactions or discontinuous dynamics cannot be adequately captured by classical local differential operators. See [5] for a detailed explanation on applications of jump process and non-local operators in financial modeling.

One of the most important examples of a nonlocal operator is the fractional Laplacian. For $s \in (0,1)$, it is defined by

$$(-\Delta)^s u(x) = C_{n,s}\,\text{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy,$$

where P.V. denotes the Cauchy principal value and $C_{n,s}$ is a normalization constant. The fractional Laplacian generalizes the classical Laplacian and describes diffusion processes with jumps of arbitrary size.

More generally, many nonlocal operators can be written in the form

$$Lu(x) = \int_{\mathbb{R}^n} \delta u(x,y)K(y)\,dy,$$

where

$$\delta u(x,y) = u(x+y) + u(x-y) - 2u(x),$$

and the kernel K satisfies appropriate symmetry (i.e., $K(y) = K(-y)$) and integrability conditions. When $K(y) \sim |y|^{-n-2s}$, the operator behaves similarly to a fractional Laplacian of order $2s$.

The study of nonlocal operators has developed rapidly in recent years, particularly, in the context of integro-differential equations. Many classical PDE techniques must be adapted to account for the global nature of the operators, and new tools have been developed to study regularity, maximum principles, and boundary behaviour of solutions.
In this chapter we will define notion of ellipticity, and survey viscosity solutions and their properties for nonlocal operators.

4.2 Uniformly Elliptic Integral operators

Definition 4.2.1 (Integro-differential operator). *Let $0 < s < 1$. A linear integro-differential operator acting on a function $u : \mathbb{R}^n \to \mathbb{R}$ is given by*

$$Lu(x) = \int_{\mathbb{R}^n} \delta u(x,y)K(y)\,dy,$$

where

$$\delta u(x,y) = u(x+y) + u(x-y) - 2u(x)$$

is the second difference operator and $K(y)$ *is a measurable function called the kernel. The kernel typically satisfies*

$$\lambda \frac{1}{|y|^{n+2s}} \leq K(y) \leq \Lambda \frac{1}{|y|^{n+2s}} \qquad \forall y \neq 0,$$

for constants $0 < \lambda \leq \Lambda$.

The kernel says how strong the long range interactions are going to be, the bounds on the kernel are the ellipticity bounds.

Definition 4.2.2 (Fully nonlinear integro-differential operator)**.** *Let* $0 < s < 1$. *A fully nonlinear uniformly elliptic integro-differential operator* I *acting on a function* $u : \mathbb{R}^n \to \mathbb{R}$ *is an operator of the form*

$$Iu(x) = \inf_{\alpha} \sup_{\beta} L_{\alpha\beta} u(x),$$

where each $L_{\alpha\beta}$ *is a linear integro-differential operator given by*

$$L_{\alpha\beta} u(x) = \int_{\mathbb{R}^n} \delta u(x,y)\, K_{\alpha\beta}(y)\, dy,$$

and

$$\delta u(x,y) = u(x+y) + u(x-y) - 2u(x).$$

The kernels $K_{\alpha\beta}(y)$ *are measurable functions satisfying the ellipticity bounds*

$$\lambda \frac{1}{|y|^{n+2s}} \leq K_{\alpha\beta}(y) \leq \Lambda \frac{1}{|y|^{n+2s}}$$

for constants $0 < \lambda \leq \Lambda$.

From now on, when we say uniformly elliptic nonlocal operator or integral operator, it will be in the sense of Definition 4.2.2.

Now we define viscosity solutions. For simplicity of the exposition, we will assume the solutions to be bounded.

Definition 4.2.3 (Viscosity subsolution)**.** *Let* I *be a fully nonlinear integro-differential operator and consider the equation*

$$Iu(x) = f(x) \quad \text{in } \Omega.$$

A bounded upper semicontinuous function $u : \mathbb{R}^n \to \mathbb{R}$ *is called a viscosity subsolution to the equation if for every* $x_0 \in \Omega$ *and every test function* $\phi \in C^2(B_r(x_0))$ *such that*

$$\phi(x_0) = u(x_0), \qquad \phi(x) \geq u(x) \text{ in } B_r(x_0),$$

the function

$$v(x) = \begin{cases} \phi(x) & \text{for } x \in B_r(x_0), \\ u(x) & \text{for } x \notin B_r(x_0), \end{cases}$$

satisfies

$$Iv(x_0) \geq f(x_0).$$

Definition 4.2.4 (Viscosity supersolution)**.** *A bounded lower semicontinuous function* $u : \mathbb{R}^n \to \mathbb{R}$ *is called a viscosity supersolution of the equation if for every* $x_0 \in \Omega$ *and every test function* $\phi \in C^2(B_r(x_0))$ *such that*

$$\phi(x_0) = u(x_0), \qquad \phi(x) \leq u(x) \text{ in } B_r(x_0),$$

the function

$$v(x) = \begin{cases} \phi(x) & \text{for } x \in B_r(x_0), \\ u(x) & \text{for } x \notin B_r(x_0), \end{cases}$$

satisfies

$$Iv(x_0) \leq f(x_0).$$

Definition 4.2.5 (Viscosity solution)**.** *A function* u *is called a viscosity solution of the equation*

$$Iu(x) = f(x) \quad \text{in } \Omega,$$

if it is both viscosity subsolution and viscosity supersolution.

4.3 Results which are counterparts to the local case

. In this section, we will state theorems which are analogous to the theorems we have seen in the local case and their proofs have the same idea but require more technical analysis due to the nonlocal nature. The proofs of the theorems here can be found in [4]. Comparison principle can be proven either using the Ishii-Lions method or by exploiting the fact that supersolutions curve downwards and vice versa, which we will do for the fractionally convex operator.

Theorem 4.3.1 (Comparison principle)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let I be a fully nonlinear integro-differential operator, uniformly elliptic in the sense of Definition 4.2.2. Consider the Dirichlet problem

$$Iu(x) = f(x) \quad \text{in } \Omega.$$

Suppose u is a viscosity subsolution and v is a viscosity supersolution in Ω, and assume that

$$u(x) \leq v(x) \quad \text{for } x \in \mathbb{R}^n \setminus \Omega.$$

Then

$$u(x) \leq v(x) \quad \text{for all } x \in \Omega.$$

Proof. For a proof we refer to [4]. □

Once we have comparison principle, we can use Perron's method as before to prove existence and uniqueness of the solutions. As in the local case, this involves proving that supremum of all subsolutions is both a subsolution and a supersolution.

Theorem 4.3.2 (Existence and uniqueness via Perron's method)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let I be a fully nonlinear integro-differential operator, uniformly elliptic in the sense of definition 4.2.2. Consider the Dirichlet problem

$$\begin{aligned} Iu(x) = f(x) \quad &\text{in} \;\; \Omega, \\ u = g \quad &\text{in} \;\; \mathbb{R}^n \setminus \Omega. \end{aligned}$$

Assume that a viscosity subsolution and a viscosity supersolution exist that satisfy the boundary condition, that is, a subsolution u' such that $u'(x) \leq g(x)$ in Ω^c and a supersolution $\tilde{u}$ such that $\tilde{u} \geq g(x)$ in Ω^c and that the comparison principle holds for I. Then there exists a unique viscosity solution u of the above Dirichlet problem.

Idea of proof. Let

$$\mathcal{S} = \{w : w \text{ is a viscosity subsolution and } w \leq g \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

Define

$$u(x) = \sup_{w \in \mathcal{S}} w(x).$$

The function u is called the Perron envelope. One shows that the upper semicontinuous envelope of u is a viscosity subsolution and the lower semicontinuous envelope is a viscosity supersolution. Using the comparison principle, these envelopes coincide and u is a viscosity solution. Uniqueness follows from the comparison principle. For the full proof refer to [4]. □

The Hölder continuity is proved using a nonlocal ABP estimate, see [4], similar to the local case. But due to the technical nature of the statement, we will not present the nonlocal ABP estimate here. However, we note that the ABP estimate of [4] uses supremum norm of f, unlike the L^n norm as obtained by Theorem 2.3.7. Therefore, this estimate can not be used to derive narrow domain maximum principle for fully nonlinear nonlocal operators. It still remains unclear where an ABP estimate with L^p norm of f is possible or not. One should also note that, as $s \to 1$, this ABP estimate in [4] converges to the local ABP estimate. We will now state the Hölder continuity result for the nonlocal equations.

Theorem 4.3.3 (Hölder continuity)**.** Let u be a bounded viscosity solution of

$$Iu(x) = f(x) \quad \text{in } B_1,$$

where I is a fully nonlinear integro-differential operator which is uniformly elliptic in the sense of definition 4.2.2 Assume

$$f \in L^\infty(B_1).$$

Then there exist constants $\alpha \in (0,1)$ and $C > 0$, depending only on n, λ, Λ, and s, such that

$$\|u\|_{C^\alpha(B_{1/2})} \leq C \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1)} \right).$$

Proof. The proof is very similar to the local case, where you show a supersolution is in L^ϵ so some $\epsilon \geq 0$, and a lemma on the growth of a subsolution. For the detailed proof, please check [4]. □

Chapter 5

Fractional Convexity

5.1 Introduction

We call $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is convex open set, a convex function if for any two points $x, y \in \Omega$, we have

$$u(tx + (1-t)y) \leq tu(x) + (1-t)u(y)$$

for $t \in [0,1]$. Notice that, the function $v(t) = tu(x) + (1-t)u(y)$ solves the differential equation

$$\frac{d^2v}{dt^2} = 0$$

with the boundary conditions $v(0) = u(x)$ and $v(1) = u(y)$. The operator $\frac{d^2}{dt^2}$ is the one dimensional Laplacian.

With this as motivation, Del Pezzo, Quass, and Rossi introduce fractional convexity in their paper [8],. They define it as follows:

We call $u : \mathbb{R}^n \to \mathbb{R}$ an s-convex function in Ω, where $\Omega \subset \mathbb{R}^n$ is convex open set, if for any two points $x, y \in \Omega$, we have

$$u(tx + (1-t)y) \leq v(t)$$

for $t \in [0,1]$, where $v(t)$ is the unique solution to the Dirichlet problem

$$\begin{aligned} (-\Delta)^s v(t) &= 0 \qquad \text{for } t \in [0,1], \\ v(t) &= u(tx + (1-t)y) \qquad \text{for } t \in [0,1]^c. \end{aligned}$$

As the fractional Laplacian is a nonlocal operator, we have to define the boundary condition on the whole complement.

In the context of regularity theory for local operators, we have seen how important the notion of convex envelope and convexity in general are. Due to the theorem of Alexandrov, convex envelope of a function gives us the natural candidate for the second derivative of functions at the contact points with the function, which we exploit to obtain comparison principle, existence (the use of Jets to prove existence for degenerate elliptic equations heavily relies on convex analysis, see [6]) and as we have seen in Chapter 2, regularity results.

Note that convexity is a local notion. If we are to adapt the techniques developed for local operators to nonlocal operators, we need to have a notion of nonlocal convexity. Therefore, trying to define fractional convexity and exploring its properties seems to be a fruitful direction of research.

5.2 Fractional Convex Envelope

In Chapter 3, we have seen how we can frame the problem of obtaining the convex envelope as a PDE. In this section we will do the same, we will define fractional convex envelope and show that obtaining the fractional convex envelope is equivalent to solving an integro-differential equation. We are following the work presented in [8],.

Definition 5.2.1. *Let Ω be a convex domain, and $g : \Omega^c \to \mathbb{R}$ be a bounded function, we call ϕ^s_g the s-convex envelope of g where:*

$$\phi^s_g = \sup\{\phi \mid \phi \textit{ is s-convex in } \Omega \textit{ and}, \phi \leq g \textit{ on } \Omega^c\}.$$

In Chapter 3 we showed that the convex envelope satisfies the PDE

$$\begin{aligned} \lambda_1(u(x)) &= 0 \qquad && x \in \Omega, \\ u(x) &= g(x) \qquad && x \in \partial\Omega. \end{aligned}$$

Note that

$$\lambda_1 u(x) = \inf_{z \in \mathbb{S}^n} \frac{d^2 u(x+tz)}{dt^2}.$$

We use this to define the required nonlocal equation for fractional s-convex envelope. We define the Dirichlet problem for the first fractional eigenvalue:

$$\begin{cases} \Lambda_1^s(u(x)) = \inf\left(\int_{\mathbb{R}} \frac{u(x+tz)-u(x)}{|t|^{2s+1}}dt : z \in \mathbb{S}^{n-1}\right) = 0 & x \in \Omega, \\ u(x) = g(x) & x \in \Omega^c. \end{cases} \tag{5.1}$$

We claim that the s-convex envelope of g is the unique solution to (5.1). As we did in Chapter 3 for the classical convex case, we will first show a function u is s-convex in Ω if and only if $\Lambda_1^s u \geq 0$ in Ω. To do so, we first define a new notion of viscosity solution for this operator. We quote this definition as presented in [8],.

Definition 5.2.2. *"We call a function $u : \mathbb{R}^n \to \mathbb{R}$ to be a* viscosity subsolution *of*

$$\Lambda_1^s u(x) = 0 \quad \text{in } \Omega \tag{5.2}$$

if for all $x \in \Omega$, $z \in \mathbb{S}^{n-1}$, and any open interval $I \ni 0$ such that $x + tz \in \Omega$ for all $t \in I$, and any test function $\phi \in C^2(\mathbb{R})$ such that $\phi(0) = \tilde{w}(0)$ and $\phi(t) \geq \tilde{w}(t)$ in I we have

$$\Delta_1^s \hat{\phi}(0) \geq 0,$$

where

$$\hat{\phi}(t) = \begin{cases} \phi(t) & \text{for } t \in I, \\ \tilde{w}(t) & \text{for } t \in \mathbb{R} \setminus I, \end{cases}$$

and $\tilde{w}(t)$ denotes the upper semicontinuous envelope of $w(t) = u(x + tz)$.

We call the function $u : \mathbb{R}^n \to \mathbb{R}$ is a viscosity supersolution *of (5.2) if for any $x \in \Omega$, any $\phi \in C^2(\mathbb{R}^n)$ such that $\phi(x) = \tilde{w}(x)$ and $\phi(y) \leq \tilde{w}(y)$ in $\mathbb{R}^N$ we have*

$$\Delta_1^s \hat{\phi}(x) \leq 0,$$

and $\tilde{w}(y)$ denotes the lower semicontinuous envelope of u in $\mathbb{R}^n$.

Finally, we say that u is a viscosity solution *of (5.2) when it is both a viscosity subsolution and a viscosity supersolution of (5.2). "*

In this new definition, we have used 1-d test functions for subsolutions but $\mathbb{R}^n$ test functions for supersolutions. One may also consider the usual definition for subsolution. In the paper [8],, the authors show that these two notions of test functions are equivalent. Using this definition, we will show that the s-convex envelope of g is the unique solution to

(5.1).

Definition 5.2.3. *We define* $\mathcal{L}_s$ *as*

$$\mathcal{L}_s(\mathbb{R}^N) := \left\{ v \in L^\infty_{\text{loc}}(\mathbb{R}^N) : \ t \mapsto v(x+tz) \in \mathcal{L}_s(\mathbb{R}) \ \ \forall x \in \Omega, \ \forall z \in \mathbb{S}^{n-1} \right\}.$$

Theorem 5.2.4. Let $u \in \mathcal{L}_s(\mathbb{R}^n)$ be a viscosity subsolution to

$$\Lambda_1^s u(x) = 0 \quad \text{in } \Omega,$$

as defined in Definition 5.2.2. Then, u is s-convex in Ω.

Proof. Let v be a subsolution of (5.1) and $x, y \in \Omega$. We write

$$tx + (1-t)y = t|x-y|z + y,$$

where

$$z = \frac{x-y}{|x-y|} \in \mathbb{S}^{n-1}.$$

By definition, we have that

$$w(t) = u(y + t|x-y|z)$$

is a viscosity subsolution of

$$\Delta_1^s w(t) = 0 \quad \text{in } (0,1).$$

By applying the comparison principle we have that

$$u(tx+(1-t)y) = w(t) \le v(tx+(1-t)y) \qquad \forall t \in (0,1).$$

Hence, u is s-convex and we are done. □

We now prove the reverse direction.

Theorem 5.2.5. If $u : \mathbb{R}^n \to \mathbb{R}$ is s–convex in Ω, then u is a viscosity subsolution to

$$\Lambda_1^s u(x) = 0 \qquad \text{in } \Omega,$$

as defined by Definition 5.2.2.

Proof. We proceed by the method of contradiction. Let there be $x \in \Omega$, $z \in S^{n-1}$, an open interval $I \ni 0$ such that $x + tz \in \Omega$ for all $t \in I$. Let $\tilde{w}(t) = u(x+tz)$, and let $\phi \in C^2(\mathbb{R})$ be a test function such that $\phi(0) = \tilde{w}(0)$ and $\phi(t) \geq \tilde{w}(t)$ in I such that

$$\Delta_1^s \hat{\phi}(0) < 0.$$

As $u(x+tz) \in \mathcal{L}_s(\mathbb{R})$, we have that $\Delta_1^s \hat{\phi}(t)$ is a continuous function (See [18]). Therefore, there is $\delta > 0$ such that

$$\Delta_1^s \hat{\phi}(t) < 0 \quad \text{in } (-\delta, \delta) \subset I.$$

Consider

$$x_0 = x + \delta z, \qquad y_0 = x - \delta z.$$

As v is the viscosity solution of

$$\begin{cases} \Delta_1^s v(tx_0 + (1-t)y_0) = 0 & t \in (0,1), \\ v(tx_0 + (1-t)y_0) = u(tx_0 + (1-t)y_0) & t \in \mathbb{R} \setminus (0,1), \end{cases}$$

$v(tx_0 + (1-t)y_0) \in C^\infty(0,1)$ (See [18]) and

$$u(tx_0 + (1-t)y_0) \leq v(tx_0 + (1-t)y_0) \quad \text{in } (0,1),$$

by definition of s-convexity. Therefore,

$$\tilde{w}(0) = \inf_{r>0} \{\sup\{u(tx_0 + (1-t)y_0) : t \in (-r,r)\}\} \leq v(x). \tag{5.3}$$

Consider the function $z(t)$

$$z(t) = v\left(\frac{t+\delta}{2\delta} x_0 + \left(1 - \frac{t+\delta}{2\delta}\right) y_0\right).$$

Then $z(t)$ is a viscosity solution to

$$\begin{cases} \Delta_1^s z(t) = 0 & t \in (-\delta, \delta), \\ z(t) = u(x+tz) & t \in \mathbb{R} \setminus (0,1). \end{cases}$$

From the strong maximum principle, we have that

$$z(t) < \hat{\phi}(t) \qquad \forall t \in (-\delta, \delta).$$

Therefore, using (5.3), we have that

$$\tilde{w}(0) \leq v(x) = z(0) < \hat{\phi}(0) = \tilde{w}(0),$$

a contradiction. This finishes the proof. □

We state the final result of this chapter

Theorem 5.2.6. Let Ω be a convex set. Let $g : \Omega^c \to \mathbb{R}$ be a bounded function and ϕ_g^s be its s-convex envelope in Ω. Then ϕ_g^s solves (5.1) in viscosity sense.

Proof. We will prove the comparison principle for Λ_1^s in the next chapter. By the comparison principle and Perron's method we see that the solution to (5.2) is

$$u(x) := \sup\{v(x) \mid v \in USC(\overline{\Omega});\ \Lambda_1^s(v) \geq 0 \text{ in } \Omega, \quad u \leq g \text{ on } \Omega^c\}.$$

But by the previous lemma, $\Lambda_1^s(v) \geq 0$ if and only if v is s-convex. Therefore,

$$u(x) = \sup\{v \mid v \text{ is s-convex in } \Omega,\ u \leq g \text{ on } \partial\Omega\}$$

That is $u(x) = \phi_g(x)$, and we are done. □

In this chapter, we have seen the relationship between a nonlocal operator and s-convex envelope, as in the local case. In the next chapter we will explore the properties of this operator.

Chapter 6

Results

6.1 Comparison principle

In this section, we will prove comparison principle for the operator Λ_1^s. We note the comparison principle for this operator has been proved in [8],, but we prove it using different methods.

6.1.1 Idea

Let's consider one dimensional Laplacian $\Delta u(x) = \frac{d^2u(x)}{dx^2}$. If a function is a supersolution to Δ, that is, $\Delta u \leq 0$, then we know that it is curving downward and vice versa for the case of sub-solution. This property of having supersolutions curving 'downwards' and subsolution curving 'upwards', holds for elliptic operators. Suppose we have a subsolution to our elliptic operator I such that it's graph looks like a well (it curves upwards and has minimum in the definition of the domain). A supersolution which satisfies the same boundary conditions as this subsolution should not touch this function, because it has the tendency to curve 'upwards', see the figures (6.1),(6.2). This fact is exploited to prove the comparison principle for elliptic operators in the next section.

6.1.2 Results

We will state certain preliminary results about approximating viscosity solutions by inf-sup convolutions. This method has become popular due to the works of Ishii and Jensen, see [6]. For the rest of this subsection, we will follow the method introduced by Caffarelli and Silvestre in [4]. We first quote the definitions introduced in [4]

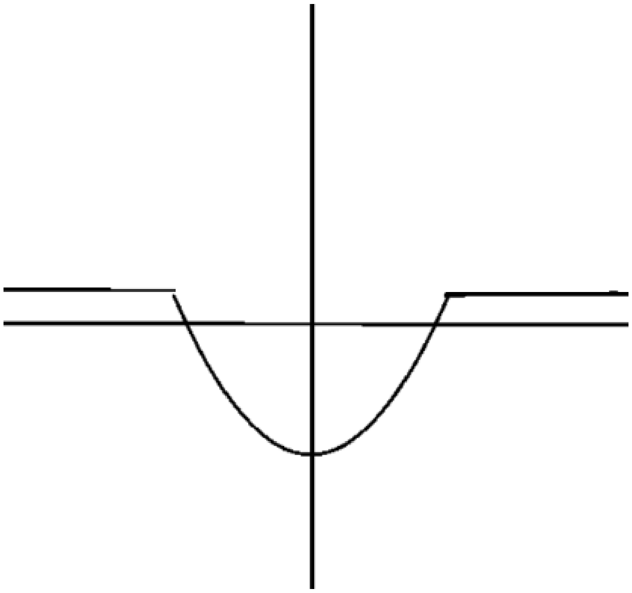

Figure 6.1: Subsolution which is well like.

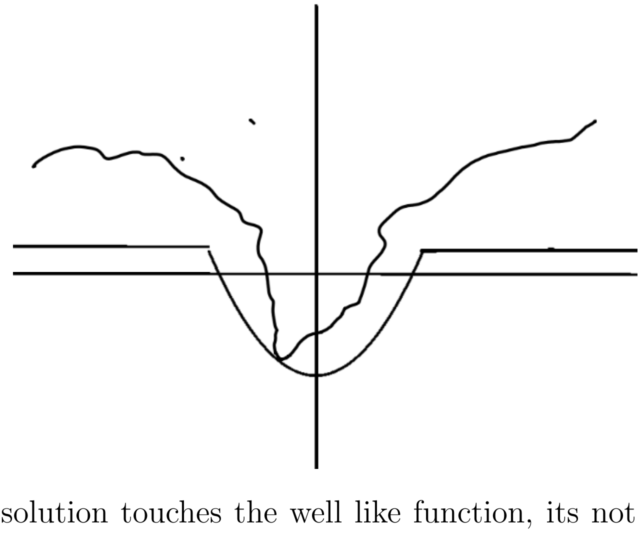

Figure 6.2: If a supersolution touches the well like function, its not curving downwards, a contradiction.

Definition 6.1.1 (Γ-convergence)**.** *"A sequence of lower-semicontinuous functions u_k Γ-converges to u in a set Ω if the following two conditions hold:*

- *For every sequence $x_k \to x$ in Ω,*

$$\liminf_{k\to\infty} u_k(x_k) \geq u(x).$$

- *For every $x \in \Omega$, there exists a sequence $x_k \to x$ in Ω such that*

$$\limsup_{k\to\infty} u_k(x_k) = u(x).''$$

Definition 6.1.2. *" Let u be an upper-semicontinuous function. Then,* ***sup-convolution*** *approximation u^ϵ is given by*

$$u^\epsilon(x) = \sup_y \left\{ u(x+y) - \frac{|y|^2}{\epsilon} \right\}.$$

Similarly, if u is lower-semicontinuous, the ***inf-convolution*** *u^ϵ is given by*

$$u_\epsilon(x) = \inf_y \left\{ u(x+y) + \frac{|y|^2}{\epsilon} \right\}.''$$

It is well known that u^ϵ is $C^{1,1}$ from below, and u_ϵ is $C^{1,1}$ from above. See [3],. The proof of the next theorem will be skipped, proof can be found in [3],.

Theorem 6.1.3. "Suppose u is bounded and lower-semicontinuous in $\mathbb{R}^n$. Then u_ϵ Γ-converges to u. Similarly, if u is bounded and upper-semicontinuous in $\mathbb{R}^n$, then $-u^\epsilon$ Γ-converges to $-u$."

Theorem 6.1.4. Let v be a supersolution to $\Lambda_1^s \tilde{u}(x) = 0$ in Ω, and u be a subsolution to $\Lambda_1^s \tilde{u}(x) = 0$. Then $\Lambda_1^s(v-u)(x) \leq 0$ in the viscosity sense.

Proof. We will use inf-sup convolutions. For the fractional Laplacian Δ^s (see [4]) it is known that if $\Delta^s u \geq f$ in viscosity sense, then $\Delta^s u^\epsilon \geq f - d_\epsilon$, and if $\Delta^s v \leq f$ in viscosity sense, then $\Delta^s v_\epsilon \leq f + d_\epsilon$. If $\Lambda_1^s u \geq 0$, then in all directions, the one dimensional fractional Laplacian is greater than zero, therefore $\Lambda_1^s u^\epsilon \geq -d_\epsilon$. If $\Lambda_1^s v \leq 0$, there exists a direction z for each point x in the domain such that $\Delta^s v(x+tz) \leq \epsilon$. Therefore, $\Delta^s v_\epsilon \leq (x+tz) \leq \epsilon + d_\epsilon$, which implies $\Lambda_1^s v_\epsilon \leq \epsilon + d_\epsilon$. Due to the semiconvexity of u^ε and $-v_\varepsilon$, there is a parabola touching them at x from below (definition of $C^{1,1}$ from below). If a C^2 function touches

$v_\varepsilon - u^\varepsilon$ from below at x, then both u^ε and $-v_\varepsilon$ are $C^{1,1}$ at x and we can evaluate $\Lambda_1^s(v_\epsilon - u^\epsilon)$ classically. Therefore, $\Lambda_1^s(v_\epsilon - u^\epsilon) \leq \Lambda_1^s v_\epsilon - \Lambda_1^s u^\epsilon \leq 2d_\epsilon + \epsilon$ where the first inequality follows from the infimum structure of Λ_1^s. Due to stability of viscosity solutions under Γ convergence, we have

$$\Lambda_1^s(v-u) \leq 0$$

. $\square$

Now we will construct a particular subsolution, which will help us prove the comparison principle. For a function u, we define $I_s^z(u(x))$ as

$$I_s^z(u(x)) = \int_{\mathbb{R}} \frac{u(x+tz)-u(x)}{|t|^{2s+1}} dt$$

Lemma 6.1.5. Let $u : \mathbb{R} \to \mathbb{R}$ be defined as

$$\chi(t) = \begin{cases} -(1-t^2)^s & \text{for } t \in (-1,1), \\ 0 & \text{for } t \in (-1,1)^c, \end{cases}$$

Then, $-(-\Delta)^s \chi(t) = \Gamma(2s+1)$ for $t \in (-1,1)$.

Proof. See [9].

Corollary 6.1.5.1. Let $h : \mathbb{R}^n \to \mathbb{R}$ be defined as $h = \sum_i^n \chi_i$, where

$$\chi_i(x) = \begin{cases} -(1-x_i^2)^s & \text{when } x_i \in (-1,1), \\ 0 & \text{for } x_i \in (-1,1)^c, \end{cases}$$

and $x = (x_1, x_2 ... x_n)$. Then $\Lambda_1^s h(x) > 0$ for all $x \in B(0,1)$.

Proof. If we prove $I_s^z(h(x)) > 0$ for all $x \in B(0,1)$ and $z \in \mathbb{S}^{n-1}$, we are done. As h is C^2 inside $B(0,1)$ we can evaluate the fractional Laplacian of h classically.

$$I_s^z(h(x)) = \Delta^s \sum_{i=0}^{n} \chi(x_i + z_i t)$$

The fractional Laplacian is translation invariant and $\Delta^s v(cx) = |c|^{2s}(\Delta^s v)(cx)$. It is also true that $x_i, z_i \in (-1,1)$ and at least for one i , $|z_i| > 0$, therefore $\Delta^s \chi(x_i + tz_i) > 0$ for that i and otherwise $\Delta^s \chi(x_i + tz_i) \geq 0$. Therefore,

$$I_s^z(h(x)) > 0$$

for all x and z. Hence,

$$\Lambda_1^s h(x) > 0$$

□

Theorem 6.1.6. Let v, u be such that $\Lambda_1^s v \leq 0$ and $\Lambda_1^s u \geq 0$ in $B(0,1)$ and $v \geq u$ on $B(0,1)^c$. Then $v \geq u$ in $\mathbb{R}^n$.

Proof. By (6.1.4), $\Lambda_1^s(v-u) \leq 0$. We have that $\Lambda_1^s h > 0$ and h is C^2 in $B(0,1)$ by Corollary 6.1.5.1. Consider,

$$\phi_M(x) = M + \epsilon h(x).$$

Let M_0 be the largest M such that ϕ_M touches $v-u$ from below. We will prove that $M_0 > 0$. Suppose not, then ϕ_{M_0} touches $v-u$ in $B(0,1)$ from below at x_0. But by definition of viscosity supersolution, $\Lambda_1^s \phi_{M_0} \leq 0$ which is a contradiction as $\Lambda_1^s h(x_0) > 0$. Therefore, $M_0 > 0$. By comparison principle,

$$v - u \geq \phi_{M_0} = M_0 + \epsilon h.$$

By letting $\epsilon \to 0$, we get our result

$$v - u \geq 0 \quad \text{in } \mathbb{R}^n.$$

□

6.2 A regularity result

6.2.1 Idea behind blow-up arguments

Hölder regularity is a local property, that is, it only depends on the behaviour of the function in a neighbourhood. So, to prove Hölder regularity of functions, we just need to know what kind of properties they locally satisfy, for example, what kind of PDE they locally satisfy. To understand this, we choose a point and 'zoom-in' infinitely (blow-up is used here because this process 'blows up' a neighbourhood of the point to the whole space). If we have theorems which classify global solutions to the PDE which are obtained by zooming in (the operator approximately satisfies locally, and after the blow-up process, on the whole space), we can obtain regularity results.

Figure 6.3: Function which is Lipschitz regular

The theorems which classify global solutions are called Liouville type theorems. Roughly, they say that if a function satisfies a PDE in the whole space, they cannot grow very slowly. This growth is of course, determined by the type of PDE we are dealing with. How does this help us prove regularity? We assume that our function is not α Hölder regular. When we apply the blow-up procedure, this translates to the function growing slower than $|x|^\alpha$. By applying our Liouville type theorem, we conclude that this is a contradiction.

This has been visualized in (6.3) and (6.4).

6.2.2 Results

We state a Liouville-type result by Biswas (from his personal notes).

Theorem 6.2.1. Let $s > \frac{1}{2}$. Denote $\kappa = 2s - 1$. If $u \in C(\mathbb{R}^n)$ is a non-constant fractionally s-convex function, that is, $\Lambda_1^s u \geq 0$ in $\mathbb{R}^n$, then we have

$$\lim_{R \to \infty} \sup_{B_R^c} \frac{|u(x)|}{1 + |x|^\kappa} > 0.$$

This result is a corollary of the following result (again proved in a personal note of Biswas)

Theorem 6.2.2. Let $s \geq \frac{1}{2}$. If u is fractionally convex in B_1 and $u \in L^\infty(\mathbb{R}^n)$, then $u \in C^\kappa(B_{\frac{1}{4}})$.

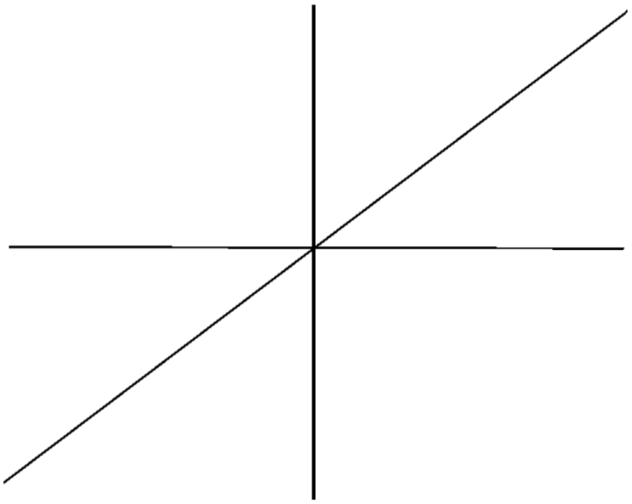

Figure 6.4: Function growing linearly when zoomed in

The proof of this theorem is done by explicitly showing that $|x|^\kappa$ is a supersolution for Λ_1^s everywhere except the origin, and using the definition of viscosity solutions and the comparison principle .

We follow a different method of Ros-Oton and Serra, introduced in [17]. The following is a Liouville-type result for the difference quotients.

Corollary 6.2.2.1. Let $\Lambda_1^s(u(.+h)-u) \geq 0$. If $[u]_{C^\alpha(B_R)} \leq CR^\beta$, for all $R \geq 1$ (where $\beta < \kappa$), then u is a polynomial of degree less than $\lfloor 2s-1+\alpha \rfloor$.

Proof. By previous theorem, $u(x+h)-u(x)$ is a constant for all h. Now, we claim that if $u(x+h)-u(x)$ is a polynomial for all h, then u is also a polynomial. Indeed, let $\Delta_h u(x) = u(x+h)-u(x) = P(x)$, where P is a polynomial. $\Delta_h P(x) = P(x+h)-P(x)$ is a polynomial of degree strictly lesser than $P(x)$, as

$$\begin{aligned}
P(x) &= a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \\
P(x+h) &= (a_d(x+h)^d) + a_{d-1}(x+h)^{d-1} + \cdots + a_0 \\
&= (a_d x^d + a_d d h x^{d-1} + a_d \binom{d}{2} h^2 x^{d-2} + \cdots + a_d h^d) + a_{d-1}(x+h)^{d-1} + \cdots + a_0
\end{aligned}$$

$$P(x+h)-P(x) = a_d d h x^{d-1} + \text{lower order terms}$$

Therefore, if P is a polynomial of degree d, $\Delta_h^{d+1}P = 0$ for all h. Therefore, $\Delta_h^{d+2}u = 0$ for all h. It is a well known fact that a function whose $(d+2)$-st forward difference is zero is a polynomial of degree at most $d+1$ (see [13]). Therefore, u is a polynomial. Because of the growth rate bound, u is a polynomial of degree less than $\lfloor 2s-1+\alpha \rfloor$. □

Now, we prove a regularity result which is an application of our previous Liouville theorem. We define

$$\Lambda_s^n(u(x)) = \sup\left(\int_{\mathbb{R}} \frac{u(x+tz)-u(x)}{|t|^{2s+1}}dt : z \in \mathbb{S}^{n-1}\right)$$

Theorem 6.2.3. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a C_c^∞ function such that $w(t) = u(x+tz) \in L_s(\mathbb{R})$. If u satisfies $\Lambda_s^1 u(x) > -c$ and $\Lambda_s^n u(x) < c$, that is, $-c < I_s^z u(x) < c$ for all $x \in B_1, z \in S^{n-1}$, then for any $0 < \alpha < 2s-1$ such that $2s-1+\alpha$ is not an integer,

$$[u]_{C^{2s-1+\alpha}(B_{\frac{1}{2}})} < C\left(c + \|u\|_{C^{2s-1}(B_{\frac{1}{2}})}\right), \tag{6.1}$$

Proof. We proceed by method of contradiction. Suppose (6.1) is not true. Then for some c, there exists w_k such that

$$\begin{aligned} -c < I_s^z w_k(x) < c,& \\ ||w_k||_{C^{2s-1}(B_{\frac{1}{2}})} \leq 1,& \\ ||w_k||_{C^{2s-1+\alpha}(B_{\frac{1}{2}})} \geq k.& \end{aligned} \tag{6.2}$$

Let $\mu = \lfloor 2s-1+\alpha \rfloor$. Since $\mu < \alpha + 2s - 1$, we have

$$\sup_k \sup_{\xi \in B_{\frac{1}{2}}} \sup_{r>0} r^{-\alpha}[w_k]_{C^{2s-1}(B_r(\xi))} = +\infty.$$

Next, we define

$$\theta(r) := \sup_k \sup_{\xi \in B_{\frac{1}{2}}} \sup_{r'>r} r^{-\alpha}[w_k]_{C^{2s-1}(B'_r(\xi))}.$$

The function θ is monotone nonincreasing, and we have $\theta(r) < +\infty$ for $r > 0$ as we are assuming that $||w_k||_{C^{2s-1}(B_{\frac{1}{2}})} \leq 1$. In addition, we have $\theta(r) \to +\infty$ as $r \to 0$.

As θ is nonincreasing for every positive integer m, there exists $r'_m \geq \frac{1}{m}$, k_m, and $\xi_m \in B_{\frac{1}{2}}$, for which

$$(r'_m)^{-\alpha}[w_{k_m}]_{C^{2s-1}(B_{r'_m}(\xi_m))} \geq \frac{1}{2}\theta(1/m) \geq \frac{1}{2}\theta(r'_m) \tag{6.3}$$

Note that $r'_m \to 0$.

Let $p_{k,\xi,r}(.-\xi)$ be the polynomial of degree at most μ in the variables $(x-\xi)$ which best fits u_k in $B_r(\xi)$ by least squares. That is,

$$p_{k,\xi,r} := \min_{p\in\mathbb{P}_\mu} \int_{B_r(\xi)} \left(w_k(x) - p(x-\xi)\right)^2 dx,$$

where $\mathbb{P}_\mu$ denotes the vector space of polynomials with degree less that μ. For simplicity, we will denote

$$p_m = p_{k_m,\xi_m,r'_m}$$

We consider the blow up sequence

$$v_m = \frac{w_{k_m}(\xi_m + r'_m x) - p_m(r'_m x)}{(r'_m)^{2s-1+\alpha}\theta(r'_m)}.$$

Note that, for all $m \geq 1$;

$$\int_{B_1(0)} v_m(x)q(x)dx = 0 \qquad \text{for all } q \in \mathbb{P}_\mu \tag{6.4}$$

This says that v_m is orthogonal to $\mathbb{P}_\mu$. (6.3) implies nondegeneracy ;

$$[v_m]_{C^{2s-1}} \geq \frac{1}{2} \tag{6.5}$$

Now we show growth control, which is required in the end to apply Liouville theorem.

$$\begin{aligned}[v_m]_{C^{2s-1}(B_R)} &= \frac{1}{\theta(r'_m)(r'_m)^\alpha}[w_{k_m}]_{C^{2s-1}(B_{r'_m}(z_m))} \\ &= \frac{R^\alpha}{\theta(r'_m)(Rr'_m)^\alpha}[w_{k_m}]_{C^{2s-1}(B_{r'_m}(z_m))}\end{aligned}$$

By monotonicity and definition of θ;

$$[v_m]_{C^{2s-1}(B_R)} \leq CR^\alpha \tag{6.6}$$

for all $R \geq 1$.

By (6.4) and (6.6) we have that

$$||v_m||_{L^\infty(B_1)} \leq C. \tag{6.7}$$

Now, using (6.6) and (6.7) we obtain,

$$[v_m]_{C^\gamma(B_r)} \leq CR^{2s-1+\alpha-\gamma} \qquad \text{for all } \gamma \in [0, 2s-1]. \tag{6.8}$$

The result for all $\gamma \in [0, 2s-1]$ follows by interpolation. On compact subsets of $\mathbb{R}^n$, by Arzelà-Ascoli theorem we have $v_m \to v$ (taking a subsequence if necessary). Moreover, passing to the limit (6.8) for all $\alpha < \gamma \leq 1$ such that $\gamma \leq 2s-1$, we find that

$$[v]_{C^\gamma(B_R)} \leq \bar{C}R^\beta \tag{6.9}$$

Thus, v satisfies the growth condition of (6.2.2.1) as β can be chosen such that $\beta < 2s-1$. Now, note that, since $\mu \leq 1$,

$$\delta^2 p(x+h, y) - \delta^2 p(x, y) = 0$$

for all $p \in \mathbb{P}_\mu$ and for all x, y, h in $\mathbb{R}^n$.
Namely, using the definition of v_m and the assumption $-c < I_s^z w_k(x) < c$, we obtain

$$\frac{1}{(r'_m)^{2s}} |\Lambda_1^s \left((r'_m)^{2s-1+\alpha} \theta(r'_m) \{v_m(.+h) - v_m\} \right)(x)| \leq c$$

whenever $|x| \leq \frac{1}{2r'_m}$, and thus

$$|\Lambda_1^s(v_m(.+h) - v_m)(x)| \leq c\frac{(r'_m)^{1-\alpha}}{\theta(r'_m)}$$

Passing to the limit (we can pass the limit on balls of radius R and then conclude for $\mathbb{R}^n$)

$$\Lambda_1^s(v(.+h) - v) = 0$$

in all of $\mathbb{R}^n$. As

$$[v]_\alpha \leq \tilde{C}R^{2s-1-\epsilon}$$

By (6.9), v is a polynomial of degree μ. On the other hand, passing (6.4) to the limit, we obtain that v is orthogonal to all polynomials of degree μ, so v has to be 0. This is a contradiction to (6.5). Therefore, we are done. □

Approximating any function $u \in C^{2s-1}(\mathbb{R}^n)$ by C_c^∞ functions, previous theorem will hold for any $u \in C^{2s-1}(\mathbb{R}^n)$ satisfying $-c \leq I_s^z u \leq c$ in B_1.

Now, we state another result by Biswas.

Theorem 6.2.4. Let $s > \frac{1}{2}$. Let u be fractionally convex in $\mathbb{R}^n$ and $\sup_{B_R} |u| < CR^{2s-1}$. Then $u \in C^{2s-1}(\mathbb{R}^n)$.

Suppose that we have a fractionally convex function u in $\mathbb{R}^n$, such that $\Lambda_s^n u \leq C$ in B_1. Then by (6.2.4), this function satisfies the hypothesis of (6.2.3). This means that $u \in C^{2s-1+\alpha}(B_{\frac{1}{2}})$ for all $0 < \alpha < 2s-1$, which is an analogous result to the local case,i.e, to the fact that a convex function is twice differentiable almost everywhere.

Proving regularity for fractional convexity operator is hard because of the combined effects of degeneracy and non-local influence. Non-degeneracy does not allow us to conclude regularity in all directions, and due to non-local nature of the operator, one cannot construct approximate solutions which depend only on the neighbourhood of the point.

Bibliography

[1] A. D. Aleksandrov, Almost everywhere existence of the second differential of a convex function and some properties of convex functions, Leningrad. Univ. Ann. (Math. Ser.) 37 (1939), 3–35. (Russian)

[2] G. Barles and P. E. Souganidis, Convergence of approximation schemes for fully nonlinear second order equations, Asymptotic Analysis 4, (1991) (3), 271–283

[3] L. A. Caffarelli and X. Cabré, Fully nonlinear elliptic equations, American Mathematical Society Colloquium Publications, vol. 43, Providence, R.I.: American Mathematical Society, (1995)

[4] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math., 62 (2009), pp. 597-638

[5] R. Cont and P. Tankov, Financial modelling with jump processes. Chapman and Hall/CRC, 2004

[6] M. G. Crandall, H. Ishii, and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bulletin of the American Mathematical Society, New Series, 1992

[7] M.G. Crandall and P. L. Lions, Viscosity solutions of Hamilton-Jacobi equations, Transactions of the American Mathematical Society 277, (1983) (1): 1–42

[8] L. M. Del Pezzo, A. Quaas, and J. D. Rossi, Fractional convexity, Math. Ann. 383 (2022), no. 3–4, 1687–1719

[9] B. Dyda. Fractional calculus for power functions and eigenvalues of the fractional Laplacian. Fract. Calc. Appl.Anal. 15 (2012), no. 4, 536–555

[10] L. C. Evans and R. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Math., CRC Press, Ann Arbor, 1992.

[11] X. Fernández-Real and X. Ros-Oton Regularity Theory for Elliptic PDE Zurich Lectures in Advanced Mathematics, vol. 28, 978-3-98547-028-0, EMS Press, Berlin (2022)

[12] R. Jensen, The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations, Archive for Rational Mechanics and Analysis, 101 (1988) (1): 1–27

[13] D. Leviatan, Finite Difference Methods, Springer, 1986

[14] A. M. Oberman, The convex envelope is the solution of a nonlinear obstacle problem. Proc. Amer. Math. Soc., 135(6) (2007), 1689–1694

[15] A. M. Oberman and L. Silvestre, The Dirichlet problem for the convex envelope. Trans. Amer. Math. Soc. 363 (2011), no. 11, 5871–5886

[16] C. Pucci, Maximum and minimum first eigenvalues for a class of elliptic operators, Proc. Amer. Math. Soc. 17 (1966), 788–795

[17] X. Ros-Oton and J. Serra, Regularity theory for general stable operators, J. Differential Equations 260 (2016), 8675-8715.

[18] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator. Comm. Pure Appl. Math., 60(1):67–112, 2007