Multiplier theorem on the Heisenberg group

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by

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CERTIFICATE

This is to certify that this dissertation entitled **Multiplier theorem on the Heisenberg Group** towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by **Moksh Juneja** at Indian Institute of Science Education and Research, Pune under the supervision of **Dr. Divyang Bhimani**, Assistant Professor, **Department of Mathematics**, during the academic year 2025-2026.

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This thesis is dedicated to my grandmothers

for whom knowledge was always the goal.

DECLARATION

I hereby declare that the matter embodied in the report entitled **"Multiplier theorem on the Heisenberg group"** is the result of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research (IISER) Pune, under the supervision of **Dr. Divyang Bhimani**, and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgment of collaborative research and discussions.

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Abstract

The question of boundedness of operators is of importance in analysis. We look into the operators which act on the *frequency* space of functions, that is those operators which act on the Fourier transform of an L^p function, $1 < p < \infty$. Our operators are given by multiplication with a certain function called the *multiplier*. The boundedness conditions of these operators, on the Euclidean space have been solved. This thesis asks the boundedness question for the case of a non-abelian group called the *Heisenberg group* (H^n). Our main theorem gives a smoothness condition of the multiplier, constrained to the dimension of H^n. This is smoothness condition is similar to *Mikhil's condition* for boundedness of multipliers on $\mathbb{R}^n$.

We use the tools developed by **Stein**, improving the theory given by **Littlewood** and **Paley**. These tools are the $g-$functions, which we can bound and so get the bounds for our multiplier. To use this on H^n we have to move all the required ideas and machinery from $\mathbb{R}^n$ to H^n. To move all the required machinery we need some understanding of certain family of functions like the *Hermite functions* and the *Laguerre functions.*

In essence, this thesis uses the tools and ideas developed for the Euclidean space, to tackle the problem of boundedness of multipliers and then builds analogues of it for the Heisenberg group. These tools are then put to use to show that under certain conditions, the boundedness of a multiplier can be established. We can further expand on this study by looking into the unsolved problems in $\mathbb{R}^n$ and ask the same in H^n.

Contents

Notations

Notation	Description
τ^z	Translation by z
δ^h	Scaling by h
$\hat{f}$	Fourier transform of f
$f^\vee$	Inverse Fourier transform
T^*	Adjoint of the operator T
ρ_h	Schrodinger representation of H^n corresponding to $h \neq 0$
ρ	Schrodinger representation corresponding to $h = 1$
$\circledast_\lambda$	Lambda twisted convolution
$\mathcal{S}(\mathbb{R}^n)$	Schwartz space on $\mathbb{R}^n$
$\mathcal{S}'(\mathbb{R}^n)$	Space of tempered distributions on $\mathbb{R}^n$
C^1	Space of continuous differentiable functions
$C^\infty(X)$	Space of smooth functions on the space X
C_0^∞	Space of smooth functions with compact support
$\mathbb{R}^*$	$\mathbb{R} \setminus \{0\}$
$\mathcal{M}_{p,q}(\mathbb{R}^n)$	Space of bounded linear operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$
$\mathcal{M}_p(\mathbb{R}^n)$	Space of L^p multipliers on $\mathbb{R}^n$
h^n	Heisenberg Lie algebra
H^n	Heisenberg Lie group
H^n_{pol}	Polarized Heisenberg Lie group
H^n_{red}	Reduced Heisenberg Lie group
$\mathcal{S}(\mathsf{H}^n)$	Space of Schwartz functions on H^n
S_2	Space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$
$Sp(n, \mathbb{R})$	Set of linear transformations of $2n$ dimensional vector space over the field $\mathbb{R}$ which preserve a non-degenerate, skew-symmetric, bilinear form
$L^2(\mathbb{R}^*, S_2, d\mu)$	Space of square integrable functions on $\mathbb{R}^*$, taking values in S_2 with measure $d\mu$
Bf	Bargmann transform of function f
$\mathcal{F}_n$	Fock space
h_α	Hermite function of order α
h_α^λ	$\lambda-$scaled Hermite function of order α
H_α	Hermite polynomials of degree α
$\Phi_{\alpha,\beta}$	Special Hermite functions
L_m^α	Laguerre polynomial of type α
Q_k	Projection operator on $L^2(\mathbb{R}^n)$, onto the kth eigenspace spanned by $\{h_\alpha : \lvert\alpha\rvert = k\}$
$\Theta(G)$	Linear space of entry functions on G spanned by the entries of finite-dimensional irreducible representations of G

1 Introduction

Joseph Fourier studied the problem of the solutions of the heat equation in 1807, which marked the beginning of Fourier transform, and in general harmonic analysis. Since then a lot of developments have been made in the domain and still, a lot of questions remain unanswered.

We assume some familiarity with topics in analysis, like Measure theory, theory of distribution and basic Fourier analysis. Our goal in this section is to have a ground level understanding of what multipliers are, and develop sufficient tools to study their behavior.

1.1 Preliminaries

To begin our study we first recall certain definitions and theorems from the Euclidean harmonic analysis and harmonic analysis on locally compact abelian groups. These results can be found in any standard textbook on harmonic analysis. The results here have be taken from the books *Classical Fourier Analysis* by L. Grafakos and *A First Course in Harmonic Analysis* by Anton Deitmar.

Definition 1.1 (Distribution function)**.** *For a measurable function f on X the distribution function of f is defined as*

$$d_f(k) = \mu(\{x \in X : |f(x)| > k\})$$

We have the following alternative for the L^p norm of a function, using the distribution function.

Proposition 1.1. *For $f \in L^p(\mathbb{R}^n, \mu)$ with $0 < p < \infty$, we have*

$$\| f \|_{L^p}^p \leq p \int_0^\infty k^{p-1} d_f(k)\, dk$$

Proof. We can write the distribution function as the integral of the characteristic function of the corresponding set and by interchange of limits, made possible by Fubini's theorem we get

$$\begin{aligned} p \int_0^\infty k^{p-1} d_f(k\, dk) &= \int_{\mathbb{R}^n} \int_0^{|f(x)|} pk^{p-1}\, dk\, d\mu(x) \\ &= \int_{\mathbb{R}^n} |f(x)|^p \, d\mu(x) \\ &= \| \ f \ \|_{L^p}^p \end{aligned}$$

□

We now define the weak L^p space as

Definition 1.2. *For any p in $0 < p < \infty$, we define the weak $L^p(\mathbb{R}^n)$ space as the set of all μ-measurable functions such that*

$$\begin{aligned} \| \ f \ \|_{L^{p,\infty}} &= \inf \left\{ C > 0 : d_f(k) \leq \frac{C^p}{k^p}, \ \forall k > 0 \right\} \\ &= \sup\{\alpha \, d_f(\alpha)^{1/p}\} \end{aligned}$$

is finite. By definition, weak $L^\infty(\mathbb{R}^n)$ is $L^\infty(\mathbb{R}^n)$.

We have the following containment for the *weak* $L^p(\mathbb{R}^n)$ space.

Proposition 1.2. *For any $f \in L^p(\mathbb{R}^n)$ and $0 < p \leq \infty$ we have*

$$\| \ f \ \|_{L^{p,\infty}} \leq \| \ f \ \|_{L^p}$$

Which means that we have $L^p(\mathbb{R}^n) \subset L^{p,\infty}(\mathbb{R}^n)$ for $p \neq \infty$.

Proof. By Chebyshev's inequality we have

$$k^p \, d_f(k) \leq \int_{\{x:|f(x)|>k\}} |f(x)|^p \, dx$$

for which LHS is $\| f \|_{L^{p,\infty}}$ by definition and RHS is at most $\| f \|_{L^p}^p$. □

This containment of *weak* L^p in L^p space leads us to define

Definition 1.3. *Operators that map L^p to L^q are called strong type (p, q). Operators that map L^p to $L^{q,\infty}$ are called weak type (p, q).*

We next have two interpolation theorems which significantly ease the task of showing boundedness for operators acting on any $L^p(\mathbb{R}^n)$.

Theorem 1.1 (Marcinkiewicz Interpolation Theorem)**.** *Let (X, μ) and (Y, ν) be measure spaces and let $0 < p_0 < p_1 \leq \infty$ and T be a sublinear operator on $L^{p_0}(X)+L^{p_1}(X)$, taking values in the space of measurable functions on Y. Assume that we have the bounds*

$$\begin{aligned} \| T(f) \|_{L^{p_0,\infty}(Y)} \; &\leq \; A_0 \| f \|_{L^{p_0}(X)} \quad \forall f \in L^{p_0}(X) \\ \| T(f) \|_{L^{p_1,\infty}(Y)} \; &\leq \; A_1 \| f \|_{L^{p_1}(X)} \quad \forall f \in L^{p_1}(X) \end{aligned}$$

where A_0 and A_1 are positive constants. Then for any $p_0 < p < p_1$ and for any $f \in L^p(X)$ we have the estimate

$$\| T(f) \|_{L^p(Y)} \leq A \| f \|_{L^p(X)}$$

where we have A depending on p, p_0, p_1, A_0 and A_1.

Note that if the operator is sub-linear and of *weak type* (p_0, p_0) and *weak type*(p_1, p_1) then it is of *strong type*(p,p) for $p_0 < p < p_1$. This assumes weak estimates and compensates it by being valid for the larger class of sub-linear functions.

Our next theorem is only valid for linear operators and assumes stronger end-point estimates, but gives a more natural bound for the intermediate spaces.

Theorem 1.2 (Riesz-Thorin Interpolation theorem). *Let (X,μ) and (Y,ν) be two measure spaces and let T be a linear operator defined on the set of all simple functions on X and taking values in the set of all measurable functions on Y. Let $1 \leq p_0, q_0, p_1, q_1 \leq \infty$ and assume the following bounds*

$$\| T(f) \|_{L^{q_0}} \leq M_0 \| f \|_{L^{p_0}}$$
$$\| T(f) \|_{L^{q_1}} \leq M_1 \| f \|_{L^{p_1}}$$

for all simple functions f on X. Then for any $0 < \epsilon < 1$ such that

$$\frac{1}{p} = \frac{1-\epsilon}{p_0} + \frac{\epsilon}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\epsilon}{q_0} + \frac{\epsilon}{q_1}$$

we have the bound

$$\| T(f) \|_{L^q} \leq M_0^{1-\epsilon} M_1^{\epsilon} \| f \|_{L^p}$$

for any simple function f on X. The operator T has a unique bounded extension which acts on $L^p(X,\mu)$ and gives functions in $L^q(Y,\nu)$ because of density of simple functions.

The proof of both the theorems are omitted and can be consulted in [3].

Next we define a smoothing operation called the convolution of two functions

Definition 1.4 (Convolution). *Let $f, g \in L^1(\mathbb{R}^n)$, then the convolution of f and g is given by*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x-y)\, dy$$

First it is easy to see that the integral is bounded by the L^1 norms of f and g and so it makes sense. Next we see that it is a smoothening operation as we have

Theorem 1.3 (Minkowski's inequality). *For $1 \leq p \leq \infty$ and for $g \in L^p(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$ we have*

$$\| f * g \|_{L^p(\mathbb{R}^n)} \leq \| f \|_{L^1(\mathbb{R}^n)} \| g \|_{L^p(\mathbb{R}^n)}$$

We have a more general inequality,

Theorem 1.4 (Young's inequality)**.** *Let* $1 \le p, q, r \le \infty$ *be such that*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

Then for any $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$ *we have*

$$\| f * g \|_{L^r(\mathbb{R}^n)} \le \| f \|_{L^p(\mathbb{R}^n)} \| g \|_{L^q(\mathbb{R}^n)}$$

Now we move to define the notion of Fourier transform. Before that we need to define a space with which we can easily work.

Definition 1.5 (Schwartz function)**.** *A smooth complex valued function* f *on* $\mathbb{R}^n$ *is called a Schwartz function if for any multi-indices* α, β *we have a positive constant* $K_{\alpha,\beta}$ *satisfying*

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| = K_{\alpha,\beta} < \infty$$

We denote the Schwartz space on $\mathbb{R}^n$ *by* $\mathcal{S}(\mathbb{R}^n)$*.*

Example 1.1. *The Gaussian function* e^{-x^2} *is a prototypical example of a Schwartz space function.*

Definition 1.6 (Fourier transform)**.** *For a function* $f \in \mathcal{S}(\mathbb{R}^n)$*, define its Fourier transform at* ξ *as*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi x \cdot \xi} \, dx$$

The Fourier transform of a Schwartz function is a Schwartz function, that is; if $f \in \mathcal{S}(\mathbb{R}^n)$ then $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$.

The following properties of the Fourier transform are an easy check

Proposition 1.3. *For* $f, g \in \mathcal{S}(\mathbb{R}^n)$, $x, y, \xi \in \mathbb{R}^n$, $b \in \mathbb{C}$, α *be a multi-index and* $t > 0$ *we have*

1. $\| \hat{f} \|_{L^\infty} \leq \| f \|_{L^1}$

2. $\widehat{f+g} = \hat{f} + \hat{g}$

3. $\widehat{bf} = b\hat{f}$

4. $\widehat{\tau^y(f)}(\xi) = e^{-2\pi i y\cdot\xi}\hat{f}(\xi)$

5. $(e^{2\pi i x\cdot y}f(x))\hat{}(\xi) = \tau^y(\hat{f})(\xi)$

6. $(\delta^t(f))\hat{} = t^{-n}\delta^{t^{-1}}(\hat{f})$

7. $(\partial^\alpha f)\hat{}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi)$

8. $\partial^\alpha \hat{f}(\xi) = ((-2\pi i x)^\alpha f(x))\hat{}(\xi)$

9. $\widehat{f*g} = \hat{f}\hat{g}$

We now define the inverse of this transform.

Definition 1.7 (Inverse Fourier transform)**.** *For $f \in \mathcal{S}(\mathbb{R}^n)$ we define the inverse Fourier transform as*

$$f^\vee(x) = \hat{f}(-x)$$

As the name suggests, this operation should be the reverse of Fourier transform, and so we have

Theorem 1.5. *For functions $f, g, h \in \mathcal{S}(\mathbb{R}^n)$ we have*

1. $\int_{\mathbb{R}^n} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^n} \hat{f}(x)g(x)\,dx$

2. ***(Fourier Inversion)*** $(\hat{f})^\vee = f = (f^\vee)\hat{}$

3. ***(Planceherel's identity)*** $\| f \|_{L^2} = \| \hat{f} \|_{L^2} = \| f^\vee \|_{L^2}$

4. $\int_{\mathbb{R}^n} f(x)g(x)\,dx = \int_{\mathbb{R}^n} \hat{f}(x)g^\vee(x)\,dx$

We can extend this definition of Fourier transform from functions in $\mathcal{S}(\mathbb{R}^n)$ to functions in $L^1 \cap L^2(\mathbb{R}^n)$ and so to functions in $L^2(\mathbb{R}^n)$, which gives us the nice result

Theorem 1.6 (Plancherel theorem)**.** *The Fourier transform is an isometric, isomorphism on $L^2(\mathbb{R}^n)$. That is, we have*

$$\| f \|_{L^2} = \| \hat{f} \|_{L^2}$$

This covers the very basic requirements for Fourier transform on $\mathbb{R}^n$. We now change the space from $\mathbb{R}^n$ to any Locally compact abelian group (LCA group).

To define the Fourier transform on LCA groups we need the notion of the dual group and the characters of a group.

Definition 1.8 (Character)**.** *A character χ of a finite abelian group G is a group homomorphism to the circle group $\mathbb{T}$ (the unit torus). The set of all characters of the group G is given by $\widehat{G}$.*

Definition 1.9 (Dual group)**.** *The set $\widehat{G}$, along with the pointwise product forms an abelian group, called the dual group of G or the Pontryagin dual of G.*

Now we can define the Fourier transform for functions $f \in l^2(G)$, for a finite group G.

Definition 1.10. *The Fourier transform for a function $f \in l^2(G)$ is given by*

$$\hat{f}(\chi) = \frac{1}{\sqrt{|G|}} \langle f, \chi \rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} f(g) \overline{\chi(g)}$$

Note that we have an additional factor of $\frac{1}{\sqrt{|G|}}$ which doesn't come up in the definition of the Fourier transform on $\mathbb{R}^n$. This is because of the choice of our *character* on $\mathbb{R}^n$, if instead of $e^{2\pi i x \cdot \xi}$ we use, $e^{ix \cdot \xi}$ we will get a normalizing factor of $\frac{1}{\sqrt{2\pi}}$.

Similar to $L^2(\mathbb{R}^n)$ isomorphism of Fourier transform, we have

Theorem 1.7. *The Fourier transform gives an isomorphism of the Hilbert spaces $l^2(G) \to$*

$l^2(\hat{G})$. *The same can also be done for* $\hat{G}$ *to get the map* $f \mapsto \hat{\hat{f}}$, *for which we have*

$$\hat{\hat{f}}(\delta_g) = f(g^{-1})$$

Similarly, we define the notion convolution product on finite abelian groups as

Definition 1.11 (Convolution product for groups). *For* $f, g \in l^2(G)$, *we define the convolution product as*

$$(f * g)(a) = \frac{1}{\sqrt{|G|}} \sum_{b \in G} f(b) g(b^{-1}a)$$

Similar to the Euclidean case, we have that Fourier transform turns convolution product into pointwise product, that is

Theorem 1.8. *For* $f, g \in l^2(G)$, *we have*

$$\widehat{f * g} = \hat{f}\hat{g}$$

We can move our setup from finite abelian group to locally compact abelian groups.

Definition 1.12 (LCA group). *A metrizable, σ-locally compact abelian group is called an LCA group.*

Example 1.2.

- *The group* $(\mathbb{R}, +)$ *with the usual topology on* $\mathbb{R}$ *makes it an LCA group*
- *Using the discrete metric on a finite abelian group, we can make it an LCA group.*

Parallel to the development of characters and dual groups for finite abelian groups we have *characters* and *dual groups* for LCA groups as well.

Definition 1.13 (Character on LCA group). *We define the character of a metrizable* *abelian group G, as the continuous group homomorphism from G to the circle group $\mathbb{T}$. That is, the map $\chi : G \to \mathbb{T}$ satisfying*

$$\chi(ab) = \chi(a)\chi(b)$$

The set of characters of the group G is given as $\hat{G}$.

Example 1.3.

- *For the group $\mathbb{Z}$, we have the maps $k \mapsto e^{2\pi ikx}$ as the characters for $x \in \mathbb{R} \setminus \mathbb{Z}$.*
- *For $\mathbb{R}$, the characters are given by the maps $x \mapsto e^{2\pi ixy}$ for $y \in \mathbb{R}$.*

Here too, the pointwise product makes $\hat{G}$ into an abelian group called the dual group or the Pontryagin dual of G.

From previous example we can easily see that $\widehat{\mathbb{Z}}$ is isomorphic to $\mathbb{R}\setminus\mathbb{Z}$ and $\widehat{\mathbb{R}}$ is isomorphic to $\mathbb{R}$.

One last thing we need before we can define the Fourier transform on an LCA group is the notion of integration. This integration is called the *Haar integration.*

Definition 1.14 (Integral). *We call a linear functional I, an integral on $C_0(G)$, if for any $f \in C_0(G)$, we have*

$$f \geq 0 \quad \implies I(f) \geq 0$$

The integral is called *left invariant* if we have

$$I(L_s f(x)) = I(f(s^{-1}x)) = I(f(x))$$

We will always have such a functional which we can call the *integral* of a group G. This is due to the following theorem

Theorem 1.9. *For any LCA group G, there exists a non-zero left-invariant integral G,* *unique up to scaling.*

This theorem ensure that we can define convolution and Fourier transform on any suitable LCA group.

Example 1.4.

- *The Haar integral on* $\mathbb{R}$ *is given by the usual Lebesgue integration, that is*

$$I(f) = \int_{\infty}^{\infty} f(x)\, dx$$

- *For* $\mathbb{R} \setminus \mathbb{Z}$*, the Haar integral is given by*

$$I(f) = \int_{0}^{\infty} f(x)\, dx$$

Definition 1.15 (Fourier transform for LCA groups)**.** *Let* $\hat{G}$ *be the dual group of* G *then the Fourier transform of a function* f *on* G *is defined to be the function* $\hat{f} : \hat{G} \to \mathbb{C}$ *given by*

$$\hat{f}(\chi) = \int_{G} f(x)\overline{\chi(x)}\, dx$$

where χ *is a character.*

It is easy to see that this definition encapsulates the cases of finite abelian groups and the case of $\mathbb{R}^n$. For convolution also we have a similar result which would capture the ideas previously established.

Theorem 1.10. *For functions* $f, g \in L^1(G)$ *which are bounded and continuous we have*

$$(f * g)(x) = \int_{G} f(xy^{-1})g(y)\, dy$$

exists for any $x \in G$ *and we have*

$$\widehat{f * g}(\chi) = \hat{f}(\chi)\hat{g}(\chi)$$

for any $\chi \in \widehat{G}$.

The only thing now remaining is the L^2 correspondence we had for $\mathbb{R}^n$, for any LCA group G, that is the Plancherel theorem equivalent for any LCA group.

Theorem 1.11. *For an LCA group* G*, we have a unique Haar measure on* $\widehat{G}$ *such that for any bounded, continuous and measurable function* f *on* G*, we have*

$$\| f \|_{L^2} = \| \hat{f} \|_{L^2}$$

That is, Fourier transform gives an isomorphism between $L^2(G)$ *and* $L^2(\widehat{G})$.

1.2 Motivation

Operators that commute with translation play a central role in our study and in general as well. We begin by defining **commuting with translation**.

The translation operator is given by τ_z, for $z \in \mathbb{R}^n$ acting on the space of functions on $\mathbb{R}^n$, and is given by

$$(\tau^z f)(x) = f(x - z)$$

Definition 1.16 (Translation)**.** *A vector space,* X*, of measurable functions on* $\mathbb{R}^n$*, is closed under translation if we have* $\tau_z(f) \in X$, $\forall z \in \mathbb{R}^n$ *and* $f \in X$.

Definition 1.17 (Scaling)**.** *A vector space* X*, of measurable functions on* $\mathbb{R}^n$*, is closed under scaling if we have* $\delta^h(f) \in X, \forall h \in \mathbb{R}^n$ *and* $f \in X$*, where*

$$\delta^h f(x) = f(hx) \quad \textit{where}\, h, x \in \mathbb{R}^n$$

Definition 1.18. *For vector spaces* V *and* W*, of measurable functions on* $\mathbb{R}^n$*, that are closed under translation. And if* T *is an operator from* V *to* W *then* T *commutes with translation if*

$$T(\tau^z(f))(x) = \tau^z(T(f))(x) \qquad \forall f \in X, \forall x, z \in \mathbb{R}^n$$

An automatic consequence of this definition is that any convolution operator commutes with translation. A more remarkable result holds true.

Theorem 1.12. *Let $1 \leq p, q \leq \infty$ and T be a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ that commute with translation. Then we have a unique tempered distribution ν such that*

$$T(f)(x) = (f * \nu)(x) \qquad \forall f \in \mathcal{S}$$

Before we can move to prove this we need the following lemmas

Lemma 1.1. *Under the hypothesis of the theorem, for $f \in \mathcal{S}(\mathbb{R}^n)$, the distributional derivatives of $T(f)$ are L^q functions that satisfy*

$$\partial^\alpha T(f) = T(\partial^\alpha f) \qquad \textit{for all multi-indices}\, \alpha$$

Proof. Let α be the vector $(0, \cdots, 1, \cdots, 0)$, jth entry is 1. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, then we already know

$$\frac{\tau^{-he_j}(g) - g}{h} - \partial_j g \to 0 \qquad \text{in } \mathcal{S} \text{ as } h \to 0$$

and so this convergence holds in L^p as well as

$$T\left(\frac{\tau^{-he_j}(g) - g}{h} - \partial_j g\right) \to 0 \qquad \text{in } L^q \text{as } h \to 0$$

This gives us that

$$\begin{aligned}
\langle \partial_j T f, g \rangle &= -\int_{\mathbb{R}^n} T(f) \partial_j g \, dx \\
&= -\lim_{h \to 0} \int_{\mathbb{R}^n} T(f) \left(\frac{\tau^{-he_j} - I}{h} g \, dx\right) \\
&= \lim_{h \to 0} \int_{\mathbb{R}^n} \left(\left(\frac{I - \tau^{he_j}}{h}\right) \circ T(f)\right) \, dx \\
&= \lim_{h \to 0} \int_{\mathbb{R}^n} T\left(\frac{I - \tau^{he_j}}{h} f\right) g \, dx \\
&= \int T(\partial_j f) \, dx
\end{aligned}$$

where the second last equality is due to commutation with translation and last one is because of the L^p convergence shown above. □

Lemma 1.2. *For* $1 \le q \le \infty$ *and* $h \in L(\mathbb{R}^n)$*, if all the distributional derivatives of* h *are also in* L^q*, then* h *is almost everywhere equal to a continuous function* H *satisfying*

$$H(0) \le C_{n,q} \sum_{|\alpha| \le n+1} \| \partial^\alpha h \|_{L^q}$$

Proof. Fix φ_R to a be a smooth function with compact support, such that $\varphi_R = 1$ for $|x| \le R$ and $\varphi = 0$ for $|x| \ge 2R$. For $h \in L^q$ then $\varphi_R h \in L^1$. Our idea of proof is the existence of inverse Fourier Transform of an L^1 function, and so the original function is a.e. equal to a continuous function. □

We have

$$1 \le C_n(1+|x|)^{-(n+1)} \sum_{|\alpha| \le n+1} |(-2\pi i x)^\alpha|$$

which follows from the inequality

$$|x|^k \le C_{n,k} \sum_{|\beta|=k} |x|^\beta$$

and so we have

$$\begin{aligned}
|\widehat{\varphi_R h}(x)| &\le C_n\,(1+|x|)^{-(n+1)} \sum_{|\alpha| \le n+1} |(-2\pi i x)^\alpha \widehat{\varphi_R h}(x)| \\
&\le C_n\,(1+|x|)^{-(n+1)} \sum_{|\alpha| \le n+1} \| (\partial^\alpha(\varphi_R h))\widehat{}\, \|_{L^1} \\
&\le C_n\,(2^n R^n \nu_n)^{1/q'} (1+|x|)^{-(n+1)} \sum_{|\alpha| \le n+1} \| (\partial^\alpha(\varphi_R h))\widehat{}\, \|_{L^q} \\
&\le C_{n,r}\,(1+|x|)^{-(n+1)} \sum_{|\alpha| \le n+1} \| \partial^\alpha h \|_{L^q}
\end{aligned}$$

using the boundedness of $\partial^\alpha \varphi_R$ and the Leibniz' rule. This gives us that $\| \widehat{\varphi_R h} \|_{L^1}$ is bounded and so the Fourier inversion holds. As $R > 0$ were arbitrary and $\varphi_R = 1$ for $|x| \leq R$, we get h is a.e. equal to a continuous function. For the inequality regarding $|H(0)|$, now it becomes trivial as $|H(0)| \leq \| \widehat{\varphi_1 h} \|_{L^1}$.

We can now finish the proof of our theorem. Define a linear functional, u on $\mathcal{S}$ by

$$\langle u, f \rangle = (Tf)(0)$$

then we have

$$\begin{aligned}
|Tf(0) = |\langle u, f \rangle| &\leq C_{n,q} \sum_{|\alpha| \leq n+1} \| \partial^\alpha T(f) \|_{L^q} \\
&\leq C_{n,q} \sum_{|\alpha| \leq n+1} \| T(\partial^\alpha f) \|_{L^q} \\
&\leq C_{n,q} \| T \|_{L^p \to L^q} \sum_{|\alpha| \leq n+1} \| \partial^\alpha f \|_{L^p} \\
&\leq C_{n,q} \| T \|_{L^p \to L^q} \sum_{|\alpha|, |\beta| \leq N} s_{\alpha,\beta}(f)
\end{aligned}$$

where $s_{\alpha,\beta}$ are the semi-norms of f we have used the boundedness of T and the fact that the αth derivatives of $f \in \mathcal{S}$ are bounded. Hence u is a tempered distribution. Setting $\nu(x) = u(-x)$ and using the fact that convolution of a Schwartz function with a tempered distribution is in C^∞ , we can write $T(f) = f * \nu$, to see this

$$\begin{aligned}
(f * \nu)(x) &= \langle \nu, \tau^x(\tilde{f}) \rangle \\
&= T(\tau^{-x}(f))(0) \\
&= \tau^{-x}(T(f))(0) \qquad \text{(T commutes with translation)} \\
&= T(f)(x)
\end{aligned}$$

this hold for $f \in \mathcal{S}$.

We denote the set of all bounded linear operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ that commute with translation by $\mathcal{M}_{p,q}(\mathbb{R}^n)$. In fact this space is a Banach space with respect to the operator norm and owing to the characterisation of $\mathcal{M}_{p,q}$ given by the above theorem, studying the properties of the tempered distribution suffices to understand T.

We next see that the only case to consider is for $1 \leq p \leq q \leq \infty$.

Theorem 1.13. $\mathcal{M}_{p,q} = \{0\}$ *whenever* $1 \leq q < p < \infty$.

Proof. For $f \in \mathcal{C}_0^\infty$ and $h \in \mathbb{R}^n$ we have,

$$\| \tau^h(T(f)) + T(f) \|_{L^q} = \| T(\tau^h(f) + f) \|_{L^q} \leq \| T \| \| \tau^h(f) + f \|_{L^p}$$

Letting $h \to \infty$, we get

$$2^{1/q} \| T(f) \|_{L^q} \leq \| T \| \, 2^{1/p} \| f \|_{L^p}$$

which holds if and only if $T = 0$. □

Next we will see the usefulness of the adjoint of an operator which will help us get results for the dual of the spaces $\mathcal{M}_{p,q}(\mathbb{R}^n)$. This works because of the following theorem

Theorem 1.14. *For* $1 < p \leq q < \infty$ *and* $T \in \mathcal{M}_{p,q}(\mathbb{R}^n)$, *we can define* T *on* $L^{q'}(\mathbb{R}^n)$, *so that it maps* $L^{q'}(\mathbb{R}^n)$ *with norm*

$$\| T \|_{L^{q'} \to L^{p'}} = \| T \|_{L^p \to L^q}$$

Which means we have the identification

$$\mathcal{M}^{q',p'}(\mathbb{R}^n) = \mathcal{M}_{p,q}(\mathbb{R}^n)$$

Proof. The proof follows from the fact that if T is given by convolution with $u \in \mathcal{S}'$ then

T^* is also given by convolution, but with $\bar{\tilde{u}} \in \mathcal{S}'$. We also require the identity

$$\overline{f * \bar{\tilde{u}}} = (\bar{\tilde{f}} * u) \qquad f \in L^{q'}$$

which show that T is well defined on $L^{q'}$. The only thing left is to show that T and T^* have the same norm in $L^{p'}$. This is easily seen from the above equality and so we get that

$$\| T \|_{L^p \to L^q} = \| T^* \|_{L^{q'} \to L^{p'}} = \| T \|_{L^{q'} \to L^{p'}}$$

□

We now study the specific cases of $p = 1 = q$ and $p = 2 = q$ and characterise them in terms of the properties of the associated distribution. For other cases of $p = q$ the characterisation is still unknown.

Theorem 1.15. *An operator T is in $\mathcal{M}^{1,1}(\mathbb{R}^n)$ if and only if it is given by the convolution with a finite Borel (complex valued) measure. The norm of the operator is equal to the total variation of the measure.*

We will not provide a proof for this case but it can be looked up in [3], the theorem is stated here to show what we mean by characterisation of an operator. We will now look into the case of $p = 2 = q$.

Theorem 1.16. *The operator T lies in $\mathcal{M}^{2,2}(\mathbb{R}^n)$ if and only if it is given by a convolution with $u \in \mathcal{S}'$ whose Fourier transform $\hat{u}$ is an L^∞ function. The norm of $T : L^2 \to L^2$ is equal to $\| \hat{u} \|_{L^\infty}$.*

Proof. If $\hat{u} \in L^\infty$, Plancherel's theorem gives

$$\int_{\mathbb{R}^n} |f * u|^2 \, dx = \int_{\mathbb{R}^n} |\hat{f}(\xi)\hat{u}(\xi)|^2 \, d\xi \ \leq \| \hat{u} \|_{L^\infty}^2 \| \hat{f} \|_{L^2}^2$$

and so $\| T \|_{L^2 \to L^2} \leq \| \hat{u} \|_{L^\infty}$, giving T in $\mathcal{M}^{2,2}(\mathbb{R}^n)$. For the converse claim, we show that $\hat{u}$ is a bounded function. For $R > 0$, we define $\varphi_r \in \mathcal{C}_0^\infty$ which is supported on the ball $B(0, 2R)$ and is equal to 1 on the ball $B(0, R)$. The product $\varphi_R \hat{u} = ((\varphi_R)^\vee * u)\hat{} = T(\varphi_R^\vee)\hat{}$, is an L^2 function and on the ball $B(0, R)$, $\varphi_R \hat{u} = \hat{u}$. Hence $\hat{u}$ is in $L^2(B(0, R))$ for any arbitrary R hence is locally square integrable.

For $f \in L^\infty(\mathbb{R}^n)$ having compact support, $f\hat{u} \in L^2$ and we can use Plancherel's theorem again along with boundedness of T to get

$$\int_{\mathbb{R}^n} |f(x)\hat{u}(x)|^2 \, dx = \int_{\mathbb{R}^n} |T(f^\vee(x))|^2 \, dx \ \le \| \, T \, \|_{L^2 \to L^2} \int_{\mathbb{R}^n} |f(x)|^2 \, dx$$

Hence for any bounded function with compact support f we have

$$\int_{\mathbb{R}^n} (\| \, T \, \|^2_{L^2 \to L^2} \ - |\hat{u}(x)|^2) |f(x)|^2 \, dx \ge 0$$

For $f = |B(x, r)|^{-1} \chi_{B(x,r)}$ for $r > 0$ giving $\| \, T \, \|_{L^2 \to L^2} \ge \ |\hat{u}(x)|^2$ for almost all x. Using this with the previous bound $\| \, T \, \|_{L^2 \to L^2} \le \| \, \hat{u} \, \|_{L^\infty}$, we get the bounds of T. □

Definition 1.19. *For* $1 \le p < \infty$*, denote by* $\mathcal{M}_p(\mathbb{R}^n)$ *the space of all bounded functions* m *on* $\mathbb{R}^n$ *such that*

$$T_m(f) = (\hat{f} m)^\vee \qquad f \in \mathcal{S}$$

is bounded on $L^p(\mathbb{R}^n)$ *(Or is a bounded extension of an operator which was initially defined on a dense subspace). The norm of* m *is defined by*

$$\| \, m \, \|_{\mathcal{M}_p} = \| \, T_m \, \|_{L^p \to L^p}$$

The definition implies that $m \in \mathcal{M}_p \Leftrightarrow T \in \mathcal{M}^{p,p}$. The elements of $\mathcal{M}_p$ are called *L^p multipliers* or *L^p Fourier multipliers*. By the previous two theorems we know that $\mathcal{M}_2 = L^\infty$ and $\mathcal{M}_1$ is the space of finite Borel measures. We also get that $m \in \mathcal{M}_p \Leftrightarrow m \in \mathcal{M}_{p'}$

and we have the bound

$$\| m \|_{\mathcal{M}_p} = \| m \|_{\mathcal{M}_{p'}} \qquad 1 < p < \infty$$

We also have these spaces to be nested, this is because of *Riesz-Thorin Interpolation* theorem which is, for $1 \leq p \leq q \leq 2$, we have

$$\mathcal{M}_1 \subseteq \mathcal{M}_p \subseteq \mathcal{M}_q \subseteq \mathcal{M}_2 = L^\infty$$

Example 1.5. *Consider the function $m(\xi) = e^{2\pi i \xi \cdot b}$, then for all $b \in \mathbb{R}^n$ we have,*

$$\begin{aligned} T_m f(x) &= (\hat{f}(\xi) m(\xi))^\vee (x) \\ &= (e^{2\pi i \xi \cdot b} f(\xi))^\vee \\ &= f(x+b) \end{aligned}$$

And so $m \in L^p$ with $\| m \|_{L^p} = 1$.

Proposition 1.4. *For $1 \leq p < \infty$ the space $\mathcal{M}_p$ along with the norm $\| \cdot \|_{\mathcal{M}_p}$ is a Banach space and is closed under pointwise multiplication.*

Proof. The case $1 \leq p \leq 2$ is sufficient. For pointwise product we just correspond each multiplier to its operator and then composition of these operator will give us the closure under pointwise multiplication. what we mean is

$$\| m_1 m_2 \|_{\mathcal{M}_p} = \| T_{m_1} T_{m_2} \|_{L^p \to L^p} \leq \| m_1 \|_{\mathcal{M}_p} \| m_2 \|_{\mathcal{M}_p}$$

Hence, $\mathcal{M}_p$ is an algebra. For completeness we use the completeness of L^∞, which contains all $\mathcal{M}_p$. Let m be the limit, in L^∞ norm, of the Cauchy sequence $\{m_j\}_j$. Then for a fixed

$f \in \mathcal{S}$, we have

$$T_{m_j}(f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) m_j(\xi) e^{2\pi i x \cdot \xi} \, d\xi \to \int_{\mathbb{R}^n} \hat{f}(\xi) m(\xi) e^{2\pi i x \cdot \xi} \, d\xi = T_m(f)(x)$$

a.e by Lebesgue dominated theorem. $\{m_j\}_j$ is bounded in $\mathcal{M}_p$, because it is Cauchy in $\mathcal{M}_p$ and so $\sup_j \parallel m_{j \parallel_{\mathcal{M}_p}} \leq C$. We can now use Fatou's lemma to get

$$\begin{aligned} \int_{tn} |T_m(f)|^p \, dx &= \int_{\mathbb{R}^n} \liminf_{j \to \infty} |T_{m_j}(f)|^p \, dx \\ &\leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} |T_{m_j}(f)|^p \, dx \\ &\leq \liminf_{j \to \infty} \parallel m_j \parallel_{\mathcal{M}_p}^p \parallel f \parallel_{L^p}^p \\ &\leq C^p \parallel f \parallel_{L^p}^p \end{aligned}$$

giving $m \in \mathcal{M}_p$. We also get that for $\mu_j \in \mathcal{M}_p$ and $\mu_j \to \mu$ a.e., then μ is in $\mathcal{M}_p$ and satisfies

$$\parallel \mu \parallel_{\mathcal{M}_p} \leq \liminf_{j \to \infty} \parallel \mu_{j \parallel_{\mathcal{M}_p}}$$

From this we can see that the Cauchy sequence $\{m_j\}_j$ goes to m in $\mathcal{M}_p$. □

Finally we note some properties of multipliers.

Proposition 1.5. *For all $m \in \mathcal{M}_p$, $1 \leq p < \infty$, $x \in \mathbb{R}^n$, and $h > 0$ we have*

1. $\parallel \tau^x(m) \parallel_{\mathcal{M}_p} = \parallel m \parallel_{\mathcal{M}_p}$
2. $\parallel \delta^h(m) \parallel_{\mathcal{M}_p} = \parallel m \parallel_{\mathcal{M}_p}$
3. $\parallel \tilde{m} \parallel_{\mathcal{M}_p} = \parallel m \parallel_{\mathcal{M}_p}$
4. $\parallel e^{2\pi i (\cdot) \cdot x} m \parallel_{\mathcal{M}_p} = \parallel m \parallel_{\mathcal{M}_p}$
5. $\parallel m \circ A \parallel_{\mathcal{M}_p} = \parallel m \parallel_{\mathcal{M}_p}$ *A is an orthogonal matrix.*

The proof of these properties are straightforward and hence omitted.

We state a multiplier theorem, due to, Mikhlin which will be similar to our multiplier theorem.

Theorem 1.17 (Mikhlin)**.** *Let m be a complex valued bounded function on $\mathbb{R}^n \setminus \{0\}$ such that it satisfies*

$$|\partial_y^\alpha m(y)| \;\leq\; A|y|^{-|\alpha|}$$

for all multi-indices $|\alpha| \;\leq\; \left[\frac{n}{2}\right] + 1$. Then we have for all $1 < p < \infty$, m lies in $\mathcal{M}_p(\mathbb{R}^n)$ and has the estimate

$$\| \, m \, \|_{\mathcal{M}_p} \leq \; C_n \max{(p, (p-1)^{-1})} (A + \| \, m \, \|_{L^\infty})$$

Moreover, the operator sending f to $(\hat{f}m)^\vee$ is weak (1,1).

We will look into a similar theorem which will state the conditions in terms of the partial derivatives of the multiplier, but instead of $\mathbb{R}^n$, it will be on H^n. But first we need to have some knowledge of the Heisenberg group which will be our next theme.

2 The Heisenberg Group

The novel idea of quantization for explanation of the physical world and establishing the quantization of the Poisson brackets, the Heisenberg's canonical commutation relations gave these ideas a mathematical footing. These commutation relations come up as the structure equations of the Lie algebra over the Heisenberg group, and hence the name.

To work with the Heisenberg group we need its representations, atleast the irreducible ones. The nice part of this is that it admits only one faithful, unitary and irreducible representations, upto unitary equivalence. This simplifies our work by letting us focus on only one representation. This is the result of "Stone-von Neumann" theorem. It also allows us to use the pleasant features of each 'area' without worrying much about the other representations, provided we can interchange the two. We'll see this when we will look into the representation of Heisenberg group onto the Fock space.

2.1 Heisenberg's commutation relations

Hamiltonian Mechanics

Newton gave us the laws governing the motion of any particle, which then in turn lead to the mathematisation of physics. The forces acting on any particle is involved in the second order differential equation, the knowledge of which, then gives us the motion of the particle being completely determined by its velocity and position at any time of our choosing. Hence, we can say that the 'state' of a particle is completely determined its velocity and position.

This notion is changed slightly in the Hamiltonian Mechanics by replacing the velocity of the particle with its momentum, and so the we get the state space namely, the **Phase Space** $\mathbb{R}^{2n}$, with coordinates

$$(p, q) = (p_1, p_2, ..., p_n, q_1, q_2, ..., q_n) \tag{1}$$

for a particle with p as its momentum vector and q as its position vector. Every observable is real-valued function of the phase space and so is a function of momentum and position.

Next we are tasked with the time evolution of the system and how it is represented. The time evolution of the system is given by certain transformations of the phase space. The distinguishing feature of Hamiltonian mechanics is that these transformations leave the following differential form invariant.

$$\Omega = \sum_{j=1}^{n} dp_j \wedge dq_j$$

where Ω is a translation-invariant bilinear form on the tangent space $\mathbb{R}^{2n}$ of the phase space $\mathbb{R}^{2n}$. {When $\mathbb{R}^{2n}$ is identified with the tangent space at any point of $\mathbb{R}^{2n}$, then Ω is the **symplectic form** on $\mathbb{R}^{2n}$, which is denoted by the brackets $[(p, q)(p', q')] = pq' - p'q$ }

We call the diffeomorphisms which preserve the above symplectic form as **symplectomorphisms**, these form a group of linear transformations, that is, the group of all $T \in GL(2n, \mathbb{R})$ such that we have

$$[T(p, q), T(p', q')] = [(p, q), (p', q')] \tag{2}$$

and we call it the **symplectic group** $Sp(n, \mathbb{R})$. The form Ω is degenerate and so we can identify the tangent vectors with cotangent vectors, and so we will have to each tangent vector X at (p, q), the cotangent vector ω_X at (p, q) such that

$$\omega_X(Y) = \Omega(Y, X) \tag{3}$$

For a function on $\mathbb{R}^{2n}$ i.e. a smooth observable, we can associate a vector field X_f to its one form df, we call this vector field the **Hamiltonian vector field** of f and is given by

$$df(Y) = \Omega(Y, X_f) \tag{4}$$

or

$$X_f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j} \right) \tag{5}$$

And now we can define the **Poisson bracket** of two smooth observables f and g as

$$\{f, g\} = \Omega(X_f, X_g) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right)$$

as is easily seen that it is skew-symmetric and as usual this satisfies the **Jacobi identity**.

Hence we have that, the space of smooth observables is a Lie algebra, and the map $f \to X_f$ is a Lie algebra homomorphism and the following relation holds

$$[X_f, X_g] = X_{\{f,g\}} \tag{6}$$

We can easily see that the kernel of the above mentioned Lie algebra homomorphism has the kernel which consists of constant functions and the Poisson bracket of the coordinates of the phase space satisfies the relations

$$\{p_j, p_k\} = \{q_j, q_k\} = 0 \qquad \{p_j, q_k\} = \delta_{jk}$$

The coordinates for which above commutation relations hold are called the **canonical coordinates** and the canonical transformations of $\mathbb{R}^{2n}$ map one canonical coordinate system to another.

Hence, we get the Hamiltonian Mechanical formulations that canonical commutations equations are given by

$$[X_j, X_k] = [Y_j, Y_k] = 0, \qquad [X_j, Y_k] = \delta_{j,k} K \tag{7}$$

for X_j, X_k, Y_j, Y_k are the basic observables and K is a constant such that $K^2 = -1$. also the condition that the only allowed transformations of $\mathbb{R}^{2n}$ are those that preserve the symplectic form Ω.

Using these as our building blocks we now look at the basic construction of Heisenberg Group.

2.2 The Heisenberg group

We can use the commutation relations to define the Heisenberg Lie algebra using which we would be able to define the Heisenberg group.

First, we define a map $m : \mathbb{R}^n \longrightarrow M_{n+2}(\mathbb{R})$ as,

$$m(x, y, k) = \begin{pmatrix} 0 & x_1 & \cdots & x_n & k \\ 0 & 0 & \cdots & 0 & y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and also define $M : \mathbb{R}^n \longrightarrow M_{n+2}(\mathbb{R})$

$$M(x, y, k) = I + m(x, y, k)$$

We can see by trivial matrix multiplication that

$$m(x, y, k)m(x', y', k') = m(0, 0, xy') \tag{8}$$

and

$$M(x,y,k)M(x',y',k') = M(x+x', y+y', k+k'+xy') \tag{9}$$

We can now use the commutator of the matrices to define a Lie bracket on the space of matrices and hence give it a Lie algebra structure. Using the commutator, we have the following

$$[m(x,y,k), m(x',y',k')] = m(0,0,xy'-yx') \tag{10}$$

and because of (10) we have

$$m(x,y,k)^2 = m(0,0,xy) \quad \& \quad m(x,y,k)^t = 0 \qquad for \quad t \geq 3$$

and hence we have

$$e^{m(x,y,k)} = I + m(x,y,k) + \frac{1}{2}m(x,y,k)^2 = M(x,y,k+\frac{1}{2}xy) \tag{11}$$

The Heisenberg group has $2n+1$ one parameter subgroups, each given corresponding to a coordinate, that is

$$\beta_i = \{(ke_i, 0, 0) : k \in \mathbb{R}\}, \text{ and } \beta_{n+i} = \{(0, ke_i, 0) : k \in \mathbb{R}\}$$

for $1 \leq j \leq n$ and $\beta_{2n+1} = \{(0,0,t) : t \in \mathbb{R}\}$, where e_i are the canonical basis vectors in $\mathbb{R}^n$. Corresponding to these subgroups, we have the left invariant vector fields on H^n which are given by

$$X_j = \left(\frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}\right) \qquad 1 \leq j \leq n$$

and

$$Y_j = \left(\frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}\right) \qquad 1 \leq j \leq n$$

and

$$T = \frac{\partial}{\partial t}$$

as can be verified easily, these vector fields satisfy the commutation relations of Hamiltonian mechanics and so are aptly capturing the physical nature and presenting them in a mathematical setting. Even more so, we have that the Heisenberg Lie algebra is isomorphic to the Lie algebra of matrices $m(p, q, t)$ defined above.

Definition 2.1 (Polarized Heisenberg group)**.** *The polarized Heisenberg group is the space* $\mathbb{R}^{2n+1}$ *with the identification* $(p, q, t) = M(p, q, t)$ *and the group operation is given by the multiplication of the corresponding matrices. It is denoted by* H_n^{pol}*.*

We can use this as the definition of Heisenberg Group but use a slightly different notion of group operation which uses the symplectic form on $\mathbb{R}^{2n}$. To that extent, we can easily verify that

$$\exp(m(x, y, k)) \exp(m(x', y', k')) = \exp(m(x + x', y + y', t + t' + \frac{1}{2}(xy' - yx'))) \qquad (12)$$

If we use this group law along with the identification of $(x, y, k) \in \mathbb{R}^{2n+1}$ with the exponential $e^{m(X)}$, we get a group namely, the **Heisenberg Group** and denote it by H_n. Hence we get two correspondences of $\mathbb{R}^{2n+1}$, that are:-

$$1. X \longrightarrow m(X) \longrightarrow e^{m(X)}$$
$$2. X \longrightarrow M(X)$$

and each have their own corresponding group law. The correspondence $X \longrightarrow m(X)$ is a Lie algebra homomorphism from h_n to $\{m(X) : X \in \mathbb{R}^{2n+1}\}$ and the exponential map is the bijective correspondence between $\{m(X) : X \in \mathbb{R}^{2n+1}\}$ and $\{M(X) : X \in \mathbb{R}^{2n+1}\}$. We also see that

$$(x, y, k) \longrightarrow (x, y, k + \frac{1}{2}xy)$$

is the isomorphic map between H_n and $\mathsf{H}_\mathsf{n}^\mathsf{pol}$, and is also the exponential map from h_n.

A simple calculation shows that for H^n the identity is $(0,0,0)$ and the inverse of (p,q,t) is $(-p,-q,-t)$ and for H^n_{pol} the identity is $(0,0,0)$ and inverse of (p,q,t) is $(-p,-q,-t+pq)$.

We have $C = \{(0,0,t) : t \in \mathbb{R}\}$ as the center of both H_n and $\mathsf{H}_\mathsf{n}^\mathsf{pol}$.

We can also identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ then the symplectic form becomes $Im(z\cdot\overline{w})$, for $z = p+iq$ and $w = p' + iq'$. This doesn't change much of what we have seen, it just changes the identification of $\mathsf{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the group operation

$$(p,q,t)(p',q',t') = (p+p', q+q', t+t' + \frac{1}{2}(pq' - p'q))$$

to the identification $\mathsf{H}^n = \mathbb{C}^n \times \mathbb{R}$ with the group operation

$$(z,t)(w,s) = (z+w, t+s+\frac{1}{2}Im(z\cdot\overline{w}))$$

and $z \cdot \overline{w} = \sum_{i=1}^n z_i \overline{w_i}$.

The Automorphisms of Heisenberg Group

Definition 2.2 (Automorphism). *In an algebraic structure such as a group, a ring, or vector space, an automorphism is simply a bijective homomorphism of an object into itself.*

In essence, an automorphism is a bijective map, from a set, say X, to itself which preserves the algebraic property of the set.

Let $Aut(\mathsf{H}^n)$ and $Aut(\mathsf{h}^n)$ denote the automorphism groups of H^n and $\mathsf{H}^\mathsf{n}_\mathsf{pol}$ as a group and as a Lie algebra, respectively. The important thing to note is the underlying space is same for both $Aut(\mathsf{H}^n)$ and $Aut(\mathsf{h}^n)$, that is $\mathbb{R}^{2n+1}$ and any automorphism first has to be an isomorphism of $\mathbb{R}^{2n+1}$ and then preserve the algebraic properties. To this extent, we have the following:

Theorem 2.1. $Aut(\mathsf{H}^n) = Aut(\mathsf{h}^n)$

Proof. For $T \in Aut(\mathsf{h}^n)$, we have T linear and preserves the Lie bracket, now use this fact with

$$XY = X + Y + [X, Y]$$

for $X, Y \in \mathbb{R}^{2n+1}$ to get that $T(XY) = T(X)T(Y)$.

For $S \in Aut(\mathsf{H}^n)$, and for elements x, y which have $[x, y] = 0$ using the above equality we get that S commutes with constants. Hence we get

$$cS(xy) = cS(x) + cS(y) + c^2 S([x, y])$$

and from $\lim s \to 0$ we get that S is linear and also preserves the Lie bracket, hence $S \in Aut(\mathsf{H}^n)$ □

We have seen that that the automorphism groups are equal for the Heisenberg group and the Heisenberg Lie algebra, we now want to classify, if we can, the possible elements of this Automorphism group. For that we first see that we have the following as the possible elements of $Aut(\mathsf{H}^n)$ or of $Aut(\mathsf{h}^n)$.

1. **Dilation maps** : Any map D_a such that

$$D_a(x, y, k) = (ax, ay, a^2 k)$$

   is an automorphism of H^n, because of distributive property of multiplication.

2. **Conjugation maps** : As H^n is a group, the inner automorphism maps

$$(l, m, n)(x, y, k)(l, m, n)^{-1} = (x, y, k + ly - mx)$$

are group homomorphisms and lie in $Aut(\mathsf{H}^n)$.

3. **Symplectic maps** : Maps which preserve the Symplectic form on $\mathbb{R}^{2n}$. Let $S \in \mathsf{Sp}(\mathbb{R}^{2n})$ be arbitrary, that is S is an automorphism of $\mathbb{R}^{2n}$ which preserves the symplectic form. Then the maps of the form

$$(x, y, k) \mapsto (S(x, y), k)$$

are in $Aut(\mathsf{H}^n)$.

4. **Inversion maps** : The maps for which we have

$$(x, y, k) \mapsto (y, x, -k)$$

are also in $Aut(\mathsf{H}^n)$ because of interchange of the first two coordinates and the distributivity of negative sign.
Also, this map is not inversion in the usual sense (that is $(x, y, z,) \mapsto (-x, -y, -z))$ but rather it inverts only the last coordinate and rotate $\mathbb{R}^{2n}$ about the axis by $\frac{\pi}{2}$rads in the clockwise direction.
Lastly, this maps has only two elements as

$$(x, y, k) \mapsto (y, x, -k) \mapsto (x, y, k)$$

There can be more automorphisms of H^n, but as our next result shows these will be only those that are a combination of those given above.

Theorem 2.2. *Every automorphism of H^n can be uniquely written as $T_1T_2T_3T_4$ where T_1 is a **Symplectomorphism**, T_2 is a **Conjugation**, T_3 is a **Dilation** and T_4 is **Inversion**.*

Proof. Any automorphism of H^n has to

- be linear

- send the center to center
- be a group homomorphism

Using these we can see that any $T \in Aut(\mathsf{H}^n)$ sends $(x, y) \in \mathbb{R}^{2n}$ to $A(x, y)$ where $A \in GL(n, \mathbb{R})$, because A is linear and is an isomorphism of $\mathbb{R}^{2n}$ because $\det(A) \neq 0$.

T also sends the set $C = \{(0, 0, k) : k \in \mathbb{R}\}$ to itself, is a group homomorphism and is linear so, using the above observation we have

$$T(x, y, k) = (A(x, y), lx + my + nk)$$

which we can reduce to the map $(x, y, k) \mapsto (B(x, y), k)$, where $B \in GL(n, \mathbb{R})$, using the automorphisms T_4, T_3, T_2. Now the above map is an H^n automorphisms if and only if B is in $\mathsf{Sp}(\mathbb{R}^{\mathsf{2n}})$. Uniqueness follows from the order of T_1, T_2, T_3 and T_4, and hence the theorem.

□

2.3 The Schrödinger Representation of H^n

Now that we have a group, we want to do some analysis on it and for that we need the group's representations. The Schrödinger representations of H^n are the most useful and easy to understand. For this we define the function $\rho_h : \mathsf{H}^n \longrightarrow U(L^2(\mathbb{R}^n)$ as

$$\rho_h(p, q, t)f(x) = (\rho(hp, q, ht)f)(x) =)e^{2\pi it}e^{(2\pi iq\cdot x)+(\pi ip\cdot q)f(x+p)} \tag{13}$$

here $U(L^2(\mathbb{R}^n))$ is the space of unitary operators on the $L^2(\mathbb{R}^n)$. These operators are continuous on the Schwartz space and can be extended to continuous operators on the space of tempered distributions. As can be seen easily, ρ_h is an operator on $L^2(\mathbb{R}^n)$, for a fixed $h \in \mathbb{R}$. We claim that it is in fact a unitary operator on $L^2(\mathbb{R}^n)$.

Proposition 2.1. *For every $h \in \mathbb{R}$, ρ_h is a unitary operator on $L^2(\mathbb{R}^n)$.*

Proof. We show the claim is true for $h = 1$ and other cases follow easily. Also $e^{2\pi it}$ doesn't have much role to play so we can ignore it for the time being and let $\rho(p,q)$ correspond to $\rho(p,q,t)$ without any consequences in the evaluation of the norm. Now we show that

$$\| \rho(p,q)f \|_{L^2(\mathbb{R}^n)} = \| f \|_{L^2(\mathbb{R}^n)}$$

and that $\rho(p,q)$ is onto to complete our claim.

We first show that $\rho(p,q)$ is onto; let $g \in L^2(\mathbb{R})^n$ and define f on $\mathbb{R}^n$ as

$$f(x) = e^{2\pi i q\cdot x + \pi i q\cdot p}(g-p)$$

Then a simple calculation shows that $(\rho(p,q)f)(x) = g(x)$.

We observe that for $f(x) = (\rho(-q,-p)g)(x)$, and by definition of ρ_h, we have

$$\begin{aligned} \| f \|_{L^2(\mathbb{R}^n)} &= \| (\rho(-q,-p)g)(x) \|_{L^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} |e^{(-2\pi i p\cdot x)-(\pi i p\cdot q)} g(x-q)|^2 \, dx \\ &\leq \int_{\mathbb{R}^n} |g(x-q)|^2 \, dx \\ &= \| g \|_{L^2(\mathbb{R}^n)} \end{aligned} \tag{14}$$

Similarly, for $g(x) = (\rho(p,q)f)(x)$ we get $\| g \|_{L^2(\mathbb{R}^n)} \leq \| f \|_{L^2(\mathbb{R}^n)}$

and so we get

$$\| f \|_{L^2(\mathbb{R}^n)} = \| (\rho(p,q)f)(x)) \|_{L^2(\mathbb{R}^n)}$$

$\square$

The essential thing in the proof was that irrespective of the inputs (p,q) of ρ, the inequalities hold and so the norm of the image function could be bounded by norm of f.

And so for any h we have ρ_h is a unitary representation of H_n on $L^2(\mathbb{R}^n)$ and the corresponding representation $d\rho_h$ of h_n given by

$$d\rho_h = 2\pi i(hpD + qX + tI)$$

where $Xf(x) = xf(x)$ and $Df(x) = \frac{h}{2\pi i}\frac{\partial f(x)}{\partial x}$. The representation ρ_h is called the **Schrödinger Representation** of H_n with parameter h.
A more elegant structure comes out when we take the symplectic form rather than the Euclidean inner product to pair (p, q) and (D, X) in the exponent. So we define

$$\rho'(p, q, t) = \rho(-q, p, t) = e^{2\pi it}e^{2\pi i(pX - qD)}$$

and so we get

$$e^{2\pi i(pX-qD)}f(x) = e^{2\pi ipx - \pi ipq}f(x - q)$$
$$[e^{2\pi i(pX-qD)}f]^\wedge(\xi) = e^{-2\pi q\xi + \pi ipq}\hat{f}(\xi - p)$$

The physical interpretation of the above equations says that the average value of the position (momentum) of $e^{2\pi i(pX-qD)}f$ is just the translation of the average value of the position (momentum) by q (p). Nonetheless we stick to the usual ρ, but no harm comes in knowing how to go from one representation to another.

The automorphism map $\rho(p, q, t) \longrightarrow \rho(-q, p, t)$ gives us the Symplectic version from the Euclidean version, and for the operators we have the Fourier transform as

$$\mathcal{F}\rho(p, q, t,)\mathcal{F}^{-1} = \rho'(p, q, t)$$

To see if the representation is faithful or not we look at the kernel of the map ρ, which we see is the set $\{(0,0,s) : s \in \mathbb{Z}\}$. Now using this kernel we define the **reduced Heisenberg Group** as the quotient

$$\mathsf{H}_n^{red} = \mathsf{H}_n / \{(0,0,s) : s \in \mathbb{Z})\}$$

There is no change in notation, but it goes without mention that $t \in [0,1)$

Integrated Representation

We defined ρ_h as a unitary operator on $L^2(\mathbb{R}^n)$ and our main focus was on $h = 1$, but h can be any real number and so we need a continuous version of the representations of H_n, which we see now.

For a function $\Phi \in L^1(\mathsf{H}_{red}^n)$, we can define

$$\rho(\Phi) = \int_{\mathsf{H}_{red}^n} \Phi(X)\rho(X)\,dX = \iiint \Phi(p,q,t)\rho(p,q,t)\,dp\,dq\,dt \tag{15}$$

with $\| \rho(\Phi) \| \leq \| \Phi \|_1$. The question of how $\rho(\Phi)$ acts on $L^2(\mathbb{R}^n)$ can be answered by using the definition of the ρ acting on $L^2(\mathbb{R}^n)$ and is given by

$$(\rho(\Phi)f)(x) = \int_{\mathsf{H}_{red}^n} \Phi(X)(\rho(X)f)(x)\,dX$$

Because the variable t doesn't have much role to play, we consider the Fourier series of $\Phi \in \mathsf{H}_{red}^n$ about the variable t, that is

$$\Phi(p,q,t) = \sum_{j=-\infty}^{\infty} \Phi_j(p,q)e^{2\pi i j\cdot t} \tag{16}$$

The assumption here is that the series is interpreted as a Cesàro or Abel mean. This expansion simply gives

$$\begin{aligned}\rho(\Phi) &= \sum_{\infty}^{\infty} \iiint \Phi_k(p,q)\rho(p,q)e^{2\pi i(k+1)t}\,dp\,dq\,dt \\ &= \iint \Phi_{-1}(p,q,t)\rho(p,q,t)\,dp\,dq\end{aligned} \tag{17}$$

because of the integral in t being from 0 to 1 and so the only surviving term will be for $k+1=0$.

Hence we can instead consider ρ to be a representation of $L^1(\mathbb{R}^{2n})$ with a special convolution structure called the **twisted convolution**. Hence we can define for $F \in L^1(\mathbb{R}^{2n})$, we have

$$\begin{aligned}(\rho(F)f)(x) &= \iint F(p,q)(\rho(p,q,)f)(x)\,dp\,dq \\ &= \iint F(y-x,q)e^{\pi i q(x+y)}f(y)\,dy\,dq\end{aligned} \tag{18}$$

From the theory of integral operators we get that the kernel of the integral operator $\rho(F)$ is given by

$$\begin{aligned}K_F(x,y) &= \int F(y-x,q)e^{\pi i q(x+y)}\,dq \\ &= \mathcal{F}_2^{-1}F\left(y-x, \frac{x+y}{2}\right)\end{aligned} \tag{19}$$

and so we get the following result

Theorem 2.3. *The map ρ from $L^1(\mathbb{R}^n)$ to the space of bounded operators on $L^2(\mathbb{R}^n)$, extends uniquely to a bijection from $\mathcal{S}'(\mathbb{R}^{2n})$ to the space of continuous linear maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R})^n$. It also maps $L^2(\mathbb{R}^n)$ unitarily onto the space of* ***Hilbert-Schmidt operators*** *on $L^2(\mathbb{R}^n)$, and $\rho(F)$ is compact operator on $L^2(\mathbb{R}^n)$ fro all $F \in L^1(\mathbb{R}^n)$*

Before we delve into the proof we look at what this theorem says; the first claim says that we can extend, uniquely, the map ρ to the **Schwartz Space on** $\mathbb{R}^{2n}$ with the image of ρ giving a bounded linear map from Schwartz Space on $\mathbb{R}^n$ to the space of tempered distributions on $\mathbb{R}^n$.

The next claim brings out the compactness of the operator $\rho(F)$ and gives us a way to construct Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ from $L^2(\mathbb{R}^{2n})$ functions.

Proof. We need the kernel K_F to make sense, for F an arbitrary tempered distribution, the kernel (24) makes sense and so we get K_F as a tempered distribution. Now the first claim is just a simple application of **Schwartz Kernel theorem**(**Appendix D**). Explicitly for this case, we have the map

$$f \longrightarrow \int K_F(\cdot, y) f(y)\, dy$$

which connects $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R})^n$.

As for the second claim, we use the fact that on $L^2(\mathbb{R}^n)$, Fourier transform is unitary and so for any $F \in L^2(\mathbb{R}^n)$, $\rho(F)$ is a Hilbert-Schmidt operator with the kernel K_F. Finally, for $F \in L^1 \cap L^2$, $\rho(F)$ is compact and so for $F \in L^1, \rho(F)$ is compact as the norm limit of compact operators is compact and $\| \rho(F) \| \leq \| F \|_1$, where the norm on $\rho(F)$ is the operator norm. $\square$

Now we look at how the integrated representation interact with the original representation of H_n

Proposition 2.2. *For $F \in \mathcal{S}'(\mathbb{R}^{2n})$ and $p, q \in \mathbb{R}^n$, we have*

$$\rho(F)\rho(p,q) = \rho(G) \qquad \& \qquad \rho(p,q)\rho(F) = \rho(H)$$

where

$$G(a,b) = e^{\pi i(pb-qa)} F(a-p, b-q) \qquad \& \qquad H(a,b) = e^{\pi i(aq-bp)} F(a-p, b-q)$$

The proof is straight forward so we omit it.

We turn to the fact that convolution on $L^1(\mathsf{H}^n_{red})$ makes it into a Banach algebra and so we have

$$(F * G)(X) = \int F(Y)G(Y^{-1}X)\,dY = \int F(XY^{-1})G(Y)\,dY$$

and when the integrated representation of the convolution product is take we get that it is the product of the integrated representations, what we mean is

$$\rho(F * G) = \iint F(XY^{-1})G(Y)\rho(X)\,dX\,dY = \rho(F)\rho(G)$$

Now we want to transfer this algebraic structure to $L^1(\mathbb{R}^{2n})$ and from as previously seen the only remaining term is for $k = -1$ we get that only for the $k = -1$ it works and so using that as a starting point we have for $\phi \in L^1(\mathbb{R}^n)$

$$\rho(\phi) = \rho(\phi_0) \quad where \quad \phi_0(p,q,t) \in L^1(\mathsf{H}^n_{red}), \quad \phi_0(p,q,t) = \phi(p,q)e^{-2\pi t}$$

and so we get for $\phi, \psi \in L^1(\mathbb{R}^{2n})$

$$(\phi_0 * \psi_0)(p,q,t,) = e^{-2\pi i t} \iint \phi(a,b)\psi(p-a,q-b)e^{\pi i(aq-bp)}\,da\,db \tag{20}$$

That is, if we define a new "convolution" operation, $\circledast$, such that the above equation holds we get

$$(\phi_0 * \psi) = (\phi \circledast \psi)_0$$

Definition 2.3 (Twisted convolution)**.** *The twisted convolution is the operation given by*

$$(\phi \circledast \psi)(p,q) = \iint \phi(a,b)\psi(p-a,q-b)e^{\pi i(aq-bp)}\,da\,db$$

where ϕ, ψ are in $L^1(\mathbb{R}^{2n})$.

It is set up in such a way that we get the relation,

$$\rho(\phi \circledast \psi) = \rho(\phi)\rho(\psi)$$

Due to the non-commutativity of H^n, we see that the twisted convolution is also not commutative, but it has an even better result as compared to ordinary convolution

Proposition 2.3. *For $\phi, \psi \in L^2(\mathbb{R}^{2n})$ we have $\| \phi \circledast \psi \|_2 \leq \| \phi \|_2 \| \psi \|_2$*

It simply follows from **Theorem 2.3**, the fact that $\rho(\phi \circledast \psi) = \rho(\phi)\rho(\psi)$ and the Schwartz inequality.

2.4 Fourier-Wigner Transform

Now that we have a representation of H_n, the question is to ask what its matrix coefficients are and the closely related what the characters are of the representation ρ_h. First we define what, irreducible representations and matrix coefficients are

Definition 2.4 (Irreducible representations)**.** *An irreducible representation, say* Π*, of a group on the vector space* V*, is a group representation that has no nontrivial, invariant subspaces of* V*.*

What this means is that the only Π-invariant subspaces are the trivial subspace $\{0\}$ and the whole space V.

Definition 2.5. *Matrix coefficient of a group G is a complex valued function $f_{v,F}$ on G such that we have*

$$f_{v,F} = F \circ \rho \quad \textit{that is} \quad f_{v,F}(g) = F(\rho(g)v)$$

where ρ is a linear representation of the group G and F is a continuous linear functional on G

For our case let $f, g \in L^2(\mathbb{R}^n)$, then define the function

$$V(f,g)(p,q) = \langle \rho(p,q)f, g \rangle = \int e^{2\pi i qx + \pi i pq} f(x+p)\overline{g(x)}\, dx \tag{21}$$

We have omitted $e^{2\pi i t}$ as $(\rho(p,q,t)f)(x) = e^{2\pi i t}(\rho(p,q,0)f)(x)$ and so it doesn't add or remove any information available to us, hence for simplicity we choose to omit it. We call $V(f,g)(p,q)$ the **Fourier-Wigner transform**.

Proposition 2.4. *For $f, g \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$V(f,g)(p,q) = \int e^{2\pi i q\cdot x} f(x+\frac{p}{2})\overline{g(x-\frac{p}{2})}\,dx \tag{22}$$

Proof. Put $x = y - \frac{p}{2}$ in (21) to get (22) □

Proposition 2.5. *$V : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$ is a bilinear mapping.*

Proof. The bilinearity is a simple check, which is due to the inner product on $L^2(\mathbb{R}^n)$, as for the mapping from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$, we use the fact that for $h \in \mathcal{S}(\mathbb{R}^{2n})$, the function H on $\mathbb{R}^{2n}$ defined by

$$H(p,q) = \int_{\mathbb{R}^n} e^{2\pi i q\cdot y} h(p,y)\,dy$$

is in $\mathcal{S}(\mathbb{R}^{2n})$.
And so, setting $h(p,y) = f(y+\frac{p}{2})\overline{g(p-\frac{p}{2})}$ gives our claim □

This proposition then gives us the following.

Theorem 2.4. *V maps $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^{2n})$ and has a unique extension from $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^{2n})$. We also have that V is sesquilinear and unitary on $L^2(\mathbb{R}^n)$ and we have*

$$\langle V(f,g), V(\phi,\psi)\rangle = \langle f,\phi\rangle\overline{\langle g,\psi\rangle} \tag{23}$$

Now that we have a relation between the Fourier-Wigner transform and the function $f, g \in L^2$ we can easily show that ρ_h is an irreducible representation.

Theorem 2.5. *For $h \in \mathbb{R}\backslash\{0\}$, the representation ρ_h is irreducible.*

Proof. Let ρ_h be a reducible representation, then $\tilde{V}$ which is closed, non-zero and invariant under ρ_h is a proper subspace of L^2 and let $f \in \tilde{V}$ be a non-zero element. Then we have some $g \notin \tilde{V}$. If $g \perp \tilde{V}$ then $g \perp f$ and so $g \perp \rho_h(p,q)f$, that is $V(f,g)(p,q) = 0 \quad \forall p,q \in \mathbb{R}^n$. Also, by (23) we have

$$0 = \| V(f,g)(p,q) \|_2^2 = \| f \|_2^2 \| g \|_2^2$$

That is $g = 0$ and so $\tilde{V} = L^2(\mathbb{R}^n)$ □

Proposition 2.6. *For any $a,b,c,d \in \mathbb{R}^n$ we have*

$$V(\rho(a,b)f, \rho(c,d)g)(p,q) = e^{\pi i(dp+da-aq-cq-cb+bp)} V(f,g)(p+a-c, q+b-d) \tag{24}$$

Proof. We have

$$\begin{aligned}\langle \rho(p,q)\,\rho(a,b)f, \rho(c,d)g\rangle &= \langle \rho(-c,-d)\,\rho(p,q)\,\rho(a,b)f, g\rangle \\ &= \langle \rho(p+a-c, q+b-d, \frac{1}{2}(dp+da-aq-cq-cb+bp)f, g\rangle\end{aligned}$$

which gives us (24). □

As for matrix elements of the integrated representation we have

$$\begin{aligned}\langle \rho(\Phi)f, g\rangle = \int \rho(\Phi)f(x)\overline{g(x)}\,dx &= \int \left[\int \Phi(p,q)\rho(p,q)f(x)\,dp\,dq\right]\overline{g(x)}\,dx \\ &= \iint \Phi(p,q)V(f,g)(p,q)\,dp\,dq\end{aligned} \tag{25}$$

If $\Phi = \overline{V(f,g)}$, then we have

$$\langle \rho(\Phi)f, g\rangle = \iint \overline{V(f,g)(p,q)}V(f,g)(p,q)\,dp\,dq = \langle f, f\rangle\overline{\langle g, g\rangle}$$

by (21) and (23). So for $\phi \in L^2(\mathbb{R}^n)$ we have

$$\begin{aligned}
\langle \rho(\Phi)\phi, \psi \rangle &= \iint \overline{V(f,g)(p,q)} V(\phi,\psi)(p,q)\, dp\, dq \\
&= \langle \phi, f \rangle \overline{\langle \psi, g \rangle} \\
&= \langle \phi, f \rangle \langle g, \psi \rangle
\end{aligned}$$

which gives

$$\rho(\Phi) f = \langle \phi, f \rangle g \tag{26}$$

where $\Phi = \overline{V(f,g)}$

This means that if $\overline{\Phi}$ is a Fourier-Wigner transform then the range of operators $\rho(\Phi)$ on L^2 is one dimensional. This becomes useful when we want to get the twisted convolution of the Fourier-Wigner transforms.

For $\Phi_i = \overline{v(\phi_i, \psi_i)} \quad i = 1, 2$ we have

$$\overline{V(\phi_1, \psi_1)} \circledast \overline{V(\phi_2, \psi_2)} = \langle \psi_2, \phi_1 \rangle \overline{V(\phi_2, \psi_1)} \tag{27}$$

We'll now calculate the Fourier-Wigner transform of Gaussian, as they play an integral role in our theory.

Proposition 2.7. *For $\phi(x) = 2^{n/4} e^{-\pi x^2}$, $\Phi = V(\phi, \phi)$ and $\Phi^{ab} = V(\phi, \rho(a,b)\phi)$ we have*

i.) $\Phi(p,q) = e^{\frac{-\pi}{2}(p^2+q^2)}$

ii.) $\Phi^{ab}(p,q) = V(\phi, \rho(a,b)\phi)(p,q) = e^{\pi i(bp-aq)} e^{\frac{-\pi}{2}[(p-a^2)+(q-b)^2]}$

iii.) $\rho(\Phi)\rho(a,b)\rho(\Phi) = e^{\frac{-\pi}{2}(a^2+b^2)} \rho(\Phi)$

iv.) $\Phi \circledast \Phi^{ab} = e^{\frac{-\pi}{2}(a^2+b^2)} \Phi$

Proof. i.)

$$\Phi(p,q) = 2^{n/2} \int e^{2\pi iqx} e^{-\pi[x+(p/2)]^2 - \pi[x-(p/2)]^2}\, dx$$
$$= 2^{n/2} e^{(\frac{-\pi}{2})p^2} \int e^{2\pi iqy} e^{-2\pi x^2}\, dx = e^{\frac{-\pi}{2}(p^2+q^2)}$$

This comes directly from Appendix A

ii.) Follows from *i.)* and (24)

iii.) As Φ is the conjugate of a Fourier-Wigner transform we use (27).

iv.) $\rho(\Phi^{ab}) = e^{\pi i(bp-aq)} V(\phi,\phi)(p-a, q-b) = \rho(a,b)\rho(\Phi)$ by **Proposition 2.2**, and so $\rho(\Phi \circledast \Phi^{ab}) = \rho(\Phi)\rho(\Phi^{ab}) = \rho(\Phi)\rho(a,b)\rho(\Phi)$. Hence by *iii.)* we get

$$\rho(\Phi \circledast \Phi^{ab}) = e^{\frac{-\pi}{2}(a^2+b^2)} \rho(\Phi)$$

and as ρ is faithful, we get *iv.)*.

□

The Fourier-Wigner transform of the functions $f, g \in L^2(\mathbb{R}^n)$ can be seen as the inverse Fourier transform, about q of the function $f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)}$ and so we can ask about the Fourier transform of the Fourier-Wigner transform.

Definition 2.6 (Wigner transform)**.** *The Fourier transform of the Fourier-Wigner transform of two functions $f, g \in \mathcal{S}(\mathbb{R}^n)$, denoted by $\boldsymbol{W}$, is the transform*

$$W(f,g)(x,\xi) = \widehat{V(f,g)}(x,\xi) = (2\pi)^{-n/2} \int e^{-i\xi\cdot p} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)}\, dp$$

The Wigner transform satisfies the following identity

Theorem 2.6 (Moyal's Identity). *For $f, g, \phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\langle W(f,g), W(\phi,\psi)\rangle = \langle f,g\rangle\overline{\langle \phi,\psi\rangle}$$

This is the same identity as (23) but for Wigner transform.

Proof. If we put $F(a,b) = f(a)\overline{g(b)}$ and $G(a,b) = \phi(a)\overline{\psi(b)}$ for $a, b \in \mathbb{R}^n$ and define $U : \mathcal{S}(\mathbb{R}^{2n}) \to \mathcal{S}(\mathbb{R}^{2n})$ as

$$(UH)(x,\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^{2n}} e^{-i\xi\cdot p} H\left(y+\frac{p}{2}, y-\frac{p}{2}\right)\, dp \quad x,\xi \in \mathbb{R}^n$$

Then by using the Plancherel theorem, the inner product $\langle UF, UG\rangle$ can be simplified to get

$$\langle UF, UG\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F\left(x+\frac{p}{2}, x-\frac{p}{2}\right)\overline{G\left(x+\frac{p}{2}, x-\frac{p}{2}\right)}\, dp\, dx$$

Now the proof follows from the simple substitution $u = x + \frac{p}{2}$ and $v = x - \frac{p}{2}$. $\square$

We have a result which is parallel to **Proposition 2.6** for the Wigner transform

Proposition 2.8. 1. *For $a, b, c, d \in \mathbb{R}^n$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\begin{aligned} &W(\rho(a,b)f, \rho(c,d)g)(x,\xi) \\ &\qquad = e^{i\{(a-c)\cdot x+(b+d)\cdot\xi\}} e^{\frac{1}{2}i(a\cdot d-b\cdot c)} W(f,g)\left(x+\frac{b+d}{2}, \xi-\frac{a+c}{2}\right)\end{aligned}$$

2. *$W(g,f) = \overline{W(f,g)}$ $f, g \in \mathcal{S}(\mathbb{R}^n)$.*

The proofs of these two claims are pretty straight forward.

2.5 Stone-von Neumann Theorem

We have shown that ρ_h is a unitary (**Proposition 2.1**), irreducible (**Theorem 2.5**) representation of H^n for $h \neq 0 \in \mathbb{R}$. Now we can ask, if for some other irreducible, unitary representation, Π, of H^n do we have a relation connecting Π to ρ_h. This becomes the main theme of this section, the Stone-von Neumann theorem shows that if such a Π exists, whose image of the center of H^n is non-trivial, then it is unitarily equivalent to ρ_h.

Theorem 2.7 (Stone-vone Neumann Theorem)**.** *For a unitary representation* Π *of* H^n *on a Hilbert space* $\mathcal{H}$*, such that* $\Pi(0,0,k) = e^{2\pi i h k} I$*, for some* $h \in \mathbb{R}\backslash\{0\}$*. Then* $\mathcal{H} = \bigoplus \mathcal{H}_\alpha$*, where* $\mathcal{H}_\alpha$*'s are mutually orthogonal, invariant subspaces of* $\mathcal{H}$*, such that* $\Pi|_{\mathcal{H}_\alpha}$ *are unitary equivalent to* ρ_h *for each* α*.*

Proof. The idea of the proof is to construct elements in $\mathcal{H}$ that mimic the Gaussian function in the $L^2(\mathbb{R}^n)$ then these elements can be unitarily mapped to the Gaussian, which will give the map intertwining the two representations.

As before we have the following notation

$$\phi(x) = 2^{1/4} e^{-\pi x^2}, \quad \phi^{ab}(x) = \rho(a,b)\phi(x), \quad \Phi = V(\phi, \phi), \quad \Phi^{ab} = V(\phi, \phi^{ab})$$

and by **Proposition 2.7**, we have

$$\langle \phi^{pq}, \phi^{ab} \rangle = \Phi^{ab}(p,q), \quad \Phi \circledast \Phi^{ab} = e^{\frac{-\pi}{2}(a^2+b^2)} \Phi$$

For Π too, we can set $\Pi(p,q) = \Pi(p,q,0)$ and get $\Pi(p,q)\Pi(r,s) = e^{\pi i(ps-qr)}$ $\Pi(p+r, q+s)$.We also consider the integrated version of Π

$$\Pi(F) = \iint F(p,q)\Pi(p,q)\, dp\, dq \qquad\qquad (F \in L^1(\mathbb{R}^n))$$

and similarly we have

$$\Pi(F)\Pi(G) = \Pi(F \circledast G), \tag{28}$$

$$\Pi(F)\Pi(a,b) = \Pi(G) \qquad G(p,q) = e^{\pi i(aq-bp)}F(p-a,q-b) \tag{29}$$

$$\Pi(s,b)\Pi(F) = \Pi(H) \qquad H(p,q) =^{\pi i(bp-aq)} F(p-a,q-b) \tag{30}$$

As for $\Pi(F) = 0$, we have for any $u, v \in \mathbb{R}^n$,

$$0 = \langle \Pi(a,b)\Pi(F)\Pi(-a,-b)u, v\rangle = \iint e^{2\pi i(bp-aq)}F(p,q)\langle \Pi(p,q)u, v\rangle \, dp\, dq$$

which gives $F(p,q)\langle \Pi(p,q)u, v\rangle = 0$ for all u, v and a.e. (p,q), or $F = 0$ a.e., which means that Π is faithful

Now using the above mentioned properties we get the following

1. $\Pi(\Phi)\Pi(a,b)\Pi(\Phi) = \Pi(\Phi \circledast \Phi^{ab}) = e^{\frac{-\pi}{2}(a^2+b^2)}\Pi(\Phi)$, for $F = \Phi$ defined previously.

2. Φ is real and even and for $(a,b) = (0,0)$, we have that $\Pi(\Phi)^2 = \Pi(\Phi)$,

3. As $\Pi(F)$ is unitary, we have $\langle \Pi(F)f, g\rangle = \langle f, \Pi(F)g\rangle$ that is, $\Pi(F)$ is self adjoint.

So we get that $\Pi(\Phi)$ projects orthogonally onto $\mathcal{H}$, say $\mathcal{I}$ and is non-zero as it is faithful. We use the nice property that, for any (a,b) in range of $\mathcal{I}$, $a = \Pi(\Phi)a$, $b = \Pi(\Phi)b$. This helps us compute the inner product on the $\mathcal{I}$, using (28), (29) and (30), above.

So if we have an orthonormal basis for the $\mathcal{I}$, say f_i and let $\mathcal{I}_i$ be the closed linear span of $\Pi(\Phi)f_i$, then for $i \neq j$, we have

$$
\begin{aligned}
\langle \Pi(p,q)f, \Pi(r,s)g\rangle &= \langle \Pi(-r,-s)\Pi(p,q)\Pi(\Phi)f_i, \Pi(\Phi)f_j\rangle \\
&= e^{\pi i(ps-qr)}e^{\frac{-\pi}{2}[(p-r)^2+(q-s)^2]}\langle f_i, f_j\rangle \qquad (31)\\
&= 0
\end{aligned}
$$

Also every $\mathcal{I}_i$, is Π invariant by definition and so we can take $\mathcal{P} = (\bigoplus \mathcal{I}_i)^{\perp}$, which is also invariant under Π and $\mathcal{P} = 0$, using the previous argument repetitively.

We see that similarity in (31) and in **Proposition 2.7, (iv)**, giving the desired map $\Pi(a,b)f_i \longrightarrow \phi^{ab}$. Now it is straight forward that for any $u \in \mathcal{H}$ and $g \in L^2(\mathbb{R}^n)$, $\| u \|_{\mathcal{H}} = \| g \|_{L^2(\mathbb{R}^n)}$. We can extend the above map by linearity and continuity to get a unitary map from $\mathcal{H}$ to $L^2(\mathbb{R}^n)$.

The only thing remaining is to check that $\Pi|_{\mathcal{I}_i}$ is equivalent to ρ for any i. This observation is a simple consequence of the equality

$$\langle f^{ab}, f^{cd}\rangle = \langle \phi^{ab}, \phi^{cd}\rangle \qquad \text{for any } a, b, c, d$$

□

We'll now see how **Stone-von Neumann Theorem** helps us in classifying the irreducible, unitary representations of H_n. Before we give the proper characterisation, we define what an intertwining operator is.

Definition 2.7. *Given two unitary representations, say ρ_1 and ρ_2 of a group G, on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. An intertwining operator is a map $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, such that*

$$\rho_2(g)T = T\rho_1(g) \qquad \textit{for all} \quad g \in G$$

The set of all such T, intertwining ρ_1 to ρ_2, is denoted by $\mathcal{C}(\rho_1, \rho_2)$, and if $\mathcal{C}(\rho_1, \rho_2)$ has a unitary representation then ρ_1 and ρ_2 are said to be unitary equivalent.

The hypothesis that $\Pi(0,0,t) = e^{2\pi iht}I$, can be derived from **Schur's lemmas**(Appendix D); For Π an irreducible, unitary representation of H_n, on a Hilbert space $\mathcal{H}$, we have $\mathcal{C}(\Pi,\Pi) = \{cI : c \in \mathbb{C}\}$. It is also a unitary representation and so we have that $\Pi(X)\Pi(X)^* = I$. Also, we have that $\Pi(X) \in \mathcal{C}(\Pi,\Pi)$ for any $X \in \mathcal{Z}$ (the center of H_n).

Hence we get that $\Pi(X)$, $X \in \mathcal{Z}$, maps into $\{cI : c \in \mathbb{C},\ |c| = 1\}$ homeomorphically and so $\Pi(0,0,k) = e^{2\pi ikh}$ for some $h \in \mathbb{R}$.

- For $h \neq 0$, we have by **Stone-von Neumann theorem**, that Π is unitary equivalent to some ρ_h.

- For $h = 0$, we get that Π maps $\mathcal{Z}$ into the 0-operator, which basically means that Π maps $\mathsf{H}_n/\mathcal{Z} \simeq \mathbb{R}^{2n}$ into $\{cI : c \in \mathbb{C},\ |c| = 1\}$. Now we know that the irreducible representations of $\mathbb{R}^{2n}$ are just homomorphisms into the circle group, that is $(p,q) \longrightarrow e^{2\pi i(pa+bq)}$.

These observations, when written succinctly give us the following theorem

Theorem 2.8. *Every irreducible representation of H_n is unitarily equivalent to one and only one of the following*

1. $\rho_h : (h \in \mathbb{R}\backslash\{0\})$, *acting on* $L^2(\mathbb{R}^n)$

2. $\sigma_{ab}(p,q,t) = e^{2\pi i(ap+bq)}$: $(a, b \in \mathbb{R}^n)$ *acting on* $\mathbb{C}$

2.6 Fourier Transform on Heisenberg Group

We have completely classified the irreducible, unitary representation of H_n. Now we can ask the question of 'What will be the group Fourier transform on H^n?' For this we need a measure and a function space on H^n, as a set $\mathsf{H}^n = \mathbb{C}^n \times \mathbb{R}$, we can take the Lebesgue measure $(dz\,dx)$ to be Haar measure on H^n as it is left and right invariant, making H^n unimodular. The function space we take is the usual $L^p(\mathsf{H}^n)$.

The Fourier transform of $f \in L^1(\mathsf{H}^n)$ is the operator acting on $L^2(\mathbb{R}^n)$ as

$$\hat{f}(\lambda)g(x) = \int_{\mathsf{H}^n} f(p,q,t)[\rho(p,q,t)g](x)\,dp\,dq\,dt$$

where $g \in L^2(\mathbb{R}^n)$ and $\rho(p,q,t)$ is the Schrödinger representation.

The operator is bounded on $L^2(\mathbb{R}^n)$ as

$$|\langle \rho(p,q,t)g,h\rangle| \leq \| g \|_{L^2} \| h \|_{L^2}$$

and so

$$|\langle \hat{f}(\lambda)g,h\rangle| \leq \| f \|_{L^1} \| g \|_{L^2} \| h \|_{L^2}$$

Thus $\hat{f}$ satisfies the norm

$$\| \hat{f} \| \leq \| f \|_{L^1}$$

For a function on H^n, we can split it and write

$$f^\lambda(z) = \int_{-\infty}^{\infty} f(z,t)e^{i\lambda t}\,dt, \qquad z = p + iq$$

that is f^λ is the inverse Fourier transform of f in the t variable. We can plug this in the definition of $\hat{f}$ to get

$$\hat{f}(\lambda)g = \int_{\mathbb{C}^n} f^\lambda(z)\rho_\lambda(z)g\,dz$$

and so our operators of interest take the form

$$W_\lambda(f) = \int_{\mathbb{C}^n} f(z)\rho_\lambda(z)\,dz$$

for f being a suitable function on $\mathbb{C}^n$. For $\lambda = 1$, this is called the **Weyl transform**. The Weyl transform also arises in the study of pseudo-differential operators and so is an important bridge between our study and the study of pseudo-differential operators.

The two important properties of Weyl transform that we need are:

1. W_1 is a continuous map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

2. We have the relation

$$W_1(f)g(y) = \int_{\mathbb{C}^n} f(z)\,(\pi_1(z)g)\,(y)\,dz$$

   for $g \in L^2(\mathbb{R}^n)$ and f a function on $\mathbb{C}^n$.

From (1) above, we can extend the Weyl transform to $L^2(\mathbb{C}^n)$ and (2) tells us that the Weyl transform is an integral operator, which can be written explicitly as

$$W_1(g)\phi(y) = \int_{\mathbb{R}^{2n}} g(p+iq)e^{i(p\cdot y+\frac{1}{2}(p\cdot q))}\phi(y+q)\,dp\,dq$$

so the kernel of W_1 is given by

$$K_f(y,\eta) = \int_{\mathbb{R}^n} f(p,\eta-y)e^{\frac{i}{2}x\cdot(\eta+y)}\,dp$$

where $f(p,q) = f(p+iq)$. For sufficiently nice f, that is $f \in L^1 \cap L^2(\mathbb{C}^n)$, the kernel K_f is in $L^2(\mathbb{R}^n)$ and so $W_1(f)$ is a Hilbert-Schmidt operator.

Moving forward we will use $\rho(z,t)$ where $z = p + iq$ which we can split even further as $\rho_\lambda(z,0) = \rho_\lambda(z)$.

Let S_2 be the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ with inner product given by $\langle T,S\rangle = tr(TS^*)$. $L^2(\mathbb{R}^*, S_2, d\mu(\lambda))$ is the space of functions on $\mathbb{R}^*$, taking values in S_2 and are square integrable with respect to the measure $d\mu(\lambda)$ where $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n\,d\lambda$. Similar to the case in $\mathbb{R}^n$, the group Fourier transform on H^n gives an isomorphism between $L^2(\mathsf{H}^n)$ and the space $L^2(\mathbb{R}^*, S_2, d\mu(\lambda))$.

Theorem 2.9. *(Plancherel Theorem) The group Fourier transform is an isometric isomorphism between $L^2(\mathsf{H}^n)$ and $L^2(\mathbb{R}^*, S_2, d\mu(\lambda))$.*

Proof. We show the claim for $f \in L^1 \cap L^2(\mathsf{H}^n)$ and then by denseness we can extend it to all of $L^2(\mathsf{H}^n)$. We have for $f \in L^2(\mathsf{H}^n)$,

$$|\lambda|^{-1} \parallel \hat{f}(\lambda) \parallel_{HS}^2 = (2\pi) \int_{\mathbb{C}^n} |f^\lambda(z)|^2 \, dz$$

This can be derived from the Plancherel Theorem. But by definition of f^λ, we have

$$\int \parallel \hat{f}(\lambda) \parallel_{HS}^2 \; d\mu(\lambda) = \frac{1}{2\pi} \int \int_{\mathbb{C}^n} |f^\lambda(z)|^2 \, dz \, d\lambda$$

Again by Euclidean Plancherel theorem we get

$$\parallel \hat{f} \parallel_{L^2(\mathbb{R}^*, S_2, d\mu)} = \parallel f \parallel_{L^2(\mathsf{H}^n)}$$

$\square$

We can ask if there is an inversion formula also, the answer is positive. We will look at the case for $\lambda = 1$, the other cases will be a simple application of change of variable and Euclidean Fourier inversion. To this we observe that

$$\begin{aligned} \rho(w)^* W(g) &= \int_{\mathbb{C}^n} g(z) \rho(-w) \rho(z) \, dz \\ &= \int_{\mathbb{C}^n} g(z) e^{-i/2 \mathrm{Im}(w \cdot \overline{z})} \rho(z - w) \, dz \\ &= \int_{\mathbb{C}^n} g(z + w) e^{-i/2 \mathrm{Im}(w \cdot \overline{z})} \rho(z) \, dz \end{aligned}$$

where we have used $\rho(-w)\rho(z) = e^{-\frac{i}{2}\mathrm{Im}(w \cdot \overline{z})} \rho(z - w)$, in the second equality and a change of variables in the third one.

This shows that $\rho(w)^* W(g) = W(G_w)$, where

$$G_w(z) = g(z + w) e^{-\frac{i}{2}\mathrm{Im}(w \cdot \overline{z})}$$

and so the kernel of $W(G)$ is given by

$$K_w(\xi, \xi) = e^{\frac{i}{2}p\cdot q} e^{-ip\cdot\xi} \int_{\mathbb{R}^n} e^{ix\cdot(\xi - \frac{q}{2})} g(x, q)\, dx$$

Hence the formula $tr(\rho(w)^* W(g)) = \int_{\mathbb{R}^n} K_w(\xi, \xi)\, d\xi$ gives us

$$tr(\rho(w)^* W(g)) = e^{\frac{i}{2}p\cdot q} \int_{\mathbb{R}^n} e^{-p\cdot\xi} \left(\int_{\mathbb{R}^n} e^{ix\cdot(\xi - \frac{q}{2})} g(x, q)\, dx \right) d\xi$$

To define the trace of an operator on an Hilbert space, we need it to be of *trace class*, for the trace of an arbitrary operator need not be finite, much less be of the above form.

Definition 2.8 (Trace class)**.** *For a separable Hilbert space $\mathcal{H}$, with $\{e_i\}_{i=1}^{\infty}$ as the orthonormal basis, and for an operator $T : \mathcal{H} \to \mathcal{H}$ which is bounded on $\mathcal{H}$, then the trace of T is given by*

$$Tr(T) = \langle Te_i, e_i \rangle$$

and T is of trace class if we have

$$Tr(|T|) < \infty$$

where $|T| = \sqrt{T^*T}$ is the positive semi-definite Hermitian square root

For our case $\rho(w)$ and $W(g)$ are Hilbert-Schmidt operator and so their product is of trace class.

Now that the trace makes sense, we can combine it with the Euclidean Fourier inversion gives us

$$tr(\rho(w)^* W(g)) = (2\pi)^n g(w)$$

and so we have

Theorem 2.10. *(Fourier Inversion) For all $f \in \mathcal{S}(\mathsf{H}^n)$, we have the inversion formula*

$$f(z, t) = \int_{-\infty}^{\infty} tr(\rho(z, t)^* \hat{f}(\lambda))\, d\mu(\lambda)$$

As in the Euclidean case, the group Fourier transform takes convolution into products, and the space $L^1(\mathsf{H}^n)$ becomes a (non commutative) Banach algebra.

We will next introduce a different representation of H^n, though we'll stick to the Schrodinger representation, and see how it gives rise to functions which play a central role in our study.

3 Hermite functions and the sub-Laplacian

We saw the representations of H^n as operators on $L^2(\mathbb{R}^n)$. Parallelly, we can have a representation on the space of functions acting on the complex plane, which will give us much nicer functions to work with. This new space, called the **Fock space**, has the advantage that we can use complex analytic techniques on it to get fruitful information from it.

3.1 Bargmann transform and the Fock space

We have for $\phi_0(x) = 2^{\frac{n}{2}} e^{-\pi x^2}$, $\| \phi_0 \|_2 = 1$ and by **Proposition 2.7**, $f \longrightarrow V(f, \phi_0)$ is an isometry, that is

$$\begin{aligned} V(f,\phi_0)(p,q) &= \langle f, \rho(-p,-q)\phi_0 \rangle \\ &= 2^{\frac{n}{4}} \int f(x) e^{2\pi i q x - \pi i p q} e^{-\pi(x-p)^2}\, dx \\ &= 2^{\frac{n}{4}} e^{\frac{-\pi}{2}(p^2+q^2)} \int f(x) e^{2\pi x(p+iq) - \pi x - \frac{\pi}{2}(p+iq)}\, dx \end{aligned} \tag{32}$$

If we consider the realisation $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, and setting $z = p + iq$ in above (32), we can define

$$Bf(z) = 2^{\frac{n}{4}} \int f(x) e^{2\pi x z - \pi x^2 - \frac{\pi}{2} z^2}\, dx$$

Using the above properties we see that B is an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{C}^n, e^{-\pi|z^2|}\, dz)$ and for $f \in L^2(\mathbb{R}^n)$, $\quad Bf$ converges uniformly in any compact subset of $\mathbb{C}^n$, this gives that Bf is an entire function on $\mathbb{C}^n$. Here dz is the Lebesgue measure on $\mathbb{C}$ and so $e^{-\pi|z^2|}\, dz$ is a measure with a nice weight.

This function Bf on $\mathbb{C}^n$ is called the **Bargmann Transform** of the function f, and the image space of Bf, that is the space $L^2(\mathbb{C}^n, e^{-\pi|z|^2}\, dz)$ is called the **Fock space**.

$$\mathcal{F}_n = \left\{ F : F \,\text{is entire on}\, \mathbb{C}^n, \,\&\, \| F \|_{\mathcal{F}}^2 = \int |F(z)|^2 e^{-\pi|z|^2}\, dz < \infty \right\}$$

Now that we have made a connection between $L^2(\mathbb{R}^n)$ and the Fock space, which consists of entire functions on $\mathbb{C}^n$, we can use these nicer properties of the entire functions and get new properties for the representations of H_n. One of them is the existence of an orthonormal basis for $\mathcal{F}_n$.

Theorem 3.1. *Let*

$$\zeta_\alpha(z) = \sqrt{\frac{\pi^{|\alpha|}}{\alpha!}} z^\alpha$$

Then the family of functions $\{\zeta_\alpha : |\alpha| \geq 0\}$ *forms an orthonormal basis for* $\mathcal{F}_n$

Proof. The proof uses the Gamma function, for which we have

$$(z-1)! = \Gamma(z) = \int_o^\infty s^{z-1} e^{-s}\, ds, \qquad \text{for} z \in \mathbb{Z}^+$$

Using the standard inner product on $L^2(\mathbb{C}^n)$, we have

$$\begin{aligned}\sqrt{\frac{\alpha!\beta!}{\pi^{|\alpha|+|\beta|}}}\langle \zeta_\alpha, \zeta_\beta\rangle_{\mathcal{F}} &= \prod_1^n \int_{\mathbb{C}} z_j^{\alpha_j} \overline{z_j}^{\beta_j} e^{-\pi|z_j|^2}\, dz_j \\ &= \prod_1^n \int_0^\infty \int_0^{2\pi} e^{i\theta(\alpha_j-\beta_j)} r^{\alpha_j+\beta_j+1} e^{-\pi r^2}\, d\theta\, dr\end{aligned}$$

The integral is non zero in θ only for $\alpha_j = \beta_j$, for every j and so we are left with RHS being equal to

$$(2\pi)^n \prod_1^n \int_0^\infty r^{2\alpha_j+1} e^{-\pi r^2}\, dr = \prod_0^n \int_0^\infty \left(\frac{t}{\pi}\right)^{\alpha_j} e^{-\pi t}\, dt = \frac{\alpha!}{\pi^{|\alpha|}}$$

that is, we have $\frac{\alpha!}{\pi^{|\alpha|}} \parallel \zeta_\alpha \parallel_{\mathcal{F}}^2 = \frac{\alpha!}{\pi^{|\alpha|}}$, which gives the orthonormality of the set.

If we restrict our domain to just balls of radius r centered at 0, $B_r = \{z \in \mathbb{C}^n : |z| \leq r\}$ and let $a_{r,\alpha} = \pi^{-|\alpha|} \prod_1^n \int_0^r t^{\alpha_j} e^{-t}\, dt$, then $\{a_{r,\alpha}^{-1/2} z^\alpha\}$ is an orthonormal set in $L^2(\mathbb{C}^n, e^{-\pi|z|^2}\, dz)$. Now we use the fact that $F \in \mathcal{F}_n$ is entire and analytic. As F is analytic on B_r, we can write its Taylor series about 0, let it be $F(z) = \sum b_\alpha z^\alpha$, this series converges uniformly to F, on B_r for any radius r. We also have the α^{th} Fourier coefficient of F as

$$\langle F, a_{r,\alpha}^{-1/2} z^\alpha \rangle_{\mathcal{F}} = a_{r,\alpha}^{-1/2} b_\alpha \langle z^\alpha, z^\alpha \rangle = a_{r,\alpha}^{-1/2} b_\alpha$$

So by Parseval's identity, we have

$$\int_{B_r} |F(z)|^2 e^{-\pi|z|^2} \, dz = \sum a_{r,\alpha} |b_\alpha|^2$$

As $r \to \infty$, we have $\| F \|_{\mathcal{F}}^2 = \sum |\langle f, \zeta_\alpha \rangle_{\mathcal{F}}|^2$ □

A nice observation pops up in the proof of the theorem, it gives us that the Fourier series of $F \in \mathcal{F}_n$, with respect to the basis ζ_α, is the Taylor series of F. And we have, by Schwartz inequality for integrals, that

$$\begin{aligned} |F(z)| &= \left| \sum b_\alpha (\pi^{|\alpha|}/\alpha!)^{1/2} z^\alpha \right| \\ &\leq \left(\sum |b_\alpha|^2 \right)^{1/2} \left(\sum (\pi^{|\alpha|}/\alpha!) |z|^{2\alpha} \right)^{1/2} = \| F \|_{\mathcal{F}} \, e^{\frac{\pi}{2}|z|^2} \end{aligned} \tag{33}$$

This inequality shows that $F \longrightarrow F(z)$ is a bounded linear functional on $\mathcal{F}_n$, and so by **Riesz Representation Theorem** we have $E_z \in \mathcal{F}_n$ (the evaluation map) such that

$$F(z) = \langle F, E_z \rangle_{\mathcal{F}}$$

for which E_z can be easily obtained as

$$E_z(w) = \sum \langle E_z, \zeta_\alpha \rangle_{\mathcal{F}} \zeta_\alpha(w) = \sum \overline{\zeta_\alpha(z)} \zeta(w) = \sum (\pi^{|\alpha|} \overline{z}^\alpha w^\alpha / \alpha!) = e^{\pi w \overline{z}} \tag{34}$$

We have an observation which eases the determination of entire functions.

Proposition 3.1. *An entire function $E(z, \overline{w})$ of z and $\overline{w}$ is uniquely determined by its restriction to the diagonal $z = w$.*

Proof. Substituting u and v as functions of z and $\overline{w}$, we get $u = \frac{1}{2}(z + \overline{w})$ and $v = -\frac{1}{2} i (z - \overline{w})$ which gives us a function $G(u, v) = K(z, \overline{w})$. Hence, G is entire and given its values for real u and v, which is possible only for $w = z$, hence the claim. □

This can now be extended to a bounded operator on $\mathcal{F}_n$, for which we have that a bounded operator T, can be uniquely determined by $K_T(z, \overline{z}) = \langle TE_z, E_z \rangle$.

We now bring this property back to the Schrödinger representations. Hence, we have the transferred representation β as

$$\beta(p + iq, t)B = B\rho(p, q, t)$$

Calculating for β gives us

$$\beta(z, t)F(x) = e^{-\frac{\pi}{2}|z|^2 - \pi x\overline{z} + 2\pi it}F(x + z)$$

The magic comes for ϕ_0 (Gaussian) ,

$$B\phi_0(x) = 2^{n/2}e^{-\frac{\pi}{2}|x|^2} \int e^{-2\pi xy - 2\pi y^2}\, dx = 1 = E_0(x)$$

which gives for $z = r + is$

$$B(\rho(r, s)\phi_0)(x) = e^{-\frac{\pi}{2}|z|^2}E_{-z}(x)$$

giving E_w to lie in the range of B and so $B(L^2(\mathbb{R}^n)) = \mathcal{F}_n$ for we have $\langle F, E_w \rangle_{\mathcal{F}} = 0$ only for $F = 0$.

The important role of Fock-Bargmann representations are displayed when we take the inverse **Bargmann transform** for the functions that form the orthonormal basis for the Fock space, that is $B^{-1}\zeta_\alpha(x)$. These Bargmann inverted functions are called the *αth, normalized, n-dimensional* **Hermite functions** and denote them by $\tilde{h}_\alpha$.

Hermite functions play a huge role in the theoretical development of the Heisenberg group, they have been extensively studied for they have a very special property of giving an orthonormal basis for $L^2(\mathbb{R}^n)$ and the fact that they are eigenfunctions of the Fourier transform. Now we have motivation to define the inverse of the Bargmann transform.

We know that B is unitary and for a function in $F \in \mathcal{F}_n$, and $f \in L^2(\mathbb{R}^n)$ we have

$$\langle B^{-1}F, f\rangle = \langle F, Bf\rangle_{\mathcal{F}_n} = 2^{n/4} \iint F(z)\overline{f(z)} e^{2\pi x\overline{z} - \pi x^2 - (\pi/2)\overline{z}^2 - \pi |z|^2} \, dx \, dz$$

giving

$$B^{-1}F(x) = 2^{n/4} \int F(z) e^{2\pi x \overline{z}} \cdot e^{(\pi/2)\overline{z}^2 - \pi|z|^2} \, dz$$

if the integral makes sense, that is absolutely convergent. For F being a polynomial or satisfying the bound $|F(z)| \leq Ce^{k|z|^2}$, for some $k < \frac{\pi}{2}$; this need not be the case for a general function in the Fock space for those we can approximate the function by the Taylor series, apply the inverse transform to the terms of the series and then take the limit, in L^2 norm of the resulting functions. With this we can define the **Bargmann Kernel** as

$$B(z, x) = 2^{n/4} e^{2\pi x z - \pi x^2 - (\pi/2) z^2} \qquad z \in \mathbb{C}^n, \, x \in \mathbb{R}^n$$

and so using this we can simplify our transforms as

$$Bf(z) = \int B(z, x) f(x) \, dx, \qquad B^{-1}F(z) = \int B(\overline{z}, x) F(z) e^{-\pi |z|^2} \, dz$$

3.2 Hermite Functions

We now come to the **Hermite functions** which we motivated by the inverse Bargmann transform of the basis of the Fock space.

We will utilise the operators

$$A_j = \sqrt{\pi} B(X_j + iD_j) B^{-1} \qquad A_j^* = \sqrt{\pi}(X_j - iD_j) B^{-1}$$

Then for $w = r + is, \in \mathbb{C}$ we have define

$$\frac{\partial}{\partial w_j} = \frac{1}{2}\left(\frac{\partial}{\partial r_j} - i\frac{\partial}{\partial s_j}\right), \quad \frac{\partial}{\partial \overline{w_j}} = \frac{1}{2}\left(\frac{\partial}{\partial r_j} + i\frac{\partial}{\partial s_j}\right)$$

Hence, we have

$$\begin{aligned} A_j F &= \frac{\sqrt{\pi}}{2\pi i} B\left(\frac{\partial}{\partial s_j} + i\frac{\partial}{\partial s_j}\right)\rho(r,s)B^{-1}F|_{r=0=s} \\ &= \frac{1}{\sqrt{\pi}}\frac{\partial}{\partial w_j}\beta(w)F|_{w=0} \\ &= \frac{1}{\sqrt{\pi}}\frac{\partial F}{\partial z_j} \end{aligned}$$

and similarly for A^* we get

$$A^* F = \sqrt{\pi} z_j F$$

The domains of A_j and A_j^* are $\left\{F \in \mathcal{F}_n : \frac{\partial F}{\partial z_j} \in \mathcal{F}_n\right\}$ and $\{F \in \mathcal{F}_n : z_{jF \in \mathcal{F}_n}\}$, respectively. Moreover they satisfy commutation relations which are somewhat similar to our initial ones.

$$[A_j, A_k] = [A_j^*, A_k^*] = 0, \qquad [A_j, A_k^*] = \delta_{jk} I$$

Let e_j be the multi-index whose jth entry is 1 and all other entries are 0 then we have

$$A_j \zeta_\alpha = \sqrt{\alpha_j}\zeta_{\alpha - e_j}, \qquad A_j^* \zeta_\alpha = \sqrt{\alpha_j + 1}\zeta_{\alpha + e_j}$$

and $\zeta_{\alpha - e_j} = 0$ if $\alpha_j = 0$. Especially, we get

$$\zeta_\alpha = \frac{1}{\sqrt{\alpha!}}(A_{1^*})^{\alpha_1} \cdots (A_n^*)^{\alpha_n} \zeta_0$$

Using these we define operators

$$Z_j = \frac{1}{\sqrt{\pi}} B^{-1} A_j B, \qquad Z_j^* = \frac{1}{\sqrt{\pi}} B^{-1} A_j^* B$$

and for a multi-index α, we have $Z^\alpha = Z_1^{\alpha_1}, \cdots, Z_n^{\alpha_n}$ and similarly for Z^*. We then have

$$Z_j^* f(x) = x_j f(x) - \frac{1}{2\pi}\frac{\partial f}{\partial x_j} = -\frac{1}{2\pi} e^{\pi x^2} \frac{\partial}{\partial x_j}(e^{-\pi x^2} f(x))$$

and so

$$Z^{*\alpha} f(x) = \left(-\frac{1}{2\pi}\right)^{|\alpha|} e^{\pi x^2} \left(\frac{\partial}{\partial x}\right) (e^{-\pi x^2} f(x))$$

This can be further simplified by the fact that $\tilde{h}_0(x) = 2^{n/4} e^{-\pi x^2}$ and so we get

$$\tilde{h}_\alpha(x) = \sqrt{\frac{1}{\alpha!}}(B^{-1} A^{*\alpha} \zeta_0)(x) = \frac{2^{n/4}}{\sqrt{\alpha!}} \left(-\frac{1}{2\sqrt{\pi}}\right)^{|\alpha|} e^{\pi x^2} \left(\frac{\partial}{\partial x}\right)^\alpha (e^{-2\pi x^2})$$

In comparison to the classical Hermite function, this function varies by a constant and some scaling to the argument. More precisely, for $\tilde{h_j}(x) = (-1)^j e^{x^2/2} \frac{d^j}{dx^j} e^{-x^2}$, we have the relation

$$\tilde{h_j}(x) = \frac{2^{1/4}}{\sqrt{2^j j!}} h_j(\sqrt{2\pi} x)$$

Some properties of the Hermite functions are in order

1. We have for $x \in \mathbb{R}^n$ and α, a multi-index

$$\tilde{h}_\alpha(x) = \tilde{h}_{\alpha_1}(x_1) \cdots \tilde{h}_{\alpha_n}(x_n)$$

2. $H_\alpha(x) = 2^{n/4+|\alpha|}\sqrt{\frac{\pi^{|\alpha|}}{\alpha!}} x^\alpha +$ (terms with degree less than α). Every polynomial of degree $\leq k$ on $\mathbb{R}^n$ is a linear combination of Hermite polynomials of degree $\leq k$.

3. We have

$$Z_j \tilde{h}_\alpha = \sqrt{\frac{\alpha_j}{\pi}} h_{\alpha-1}, \qquad Z_j^* \tilde{h}_\alpha = \sqrt{\frac{\alpha_j + 1}{\pi}} h_{\alpha+1}$$

4.

$$2\pi(D^2 + X^2) = 2\pi x^2 - \frac{1}{2\pi}\frac{d^2}{dx^2}$$

is called the Hermite operator corresponding to our case, the standard Hermite operator is given by

$$x^2 - \left(\frac{d}{dx}\right)^2$$

5. The Hermite function, $\tilde{h}_k$ is the eigenfunction of the Hermite operator with eigenvalue $(2k+1)$. This gives us the eigenfunctions and eigenvalues of the n-dimensional Hermite operator as

$$2\pi(D^2 + X^2)\tilde{h}_\alpha = (2|\alpha| + n)\tilde{h}_\alpha$$

6. $\{\tilde{h}_\alpha\}$ form an orthonormal base of $L^2(\mathbb{R}^n)$.

7. Hermite functions are also the eigenfunctions of the Fourier transform, that is

$$\mathcal{F}\tilde{h}_\alpha = (-1)^{|\alpha|}\tilde{h}_\alpha$$

We will see that these Hermite functions will be the basic tools that we will be using for our study of functions and operators on the Heisenberg Group.

We now use the Fourier-Wigner transform to define the **special Hermite functions**

$$\begin{aligned}\Phi_{\alpha,\beta}(z) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi} h_\alpha\left(\xi + \frac{y}{2}\right) h_\beta\left(\xi - \frac{y}{2}\right)\, d\xi \\ &= V(h_\alpha, h_\beta)(z) \qquad \text{for } z = x + iy\end{aligned} \tag{35}$$

Proposition 3.2. *The functions $\Phi_{\alpha,\beta}$ form a complete orthonormal system for $L^2(\mathbb{C}^n)$.*

Proof. The orthonormality is a direct consequence of the Hermite functions and the Fourier-Wigner transform. The completeness uses the fact that h_α forms an orthonormal basis of $L^2(\mathbb{R}^n)$, the Weyl transform and the Plancherel theorem for the Weyl transform. □

Next we define a Laguerre polynomial of type $\alpha > -1$ as

$$L_m^\alpha(t)e^{-t}t^\alpha = \frac{1}{m!}\left(\frac{d}{dt}\right)^m (e^{-t}t^{m+\alpha})$$

and more generally we have

$$L_m^\alpha(z) = \prod_{i=1}^n L_{m_i}^{\alpha_i}\left(\frac{1}{2}|z_i|^2\right)$$

and so the special Hermite functions can be written in terms of these Laguerre functions as

$$\Phi_{\alpha,\alpha}(z) = (2\pi)^{-n/2}\prod_{i=1}^n L_{\alpha_j}\left(\frac{1}{2}|z_j|^2\right) e^{\frac{-1}{4}|z_j|^2}$$

The reason to introduce these new families of functions is to use the spectral decomposition of certain nice operators. The special Hermite functions are eigenfunctions of the operator

$$L = -\Delta_z + \frac{1}{4}|z|^2 - i\sum_{i=1}^n\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)$$

here Δ_z is the standard Laplacian on $\mathbb{C}^n$. The special Hermite functions then satisfy

$$L(\Phi_{\alpha,\beta}) = (2|\beta| + n)\Phi_{\alpha,\beta}$$

The important thing to note here is the fact that the eigenvalue only depends on the value of $|\beta|$ and is independent of α. So for a fixed $|\beta| = k$ we can define it's $k-$th eigenspace as the span of $\{\Phi_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}^n, |\beta| = k\}$. Obviously, this is an infinite dimensional space.

We can now expand functions in $L^2(\mathbb{C}^n)$ using the special Hermite expansion as

$$f(z) = \sum_\alpha\sum_\beta\langle f, \Phi_{\alpha,\beta}\rangle\Phi_{\alpha,\beta}$$

This series converges in the L^2 norm.

The Laguerre functions on the other hand are related to the sub-Laplacian on H^n as we will see and using its eigenfunctions we will get a similar decomposition of functions in $L^2(\mathsf{H}^n)$.

3.3 Laguerre functions

We saw that special Hermite functions form an orthonormal system in $L^2(\mathbb{C}^n)$ and they can be expressed in the form of Laguerre functions, so it is natural to assume that Laguerre functions would satisfy the same. We first give the definition of the Laguerre function.

Definition 3.1 (Laguerre functions)**.** *The family of functions defined recursively by*

$$L_0(x) = 1, \qquad L_1(x) = 1 - x$$
$$L_{i+1}(x) = \tfrac{(2i+1-x)L_i(x) - iL_{i-1}(x)}{i+1}$$

We can also write it in the closed form as

$$L_k(x) = \sum_{i=0}^{n} \binom{k}{i} \frac{(-1)^i}{i!} x^i$$

The Laguerre polynomials are the solutions to the second order linear differential equation, the **Laguerre differential equation** given by

$$xy'' + (1-x)y' + ny = 0$$

Lemma 3.1. *We have the orthogonality relation*

$$\int_0^\infty L_k^\alpha(t) L_j^\alpha(t) e^{-t} t^\alpha \, dt = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)} \delta_{jk}$$

Proof. We first integrate $L_k^\alpha(t)t^\alpha e^{-t}$ with a polynomial function, f. Then by definition of $L_k^\alpha(t)$ and repeated application of integration by parts we get

$$\int_0^\infty f(t) L_k^\alpha(t) e^{-t} t^\alpha \, dt = \frac{1}{k!} \int_0^\infty f^{(k)}(t) e^{-t} t^{k+\alpha} \, dt$$

If the degree of f is less than k then the integral is 0. Hence we can choose $\tilde{k} = \max(j,k)$ and get value for $j \neq k$.

For $j = k$ we put $f(t) = L_k^\alpha(t)$ and then the $k-$th derivative of $f(t)$ is $(-1)^k$. Hence giving

$$\int_0^\infty L_k^\alpha(t) L_j^\alpha(t) e^{-t} t^\alpha \, dt = \frac{1}{k!} \int_0^\infty t^{k+\alpha} e^{-t} \, dt$$

and we are done. □

We also have a generating function identity, for $|w| < 1$, for the Laguerre functions

$$\sum_{k=0}^\infty L_k^\alpha(t) w^k = \left(\frac{1}{1-w}\right)^{\alpha+1} e^{-\frac{w}{1-w}t}$$

This can be seen from the definition of $L_k^\alpha(t)$ and the binomial expansion of $(1-w)^{-(\alpha+1)}$.

Using this identity we have

$$\sum_{|\beta|=k} \Phi_{\beta,\beta}(z) = (2\pi)^{-n/2} L_k^{n-1}\left(\frac{1}{2}|z|^2\right) e^{-|z^2|/4}$$

if we let

$$\varphi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right) e^{-|z|^2/4}$$

then we have

Proposition 3.3. *For a function $f \in L^2(\mathsf{H}^n)$, we have*

$$f = (2\pi)^{-n} \sum_{k=0}^\infty f \circledast \varphi_k$$

Proof. For $f, g \in L^2(\mathbb{R}^n)$, we have

$$\langle V(\overline{\Phi}_{\alpha,\beta}) f, g\rangle = (2\pi)^{n/2} \langle V_f(g), \Phi_{\alpha,\beta}\rangle = (2\pi)^{n/2} \langle f, h_\alpha\rangle \langle h_\beta, g\rangle$$

and so we have

$$\begin{aligned}\langle W(\overline{\Phi}_{\alpha,\beta} \circledast \overline{\Phi}_{\mu,\nu})f, g\rangle &= (2\pi)^{n/2}\langle W(\overline{\Phi}_{\mu,\nu})f, h_\alpha\rangle\langle h_\beta, g\rangle \\ &= (2\pi)^{n/2}\langle f, h_\mu\rangle\langle h_\nu, h_\alpha\rangle\langle h_\beta, g\rangle\end{aligned}$$

because of orthogonality of $\{h_\alpha\}$, we get

$$\overline{\Phi}_{\alpha,\beta} \circledast \overline{\Phi}_{\mu,\nu} = (2\pi)^{n/2}\delta_{\beta,\mu}\Phi_{\alpha,\nu}$$

or

$$\Phi_{\alpha,\beta} \circledast \Phi_{\mu,\nu} = (2\pi)^{n/2}\delta_{\alpha,\nu}\Phi_{\beta,\mu}$$

Let the projection of f onto the eigenspace spanned by $\{\Phi_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}^n, |\beta| = k\}$, given by $Q_k f$ is

$$Q_k f = \sum_{|\beta|=k} \sum_{\alpha} \langle f, \Phi_{\alpha,\beta}\rangle\Phi_{\alpha,\beta}$$

If we now use the fact that $\Phi_{\alpha,\beta} \circledast \Phi_{\mu,\nu} = (2\pi)^{n/2}\delta_{\alpha,\nu}\Phi_{\beta,\mu}$ on f and its projection, we get

$$f \circledast \Phi_{\beta,\beta} = \sum_{\alpha} \langle f, \Phi_{\alpha,\beta}\rangle\Phi_{\alpha,\beta}$$

which in view definition of $\varphi_k(z)$ gives the claim. □

We also have

$$W(\Phi_{\alpha,\alpha})f = (2\pi)^{n/2}\langle f, h_\alpha\rangle h_\alpha$$

and so we can say that the projection of $L^2(\mathbb{R}^n)$ onto the $k-$th eigenspace ($\{h_\alpha : |\alpha| = k\}$) will be given by the Weyl transform of ϕ_k acting on the function, that is

$$W(\varphi_k)f = (2\pi)^{n/2} \sum_{|\alpha|=k} \langle f, h_\alpha\rangle h_\alpha = (2\pi)^n Q_k f$$

For radial functions, the situation becomes even simpler, for them the Weyl transform is nothing but the Laguerre transform. Consider

$$\psi_k(r) = \left(\frac{2^{1-n}k!}{(k+n+1)!}\right)^{1/2} L_k^{n-1}\left(\frac{1}{2}r^2\right) e^{-r^2/4}$$

These functions form an orthonormal basis for the space $L^2(\mathbb{R}^+, r^{2n-1}dr)$, and so we can expand $f(z) = f(r), r = |z|$ as

$$f(r) = \sum_{k=0}^{\infty} \left(\int_0^{\infty} f(s)\psi_k(s)s^{2n-1}\, ds\right) \psi_k(r)$$

Parallelly, we can write

$$f(z) = \sum_{k=0}^{\infty} R_{k(f)}\varphi_k(z)$$

where

$$R_k(f) = \frac{2^{1-n}k!}{(k+n-1)!} \int_0^{\infty} f(s)\varphi_k(s)s^{2n-1}\, ds$$

Orthogonality simplifies it even further to give

$$f \circledast \varphi_k = (2\pi)^n \sum_{k=0}^{\infty} R_k(f)P_k$$

This result and the fact that Weyl transform turn twisted convolutions to multiplication motivates the claim.

Theorem 3.2. *For f a radial, in the complex variable, function on H^n ($f \in L^1(\mathsf{H^n})$) we have*

$$\hat{f}(\lambda) = \sum_{k=0}^{\infty} R_k(\lambda, f)Q_k(\lambda)$$

where $R_k(\lambda, f) = R_k(f^\lambda)$

Proof.

$$\hat{f}(\lambda) = \int_{\mathsf{H}^n} f(w,t)\rho_\lambda(w,t)\, dw\, dt$$

Putting $\varphi_k^\lambda(z) = L_k^{n-1}\left(\frac{1}{2}|\lambda||z|^2\right)e^{-|z|^2/4}$ and $f^\lambda(z) = \int f(z,t)e^{i\lambda t}\,dt$ we see that

$$Q_k(\lambda) = \int_{\mathbb{C}^n} \varphi_k^\lambda(z)\rho_\lambda(z)\,dz$$

Now that we have all the ingredients we just have to put them together to get the claim.

□

Now that we have some working knowledge of the special Hermite functions and Laguerre functions, we can now move to study the sub-Laplacian.

3.4 The sub-Laplacian on H^n

The Laplacian on $\mathbb{R}^n$, Δ, is a translation and rotation invariant differential operator that is homogeneous of degree 2.

Definition 3.2. *An operator T is of homogeneous of degree α if we have*

$$T(f(rz)) = r^\alpha T(f(z)) \qquad r > 0, z \in \mathbb{R}^n$$

Definition 3.3. *A Laplacian is a second order differential operator Δ on a group G, such that*

1. *It is bi-invariant.*
2. *It is elliptic.*
3. *It is formally self-adjoint.*
4. *It maps constant functions to zero.*

For the Heisenberg group, left translations are given by

$$L_g f(w) = f(g^{-1}w) \qquad g, h \in \mathsf{H}^n$$

and rotations by

$$R_k f(w,t) = f(kw,t) \qquad k \in U(t)$$

where $U(t)$ is the group of $n \times n$ unitary matrices,and dilations by

$$\delta_r f(w,t) = f(rw, r^2 t)$$

clearly rotations and dilations are automorphisms of H^n. A differential operator is said to be left invariant if it commutes with L_g, for any element g of the Heisenberg group. Similarly, it is said to be rotation invariant if it commutes with R_k, for any k in H^n. The homogeneity definition remains the same but as the dilation is non-isotropic on H^n, we say a differential operator, T, is of homogeneous degree α if $T(f(\delta_r g)) = r^\alpha T(f(g))$. The sub-Laplacian (or Kohn-Laplacian) is the unique, up to a constant multiple, left invariant, rotation invariant, differential operator of homogeneous degree 2.

The sub-Laplacian is explicitly given by

$$\mathcal{L} = -\sum_{j=1}^{n} (X_j^2 + Y_j^2)$$

where X_j and Y_j are the left-invariant vector fields introduced earlier. The expression of sub-Laplacian can be further simplified to get

$$\mathcal{L} = -\Delta_z - \frac{1}{4}|z|^2 \partial_t^2 + R\partial_t$$

where Δ_z is the Laplacian on $\mathbb{C}^n$ and

$$R = \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right)$$

We will look at the joint spectrum of $\mathcal{L}$ and $T = \partial_t$, which commutes and so the joint spectrum is well defined. The definition and the properties of joint spectrum are out of the scope our study, we will just take it as $(\mathcal{L}, \partial_t)$.

For X, a left invariant vector field on H^n, and ρ is a representation , we can take $\rho(X)$ to be the skew-adjoint operator on C^∞ vectors of the representation. If ρ acts on a Hilbert space $\mathcal{H}$ then for an element, P, of the universal enveloping algebra of left invariant differential operators on H^n, we have

$$P(\langle \rho(g)u, v\rangle) = \langle \rho(g)\rho(A)u, v\rangle \qquad g \in \mathsf{H}^n \text{ and } u, v \in \mathcal{H}$$

Using this, we consider the eigenfunctions of P which can be split into the cases when P is in the center of the universal enveloping algebra and when it is not.

For P in the center, by Schur's relations we have $\rho(P) = \lambda I$, whenever ρ is irreducible. Hence any element $\langle \rho(g)u, v\rangle$ is an eigenfunction of of P.

For P not in the center we have, $\langle \rho(g)u, v\rangle$ is the eigenfunction whenever u an eigenfunction of $\rho(p)$.

The sub-Laplacian is left-invariant differential operator and so we can take $P = \mathcal{L}$ for the above case and then calculate $\rho(\mathcal{L})$. In the Schrodinger representation we have

$$\rho_\lambda(X_j)f(x) = -i\lambda x_j f(x)$$

and

$$\rho_\lambda(Y_j)g(x) = \frac{\partial}{\partial x_j} g(y)$$

hence giving

$$\rho_\lambda(\mathcal{L}) = -\Delta + \lambda^2|x|^2 = H(\lambda)$$

$H(\lambda)$ is called the scaled Hermite operator. It has

$$h_\alpha^\lambda(x) = |\lambda|^{n/4} h_\alpha(|\lambda|^{1/2}x)$$

as the eigenfunctions with eigenvalues $(2|\alpha| + n)|\lambda|$.

Hence we get that $\langle \rho_\lambda(w,t)h_\alpha^\lambda, h_\beta^\lambda\rangle$ are the eigenfunctions of the sub-Laplacian, with the same eigenvalues, $(2|\alpha|+n)|\lambda|$. Now we bring in the special Hermite functions which we defined before, to get

$$\langle \rho_\lambda(w,t)h_\alpha^\lambda, h_\beta^\lambda\rangle = (2\pi)^{n/2} e^{i\lambda t}\Phi_{\alpha,\beta}(\sqrt{\lambda}w) \qquad \text{for } \lambda > 0$$

and

$$\langle \rho_\lambda(w,t)h_\alpha^\lambda, h_\beta^\lambda\rangle = (2\pi)^{n/2} e^{i\lambda t}\overline{\Phi}_{\alpha,\beta}(\sqrt{-\lambda}w) \qquad \text{for } \lambda < 0$$

At the same time we see that,

$$iT(\langle \rho_\lambda(w,t)h_\alpha^\lambda, h_\beta^\lambda\rangle) = -\lambda\langle \rho_\lambda(w,t)h_\alpha^\lambda, h_\beta^\lambda\rangle$$

that is, $\langle \rho_\lambda(w,t)h_\alpha^\lambda, h_\beta^\lambda\rangle$ are also the eigenfunctions of iT.

We substitute

$$\varphi_k(w) = (2\pi)^{n/2}\sum_{|\alpha|=k}\Phi_{\alpha,\alpha}(w)$$

For radial functions, we then have $e^{i\lambda t}\varphi_k(\sqrt{|\lambda|}w)$ as the radial eigenfunctions of $\mathcal{L}$, and so we substitute further

$$e_k^\lambda(w,t) = e^{i\lambda t}\varphi_k(\sqrt{|\lambda|}w)$$

After all this, we have that $e_k^\lambda(w,t)$ are the joint eigenfunctions of $\mathcal{L}$ and T. The joint spectrum is called the **Heisenberg fan**, given by

$$S = \{(\lambda, (2k+n)|\lambda|) : \lambda \in \mathbb{R}^+, k \in N\}$$

Theorem 3.3. *For $f \in L^2(\mathsf{H}^n)$ we have*

$$f(w,t) = \sum_{k=0}^{\infty} f * e_k^\lambda(w,t)\, d\mu(\lambda)$$

$d\mu(\lambda) = |\lambda|^{-n-1}(2\pi)^{-n-1}$ is the Plancherel measure on H^n.

The proof doesn't introduce any novel ideas, it is a collective application of the previously defined theorems and decomposition.

Proof. By Fourier inversion we have

$$f(w,t) = \int_{\infty}^{\infty} tr(\rho_\lambda(w,t)^* \hat{f}(\lambda))\, d\mu(\lambda)$$

Let $\lambda > 0$, for $\lambda < 0$ the proof is similar, then we have due to orthonormality of h_α^λ

$$tr(\rho_\lambda(w,t)^* \hat{f}(\lambda)) = e^{i\lambda t} \sum_\alpha \langle \rho_\lambda(w)^* \hat{f}(\lambda) h_\alpha^\lambda, h_\alpha^\lambda \rangle$$

Plugging in $\hat{f}$ and expanding $\rho_\lambda(w) h_\alpha^\lambda(x)$ in terms of h_β^λ and then simplifying using special Hermite functions we get

$$tr(\rho_\lambda(w,t)^* \hat{f}(\lambda)) = (2\pi)^n e^{i\lambda t} \sum_\alpha \sum_\beta \left(\int_{\mathbb{C}^n} f^\lambda(z) \Phi_{\alpha,\beta}(\sqrt{\lambda} z)\, dz \right) \overline{\Phi}_{\alpha,\beta}(\sqrt{\lambda} w)$$

This is the especial Hermite expansion of $F^\lambda(z)$ and so we can write it as

$$\sum_\alpha \sum_\beta \left(\int_{\mathbb{C}^n} f^\lambda(z) \Phi_{\alpha,\beta}(\sqrt{\lambda} z)\, dz \right) \overline{\Phi}_{\alpha,\beta}(\sqrt{\lambda} w) = (2\pi)^{-n} \sum_{k=0}^{\infty} f^\lambda *_\lambda \varphi_k^\lambda(w)$$

Hence we get

$$tr(\rho_\lambda(w,t)^* \hat{f}(\lambda)) = \sum_{k=0}^{\infty} f * e_k^\lambda(w,t)$$

Now just plug it back into the Fourier inversion formula to get the claim. □

Similar to the Euclidean case, we also have an equality connecting the L^2 norm of f and the "Fourier coefficients", that is the Plancherel theorem for the previous expansion

Theorem 3.4. *For $f \in L^2(\mathsf{H}^n)$ we have the following equality*

$$\| f \|_2^2 = (2\pi)^{-2n-1} \sum_{k=0}^{\infty} \int_{\infty}^{\infty} \int_{\mathbb{C}^n} |f * e_k^\lambda(z,0)|^2 \lambda^{2n}\, dz\, d\lambda$$

Proof. We show the claim holds for $\lambda > 0$, for $\lambda < 0$ a similar calculation holds. We have,

$$tr(\hat{f}(\lambda)^* \hat{f}(\lambda)) = \int_{\mathbb{C}^n} \overline{f}(z,t) tr(\rho_\lambda(z,t)^* \hat{f}(\lambda))\, dz\, dt$$

and in view of previous theorem along with the special Hermite expansion of f^λ we get

$$tr(\hat{f}(\lambda)^* \hat{f}(\lambda)) = (2\pi)^{-n} \lambda^n \sum_{k=0}^{\infty} \int_{\mathbb{C}^n} |f^\lambda *_\lambda \varphi_k^\lambda(z)|^2\, dz$$

Now we apply the Plancherel theorem to get

$$\| f \|_2^2 = (2\pi)^{-2n-1} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} |f^\lambda *_\lambda \varphi_k^\lambda(z)|^2 \lambda^{2n}\, dz\, d\lambda$$

and now use

$$f^\lambda *_\lambda \varphi_k^\lambda(w) = f * e_k^\lambda(w, 0)$$

□

4 Littlewood-Paley-Stein theory

Now that we have some idea of the problem we will look into the machinery required to tackle it.

For the case of L^2 we have nice orthogonality relations which help us compare the norm of the function with the norm of its Fourier transform. For $p \neq 2$ this comparison might not be even possible. For us to get the estimates we look at the theory of Littlewood and Paley.

Example 4.1. *Let ζ be a Schwartz space function whose Fourier transform is positive and supported in $\left(-\frac{1}{4}, \frac{1}{4}\right)$. For a function, g_m whose Fourier transform is translated by m from that of the Fourier transform of ζ, that $\hat{g}_m(\xi) = \hat{\zeta}(\xi - m)$. Hence we have $g_m(x) = e^{2\pi i m\cdot x}\zeta(x)$ and we can have $\hat{g}_m$ to have disjoint supports and so for sufficiently large N we'll have*

$$\sum_{m=0}^{N} \| g_m \|_{L^p}^p = (N+1) \| \zeta \|_{L^p}^p$$

Also,

$$\begin{aligned}
\| \sum_{m=0}^{N} g_m \|_{L^p}^p &= \int_{\mathbb{R}^n} \left| \frac{e^{2\pi i(N+1)x} - 1}{e^{2\pi i x} - 1} \right|^p |\zeta(x)|^p \, dx \\
&\geq c \int_{|x| < \frac{1}{10}(N+1)^{-1}} \frac{(N+1)^p |x|^p}{|x|^p} |\zeta(x)|^p \, dx \\
&= C_\zeta (N+1)^{p-1}
\end{aligned}$$

ζ is non-zero in a neighbourhood of 0 and so

$$\| \sum_m g_m \|_{L^p}^p \leq \; C_p \sum_m \| g_m \|_{L^p}^p$$

will not hold for g_m for $p > 2$.

Definition 4.1. *Let Ψ be an integrable function on $\mathbb{R}^n$ and $j \in \mathbb{Z}$. Then we define the*

Littlewood-Paley operator δ_j associated to Ψ as

$$\delta_j(f) = f * \Psi_{2^{-j}}, \qquad \textit{where } \Psi_{2^{-j}}(x) = 2^{jn}\Psi(2^j x)$$

Thus we have

$$\widehat{\Psi_{2^{-j}}}(\xi) = \hat{\Psi}(2^{-j}\xi)$$

for all ξ in $\mathbb{R}^n$. If Ψ is a Schwartz space function and f is a tempered distribution, then $\delta_j(f)$ is well defined.

Say $\hat{\Psi}$ is supported in some annulus $0 < a < |\xi| < b < \infty$ then Fourier transform of the Littlewood-Paley operator will be supported in the annulus $a2^j < |\xi| < b2^j$. This means that δ_j is removing all frequencies except for those contained in $a2^j < |\xi| < b2^j$. We can bring a and b closer to get the values for frequencies close to 2^j.

The map

$$f \mapsto \left(\sum_{j\in\mathbb{Z}} |\delta_j(f)|^2 \right)^{1/2}$$

is called the square function associated with the Littlewood-Paley operator. Using this map we can get the L^p orthogonality relations which are given by the Littlewood-Paley theorem.

Theorem 4.1 (Littlewood-Paley)**.** *For Ψ an integrable $\mathcal{C}^1$ function on $\mathbb{R}^n$, whose mean value is 0 and which satisfies the condition*

$$|\Psi(x)| + |\nabla\Psi(x)| \;\leq\; B(1+|x|)^{-n-1}$$

Then we have the following

1. *For any function f in L^p and p, $1 < p < \infty$, we have a constant $C_n < \infty$ such that*

$$\| \left(\sum_{j\in\mathbb{Z}} |\delta_j(f)|^2 \right)^{1/2} \|_{L^p} \;\leq\; C_n B \max\left(p, (p-1)^{-1} \| \; f \; \|_{L^p(\mathbb{R}^n)}\right)$$

2. For any f in $L^1(\mathbb{R}^n)$ we have a constant $C'_n < \infty$ such that

$$\| \left(\sum_{j \in \mathbb{Z}} |\delta_j(f)|^2 \right)^{1/2} \|_{L^{1,\infty}(\mathbb{R}^n)} \leq \; C'_n B \parallel f \parallel_1$$

Conversely, let Ψ be a Schwartz space function such that it

1. $\hat{\Psi}$ is compactly supported away from the origin and

$$\sum_{j \in \mathbb{Z}} |\hat{\Psi}(2^{-j}\xi)|^2 = 1 \qquad \xi \in \mathbb{R}^n \setminus \{0\}$$

or

2. $\hat{\Psi} = 0$ and

$$\sum_{j \in \mathbb{Z}} |\hat{\Psi}(2^{-j}\xi)|^2 = 1 \qquad \xi \in \mathbb{R}^n \setminus \{0\}$$

Then for any given tempered distribution f such that the function $\left(\sum_{j \in \mathbb{Z}} |\delta_j(f)|^2 \right)^{1/2}$ is in L^p for some $1 < p < \infty$, we have a unique polynomial P such that $f - P$ coincides with an L^p function and satisfies the bound

$$\parallel f - P \parallel_{L^p(\mathbb{R}^n)} \leq \; CB \max\left(p, (p-1)^{-1}\right) \| \left(\sum_{j \in \mathbb{Z}} |\delta_j(f)|^2 \right)^{1/2} \|_{L^p(\mathbb{R}^n)}$$

where C is a constant that depends on n and Ψ.

The proof of this theorem is quite involved and doesn't help us for we are working in a different space. But we'll prove the Littlewood Paley theorem for compact groups, after developing the required machinery. The proof of this theorem can be found in any standard text.

4.1 Littlewood-Paley Theorem

Before we state and proof the Littlewood-Paley theorem for compact groups we need some definitions.

Moreover, Δ may be taken to be of the form $\sum_{i,j=1}^{n} a_{ij} X_i X_j$, for (a_{ij}) being a strictly positive-definite, constant matrix and $X_1 \cdots X_n$ form the basis of the Lie algebra of G.

Note that the uniqueness is not a part of the definition. For $\mathbb{R}^2$ also, $\frac{\partial^2}{\partial x^2} + 5\frac{\partial^2}{\partial y^2}$ can be taken as the Laplacian instead of the usual $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Definition 4.2. *Consider a Hilbert space $\mathcal{H}$ and let $\{T(t) : t \in \mathbb{R}^+\}$ be a collection of linear operators on $\mathcal{H}$ satisfying the properties*

1. *$T(t+s) = T(t)T(s)$ for $t, s \in \mathbb{R}^+$.*

2. *$T(0) = I$, I is the identity operator.*

3. *Map $t \mapsto T(t)$ is continuous on $\mathbb{R}^+$.*

Then $\{T(t)\}_{t \geq 0}$ is a strongly-continuous semi-groups.

Definition 4.3. *A contraction semigroup is a semigroup that satisfies*

$$\| T(s) \| \leq 1$$

Definition 4.4. *The infinitesimal generator of a semigroup is the operator $\mathcal{J}$ such that*

$$\mathcal{J} f = \lim_{s \to 0} \frac{T(s)f - f}{s}$$

The domain of $\mathcal{J}$ is $\{f \in \mathcal{H} : \lim_{s \to 0} \frac{T(s)f - f}{s}$ exists$\}$.

We now have a precursor theorem which will give the existence of these nice semigroups and using which we will get the main theorem.

Theorem 4.2. *There exist operators $T(s), 0 < s < \infty$ such that*

1. *Each $T(s)$ is a bounded operator on $L^p(G), 1 \leq p < \infty$ of norm* 1.

2. *$\{T(s)\}$ form a strongly continuous semigroup.*

3. *$T(s)$ is self adjoint for any $s > 0$, on $L^2(G)$.*

4. *$T(s)$ is positive (i.e. $f \geq 0 \implies T(s)f \geq 0$) and $T(s)1 = 1$.*

5. *The Laplacian Δ is the infinitesimal generator of the semigroup.*

6. *There exists a C^∞ function, K_s on $(G \times (0, \infty))$ such that*

$$T(s)f(x) = K_s * f(x) \quad s > 0$$

Proof. We stray away from the convention, for an eigenfunction ϕ of Δ, with eigenvalue λ, we have $\Delta\phi = -\lambda\phi$. This has the advantage of giving all eigenvalues of Δ as non-negative because of $\langle \Delta f, f \rangle \leq 0$, for any $f \in C^\infty(G)$ and $\langle \cdot, \cdot \rangle$ is the usual inner product on $L^2(G)$. This holds due to the strict positivity of the coefficients, (a_{ij}) in the expansion of Δ.

By Peter-Weyl decomposition, we only need to work with the space of *entry functions* Θ as they are dense in $L^2(G)$. Let $\{\chi_{ij}(x)\}$ be a representation class α, i.e. χ_{ij} is an entry function, of the group G and put $\eta_{ij} = \Delta\chi_{ij}$. We have the relation

$$\chi_{ij}(ax) = \sum_k \chi_{ik}(a)\chi_{kj}(x)$$

and similarly

$$\chi_{ij}(xa) = \sum_k \chi_{ik}(x)\chi_{kj}(a)$$

This gives us $\eta_{ij}(ax) = \sum_k \eta_{ik}(a)\chi_{kj}(x)$ by right invariance of Δ and $\eta_{ij}(xa) = \sum_k \eta_{ik}(x)\chi_{kj}(a)$ by left invariance. Set $x = 1$ to get

$$\eta_{ij}(y) = \sum_k \eta_{ik}(1)\chi_{kj}(y) \quad \text{and} \quad \eta_{ij}(y) = \sum_k \chi_{ik}(y)\eta_{kj}(1)$$

Hence by Schur's lemma we get that the matrix $\{\eta_{ij}(1) = -\lambda_\alpha I\}$ and so $\Delta\chi = -\lambda_\alpha\chi$ for any χ in the vector space spanned by the entries of $\{\chi_{ij}\}$.

By Peter-Weyl theorem we also get the decomposition $L^2(G) = \sum_{\alpha\in\Lambda} \oplus H_\alpha$ where H_α is vector space spanned by the entries of $\{\chi_{ij}\}$ of class α. Our previous argument then show that this decomposition is and eigenfunction decomposition with respect to Δ. So we pick an orthonormal bases for all H_α say $1, \chi_1(x), \chi_2(x \cdots)$ then each one is an eigenfunction of Δ with eigenvalue $\mu_i > 0$.

Θ is the space of all finite linear combinations of χ_i and for any function $f = \sum_i a_i\chi_i$ in Θ we have $\Delta f = \sum_i -\mu_i\alpha_i\chi_i$ which indicates to putting $T(s) = e^{s\Delta}$ for functions in Θ. The extension of $T(s)$ from Θ to $L^2(G)$ as a bounded linear operator is obvious form the fact

$$\| T(s)f \|_2^2 = \sum_i |e^{-\mu_i t}\alpha_i|^2 \;\leq\; \sum_i |\alpha_i|^2 = \| f \|_2^2$$

Hence $T(s)$ is a strongly continuous semigroup of self adjoint operators on L^2 which have $T(s)1 = 1$.

The resolvent $R(\lambda, \Delta)$ is the operator $(\lambda I - \Delta)^{-1}$ for $\lambda > 0$. We will use a fact from semigroup theory that each $T(s)$ is positive if and only if $R(\lambda, \Delta)$ is positive for $\lambda > 0$. Using this we show that $T(s)$ is positive on Θ.

Let $\phi \in \Theta$ and $\phi \geq 0$ and $R(\lambda, \Delta)\phi = \psi \in \Theta$, then $\lambda\psi - \Delta\psi = \phi \geq 0$. If $\phi < 0$ then we have a point $x_0 \in G$ such that ψ is minimum due to G being compact and $\psi(x_0) < 0$, also $\Delta\psi(x_0) \geq 0$ as ψ is minimum at x_0. Hence $\lambda\psi(x_0) - \Delta\psi(x_0) < 0$ but this contradicts the positivity of ϕ. Hence giving $T(s)$ to be positive.

We can extend $T(s) : \Theta \to \Theta$ which maps 1 into 1 to $T(s) : C(G) \to C(G)$ with norm 1. By Riesz representation theorem we have, for any $x_0 \in G$ and $s > 0$ we can define a positive linear functional on $C(G)$ as

$$(T(s)f)(x_0) = \int_G f(x)d\mu_{x_0}^t(x)$$

where $d\mu^t_{x_0}$ is a positive measure of total mass 1. Also, by bi-linearity $\mu^t_{x_0}(E) = \mu^t_1(x_0^{-1}E)$ and so we can write

$$(T(s)f)(x_0) = f * \mu^t_1(\times_0) \qquad \forall f \in C(G)$$

Now it is easy to see that $T(s)f$ is L^p bounded. Hence we have shown the claim for all except 6. For which we will need a form of the Sobolev lemma.

Lemma 4.1 (Sobolev)**.** *For $f \in C_0^\infty(\mathbb{R}^n)$ we have*

$$\sup_{x\in\mathbb{R}^n} |f(x)| \;\le\; A \sum_{0\,\le\,|\alpha|\,\le\,N} \| \frac{\partial}{\partial x^\alpha} f \|_2$$

where N is any integer greater than $\frac{n}{2}$ and A depends only on n.

The proof is quite a straightforward application of Plancherel theorem and the property of Fourier transform to change multiplication into differentiation.

For our case, we need the above lemma for a compact Lie group G, instead of $\mathbb{R}^n$.

Lemma 4.2. *Let $f \in C^\infty(G)$, then*

$$\| f \|_\infty \le \; A \sum_{l=0}^{N} \| \Delta^l f \|_2$$

where N is any integer greater than $\frac{n}{2}$ and A depends only on G.

The splitting of our Laplacian into the basis of the Lie algebra $\mathfrak{g}$ helps us give the inequality

$$\| X_k f \|_2 \le \; C \| \Delta f \|_2^{1/2} \| f \|_2^{1/2}$$

where X_k is in the basis of $\mathfrak{g}$. Which can be further bounded by

$$C(\| \Delta f \|_2 + \| f \|_2)$$

by using the inequality $2ab \leq a^2 + b^2$. Doing this for all basis elements X_i of $\mathfrak{g}$ we get

$$\| D(X_1 \cdots X_n) f \|_2 \leq \tilde{C}_p \sum_{l=0}^{N} \| \Delta^l f \|_2$$

where $D(x_1 \cdots X_n)$ is in the universal enveloping algebra of G, of degree $\leq N$. From the Sobolev lemma we can show

$$\| f \|_\infty \leq \| D(X_1 \cdots X_n) f \|_2$$

holds for f with small enough support and in the canonical coordinates. By a partition of unity argument this can be extended to the general case, and so we are done.

We use this lemma on the eigenfunctions, χ_i which gives

$$\sup_{x \in G} |\chi_i(x)| \leq C(1 + \mu_i)^N$$

for any $N > \frac{1}{2}\text{dim } G$. Hence, for $g \in \Theta$ and $u = T(s)g$ we have

$$\Delta^l u(x,t) = \sum (-\mu_k)^l e^{-\mu_k t} a_k \chi_k(x)$$

and so we have the bound

$$\| \Delta^l u(t) \|_2^2 = \sum \mu_k^{2l} e^{-2\mu_k t} |a_k|^2 \leq Ct^{-2l} \sum |a_k|^2 = Ct^{-2l} \| f \|_2^2$$

Now using above bound with the fact that $\left(\frac{\partial}{\partial t}\right) u(x,t) = \Delta u(x,t)$ we get

$$\| \frac{\partial^k}{\partial t^k} \Delta^l u(x,t) \|_{L^2(G)} \leq Ct^{-M} \| f \|_2$$

where C and M depend on only k and l.

Now we use the Sobolev lemma and get $u(x,t) \in C^\infty(G \times (0,\infty))$ for any $g \in L^2(G)$. For $g \in L^1$ we can take $T(s) = T\left(\frac{s}{2}\right) T\left(\frac{s}{2}\right)$ and as it is a composition of continuous operators we get that $T(s)$ maps L^1 to $C^\infty(G \times (0, \infty))$. $\square$

The theorem gives the existence of a semigroup which satisfies the heat equation, rather we want the semigroup which is roughly defined as the solution of the Laplace's equation. What we mean is

$$P(s)f(x) = u(x,t)$$

is the solution of the differential equation

$$\left(\left(\frac{\partial^2}{\partial t^2}\right) + \Delta\right) u = 0$$

with the boundary condition that

$$u(x,0) = f(x)$$

Let $f \in \Theta$ then we can decompose it into the spectral projections of the Δ and so get $f = \sum_i a_i \chi_i \in \Theta$. Then putting

$$P(s)f = \sum_i e^{-\mu_i^{1/2} t} a_i \chi_i \qquad t \geq 0$$

will satisfy Laplace's equation. Or formally we can say,

$$P(s) = e^{-s\Delta_i^{1/2}}$$

With regards to $P(s)$ we have a similar theorem as before which gives us sufficient knowledge of the semigroup $\{P(s) : s \geq 0\}$.

Theorem 4.3. *For $P(s)$ as defined above, on Θ we have*

1. *A bounded extension from Θ to L^p, $1 \leq p \leq \infty$, and on $C(G)$.*

2. $\| P(s)f \|_p \leq \| f \|_{p'}$ *for all f and $s \geq 0$.*

3. *$P(s)$ is a self adjoint operator on $L^2(G)$.*

4. *$P(s)$ is positive, that is $f \geq 0 \implies P(s)f \geq 0$.*

5. *$\Delta^{1/2}$ is the infinitesimal generator of $P(s)$, that is $\lim_{s\to 0} \frac{P(s)f-f}{s} = (-\Delta)^{1/2} f$ for $f \in \Theta$.*

Proof. We will use the properties of $T(s)$ and the identity

$$e^{-\sigma} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\sigma^2/4u}\, du \qquad \sigma > 0$$

Hence we get

$$P(s)f = \frac{1}{\sqrt{\pi}} = \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T^{s^2/4u} f\, du \qquad s > 0$$

Hence the L^p norm of $P(s)f$ can now be bounded by the boundedness of $T(s)$ and so the theorem holds. □

The image function in $C^\infty(G \times (0,\infty))$ is called the *Poisson integral* of f.

We define the **Littlewood-Paley** $g-$function as

$$g(f)(x) = \left(\int_0^\infty s|\nabla u(x,s)|^2\, ds \right)^{1/2} \qquad f \in L^p(G)$$

where $|\nabla f|^2(x,s) = \left(\frac{\partial f}{\partial t}\right)^2 + |\nabla_x f|^2$ and $|\nabla f|^2(x) = \sum_{ij} a_{ij}(X_j f)(X_j f)$.

This $g-$function will be very useful in achieving our goal and it lies in $L^p(G)$, as we show next.

Theorem 4.4. *For $f \in L^p(G)$, $1 < p < \infty$, we have $g(f)$ also in $L^p(G)$ and the bound*

$$\| g(f) \|_p \leq \; A_p \| f \|_p$$

where A_p only depends on p.

Proof. Before we tackle the proof we require a series of lemmas which will be used in our proof.

Lemma 4.3 (Lemma A). *For any function f, define the maximal function, $\tilde{F}$ as $\tilde{F}(x) = \sup_{s>0} |P(s)f(x)|$. Then for $1 < p \leq \infty$ we have the bound*

$$\| \tilde{F} \|_p \leq C_p \| f \|_p$$

for any $f \in L^p$ and C_p independent of f.

Lemma 4.4 (Lemma B). *For a positive, harmonic function $u(x,s)$ on $G \times (0,\infty)$, that is $\Delta u = 0$ and $u > 0$. We have*

$$\Delta u^p = p(p-1)u^{p-2}|\nabla u(x,s)|^2$$

for any $p \geq 1$.

Lemma 4.5 (Lemma C). *Let $F(x,s) \in C^\infty(G \times (0,\infty))$ and let $t|\frac{\partial F}{\partial t}| \to 0$ as $t \to 0$ or as $t \to \infty$. Then we have*

$$\int_0^\infty \int_G \Delta F(x,s)\, dx\, ds = \int_G F(x,0)\, dx \; - \int_G F(x,\infty)\, dx$$

We for the time being assume these lemmas and give the proof after we prove our main theorem.

We prove the theorem in 2 parts. The first part will deal with $1 < p \leq 2$ and will be used in the second part

Part 1 Consider $f \in C^\infty(G)$ which is strictly positive and its Poisson integral $u(x,s) \in C^\infty(G \times [0,\infty])$ and $u(x,s) > 0$. So we get $(u(x,s))^p \in C^\infty(G \times [o,\infty])$ and $t\frac{\partial F}{\partial t} \to 0$ as $t \to 0$ or $t \to \infty$. So by [Lemma B] we get that $\Delta F \geq 0$.

Also,

$$(g(f)(x))^2 = \int_0^\infty s|\nabla u|^2\, ds = \frac{1}{p(p-1)} \int_0^\infty s\, u(x,s)^{2-p} \Delta u^p(x,s)\, ds$$

holds because of [Lemma B] and this can now be bounded by $\tilde{F}$, the maximal function

$$(g(f)(x))^2 \leq A_p(\tilde{F}(x))^{2-p} \int_0^\infty s\,\Delta F(x,s)\,ds$$

Now by [Lemma C] we have,

$$\begin{aligned}
\int_G \int_0^\infty s\,\Delta F(x,s)\,ds\,dx &= \int_G F(x,0)\,dx - \int_G F(x,\infty)\,dx \\
&\leq \int_G F(x,0)\,dx \\
&= \int_G |f(x)|^p\,dx
\end{aligned}$$

Hence we get

$$\begin{aligned}
\| g(f) \|_p^p &\leq A \int_G (\tilde{F}(x))^{(2-p)/p} \left(\int_0^\infty s\,\Delta F(x,s)\,ds \right)^{p/2} dx \\
&\leq A \left(\int_G |\tilde{F}(x)|^p\,dx \right)^{(2-p)/2} \left(\int_G \int_0^\infty s\,\Delta F(x,s)\,ds\,dx \right)^{p/2} \\
&\leq B \| f \|_p^{p(2-p)/2} \cdot \| f \|_p^{p^2/2} \\
&= C \| f \|_p^p
\end{aligned}$$

where 2nd inequality holds because of Holder's inequality and so we are done for $1 < p \leq 2$.

Part 2 Let $p \geq 4$ and q be conjugate to $\frac{p}{2}$, that is $\frac{1}{q} + \frac{2}{p} = 1$, and so $1 < q \leq 2$. Let f have L^p norm 1. We can get

$$\| g(f) \|_p^2 = \sup \int_G (g(f)(x))^2 \psi(x)\,dx$$

where supremum is taken over all $\psi \in C^\infty$ such that $\| \psi \|_q \leq 1$. The idea is to reduce the $g(f)^2$ in terms of $g-$functions and $\tilde{F}$ which we can bound easily. So our goal reduces to show

$$\int_G g(f)^2(x)\psi(x)\,dx \;\le\; A_p\left[\int_G f(x)^2\psi(x)\,dx + \int_G \tilde{F}(x)g(f)g(\psi)(x)\,dx\right] \quad (1)$$

The bi-invariance of Δ helps it commute with any element of the basis, X_j, of the vector field. So any function will also commute with X_j, what we are getting at is $T(s) = e^{s\Delta}, s > 0$ and $P(s)$ commute with X_j. This holds for Θ and passing to a general C^∞ function is a usual limiting argument.

$\{P(s) : s > 0\}$ is a semigroup and so $u(x, s_1 + s_2) = P(s_1)u(x, s_2)$. Now using commutativity we get

$$X_j\,u(x,t) = P(t/2)X_j u(x,t/2) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,t) = P(t/2)\frac{\partial u}{\partial t}u(x,t/2)$$

By change definition of $|\nabla u|^2$ and change of basis of the Lie algebra we can write

$$|\nabla u|^2 = \sum_i (X_i u)^2 + \left(\frac{\partial u}{\partial t}\right)^2$$

To show that (1) holds we need the estimate

$$\begin{aligned}\int_G (g(f)(x))^2\psi(x) &= \int_0^\infty \int_G s|\nabla u(x,s)|^2\psi(x)\,dx\,ds \\ &\le A\int_0^\infty \int_G s|\nabla u(x,s)|^2\psi(x,s)\,ds \quad (2)\end{aligned}$$

where $\psi(x, s)$ is the Poisson integral of ψ.

In view of the commutative argument before we just need to show that $|X_j u(x, s)|^2 \le$

$P(s/2)(|X_j u(x,t/2)|^2)$ and similarly for $(\frac{\partial u}{\partial t})^2$. Which means that

$$\begin{aligned}
X_j u(x,s) = P(s/2)X_j u(x,s) &= \int_G p_{s/2}(xy^{-1})X_j u(y,s/2)\,dy \\
&\leq \left(\int_G p_{s/2}(xy^{-1})|X_j u(y,s/2)|^2\,dy\right)^{1/2} \\
&= (P(s/2)(X_j u)^2)^{1/2}
\end{aligned}$$

where p_t is the Poisson kernel and the inequality follows from Holder's, and so we get

$$|\nabla u(x,s)|^2 \;\leq\; P(s/2)(|\nabla u(x,s/2)|^2)$$

Finally, we combine all the above arguments to get (2), that is

$$\begin{aligned}
\int_G (g(f)(x))^2\psi(x) \;&\leq\; \int_0^\infty \int_G sP(s/2)(|\nabla u(x,s/2)|^2)P(s/2)\psi(x)\,dx \\
&= \int_0^\infty \int_G s|\nabla u(x,s/2)|^2\psi(x,s/2)\,dx\,ds
\end{aligned}$$

where first inequality holds due to self-adjointness of $P(s/2)$, and so we get our claim (2).

Replace $|\nabla u(x,s)|^2$ by $\frac{1}{2}\Delta u^2(x,s)$ in (2) to get

$$\int_G (g(f)(x))^2\psi(x) \;\leq\; A\int_0^\infty \int_G s(\Delta u^2(x,s))\psi(x,s)\,dx\,ds$$

We can use the product rule on the right hand side if we take $M = (u(x,s))^2$ and $N = \psi$ in

$$\Delta(MN) = (\Delta M)N \,+\, M(\Delta N) \,+\, 2\sum_{ij} a_{ij}(X_i M)(X_j N) \,+\, 2\left(\frac{\partial M}{\partial t}\right)\left(\frac{\partial N}{\partial t}\right)$$

which gives

$$\begin{aligned}
\int_G (g(f)(x))^2 \psi(x) \;&\leq\; A \int_0^\infty \int_G s\Delta(fg)\, dx\, ds \\
&- 2A \int_0^\infty \int_G \sum_{ij} a_{ij} X_j(u^2(x,s)) X_i \psi(x,s)\, dx\, ds \\
&- 2A \int_0^\infty \int_G \left(\frac{\partial u^2(x,s)}{\partial s}\right)\left(\frac{\partial \psi(x,s)}{\partial s}\right) dx\, ds \\
&= A(I_1 - I_2)
\end{aligned}$$

The estimate for I_1 is given by [Lemma C], which gives

$$\begin{aligned}
I_1 &= \int_G M(x)N(x)\, dx - (\text{positive term}) \\
&= \int_G f^2(x)\psi(x)\, dx - (\text{positive term}) \\
&\leq \int_G f^2(x)\psi(x)\, dx
\end{aligned}$$

For I_2, we use the fact $X_j u^2 = 2uX_j u$ and $\frac{\partial u^2}{\partial s} = 2u\frac{\partial u}{\partial s}$ to get

$$\begin{aligned}
I_2 &= 4\int_0^\infty \int_G s\, u(x,s) \left[\sum_{ij} a_{ij}(X_j u(x,s) X_i \psi(x,s)) + \left(\frac{\partial u(x,s)}{\partial s}\right)\left(\frac{\partial \psi(x,s)}{\partial s}\right)\right] dx\, ds \\
&\leq 4\int_0^\infty \int_G s\, u(x,s) |\nabla u(x,s)| |\nabla \psi(x,s)|\, dx\, ds \\
&\leq 4\int_G \tilde{F}(x)(g(f))(x)(g(\psi))(x)\, dx
\end{aligned}$$

where the last inequality is due to Schwartz inequality. This altogether gives us (1).

Now apply Holder's inequality to (1) and [Lemma A] to get the estimate

$$\begin{aligned}
\int_G (g(f)(x))^2 \psi(x)\, dx \;&\leq\; A_p[\| f \|_p^2 \;+\; \| \tilde{F} \|_p \| g(\psi) \|_q] \\
&\leq\; B_p[\| f \|_p^2 \;+\; \| \tilde{F} \|_p \| g(f) \|_p] \\
&\leq\; C_p[\| f \|_p^2 \;+\; \| f \|_p \| g(f) \|_q]
\end{aligned}$$

where we have used that $\| g(\psi) \|_q \leq A_q$ for $\| \psi \|_q \leq 1$. Now we can finally take the supremum over all ψ whose norm is less than or equal to 1, and get our claim for $1 < p < \infty$. $\square$

We are still left with the proofs of our lemmas which we have used very frequently.

Lemma A. To prove this lemma we have to go out of our area and take the following theorem for granted.

Theorem 4.5 (Hopf-Dunford-Schwartz). *For $T(t)$, a measurable semigroup of operators on $L^p(\mathfrak{M}, d\mu)$, which are bounded operators on L^p for each $p \in [1, \infty]$ then the maximal function defined as*

$$Mf(x) = \sup_{s>0} \left(\frac{1}{s} \int_0^s |(T(t)f)(x)|\, dt \right)$$

satisfies the inequalities

1. $\| Mf \|_p \leq A_p \| f \|_p$ *for each* p *with* $1 < p < \infty$
2. $\mu(\{x \in \mathfrak{M} : Mf(x) > \alpha\}) \leq \dfrac{A}{\alpha} \| f \|_1$ *for each* $\alpha > 0$ *and* $f \in L^1$

where A is independent of f and α.

We will only use the first result for the proof of our lemma as it controls $T(s)$ and in turn $P(s)$, which is nothing but the average of $T(s)$ with certain weight. We can re-write $P(s)$ as

$$P(s)f = s^{-2} \int_0^\infty g\left(\frac{y}{t^2}\right) T^y f\, dy$$

where $g(x) = \frac{1}{2\sqrt{\pi}} e^{-1/4y} y^{-3/2}$, and we have $g(x)$ and $xg(x)$ in $L^1(0, \infty)$. We now integrate by parts $P(s)f$ to get

$$\begin{aligned}
|P(s)f(x)| &= \left| -t^2 \int_0^\infty \left(\frac{1}{y} \int_0^y T(s)f(x)\, ds \right) \left(y \frac{d}{dy} g\left(\frac{y}{t^2}\right) \right) dy \right| \\
&\leq \sup_{y>0} \left| \frac{1}{y} \int_0^y T(s)f(x)\, ds \right| \cdot \left(t^{-2} \int_0^\infty \left| y \frac{dg}{dy} \left(\frac{y}{t^2}\right) \right| dy \right) \\
&\leq A \sup_{y>0} \left| \frac{1}{y} \int_0^\infty T(s)f(x)\, ds \right|
\end{aligned}$$

where $A = \parallel xg'(x) \parallel_1$ and so $(s)f$ is bounded by the maximal function of f for each $s > 0$, and so the lemma holds by **Hopf-Dunford-Schwartz**. □

Lemma B. The proof is straightforward and follows from the fact that X_j correspond to $\partial/\partial t$ in canonical coordinates, which give

$$X_j u^p = p u^{p-1} X_j u \quad \text{and} \quad \frac{\partial}{\partial t} u^p = p u^{p-1} \frac{\partial}{\partial t} u$$

Next step is to do this again and then add the two identities, which will give our claim. □

Lemma C. For any $f \in C^\infty(G)$ we can write

$$\int_G X_j f(x)\, dx = \int_g 1 \cdot X_j f(x)\, dx = -\int_G (X_j 1) f(x\, dx)$$

Also observe that,

$$\begin{aligned}
\int_0^\infty t \frac{\partial^2}{\partial t^2} F(x,t)\, dt &= t \frac{\partial F}{\partial t} \bigg|_0^\infty - \int_0^\infty \frac{\partial F}{\partial t}(x,t)\, dt \\
&= F(x,0) - F(x,\infty)
\end{aligned}$$

Now our claim can be shown from the fact that

$$\int_0^\infty s \int_G \Delta F(x,s)\, dx\, ds = \int_0^\infty s \int_G \frac{\partial^2 F}{\partial t^2}(x,s)\, dx\, ds$$

□

We have shown that $g-$functions are L^p bounded for functions on the compact group G. This in itself is a big result and helps show a lot of nice results for the case of compact groups. But we need this kind of theorem for an even more general space, which luckily we have.

Theorem 4.6. *If $f \in L^p(M, dx)$ then $g_1(f) \in L^p(M, dx)$ and we have the inequality*

$$\| g_1(f) \|_p \leq \ A_p \| f \|_p$$

for $1 < p < \infty$.

Here (M, dx) is a sigma finite measure space.

The proof of this theorem would require even more theory to be developed which, at this point, we cannot achieve and so we take it for granted that the theorem holds, and hope to prove it at a later time.

With this theorem, we can tackle our problem of the boundedness of multiplier on the Heisenberg group.

We had stated our motivation for studying the problem at the beginning, we wanted to look into the problem of multipliers on Heisenberg Group. The Heisenberg group being a Nilpotent group gives us a very nice entry point to study Fourier Transform in a general setting.

One way to study the topic is to look at the problems in Euclidean space and then ask same (or at least similar) questions for our space. We look into the **necessary conditions** for the boundedness of the multipliers on Heisenberg Group.

4.2 The theorem

Our main theorem of interest is

Theorem 4.7. *Let m be a β times differentiable function that satisfies*

$$|m^k(t)| \leq C|1+t|^{-k} \text{ for } k = 0, 1, 2 \cdots, \beta$$

for β satisfying

$$\beta = \begin{cases} n+2 & \text{for } n \text{ is even} \\ n+3 & \text{for } n \text{ is odd} \end{cases}$$

Then $m(\mathcal{L})$ is a bounded operator on $L^p(\mathbb{H}^n)$, for $1 < p < \infty$.

The way we tackle this problem is using the g and g^* given by the **Littlewood-Paley-Stein** theory. The proof for the theorem is simple but requires certain propositions which make our life simpler even more. We'll begin with the preliminary requirements to develop, or at least motivate the tools.

We need a contraction semi-group corresponding to the sub-Laplacian ($\mathcal{L}$) to get the aforementioned **g** functions. This can be done thanks to a theorem of Folland, we'll only use part of its implications.

Theorem 4.8 (Folland)**.** *There is a unique semi-group $\{T^t : t > 0\}$ of linear operators on $(L^1 + L^\infty)(\mathsf{H})$ that satisfy*

1. $T^t f = f * h_t$ *for* $h_t(x) = h(x,t) \in C^\infty(\mathsf{H}^\mathsf{n}, (0,\infty))$ *with* $h(x,t) \geq 0 \,\forall\, t, x$ *and* $\int_{\mathsf{H}} h_t(x)\, dx = 1 \quad \forall\, t > 0$

2. $\{T^t\}$ *forms a contraction semi-group on* $L^p(\mathsf{H}^n)$, *for* $1 \leq p \leq \infty$ *and is strongly continuous for* $p \neq \infty$.

3. T^t *is self adjoint, positive and conservative operator.*

where positive means $T^t f \geq 0 \, for \, f \geq 0$ *and conservative means* $T^t 1 = 1$.

For the proof we require a theorem of Hunt and the result of Hörmander, which are given in the **Appendix D**.

Proof. Let $\tilde{C}$ be the space of smooth functions which are constant outside of a compact set and choose a basis for the Lie algebra of H^n, let it be $\{Y_i\}_{i=0}^N$. Let the completion of the above space be given by $\tilde{C_1}$ and $\tilde{C_2}$ corresponding to the norms

$$\| f \| = \| f \|_\infty + \sum_{j=1}^{N} \| X_j f \|_N + \sum_{j,k}^{N} \| X_j X_k f \|_\infty$$

and the uniform norm, respectively. Using **Hunt's theorem**, we get strongly continuous semi-group $\{T^t\}$ on $\tilde{C_2}$ satisfying,

1. There exists a probability measure ν_t on H^n, for every $t > 0$ such that $T^t f(x) = \int_{\mathsf{H}^n} f(xy^{-1})\, d\nu_t(y)$ (which is equal to $f * d\nu_t(x)$, for nicely defined $d\nu_t$).

2. The infinitesimal generator of $\{T^t\}$ is defined on $\tilde{C_2}$ and coincides with $\mathcal{L}$.

Next, we have $\lim_{t\to 0} \nu_t(E) = 0$ for $0 \in E \subseteq \mathsf{H}^n$, this is due to $y^{-1} = 0 \cdot y^{-1}$ and $T^t f(0) = \int_{\mathsf{H}^n} f(y^{-1})\, d\nu_t(y)$. Using the symmetricity of $\mathcal{L}$, we have $d\nu_t(y) = d\nu_t(y^{-1})$. The action of $\mathcal{L}$ on $\tilde{D}$ determines the semi-group $\{T^t\}$, due to denseness and by the definition of $\mathcal{L}$ annihilating the constants.

We next define the function $h_t(x) = h(x,t)$ as the distribution on $\mathsf{H}^n \times (0,\infty)$ given by

$$\langle h, u \otimes v \rangle = \int_0^\infty \int_{\mathsf{H}^n} u(x)v(t)\, d\nu_t(x)\, dt \qquad u \in \tilde{C}(\mathsf{H}^n),\ v \in \tilde{D}((0,\infty))$$

By (2.) above, we have $\langle h, \mathcal{L}u \otimes v \rangle = \langle h, u \otimes \frac{dv}{dt} \rangle$, giving us a solution to the equation $\left(\mathcal{L} + \frac{d}{dt}\right) f = 0$. Now by **Hörmander** we have that $\left(\mathcal{L} + \frac{d}{dt}\right)$ is hypoelliptic. Hence, we get that $h \in C^\infty(\mathsf{H}^n \times (0,\infty))$, $d\nu_t(x) = h(x,t)$, $h(x,t) \geq 0\, \forall (x,t)$ and $\int h(x,t)\, dx = 1$. The self-adjointness follows from the fact that $d\nu_t(y) = d\nu_t(y^{-1})$. The first and the third claim are done.

For the strongly continuous part we use (1.) and **Young's inequality**. $\square$

Now that we have a contraction semi-group corresponding to the sub-Laplacian, we can define the g functions and then use them to prove our claim.

$$1.)\quad g(f,w)^2 = \int_0^\infty t^{2k-1}|\partial_t f(w)|^2\,dt$$
$$2.)\quad g^*(f,w)^2 = \int_{\mathsf{H}^n}\int_0^\infty t^{-n}(1+t^{-2}|v|^4)^{-k}|\partial_t f(v^{-1}w)|^2\,dt\,dw$$

Now we can show that these g functions satisfy the property

Theorem 4.9.

1. *For* $k \geq 1, \| g_k(f) \|_2 = 2^{-k} \| f \|_2$
2. *For* $1 < p < \infty, A \| f \|_p \leq \| g_k(f) \|_p \leq B \| f \|_p$.
3. *If* $k > \frac{n+1}{2}$ *and* $p > 2$, *we have* $\| g_k^*(f) \|_p \leq C \| f \|_p$

Before we move to the proof of the theorem we see that (3.) directly gives the boundedness of g^* and so we just need to have $g(m(\mathcal{L})f)$ be bounded by some $g^*(f)$ and then by (3.), our main theorem is proved.

Proof. To show (1.), we use the **Plancherel theorem** for the simple case of $k = 1$, other cases will follow suit and so will be omitted.

$$\| g_1(f) \|_2^2 = \int_0^\infty \int_{\mathsf{H}^n} t|\partial_t T^t f(w)|^2\,dw\,dt$$

The integral corresponding to w can be written using the Plancherel theorem and so we have

$$\int_{\mathsf{H}^n} |\partial_t T^t f(w)|^2\,dw = \int \| (\partial_t \hat{T^t f})(\lambda) \|_{HS}^2\; d\mu(\lambda)\,dt$$

The Hilbert-Schmidt operator can be written in terms of the λ-scaled Laplacian, i.e

$$(\partial_t \hat{T^t f})(\lambda) = -\hat{f}(\lambda)\exp(-tH(\lambda))H(\lambda)$$

and so the Hilbert-Schmidt operator norm will be given by

$$\sum (2N+n)^2|\lambda|^2 \exp(-2t(2N+n)|\lambda|) \parallel \hat{f}(\lambda)\Phi_N^\lambda \parallel_2^2$$

which when integrated against $t\,dt$ gives us the first claim.

The second claim follows from the theory g functions and we have seen its proof earlier. The third claim also makes sense once we see that for $k > \frac{n+1}{2}$, $(1+t^{-2}|w|^4)^{-k}$ becomes integrable and so we can bound the L^p norm of g^* function by L^p norm of f, because of Hölder's inequality on L^2 and then we can use our first claim. □

4.3 The proof

Now that we have shown our claim about the g functions to hold true, we claim that the following inequality holds for the above mentioned g functions

$$g_{k+1}(m(\mathcal{L})f) = g_k^*(f)$$

for some $k > \frac{n+1}{2}$ and $p > 2$. For $p < 2$, we can use duality argument to conclude the boundedness of $m(\mathcal{L})$ on $L^p(\mathsf{H}^n)$.

Set $s_t(w) = T^t f(w)$ and $S_t(w) = T^t(m(\mathcal{L})f)(w)$, then we have the easily verifiable equality

$$S_{t+u}(w) = (G_t * s_u)(w)$$

where we have $\hat{G}_t(\lambda) = \sum_{N=0}^{\infty} e^{-t(2N+n)|\lambda|} m(|\lambda|(2N+n))Q_N(\lambda)$.

We now differentiate the above equality, once with respect to u and k times with respect to t; this gives us the forms inside the integral of the g functions.

$$\partial_t^k \partial_u [T^{2t}(m(\mathcal{L})f)] = \partial_t^k G_t * \partial_u S_u$$

$$\partial_t^{k+1} S_{t+u} = \partial_t^{k+1}[T^t(m(\mathcal{L})f)] = F_t * \partial_t T^t f \qquad \text{for u = t}$$

and for F_t, we have $\hat{F}_t(\lambda) = (-1)^k \sum_{N=0}^{\infty}((2N+n)|\lambda|)^k e^{-t(2N+n)|\lambda|} m((2N+n)|\lambda|)Q_N(\lambda)$. This is a result of the action of Fourier transform on differentiation and multiplication.

And so we have by the boundedness of convolution and the Cauchy-Schwartz inequality

$$|\partial_t^{k+1} T^{2t}(m(\mathcal{L})f)(w)| \le \int |F_t(v)||\partial_t T^t f(v^{-1}w)|\, dw \le G_t^1 \cdot G_t^2(w)$$

where we have

$$\begin{aligned} &1. \quad G_t^1 = \int |F_t(v)|(1+t^{-2}|v|^4)^k\, dv \\ &2. \quad G_T^2(w) = \int (1+t^{-2}|v|^4)^{-k} |\partial_t T^t(v^{-1}w)|^2\, dv \end{aligned}$$

Observe that $G_t^2(w)$ looks very similar to g^*, for which we have a very nice estimate, and so if for the product of $G_t^1 \cdot G_t^2(w)$, we have the bound $K \cdot t^{-n-2k-1} \cdot G_t^2(w)$, then integrating it against the measure $t^{2k+1}\, dt$, we'll get the inequality

$$g_{k+1}(m(\mathcal{L})f, w) \le g_k^*(f, w)$$

and so we will be done with our claim of the Multiplier theorem. So the only thing remaining to show is the following

Lemma 4.6. *Under the assumptions of our main theorem, we have the estimate $K \cdot t^{-n-sk-1}$ for the function G_t^1 when k is the smallest integer greater than $\frac{n+1}{2}$*

Proof. We split the above integral into two parts, idea behind it? $(1+t^{-2}|v|^4)^k$ can easily be bounded by say 2^k, if we have $t^{-2}|v|^4 \le 1$, and so the only remaining part is the one with $|F_t(w)|^2$, for which we use the Plancherel theorem (again), we still have to look for the appropriate powers powers of t; those can be derived from the Fourier transform of the function F_t.

The thing to note is the absence of a factor of t^{2k} in the coefficient of $e^{(-2t(2N+n)|\lambda|)}m((2N+n)|\lambda|)$ in $|\hat{F}_t(\lambda)|^2$, which when plugged into the integral and substituted appropriately would gives us $t^{-n-2k-1} \cdot C$ as the bound for the integral. Hence, we are done with the first part for which $|w| \leq \sqrt{t}$.

We still have to deal with the part that has $|w| > \sqrt{t}$, which will require a little more effort than before. The idea of the proof sits on the decomposition of Fourier transform of functions in terms of Laguerre Polynomials, for which we have certain nice estimates and on the fact that the differential operator, along with the hypothesis of our theorem "brings down" a factor when it acts on our multiplier m.

Before we go into the analysis of the second case, we simplify certain notations;

1. $\int_{|w|>\sqrt{t}}(1+t^{-2}|v|^4)^k|F_t(w)|^2\,dw = J$

2. For $w = (z, s)$,

$$J \leq C_1 t^{-2k}\int_0^\infty \int_{\mathsf{H}^n} (s^2 + |z|^4)^k |F_t(z,s)|^2\,dz\,ds$$

$$= C_1 t^{-2k}\int_0^\infty \int_{\mathsf{H}^n} |(is - |z|^2)^k F_t(z,s)|^2\,dz\,ds$$

3. $G = (is - r^2)^k F_k(z, s)$

Then the Fourier transform of G can be decomposed in terms of its projection on the Heisenberg fan, given by

$$\hat{G}(\lambda) = \sum_{N=0}^{\infty} R_n(\lambda, is - |z|^2)^k F_t) Q_N(\lambda)$$

Then if

$$\int \sum_{N=0}^{\infty} |R_n(\lambda, (is - r^2)^k F_t)|^2 \frac{(N+n-1)!}{N!} \, d\mu(\lambda) \;\leq\; C_2 t^{-n-1} \quad \text{for } |z|^2 = r^2$$

we are done. So put $\Phi(N, \lambda) = (-1)^k [(2N+n)|\lambda|]^k e^{-t(2N+n)|\lambda|} m((2N+n)|\lambda|)$, this means that $R_N(\lambda, f) = \Phi(N, \lambda)$ and for $R_N(\lambda, (is - r^2)^k F_t) = \Phi_k(N, \lambda)$. This reduces the problem to the bounds of $\Phi_k(N, \lambda)$. The essence of the idea is similar to the previous one, concerning the absence of the factor of t^{2k}, and for that we have the following lemma. □

Lemma 4.7. *Under the hypothesis of our main theorem, we have for some $\epsilon > 0$,*

$$|\Phi_k(N, \lambda)| \;\leq\; C_3 \exp[-\epsilon(2N+n)|\lambda|]$$

Proof. The proof uses the expansion of $R_N(\lambda, f)$ for a radial function f, which is given by

$$R_N(\lambda, f) = C_n \frac{N!}{(N+n-1)!} \int_0^{\infty} \tilde{f}(r, \lambda) L_N^{n-1}(2|\lambda| r^2) e^{-|\lambda| r^2} \, r^{2n-1} \, dr$$

where $\tilde{f}(r, \lambda)$ is the Euclidean Fourier transform of f in the real variable(s). We will now look for the case of $\lambda > 0$.

Recall the property of Fourier transform

$$(ixf(x))\hat{\ }(\lambda) = \frac{d}{d\lambda} \hat{f}(\lambda)$$

Using this for $(isf)\hat{\ }(\lambda)$ we get

$$R_N(\lambda, isf) = \frac{d}{d\lambda} R_N(\lambda, f) - C_n \frac{N!}{(N+n-1)!} \int_0^{\infty} \tilde{f}(z, \lambda) \cdot \frac{d}{d\lambda} [L_N^{n-1}(2\lambda r^2) e^{-\lambda r^2}] r^{2n-1} \, dr \tag{36}$$

Now we use the recursion formula for Laguerre Polynomials, which says that

$$\frac{d}{dx}L_n^\alpha(x) = \frac{1}{x}\left[nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x)\right] = -L_{n-1}^{\alpha+1}(x)$$

Using this, computing the derivatives and the product rule we get

$$R_N(\lambda, isf) = \frac{d}{d\lambda}R_N(\lambda, f) - \frac{N}{\lambda}[R_N(\lambda, f) - R_{N-1}(\lambda, f)] + R_N(\lambda, r^2 f)$$

which gives us $\Phi_1(N, \lambda)$ as

$$\Phi_1(N, \lambda) = \frac{\partial \Phi}{\partial \lambda} - \frac{N}{\lambda}[\Phi(N, \lambda) - \Phi(N-1, \lambda)]$$

Now, we know that $\Phi(N, \lambda)$ is a function of the product $\Phi((2N+n)\lambda)$, and so if we keep this product constant, we can change the derivatives above from λ to N, to get

$$\Phi_1(N, \lambda) = \frac{1}{2}\frac{n}{\lambda}\frac{\partial \Phi}{\partial N} + \frac{N}{\lambda}\left[\frac{\partial \Phi}{\partial N} + \Phi(N, \lambda) - \Phi(N-1, \lambda)\right]$$

We can put $\Delta\Phi = \Phi(N, \lambda) - \Phi(N-1, \lambda)$ and

$$S\Phi = \frac{\partial}{\partial N}\Phi, \qquad D\Phi = \frac{\partial}{\partial N}\Phi - \Delta\Phi, \qquad T\Phi = ND\Phi$$

and using these we can re-write the above equation as

$$\Phi_1(N, \lambda) = \frac{1}{\lambda}\left[\frac{n}{2}S + T\right]\Phi(N, \lambda)$$

For any k, we see that $\Phi_k(N, \lambda)$ can be obtained by a combination of actions of differentiation and the operator Δ, and hence we can write

$$\Phi_k(N, \lambda) = \lambda^{-k}\sum_{i+j+m=k} a_{ijm}S^i T^j S^m \Phi(N, \lambda)$$

So we only have to study the action of operators S and T, defined above, on $\Phi(N, \lambda)$ to get the appropriate bounds.

The action of S^m on Φ can be better understood from the fact that $\Phi(N, \lambda)$ is a function of product of $(2N+n)$ and λ and so $S^m(\Phi(N,\lambda)) = \Phi^m((2N+n)\lambda)(w\lambda)^m$ and because of the hypothesis of the theorem, we know that a factor of $(2N+n)^{-m}$ can be pulled.

We ask if the same thing can be done for the operator T, if so, then we can pull out a factor of $(2N+n)^{-m-j-i} = (2N+n)^{-k}$ and so we'll get a bound for $|\Phi_k|^2$ and because $k > \frac{n+1}{2}$, we get our claim. Hence, the last thing to study is the action of the operator T □

Lemma 4.8. *For the aforementioned T^j we have*

$$T^j\Phi = \sum C_{pqm} N^p D^q \{\Delta^m \Phi\}$$

and the sum has the limits extending for all p, q, m such that $j + p \leq 2q + m \leq 2j$

Proof. We'll use Principle of mathematical induction on j. The base case of $j = 1$ follows trivially from the definition of T. Now assume that the equality holds for some j, now consider the case $j+1$, that is

$$T^{j+1}\Phi = \sum C_{pqm} ND(N^p D^q(\Delta^m \Phi))$$

where $j + p \leq 2q + m \leq 2j$ We have to use the nice property of Δ that

$$\Delta(fg)(N) = \Delta f(N)g(N) + F(N-1)\Delta g(N)$$

Using this we can get $\Delta(N^p D\Phi) = \Delta(N^p)D\Phi + (N-1)^p D(\Delta\Phi)$, because D and Δ commute.

We also have the following equalities

1. $\Delta(N^p) = N^p - (N-1)^p = pN^{p-1} - \sum_{i=0}^{p-2} b_i N^i$

2. $(N-1)^p D(\Delta\Phi) = N^p D(\Delta\phi) - \sum_{i=0}^{p-1} a_i N^i D(\Delta\Phi)$.

3. $\frac{\partial}{\partial N}(N^p D\Phi) = pN^{p-1}D\Phi + N^p D\left(\frac{\partial\Phi}{\partial N}\right)$

Now $D = \frac{\partial}{\partial N} + \Delta$ and so

$$\begin{aligned} D(N^p D\Phi) &= \left(\frac{\partial}{\partial N} + \Delta\right)(N^p D\Phi) \\ &= pN^{p-1}D\Phi + N^p D\left(\frac{\partial\Phi}{\partial N}\right) - pN^{p-1}D\Phi \\ &+ \left(\sum_{i=0}^{p-2} b_i N^i\right) D\Phi - N^p D(\Delta\Phi) + \sum_{i=0}^{p-1} a_i N^i D(\Delta\Phi) \\ &= N^p D\left(\frac{\partial}{\partial N} + \Delta\right)\Phi + \sum_{i=0}^{p-1} a_i N^i D(\Delta\Phi) + \sum_{i=0}^{p-2} b_i N^i D\Phi \\ &= N^p D^2\Phi + \sum_{i=0}^{p-1} a_i N^i D(\Delta\Phi) + \sum_{i=0}^{p-2} b_i N^i D\Phi \end{aligned} \tag{37}$$

Now we can plug this into the case $j+1$ and get that

$$\begin{aligned} T^{j+1} &= \sum_{p,q,m} C_{pqm} N^{p+1} D^{q+1}(\Delta^m\phi) + \sum_{p,q,m} C_{pqm} \sum_{i=0}^{p-1} a_i N^{i+1} D^q(\Delta^{m+1}\Phi) \\ &+ \sum_{p,q,m} C_{pqm} \sum_{i=0}^{p-2} b_i N^{i+1} D^q(\Delta^m\Phi) \end{aligned}$$

which is the required form of the lemma.

Using Taylor's integral form we can write

$$D\Phi(N) = \int_0^1 t \frac{\partial^2}{\partial N^2}\Phi(N-1+t,\lambda)\,dt$$

This shows that the action of D brings down a factor of N^{-2} and so D^q will bring a factor of N^{-2q}, when applied to Φ. The action of Δ^m, on Φ, will bring a factor of N^{-m}, and so with the previous lemma giving us the form of T, we have that T brings down, a factor of N^{p-2q-m} which means it essentially brings down a factor of N^{-j}. $\square$

Hence, we have that $S^i T^j S^m$ when acts on $\Phi(N,\lambda)$, brings a factor of $N^{-i-j-k} = N^{-k}$, and so we get the bound for $\Phi_k(N,\lambda)$, giving us the theorem.

5 Appendices

This Appendix will contain all the important results which we have used, either with proof or just their statements. These have been compiled here without any motivation, which if we delve into would be another extensive thesis in itself.

5.1 Appendix A (Gamma Function)

For a complex number z with $Re(z) > 0$, we have the following function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$$

We can use integration by parts to evaluate this integral and get a recurring relation for the **gamma function**. which comes out to be

$$\Gamma(z) = z\Gamma(z-1)$$

and $\Gamma(n) = (n-1)!$, where z has positive real part and $n \in \mathbb{Z}^+$.

This function also shows up in evaluating the integral of the Gaussian function $e^{-|x|^2}$, and gives a rather nice identity

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

The gamma function has an intimate connection with the *volume of a unit ball* and the *surface of a unit sphere*. This is happens because we can easily switch into the polar coordinates. Let ω_{n-1} be the surface area of the unit ball in n-dimensions and ν_n be the volume of the unit ball in n-dimensions. Then we have

$$\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

and

$$\nu_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} = \frac{\omega_{n-1}}{n}$$

5.2 Appendix B (Spherical Coordinates)

We can change the Cartesian coordinates into spherical coordinates for ease of integration, and similarly we should be able to reverse this change, that is, go to Cartesian coordinates from spherical coordinates. This change is given by

$$\int_{RS^{n-1}} F(x)\,d\mu(x) = \int_{y_1=0}^{\pi} \cdots \int_{y_{n-2}=0}^{\pi} \int_{y_{n-1}=0}^{2\pi} F(x(y))J(n,R,y)\,dy_{n-1}\,dy_{n-2}\cdots dy_1$$

for $0 \le y_1, y_2, \cdots, y_{n-2} \le \pi$ and $0 \le \theta = y_{n-1} \le 2\pi$ where $J(n,R,y)$ is the Jacobian of the transformation, $x(y) = x_1(y_1,\cdots,y_{n-1}),\cdots,x_{n-1}(y_1,\cdots,y_{n-1})$ and

$$\begin{aligned}
x_1 &= R\cos y_1 \\
x_2 &= R\sin x_1 \cos y_2 \\
&\vdots \\
x_{n-1} &= R\sin y_1 \sin y_2 \cdots \sin y_{n-2} \cos y_{n-1} \\
x_n &= R\sin y_1 \sin y_2 \cdots \sin y_{n-2} \cos y_{n-1}
\end{aligned}$$

For $n \ge 2$, we have a useful formula which helps in computing the integrals over a sphere S^{n-2}. Let f be a function over S^{n-1}, then we have

$$\int_{RS^{n-1}} f(x)\,d\mu(x) = \int_{-R}^{R} \int_{\sqrt{R^2-s^2}S^{n-2}} f(s,y)\,dy\,\frac{Rds}{\sqrt{R^2-s^2}}$$

To achieve this we put, $y' = (y_2, y_3, \cdots, y_{n-1})$ and $x' = x'(y')$. Now we can use our previous result and write the integral on left as

$$\begin{aligned}\int_{RS^{n-1}} f(x)d\mu(x) &= \int_{y_1=0}^{\pi}\left[\int_{y_2=0}^{\pi}\cdots\int_{y_{n-1}=0}^{2\pi} f(R\cos y_1, R\sin y_1 x')J(n-1,1,y')\,dy'\right]\frac{Rdy_1}{(R\sin y_1)^{2-n}}\\ &= \int_{y_1=0}^{\pi}\int_{S^{n-2}} f(R\cos y_1, R\sin y_1 x')\,d\mu(x')R^{n-1}(\sin y_1)^{n-2}\,dy_1\end{aligned}$$

Again, changing the variables to $s = R\cos y_1$ for $y_1 \in (0,\pi)$, we get the following

$$\int_{RS^{n-1}} f(x)\,d\mu(x) = \int_{-R}^{R}\left\{\int_{S^{n-2}} f(s,\sqrt{R^2-s^2}\theta)\,d\theta\right\}(\sqrt{R^2-s^2})^{n-2}\frac{Rds}{\sqrt{R^2-s^2}}$$

which can now be rescaled to fit our claim.

5.3 Appendix C (Schur's Orthogonality Relations)

Lemma 5.1 (Schur's Lemma 1). *Let π_1 and π_2 be irreducible, finite dimensional representations of G onto the vector space V_1 and V_2 respectively. Suppose that $A : V_1 \to V_2$ is a linear transformation that satisfies $\pi_1(g)A = A\pi_2(g)$ for all $g \in G$. Then either $A = 0$ or A is an isomorphism.*

Proof. From the hypothesis it is straightforward that

1. $\ker(A)$ is a π_2-invariant subspace of V_2.

2. $\mathrm{Im}(A)$ is a π_1-invariant subspace of V_1.

Hence we get $\ker(A) = 0$ or V_2 and $\mathrm{Im}(A) = 0$ or V_1. □

Lemma 5.2 (Schur's Lemma 2). *Let π be a finite dimensional representation of G on a vector space V. If $A : V \to V$ satisfies $A\pi(G) = \pi(g)A$ for all $g \in G$, then A is a multiple of the identity operator.*

Proof. As $\pi(g)(A-\lambda I) = (A-\lambda I)\pi(g)$ for any $g \in G, \lambda \in \mathbb{C}$, by previous Lemma we have $A - \lambda I = 0$ or else is invertible. As λ was arbitrary we can choose it to be the eigenvalue of A, which forces $A = \lambda I$. $\square$

Theorem 5.1 (Schur). *Let π_1 and π_2 be irreducible finite dimensional unitary representations of G. Let $\pi_1(g) = \{\pi^1_{ij}(g)\}$ and similarly $\pi_2(g) = \{\pi^2_{ij}(g)\}$. Then if π_1 and π_2 are equivalent we have*

1. $\int_G \pi^1_{ij}(g)\overline{\pi^2_{kl}(g)}\, dg = 0 \qquad$ *for each i, j, k, l.*

2. $\int_G \pi^1_{ij}(g)\overline{\pi^1_{kl}(g)}\, dg = \frac{\delta^{kl}_{ij}}{\deg(\pi_1)} \qquad \delta^{kl}_{ij}$ *is 1 for $i = k$ and $j = l; 0$ elsewhere.*

Proof. Let B be any linear transformation from $V_2 \to V_1$ and putting

$$A = \int_G \pi_1(g) B \pi_2(g^{-1})\, dg$$

we see that $A\pi_2 = \pi_1 A$ and as π_1 and π_2 are not equivalent A is not an isomorphism and so $A = 0$. As B was arbitrary we can take it to be with only a single non-zero entry and so the first claim holds.

Again let B be a linear transformation but now from V_1 to V_2 and set

$$A = \int_G \pi_1(g) B \pi_2(g^{-1})\, dg$$

Again $\pi_1 A = A\pi_1$ and $A = \lambda I$, so $\int_G \pi_1(g) B \pi_2(g^{-1})\, dg = \lambda I$. The trace of the two matrices will be equal and so we get $tr(B) = \lambda \cdot \deg(\pi_1)$. Again if B is taken to have 1 as its only non-zero value, then we get the trace of B to be either 0 or 1, depending upon the location of the non-zero entry. Finally, we use the fact that

$$\overline{\pi^1_{kl}(g)} = \pi^1_{kl}(g^{-1})$$

plugging this in gives us the second claim. $\square$

5.4 Appendix D

This appendix will solely contain the theorems which we will use in our study and whose proofs are omitted.

Theorem 5.2. *Let A be an $n \times n$ complex matrix such that $A = A^\dagger$ and $\mathrm{Re}(A)$ is positive definite. Then for any $z \in \mathbb{C}^n$,*

$$\int e^{-\pi x A x - 2\pi i z x}\, dx = \frac{1}{\sqrt{\det(A)}} e^{-\pi A^{-1} z}$$

where branch of the square root is determined by the condition $\frac{1}{\sqrt{\det(A)}} > 0$, when A is real and positive definite.

Theorem 5.3 (Schwartz-Kernel Theorem)**.** *Let U and V be open sets in $\mathbb{R}^n$, then every distribution $f \in \mathcal{D}'(X \times Y)$ defines a continuous linear map $F : \mathcal{D}(Y) \to \mathcal{D}'(X)$ such that*

$$\langle f, u \otimes v \rangle = \langle Fv, u \rangle$$

for every $u \in \mathcal{D}(U), v \in \mathcal{D}(V)$. Conversely,for every such continuous linear map F there exists one and only one distribution $f \in \mathcal{D}'(U \times V)$ such that the above equality holds. The distribution k is nothing but the kernel of the map K.

For the next theorem, we need some prior notation For $X_0, X_1, \cdots, X_r$ be the first order homogeneous differential operators in an open set $\Omega \subset \mathbb{R}^n$ with C^∞ coefficients , and $c \in C^\infty(\Omega)$. Then let P be of the form

$$P = \sum_1^r X_j^2 + X_0 + c$$

Theorem 5.4 (Hörmander)**.** *Let P be of the above form and assume that the operators $X_{j_1}, [X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \cdots, [X_{j_1}[X_{j_2}[X_{j_3} \cdots X_{j_k}]]]$ where $j_i = 0, 1, \cdots, r$ there exist n which are linearly independent at any given point in Ω. Then we have that P is hypoelliptic.*

Theorem 5.5 (Hunt)**.** *Let (S_t) be a positive semi-group on the Lie group $\mathcal{G}$. Its infinitesimal generator M is defined on at least $\mathcal{C}^2$ and has there the representation*

$$
\begin{aligned}
Mf(\tau) &= \sum a_i X_i f(\tau) + \sum a_{ij} X_i X_j f(\tau) \\
&\quad + \int_{\mathcal{G}_\rfloor \setminus \{\infty\}} \left[f(\tau\sigma) - f_(\tau) - \sum X_i f(\tau) g_i(\sigma) \right] G(d\sigma)
\end{aligned}
$$

Here the matrix (a_{ij}) is symmetric positive semi-definite and G is a positive measure on $\mathcal{G}_c \setminus \{1\}$ for which the integral $\int \phi(\sigma) G(d\sigma)$ is finite. The restriction of M to $\mathcal{C}^2$ determines the semi-group (S_t). The measure G and the operator $\Delta_2 \equiv \sum a_{ij} X_i X_j$ are determined by M, independently of the choice of the basis $X_1, X_2, \cdots, X_n$ and the functions g_i. Conversely, if the transformation is defined as above, and if the matrix (a_{ij}) and the measure G satisfy the conditions above, then M is the restriction to $\mathcal{C}^2$ of the infinitesimal generator of exactly one probability semi-group.

Theorem 5.6 (Spectral Theorem)**.** *To every self-adjoint operator A in H corresponds a unique resolution E of the identity, on the Borel subsets of the real line, such that*

$$
\langle Ax, y \rangle = \int_{\infty}^{\infty} t \, dE_{x,y}(t) \qquad (x \in \mathcal{D}(A), , y \in H)
$$

Moreover, E is concentrated on the spectrum of A, $\sigma(A) \subset (-\infty, \infty)$, in the sense that $E(\sigma(A)) = I$

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