Minimizing Laser Linewidth under Finite Resource Constraints: A Quantum Metrology Approach

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by

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Certificate

This is to certify that this dissertation entitled Minimizing Laser Linewidth under Finite Resource Constraints: A Quantum Metrology Approachtowards the partial fulfillment of the BS-MS dual degree program at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Aanandita Kottisa at Indian Institute of Technology, Bombay under the supervision of Professor Sai Vinjanampathy, Department of Physics during the academic year 2025-2026.

Professor Sai Vinjanampathy

Committee:

Professor Sai Vinjanampathy

Professor M S Santhanam

This thesis is devoted to my Amma and Nanna

Declaration

I hereby declare that the matter embodied in the report entitled Minimizing Laser Linewidth under Finite Resource Constraints: A Quantum Metrology Approach are the results of the work carried out by me at the Department of Physics, Indian Institute of Technology, Bombay, under the supervision of Professor Sai Vinjanampathy and the same has not been submitted elsewhere for any other degree.

K. Aanandita

Aanandita Kottisa

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Contributions

Generative AI tools, including ChatGPT (GPT-5.3, OpenAI) and Google Gemini, were used in a limited capacity for language refinement and to improve the clarity of selected passages during the preparation of this thesis. These tools were also occasionally employed for brainstorming and structuring ideas. All scientific content, derivations, and conclusions have been independently verified by the author and reviewed in consultation with the supervisor.

Abstract

The coherence of laser light is a fundamental property that plays a crucial role in precision measurements, interferometry, and quantum technologies. Understanding the physical mechanisms that determine laser linewidth and phase stability is therefore an important problem in quantum optics. In this thesis, we study how laser systems can be modeled and how their coherence properties can be analyzed within a quantum mechanical framework. We first review the basic principles of quantum optics and laser theory, focusing on the description of phase diffusion that leads to the finite linewidth of a laser. We then introduce the tools of estimation theory and use them to analyze the phase diffusion process in terms of Fisher Information and estimation precision. This perspective allows us to connect the coherence properties of lasers with fundamental limits on parameter estimation. In particular, we discuss the emergence of Heisenberg-limited scaling in models of laser coherence and examine the conditions required to achieve such limits. Finally, we analyze these limits from a resource-accounting perspective, highlighting how the structure of the Hamiltonian and the available physical resources constrain the realization of Heisenberg-limited lasers.

Contents

Chapter 1

Introduction

The coherence properties of laser light play a central role in many areas of modern physics, including precision spectroscopy, interferometry, and quantum information processing. A key quantity characterizing a laser is its linewidth, which determines how stable the phase of the emitted optical field remains over time. Understanding the fundamental limits that govern laser coherence is therefore an important problem in quantum optics. While conventional laser theory explains many aspects of linewidth and phase diffusion, more recent developments have shown that these limits can also be analyzed using tools from quantum parameter estimation and quantum metrology.

Quantum metrology provides a general framework for studying how precisely physical parameters can be estimated from measurements performed on quantum systems. In this context, the concept of FI plays a central role in quantifying the sensitivity of measurement outcomes to small changes in a parameter. When the parameter is encoded through unitary evolution generated by a Hamiltonian, the achievable precision is closely related to the variance of the generator. These ideas lead naturally to the notion of the Heisenberg limit, which describes the ultimate scaling of estimation precision with respect to the resources used in the measurement process.

In the context of laser physics, these concepts can be used to analyze the limits on phase stability and coherence. In particular, the work of Wiseman and collaborators has explored the structure of the laser model that restrict the achievable linewidth using arguments from estimation theory and data processing. Understanding these limits provides insight into what would be required to realize lasers that approach Heisenberg-limited coherence.

Motivated by these developments, the present thesis examines the fundamental limits of laser coherence from the perspective of quantum estimation theory and resource accounting. We first review the relevant concepts from quantum optics and laser theory in order to establish the physical description of optical coherence and phase diffusion. We then introduce the framework of quantum parameter estimation, focusing on the role of Fisher Information and the scaling behaviour associated with different resource assumptions. Building on this background, we discuss the analysis presented in Wiseman's work on the limits of laser coherence. Finally, we explore a simplified resource-counting perspective and examine how the structure of the Hamiltonian governing the system influences the possibility of achieving Heisenberg-limited behaviour.

The remainder of this thesis is organized as follows. Chapter 2 reviews the basic concepts of quantum optics relevant to the description of laser fields and coherence. Chapter 3 introduces the formalism of quantum parameter estimation and discusses the role of FI in determining precision limits. Chapter 4 presents a discussion of the limits on laser coherence derived in the work of Wiseman and collaborators. Finally, Chapter 5 describes our analysis of resource accounting and discusses its implications for the realization of Heisenberg-limited lasers.

Chapter 2

Quantum Optics and Laser Theory

This chapter introduces the basic concepts of quantum optics that are required for the analysis in the later chapters.

2.1 Quantization of Field

The quantization of the electromagnetic field (EM) and the associated quantum fluctuations are discussed in many quantum optics textbooks such as [1], [3], and [2]. Here we follow the procedure and methods detailed in [1].
We have read through the Maxwell Equations and understood light from a continuous wave perspective. Following quantum mechanics we were introduced to discreet energy levels which were formerly thought of as continuous. Quantization of light deals with exactly this discreet energy levels corresponding to light, which are commonly called photons. We study the limitations of the classical field in the later chapters and why quantization is essential. We first address on how exactly can one make the connection from classical electromagnetic field to quantized light.
We begin with the case of studying the quantization of single-mode field. The setup is modeled as a single mode field in a conducting cavity of length L along z-axis with the walls at $z = 0$ and $z = L$. From the Maxwell equations we know that the boundary condition is that the electric field vanishes at the boundaries implying that the electric field will be a standing wave. The field is assumed to be polarized along one direction, we assume it be

along the x-axis, and given as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = \underbrace{\left(\frac{2\omega^2}{V\epsilon_0}\right)^{1/2} q(t) sin(kz)}_{E_x(z,t)} \ \boldsymbol{e}_x \tag{2.1}$$

where $\boldsymbol{e}_x$ represents a unit polarization vector. The cavity is assumed to have no other sources. The above result is derived following a solution of the Maxwell's equation along with the appropriate boundary conditions. Here V denotes the volume of the cavity, and k is the wave number defined in terms of frequency ω as $k = \omega/c$. The boundary condition at $z = L$ leads to a discrete set of allowed frequencies given by $\omega_m = c(m\pi/L)$, where $m = 1, 2, \ldots$. In the following discussion we consider only one of these allowed modes with frequency ω and neglect the remaining modes.

Further, the magnetic field is defined as

$$\boldsymbol{B}(\boldsymbol{r},t) = \underbrace{\left(\frac{\mu_0\epsilon_0}{k}\right)\left(\frac{2\omega^2}{V\epsilon_0}\right)^{1/2} \dot{q}(t) cos(kz)}_{B_y(z,t)} \boldsymbol{e}_y \tag{2.2}$$

where $q(t)$ is a time-dependent field variable with the dimensions of length. It plays the role of a canonical position variable in the oscillator description of the field. Thereby, $\dot{q}(t)$ is canonical momentum for a particle of unit mass ($p(t) = \dot{q}(t)$). This allows the electromagnetic field within the cavity be written in terms of its spatial mode structure.

Finally, the Hamiltonian for this system is given by

$$H = \frac{1}{2}\int \left[\epsilon_0 \boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{E}(\boldsymbol{r},t) + \frac{1}{\mu_0}\boldsymbol{B}(\boldsymbol{r},t)\cdot\boldsymbol{B}(\boldsymbol{r},t)\right] \mathrm{d}V \tag{2.3}$$

From solutions of EM field we can see that

$$H = \frac{1}{2}(p^2 + \omega^2 q^2). \tag{2.4}$$

The above expression is analogous to a harmonic oscillator of unit mass. In this expression the EM field amplitudes play the role of position and momentum variables.

In the current context of quantum mechanics, the classical variables p and q are promoted to be operators $\hat{p}$ and $\hat{q}$ that correspond to observable quantities. Thereby, the following canonical commutation condition is satisfied:

$$[\hat{q}, \hat{p}] = i\hbar\hat{I}. \tag{2.5}$$

Employing this transformation, the electrical and magnetic fields in Eqs. (2.1) and (2.2) and the Hamiltonian in Eq. (2.3) can be trivially modified to give $\hat{E}_x$ and $\hat{B}_y$ in terms of the quantum operators. More clearly, the Hamiltonian in Eq. (2.3) takes the form of a 1D quantum harmonic oscillator.

From elementary quantum mechanics we know the solution of QHO in terms of annihilation and creation operators. The annihilation operator is defined as

$$\hat{a} = (2\hbar\omega)^{-1/2}(\omega\hat{q} + i\hat{p}) \tag{2.6}$$

and the creation operator is the complex conjugate of the above expression. Further, these operators satisfy the standard commutation relation.

The EM field can be redefined in terms of the annihilation and creation operators presenting with the Hamiltonian operator as follows

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}). \tag{2.7}$$

The Heisenberg's equation defined over an arbitrary operator $\hat{O}$ is

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{O}]. \tag{2.8}$$

Using the Eqs. ((2.8)) and ((2.7)) the annihilation operator can be given as:

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t}. \tag{2.9}$$

From the QHO we know that the energy eigenvalues of the Hamiltonian are the eigenvalues of the operator $\hat{a}^\dagger\hat{a}$, called the number operator $\hat{n}$ with eigenket $|n\rangle$ with energy E_n such that

$$\hat{H}|n\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})|n\rangle = E_n|n\rangle \,. \tag{2.10}$$

The state $|n\rangle$ corresponds to a Fock state containing n photons.

To study the creation operator $\hat{a}^\dagger$ acting on $|n\rangle$, we apply $\hat{a}^\dagger$ to Eq. ((2.10)). Using the commutation relations, it follows that the resulting state is again an eigenstate of the Hamiltonian, but with a modified eigenvalue.

$$\hat{H}(\hat{a}^\dagger|n\rangle) = (E_n + \hbar\omega)(\hat{a}^\dagger|n\rangle). \tag{2.11}$$

The state obtained by applying the operator $\hat{a}^\dagger$ to $|n\rangle$ has an energy that exceeds E_n by an amount $\hbar\omega$. Thus $\hat{a}^\dagger|n\rangle$ corresponds to the next higher energy eigenstate, indicating that the operator increases the excitation number by one quantum. In the context of the electromagnetic field, this operation can be interpreted as the creation of a photon with energy $\hbar\omega$. For this reason, $\hat{a}^\dagger$ is referred to as the creation operator.

A similar analysis can be carried out for the operator $\hat{a}$. Acting with $\hat{a}$ on $|n\rangle$ produces a state whose energy eigenvalue is $E_n - \hbar\omega$. This operation reduces the excitation number by one and corresponds to the removal of a photon of energy $\hbar\omega$. Consequently, $\hat{a}$ is known as the annihilation operator.

Since the Hamiltonian of the quantum harmonic oscillator is composed of positive operators, its energy spectrum must be bounded from below. Therefore there exists a lowest energy eigenstate, denoted by $|0\rangle$, with eigenvalue $E_0 > 0$. This ground state satisfies the condition

$$\hat{a}|0\rangle = 0.$$

The eigenvalue equation for the ground state then leads to

$$\hat{H}|0\rangle = 1/2\hbar\omega|0\rangle \tag{2.12}$$

From the Eq(2.11) we can deduce that $E_{n+1} = E_n + \hbar\omega$, hence the following relation

$$E_n = \hbar\omega(n + \frac{1}{2}), \; n = 0, 1, 2...$$

Thus we can say that

$$\hat{n}\,|n\rangle = n\,|n\rangle$$

Since these kets are energy eigenstates they are normalized to one $\langle n|n\rangle = 1$
We look into the exact scaling of state $|n\rangle$ upon the action of $\hat{a}$. We have

$$\hat{a}\,|n\rangle = c_n\,|n-1\rangle \tag{2.13}$$

The constant c_n can be calculated by taking the inner product

$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle = |c_n|^2 \tag{2.14}$$
$$n = |c_n|^2 \tag{2.15}$$

and thus we can take $c_n = \sqrt{n}$. This gives

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n\rangle\,. \tag{2.16}$$

Similarly we can deduce that

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n\rangle\,. \tag{2.17}$$

Since $\hat{H}$ and $\hat{n}$ are hermitian operators, the different eigenstates are orthogonal and form a complete set,

$$\sum_{n=0}^{\infty} |n\rangle\,\langle n| = 1. \tag{2.18}$$

2.2 Fluctuations in Quantized Field

In this section we examine the connection between photon number states and the electric field of the quantized electromagnetic mode. The discussion presented here follows the treatment in [1].

To understand how a photon number state influences the electric field, we evaluate the expectation value of the electric field operator in the Fock state $|n\rangle$,

$$\langle n|\hat{E}_x(z,t)|n\rangle = \mathcal{E}_0 \sin(kz) \left[\langle n|(\hat{a}+\hat{a}^\dagger)|n\rangle\right] = 0. \tag{2.19}$$

Thus the average electric field associated with a number state vanishes. However, the energy density calculated by the mean squared value of the field is non-zero. Carrying out the algebra gives

$$\langle n|\hat{E}_x^2(z,t)|n\rangle = 2\mathcal{E}_0 \sin^2(kz)\left(n+\frac{1}{2}\right). \tag{2.20}$$

The fluctuations are determined by the variance,

$$\langle \Delta\hat{E}_x^2(z,t)\rangle = 2\mathcal{E}_0 \sin^2(kz)\left(n+\frac{1}{2}\right). \tag{2.21}$$

These fluctuations remain finite in no field (vacuum state $|0\rangle$). Therefore, the electromagnetic field exhibits intrinsic fluctuations even in the absence of photons. These residual fluctuations are referred to as *vacuum fluctuations*.

2.2.1 Quadrature operators

To detail the measurable components of the EM field, we introduce the quadrature operators. These operators correspond to real field observables. The discussion presented here follows the treatment given in [1].

Including the explicit time dependence, the electric field operator becomes

$$\hat{E}_x = \mathcal{E}_0(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) sin(kz). \tag{2.22}$$

Quadrature operators are define as

$$\hat{X}_1 = \frac{1}{2}(\hat{a}+\hat{a}^\dagger) \tag{2.23}$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a}-\hat{a}^\dagger). \tag{2.24}$$

The electric field in terms of the quadrature operators is

$$\hat{E}_x(z,t) = 2\mathcal{E}_0 sin(kz)[\hat{X}_1 cos(\omega t) + \hat{X}_2 sin(\omega t)]. \tag{2.25}$$

From the expression above it is evident that the operators $\hat{X}_1$ and $\hat{X}_2$ represent two field amplitudes that oscillate with a relative phase shift of $\pi/2$. These operators play a role analogous to the canonical position and momentum variables of the harmonic oscillator. The corresponding commutation relation between the quadrature operators is given by

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}. \tag{2.26}$$

Using the commutation relation between $\hat{a}$ and $\hat{a}^\dagger$ we derive the commutation relationship between $\hat{X}_1$ and $\hat{X}_2$ to be

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}. \tag{2.27}$$

The Heisenberg uncertainty relationship for the quadratures then follows to be

$$\langle(\Delta\hat{X}_1)^2\rangle \langle(\Delta\hat{X}_2)^2\rangle \geq \frac{1}{16}. \tag{2.28}$$

The minimum uncertainty is attained for special states called coherent states which are defined in the next section.

2.3 Phase Operator and Number Operator

In addition to the quadrature operators, which correspond to measurable field observables, it is also useful to examine the phase operator and its conjugate variable. The motivation and discussion presented in this section are largely based on the treatment given in [4]. For a simple harmonic oscillator described by Eq. (2.4), the classical solution is well known and is given by

$$X(t) = Ae^{i\phi} + A^* e^{-i\phi}, \quad \phi = \omega t \tag{2.29}$$

$$P(t) = i\omega[Ae^{i\phi} - A^* e^{-i\phi}]. \tag{2.30}$$

Comparing this equation with the Quadrature definitions Eq (2.23) which are scaled to be dimensionless position and momentum operators. We can think analogously and assume the form of $\hat{a}^\dagger$ and $\hat{a}$ to be $\hat{R}e^{i\hat{\phi}}$ and $e^{-i\hat{\phi}}\hat{R}$ respectively, where $\hat{R}$ is Hermitian and $e^{i\hat{\phi}}$ is unitary implying that $\hat{\phi}$ is Hermitian. Since $\hat{n} = \hat{a}^\dagger\hat{a}$ it can be seen that

$$\hat{n} = \hat{R}^2 \tag{2.31}$$

$\hat{R}$ is hermitian and hence diagonal in some basis with eigenvalues as real. Now in the same basis the $\hat{R}^2$ would be diagonal with positive eigenvalues. From equation Eq (2.31) we can see that $\hat{R}$ is diagonal in the number basis and therefore

$$\hat{R} = \sqrt{\hat{n}} \tag{2.32}$$

$$\hat{a}^\dagger = \sqrt{\hat{n}}e^{-i\hat{\phi}} \tag{2.33}$$

$$\hat{a} = e^{i\hat{\phi}}\sqrt{\hat{n}}. \tag{2.34}$$

These equations will lead to the following commutation relation

$$[\hat{n}, e^{i\hat{\phi}}] = e^{i\hat{\phi}} \tag{2.35}$$

along with the property that $e^{i\hat{\phi}}$ is unitary gives

$$[\hat{n}, \hat{\phi}] = -i. \tag{2.36}$$

However taking matrix elements for the above relationship gives

$$(n - m)\langle n|\hat{\phi}|m\rangle = -i\delta_{nm} \tag{2.37}$$

This is obviously impossible cause for $n = m$ the right hand side is zero while the left hand side is $-i$. An important assumption that we made which was wrong is that $e^{i\hat{\phi}}$ is unitary. It is not a unitary matrix and $\hat{\phi}$ is not hermitian. Over several years many different phase operators have been defined but we shall define here one of the most commonly used phase operator called the Susskind-Glogower operators which are denoted as $\hat{e}^{\pm i\phi}$. They are defined as

$$\hat{e}^{-i\phi} = \hat{a}^\dagger(\hat{n}+1)^-1/2 \tag{2.38}$$

$$\hat{e}^{i\phi} = (\hat{n}+1)^-1/2\hat{a} \tag{2.39}$$

From these definitions we can see that

$$\hat{e}^{i\phi}\hat{e}^{-i\phi} = 1$$
$$\text{but} \quad \hat{e}^{-i\phi}\hat{e}^{i\phi} \neq 1.$$

This definition gives the same operator as $\hat{a}$ and $\hat{a}^\dagger$ just without the scaling factors.

$$\hat{e}^{i\phi}|m\rangle = |m-1\rangle \tag{2.40}$$
$$\hat{e}^{-i\phi}|m\rangle = |m+1\rangle . \tag{2.41}$$

We can use these operators to define trigonometric phase functions which are hermitian operators and hence become observables. In the large amplitude limit these functions resemble the classical functions of phase,

$$cos\hat{\phi} = \frac{1}{2}[\hat{e}^{i\phi} + \hat{e}^{-i\phi}] \tag{2.42}$$
$$sin\hat{\phi} = \frac{1}{2i}[\hat{e}^{i\phi} - \hat{e}^{-i\phi}]. \tag{2.43}$$

The eigenvectors for the phase operator can be readout from

$$\hat{e}^{i\phi}\sum_{m=0}^{\infty} e^{im\theta}|m\rangle = e^{i\theta}\sum_{m=0}^{\infty} e^{im\theta}|m\rangle . \tag{2.44}$$

Here $|\theta\rangle = \sum_{m=0}^{\infty} e^{im\theta}|m\rangle$ is not normalizable but sum to identity.

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta\,|\theta\rangle\langle\theta| = 1. \tag{2.45}$$

Derived from

$$\int_0^{2\pi} d\theta e^{i(m-m')\theta} = 2\pi\delta_{m,m'}. \tag{2.46}$$

So for any state

$$|\psi\rangle = \sum_n C_n |n\rangle , \tag{2.47}$$

We may associate a phase distribution $\mathcal{P}(\theta)$ defined as

$$\begin{aligned}\mathcal{P}(\theta) &= \frac{1}{2\pi}|\langle\theta|\psi\rangle|^2 \\ &= \frac{1}{2\pi}\Big|\sum_n e^{-in\theta}C_n\Big|\end{aligned} \tag{2.48}$$

$\mathcal{P}(\theta)$ is clearly always positive and integrating over θ we can see that its normalized

$$\int_0^{2\pi}\mathcal{P}(\theta)d\theta = \langle\psi|\psi\rangle = 1. \tag{2.49}$$

2.4 Coherent States

We now examine additional properties of coherent states. As discussed earlier, these states saturate the minimum uncertainty relation between the field quadratures. Beyond this defining feature, coherent states possess several important characteristics.

A detailed discussion of coherent states and their applications can be located in multiple references, including [1] and [3]. In the present section, the presentation and pedagogical approach largely follow the treatment given in [2]. A coherent state is defined as

$$|\alpha\rangle = e^{|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,. \tag{2.50}$$

When apply $\hat{a}$ on a coherent state $|\alpha\rangle$ we get

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\,. \tag{2.51}$$

Hence coherent states are eigenvalues of $\hat{a}$. Coherent states are also normalized states.

$$\begin{aligned}\langle\alpha|\alpha\rangle &= e^{-|\alpha|^2}\sum_{m,n=0}^{\infty}\frac{\alpha^m(\alpha^*)^n}{\sqrt{m!n!}}\delta_{m,n} \\ &= e^{-|\alpha|^2}\sum_{m=0}^{\infty}\frac{(|\alpha|^2)^m}{m!} = 1\end{aligned} \tag{2.52}$$

With some algebra, we can see that coherent states satisfy the minimum uncertainty relationship.

$$\langle\alpha|\hat{X}_1|\alpha\rangle = \frac{1}{2}(\alpha + \alpha^*) \tag{2.53}$$

$$\langle\alpha|\hat{X}_2|\alpha\rangle = \frac{1}{2i}(\alpha - \alpha^*) \tag{2.54}$$

Now lets calculate $\hat{X}_1^2$ and $\hat{X}_2^2$ expectation value

$$\langle\alpha|\hat{X}_1^2|\alpha\rangle = \frac{1}{4}\langle\alpha|(\hat{a}^2 + (\hat{a}^\dagger)^2 + 2\hat{n} + 1)|\alpha\rangle = \frac{1}{4}((\alpha + \alpha^*)^2 + 1) \tag{2.55}$$

$$\langle\alpha|\hat{X}_2^2|\alpha\rangle = \frac{-1}{4}\langle\alpha|(\hat{a}^2 + (\hat{a}^\dagger)^2 - 2\hat{n} - 1)|\alpha\rangle = \frac{-1}{4}((\alpha - \alpha^*)^2 - 1) \tag{2.56}$$

Combining these sets of equations

$$\langle\Delta\hat{X}_1^2\rangle = \langle\hat{X}_1^2\rangle - \langle\hat{X}_1\rangle^2 = 1/4 = \langle\Delta\hat{X}_2^2\rangle \tag{2.57}$$

Hence it satisfies the minimum uncertainty principle that vacuum states also show. When we calculate the average photon number with a coherent state we get

$$\begin{aligned}\langle\alpha|\hat{n}|\alpha\rangle &= e^{-|\alpha|^2}\sum_{n,m=0}^{\infty} n\frac{(\alpha^*)^m(\alpha)^n}{\sqrt{n!}\sqrt{m!}}\delta_{nm} \\ &= |\alpha|^2.\end{aligned} \tag{2.58}$$

We can also calculate the fluctuations in the photon count which gives

$$\langle\Delta\hat{n}\rangle^2 = |\alpha|^2 = \langle\hat{n}\rangle \tag{2.59}$$

This shows that the photon statistics for a coherent state are Poissonian in nature. Also the fractional uncertainity in the photon number decreases with increasing $\langle\hat{n}\rangle$. Now lets look at its phase distribution by taking $\alpha = |\alpha|e^{i\phi}$

$$\begin{aligned}\mathcal{P}(\theta) &= \frac{1}{2\pi}|\langle\theta|\alpha\rangle|^2 \\ &= \frac{1}{2\pi}e^{-|\alpha|^2}\left|\sum_{n=0}^{\infty} e^{(in(\theta-\phi)}\frac{|\alpha|^n}{\sqrt{n!}}\right|\end{aligned} \tag{2.60}$$

The distribution tends to gaussian distribution for large $|\alpha|^2$, hence the above sum can be approximated to

$$\mathcal{P}(\theta) = \left(\frac{2|\alpha|^2}{\pi}\right)^{1/2} exp[-2|\alpha|^2(\theta-\phi)^2] \tag{2.61}$$

From this we can see that the Gaussian curve has peak at $\theta = \phi$. Further the peak thickness turns narrower with increasing $\langle\hat{n}\rangle = |\alpha|^2$. This means the phase gets more localized with increasing $\langle\hat{n}\rangle$. From the quadrature calculation we can see that the EM field expectation values are periodic in time and space for a coherent state.
From the discussion above, it follows that coherent states are quantum states that most closely resemble classical monochromatic electromagnetic fields, particularly in the limit of large mean photon number $\langle\hat{n}\rangle$. A property of coherent states is the absorption of photons from the field does not alter the form of the state. In other words, repeated photon absorption leaves the state unchanged up to a proportionality factor.

Although coherent states are not eigenstates of any physical observable, they are closely related to practical measurement processes. In many optical experiments the detection mechanism is based on photoelectric measurements, which involve the annihilation operator. Because coherent states are eigenstates of the $\hat{a}$ operator, they naturally arise in the description of such measurements. This provides an intuitive connection between the quantum description of the EM fields and its classical counterpart.

Coherent states may also be expressed as vacuum states that have been displaced in phase space. The operator responsible for generating this transformation is known as the displacement operator. Thus a coherent state $|\alpha\rangle$ can be written as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n(\hat{a}^\dagger)^n}{n!}|0\rangle \tag{2.62}$$

$$= e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}|0\rangle\,. \tag{2.63}$$

We can symmetrize the equation by inserting the operator $e^{-\alpha^*\hat{a}}$ just before the $|0\rangle$ in Eq (2.62). Thus the equation changes to

$$|\alpha\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|0\rangle\,. \tag{2.64}$$

Now using the Campbell-Baker-Hausdorff operator identity as given in [1] We can re-write the equation for displacement as

$$|\alpha\rangle = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}\,|0\rangle\,. \tag{2.65}$$

This $e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ is the displacement operator $\hat{D}(\alpha)$.Coherent states under time evolution by the Hamiltonian of the form $\hat{H} = \hbar\omega(\hat{n} + 1/2)$ evolves into an other coherent state with a phase factor.

$$\begin{aligned}|\psi(t)\rangle &= e^{-i\omega t/2}e^{-i\omega t\hat{n}}\,|\alpha\rangle \\ &= e^{-i\omega t/2}\,|\alpha e^{-i\omega t}\rangle\,. \end{aligned}\tag{2.66}$$

Coherent states thus undergo continuous, periodic evolution into other coherent states over time, with period $2\pi/\omega$.
Coherent states are normalized but not orthogonal. We can see this

$$\begin{aligned}\langle\alpha_2|\alpha_1\rangle &= e^{-(|\alpha_1|^2+|\alpha_2|^2)/2}\sum_n \frac{(\alpha_1^*\alpha_2)^n}{n!} \\ &= e^{-|\alpha_1-\alpha_2|^2/2}e^{(\alpha_2^*\alpha_1-\alpha_2\alpha_1^*)/2}. \end{aligned}\tag{2.67}$$

There exist no values of α_1 and α_2 such that the above term vanishes. Hence the states are not orthogonal.
Despite their non-orthogonality coherent states span the entire Hilbert space and form a convenient basis. In order to see that they span the space we can check that

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\,d^2\alpha = 1 \tag{2.68}$$

over the entire complex plane. We can start by writing $\alpha = re^{i\theta}$ so that $d^2\alpha = rdrd\theta$.

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\,d^2\alpha = \frac{1}{\pi}\int_0^\infty dr\int_0^{2\pi} d\theta\sum_n\sum_m e^{-r^2}\frac{r^{n+m+1}}{\sqrt{n!m!}}e^{i(n-m)\theta}\,|n\rangle\langle m|\,. \tag{2.69}$$

Now on interchanging the order of integration and summations, the angular part leads to a factor $2\pi\delta_{nm}$ this leads to the double summation to become a single sum

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, d^2\alpha = \sum_n \frac{1}{n!}|n\rangle\langle n| \int_0^\infty 2e^{-r^2} r^{2n+1} dr \tag{2.70}$$

$$= \sum_n |n\rangle\langle n| \tag{2.71}$$

$$= 1. \tag{2.72}$$

Since the Fock states span the entire Hilbert space the completeness relationship for the coherent states also follows. Hence any state can be written as sum of coherent states. Thus any state $|\psi\rangle$ can be written as

$$|\psi\rangle = \frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\psi\rangle\, d^2\alpha. \tag{2.73}$$

Coherent states are linearly dependent too. Hence any state doesn't have a unique representation in terms of coherent states. Because of this coherent states form an over-complete basis.

2.5 Different Representations

In quantum optics states can be represented in different basis like the Fock space or in therms of coherent states etc. These different kinds of representation of states and operators in different basis help to make calculations simpler as well as some representations show natural resemblance in structure to classical forms, which have been well developed. The description we provide here is motivated from [1] and [2]

2.5.1 P representation

P-representation also known as the Glauber-Sudarshan P function is defined as

$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha|\, d^2\alpha \tag{2.74}$$

The P-function provides a representation that is analogous to a phase-space distribution in the space of coherent states. In this representation the complex variable α serves as the phase-space coordinate. Since the density operator $\hat{\rho}$ is Hermitian, the function $P(\alpha)$ must be real. However, for certain quantum states of the field the P-function can take negative values, distinguishing it from a classical probability distribution.

Expectation values of observables that are written in normal order-where the annihilation operators a appear to the right of the creation operators $a^\dagger$-can be conveniently expressed using the P-representation:

$$G^{\hat{(N)}}(a, a^\dagger) = \sum_n \sum_m C_{nm}(a^\dagger)^n (a)^m \tag{2.75}$$

the expectation of this function is:

$$\langle G^{\hat{(N)}}(a, a^\dagger) \rangle = \int P(\alpha) G^{(N)}(\alpha, \alpha^*) d^2\alpha \tag{2.76}$$

This is the optical equivalence theorem. So the P representation naturally becomes the easier choice when the operators are normally ordered.

2.6 Lindblad Master Equation

Lindblad master equation models the dynamics of a quantum system interacting with a reservoir, under a set of standard assumptions. The treatment and methodology followed in this section are based on the discussion presented in [3]. The fundamental assumptions are as follows

i) The reservoir possesses a much larger number of degrees of freedom than the system and hence remains effectively unchanged by the interaction (Born approximation).

ii) The bath correlation time is significantly shorter than the system's evolution timescale, allowing the dynamics to be treated as memoryless (Markov approximation).The bath state is assumed to be time-independent.

iii) The fast oscillating terms are neglected through the rotating-wave approximation(RWA), ensuring physical consistency and positivity of the reduced dynamics. Under these assumptions, the evolution of the system's state is described by the Lindblad form of the master equation.

$$\dot{\rho}_S(t) = \frac{-i}{\hbar}\,\mathrm{Tr}_R\left[V(t), \rho_S(t_i)\otimes\rho_R(t_i)\right] - \frac{1}{\hbar^2}\,\mathrm{Tr}_R\int_{t_i}^{t} dt'\left[V(t), [V(t'), \rho_S(t)\otimes\rho_R(t_i)]\right]. \tag{2.77}$$

Here the ρ_S and ρ_R are system and reservoir density matrices. $V(t)$ gives the system-reservoir interaction energy. Here t_i is the initial time the interaction starts.
To illustrate the lindblad equation lets study the evolution of the density matrix of a oscillator in a reservoir as in [3]. The operators for the field of interest are a and $a^\dagger$ while the reservoir operators are b_k and $b_k^\dagger$. The Hamiltonian for the system, reservoir and the interaction is respectively

$$H = H_0 + H_1 \tag{2.78}$$

$$H_0 = \hbar\nu a^\dagger a + \sum_k \hbar\nu_k b_k^\dagger b_k \tag{2.79}$$

$$H_1 = \hbar\sum_k g_k(b_k^\dagger a + a^\dagger b_k) \tag{2.80}$$

The H_0 is the bare Hamiltonian of the individual system and reservoir while H_1 is the interaction Hamiltonian with g_k being the interaction strength of each mode of the reservoir with the mode of the system. The interaction hamiltonian is then

$$V(t) = \hbar\sum_k g_k[b_k^\dagger a e^{-i(\nu-\nu_k)t} + a^\dagger b_k e^{i(\nu-\nu_k)t}]. \tag{2.81}$$

When the b_k modes are initially in thermal equilibrium, implying the reservoir state is

$$\rho_R = \prod_k (1 - exp(-\frac{\hbar\nu_k}{k_B T}))exp(-\frac{\hbar\nu_k b_k^\dagger b_k}{k_B T}), \tag{2.82}$$

where k_B is the Boltzmann constant and T is the temperature.

Average thermal boson number at frequency ν_k is

$$\bar{n}_k = \frac{1}{exp(\frac{\hbar\nu_k}{k_B T}) - 1}. \tag{2.83}$$

Using the above the equation of motion of the density matrix is defined as:

$$\begin{aligned}\dot{\rho} =& \frac{-\mathcal{C}}{2}\bar{n}_{th}(aa^\dagger\rho - 2a^\dagger\rho a + \rho aa^\dagger) \\ & \frac{-\mathcal{C}}{2}(\bar{n}_{th}+1)(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a).\end{aligned} \tag{2.84}$$

Here the decay constant is given by $\mathcal{C}$. The mean number of quanta at frequency ν in the thermal reservoir is given by $n_{th} = \bar{n}_k$ at ν(given by Eq. (2.83)).This is a thermal master equation and a basic example to study the Lindblad master equation. The above equation can be rewritten as

$$\dot{\rho} = \bar{n}_{th}\mathcal{D}(a^\dagger) + (\bar{n}_{th}+1)\mathcal{D}(a) \tag{2.85}$$

Where $D(A)\rho = A\rho A^\dagger - \frac{1}{2}\{A^\dagger A, \rho\}$. The above equation can be interpreted as the operator $\mathcal{D}(a)$ causing dissipation of bosons of the system while the $\mathcal{D}(a^\dagger)$ causes the addition of bosons(jump operator) into the system. Hence the lindblad master equation comprises of dissipation and jump operators with corresponding rates to give the total evolution of the density matrix. This is the general structure of Lindblad master equation with rates being controlled by the state of the reservoir.

2.7 Langevin Equation

Having examined the time evolution of the density operator, we now turn to an alternative description in which the operators evolve while the state of the system is taken to be time independent. This corresponds to the Heisenberg-picture treatment of open quantum systems. The formal procedure used here follows the approach presented in [3]. The langevin equation and lindblad equations are same as the Heisenberg picture vs Schrodinger picture for open quantum systems. Instead of evolving the density operator ρ, we study the evolution of an observable O. The time derivative of its expectation value is

$$\frac{d}{dt}\langle O\rangle = \mathrm{Tr}\left(O\frac{d\rho}{dt}\right). \tag{2.86}$$

Substituting the Lindblad master equation of the general form where L_k are the dissipators or jump operators,

$$\frac{d\rho}{dt} = -i[H, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right), \tag{2.87}$$

we obtain

$$\frac{d}{dt}\langle O \rangle = -i\langle [O, H] \rangle + \sum_k \left(\langle L_k^\dagger O L_k \rangle - \frac{1}{2}\langle \{L_k^\dagger L_k, O\} \rangle \right). \tag{2.88}$$

This corresponds to the following Heisenberg equation of motion for the operator O:

$$\frac{dO}{dt} = i[H, O] + \sum_k \left(L_k^\dagger O L_k - \frac{1}{2}\{L_k^\dagger L_k, O\} \right). \tag{2.89}$$

The above equation describes the deterministic evolution of system operators in the presence of dissipation and jump operators.

To illustrate the emergence of the Langevin equation and the relation to the fluctuation-dissipation theorem we take the model of the harmonic oscillator with mode of interest a interacting with a reservoir as in [3]. Lets consider the same example we studied before (2.78) The Heisenberg equation of motion for the operators are

$$\dot{a} = \frac{i}{\hbar}[H, a] = -i\nu a(t) - i\sum_k g_k b_k(t) \tag{2.90}$$

$$\dot{b_k} = -i\nu_k b_k(t) - i g_k a(t) \tag{2.91}$$

We are interested in studying the evolution of the system mode so we integrate over the reservoir mode which gives

$$b_k(t) = b_k(0)e^{-i\nu_k t} - i g_k \int_0^t dt' a(t') e^{-i\nu_k(t-t')}. \tag{2.92}$$

The first and second terms are the free evolution of the reservoir modes and their interaction with the system, respectively. The above equation substituted in evolution equation of system mode gives

$$\dot{a} = -i\nu a - \sum_k g_k^2 \int_0^t dt' a(t') e^{-i\nu_k (t-t')} + f_a(t) \tag{2.93}$$

$$f_a(t) = -i \sum_k g_k b_k(0) e^{-i\nu_k t}. \tag{2.94}$$

The term $f_a(t)$ is the noise term coming purely from the reservoir modes at $t = 0$. It varies rapidly due to multiple modes being present. The above equation can be solved in the regime where noise modes vary rapidly. The approximation called Weisskopf-Wigner can be applied which leads to

$$\sum_k g_k^2 \int_0^t dt' a(t') e^{-i(\nu_k - \nu)(t-t')} \approx \frac{1}{2} \mathcal{C} a(t) e^{i\nu t}. \tag{2.95}$$

Here $\mathcal{C}$ is the same as the decay constant in Eq. (2.84),which comes from the integration of the interaction mode strengths with the system mode under the Weisskopf-Wigner approximation. Hence the final equation comes to

$$\dot{a} = -i\nu a - \frac{\mathcal{C}}{2} a + f_a(t). \tag{2.96}$$

Till now we have not assumed any properties for the reservoir modes. Considering the reservoir modes in thermal equilibrium following the bath correlation functions (or white noise operator expectations) gives

$$\langle f_a^\dagger(t) f_a(t) \rangle = \mathcal{C} n_{th} \delta(t - t'). \tag{2.97}$$

Where $n_{th} = \bar{n}_k$ at ν given by Eq. (2.83).

The terms in the evolution equation correspond to:

- Hamiltonian evolution: $-i\nu a$
- Dissipation: $-\frac{\mathcal{C} n_{th}}{2} a$
- Noise arising from the environment: $f_a(t)$

The quantum Langevin equation provides a stochastic differential equation describing the dynamics of system operators including environmental coupling. The deterministic terms account for coherent evolution and damping, while the noise term captures fluctuations introduced by the bath.

The noise term ensures that the dynamics satisfy the fluctuation-dissipation relation and preserves the canonical commutation relations of the system operators.

2.8 Laser Theory

Lasers produce coherent electromagnetic radiation through stimulated emission occurring in an active medium that is placed within an optical cavity. The cavity acts as a feedback mechanism, causing the optical field to circulate and interact multiple times with the gain medium, thereby enhancing the amplification of the radiation. When the gain supplied by the medium balances the intrinsic losses of the cavity, the system reaches the lasing threshold and a steady output of coherent light is produced. In the following discussion we first consider the laser model with linear gain and damping. The analysis and formal procedure for this description follow [7]. The subsequent treatment of gain saturation, which introduces a nonlinear dependence of the gain on the field intensity, is based on the arguments and methods presented in [7] and [6].Finally, we examine the role of quantum phase noise in determining the spectral properties of laser radiation, including the fundamental linewidth limit [5].

A key feature of laser operation is the presence of population inversion in the gain medium, which enables stimulated emission to dominate over absorption. A two level atom cannot have population inversion in steady state since the rate of stimulated emission(W) and absorption are equal. Assume the population in the higher level to be N_2 and the lower level to be N_1. We can write the detailed balance equation, $dN_2/dt = WN_1 - (W+\tau)N_2$. Here τ is the spontaneous emission rate. Now for $N = N_1 + N_2$ in steady state ($dN_2/dt = 0$) the population of $N_2 = WN/(2W+\tau)$ and $N1 = (W+\tau)N/(2W+\tau)$. This gives $N_2/N_1 = W/(W+\tau) < 1$ implying that $N_2 > N_1$ can never occur hence a three-level system is needed for population inversion to attained and sustained.

2.8.1 Linear Gain and Damp

The standard laser master equation arises for a wide range of gain media when certain conditions are satisfied, particularly when the effects of thermal photon noise and photon reabsorption can be neglected.The procedure followed here are based on [7]. In such a model, the amplification mechanism originates from the interaction between the field mode corresponding to the laser and a single atomic transition. This interaction is commonly described using the Jaynes-Cummings Hamiltonian

$$H = i\Omega(\sigma a^\dagger - \sigma^\dagger a), \tag{2.98}$$

Here $\sigma = |l\rangle\langle u|$ represents the atomic lowering operator, and Ω is the single-photon Rabi frequency.

Lets start by defining $\epsilon = \Omega\tau$. Consider an interaction time τ such that the condition $\epsilon\sqrt{\bar{n}} \ll 1$, where $\bar{n}$ denotes the average number of photons. Under this assumption, the unitary evolution operator $e^{-iH\tau}$ acting on the initially state factorized such that the atom is excited state $R = \rho \otimes |u\rangle\langle u|$, may be expanded up to ϵ^2 in the small parameter ϵ(ignoring higher powers). This expansion yields the resulting atom-field entangled state.

$$R = \rho \otimes |u\rangle\langle u| + \epsilon(a^\dagger\rho \otimes |l\rangle\langle u| + H.c.) + \epsilon^2(a^\dagger\rho a \otimes |l\rangle\langle l| - \frac{1}{2}aa^\dagger, \rho \otimes |u\rangle\langle u|). \tag{2.99}$$

Assume that the atomic state is measured immediately following its interaction with the field. When the measurement result corresponds to the atom being found in the upper state, the field is projected into a conditional state whose norm determines the probability of obtaining this outcome. To first order in ϵ^2, the resulting field state is given by,

$$\begin{aligned}\tilde{\rho_u} = \langle u| R |u\rangle &= (1 - \epsilon^2\mathcal{A}[a^\dagger])\rho \\ &= exp(-\epsilon^2 aa^\dagger/2)\ \rho\ exp(-\epsilon^2 aa^\dagger/2),\end{aligned} \tag{2.100}$$

here the superoperator $\mathcal{A}$ is defined as

$$\mathcal{A}[A]B = \frac{1}{2}(A^\dagger AB + BA^\dagger A). \tag{2.101}$$

A and B being arbitrary operators. Likewise the atom when detected in the lower state, the field state is given by

$$\tilde{\rho}_l = \langle l| R |l\rangle = \epsilon^2 \mathcal{T}[a^\dagger]\rho, \tag{2.102}$$

here the superoperator $\mathcal{T}$ is defined as

$$\mathcal{T}[A]B = ABA^\dagger. \tag{2.103}$$

We now assume that emission events within the cavity are statistically independent and that the injection of excited atoms occurs according to a Poisson process with rate Γ, where $\Gamma \ll \tau^{-1}$. Under these conditions, the evolution equation becomes

$$\rho(t+dt) = \Gamma dt(\tilde{\rho}_l + \tilde{\rho}_u) + (1 - \Gamma dt)\rho \tag{2.104}$$

Here the linear loss by the cavity occurs with the rate κ.

$$\dot{\rho} = \Gamma\epsilon^2 \mathcal{D}[a^\dagger]\rho + \kappa\mathcal{D}[a]\rho. \tag{2.105}$$

The notation

$$\mathcal{D}[A] = \mathcal{T}[A] - \mathcal{A}[A]. \tag{2.106}$$

$$\frac{d\langle n\rangle}{dt} = Tr[n\dot{\rho}] = \Gamma\epsilon^2 Tr[a^\dagger a\mathcal{D}[a^\dagger]\rho] + \kappa Tr[a^\dagger a\mathcal{D}[a]\rho] \tag{2.107}$$

Steady state of photon flux implies $d\langle n\rangle/dt = 0$.

$$\bar{n} = \langle n\rangle_{ss} = \frac{\Gamma\epsilon^2}{\kappa - \Gamma\epsilon^2}. \tag{2.108}$$

For steady state to exist we need $\kappa > \Gamma\epsilon^2$. From the standpoint of quantum statistics, a laser field can be considered coherent only when the mean photon number satisfies $\bar{n} \gg 1$, since the phase uncertainty scales as $\delta\phi \sim 1/\sqrt{\bar{n}}$. However, for Eq. (2.105), the photon number variance satisfies $\langle n^2\rangle - \langle n\rangle^2 \sim \bar{n}^2$. Such a large variance produces significant low-frequency fluctuations in the output intensity (of order $\sim \kappa/\bar{n}$), which degrades the second-order coherence of the laser. As a result, the photon statistics deviate from a Poisson distribution, indicating that the field is not in a coherent state.

The origin of this difficulty in Eq. (2.105) lies in the nature of stimulated emission. When stimulated emission occurs from an effectively undepleted source, for example a reservoir of excited atoms, it introduces additional intensity noise, as reflected in Eq. (2.105). Since stimulated emission requires $\bar{n} \gg 1$, this leads to the gain becoming proportional to the field intensity. Consequently, any fluctuation that raises the intensity above its average value enhances the gain and further amplifies the deviation. Conversely, if the intensity falls below its mean value, the gain correspondingly decreases, reinforcing the downward fluctuation. To suppress this feedback mechanism and recover second-order coherence in the output field, it is necessary for the photon gain to depend nonlinearly on the intensity.

2.8.2 Gain Saturation

In most lasers, the gain nonlinearity as a function of intensity naturally provides a physically realistic model that preserves second-order coherence by incorporating essential components. This nonlinearity becomes pronounced due to gain saturation in the medium.

Key model elements are:

(i) An optical cavity that supports the laser mode of the bosonic field and enables output beam formation.

(ii) A reservoir or source of bosons (photons) coupled to the cavity mode, supplying bosons for amplification.

(iii) An irreversible mechanism that preferentially transfers bosons from the source into the laser mode, thereby providing gain.

(iv) A dissipative channel or sink that removes bosons from the system.

(v) A pumping process that replenishes the source, maintaining a steady supply of particles.

The modeling framework and analysis used here follow the treatment presented in [6]. Firstly, we define the annihilation operators corresponding to laser cavity, intermediate (source to laser cavity; irreversible transfer) and source as a, b and c, respectively. With these definitions, the evolution of the complete system is described by:

$$\dot{W} = \sum_{i=1}^{5} \mathcal{L}_i W, \tag{2.109}$$

where $\mathcal{L}_i$ are the Louville superoperators defined below.

$$\mathcal{L}_1 W = \kappa \mathcal{D}[a] W, \tag{2.110}$$

$$\mathcal{L}_2 W = -i[g(c^\dagger ab + ca^\dagger b^\dagger), W], \tag{2.111}$$

$$\mathcal{L}_3 W = \lambda \mathcal{D}[b] W, \tag{2.112}$$

$$\mathcal{L}_4 W = \gamma(N+1)\mathcal{D}[c] W, \tag{2.113}$$

$$\mathcal{L}_5 W = \gamma N \mathcal{D}[c^\dagger] W. \tag{2.114}$$

The superoperator $\mathcal{L}_1$ represents the damping of the cavity (laser) mode due to photon leakage through the cavity mirrors, occurring at a rate κ.

The second contribution, $\mathcal{L}_2$, corresponds to a coherent Hamiltonian interaction. This couples the source, intermediate and laser modes.

The term $\mathcal{L}_3$ describes the decay of the intermediate mode at a rate λ. This decay is assumed to be very strong, such that λ is much larger than the other dynamical rates. In this limit, the coupling induced by $\mathcal{L}_2$ effectively becomes unidirectional, transferring excitations from the source towards the laser mode.

Finally, the superoperators $\mathcal{L}_3$ and $\mathcal{L}_4$ account for interactions with a broad reservoir containing N bosons per mode. The term $\mathcal{L}_5$ acts as the pumping mechanism providing amplification N bosons per mode, while $\mathcal{L}_4$ represents dissipative loss from the source.

To characterize this three-mode master equation into a single-mode master equation by eliminating the first mode b followed by mode c. We employ the adiabatic elimination for mode b under the assumption $\lambda \gg g$. Among the five superoperators the damping term $\mathcal{L}_3$ and reversible coupling Hamiltonian term $\mathcal{L}_2$ participate in the langevin equation of mode b. Since the Eq. (2.113) only gives the expectation value we need to add the white noise term

$b_{in}(t)$.

$$
\begin{aligned}
\dot{b} &= \lambda(\mathcal{D}^\dagger[b]b) + i[g(c^\dagger ab + ca^\dagger b^\dagger), b] + \sqrt{\lambda}b_{in}(t) \\
\dot{b} &= -\frac{\lambda}{2}b - igca^\dagger + \sqrt{\lambda}b_{in}(t)
\end{aligned} \tag{2.115}
$$

Now since $\lambda \gg g$ we can adiabatically eliminate mode b by the argument that mode b decays very fast relative to the reversible coupling and hence reaches steady state in time scales that do not significantly affect the source and laser mode. Hence taking $\dot{b} = 0$ we get

$$
b(t) = -i\frac{2g}{\lambda}ca^\dagger + \frac{2}{\sqrt{\lambda}}b_{in}(t). \tag{2.116}
$$

Now by putting Eq (2.116) into the langevin equations corresponding to operator a and c and averaging over the noise terms using ito calculus we eliminate mode b entirely. Now from langevin equations of operator a and c we go back to the lindblad equation, then Eq (2.109) modifies into

$$
\dot{R} = (\mathcal{L}_1 + \mathcal{L}_e + \mathcal{L}_4 + \mathcal{L}_5)R \tag{2.117}
$$

Where

$$
\mathcal{L}_e = \eta\mathcal{D}[ca^\dagger]R. \tag{2.118}
$$

Here R denotes the reduced density operator describing the combined state of the source mode and the laser mode. The parameter $\eta = g^2/\lambda$ characterizes the effective irreversible coupling between these two modes that results after eliminating the intermediate mode.

To further simplify the description, we now eliminate the source mode c. For this purpose we assume that the reservoir occupation satisfies $N \ll 1$. Under this condition the density operator R can be expressed as

$$
R = \sum_{i=0}^{1}\sum_{j=0}^{1}\rho_{ij} \otimes |i\rangle\langle j| \tag{2.119}
$$

Where $|i\rangle$ is the state of source mode and ρ_{ij} are the conditioned laser mode unnormalised density matrices. Since R is a density matrix $\rho_{10}^{\dagger} = \rho_{01}$ and $Tr(\rho_{00} + \rho_{11}) = 1$. Now putting the Eq (2.119) into Eq (2.117) we get the following equations

$$\dot{\rho_{00}} = \mathcal{L}_1\rho_{00} + \eta\mathcal{T}[a^{\dagger}]\rho_{11} + \gamma(N+1)\rho_{11} - \gamma N\rho_{00} \tag{2.120}$$

$$\dot{\rho_{11}} = \mathcal{L}_1\rho_{11} - \eta\mathcal{A}[a^{\dagger}]\rho_{11} - \gamma(N+1)\rho_{11} + \gamma N\rho_{00} \tag{2.121}$$

$$\dot{\rho_{10}} = \mathcal{L}_1\rho_{10} - \frac{\eta}{2}(aa^{\dagger}\rho_{10} + H.c) - \gamma(N+1)/2\rho_{10} - \gamma N\rho_{10} \tag{2.122}$$

$$\dot{\rho_{01}} = \dot{\rho_{10}}^{\dagger} \tag{2.123}$$

From these equations we can see that Eq (2.120) and (2.121) are decoupled from (2.122) and (2.123) since the pump and sink of source mode is a thermal bath which is incoherent. Now equation (2.122) has no source terms only decay terms and hence will not contribute significantly to the final master equation of the laser mode.

Now turning our attention towards Eq (2.120) and (2.121). Since $N \ll 1$ the source term in (2.121) is very weak and the damp term is strong so ρ_{11} relaxes faster than ρ_{00}. Assuming $\gamma \gg \kappa$ we can neglect $\mathcal{L}_1$ and the the steady state $\dot{\rho_{11}} = 0$ from Eq (2.121) gives

$$\rho_{11} \simeq \gamma N(\gamma + \eta\mathcal{A}[a^{\dagger}])^{-1}\rho_{00} \tag{2.124}$$

We can clearly see that $\rho_{11} \sim N$ hence the final density matrix $\rho \simeq \rho_{00}$. We show in the next equation that it varies in time scales of κ. Substituting Eq (2.124) into Eq (2.120) we get the master equation of the laser mode which has nonlinearity in the intensity embedded in it.

$$\dot{\rho} = \kappa\mathcal{D}[a]\rho + Gn_s\mathcal{D}[a^{\dagger}](n_s + \mathcal{A}[a^{\dagger}])^{-1}\rho \tag{2.125}$$

Where $n_s = \gamma/\eta$ and $G = \eta N$. This is the main result of gain saturation model in the paper [6].

The gain term in the above equation is the second term. The gain rate from the master equation varies as

$$Gn_sTr[aa^{\dagger}\mathcal{D}[a^{\dagger}](n_s + \mathcal{A}[a^{\dagger}])^{-1}\rho] = Gn_s\left\langle \frac{aa^{\dagger}}{n_s + aa^{\dagger}} \right\rangle \tag{2.126}$$

Now writing this in terms of photon number equation [28] becomes

$$Gn_s \left\langle \frac{n+1}{n_s+n+1} \right\rangle .$$

The number operator change is

$$\frac{d\langle n \rangle}{dt} = Gn_s \left\langle \frac{n+1}{n_s+n+1} \right\rangle - \kappa \langle n \rangle . \tag{2.127}$$

The $n_s + n + 1$ in the denominator causes the gain to saturate as the photons increase and shows the nonlinearity that helps to reduce the scaling of intensity fluctuations relative to the linear gain model and preserves the second-order correlation function.
The gain rate can also be derived from the steady state occupation number of source mode(c). Let ρ_c be the source mode density matrix. Then rate of change of occupation number of source mode and the steady state occupation number is given below respectively

$$Tr_c[c^\dagger c\dot{\rho}_c] = \gamma(N+1)Tr_c[c^\dagger c\mathcal{D}[c]\rho_c] + \eta Tr_c[c^\dagger c \; Tr_a[\mathcal{D}[ca^\dagger]\rho]] \tag{2.128}$$

$$\langle n_c \rangle_{ss} = \left\langle \frac{\gamma N}{\gamma + \eta(n+1)} \right\rangle \tag{2.129}$$

Here ρ is the state of source mode (c) and laser mode (a). Now the steady state occupation number transfers its bosons to laser mode with the interaction strength η hence the gain rate is

$$Gn_s \left\langle \frac{1}{n_s+n+1} \right\rangle = \eta \langle n_c \rangle_{ss} \tag{2.130}$$

We note that:

$$G/\kappa \begin{cases} < 1, & \overline{n} < n_s, \quad \text{gain is linear} \\ > 1, & \overline{n} \text{ is comparable } n_s, \quad \text{gain is nonlinear} \\ \gg 1, & \overline{n} \gg n_s, \quad \text{gain is constant} \end{cases} \tag{2.131}$$

In the far above threshold regime the Eq (2.125) can approximately be written as

$$\dot{\rho} = \kappa\mathcal{D}[a]\rho + Gn_s\mathcal{D}[a^\dagger](\mathcal{A}[a^\dagger])^{-1}\rho \tag{2.132}$$

Now let us assign $\mu = Gn_s$ then the Eq (2.132) becomes

$$\dot{\rho} = \kappa\mathcal{D}[a]\rho + \mu\mathcal{D}[a^\dagger](\mathcal{A}[a^\dagger])^{-1}\rho \tag{2.133}$$

In this limit we can see from Eq. (2.127) at steady state

$$\langle n\rangle_{ss} = \frac{\mu}{\kappa} \tag{2.134}$$

Under the same limiting conditions, the photon statistics of the laser mode become Poissonian, which is characteristic of a coherent field. In the FAT regime, the master equation can be formulated in the Fock basis, and the steady state can be calculated in Fock space from $\dot{\rho} = 0$. The form of the equations and the corresponding analysis presented here follow the treatment given in [7]. Following the algebra gives the steady state as

$$\rho_{ss} = \sum_n e^{-\mu/\kappa}\frac{(\mu/\kappa)^n}{n!}\left|n\right\rangle\left\langle n\right|. \tag{2.135}$$

The above equation implies that the stationary state is diagonal and all off diagonal terms are zero. This can be represented in a more useful form where $|\alpha e^{i\phi}\rangle$ is a coherent state with amplitude $\alpha e^{i\phi}$

$$\rho_{ss} = \int_0^{2\pi}\frac{d\phi}{2\pi}\left|\alpha e^{i\phi}\right\rangle\left\langle\alpha e^{i\phi}\right|, \tag{2.136}$$

Here $|\alpha| = \sqrt{\mu/\kappa}$. The above equation implies that the steady state solution is a sum of coherent states which have a mean photon number of $Tr[a^\dagger a\rho_{ss}] = \langle n\rangle_{ss} = \mu/\kappa$.
The FAT laser equation need not only be interpreted as the limiting behavior of a laser operating far above threshold. It can also arise from an alternative physical picture based on a conditional atomic feedback process. In this scenario, atoms are sent through the cavity one at a time. If an atom exits the cavity in the excited state, it is reintroduced into the cavity for another interaction with the field. Only those atoms that leave the cavity in the lower state are allowed to exit the system.

Let us suppose that k atoms pass through the cavity before the $(k+1)$-th atom is finally detected in the lower state. Using Eqs. (2.100) and (2.102), the resulting state transformation gives

$$\tilde{\rho}_k = \epsilon^2 \mathcal{T}[a^\dagger] exp[-k\epsilon^2 aa^\dagger/2] \rho exp[-k\epsilon^2 aa^\dagger/2]. \tag{2.137}$$

Thus the average density operator for the atom to be detected in the ground state is given by

$$\rho' = \sum_{k=0}^{\infty} \tilde{\rho}_k. \tag{2.138}$$

Now since ϵ^2 is small the sum can be changed to integral and $\beta = \epsilon^2 k$

$$\rho' = \mathcal{T}[a^\dagger] \int_0^\infty exp[-\beta aa^\dagger/2] \rho exp[-\beta aa^\dagger/2] d\beta \tag{2.139}$$

This gives

$$\rho' = \mathcal{T}[a^\dagger] \mathcal{A}[a^\dagger]^{-1} \rho. \tag{2.140}$$

Now if the atoms are pumped at the rate μ and the loss rate in the cavity is κ then the lindblad equation becomes

$$\dot{\rho} = \mu(\mathcal{T}[a^\dagger]\mathcal{A}[a^\dagger]^{-1} - 1)\rho + \kappa \mathcal{D}[a]\rho \tag{2.141}$$

$$= \mu \mathcal{D}[a^\dagger](\mathcal{A}[a^\dagger])^{-1}\rho + \kappa \mathcal{D}[a]\rho. \tag{2.142}$$

This equation is the same as FAT laser equation, hence this model shows us that by the gain of one photon by one atom independent of the photon number we can achieve the FAT laser. This interpretation provides an intuitive way of understanding the origin of the FAT laser master equation and closely follows the central idea discussed in [7].

2.8.3 P representation of the FAT Laser

First, we convert Eq. (2.132) by using the simple relations taken from [3]:

$$a^\dagger |\alpha\rangle \langle\alpha| = \Big(\frac{\partial}{\partial\alpha} + \alpha^*\Big) |\alpha\rangle \langle\alpha| \tag{2.143}$$

$$a |\alpha\rangle \langle\alpha| = \alpha |\alpha\rangle \langle\alpha| \tag{2.144}$$

$$|\alpha\rangle \langle\alpha| a^\dagger = \alpha^* |\alpha\rangle \langle\alpha| \tag{2.145}$$

$$|\alpha\rangle \langle\alpha| a = \Big(\frac{\partial}{\partial\alpha^*} + \alpha\Big) |\alpha\rangle \langle\alpha| \tag{2.146}$$

where $|\alpha\rangle$ is a coherent state, now writing $\alpha = re^{i\phi}$. We can convert the P representation into polar representation. The final equation would then be

$$\frac{\partial P}{\partial t} = Gn_s\Big(\frac{1}{4r^2}\Big(r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\phi^2}\Big) + \frac{1}{2r}\frac{\partial}{\partial r}r^2\Big)\Big[1 + r^2 + \frac{1}{2r}\frac{\partial}{\partial r}r^2\Big]^{-1} P + \frac{\kappa}{2}\Big(\frac{1}{r}\frac{\partial}{\partial r}r^2\Big)P. \tag{2.147}$$

The derivation of the P-representation from the Lindblad master equation follows the standard procedure outlined in [3]. Now at the time t when the mean photon number has stabilized or there is no sufficient change in the variable r the above equation can be simplified by assuming r to be a constant such that $r^2 = \langle n\rangle_{ss}$, which yields

$$\frac{\partial P}{\partial t} = \frac{Gn_s}{4r^2}\Big(\frac{\partial^2}{\partial\phi^2}\Big)\Big[1 + r^2\Big]^{-1} P. \tag{2.148}$$

This equation is the exactly similar to a continuous random walker (Fokker-Planck equation for P) in one direction. Since its a pure diffusion process we can see that the diffusion constant D can be written as

$$D = \frac{Gn_s}{4r^2}\Big(\frac{1}{1 + r^2}\Big) \tag{2.149}$$

The above equation can also be represented as

$$D = \frac{Gn_s}{4\langle n\rangle_{ss}}\Big(\frac{1}{1 + \langle n\rangle_{ss}}\Big) \tag{2.150}$$

Also from the Eq. (2.127) in the FAT laser regime($\langle n \rangle >> n_s$) we can see at steady state

$$\frac{Gn_s}{(\langle n \rangle_{ss} + 1)} = \kappa. \tag{2.151}$$

Hence, substituting the the above in Eq. (2.150) we get

$$D = \frac{\kappa}{4 \langle n \rangle_{ss}}. \tag{2.152}$$

We shall see in the next section how this is is proportional to the line width of the FAT Laser.

2.8.4 Schalow Townes Limit

The laser linewidth primarily stems from random phase fluctuations in the emitted field. These fluctuations arise primarily from spontaneous emission, which is measured using a Michelson interferometer where interference fringe visibility reveals the decay of phase correlations. This visibility reduction corresponds to the first-order correlation function, which governs the laser's coherence time and directly influences its linewidth. The standard quantum limit on laser linewidth-the Schawlow-Townes (ST) limit [5]-sets this boundary.

The derivation follows the procedure in [3], analyzing interference visibility gradients from Michelson interferometer output.

The laser linewidth arises from random phase fluctuations of the emitted field, primarily due to spontaneous emission. Interferometrically, this can be understood by sending the output of the laser through a Michelson interferometer: the visibility of the interference fringes decays with increasing delay between the arms, reflecting the gradual loss of phase correlation. This decay in visibility can be understood as the decay of the second-order correlation function for the field due to phase diffusion. The characteristic timescale of the decay defines the coherence time of the laser, which is inversely proportional to the linewidth. Hence the lower bound on the laser linewidth - known as the Schawlow-Townes(ST) limit [5], defines the *standard quantum limit* of laser linewidth. The procedure we follow to derive the result is based on [3].

Mathematically the first correlation function $g^{(1)}(\tau)$ is given as:

$$g^{(1)}(\tau) = \frac{\langle a^\dagger(t+\tau)a(t)\rangle}{\langle a^\dagger(t)a(t)\rangle_{ss}} \tag{2.153}$$

The first correlation function defined above is phase shift invariant. Therefore, using the Quantum Regression Theorem it can be written as follows in terms of a unitary function, say $U(t)$, from [7].

$$g^{(1)}(\tau) * \langle a^\dagger(t)a(t)\rangle_{ss} = Tr(U^\dagger(t+\tau)a^\dagger U(t+\tau)U^\dagger(t)aU(t)\rho_{ss})$$

where $\rho_{ss} = |\alpha\rangle\langle\alpha|$.

Assuming, $\mathcal{L}$ is the lindblad for FAT Laser such that $\dot{\rho} = \mathcal{L}\rho$, then the above expression gives

$$g^{(1)}(\tau) * \langle a^\dagger(t)a(t)\rangle_{ss} = Tr(a^\dagger e^{\mathcal{L}\tau}(a\rho_{ss})). \tag{2.154}$$

The second step simplifies to the third step since ρ_{ss} is a steady state of the lindblad equation give in Eq. (2.133). Let us define $\chi = a\rho_{ss}$. Before we simplify the following equation lets convert the χ into polar representation using Eq. (2.136) where α is replaced with r which is

$$\chi = a\rho_{ss}$$
$$\chi = \int_0^{2\pi} \frac{d\phi}{2\pi} re^{i\phi} |re^{i\phi}\rangle\langle re^{i\phi}| \tag{2.155}$$

Now comparing this with the P represenation of any state we know see that the above state has a constant r and $P_\chi(\phi) = \frac{re^{i\phi}}{2\pi}$. Since we are solving the equation in the regime where the fixed mean is obtained we can use the Eq. (2.148) to solve the Lindblad evolution in correlation function. This gives

$$\begin{aligned}
\frac{\partial P_\chi(\phi)}{\partial t} &= \frac{Gn_s}{4r^2}\Big(\frac{\partial^2}{\partial\phi^2}\Big)\Big[1+r^2\Big]^{-1}P_\chi(\phi) \\
&= \frac{Gn_s}{4r^2}(\frac{1}{1+r^2})\frac{\partial^2}{\partial\phi^2}\frac{re^{i\phi}}{2\pi} \\
&= \frac{Gn_s}{4r^2}(\frac{1}{1+r^2})\frac{-re^{i\phi}}{2\pi} \\
&= \frac{-Gn_s}{4r^2}(\frac{1}{1+r^2})P_\chi(\phi).
\end{aligned} \tag{2.156}$$

Now the above equation can be integrated and using the Eq. (2.151) we get

$$\begin{aligned}
P_\chi(\phi,\tau) &= \frac{re^{i\phi(0)}}{2\pi}e^{\frac{-\kappa\tau}{4\langle n\rangle_{ss}}} \\
\chi(\tau) &= \int_0^{2\pi} P_\chi(\phi,\tau)\left|re^{i\phi}\right\rangle\left\langle re^{i\phi}\right| d\phi \\
\chi(\tau) &= \int_0^{2\pi} \frac{re^{i\phi}}{2\pi}e^{\frac{-\kappa\tau}{4\langle n\rangle_{ss}}}\left|re^{i\phi}\right\rangle\left\langle re^{i\phi}\right| d\phi.
\end{aligned} \tag{2.157}$$

Now by doing the last operation

$$\begin{aligned}
g^{(1)}(\tau)\times\langle a^\dagger(t)a(t)\rangle_{ss} &= Tr(a^\dagger\chi(\tau)) \\
&= Tr(a^\dagger\int_0^{2\pi}\frac{re^{i\phi}}{2\pi}e^{\frac{-\kappa\tau}{4\langle n\rangle_{ss}}}\left|re^{i\phi}\right\rangle\left\langle re^{i\phi}\right| d\phi) \\
&= r^2 e^{\frac{-\kappa\tau}{4\langle n\rangle_{ss}}}
\end{aligned} \tag{2.158}$$

We know $\langle a^\dagger(t)a(t)\rangle_{ss} = r^2 = \langle n\rangle_{ss}$. Hence,

$$g^{(1)}(\tau) = e^{\frac{-\kappa\tau}{4\langle n\rangle_{ss}}} \tag{2.159}$$

Now from the Eq. (2.152) we can write

$$g^{(1)}(\tau) = e^{-D\tau}. \tag{2.160}$$

The Power Spectral density defined as the Fourier transform of the first correlation function given above is,

$$S(\omega) \propto Re(\int_0^\infty g^{(1)}(\tau) e^{i\omega\tau} d\tau)$$

Substituting the result in Eq. (2.159):

$$\begin{aligned} S(\omega) &\propto Re(\int_0^\infty e^{-D\tau} e^{i\omega\tau} d\tau) \\ &\propto \frac{D}{D^2 + \omega^2}. \end{aligned} \tag{2.161}$$

From this we can see that the Power Spectral Density function is a lorenztian with FWHM linewidth as $2D = \kappa/(2\langle n \rangle_{ss})$. This completes the Schalow Townes Limit for laser which is also called as the Standard Quantum Limit. The same result has been derived using different approaches in [2] and [6].

Chapter 3

Measurement Theory

Estimation theory is concerned with the problem of inferring unknown parameters from observed data. Given a set of measurements whose statistics depend on an unknown parameter, the goal is to construct an estimator that provides the best possible estimate of that parameter according to some optimality criterion, such as minimizing the mean squared error. In classical estimation theory, the measurement outcomes are described by probability distributions ($p(x|\theta)$) that depend on the parameter (θ), and estimators are functions of the observed data used to infer its value. In quantum estimation theory, the problem is generalized to quantum systems where the parameter is encoded in a quantum state (ρ_θ), and measurements must first be performed to obtain classical data. The theory therefore studies both the optimal measurement strategy and the optimal estimator, leading to fundamental limits on estimation precision such as those given by the quantum Cramér-Rao bound. The material presented in this chapter is largely based on the treatments in [8] and [11].

3.1 Classical Estimation Theory

This section provides a brief review of several fundamental concepts in classical estimation theory. The discussions in this section are based on [8]. In particular, a number of lower bounds exist that restrict the achievable variance of an estimator for a given parameter.

Parameter estimation concerns the determination of a vector of unknown parameters

$\theta = (\theta_1, \theta_2, \ldots)^T$ from a collection of observed data $\boldsymbol{x} = (x_1, x_2, \ldots)^T$. The relationship between the parameters and the observed data is described through a conditional probability distribution $p(\boldsymbol{x} \mid \theta)$, which serves as the statistical model of the physical system under consideration. The parameters to be estimated can generally be classified into two categories:

(1) The parameter vector θ is deterministic but unknown.

(2) The parameter vector θ is a random variable with a prior probability distribution.

In the first scenario, estimation methods based on the FI framework are typically employed. In the second scenario, the parameter is regarded as a random quantity and Bayesian estimation techniques are used.
We look at Fisher estimation first which deals with estimation of non random parameters. In situations where the parameter θ is non-random but unknown, one of the most widely used estimation procedures is the maximum likelihood estimator (MLE), which is given as

$$\hat{\theta} = arg\ \underset{\theta}{max}\ Pr(\theta|\mathbf{x}). \tag{3.1}$$

In this context, the quantity $Pr(\theta|\boldsymbol{x})$ represents the probability that the parameter takes the value θ conditioned on the observed data $\boldsymbol{x}$. Determining this probability requires specifying a model that describes the underlying process relating the parameter to the measurement outcomes.
Although the parameter θ itself is not random, the observed data $\boldsymbol{x}$ is treated as a random variable because of statistical fluctuations in the measurement process. Consequently, when a sufficiently large set of measurement outcomes is available, an estimator that reproduces the true value of the parameter is referred to as a perfect estimator.

3.1.1 Covariance Matrix and FI(FI)

A standard way to measure the performance of an estimator is through the variance of its estimate relative to the true value of the parameter. Accordingly, the estimation error can be defined as the deviation between the estimated quantity and the actual parameter ($\Delta\theta = \hat{\theta} - \theta$). This quantity depends explicitly on the actual parameter, which is not known in practice. It is therefore more instructive to characterize the spread of the estimates of θ through a statistical measure such as the variance of the estimator. The mathematical formulation and analysis presented here follow the procedure described in [8].

$$\delta\hat{\theta} = \hat{\theta} - E_{\mathbf{x}|\theta}[\hat{\theta}(\mathbf{x})], \tag{3.2}$$

where

$$E_{\mathbf{x}|\theta}[\hat{\theta}(\mathbf{x})] = \int d\mathbf{x} Pr(\theta|\mathbf{x})\hat{\theta}(\mathbf{x}). \tag{3.3}$$

The quantity $E_{\boldsymbol{x}|\theta}[\hat{\theta}(\boldsymbol{x})]$ denotes the expectation of the estimator $\hat{\theta}$ taken relative to the likelihood distribution $Pr(\boldsymbol{x}|\theta)$. In the limit of a sufficiently large dataset, if the estimator is unbiased, its expectation value converges to the true parameter value. Under these conditions, the estimation error $\delta\hat{\theta}$ approaches $\Delta\theta$.
Covariance matrix is goven by

$$Cov(\hat{\theta}) = E_{\theta|\mathbf{x}}[\delta\hat{\theta}\delta\hat{\theta}^T] \tag{3.4}$$

$$\tag{3.5}$$

The diagonal components of the covariance matrix correspond to the variances of the parameters θ_j, with $j \in \{1, \ldots, D\}$. It is important to emphasize that the covariance matrix is not identical to the mean-square error (MSE) matrix, which is given by

$$E_{\theta|\mathbf{x}}[(\Delta\hat{\theta})(\Delta\hat{\theta})^T] = Cov[\hat{\theta}] + b(\hat{\theta})^2 \tag{3.6}$$

The quantity $b(\hat{\theta})^2 = E_{\boldsymbol{x}|\theta}[\hat{\theta}(\boldsymbol{x})] - \theta$ is the bias times bias associated with the estimator. It follows that for an estimator which is unbiased, where $b(\theta)_j = 0$, the covariance matrix reduces to the (MSE) matrix. If that happens then the expectation of the estimator converges to the actual parameter.
In many practical situations it is difficult to determine the exact values of the covariance matrix associated with an estimator. For this reason, it is useful to derive lower bounds on quantities such as the covariance matrix, the mean-square error (MSE), and related measures of estimation uncertainty. These bounds on parameter estimation provide limits on what can be achievable.

To simplify the notation in the following discussion, we write $E[\cdot]$ instead of $E_{\boldsymbol{x}|\theta}[\cdot]$ for the expectation value. Furthermore, if X is any random vector and Y is its transpose, the matrix XY is positive semidefinite. As a consequence, its expectation value satisfies the inequality

$$E[XY] \geq 0.$$

This property will be used to construct inequalities that lead to lower bounds on the covariance matrix. Accordingly, we make the choice

$$X = f - Ag \text{ and } Y = X^T \tag{3.7}$$

with f as a real D-dim vector of the same size as θ, g as the real R-dim vector and A as a matrix $R \times D$ that only depends on θ. Upon applying the inequality above and reducing it, since the matrix A is independent of the measurement outcomes $\boldsymbol{x}$, it may be moved outside the expectation operator. we obtain

$$E[ff^T] \geq E[fg^T]A^T + AE[gf^T] - AE[gg^T]A^T. \tag{3.8}$$

if we redefine $T = E[fg^T]$ and $G = E[gg^T]$ then we get

$$E[ff^T] \geq TA^T + AT^T - AGA^T. \tag{3.9}$$

The above expression establishes a bound on the expectation value of ff^T. Because T and G are defined as expectation values, they are independent of the measurement outcomes $\boldsymbol{x}$. Consequently, one can set $A = TG^{-1}$, assuming that G^{-1} exists, which introduces a restriction on the choice of the function g. This substitution then leads to

$$E[ff^T] \geq TG^{-1}T^T \tag{3.10}$$

Choosing $f = \delta\hat{\theta}$ gives $E[ff^T] = Cov(\hat{\theta})$. The information matrix is defined as G matrix. The T matrix correspondingly gives

$$T = E[\hat{\theta}g^T] - E[\theta g^T]. \tag{3.11}$$

We impose the additional condition that the function g is chosen such that $E_{\boldsymbol{x}|\theta}[g^T] = 0$. Hence the first term in in the equation for expression of T becomes

$$E_{\mathbf{x},\theta}[\hat{\theta}g^T] = E_{\mathbf{x}}[\hat{\theta}E_{\theta|\mathbf{x}}[g^T]] = 0. \tag{3.12}$$

So different choices of the estimator $\hat{\theta}$ and g give different bounds. A common choice of g gives the FI and the "Cramer-Rao Bound".

Cramer-Rao Bound

We make the choice for g to be

$$g = \frac{\partial ln Pr(\mathbf{x}|\theta)}{\partial\theta}, \tag{3.13}$$

For this choice of $g(x, \theta)$, the first and second derivatives of $Pr(\boldsymbol{x}|\theta)$ should exist and be integrable. The above function is a convenient choice since the logarithmic form ensures additivity for independent samples, reflecting the fact that probabilities for independent events combine multiplicatively. In addition, the $\partial g/\partial\theta$ characterizes how strongly the probability distribution responds to changes in the parameter value. From an intuitive perspective, if the distribution changes significantly for small variations in θ, even slight changes in the parameter will produce observable differences in the measurement outcomes $\boldsymbol{x}$, thereby allowing more accurate estimation.

Consider a single parameter θ for example. In this situation, g becomes a scalar function, and the quantity T takes the form

$$T = E_{\mathbf{x}|\theta}[\theta\frac{\partial ln Pr(\mathbf{x}|\theta)}{\partial\theta}] = -1 \tag{3.14}$$

G matrix follows to be

$$E_{\mathbf{x}|\theta}[\Big(\frac{\partial ln Pr(\mathbf{x}|\theta)}{\partial\theta}\Big)^2] \equiv I(\theta) \text{ FI} \tag{3.15}$$

The variance in θ is constrained by

$$(\delta\theta)^2 \geq I^{-1}. \tag{3.16}$$

This is the Cramer-Rao Bound.

$$\Delta\theta = \frac{\partial ln Pr(\mathbf{x}|\theta)}{\partial\theta}. \tag{3.17}$$

The bound is saturated when the above condition satisfies. For a multi-parameter, the

Cramer-Rao Bound is a matrix given by

$$Cov[\hat{\theta}]_{ij} \geq [I(\theta)^{-1}]_{ij}. \tag{3.18}$$

3.1.2 Geometry of Estimation Theory

The FI introduced earlier arises naturally from a geometric structure on the space of probability distributions. This section contains an insightful geometric derivation from [8], that connects parameter estimation to the distinguishability of probability distributions. This geometric perspective will later help motivate the definition of the Quantum FI(QFI).

The Probability Simplex

Consider an experiment whose measurement outcomes are denoted by $\mathbf{x} = (x_1, x_2, \ldots, x_D)$. For a given value of the parameter θ, the outcomes are described by a probability distribution $P(\mathbf{x}|\theta)$. The estimation problem can therefore be interpreted as distinguishing between two probability distributions $P(\mathbf{x}|\theta_A)$ and $P(\mathbf{x}|\theta_B)$.

All probability distributions form a geometric space known as the *probability simplex*. For a single parameter θ, the distributions $P(\mathbf{x}|\theta)$ trace out a curve within this simplex. Determining the number of measurements needed to distinguish between two values of θ therefore reduces to defining an appropriate distance measure on this space.

For discrete probability distributions with probabilities $\{p_j\}$, an infinitesimal distance on the simplex can be written as

$$ds^2 = \sum_{jk} h_{jk}\, dp_j\, dp_k \tag{3.19}$$

where dp_j are infinitesimal probability changes and h_{jk} is the metric tensor on the simplex.

The metric tensor satisfies

$$h_{jk}h_{kl} = \delta_{jl}, \tag{3.20}$$

where δ_{jl} is the Kronecker delta.

The Fisher–Rao Metric and Statistical Distance

To construct a natural metric on the probability simplex, we define an inner product between random variables in terms of expectation values. For a random variable A and B, the expectation value is

$$\langle A \rangle = \sum_j A_j p_j. \tag{3.21}$$

The classical correlation follows to be

$$\langle AB \rangle = \sum_{jk} A_j B_k h_{jk} = \sum_j A_j B_j p_j. \tag{3.22}$$

Using the relation in Eq. ((3.20)), the metric tensor takes the form

$$h_{jk} = \frac{\delta_{jk}}{p_j}. \tag{3.23}$$

Substituting this into Eq. ((3.19)) gives the *Fisher–Rao metric*

$$ds^2_{\mathrm{FR}} = \sum_{jk} dp_j\, dp_k\, h_{jk} = \sum_j \frac{(dp_j)^2}{p_j}. \tag{3.24}$$

The above quantity is a measure the statistical distance between the two different probability distributions on the simplex.

For continuous probability density functions $p(x)$, the metric can be given as

$$h_{ab} = \frac{1}{4} \int_\Omega dx \, \frac{\partial_a p(x) \, \partial_b p(x)}{p(x)}, \tag{3.25}$$

where Ω denotes the domain of the distribution and $\partial_a \equiv \partial/\partial\theta_a$.

To connect this geometry with parameter estimation, consider the change in statistical distance along a path parametrized by θ. Using $p_j = p_j(\theta)$, we obtain

$$\frac{ds^2}{d\theta^2} = \sum_j \frac{1}{p_j} \left(\frac{dp_j}{d\theta} \right)^2 = \sum_j p_j \left(\frac{\partial \ln p_j}{\partial \theta} \right)^2. \tag{3.26}$$

Comparing this expression with the definition of the FI, we find

$$I(\theta) = \left(\frac{ds}{d\theta} \right)^2. \tag{3.27}$$

For continuous probability distributions, the FI becomes

$$I(\theta) = \int dx \, P(\mathbf{x}|\theta) \left(\frac{\partial \ln P(\mathbf{x}|\theta)}{\partial \theta} \right)^2. \tag{3.28}$$

Thus, the FI quantifies how rapidly the probability distribution changes along a path in the probability simplex parametrized by θ. A larger FI implies that small variations in θ lead to more distinguishable probability distributions, and therefore allow more precise parameter estimation.

3.1.3 Quantum model of precision measurements

A general quantum estimation protocol consists of three stages: probe preparation, parameter encoding, and measurement and the following description is based on [8]. Initially, a probe system is prepared in a quantum state $\rho(0)$. The parameter θ is then encoded in the probe through a physical process described by a quantum channel $\Lambda(\theta)$, which transforms the initial state into a parameter-dependent state

$$\rho(\theta) = \Lambda[\rho(0)](\theta). \tag{3.29}$$

After the encoding stage, a measurement is performed on the probe. This measurement is described by a positive operator valued measure (POVM) $\Pi(x)$, where x represents the measurement outcomes. The probability distribution of measurement outcomes is determined by Born's rule,

$$\Pr(x|\theta)\, dx = \mathrm{Tr}\left[\Pi(x)\rho(\theta)\right] dx. \tag{3.30}$$

The operators $\Pi(x)$ satisfy the completeness relation

$$\int dx\, \Pi(x) = I. \tag{3.31}$$

The probability distribution $\Pr(x|\theta)$ describes the statistical behaviour of the measurement outcomes for a given value of the parameter. The estimation problem therefore reduces to distinguishing between nearby probability distributions corresponding to different values of θ.

In practice, the measurement outcomes are processed through an estimator $\check{\theta}(x)$ that produces an estimate of the unknown parameter. Due to the probabilistic nature of quantum measurements, repeated experiments generally produce different measurement outcomes, leading to statistical fluctuations in the estimator.

The bounds derived in classical estimation theory apply directly to the probability distributions obtained from quantum measurements. However, quantum mechanics introduces additional freedom in the choice of probe states and measurement strategies, which can influence the attainable estimation precision.

3.1.4 Quantum Fisher Information(QFI)

The classical FI corresponding with the measurement outcomes is given by (θ. This quantity determines the maximum precision achievable for estimating the parameter θ from the probability distribution of the measurement outcomes. The following derivation and ideas are based on [11].
Since the probability distribution arises from a quantum measurement through Born's rule, the FI can be explicitly written as

$$I(\theta) = \int dx \, \frac{1}{\mathrm{Tr}[\Pi(x)\rho(\theta)]} \left(\frac{\partial}{\partial \theta} \mathrm{Tr}[\Pi(x)\rho(\theta)] \right)^2 . \tag{3.32}$$

The classical FI depends on the specific measurement $\Pi(x)$ performed on the quantum system. However, by optimising over all possible measurements, one obtains the maximum achievable information about the parameter encoded in the quantum state. This maximal quantity is known as *Quantum Fisher Information* (QFI),

$$I_Q(\theta) = \max_{\{\Pi(x)\}} I(\theta). \tag{3.33}$$

The QFI therefore represents the final sensitivity of the quantum state $\rho(\theta)$ to changes in the parameter θ. It depends only on the parameter-dependent state and is independent of any specific measurement choice.

Consequently, the QFI can be interpreted as a geometric property of the family of quantum states $\rho(\theta)$ in the space of density operators. In particular, it determines the statistical distinguishability between neighbouring quantum states $\rho(\theta)$ and $\rho(\theta + d\theta)$.

This relationship between distinguishability and parameter sensitivity leads to a geometric interpretation of quantum estimation, where the quantum FI defines a natural metric on the space of quantum states.

For a family of quantum states parametrised by θ, the statistical distance between neighbouring states is related to the quantum FI through

$$ds^2 = \frac{1}{4} I_Q(\theta) d\theta^2. \tag{3.34}$$

Chapter 4

Hisenberg Limited Laser

4.1 Laser without stimulated emission

The previous section discussed that conventional laser operation relies on amplification arising from stimulated emission. However, examining Eq (2.132) in the far-above-threshold (FAT) regime shows that the gain term becomes independent of the photon number and is effectively a constant. In this regime, the resulting dynamics can be interpreted as arising from a constant gain process, suggesting that stimulated emission need not be a strict requirement for laser operation.

Motivated by this observation, Wiseman proposed an alternative laser model [7] in which the mechanism of stimulated emission is eliminated. In this model, a two-fold reduction in the phase diffusion of the laser field is achieved relative to the standard quantum limit, resulting in a narrower linewidth and therefore a higher quality laser source.

The key idea behind this proposal is that stimulated emission itself contributes significantly to phase diffusion. This contrasts with the conventional view that attributes the fundamental phase noise primarily to spontaneous emission. In Wiseman's model [7], stimulated emission introduces an additional amplification component to the phase diffusion process. By removing stimulated emission from the gain mechanism, this amplification contribution to the phase noise is eliminated, thereby reducing the overall phase diffusion and producing a laser with a narrower linewidth.

When we look into the fundamental atom field interaction Hamiltonian Eq. (2.98), it involves a which is the annihilation operator. This operator a has the c number $\sqrt{n}e^{i\phi}$, n and ϕ being number of photons and their phase. For a linear light-matter coupling, the fundamental gain rate scales proportionally to the number of photons n, which map to the process of stimulated emission. In a complete quantum mechanical description, spontaneous emission contributes an additional term, resulting in a gain rate that scales as $n+1$. Consequently, even in the FAT regime the underlying mechanism of stimulated emission is still present.
Now if we replace the annihilation operator a in Eq. (2.132) with an operator that doesn't show $\sqrt{n}$ scaling we might be able to obtained a model that shows less amplification of the phase diffusion. The operator with this property is the Susskind-Glogower phase operator $e\hat{e}^{i\phi}$ as in [4]. That is we replace the operator a in Eq. (2.98) by the phase operator

$$e = \sum_{n=1}^{\infty} |n-1\rangle \langle n| \,. \tag{4.1}$$

The modified Hamiltonian follows the construction presented in [7].

$$H = i\Omega(\sigma e^{\dagger} - \sigma^{\dagger} e), \tag{4.2}$$

Now from the properties of Susskind-Glogower operators $e^{\dagger} = \sum_{n=0}^{\infty} |n+1\rangle \langle n|$ and

$$ee^{\dagger} = I, \qquad e^{\dagger}e = I - |0\rangle \langle 0| \tag{4.3}$$

With this Hamiltonian, we apply the FAT laser model described earlier, following the treatment in [7], then under the analysis Eq. (2.137) becomes

$$\tilde{\rho}_k = \mathcal{T}[e^{\dagger}]\rho \tag{4.4}$$

So the further analysis follows the final lindblad equation that becomes

$$\dot{\rho} = \mu\mathcal{D}[e^{\dagger}]\rho + \kappa\mathcal{D}[a]\rho. \tag{4.5}$$

Now for this Lindblad equation we have to check if the steady state photon distribution is indeed Poissonian and also if the linewidth shows any reduction. The above equation in the Fock space is available in [7] This gives the same steady state as the FAT laser master equation. Hence the photon distribution is Poissonian.

Now lets look at the $g^{(1)}(\tau)$ function to steady the spectral properties. Unlike in the FAT laser we cannot convert this lindblad equation into P representation easily and solve it. The fundamental reason for this is that the phase operator has eigenstates which are not normalizable, hence it cannot be easily converted interms of coherent state representation. So we device a different method to study the $g^{(1)}(\tau)$ function element wise in the Fock space and add it up.

Since the steady state is a phase average of coherent states (2.136) and the $g^{(1)}(\tau)$ function is independent of phase reference since it is a relative measure of the phase drift. Hence we can take $\rho_{ss} = |\alpha\rangle\langle\alpha| (\alpha = \sqrt{\langle n\rangle_{ss}})$ as the steady state for the calculation of $g^{(1)}(\tau)$ function or two time correlation function. So if $\rho(0) = |\alpha\rangle\langle\alpha|$ the two point correlation function becomes

$$\begin{aligned} g^{(1)}(\tau) * \langle a^\dagger(t)a(t)\rangle_{ss} &= Tr(a^\dagger e^{\mathcal{L}\tau}(a\rho_{ss})) \\ &= Tr(a^\dagger e^{\mathcal{L}\tau}(a|\alpha\rangle\langle\alpha|)) \\ &= Tr(a^\dagger \alpha(\rho(\tau))) \end{aligned} \tag{4.6}$$

Here $\mathcal{L}$ is the lindblad equation which could be the FAT laser or the Susskind-Glogower laser. But the next steps we follow are only for the Susskind-Glogower laser lindblad equation. This procedure is followed as in the Wiseman paper [7]. So we define a function

$$f_n(t) = \sqrt{n}\rho_{n-1,n}(t)/\alpha^* \tag{4.7}$$

The two point correlation function $g^{(1)}(\tau)$ is simply the summation of above for all possible values of n, such that $g^{(1)}(0) = \sum_n f_n(0) = 1$. Hence if we can compute $f_n(t)$ correctly then we can add then to get the correlation function. From the lindblad equation we can get:

$$\dot{f_n} = \mu\Big(\sqrt{\frac{n}{n-1}}f_{n-1} - f_n\Big) + \kappa\Big(-\frac{2n-1}{2}f_n + nf_{n+1}\Big) \tag{4.8}$$

Recall the Poissonian distribution for steady-state photon number. Therefore, their statistics remain largely unchanged over time, even after the phase-kicking measurement. Consequently, the two point correlation measurement primarily affects the phase of the field without significantly altering the photon number distribution. Now we can see that this parameter

$$r_n(t) = \frac{\langle n \rangle_{ss} f_n(t)}{n f_{n+1}(t)} = \frac{\mu f_n(t)}{\kappa n f_{n+1}(t)} \tag{4.9}$$

captures the Poissonian nature of the state. We can see that this parameter equals 1 for $\rho(0) = |\alpha\rangle \langle\alpha|$; derivation skipped here for the sake of brevity. This implies that the $r_n(t) \approx 1$ for any later time after the steady state has been attained. Re write Eq. (4.8) interms of r_n and employ $r_n \approx 1$ to give:

$$\dot{f_n} = \kappa(\sqrt{(n^2 - n)} - n + \frac{1}{2} \tag{4.10}$$

since for the $g^{(1)}$ function we have to add all the f_n from $n = 0$ to ∞. Now for n value small the value of f_n is small since the denominator is large($\alpha^* = \sqrt{\langle n \rangle_{ss}} e^{-i\phi}$) and for large n limit of the above function (using Taylor series expansion; skipped here on account of being trivial)

$$\dot{f_n} \approx \frac{\kappa}{8n} f_n. \tag{4.11}$$

Now since the distribution of n is Poissonian when $\langle n \rangle_{ss} >> 1$. The above equation can be approximated to

$$\dot{f_n} \approx \frac{\kappa}{8 \langle n \rangle_{ss}} f_n \tag{4.12}$$

$$f_n(\tau) \approx f_n(0) exp(\frac{\kappa\tau}{8 \langle n \rangle_{ss}}) \tag{4.13}$$

Hence the $g^{(1)}(\tau) \approx exp(\kappa\tau/(8 \langle n \rangle_{ss})) \sum f_n(0) = exp(\kappa\tau/(8 \langle n \rangle_{ss}))$. Hence the Power spectral Density function which is a Fourier transform of the $g^{(1)}(\tau)$ as seen before will be a lorenztian distribution with FWHM $= 2D \approx \kappa/(4 \langle n \rangle_{ss})$.

This result agrees with the findings of [7]. The analysis above shows that eliminating the stimulated emission contribution results in a linewidth equal to one half of the standard quantum limit. Nevertheless, such a model is difficult to realize in practice because it involves the operator e, which is not associated with a physically realizable field operator.

4.2 Heisenberg limit of laser

The standard Quantum limit as studied before the linewidth scales as $1/\langle n\rangle$. Coherence time as defined before is the inverse of the linewidth. Now we define a dimensionless number $\mathfrak{C}$ called coherence, rate of photons emission times the coherence time. For standard quantum limit, it is

$$\mathfrak{C}_{SQL} \propto \langle n\rangle^2 , \tag{4.14}$$

As the coherence time scales as $\langle n\rangle$ and the number of photons released per unit time scales as $\langle n\rangle$. Wiseman and co-workers in [9] showed in their work on the Heisenberg limit of laser coherence that the maximum achievable coherence of a laser scales as $\mathfrak{C}_{HL} \propto \langle n\rangle^4$. This bound, referred to as the Heisenberg limit, represents the constraint on laser coherence. In the following sections, we briefly outline the main ideas and the key theoretical tools used in establishing this limit. The first subsection introduces to basic conditions applies on the laser model for which the analysis works, the second subsection elaborates on how to achieve the standard quantum limit using estimation theory and the final section concludes on how to achieve the Heisenberg limit.

4.2.1 Conditions on the model

The ultimate bound on the coherence measure is obtained by considering a model that satisfies the set of assumptions outlined in [9] as well as in [10]. I elaborate on those assumptions in this section.

Description of the 1-D field

We study the operators that describe the photon field $b(t)$ as in [6]. Adopting a field-based description, we examine a beam moving with fixed speed, characterized by statistical properties that are unchanged under translations in time the standard commutation rule is given by

$$[b(t), b^\dagger(s)] = \delta(t - s). \tag{4.15}$$

Coherence of the field can be quantified using the field operators by considering the average number of photons in a single spatial mode. This quantity maximized over all possible modes $\mathfrak{u}$ in a frequency interval defines the measure of coherence as in [9] as

$$\mathfrak{C} := \underset{u \in \mathfrak{u}}{max} \left\langle b_u^\dagger b_u \right\rangle , \tag{4.16}$$

where b_u being the annihilation operator for the spatial mode u:

$$b_u = \frac{1}{\sqrt{I_u}} \int_{-\infty}^{\infty} dt u(t) b(t). \tag{4.17}$$

where $I_u = \int_{-\infty}^{\infty} dt |u(t)|^2$. The set $\mathfrak{u}$ represents all spatial mode functions that satisfy the normalization condition

$$\int |u(\mathbf{r})|^2 d^3 r = 1. \tag{4.18}$$

For a beam whose statistical properties are translationally invariant, the maximum of the coherence measure is obtained for a mode function with a flat temporal profile $|u(t)|^2$. Such a waveform, however, is not strictly normalizable. To address this, one may introduce a regularized mode function that decays slowly in time, for example

$$u(t) \propto e^{-|t|/l}, \tag{4.19}$$

and limit $l \to \infty$.

The occupation number associated with a given mode u is defined as

$$N_u = \langle b_u^\dagger b_u \rangle. \tag{4.20}$$

The coherence measure is then given by the maximum occupation number over all allowed modes,

$$\mathfrak{C} = \max_{\mathfrak{u}} N_u. \tag{4.21}$$

Substituting the definition of b_u into the expression for N_u, we obtain

$$N_u = \frac{1}{I_u} \int_{t=-\infty} \mathrm{d}t \left[\int_{t'=-\infty}^{\infty} \mathrm{d}t' u^*(t') c(t-t') \right] u(t) \tag{4.22}$$

where $c(t') := \langle b^\dagger(t+t')b(t)\rangle$ is a one dimensional time invariant field (t). The above can be simplified to give

$$N_u = \frac{1}{2\pi I_u}\int_{-\infty}^{\infty} d\omega |\tilde{u}(\omega)|^2 \tilde{c}(\omega) \tag{4.23}$$

$\tilde{u}$ and $\tilde{c}$ being Fourier transforms of the respective functions. The occupation number is maximized for a mode function whose frequency-domain profile is concentrated at the frequency where the spectral correlation function attains its maximum value. This corresponds to choosing a waveform satisfying

$$|\tilde{u}(\omega)|^2 = \delta(\omega - \omega_{\max}), \tag{4.24}$$

where $\omega_{\max} := \operatorname{argmax}_\omega \tilde{c}(\omega)$.

Employing standard relation between spectral correlation function and PSD ($P(\omega)$), the maximum coherence is

$$\mathfrak{C} := 2\pi \max_\omega P(\omega). \tag{4.25}$$

By transforming to an appropriate rotating frame, the operator $b(t)$ can always be defined such that its spectrum is centered at $\omega = 0$. In this case, the optimal waveform corresponds to a constant temporal profile, ($u(t) = \text{const.}$). Using this result,

$$\mathfrak{C} = \int_{-\infty}^{\infty} dt' \left\langle b^\dagger(t+t')b(t)\right\rangle . \tag{4.26}$$

Further,

$$\mathcal{N} = \int d\omega\, P(\omega), \tag{4.27}$$

where $\mathcal{N}$ is the "beam photon flux". For a standard laser, the spectral distribution is known to take a Lorentzian form, as discussed previously. The peak value of the spectrum is therefore given by

$$P(\omega) = \frac{2}{\pi}\frac{\mathcal{N}}{l}, \tag{4.28}$$

where l is the FWHM of the power spectral distribution.

For a conventional laser, the linewidth satisfies the scaling $l \propto 1/\langle n \rangle$, while the photon flux scales proportionally with the mean photon number, $\mathcal{N} \propto \langle n \rangle$. Substituting these relations into the expression for the coherence measure yields

$$\mathfrak{C} = \frac{4\mathcal{N}}{l} \propto \langle n \rangle^2. \tag{4.29}$$

Thus, the coherence measure scales quadratically with the mean photon number for a standard laser.

Endogenous Phase

The second condition mandates that phase generation occurs intrinsically within the laser cavity. Specifically, all phase characteristics of the output beam must originate exclusively from internal cavity dynamics, independent of any external phase standards or input fields.

The endogenous phase condition can then be expressed through the action of a phase-shift unitary acting on the cavity alone. Defining

$$\mathcal{U}_c^\theta(\rho_c) = U_c^\theta \rho_c U_c^{\theta\dagger}, \tag{4.30}$$

with rotation of cavity by θ defined by

$$U_c^\theta = e^{i\theta n_c}, \tag{4.31}$$

where $n_c$being cavity photon number operator. The endogenous-phase requirement is that the dynamics of the system respect this symmetry, leading to the equivalence condition

$$Tr_{e'}[\mathcal{U}_{ce}^{T\to T'}((\mathcal{U}_c^\theta \otimes \mathcal{I}_e)\rho_{ce}(T))] = Tr_{e'}[(\mathcal{U}_{cb}^\theta \otimes \mathcal{I}'_e)(\mathcal{U}_{ce}^{T\to T'}(\rho_{ce}(T)))], \quad \forall\, T' > T, \quad \theta \in [0, 2\pi) \tag{4.32}$$

where $\mathcal{U}_{cb}^\theta := e^{i\theta(n_c+n_b)}\rho_{cb}e^{-i\theta(n_c+n_b)}$ and n_b being photon operator for beam.

A measurement is said to be phase-covariant if its positive operator-valued measure (POVM) elements E^ϕ, labeled by the measurement outcome ϕ, transform under a phase shift according to

$$\mathcal{U}^{\theta}(E^{\phi}) = E^{\theta+\phi}. \tag{4.33}$$

This prepares the cavity in a state where measurement outcome is embedded in the optical phase.

Let ρ_c^{fid} denote the fiducial cavity state, defined as the state obtained when the measurement outcome is $\phi_F = 0$. For a general outcome ϕ_F, the corresponding cavity state is obtained by applying a phase shift generated by the cavity photon number operator n_c, giving

$$\rho_{c|\phi_F}(T') = \mathcal{U}_c^{\phi_F}(\rho_c^{fid}). \tag{4.34}$$

The proof of this is

$$\begin{aligned}
\rho_{c|\phi_F}(T') &\propto Tr_{be'}[\mathcal{U}_{ce}^{T\to T'}((\mathcal{U}_c^{\theta}\otimes\mathcal{I}_e)(\rho_{ce}(T)))(\mathcal{I}_c\otimes E_b^{\phi_F}\otimes I_{e'})] \\
&= Tr_{be'}[\mathcal{U}_{cb}^{\theta}(\mathcal{U}_{ce}^{T\to T'}(\rho_{ce}(T))(\mathcal{I}_c\otimes\mathcal{U}_b^{\phi_F}(E_b^0)\otimes I_{e'})] \\
&= Tr_{be'}[(\mathcal{U}_c^{\theta}\otimes\mathcal{U}_b^{\theta-\phi_F}\otimes\mathcal{I}_{e'})(\mathcal{U}_{ce}^{T\to T'}(\rho_{ce}(T))(\mathcal{I}_c\otimes E_b^0\otimes I_{e'})]
\end{aligned} \tag{4.35}$$

The above expression being true for any value of θ, we choose $\theta = \phi_F$. Thereby

$$\begin{aligned}
\rho_{c|\phi_F} &= \frac{Tr_{be'}[(\mathcal{U}_c^{\theta}\otimes\mathcal{I}_b\otimes\mathcal{I}_{e'})(\rho_{cbe'}(T')(\mathcal{I}_c\otimes E_b^0\otimes I_{e'})]}{Tr[\rho_b E_b^0]} \\
&= \mathcal{U}_c^{\phi_F}(\rho_c^{fid})
\end{aligned} \tag{4.36}$$

The introduced fiducial state is then $\rho_c^{fid} := Tr_{be'}[(\rho_{cbe'}(T')(\mathcal{I}_c\otimes E_b^0\otimes I_{e'})]/Tr[\rho_b E_b^0]$.

Stationarity

The beam being statistically time invariant in the long time limit, for the cavity in unique steady state ρ_c^{ss}

$$\mathcal{U}_c^{\theta}(\rho_c^{ss}) = \rho_c^{ss}\ \forall\ \theta. \tag{4.37}$$

This statement directly implies that the steady state is diagonal in the fock basis since $e^{in_c\phi}\rho_c^{ss}e^{-in_c\phi} = \rho_c^{ss}$.

Now lets prove the important result that phase encoding or embedding by the phase covariant measurement gives the same statistics as the steady state. Since we know that any state with output measurement ϕ_F is a unitary optical phase rotation of fiducial state if the fiducial state has same statistics then so do any other phase rotated state of the fiducial state.

$$\begin{aligned} P(n|\phi_F) &= \langle n|\rho_{c|\phi_F}|n\rangle \\ &= e^{i\phi_F n}e^{-i\phi_F n}\langle n|\rho_{\text{fid}}|n\rangle \\ &= \langle n|\rho_{\text{fid}}|n\rangle. \end{aligned} \tag{4.38}$$

Hence we set out to prove the result for the fiducial state that $\langle n|\rho_c^{fid}|n\rangle = \langle n|\rho^{ss}|n\rangle$. Using the POVM sum up to identity: $\int d\phi(2\pi)^{-1}E_b^{\phi} = I_b$, and stationarity assumptions:

$$\begin{aligned} \rho_c^{ss} &= \text{Tr}_{be'}\left[U_{ce}^{T-T'}\left(U_c^{\theta}\otimes I_e\,\rho_{cbe'}^{ss}(T)\right)\left(I_c\otimes\int\frac{d\phi}{2\pi}\,E_b^{\phi}\otimes I_{e'}\right)\right] \\ &= \int\frac{d\phi}{2\pi}\,\text{Tr}_{be'}\left[U_{cb}^{\theta}\otimes I_{e'}\,\rho_{cbe'}(T')\left(I_c\otimes U_b^{\phi}(E_b^0)\otimes I_{e'}\right)\right] \\ &= \int\frac{d\phi}{2\pi}\,\text{Tr}_b\left[U_c^{\theta}\otimes U_b^{\theta-\phi}\,\rho_{cb}(T')\,\left(I_c\otimes E_b^0\right)\right]. \end{aligned} \tag{4.39}$$

Choosing $\theta = \phi$ gives

$$\rho_c^{ss} = \int\frac{d\phi}{2\pi}\,U_c^{\phi}\big(\rho_c^{fid}\big)\,. \tag{4.40}$$

The fiducial state being independent of ϕ gives:

$$\begin{aligned} \langle n|\rho_c^{ss}|n\rangle &= \int\frac{d\phi}{2\pi}\,\langle n|U_c^{\phi}\big(\rho_c^{fid}\big)\,|n\rangle \\ &= \langle n|\rho_c^{fid}|n\rangle\,. \end{aligned} \tag{4.41}$$

Glauber Ideality

The first ($g^{(1)}(s,t)$) and second order ($g^{(2)}(s,s',t,t')$) Glauber functions are defined as

$$g^{(1)}(s,t) = \mathcal{N}^{-1} \langle b^\dagger(s) b(t) \rangle \tag{4.42}$$

$$g^{(2)}(s,s',t,t') = \mathcal{N}^{-2} \langle b^\dagger(s) b^\dagger(s') b(t) b(t') \rangle \tag{4.43}$$

These functions of the field should approximate those of the stochastic coherent state. $g^{(1)}(s,t)$ relates to the relative phase properties of the beam. The second order Glauber function provides information of the beam's intensity phase correlations. Since we are studying the properties of a laser we study models that keep the correlation functions as close (which will be quantified based on the system) to the ideal laser system which shows perfect coherent field. From the standard laser physics we know that $g^{(1)}(s,t) = \exp^{-D|s-t|}$ where $D = l/2$ is the diffusion constant; l being FWHM. The $g^{(2)} = 1$ for a coherent state as it has Poissonian photon distribution.

4.2.2 Ideal laser model and standard quantum limit

The standard laser model is known to satisfy the conditions discussed earlier. Within this framework, the scaling behaviour of the coherence measure $\mathfrak{C}$ is understood following phase-number uncertainty relation [1, 3]. The procedure outlined below is based on the derivation given in the Supplementary Material of [9].

The phase-number uncertainty relation implies that

$$V(\phi)V(n) \gtrsim \frac{1}{4}, \tag{4.44}$$

where $V(\cdot)$ denotes the variance of the corresponding observable. For a coherent state, $V(n) = \langle n \rangle$. Consequently, the phase variance is bounded by

$$V(\phi) \gtrsim \frac{1}{4\langle n \rangle}. \tag{4.45}$$

Assume that the coupling between the laser cavity and the output beam is linear. In this case the cavity decay is described by a linear Lindblad term.

Over an infinitesimal time interval dt, the damping process reduces the mean photon number according to

$$\langle n\rangle \to \langle n\rangle(1-\kappa dt). \tag{4.46}$$

κ being cavity damping rate.

Linear damping ensures that coherent states remain coherent under this evolution. Consequently, the phase fluctuations of the field evolve in a simple manner. In particular, the change in the phase variance during the interval dt is given by

$$dV(\phi) = \frac{\kappa dt}{4\langle n\rangle} \text{in the final steady state.} \tag{4.47}$$

Since the pump delivering energy to the cavity lacks coherence, it fails to suppress field phase fluctuations. Thus, simultaneous cavity pumping and damping inevitably increases phase variance by at least the damping-induced amount, $dV(\phi)$.

The field inside the cavity may therefore be approximated by $||\alpha|e^{i\theta(t)}\rangle$, where the amplitude is approximately fixed while the phase $\theta(t)$ undergoes stochastic fluctuations. The phase evolution can be modeled as a random walk whose variance grows according to

$$dV(\phi) = \frac{\kappa\, dt}{4\langle n\rangle}. \tag{4.48}$$

This result reproduces the diffusion constant derived in the previous section and is in agreement with the analysis presented by Wiseman [7].

Using this relation, the coherence of the emitted beam can be expressed as

$$\mathfrak{C}_{SQL} = \int_{-\infty}^{\infty} ds\, \langle b^{\dagger}(s)b(t)\rangle = \kappa\,\langle n\rangle \int_{-\infty}^{\infty} ds e^{-V[\phi(s)-\phi(t)]/2} \tag{4.49}$$

Because the phase evolves via a random walk, it can be treated as a Gaussian stochastic variable. The expectation value of the exponential for a Gaussian random variable X with variance $V(X)$ is

$$\langle e^X \rangle = e^{-V(X)/2}. \tag{4.50}$$

Using this the above equation becomes

$$\mathfrak{C}_{SQL} = \kappa \langle n \rangle \int_{-\infty}^{\infty} ds e^{-\kappa|s-t|/8\langle n \rangle} = 16 \langle n \rangle^2 = \Theta(\langle n \rangle^2). \tag{4.51}$$

Hence we proved the standard quantum limit.

4.2.3 Protocol for achieving Heisenberg limit

In this section, we outline the proof establishing coherence $\mathfrak{C} = \Theta(\langle n \rangle^4)$ for any laser model satisfying the four aforementioned conditions[9]. The proof uses two observers and three methods to estimate the laser's optical phase at time T. These methods are:

1. Filtering wherein a heterodyne measurement is done over $t = [T - \tau, T)$. This causes an embedding of a phase into the cavity say ϕ_F.
2. Retrofiltering where in another heterodyne measurement over $t = (T, T + \tau]$. This is used to estimate the phase embedded ϕ_R by the filtering process.
3. Direct measurement on the cavity at $t = T$ to estimate the phase embedded ϕ_D by the filtering process.

Since we have two estimates of the embedded phase we use the data processing inequality to relate them and find the bound of coherence.

Phase Covariant Measurement - Heterodyne Measurement

Heterodyne detection is a continuous measurement technique used to simultaneously obtain information about both quadratures of an electromagnetic field. In heterodyne detection, the signal field combines with an intense local oscillator offset from the signal frequency. This is unlike homodyne detection which measures a single field quadrature. This produces a beat signal at the difference frequency, allowing both quadratures of the field to be inferred from the measured photocurrent.

A heterodyne measurement produces a complex measurement outcome corresponding to the phase-space amplitude of the field, providing simultaneous noisy estimates of both field quadratures. Alternatively the heterodyne photocurrent can also be obtained by instead applying two homodyne measurement on the beam, with a relative phase shift between the two local oscillators.

The analysis presented here follows the treatment given in [10]. To conduct simultaneous dual homodyne measurements, field $b(t)$ first passes through a 50:50 beam splitter, with the second input port maintained in vacuum. The interaction at the beam splitter produces two output fields given by

$$b_{\pm}(t) = \frac{1}{\sqrt{2}}\left(b(t) \pm \nu(t)\right) \tag{4.52}$$

Where $\nu(t)$ represents the vacuum field entering the second port. We then mix the local oscillators with amplitudes $|\alpha|e^{i\theta}$ and $|\alpha|e^{i(\theta+\pi/2)}$ with transmittance tending to 1. Hence the two beams after interaction changes to

$$b'_{\pm}(t) = \frac{1}{\sqrt{2}}\left(b(t) \pm \nu(t)\right) + |\alpha|e^{i\psi} \tag{4.53}$$

where $\psi = \theta$ for $b'_{+}(t)$ and $\psi = \theta + \pi/2$ for $b'_{-}(t)$ have a phase difference of $\pi/2$. Respective homodyne photocurrent operators are

$$J_{\square}(t) := \lim_{|\alpha|\to\infty} \frac{\sqrt{2}\left(b'^{\dagger}_{\square}(t)b'_{\square}(t) - |\alpha|^2\right)}{|\alpha|}, \quad \square = \pm \tag{4.54}$$

Hence the complex heterodyne photocurrent to be defined is

$$J_{\text{het}}(t) := \frac{e^{i\theta}J_{+}(t) + i\,J_{-}(t)}{2} = b(t) + e^{2i\theta}\nu^{\dagger}(t). \tag{4.55}$$

The phase measurements which are phase covariant are Unitaries defined as $e^{i\phi_H}$ where

$$e^{i\phi_H} = \frac{H}{|H|} \tag{4.56}$$

where $|H| := \sqrt{H^\dagger H}$ and

$$H := \int_{t_1}^{t_2} dt u_H(t) J_{het}(t) = \int_{t_1}^{t_2} dt u_H(t) b(t) + \ a_H^\dagger. \tag{4.57}$$

Here $a_H = \int_{t_1}^{t_2} dt$ and the operator $u_H^*(t)e^{-2i\theta}\nu(t)$ serves as the ancillary operator associated with the mode entering the heterodyne detection. It obeys the commutation relation

$$[a_H^\dagger, a_H] = 1,$$

and $u_H(t)$ is a normalized filter function defined over the time interval $[t_1, t_2]$. One can verify that the operator H is Hermitian in this framework.

Filtering, Reterofiltering, Direct Measurement and Final bound

The filtering and reterofiltering as mentioned before are heterodyne measurements that are performed on the beam. Let the phase embedded by the filtering process be ϕ_F and the estimated phase estimate of the embedded phase by the retrofiltering process be ϕ_R. From the theory of the heterodyne detection the operators are

$$e^{i\hat{\phi}_\Box} = \frac{\Box}{|\Box|}, \quad \text{where } \ \Box = F, \ R \tag{4.58}$$

where $|A| = \sqrt{A^\dagger A}$ and

$$R := \int_{T}^{T+\tau} dt u_R(t) b(t) + \ a_R^\dagger \tag{4.59}$$

$$F := \int_{T-\tau}^{T} dt u_F(t) b(t) + \ a_F^\dagger \tag{4.60}$$

Here the $a_F(R)$ are the ancillary vacuum modes that enter in the heterodyne measurement protocol. Both the functions $u_F(t)$ and $u_R(t)$ are filtering functions which are normalised in thier respective intervals. From the mathematical analysis, it follows that the filtering and retrofiltering functions are phase covariant and therefore do not alter the endogenous phase condition imposed by the model. This is discussed in detail in the Supplementary Material of [9].

A convenient measure of the phase spread for a system like this is

$$1 - |\exp(i\theta)|^2, \tag{4.61}$$

which varies between zero and unity sharpness and complete uncertainity in θ. If θ represents the phase error, then the above expression is the mean square error (MSE) associated with an unbiased measurement of θ and approximately equal to $\langle\theta^2\rangle - \langle\theta\rangle^2$.
In [12], a variational approach was used to address phase estimation for a fixed mean photon number $\langle n\rangle$. For a phase measurement with mean phase $\bar{\phi}$, the corresponding phase estimate $\hat{\phi}$ has a MSE (valid for small deviations) that satisfies

$$1 - |\langle e^{i(\hat{\phi}-\bar{\phi})}\rangle|^2 \gtrsim 4|\frac{z_A}{3}|\langle n\rangle^{-2}. \tag{4.62}$$

Where z_A is the airy function. Employing stochastic methods and suitable approximations, one can demonstrate that (These derivations are presented in greater detail in the Supplementary Material of [9].)

$$1 - |\langle e^{i(\hat{\phi_R}-\hat{\phi}_F)}\rangle|^2 \sim 2\sqrt{\frac{2l}{2\mathcal{N}}}. \tag{4.63}$$

l is the FWHM and $\mathcal{N} = \int d\omega P(\omega)$. An error of the direct measurement $\mathcal{E}r$ which is defined as

$$\mathcal{E}r = 1 - |\langle e^{i(\hat{\phi}_D-\hat{\phi}_F)}\rangle|^2 \tag{4.64}$$

This error is bound as follows:

$$\mathcal{E}r \gtrsim 4|\frac{z_A}{3}|\langle n\rangle^{-2}. \tag{4.65}$$

Here we can relate the two results using the data processing inequality since the direct measurement is better that the continous measurement we get

$$1 - |\langle e^{i(\hat{\phi_R}-\hat{\phi}_F)}\rangle|^2 \gtrsim \mathcal{E}r \tag{4.66}$$

This immediately gives the inequality

$$2\sqrt{\frac{2l}{2\mathcal{N}}} \gtrsim 4|\frac{z_A}{3}| \langle n \rangle^{-2} \tag{4.67}$$

Now since the coherence $\mathfrak{C} = 4\mathcal{N}/l$, we have

$$\mathfrak{C} \lesssim \frac{2}{3}\left|\frac{3}{z_A}\right|^6 \langle n \rangle^4 \tag{4.68}$$

and hence we proved the bound.

Data Processing Inequality

Lets take a look into the data processing inequality which is the key tool here. In information theory, the *data processing inequality* (DPI) formalizes the intuitive notion that any processing applied to measurement outcomes cannot increase the information one has about the original variable; it can only preserve or reduce it. In the classical setting, if $X \to Y \to Z$ is a Markov chain of random variables, then

$$I(X;Y) \;\geq\; I(X;Z),$$

indicating that post-processing cannot increase information about X carried by the observed outcomes [13]. Analogously, in *quantum information theory*, the distinguishability between quantum states - typically quantified by the *quantum relative entropy* - is non-increasing under any completely positive trace-preserving (CPTP) map[14]. This *monotonicity property* is essentially the quantum DPI and is a cornerstone result in the analysis of quantum estimation, quantum channel coding, and continuous measurement protocols because it bounds the amount of extractable information from quantum measurement records [15].

4.3 Nonlinear Operator and Resource Requirements

Achieving the ultimate scaling of laser coherence requires dynamical processes that go beyond the linear amplification mechanisms typically assumed in standard laser models. In particular, the creation of highly coherent radiation from incoherent inputs demands carefully engineered photon-number dependent gain and loss processes inside the cavity. Such dynamics cannot be captured by simple linear operators proportional to the annihilation operator a, and instead require a nonlinear dependence.

The cavity dynamics is given by[9]

$$\dot{\rho} = \mathcal{D}[G]\rho + \mathcal{D}[L]\rho, \tag{4.69}$$

G and L being the photon gain and loss operators, respectively. These operators are carefully engineered for use in the above equation. These induce transitions between neighboring Fock states,

$$G_n = \langle n|G|n-1\rangle, \qquad L_n = \langle n-1|L|n\rangle, \tag{4.70}$$

which describe the addition or removal of a single excitation from the cavity. In contrast to conventional models, the transition amplitudes are allowed to depend on the photon number n. This photon-number dependence introduces an effective nonlinearity in the cavity dynamics.

The important point is that the operators G and L cannot generally be realized using simple linear bosonic couplings. Instead, their matrix elements must vary across the photon-number ladder. Implementing such photon-number dependent transitions requires additional controllable quantum systems coupled to the cavity.

One possible physical platform for realizing these interactions is circuit quantum electrodynamics as explored in the paper [9]. In this setting the cavity field interacts with auxiliary qubits whose frequencies can be tuned in time.

Thus, although the Heisenberg limit constitutes the quantum scaling limit under these conditions, attaining it necessitates experimental realizations that surpass the linear gain processes inherent to standard laser systems. A careful analysis of the nonlinear Hamiltonian structure and the associated physical resources is therefore essential for understanding the feasibility of realizing such enhanced coherence in practical systems.

Chapter 5

Resource keeping in Heisenberg limited laser model

Until the previous chapters we built the idea of an ideal laser and the Heisenberg limited laser. Now we delve into the resource keeping arguments that played an important role in achieving the Heisenberg limit for coherence. Since in the laser the phase is random walker we start our model from there.

5.1 The Phase Random Walk

The phase in a laser does random walk hence its a random variable. Its variance is dependent on the diffusion constant D. Hence for phase $\phi(t)$ we have the relation

$$\langle \Delta\phi^2(t) \rangle = \langle (\phi(t) - \langle \phi(t) \rangle)^2 \rangle = Dt. \tag{5.1}$$

Let r be the rate of photons emission. Then this implies

$$\langle \Delta\phi^2(t) \rangle = \frac{D_{eff}}{r} \implies D_{eff} = r \langle \Delta\phi^2(t) \rangle . \tag{5.2}$$

Now we make the following assumption that each emission is an independent event, this assumption is justified since for a laser we need Poisson statistics for the photon number.

Each independent event introduces a phase kick which is a random variable but since each event is independent and identical random variables(iid) the phase kicks generated are also iid's. So lets assume the variation of the phase kicks to be σ^2.
Here the environment is trying to learn the phase of the laser. From this model the accumulated phase variance at the end of time t is

$$\langle \Delta\phi^2(t) \rangle = r\sigma^2 t. \tag{5.3}$$

From here we can see that $D = r\sigma^2$.

5.2 Coherence Time

The coherence time of a field is commonly defined as the inverse of the linewidth of its power spectral density. In the present model, a more convenient expression is obtained by relating the linewidth to the phase diffusion constant (D). Since the linewidth is linearly proportional to the diffusion constant, the coherence time can be written as

$$\tau_c = \frac{1}{D} = \frac{1}{r\sigma^2}. \tag{5.4}$$

This refined definition captures the correct scaling behaviour of the relevant parameters within the phase diffusion model.

5.3 Measure of Coherence

We have seen multiple definitions of the measure of coherence $\mathfrak{C}$ in the previous chapter. For the current model the most relevant definition would be the total number of coherent photons(N_{coh}) which is

$$\begin{aligned} \mathfrak{C} = N_{coh} &= r\tau_c \\ &= \frac{1}{\sigma^2}. \end{aligned} \tag{5.5}$$

Now we make another assumption that the photon emission rate is given by

$$\dot{n} = \gamma n. \tag{5.6}$$

This implies that $r = \gamma n$.

5.4 Phase estimation

The environment attempts to infer the phase of the system. The precision with which the phase can be estimated depends on the amount of information accessible to the environment. In practice, this information is determined by the physical resources available to the environment, including the number of measurements performed and the strength of the interaction between the system and its surroundings. To analyze the problem systematically, we consider two distinct situations that illustrate how different levels of access and resources affect the environment's ability to estimate the phase. These two cases are:

- The environment uses a linear Hamiltonian to learn the phase.
- The environment uses a quadratic Hamiltonian to learn the phase.

5.4.1 Linear Hamiltonian

Let us assume the environment uses a Linear Hamiltonian

$$H = \omega \hat{n}. \tag{5.7}$$

Here ω is a constant with units of energy and $\hat{n}$ is the number of photons. Hence the energy is $E = \omega n$ for n photons. The power can be calculated to be

$$P = \frac{dE}{dt} = \omega \dot{n} = \omega \gamma n. \tag{5.8}$$

This implies that $n = P/(\gamma\omega)$. Now we calculate the FI due to the estimation process. The formula for quantum FI Eq. (3.33) can be simplified for pure states and optimizing the measurement using the symmetric logarithmic derivative (SLD) formalism as in [11] we get the QFI to be

$$QFI = 4(\langle H^2 \rangle - \langle H \rangle^2) = 4(\Delta^2 H). \tag{5.9}$$

Now computing this for the above model we get $QFI \propto n^2\omega^2$. Using the Cramer-Rao bound we get

$$\sigma^2 \propto \frac{1}{n^2\omega^2}. \tag{5.10}$$

Now putting this back into the formula for diffusion constant along the formula for r we get

$$D \propto \frac{\gamma}{n\omega^2}. \tag{5.11}$$

This shows the same scaling as the standard laser theory. The coherence time becomes $\tau_c \sim 1/D = n\omega^2/\gamma$ and measure of coherence $\mathfrak{C} \sim n^2\omega^2$.

5.4.2 Quadratic Hamiltonian

In this case we do the same analysis as before but the environment uses a quadratic Hamiltonian as its resource

$$H = \chi\hat{n}^2. \tag{5.12}$$

Here χ is a constant with unit of energy. So for n photons the energy is $E = \chi n^2$. The power becomes

$$P = 2\chi n\dot{n} = 2\chi\gamma n^2. \tag{5.13}$$

Now we calculate the QFI using the same formula as in Linear Hamiltonian case which gives

$$QFI \propto \chi^2 n^4. \tag{5.14}$$

Hence using the Cramer-Rao bound we get the variance to be

$$\sigma^2 \sim \frac{1}{\chi^2 n^4}. \tag{5.15}$$

Following this the diffusion constant $D = \gamma/(\chi^2 n^3)$ and the coherence time $\tau_c = \chi^2 n^3/\gamma$. Finally the measure of coherence $\mathfrak{C} \propto \chi^2 n^4$.

This result reveals the scaling associated with a Heisenberg-limited laser. The analysis indicates that achieving such a limit requires the availability of a non-linear Hamiltonian resource. In practice, the implementation of a laser that attains Heisenberg-limited coherence would involve operators that are non-linear in the field operators (a) and ($a^\dagger$), making the physical realization considerably more complex. Although the model considered here is simplified, this resource-based perspective provides useful insight into both the origin of the Heisenberg scaling and the practical challenges associated with implementing a Heisenberg-limited laser.

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