The Great Picard's Theorem And The Uniformization Theorem

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CERTIFICATE

This is to certify that this dissertation entitled "The Great Picard's Theorem and the Uniformization Theorem" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents work carried out by Lakshmi Priya M E under the supervision of Prof. Ravi S Kulkarni during the academic year 2010-2011.

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Contents

CHAPTER 1

Introduction

A holomorphic function is characterized by the property that, at points where the derivative f' is nonzero, f locally preserves angles or an equivalent property that it stretches equally in all directions. The following equation can be viewed as the mathematical formulation of the two facts mentioned above:

$$|(f \circ \gamma)'(t)| = |f'(\gamma(t))| \cdot |\gamma'(t)|$$

where f is a holomorphic function on a domain $U \subset \mathbb{C}$, $\gamma : [0,1] \to U$ is a smooth curve in U and $f'(\zeta) \neq 0$ for all $\zeta \in [\gamma]$. This simple looking equation opens up an exciting arena in the form of geometry for complex analysis. In the new setting, every nonconstant holomorphic function between domains in $\mathbb{C}$ becomes a local isometry of Riemann surfaces endowed with suitable (hermitian) metrics.

The central theme of Part I is curvature. Quoting Greene from his paper [**2**]: "The underlying idea in Riemannian geometry is that curvature controls topology; from hypothesis on curvature one hopes and expects to obtain conclusions about the topological nature of the Riemannian manifold. The natural extension of this idea to complex manifolds is that curvature should also control the complex structure." The idea that curvature controls the complex structure pervades the whole of Part I and is substantiated in many situations, the most significant one being the proof of the Great Picard's theorem.

As will be seen in chapter 5, the Uniformization theorem plays a significant role in studying holomorphic maps between Riemann surfaces, in particular domains in $\mathbb{C}$. The theory of covering spaces, in many situations, simplifies the task of studying maps between arbitrary Riemann surfaces to maps (the corresponding lifts to their covering spaces) between their universal covering spaces. Using this we give an alternate proof of the Great Picard's theorem.

The solution of the Dirichlet's problem by Perron's method and the proof of the Uniformization theorem are discussed in Part II. The Dirichlet's problem is a boundary-value problem in harmonic function theory. The solution of the Dirichlet's problem by Perron method involves constructing a Perron family, the associated Perron function being the required solution. The idea of the proof of the Uniformization theorem is: Given a Riemann surface R, we solve a certain Dirichlet's problem on R (using the Perron method) and then make use of this solution which is a harmonic function to construct a conformal map to one of the surfaces $\mathbb{D}$, $\mathbb{C}$ or $\hat{\mathbb{C}}$, by means of analytic continuation on R . Though the proof of even the most basic properties of harmonic functions are dependent on the fact that they are the real/imaginary parts of holomorphic functions and on results of Complex Analysis, harmonic function theory almost single handedly propels the proof of one of the most significant results of Complex Analysis/ Riemann Surfaces towards completion.

This thesis is expected to provide an introduction to the interaction of Complex Analysis with the other areas of Mathematics.

Part 1

The Great Picard's Theorem

CHAPTER 2

The Geometric versions of Schwarz's lemma and applications

This chapter can be considered as an introduction to *Geometric Function Theory*. We will introduce the concept of Riemann surfaces, hermitian metrics on them and curvature. We will consider (nonconstant) holomorphic maps between domains in $\mathbb{C}$ in a geometric setting, in which they become local isometries. We will also consider Ahlfors's version of Schwarz's lemma, which can be considered as an interpretation of the classical Schwarz's lemma in terms of curvature. This will set the ball rolling! Curvature is a function of the Riemann surface and the hermitian metric on it. Using a generalized version of Schwarz's lemma we will see how curvature speaks of the conformal properties of the Riemann surface. As an illustration of this, the chapter culminates with a proof of the Picard's Little theorem and a few other applications. The discussion on Riemann surfaces in section 2.1 is based on the book of Bers [**1**] and the paper of Greene [**2**]. The ensuing sections are based on the books of Krantz [**4**] and [**5**].

2.1. Riemann surfaces

In this section we will assume knowledge of manifolds and introduce the basics of the generalization of (real) manifold theory to complex manifolds, in particular Riemann surfaces.

Definition 2.1 (Riemann Surface)**.** *A Riemann surface is a connected Hausdorff topological space M together with a collection of charts $\{(U_\alpha, f_\alpha)\}_{\alpha\in\mathfrak{A}}$ with the following properties:*

1. $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ *form an open covering of M.*
2. *Each $f_\alpha : U_\alpha \to \mathbb{C}$ is homeomorphic onto an open subset of $\mathbb{C}$.*
3. *Whenever $U_\alpha \cap U_\beta \neq \phi$, the function $f_\beta \circ f_\alpha^{-1} : f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$ is holomorphic.*

Definition 2.2 (Holomorphic function between Riemann surfaces)**.** *Let M and N be Riemann surfaces. Let $\{(U_\alpha, g_\alpha)\}_{\alpha\in\mathfrak{A}}$ and $\{(V_\beta, h_\beta)\}_{\beta\in\mathfrak{B}}$ be a collection of charts on M and N respectively, satisfying the three properties in the preceding definition.*

A function $f : M \to N$ is said to be holomorphic if for every point $p \in M$ and any (U_α, g_α) and (V_β, h_β) such that $p \in U_\alpha$ and $f(p) \in V_\beta$, the function:

$$h_\beta \circ f \circ g_\alpha^{-1} : g_\alpha(U_\alpha) \to h_\beta(V_\beta)$$

is holomorphic as a map between subsets of $\mathbb{C}$.

Analogous to (real) smooth manifolds, we can also define complex manifolds. In this language, the definition of a Riemann surface becomes: A Riemann surface is a one dimensional complex manifold. Similarly analogous to the tangent space of a (real) manifold, we can also define *holomorphic tangent space* for a complex manifold, which turns out to a complex manifold.

Definition 2.3 (Conformal structure)**.** *Suppose that R is a Riemann surface. A maximal collection of charts $(U_\alpha, f_\alpha)_{\alpha\in\mathfrak{A}}$ satisfying the three conditions in definition 2.1 is said to define a conformal structure on R.*

A conformal structure on a Hausdorff topological space M makes it into a Riemann surface and distinguishes a subset of $\{f : M \to \mathbb{C} \mid f \text{ is continuous}\}$ as holomorphic functions.

Lemma 2.4. *Let R be a Riemann surface and TR its holomorphic tangent space. Then there exists a natural map $J : TR \to TR$ which satisfies:*

1. *For every $p \in R$, the restriction $J_p = J|_{T_pR}$ is a vector space isomorphism $J_p : T_pR \to T_pR$, where T_pR is considered as a real vector space.*
2. *For every $p \in R$, $J_p^2 = -I$, where I is the identity map on T_pR.*

Proof. Let $p \in R$ be an arbitrary point of R. Let (U, f) be a coordinate chart on a neighborhood of p. Suppose that $f = x + iy$, where x and y are real valued functions on R. Then for any $q \in U$, $\{\frac{\partial}{\partial x}|_q, \frac{\partial}{\partial y}|_q\}$ is a basis for T_qR over $\mathbb{R}$. We will denote $\frac{\partial}{\partial x}|_q$ and $\frac{\partial}{\partial y}|_q$ by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ respectively. Consider the linear isomorphism J_q defined by:

$$J_q\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) = a\frac{\partial}{\partial y} - b\frac{\partial}{\partial x}\ ,\ \forall\ a, b\ \in \mathbb{R}$$

It is clear that $J_q^2 = -I$. But we have defined J_q by making use of the coordinate chart (U, f). We will now show that J_q is actually independent of the coordinate chart used to define it. Suppose that (V, g) is any other coordinate chart such that $q \in V$. Suppose that $g = \tilde{x} + i\tilde{y}$, where both $\tilde{x}$ and $\tilde{y}$ are real valued functions on V. $\{\frac{\partial}{\partial \tilde{x}}|_q, \frac{\partial}{\partial \tilde{y}}|_q\}$ forms a basis for T_qR. Then consider the function G_q defined on T_qR by:

$$G_q\left(a\frac{\partial}{\partial \tilde{x}} + b\frac{\partial}{\partial \tilde{y}}\right) = a\frac{\partial}{\partial \tilde{y}} - b\frac{\partial}{\partial \tilde{x}}\ ,\ \forall\ a, b\ \in \mathbb{R}$$

We wish to show that $G_q = J_q$. Expressing one basis in terms of the other, we have:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial \tilde{x}}{\partial x}\frac{\partial}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial x}\frac{\partial}{\partial \tilde{y}} \\
\frac{\partial}{\partial y} &= \frac{\partial \tilde{x}}{\partial y}\frac{\partial}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial y}\frac{\partial}{\partial \tilde{y}}
\end{aligned}$$

From the Cauchy Riemann equations it follows that:

$$\begin{aligned}
\frac{\partial \tilde{x}}{\partial x} &= \frac{\partial \tilde{y}}{\partial y} \\
\frac{\partial \tilde{x}}{\partial y} &= -\frac{\partial \tilde{y}}{\partial x}
\end{aligned}$$

Hence we have the following:

$$\begin{aligned}
G_q\left(\frac{\partial}{\partial x}\right) &= \frac{\partial \tilde{x}}{\partial x}\frac{\partial}{\partial \tilde{y}} - \frac{\partial \tilde{y}}{\partial x}\frac{\partial}{\partial \tilde{x}} \\
&= \frac{\partial \tilde{x}}{\partial x}\left(\frac{\partial x}{\partial \tilde{y}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \tilde{y}}\frac{\partial}{\partial y}\right) - \frac{\partial \tilde{y}}{\partial x}\left(\frac{\partial x}{\partial \tilde{x}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \tilde{x}}\frac{\partial}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial \tilde{x}}{\partial x}\frac{\partial x}{\partial \tilde{y}} - \frac{\partial \tilde{y}}{\partial x}\frac{\partial x}{\partial \tilde{x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \tilde{x}}{\partial x}\frac{\partial y}{\partial \tilde{y}} - \frac{\partial \tilde{y}}{\partial x}\frac{\partial y}{\partial \tilde{x}}\right) \\
&= A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}
\end{aligned}$$

We now have:

$$\begin{aligned}
0 &= \frac{\partial \tilde{x}}{\partial \tilde{y}} \\
&= \frac{\partial \tilde{x}}{\partial x}\frac{\partial x}{\partial \tilde{y}} + \frac{\partial \tilde{x}}{\partial y}\frac{\partial y}{\partial \tilde{y}}
\end{aligned}$$

$$\therefore \frac{\partial \tilde{x}}{\partial x}\frac{\partial x}{\partial \tilde{y}} = -\frac{\partial \tilde{x}}{\partial y}\frac{\partial y}{\partial \tilde{y}} \tag{2.1.1}$$

Let us now use the Cauchy Riemann equations and equation 2.1.1 to simplify A and B.

$$\begin{aligned} A &= \frac{\partial \tilde{x}}{\partial x}\frac{\partial x}{\partial \tilde{y}} - \frac{\partial \tilde{y}}{\partial x}\frac{\partial x}{\partial \tilde{x}} \\ &= -\frac{\partial \tilde{x}}{\partial y}\frac{\partial y}{\partial \tilde{y}} - \frac{\partial \tilde{y}}{\partial x}\frac{\partial x}{\partial \tilde{x}} \\ &= \frac{\partial \tilde{y}}{\partial x}\frac{\partial x}{\partial \tilde{x}} - \frac{\partial \tilde{y}}{\partial x}\frac{\partial x}{\partial \tilde{x}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} B &= \frac{\partial \tilde{x}}{\partial x}\frac{\partial y}{\partial \tilde{y}} - \frac{\partial \tilde{y}}{\partial x}\frac{\partial y}{\partial \tilde{x}} \\ &= \frac{\partial \tilde{y}}{\partial y}\frac{\partial y}{\partial \tilde{y}} + \frac{\partial \tilde{x}}{\partial y}\frac{\partial y}{\partial \tilde{x}} \\ &= 1 \end{aligned}$$

We have shown that $G_q(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}$. In a similar way it can be shown that $G_q(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x}$. Thus $G_q = J_q$ and hence the endomorphisms J_q are well defined. The above proof also shows that $J : TR \to TR$ is a smooth function. □

Analogous to the Riemannian metric for (real) manifolds one can define a *Hermitian metric* for complex manifolds, in particular for Riemann surfaces.

Definition 2.5 (Hermitian metric)**.** *Let R be a Riemann surface. R can also be considered as a (real) manifold. A smooth Riemannian metric g on R is said to a Hermitian metric if the following holds for all $p \in R$ and for all $u, v \in T_pR$:*

$$g_p(u, v) = g_p(J_pu, J_pv)$$

where J is as in the preceding lemma.

It can be easily seen that for any Riemann surface R, there exists a Hermitian metric : R can also be considered as a (real) manifold. By making use of partition of unity we can show the existence of a Riemannian metric on *any* manifold and in particular R. Consider the new metric h defined by:

$$h(u, v) = \frac{1}{2}\left(g(u, v) + g(Ju, Jv)\right)$$

h is a Hermitian metric on the Riemann surface R.

Definition 2.6 (Conformal classes of Riemannian metrics). *Suppose that M is a (real) manifold. Then any two Riemannian metrics g_1 and g_2 on M are said to be conformally equivalent if:*

$$g_1 = \lambda g_2$$

where λ is a smooth positive function on M. The relation $\sim$ on the set of Riemannian metrics on M given by: $g \sim h$ iff g and h are conformally equivalent is an equivalence relation and the corresponding equivalence classes are called Conformal classes of Riemannian metrics on M.

Theorem 2.7. *Let M be an orientable, two dimensional, real manifold. Then the conformal classes of Riemannian metrics on M are in one-one correspondence with the conformal structures on M.*

2.2. Subsets of $\mathbb{C}$ as Riemann surfaces

Suppose that $U \subset \mathbb{C}$ is a nonempty, open, connected subset of $\mathbb{C}$. Then there is the natural conformal structure on U given by $\{(V, \phi) | V \subset U \text{ is open and } \phi : V \to \mathbb{C} \text{ is a conformal map}\}$. The holomorphic tangent space of U, $TU \cong U \times \mathbb{C}$ under the identification $\frac{\partial}{\partial z}|_p = (p, 1)$. Suppose that h is a hermitian metric on U, then the function:

$$f(p) = h((p, 1), (p, 1))$$

is a smooth positive function on U, since $X(p) = (p, 1)$ is a smooth vector field on R. Conversely assume that λ is a smooth positive function on U, then h defined in the following way defines a hermitian metric on U:

$$h_p((p, a), (p, b)) = a\bar{b}\lambda(p) \ , \ \forall \ a, b \in \mathbb{C}$$

Definition 2.8 (Metric and Length). *Let $U \subseteq \mathbb{C}$ be a domain. Then a nonnegative function on U, μ is called a metric if it satisfies the following conditions:*

(1) *μ is twice differentiable on the set $\{z \in U \mid \mu(z) > 0\}$.*
(2) *The set $\{z \in U \mid \mu(z) = 0\}$ is discrete in U.*

For $z \in U$ and $v \in \mathbb{C}$, the length of v at z denoted by $\|v\|_{\mu,z}$ is defined to be

$$\|v\|_{\mu,z} = \mu(z) \cdot \|v\|$$

where $\| \cdot \|$ is the Euclidean norm.

Suppose that μ is a metric on a domain $U \subset \mathbb{C}$. We can define a hermitian inner product on each T_pU, where p is such that $\mu(p) \neq 0$ as follows:

$$g_p((p, a), (p, b)) = \mu(p)a\bar{b} \ , \ \forall a, b \in \mathbb{C}$$

Let X and Y be smooth vector fields on U. Since μ is twice differentiable, the function $H : U \setminus \{q \in U \mid \mu(q) = 0\} \to \mathbb{C}$ given by:

$$H(p) = g_p(X_p, Y_p)$$

is also twice differentiable. Thus the metric defined above can be used to obtain g on $U \setminus \{q \in U \mid \mu(q) = 0\}$, which is a generalization of the hermitian metric.

Similar to the case of Riemannian metrics and hermitian metrics, a metric μ on U gives rise to a new distance function on the domain U.

Length of a curve $\gamma \subset U$ is defined to be:

$$l_\mu(\gamma) = \int_0^1 \parallel \dot{\gamma}(t) \parallel_{\mu,\gamma(t)} dt = \int_0^1 \mu(\gamma(t)) \parallel \dot{\gamma}(t) \parallel dt$$

For $x, y \in U$, define the set $C_{xy} = \{\gamma \subset U | \, \gamma$ is a smooth curve connecting x and $y\}$. We now define the distance between points in U by:

$$d_\mu(x, y) = inf\{l_\mu(\gamma) | \; \gamma \in C_{xy}\}$$

It is easy to see that d_μ defines a distance function on the domain U.

Definition 2.9 (Pullback metric)**.** *Suppose U and V are domains in $\mathbb{C}$ and $f : U \to V$ is a continuously differentiable function on U such that $\frac{\partial f}{\partial z}$ has isolated zeros on U. Assume that ρ is a metric on V. Then the pullback of the metric ρ via the map f, denoted $f^*\rho$ is defined to be*

$$f^*\rho(z) = \rho(f(z)) \cdot \left| \frac{\partial f}{\partial z} \right|$$

where $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$

At this juncture it is useful to make the following two observations regarding the above definition:

(1) In the above definition, if f happens to be a thrice differentiable function, then $f^*\rho$ defines a metric on U. As we will see below, the map $f : (U, f^*\rho) \to (V, \rho)$ will have interesting properties if f is a holomorphic function.
(2) Suppose f in the above definition is a nonconstant holomorphic function. Then for every $p \in U$, $df|_p$ defines a linear map between T_pU and $T_{f(p)}V$, each of which is a one dimensional complex vector space endowed with metrics (and equivalently generalized hermitian metrics) $f^*\rho$ and ρ respectively. T_pU is generated by $\frac{\partial}{\partial z}|_p$ and $T_{f(p)}V$ is generated by $\frac{\partial}{\partial z}|_{f(p)}$ over $\mathbb{C}$. Let $p \in U$ such that $f'(p) \neq 0$. The map $\partial f|_p$ is given by:

$$\begin{array}{rccc} df|_p: & T_pU & \to & T_{f(p)}V \\ & \frac{\partial}{\partial z}|_p & \to & \frac{\partial f}{\partial z}(p)\frac{\partial}{\partial z}|_{f(p)} \end{array}$$

As we already noted $TU \cong U \times \mathbb{C}$ and $TV \cong V \times \mathbb{C}$. For every $q \in U$ and $s \in V$ we identify the tangent spaces T_qU and T_sV with $\mathbb{C}$ by identifying $\frac{\partial}{\partial z}|_q$ and $\frac{\partial}{\partial z}|_s$ with $1 \in \mathbb{C}$. With this identification, the map $df|_p$ actually becomes a linear map that preserves norms (and consequently the hermitian inner product) as shown below:

$$\begin{aligned} \parallel 1 \parallel_p &= f^*\rho(p) \\ &= \rho(f(p)) \cdot |f'(p)| \\ \parallel df|_p(1) \parallel_{f(p)} &= |f'(p)| \cdot \parallel 1 \parallel_{f(p)} \\ &= |f'(p)| \cdot \rho(f(p)) \end{aligned}$$

This shows that in case of holomorphic maps, the pullback metric has very special properties. This observation serves as a motivation for the following definition.

Definition 2.10 (Isometry)**.** *Let $f : U \to V$ be a one-one, onto, continuously differentiable map between domains U and V of $\mathbb{C}$ which are equipped with metrics ρ_1 and ρ_2 respectively. f is called an isometry of the pair (U, ρ_1) with (V, ρ_2) if:*

$$f^*\rho_2(z) \ = \ \rho_1(z), \ \forall \ z \in U.$$

Proposition 2.11. *Let (U, ρ_1), (V, ρ_2) and f be as in the above definition. Suppose also that f is a holomorphic map and an isometry of (U, ρ_1) with (V, ρ_2). Then the following are true:*

(1) *Suppose $\gamma : [0, 1] \to U$ is a smooth curve, then $f \circ \gamma$ is a smooth curve in V and $l_{\rho_1}(\gamma) = l_{\rho_2}(f \circ \gamma)$*
(2) *$d_{\rho_1}(x, y) = d_{\rho_2}(f(x), f(y))$, $\forall \ x, \ y \in U$.*
(3) *f^{-1} is also an isometry.*

PROOF. (1) We will calculate the lengths of γ and $f \circ \gamma$ below:

$$
\begin{aligned}
l_{\rho_1}(\gamma) &= \int_0^1 \parallel \dot{\gamma}(t) \parallel_{\gamma(t)} dt \\
&= \int_0^1 \rho_1(\gamma(t)) \parallel \dot{\gamma}(t) \parallel dt \\
&= \int_0^1 \rho_2(f(\gamma(t))) \mid \frac{\partial f}{\partial z}(\gamma(t)) \mid \parallel \dot{\gamma}(t) \parallel dt \\
l_{\rho_2}(f \circ \gamma) &= \int_0^1 \parallel f \dot{\circ} \gamma(t) \parallel_{f(\gamma(t))} dt
\end{aligned}
$$

Since f is holomorphic it can be easily seen using the Cauchy-Riemann equations that

$$
\left| \frac{d}{dt}(f(\gamma(t))) \right| = \left| \frac{\partial f}{\partial z}(\gamma(t)) \right| \cdot \parallel \dot{\gamma}(t) \parallel
$$

Thus it follows that $l_{\rho_1}(\gamma) = l_{\rho_2}(f \circ \gamma)$.

(2) Let x and $y \in U$. Let γ be any smooth curve in U connecting x and y and let α be any smooth curve in V connecting $f(x)$ and $f(y)$. Then since f^{-1} is also holomorphic, $f^{-1} \circ \alpha$ is a smooth curve in U. It now follows from (1) that $d_{\rho_2}(f(x), f(y)) \leq d_{\rho_1}(x, y)$ as well as $d_{\rho_1}(x, y) \leq d_{\rho_2}(f(x), f(y))$. Hence f preserves distances as claimed.

(3) This directly follows from observing that f^{-1} is also holomorphic and from the definition of the pullback metric.

□

We have thus realized a conformal (or biholomorphic) map between two domains in $\mathbb{C}$ as an isometry of the domains when considered with a suitable metric.

2.3. Hyperbolic metric on the unit disc $\mathbb{D}$

In light of the previous section, we will in this section define a special metric on the unit disc $\mathbb{D} \subset \mathbb{C}$ called the *Poincaré* metric or the *Hyperbolic* metric on $\mathbb{D}$. By the end of this section it will be clear why this metric is special. From now on by $\mathbb{D}$, we will mean the unit disc in $\mathbb{C}$ which is centered at 0.

Definition 2.12 (Poincaré or Hyperbolic metric)**.** *The Poincaré or the Hyperbolic metric on $\mathbb{D}$ is given by*

$$
\rho(z) = \frac{1}{1 - |z|^2} \ , \ \forall \ z \in \mathbb{D}
$$

We recall that any conformal self map of $\mathbb{D}$ is given by

$$f(z) = e^{i\theta}\left(\frac{z-a}{1-\bar{a}z}\right)$$

where $\theta \in [0, 2\pi)$ and $a \in \mathbb{D}$. We will in the remainder of this chapter denote the function $g(z) = (\frac{z-a}{1-\bar{a}z})$ by $\phi_a(z)$.

Proposition 2.13. *Suppose f is a conformal self map of $\mathbb{D}$. Then $f : (\mathbb{D}, \rho) \to (\mathbb{D}, \rho)$ is an isometry.*

PROOF. We saw above that f is a composition of a rotation map and ϕ_a, for some $a \in \mathbb{D}$. Hence it is enough to prove that rotations and the maps ϕ_a are isometries. That rotations are isometries is easy to see.

$$\begin{aligned} \phi_a^*\,\rho(z) &= \rho(\phi_a(z))|\phi_a'(z)| \\ &= \frac{1}{1-|(\frac{z-a}{1-\bar{a}z})|^2}\left|\frac{1-|a|^2}{(1-\bar{a}z)^2}\right| \\ &= \frac{1}{1-|z|^2} \\ &= \rho(z) \end{aligned}$$

□

The above proposition is interesting in that when both the domain($\mathbb{D}$) and codomain($\mathbb{D}$) are endowed with the *same* metric ρ, it holds that *any* self conformal map turns out to be an isometry! The following theorem also suggests the origin of ρ.

Proposition 2.14. *Suppose μ is a metric on $\mathbb{D}$ which is such that any conformal self map of $\mathbb{D}$ defines an isometry of $(\mathbb{D}, \mu)$ with itself, then μ is a constant multiple of ρ.*

PROOF. Let μ be a metric on $\mathbb{D}$ such that any self conformal map of $\mathbb{D}$ is an isometry of $(\mathbb{D}, \mu)$ with itself. Consider ϕ_a for some $a \in \mathbb{D}$, then by our assumption

$$\begin{aligned} \mu(z) &= \phi_a^*\mu(z) \\ \mu(z) &= \mu\left(\frac{z-a}{1-\bar{a}z}\right)\cdot\left|\frac{1-|a|^2}{(1-\bar{a}z)^2}\right| \\ \therefore \mu(a) &= \mu(0)\cdot\frac{1}{(1-|a|^2)} \end{aligned}$$

This last equality holds for any $a \in \mathbb{D}$. Hence Proved.

□

2.4. Schwarz's Lemma

We have developed all the prerequisites required to understand some of the theorems of complex analysis in the new geometric setting. Let us begin with the Schwarz's lemma which we recall below.

Lemma 2.15 (Schwarz's Lemma). *Suppose* $f : \mathbb{D} \to \mathbb{D}$ *is holomorphic and* $f(0) = 0$. *Then the following hold:*

(1) $|f(z)| \leq |z|$ *on* $\mathbb{D}$
(2) $|f'(0)| \leq 1$
(3) *If equality holds in either of the above cases, then* $f(z) = e^{i\theta} z$ *for some* $\theta \in [0, 2\pi)$.

The following lemma is an immediate corollary of the Schwarz's lemma.

Lemma 2.16 (Schwarz-Pick Lemma). *If* $f : \mathbb{D} \to \mathbb{D}$ *is a holomorphic map then for any* $z \in D$

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}$$

Proof. Let $a \in \mathbb{D}$ be an arbitrary point. Consider the composite function $g = \phi_{f(a)} \circ f \circ \phi_a^{-1}$. Then $g : \mathbb{D} \to \mathbb{D}$ and g fixes the point 0. Applying the Schwarz's lemma to g we get $|g'(0)| \leq 1$. We also have:

$$\begin{aligned} g'(0) &= (\phi_a^{-1})'(0) \cdot f'(a) \cdot (\phi_{f(a)})'(f(a)) \\ &= (1 - |a|^2) \cdot f'(a) \cdot \frac{1}{1 - |f(a)|^2} \end{aligned}$$

We initially started with an arbitrary $a \in \mathbb{D}$, thus we have the desired result:

$$\begin{aligned} \left| (1 - |a|^2) \cdot f'(a) \cdot \frac{1}{1 - |f(a)|^2} \right| &\leq 1 \\ \therefore |f'(a)| &\leq \frac{1 - |f(a)|^2}{1 - |a|^2} \ , \ \forall \ a \in \mathbb{D} \end{aligned}$$

□

Proposition 2.17. *Suppose* $f : (\mathbb{D}, \rho) \to (\mathbb{D}, \rho)$ *is a holomorphic map. Then* f *is a distance decreasing map, i.e.,* $d_\rho(f(x), f(y)) \leq d_\rho(x, y)$, $\forall \ x, y \in \mathbb{D}$.

Proof. This follows as a direct consequence of the Schwarz-Pick lemma. We start by noting that if $g : U \to V$ is a holomorphic map between domains U and V

in $\mathbb{C}$ and μ is a metric on V, then for any curve $\gamma \subset U$ connecting points $x, y \in U$, we have

$$l_\mu(f \circ \gamma) = l_{f^*\mu}(\gamma)$$

and hence

$$d_\mu(f(x), f(y)) \leq d_{f^*\mu}(x, y). \tag{2.4.1}$$

From Schwarz-Pick Lemma we have,

$$\begin{aligned} |f'(z)| &\leq \frac{1-|f(z)|^2}{1-|z|^2} \\ \therefore |f'(z)| \frac{1}{1-|f(z)|^2} &\leq \frac{1}{1-|z|^2} \\ \therefore f^*\rho(z) &\leq \rho(z) \end{aligned}$$

Thus $d_{f^*\rho}(x, y) \leq d_\rho(x, y)$. From this and (2.4.1) the desired result follows:

$$d_\rho(f(x), f(y)) \leq d_\rho(x, y)$$

□

Remark 2.18. The preceding proposition is a direct consequence of the Schwarz-Pick lemma. Conversely Schwarz-Pick lemma can be considered as an infinitesimal version of the preceding proposition.

2.5. Schwarz's Lemma in terms of Curvature

Definition 2.19 (Curvature)**.** *Let U be a domain in $\mathbb{C}$ endowed with a metric ρ. Then for $z \in U$ such that $\rho(z) \neq 0$, the curvature of ρ at the point z, denoted $\kappa_{(U,\rho)}(z)$ is defined to be:*

$$\kappa_{(U,\rho)}(z) = \frac{-\Delta log \rho(z)}{(\rho(z))^2}$$

Lemma 2.20. *Suppose that U and V are domains in $\mathbb{C}$ and $f : U \to V$ is a conformal map. If ρ is a metric on V, then the curvature is invariant under the map $f : (U, f^*\rho) \to (V, \rho)$, i.e., $\kappa_{(U,f^*\rho)}(z) = \kappa_{(V,\rho)}(f(z))$.*

PROOF. We have

$$\begin{aligned}\kappa_{(U,f^*\rho)}(z) &= \frac{-\Delta log f^*\rho(z)}{(f^*\rho(z))^2} \\ &= \frac{-\Delta log\rho(f(z))|f'(z)|}{(\rho(f(z)))^2\cdot|f'(z)|^2} \\ &= \frac{-\Delta log\rho(f(z))}{(\rho(f(z)))^2\cdot|f'(z)|^2} \\ &= \frac{-\Delta(log\rho\circ f)(z)}{(\rho(f(z)))^2\cdot|f'(z)|^2}\end{aligned}$$

Now using the formulas for $\frac{\partial}{\partial z}(f\circ g)$ and $\frac{\partial}{\partial \bar{z}}(f\circ g)$, we get the expression for $\kappa_{(U,f^*\rho)}(z)$ to be:

$$\begin{aligned}\kappa_{(U,f^*\rho)}(z) &= \frac{(\Delta log\rho)(f(z))|f'(z)|^2}{(\rho(f(z)))^2\cdot|f'(z)|^2} \\ &= \kappa_{(V,\rho)}(f(z))\end{aligned}$$

□

Remark 2.21. Let U and ρ be as in definition 2.19. Let $S=\{z\in U|\ \rho(z)=0\}$. As seen in section 2.2, the metric ρ endows the Riemann surface $U\setminus S$ with a generalized hermitian metric. Now $U\setminus S$ along with the hermitian metric can also be thought of as a Riemannian manifold. For a Riemmanian manifold we already have a notion of curvature. This coincides with the above definition of curvature.

Note that the curvature of a metric at a point is a local property and we see that the above theorem holds if f is some nonconstant (not necessarily injective) holomorphic function on U at the points where $f'\neq 0$, which is all of U except a discrete set.

We will consider a few examples before proceeding:

(1) Let $U\subseteq\mathbb{C}$ be any domain. Let $\rho\equiv 1$ be the Euclidean metric on U. Then $\kappa_{(U,\rho)}\equiv 0$.
(2) Consider $(\mathbb{D},\rho)$, where ρ is the *Poincaré* metric on D. Then $\kappa_{(U,\rho)}\equiv -4$.
(3) On $\mathbb{C}$ consider the metric $\mu(z)=\frac{2}{1+|z|^2}$. This is often called the Spherical metric and $\kappa_{(\mathbb{C},\mu)}\equiv 1$.

Below we will prove Ahlfors' version of Schwarz's lemma and consequently obtain the classical Schwarz's lemma as a corollary.

Theorem 2.22. *Let $f : \mathbb{D} \to V$ be a holomorphic map, where V is any domain in $\mathbb{C}$. Suppose ρ is the Poincaré metric on $\mathbb{D}$ and μ a metric on V such that $\kappa_{(V,\mu)} \leq -4$ on V. Then $f^*\mu \leq \rho$.*

Proof. Let $0 < r < 1$. On $D(0,r)$ consider the metric ρ_r given by $\rho_r(z) = \frac{r}{r^2 - |z|^2}$. This is the analogue of the *Poincaré* metric for $D(0,r)$ in that the curvature of this metric on $D(0,r)$ is identically -4. Consider the function v on $D(0,r)$ given by

$$v(z) = \frac{f^*\mu}{\rho_r}$$

This is a positive function which is twice differentiable on $D(0,r)$ and hence defines a metric on it. The metric $f^*\mu$ is bounded above by a positive constant on $D(0,r)$ and by the very definition of ρ_r, $v(z) \to 0$ as $|z| \to r$. Hence v attains maximum at an interior point of $D(0,r)$. Let that point be P. We will show below that $v(P) \leq 1$. Since P is the maximum of the function v, $\Delta log v(P) \leq 0$.

$$\begin{aligned} 0 &\geq \Delta log v(P) \\ &= \Delta log f^*\mu(P) - \Delta log \rho_r(P) \\ &= -(f^*\mu(P))^2 \kappa_{(f^*\mu)}(P) + (\rho_r(P))^2 \kappa_{\rho_r}(P) \\ &\geq 4((\rho_r(P))^2 - (f^*\mu(P))^2) \end{aligned}$$

We thus have $v(P) \leq 1$ and hence $v \leq 1$ on $D(0,r)$. Since we took an arbitrary $0 < r < 1$, we get the desired result by letting $r \to 1^-$. □

Corollary 2.23 (Lemma 2.15)**.** *Schwarz's lemma.*

Proof.
(1) In the above theorem 2.22 if we take $(V,\mu) = (\mathbb{D},\rho)$, then we get $f^*\rho \leq \rho$. A closer look at proposition 2.17 shows that

$$f^*\rho \leq \rho \Rightarrow \text{distance decreasing property of } f \text{ w.r.t } \rho$$

Hence the above theorem 2.22 implies Schwarz-Pick lemma and the distance decreasing property.

If we suppose further that $f(0) = 0$ we have the following:

$$\text{distance decreasing property of } f \text{ w.r.t } \rho \Rightarrow d_\rho(f(z),0) \leq d_\rho(z,0) \Rightarrow |f(z)| \leq |z|$$

(2) Thus we have $|f(z)| \leq |z|$. In this letting $z \to 0$ yields $|f'(0)| \leq 1$.
(3) Also $f^*\rho = \rho$ iff f is an isometry. Hence the third statement in Schwarz's lemma 2.15 also follows.

□

We can indeed generalize this theorem in the following way and this generalization has many interesting applications.

Theorem 2.24. *Let $\alpha > 0$ and $A > 0$. On $D(0, \alpha)$ define the metric $\rho_\alpha^A(z) = \frac{2\alpha}{\sqrt{A}(\alpha^2 - |z|^2)}$. Suppose $f : D(0, \alpha) \to U$ is a holomorphic map and μ is a metric on U which is such that $\kappa_{(U,\mu)} \leq -B < 0$ on U. Then*

$$\frac{f^*\mu}{\rho_\alpha} \leq \frac{\sqrt{A}}{\sqrt{B}} \text{ on } D(0, \alpha)$$

The proof of the above theorem is a verbatim translation of the proof of theorem 2.22 with $\rho_r(z) = \frac{2r}{\sqrt{A}(r^2 - |z|^2)}$ and eventually we let $r \to \alpha$.

2.6. Applications

In this section, we will derive two results as a consequence of the theory developed in the previous sections. The curvature is a function of the metric. And the metric depends on the Riemann surface in consideration. The following two results illustrate how curvature gives information about the conformal nature of the Riemann surface.

Proposition 2.25. *Suppose $f : \mathbb{C} \to \Omega$ is a holomorphic function and σ is a metric on Ω such that $\kappa_{(\Omega,\sigma)} \leq -B < 0$. Then f is a constant function.*

Proof. On $D(0, \alpha)$ let ρ_α^A be the metric as in theorem 2.24. Then by theorem 2.24 it follows that

$$\begin{aligned} f^*\sigma &\leq \frac{\sqrt{A}}{\sqrt{B}} \rho_\alpha^A \\ \therefore f^*\sigma(z) &\leq \frac{\sqrt{A}}{\sqrt{B}} \frac{2\alpha}{\sqrt{A}(\alpha^2 - |z|^2)}, \forall z \in D(0, \alpha) \end{aligned}$$

Letting $\alpha \to \infty$ in the above inequality, we can conclude that $\forall z \in \mathbb{C}$, $f^*\sigma(z) = 0$. Thus $f' \equiv 0$. Hence f is a constant. □

We obtain Liouville's theorem as a corollary of this proposition.

Theorem 2.26 (Liouville's Theorem)**.** *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function which is bounded. Then f is a constant.*

Proof. Suppose that $|f| < M$ on $\mathbb{C}$. Then for any $A > 0$, the curvature of the metric $\rho_M^A(z) = \frac{2M}{\sqrt{A}(M - |z|^2)}$ on $D(0, M)$ is identically equal to $-A$. We can consider the map $f : \mathbb{C} \to D(0, M)$ and by proposition 2.25, it follows that f is constant. □

We will use proposition 2.25 to prove the following theorem as well:

Theorem 2.27 (Picard's Little theorem, Theorem 3 & Corollary 4, §2, Chapter 2, [**5**])**.** *Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and such that $f(\mathbb{C}) \subseteq \mathbb{C} \setminus \{P, Q\}$, for some $P, Q \in \mathbb{C}$, then f is a constant.*

Proof. In light of proposition 2.25, it is enough to construct a metric μ on $\mathbb{C} \setminus \{P, Q\}$ such that $\kappa_{(\mathbb{C}\setminus\{P,Q\},\mu)} \leq -B < 0$. Consider the metric μ defined below:

$$\mu(z) = \frac{(1 + |z|^{1/3})^{1/2}}{|z|^{5/6}} \cdot \frac{(1 + |z-1|^{1/3})^{1/2}}{|z-1|^{5/6}}$$

For this metric, we will calculate the curvature $\kappa(z)$

$$\kappa(z) \quad = \quad \frac{-\Delta log\mu(z)}{(\mu(z))^2}$$

Let us first calculate the numerator of the above expression. A simple calculation for $\alpha \neq 0$ gives

$$\Delta log(1 + |z|^{\alpha}) \quad = \quad \frac{\alpha^2 |z|^{\alpha-2}}{(1 + |z|^{\alpha})^2}$$

Since for every $z \neq 0$, we have $\Delta log|z| = 0$, the curvature κ is

$$\kappa(z) = -\frac{1}{18}\left[\frac{|z-1|^{5/3}}{(1 + |z|^{1/3})^3(1 + |z-1|^{1/3})} + \frac{|z|^{5/3}}{(1 + |z-1|^{1/3})^3(1 + |z|^{1/3})}\right]$$

We observe the following from the above equation:

(1) $\kappa(z) < 0, \forall z \in \mathbb{C} \setminus \{0, 1\}$
(2) $\lim_{z\to\infty} \kappa(z) = \infty$
(3) $\lim_{z\to 0} \kappa(z) = -\frac{1}{36}$
(4) $\lim_{z\to 1} \kappa(z) = -\frac{1}{36}$

Thus we have produced a metric on $\mathbb{C} \setminus \{0, 1\}$ for which the curvature is bounded above by a negative constant and hence from proposition 2.25 it follows that any holomorphic map $f : \mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$ is a constant. □

CHAPTER 3

Normal families and Great Picard's theorem

Normal families play a very significant role in the proof of Riemann Mapping theorem. Montel's theorem gives an important criterion for a family of holomorphic functions to be normal and equivalently a criterion for compactness in the space of holomorphic functions. In this chapter we will extend the notion of normal family and consider it in a geometric setting. As in the previous chapter, curvature is the main theme of this chapter too. This chapter culminates with the proof of Great Picard's theorem. The exposition in this chapter is based on the books of Krantz [**4**] and [**5**].

3.1. Introduction

We will begin by recalling a few definitions and theorems in the classical function theory.

3.1.1. Definitions and Montel's theorem.

Definition 3.1 (Normal Convergence)**.** *Let $U \subseteq \mathbb{C}$ be an open set and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on U. We say that (f_n) converges normally on U if (f_n) converges uniformly on all compact subsets of U (to a necessarily holomorphic function).*

Definition 3.2 (Normal family)**.** *Let $U \subseteq \mathbb{C}$ be an open set and let $\mathfrak{F} = \{f_\alpha\}_{\alpha\in\mathfrak{A}}$ be a family of holomorphic functions on U. We say that $\mathfrak{F}$ is a normal family if every sequence in $\mathfrak{F}$ has a subsequence that converges normally on U.*

Theorem 3.3 (Arzela-Ascoli theorem)**.** *Let K be a compact topological space. Then $C(K, \mathbb{C})$, the set of all continuous function from K to $\mathbb{C}$, is a metric space with the metric $d(f, g) = sup\{|f(x) - g(x)| \mid x \in K\}$. In this topology a subset of $C(K, \mathbb{C})$ is compact iff it is closed, bounded and equicontinuous.*

Theorem 3.4 (Montel's Theorem)**.** *Suppose that $\mathfrak{F} = \{f_\alpha\}$ is a family of holomorphic functions on an open subset U of $\mathbb{C}$. Suppose that for every compact set*

$K \subset U$, there exists $M_K > 0$ such that $|f(z)| \leq M_K$, $\forall z \in K$ and $\forall f \in \mathfrak{F}$, then $\mathfrak{F}$ is a normal family.

PROOF. We will first show that the theorem holds if U is replaced by any compact subset $K_n \subset U$ which are such that

(1) $K_1 \subset K_2 \subset K_3 \subset \cdots K_n \subset K_{n+1} \subset \cdots$
(2) $K_n \subset \overset{\circ}{K}_{n+1}$, $\forall n \geq 1$
(3) $\cup K_n = U$

We will later show that this is enough to prove the theorem. Fix a compact subset $K = K_n$. Now we consider the family $\mathfrak{F}$ as a family of functions on K. We can also consider it as a subset of $C(K, \mathbb{C})$. Then in this setting the theorem reads "if $\mathfrak{F}$ is a bounded family of holomorphic functions then, its closure is compact in $C(K, \mathbb{C})$". We already have Arzela-Ascoli theorem which gives all the compact subsets of $C(K, \mathbb{C})$. In view of this, we only have to prove that $\mathfrak{F}$ is an equicontinuous family. Let $r > 0$ be such that $\forall z \in K_n$, $D(z, r) \subset K_{n+1}$ and let $R > 0$ be such that $\forall \zeta \in K_{n+1}$, $D(\zeta, R) \subset K_{n+2}$. For any $x, y \in K_n$ such that $|x - y| < r$, let γ_{xy} represent the line connecting x and y. By our choice of r, $\gamma_{xy} \subset K_{n+1}$. Thus we have for all $f \in \mathfrak{F}$

$$\begin{aligned} f(x) - f(y) &= \oint_{\gamma_{xy}} f'(\zeta) d\zeta \\ |f(x) - f(y)| &\leq \sup_{\zeta \in L} |f'(\zeta)| \cdot |x - y| \\ &\leq \frac{M_{K_{n+2}}}{R} \cdot |x - y| \end{aligned}$$

Thus $\mathfrak{F}$ is equicontinuous and hence is normal. So assume that (f_n) is any sequence in $\mathfrak{F}$. We need to produce a subsequence which converges normally on U. Let S_1 denote the subsequence of (f_n) which converges normally on K_1, and recursively we get the sequence S_n which is the subsequence of S_{n-1} which converges normally on K_n. Now we construct the subsequence of (f_n) which is denoted $g_k = f_{n_k}$ where g_k is the k^{th} entry in the sequence S_k. Note the following about this subsequence:

(1) By our very construction $(g_k)_n^\infty \subset S_n$ and hence converges normally on K_n.
(2) Since any compact set $K \subset U$ is a subset of some K_n, we conclude that (g_k) converges normally on U.

Hence we have proved that $\mathfrak{F}$ is a normal family. □

3.1.2. Extension of the notion of normal family.

Definition 3.5 (Compact divergence). *Let $U \subseteq \mathbb{C}$ be an open set and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of holomorphic functions on U. We say that (f_n) diverges compactly* *on U if for all compact sets $K \subset U$ and $L \subset \mathbb{C}$, there exists $N \in \mathbb{N}$ such that $f_n(K) \cap L = \phi$, whenever $n \geq N$.*

Note that the above definition is equivalent to saying that the sequence $1/(f_n)_m^\infty$ (for some $m \in \mathbb{N}$) converges normally on U to the constant function 0. Now we shall extend our definition of normal families to include sequences that diverge compactly too.

Definition 3.6 (Normal family*). *Let $U \subseteq \mathbb{C}$ be an open set and let $\mathfrak{F} = \{f_\alpha\}$ be a family of holomorphic functions on U. We say that $\mathfrak{F}$ is a normal family* if every sequence in $\mathfrak{F}$ has a subsequence that converges normally or diverges compactly on U.*

In the above definition, let us for the moment, think of the functions f_α as taking values in $\mathbb{C}\cup\{\infty\}$. Then endowing $\mathbb{C}\cup\{\infty\}$ with a suitable metric, we can reformulate the definition of normal family* to read "A family of holomorphic functions taking values in $\mathbb{C}\cup\{\infty\}$ (which is equipped with some metric) is a normal family* if every sequence of functions has a subsequence that *converges normally* on U".

In the above consideration, there are a lot of terms to be made precise and the rest of this subsection will be devoted towards this.

The Riemann sphere $\mathbb{C}\cup\{\infty\}$ is a Riemann surface. This can also be considered as the sphere in $\mathbb{R}^3$. The correspondence is precisely defined by the stereographic projection p of $\mathbb{C}$ on $S^2 \subset \mathbb{R}^3$. We want to define a metric σ on $\mathbb{C}\cup\{\infty\}$ such that measurement of distances in $(\mathbb{C}\cup\{\infty\}, \sigma)$ can be thought of as being done on the sphere $S^2 \subset \mathbb{R}^3$. It is clear that this is the metric suitable for the present situation. Simple calculations lead to the *Spherical Metric* on $\mathbb{C}\cup\{\infty\}$ which is $\sigma(z) = \frac{2}{1+|z|^2}$. The Euclidean distance between the points $p(z)$ and $p(w) \in S^2$, where z, $w \in \mathbb{C}$ is given by $\frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$. Hence we have the following inequality:

$$d_\sigma(z,w) \leq \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}},\ \forall\ z,w \in \mathbb{C}$$

Since we have now extended our codomain to be $\mathbb{C}\cup\{\infty\}$ and we want to consider holomorphic maps (considered as that between complex manifolds) from U to $\mathbb{C}\cup\{\infty\}$, we can include meromorphic functions on U as well. From now on we will denote $(\mathbb{C}\cup\{\infty\}, \sigma)$ by $\hat{\mathbb{C}}$. Now the definition of a normal family becomes a concise one:

Definition 3.7 (Normal family). *A family $\mathfrak{F}$ of holomorphic functions from U to $\hat{\mathbb{C}}$ is said to be normal if every sequence of functions in $\mathfrak{F}$ has a subsequence that* *converges normally on U, i.e., for every compact set $K \subset U$ and $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d_\sigma(f_n(z), f_m(z)) < \epsilon$ whenever $z \in K$ and $n, m \geq N$.*

After having defined normal convergence in the above fashion, we are naturally lead to the following questions:

(1) What is the uniform limit of a holomorphic function, in particular, is it also holomorphic?
(2) Suppose we start with a sequence of holomorphic functions taking values in $\mathbb{C}$ which converge normally on U, then what are the possible limit functions of this sequence?

We will answer the above questions in the following lemmas.

Lemma 3.8. *Suppose that (f_n) is a sequence of holomorphic functions on U taking values in $\hat{\mathbb{C}}$ which converges normally on U. Then the limit function f is also holomorphic.*

PROOF. Let f be the limit function of (f_n). Suppose that for some $P \in U$, $f(P) \in \mathbb{C}$. Then there exists a neighborhood of P say $D(P, r)$ such that $f(\overline{D(P, r)}) \subset \mathbb{C}$ and $\overline{D(P, r)} \subset U$. Since (f_n) converges uniformly on $\overline{D(P, r)} \subset U$, we have :

$$d_\sigma(f_n(z), f(z)) < \epsilon \ , \forall n \geq N$$

where $\epsilon > 0$ is such that the $\{\zeta \in \hat{\mathbb{C}} | \ d_\sigma(\zeta, f(\overline{D(P, r)})) < \epsilon\} \cap V = \phi$, for some neighbourhood $V \ni \infty$. Note that on a compact subset K of $\hat{\mathbb{C}}$, since the spherical metric is bounded above and below, we have the constants m_K and $M_K > 0$ such that for any $z, w \in K$, $m_K|z - w| \leq d_\sigma(z, w) \leq M_K|z - w|$.

We conclude that the sequence $(f_n)_N^\infty$ of functions on $\overline{D(P, r)}$ takes values in $\mathbb{C}$ and also converges uniformly when considered as functions taking values in $\mathbb{C}$. Thus f is holomorphic on $D(P, r)$.

Suppose for some $Q \in U$ that $f(Q) = \infty$. Let $D(Q, s) \subset U$ be such that $f(z) \neq \infty$, $\forall z \in D(Q, s) \backslash \{Q\}$. By a similar argument as above it follows that f is a holomorphic function on $D(Q, s) \backslash \{Q\}$ taking values in $\mathbb{C}$. By continuity at Q, it follows that f is a meromorphic function in the usual sense and hence the limit function is a holomorphic function taking values on $\hat{\mathbb{C}}$. □

Lemma 3.9. *Suppose that (f_n) is a sequence of holomorphic functions on U, taking values in $\mathbb{C}$, which converges normally on U according to the extended definition. Then the limit function f is also a holomorphic function taking values in $\mathbb{C}$ or is identically equal to ∞.*

PROOF. Suppose that the limit function takes values in $\mathbb{C}$, then by the proof of the above lemma, it follows that f is also holomorphic taking values in $\mathbb{C}$. Now assume that for some $P \in U$ that $f(P) = \infty$, we will show that $f \equiv \infty$. We can find a neighborhood of P namely $D(P, r) \subset U$ whose closure is also in U and satisfies the property that $f(z) \neq \infty$ on $D(P, r) \setminus P$. The sequence $(\frac{1}{f_n})$ is a sequence of functions that converges normally (in the usual sense) on $D(P, s)$ for some $0 < s < r$. This sequence is nowhere vanishing and hence by Hurwitz's theorem it follows that $\frac{1}{f}$ is also nowhere vanishing or identically 0. Thus $f \equiv \infty$. $\square$

3.2. Geometric Version of Montel's theorem

Before formulating the differential form of the Montel's theorem, we take a closer look at the proof of the Montel's theorem 3.4. We note that the only step in the proof where the information that $\mathfrak{F}$ is a family of holomorphic functions is used is to prove that $|f'|$ is uniformly bounded on compact subsets of U. This boundedness implies that the family $\mathfrak{F}$ is equicontinuous and by Arzela-Ascoli's theorem, it follows that the closure of $\mathfrak{F}$ in $C(U, \mathbb{C})$ is compact and hence the desired result follows. What is the analogue of $|f'(z)|$ in case of holomorphic maps between Riemann surfaces endowed with hermitian metrics?

Suppose that $f : U \to V$ is a holomorphic function where U and V are domains in $\mathbb{C}$ considered with the Euclidean metric. Then for any $p \in U$ we have the map

$$\begin{array}{rccc} df|_p : & T_pU & \to & T_{f(p)}V \\ & \frac{\partial}{\partial z}|_p & \to & f'(p)\frac{\partial}{\partial z}|_{f(p)} \end{array}$$

Thus $|f'(p)|$ is the norm or length of the vector $df|_p(\frac{\partial}{\partial z}|_p) \in T_{f(p)}V$. In the present situation we are concerned with a holomorphic map $f : U \to \hat{\mathbb{C}}$, where U is a domain in $\mathbb{C}$. Suppose $p \in U$ is such that $f(p) \in \mathbb{C}$. Then the map $df|_p$ is:

$$\begin{array}{rccc} df|_p : & T_pU & \to & T_{f(p)}\hat{\mathbb{C}} \\ & \frac{\partial}{\partial z}|_p & \to & f'(p)\frac{\partial}{\partial z}|_{f(p)} \end{array}$$

We now have

$$\| df|_p \left(\frac{\partial}{\partial z}|_p\right) \| = |f'(p)| \cdot \| \frac{\partial}{\partial z}|_{f(p)} \| = \frac{2|f'(p)|}{1 + |f(p)|^2} = f^*\sigma(p)$$

We are now ready to state and prove the geometric version of Montel's theorem. As a final remark, in the proof of Montel's theorem 3.4, after having establised the uniform boundedness of the derivative, we made use of this to calculate the length of a particular curve connecting arbitrary points x and y and used this to prove the equicontinuity of the family $\mathfrak{F}$. In the proof of the following theorem also, we will adopt the same strategy.

Theorem 3.10. *Suppose that $\mathfrak{F}$ is a family of holomorphic functions on a complex domain U taking values in $\hat{\mathbb{C}}$ (i.e., $\mathfrak{F}$ is a family of meromorphic functions on U). Then $\mathfrak{F}$ is a normal family iff the set of pullback metrics $\{f^*\sigma | f \in \mathfrak{F}\}$ is uniformly bounded on compact subsets of U, i.e., for any compact subset $K \subset U$, there exists $M_K > 0$ such that $\frac{2|f'(z)|}{1+|f(z)|^2} \leq M_K$, $\forall f \in \mathfrak{F}$ and $\forall z \in K$.*

Proof. Assume that $\{f^*\sigma | f \in \mathfrak{F}\}$ is uniformly bounded on compact subsets of U. As in the proof of theorem 3.4, we consider compact sets K_n with the additional assumption that each $\overset{\circ}{K}_n$ is connected. Now fix $K = K_n$ and for $x, y \in K_n$ let $\gamma = \gamma_{xy}$ be a path in $\overset{\circ}{K}_{n+1}$ connecting x and y. We have the following:

$$\begin{aligned} l_\sigma(f \circ \gamma) &= \int_0^1 \| (\dot{f \circ \gamma})(t) \| \, dt \\ &= \int_0^1 \frac{2}{1 + |f(\gamma(t))|^2} \cdot |f'(\gamma(t))| \cdot |\dot{\gamma}(t)| dt \\ &\leq M_{K_{n+1}} l_\gamma \\ \therefore d_\sigma(f(x), f(y)) &\leq M_{K_{n+1}} d(x, y) \end{aligned}$$

We thus conclude that $\mathfrak{F}$ converges uniformly on K_n, $\forall n \in \mathbb{N}$. Thus $\mathfrak{F}$ is a normal family.

Now assume that $\mathfrak{F}$ is a normal family. We need to prove that $\{f^*\sigma | f \in \mathfrak{F}\}$ is uniformly bounded on all compact subsets of U. We prove this by contradiction. So assume that for some compact set K, $\{f^*\sigma | f \in \mathfrak{F}\}$ is not uniformly bounded. So $\exists (z_n) \subset K$ and $(f_n) \subset \mathfrak{F}$ such that ${f_n}^*\sigma(z_n) \geq n$. K being compact, (z_n) can be chosen such that it is convergent. $\mathfrak{F}$ being a normal family, (f_n) can be chosen such that it is converges normally on K. In a similar way as was done for holomorphic functions taking values in $\mathbb{C}$ (using Cauchy's estimates), it can be shown that if (f_n) converges to f normally on K, then $({f_n}^*\sigma)$ also converges normally to $f^*\sigma$. Thus ${f_n}^*\sigma(z_n) \to f^*\sigma(z)$. This implies that ${f_n}^*\sigma(z_n)$ is bounded which is a contradiction. □

3.3. Applications

Theorem 3.11. *If $\mathfrak{F}$ is a family of holomorphic functions taking values in $\hat{\mathbb{C}}$ such that image of each $f \in \mathfrak{F}$ is contained in $\hat{\mathbb{C}} \setminus \{P, Q, R\}$, then $\mathfrak{F}$ is a normal family.*

Proof. Without loss of generality assume that $P = 0$, $Q = 1$ and $R = \infty$. Now we need to prove that if $\mathfrak{F}$ is a family of holomorphic functions taking values in $\mathbb{C} \setminus \{0, 1\}$, it is a normal family. It is equivalent to showing that $\mathfrak{F}$ restricted to any disc $D(0, r)$ is normal. We will show that the set of pullback metrics $\{f^*\sigma | f \in \mathfrak{F}\}$ is uniformly bounded on compact subsets of $\mathbb{C}$ and hence by theorem 3.10, it will follow that $\mathfrak{F}$ is a normal family.

Let μ be the metric on $\mathbb{C} \setminus \{0,1\}$ constructed in theorem 2.27. Then $\kappa_\mu \leq \frac{-1}{36}$. Consider the metric ρ_r^A (as in theorem 2.24) on $D(0,r)$. Then by theorem 2.24 we have the following inequality:

$$f^*\mu \leq 6\sqrt{A}\rho_r^A$$

We now compare the metrics μ and σ. Since by the very construction, $\mu(z) \to \infty$ as $z \to 0, 1$ or ∞, it follows that $\frac{\sigma(z)}{\mu(z)} \to 0$ as $z \to 0, 1$ or ∞. Hence $\frac{\sigma}{\mu}$ is bounded above by a constant M. The following inequalities hold for $z \in D(0,r)$:

$$\begin{aligned} \sigma &\leq M\mu \\ \therefore f^*\sigma &\leq Mf^*\mu \leq (6\sqrt{A}M)\rho_r^A \ , \ \forall f \in \mathfrak{F} \end{aligned}$$

This proves that $\{f^*\sigma | f \in \mathfrak{F}\}$ is uniformly bounded on compact subsets of $\mathbb{C}$ and hence $\mathfrak{F}$ is normal. □

Corollary 3.12. *Suppose that $\mathfrak{F}$ is a family of holomorphic functions on U taking values in $\mathbb{C} \setminus \{P, Q\}$. Then $\mathfrak{F}$ is a normal family.*

Theorem 3.13 (The Great Picard's Theorem, Theorem 2, §4, Chapter 2, [**5**])**.** *Suppose that $f : D(0,1) \setminus \{0\} \to \mathbb{C}$ is a holomorphic function and 0 is an essential singularity of f, then in every neighborhood U of 0, f takes all values in $\mathbb{C}$ except possibly one value.*

Proof. We prove this by contradiction. Suppose that $f(D(0,1) \setminus \{0\}) \subseteq \mathbb{C} \setminus \{0,1\}$. In this case we will show that 0 is either a removable singularity or a pole of f. Consider the family of functions $\{f_n\}$ on $D(0,1) \setminus \{0\}$ which are given by $f_n(z) = f(\frac{z}{n})$. By corollary (3.12) it follows that $\mathfrak{F}$ is a normal family. So the sequence (f_n) has a subsequence that converges normally or diverges compactly on $D(0,1) \setminus \{0\}$. Say that subsequence is (f_{n_k}).

(1) Suppose that (f_{n_k}) converges normally on $D(0,1) \setminus \{0\}$. Then it converges uniformly on all compact subsets of $D(0,1) \setminus \{0\}$ and in particular on the circle $C = \{z : |z| = \frac{1}{2}\}$. Hence $f_{n_k} \leq M$ for some $M > 0$ on C. Thus f is bounded by M on the circles $\{z : |z| \leq \frac{1}{2n_k}\}$. Consider f on the annulus $A_k = \{z : \frac{1}{2n_{k+1}} \leq |z| \leq \frac{1}{2n_k}\}$, by the Maximum modulus principle f is bounded by M on every A_k. Since the sequence $n_k \to 0$, we conclude that f is bounded by M in a neighborhood of 0 and hence this would mean that 0 is a removable singularity of f contradicting our assumption.

(2) Next assume that (f_{n_k}) diverges compactly on $D(0,1) \setminus \{0\}$. Then $\frac{1}{f_{n_k}(z)} = \frac{1}{f(\frac{z}{n_k})}$ converges uniformly on $D(0,1) \setminus \{0\}$ to the constant function 0. Thus f has a pole at 0. This is a contradiction to our assumption that 0 is an essential singularity of f.

$\square$

CHAPTER 4

Covering Spaces

In this chapter we will review some basic facts about covering spaces which will play a very significant role in the chapters that follow. Many proofs are omitted as getting down to fill in all the details would take us far from our goal. The discussion in this chapter is based on §53 of the book of Munkres [**6**].

4.1. Covering Spaces and liftings

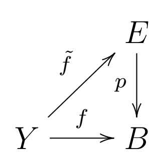
Definition 4.1 (Covering map). *A function $p : E \to B$ between two topological spaces is called a covering map if the following hold:*

(1) *p is surjective*
(2) *$\forall b \in B$, $\exists$ a neighborhood of b, $U_b \subset B$ such that $p^{-1}(U_b) = \sqcup_{\alpha \in \mathfrak{A}} V_\alpha$ and $p|_{V_\alpha} : V_\alpha \to U_b$ is a homeomorphism for every $\alpha \in \mathfrak{A}$.*

Suppose $p : E \to B$ is a covering map and $p(e) = b$, then p induces a group homomorphism p_* between the fundamental groups $\pi_1(E, e)$ and $\pi_1(B, b)$. Suppose that $f : Y \to B$ is any continuous map. The ability to "lift" the map f to a map $\tilde{f} : Y \to E$ in certain situations is the significant fact about covering spaces that is extensively used. In the remaining part of this section, we make this notion of "lifting" precise and state some results (sans proof) pertaining to the same.

In what follows let $p : E \to B$ be a covering map and let $f : Y \to B$ be any continuous map. We will also assume from now on that both B and E are path connected and locally path connected.

Definition 4.2 (Lift). *A continuous map $\tilde{f} : Y \to E$ is called a lift of the map f if it satisfies $p \circ \tilde{f} = f$, i.e., the following diagram commutes:*



It is easy to see that a necessary condition for a lift $\tilde{f}$ of f (that takes $y \in Y$ to $e \in E$) to exist is that $f_*(\pi_1(Y, y)) \subset p_*(\pi_1(E, e))$. In fact it also turns out to be a sufficient condition!

Lemma 4.3. *A lift $\tilde{f}$ of f such that $\tilde{f}(y) = e$ exists iff $f_*(\pi_1(Y, y)) \subset p_*(\pi_1(E, e))$. Also if such a lift exists it is unique.*

Definition 4.4 (Equivalent covering maps and Covering equivalence)**.** *Let $p : E \to B$ and $p' : E' \to B$ be two covering maps. They are said to be equivalent if there exists a homeomorphism $h : E \to E'$ such that $p' \circ h = p$. Such a homeomorphism h is called a covering equivalence.*

In the above definition suppose e, e' and b are such that $p(e) = p'(e') = b$ and $h(e) = e'$. Then $p'_*(\pi_1(E', e')) = p_*(\pi_1(E, e))$. Suppose $p : E \to B$ and $p' : E' \to B$ are two covering maps. The following proposition, which is a direct consequence of lemma 4.3, partially answers when the two covering maps are equivalent.

Proposition 4.5. *There exists a covering equivalence h between E and E' that takes e to e' iff $p'_*(\pi_1(E', e')) = p_*(\pi_1(E, e))$.*

Lemma 4.6. *Suppose that $p : E \to B$ is a covering map and $e_1, e_2 \in E$ are such that $p(e_1) = p(e_2) = b$ and let $\tilde{\alpha}$ be a curve in E beginning at e_1 and ending at e_2. Let $\alpha = p \circ \tilde{\alpha}$. Then $p_*(\pi_1(E, e_2)) = \alpha p_*(\pi_1(E, e_1))\alpha^{-1}$.*

We are now ready to answer the question: When are the two covering spaces (E, p) and (E', p') equivalent?

Proposition 4.7. *Suppose $e \in E$ and $e' \in E'$ are such that $p(e) = p'(e') = b$. Then the two covering maps (E, p) and (E', p') are equivalent iff the subgroups $p_*(\pi_1(E, e))$ and $p'_*(\pi_1(E', e'))$ are conjugate in $\pi_1(B, b)$.*

4.2. Regular covering and Universal covering

In this section, we will consider the set of covering equivalences of (E, p) with itself. It is easy to see that this set forms a group. This is a subgroup of $Hom(E)$. We will call this group the *group of covering transformations* of the covering map $p : E \to B$ and denote it by $\mathcal{C}(E, p, B)$. We shall call its elements *covering transformations.*

If h is a covering transformation, then for every $b \in B$ it maps the set $p^{-1}(b)$ to itself.

Theorem 4.8. *The group $\mathcal{C}(E, p, B) \cong N(H_0)/H_0$, where $H_0 = p_*(\pi_1(E, e))$ and $N(H_0)$ is the normalizer of H_0 in $\pi_1(B, b)$. (Here $p(e) = b$)*

PROOF. We will explicitly construct an isomorphism between the two groups. Let $F = p^{-1}(b)$. Consider the map $\phi : \pi_1(B,b)/H_0 \to F$. Consider a loop $[\gamma]$ in $\pi_1(B,b)$, we know there exists a unique lift $[\tilde{\gamma}]$ of this that begins at e. $\phi([\gamma])$ is defined to be the endpoint of $[\tilde{\gamma}]$. This map is clearly a bijection.

Now consider the map $\psi : \mathcal{C}(E,p,B) \to F$ defined by $\psi(h) = h(e)$. This map is injective by lemma 4.3. Consider the map $\phi^{-1}\psi : \mathcal{C}(E,p,B) \to \pi_1(B,b)/H_0$. This is a injection. We want to show that this is actually a group homomorphism and the image of $\mathcal{C}(E,p,B)$ under $\phi^{-1}\psi$ is $N(H_0)/H_0$.

(1) We will prove that $\phi^{-1}\psi$ is a group homomorphism. Let h_1, $h_2 \in G(p,q_1,q_2)$ and let $\psi(h_1) = e_1$ and $\psi(h_2) = e_2$.

$$\phi^{-1}\psi(h_1h_2) = \phi^{-1}(h_1(e_2)) = \gamma$$

where $\gamma \in \pi_1(B,b)$ is such that its lift in E that starts at e ends at $h_1(e_2)$.

$$\phi^{-1}\psi(h_1)\phi^{-1}\psi(h_2) = \phi^{-1}(e_1)\phi^{-1}(e_2) = \gamma_1\gamma_2$$

where $\gamma_1, \gamma_2 \in \pi_1(B,b)$ are such that their lifts, $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ in E that begin at e end at e_1 and e_2 respectively. We have

$$\begin{aligned} p \circ (\tilde{\gamma_1}\tilde{\gamma_2}) &= \gamma_1\gamma_2 \\ &= (p \circ \tilde{\gamma_1})(p \circ \tilde{\gamma_2}) \\ &= (p \circ \tilde{\gamma_1})((p \circ h_1) \circ \tilde{\gamma_2}) \\ &= (p \circ \tilde{\gamma_1})(p \circ (h_1 \circ \tilde{\gamma_2})) \\ &= p \circ (\tilde{\gamma_1}(h_1 \circ \tilde{\gamma_2})) \\ &= \alpha \end{aligned}$$

where $\alpha \in \pi_1(B,b)$ is such that its lift $\tilde{\alpha} = \tilde{\gamma_1}(h_1 \circ \tilde{\gamma_2})$ is a curve in E that starts at e and ends at $h_1(e_2)$.

Hence $\phi^{-1}\psi$ is a group homomorphism.

(2) Next we will show that the image of $\mathcal{C}(E,p,B)$ under $\phi^{-1}\psi$ is $N(H_0)/H_0$. This is equivalent to showing that:

$\exists h \in G(p,q_1,q_2)$ satisfying $h(e) = e_0 \Leftrightarrow \exists \alpha \in N(H_0)$ such that its lift $\tilde{\alpha}$ that starts at e ends at e_0.

($\Rightarrow$) By proposition 4.5, it follows that $p_*(\pi_1(E,e)) = p_*(\pi_1(E,e_0))$. But again by lemma 4.6, we have $p_*(\pi_1(E,e_0)) = \alpha p_*(\pi_1(E,e))\alpha^{-1}$, where the lift of α, $\tilde{\alpha}$ is a curve in E beginning at e and ending at e_0.

Thus $\alpha \in N(H_0)$ and $\tilde{\alpha}$ is a curve in E beginning at e and ending at e_0.

($\Leftarrow$) If $\exists \alpha \in N(H_0)$ such that its lift $\tilde{\alpha}$ starts at e ends at e_0, then $p_*(\pi_1(E, e)) = p_*(\pi_1(E, e_0))$. Now by proposition 4.5 the desired result follows.

□

Definition 4.9 (Regular covering). *A covering map is said to be regular if $\forall b \in B$ and $\forall e_1, e_2 \in p^{-1}(b)$, there exists a covering transformation h such that $h(e_1) = e_2$.*

It follows from theorem 4.8 that a covering map is regular iff $\forall e \in E$, $p_*(\pi_1(E, e))$ is a normal subgroup of $\pi_1(B, p(e))$.

Definition 4.10 (Properly discontinuous action). *Let X be a topological space and let $Hom(X)$ be the group of homeomorphisms from X to itself. A subgroup $G \leq Hom(X)$ is said to act properly discontinuously on X if $\forall x \in X$, there exists a neighbourhood $U_x \ni x$ such that the following holds:*

$$g_1(U_x) \cap g_2(U_x) = \phi, \ \forall g_1, g_2 \in G \text{ and } g_1 \neq g_2$$

Note that if $p : E \to B$ is a covering map and $G = G(p, q_1, q_2)$. Then G acts properly discontinuously on E. The following theorem says that starting with a topological space X and a subgroup $G \leq Hom(X)$ which acts properly discontinuously on X, the quotient map $\pi : X \to X/G$ is a regular covering for which the group of covering transformations is precisely G, where X/G is the quotient space of X under the equivalence $\sim$ given by: $x \sim y$ iff $\exists g \in G$ such that $y = g(x)$. This gives a way of constructing regular covering maps.

Theorem 4.11. *Let X be a topological space and let $G \leq Hom(X)$. The map $\pi : X \to X/G$ is a covering map iff G acts properly discontinuously on X. In this case $\mathcal{C}(X, \pi, X/G) = G$ and π is a regular covering.*

It is interesting to note that every regular covering map arises in this way as the following theorem states:

Theorem 4.12. *Suppose that $p : E \to B$ is a regular covering map and let $G = \mathcal{C}(E, p, B)$. Then there exists a homeomorphism $h : E/G \to B$ such that $h \circ \pi = p$.*

$$\begin{array}{ccc} E & & E \\ \downarrow \pi & & \downarrow p \\ E/G & \xrightarrow{h} & B \end{array}$$

Proof. It is easy to see that the map h defined locally (in evenly covered neighborhoods of E/G) by $h = p \circ \pi^{-1}$ makes sense globally as well and defines a homeomorphism $h : E/G \to B$. □

Definition 4.13 (Universal covering space). *E is called a universal covering space of B if E is simply connected.*

The following proposition is an immediate consequence of lemma 4.3:

Proposition 4.14. *Any two universal covering spaces of B are equivalent.*

An interesting example of a regular covering is the universal covering. We shall consider the theory about regular coverings developed in this section for universal coverings in the next chapter.

Definition 4.15 (Semi local simply connectedness). *A topological space X is called semilocally simply connected if every point $x \in X$ has a neighbourhood U_x such that the inclusion map induces a trivial group homomorphism $i_* : \pi_1(U_x, x) \to \pi_1(X, x)$.*

Theorem 4.16. *A topological space B has a universal covering iff it is path connected, locally path connected and semilocally simply connected.*

Thus every domain in $\hat{\mathbb{C}}$ has a universal covering space.

CHAPTER 5

Poincare Metric via Covering

In chapter 2 we defined a special metric on $\mathbb{D}$ called the *Poincaré* metric. In this chapter we will use the theory of covering spaces to define a similar such special metric - *Poincare* metric for *most* domains in $\mathbb{C}$ and eventually return to few of the theorems discussed in chapters 2 and 3. The main idea of the present chapter is this: Just as the nature of the metric that can be defined on a domain depends on the nature of the domain, it is also true that knowledge of a metric on a domain gives information about the nature of the domain. As an illustration of this, the chapter culminates with an alternate proof of the Great Picard's theorem. The exposition in this chapter is based on the book of Milnor [**7**].

5.1. Uniformization theorem and classification of Riemann surfaces

In this section we will state the Uniformization theorem and use this to classify Riemann surfaces. The Uniformization theorem and its proof are dealt with in detail in the chapters that follow.

Theorem 5.1 (Uniformization theorem)**.** *Any simply connected Riemann surface is conformally isomorphic to one of the following:*

(1) *The complex plane* $\mathbb{C}$
(2) *The unit disc* $\mathbb{D} \subset \mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$
(3) *The Riemann sphere* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ *which is topologically the one point compactification of the complex plane* $\mathbb{C}$ *with an analytic structure near* ∞ *derived from the map* $z \to \frac{1}{z}$.

All domains in $\mathbb{C}$ have a universal covering space, which according to the preceding theorem is either $\mathbb{D}$, $\mathbb{C}$ or $\hat{\mathbb{C}}$. The discussion in the previous chapter helps us to categorize domains based on their universal covering space.

We have seen in Chapter 4 that if E is the universal covering space of B, then $B \cong E/G$ where G is a subgroup of $Hom(E)$ which acts *properly discontinuously* on E. This fact aids us to specify exactly which domains in $\mathbb{C}$ have $\mathbb{D}$ or $\mathbb{C}$ or $\hat{\mathbb{C}}$ as their covering space. The following discussion leads up to the classification.

(1) The group of conformal automorphisms of $\hat{\mathbb{C}}$ is the group of Mȯbious transformations:

$$Aut(\hat{\mathbb{C}}) = \{f \mid f(z) = \frac{az+b}{cz+d} \text{ and } (ad-bc) \neq 0\}$$

Any nontrivial element $f \in Aut(\hat{\mathbb{C}})$ has atleast one fixed point. Hence there is no nontrivial subgroup of $Aut(\hat{\mathbb{C}})$ which acts properly discontinuously on $\hat{\mathbb{C}}$. Hence the only Riemann surface with $\hat{\mathbb{C}}$ as the universal covering space is $\hat{\mathbb{C}}$ itself.

(2) The group of conformal automorphisms of $\mathbb{C}$ is

$$Aut(\mathbb{C}) = \{f \mid f(z) = \lambda z + a, \lambda \in \mathbb{C} \setminus \{0\} \text{ and } a \in \mathbb{C}\}$$

Any $f(z) = \lambda z + a$ where $\lambda \neq 1$ has a fixed point. Hence a subgroup of $Aut(\mathbb{C})$ which can act properly discontinuously on $\mathbb{C}$ must necessarily be a subgroup of $G = \{f \mid f(z) = z + a, a \in \mathbb{C}\}$. Note that $G \cong (\mathbb{C}, +)$. Suppose that $H \leq G$ is any subgroup, then $H \cong L = \{h(0) \mid h \in H\}$. Thus H acts properly discontinuously on $\mathbb{C}$ iff L is a discrete subgroup of $(\mathbb{C}, +)$. We have the following 2 cases according to the number of generators of L:

(a) Suppose L is generated by a single element. In this case $L \cong \mathbb{Z}$. So $\mathbb{C}$ is the universal cover of the Riemann surface $\mathbb{C}/\mathbb{Z}$. The following map is a conformal isomorphism:

$$\begin{aligned} \phi : \quad \mathbb{C}/\mathbb{Z} &\rightarrow \mathbb{C} \setminus \{0\} \\ \bar{z} &\rightarrow e^{2\pi i z} \end{aligned}$$

Thus $\mathbb{C} \setminus \{0\}$ is the only space (upto conformal isomorphism) such that $\mathbb{C}$ is its universal covering space and the group of covering transformations is generated by a single element.

(b) Suppose L is generated by 2 generators. Then $L \cong \mathbb{Z} \oplus a\mathbb{Z}$, where $a \in \mathbb{C} \setminus \mathbb{R}$. The Riemann surface is topologically equivalent to a torus and hence in compact. So it is not conformally isomorphic to any domain in $\mathbb{C}$.

(3) From the above consideration it follows that any Riemann surface which is not conformally isomorphic to $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$ and $\mathbb{C}/L$, where $L \cong \mathbb{Z} \oplus a\mathbb{Z}$, should have $\mathbb{D}$ as its universal covering space. In particular for any domain $U \subset \mathbb{C}$ such that $U \neq \mathbb{C}$ and $U \ncong \mathbb{C} \setminus \{0\}$, $\mathbb{D}$ is the universal covering space of U.

A Riemann surface S is said to be of the following types depending on its universal covering space:

(1) *Spherical* if its universal covering space is $\hat{\mathbb{C}}$.
(2) *Euclidean* if its universal covering space is $\mathbb{C}$.
(3) *Hyperbolic* if its universal covering space is $\mathbb{D}$.

5.2. Maps between different types of Riemann surfaces

The following lemma will play a crucial role in determining the possible holomorphic maps between different types of Riemann surfaces.

Lemma 5.2. *Let U and U' be Riemann surfaces with universal covering spaces E and E' respectively. Let $p : E \to U$ and $p' : E' \to U'$ be the covering maps. Suppose $f : U \to U'$ is a holomorphic map, then it can be lifted to a holomorphic map $\tilde{f}$ between E and E', i.e., there exists $\tilde{f} : E \to E'$ such that the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ U & \xrightarrow{f} & U' \end{array}$$

Proof. The map $(f \circ p) : E \to U'$ is holomorphic and E is simply connected. Thus it follows from lemma 4.3 that $(f \circ p)$ lifts to a map $\tilde{f} : E \to E'$ and satisfies $p' \circ \tilde{f} = f \circ p$. Hence proved. □

Proposition 5.3. *Suppose that $f : S \to S'$ is a holomorphic map where S and S' satisfy one of the following:*

(1) *S is Euclidean and S' is Hyperbolic*
(2) *S is Spherical and S' is Hyperbolic*
(3) *S is Spherical and S' is Euclidean*

Then f is necessarily a constant map.

Proof. According to the previous lemma, in all the three cases f lifts to a map $\tilde{f}$ between the universal covering spaces:

(1) $\tilde{f} : \mathbb{C} \to \mathbb{D}$. By Lioville's theorem, this is a constant map and hence f is also a constant.
(2) $\tilde{f} : \hat{\mathbb{C}} \to \mathbb{D}$ restricts to a holomorphic function from $\mathbb{C}$ to $\mathbb{D}$, thus in this case also $\tilde{f}$ and hence f is also a constant.
(3) $\tilde{f} : \hat{\mathbb{C}} \to \mathbb{C}$ restricts to a holomorphic map on $\mathbb{C}$ and since $\hat{\mathbb{C}}$ is compact, image of $\tilde{f}$ is bounded in $\mathbb{C}$. Thus in this case too $\tilde{f}$ and hence f is also constant.

□

We saw in the previous section that $\mathbb{C} \setminus \{0, 1\}$ is a hyperbolic Riemann manifold and hence we obtain Picard's Little theorem as a corollary of the above proposition.

Corollary 5.4. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic map that misses at least two points in $\mathbb{C}$. Then f is a constant function.*

5.3. Poincare metric on a Hyperbolic surface

In section 2.3 we defined a metric ρ on the disc $\mathbb{D}$ which was the unique (upto a multiplicative constant) metric on $\mathbb{D}$ such that every conformal isomorphism of $\mathbb{D}$ is an isometry of $(\mathbb{D}, \rho)$. $(\mathbb{D}, \rho)$ also has a constant negative curvature. In this section we endow an arbitrary hyperbolic domain $U \subset \mathbb{C}$ with a *hyperbolic metric* using the covering map $p : \mathbb{D} \to U$. At the end of this section, we shall see the speciality of the hyperbolic metric.

We first construct the hyperbolic metric on U. We do this in the following theorem and its proof.

Theorem 5.5. *Let $U \subset \mathbb{C}$ be a hyperbolic domain and let $p : \mathbb{D} \to U$ be the universal covering map. Then there exists a metric μ on U so that the map $p : (\mathbb{D}, \rho) \to (U, \mu)$ is a local isometry.*

Proof. p being a covering map, it is a local conformal isomorphism. So given any $x \in U$ and $y \in \mathbb{D}$ such that $p(y) = x$, there exists neighborhoods $V \ni x$ and $W \ni y$ such that $p|_W : W \to V$ is a conformal isomorphism. Since W is already endowed with the metric $\rho|_W$, there exists a metric μ_y on V such that $p|_W : (W, \rho|_W) \to (V, \mu_y)$ is an isometry. μ_y is the "push down" of the metric $\rho|_W$ via p and for any $z \in W$ is given by:

$$\mu_y(p(z)) = \frac{\rho(z)}{|p'(z)|}$$

Note that the above expression is well defined since p being a local isometry, $p'(z) \neq 0$. So we get the metric μ_y in V. The natural course to take here would be to "patch" up the metric function so obtained for every point of U. We have to first check that if $a \in \mathbb{D}$ is such that $p(a) = x$, then $\mu_a(x) = \mu_y(x)$. If we succeed in proving this, then we can define the metric μ on U by $\mu(\zeta) = \mu_b(\zeta)$, where $b \in \mathbb{D}$ satisfies $p(b) = \zeta$. This metric μ clearly satisfies the condition of the theorem.

Now let x be an arbitrary point in U and let $a, y \in \mathbb{D}$ be such that $p(a) = p(y) = x$. Then we have $\mu_a(x) = \frac{\rho(a)}{|p'(a)|}$ and $\mu_y(x) = \frac{\rho(y)}{|p'(y)|}$. $\mathbb{D}$ being a universal cover, it is also a regular cover. Hence it follows from chapter 4 that there exists a covering transformation $\phi : \mathbb{D} \to \mathbb{D}$ that takes a to y, $\phi(a) = y$. We have

$$\begin{aligned} p \circ \phi &= p \\ \therefore p'(y) \cdot \phi'(a) &= p'(a) \end{aligned}$$

Since ϕ is a conformal self map of D, the composition $\phi_y \circ \phi \circ \phi_{-a}$ is a conformal self map of $\mathbb{D}$ that fixes the origin and hence is a rotation, say R. (For any $c \in \mathbb{D}$, the map $\phi_c(z) = \frac{z-c}{1-\bar{c}z}$)

We thus have the following:

$$\begin{aligned} \phi_y \circ \phi \circ \phi_{-a} &= R \\ \therefore \phi &= \phi_{-y} \circ R \circ \phi_a \\ \therefore \phi'(a) &= \phi'_{-y}(0) \cdot R'(0) \cdot \phi'_a(a) \\ \therefore |\phi'(a)| &= |\phi'_{-y}(0)| \cdot |\phi'_a(a)| \\ &= (1 - |y|^2) \cdot \frac{(1-|a|^2)}{(1-|a|^2)^2} \\ &= \frac{1-|y|^2}{1-|a|^2} = \frac{\rho(a)}{\rho(y)} \end{aligned}$$

Hence we have

$$\begin{aligned} |p'(y) \cdot \phi'(a)| &= |p'(a)| \\ \therefore |p'(y)| \cdot \frac{\rho(a)}{\rho(y)} &= |p'(a)| \\ \therefore \frac{\rho(a)}{|p'(a)|} &= \frac{\rho(y)}{|p'(y)|} \\ \therefore \mu_a(x) &= \mu_y(x) \end{aligned}$$

□

Definition 5.6 (Poincaré metric)**.** *For any hyperbolic domain $U \subset \mathbb{C}$, the metric μ in the above theorem is called the Poincare metric of U.*

The metric μ defined in the previous proof makes the map $p : (\mathbb{D}, \rho) \to (U, \mu)$ a local isometry, hence the curvature of (U, μ) is a constant ($\equiv -4$). With this fact, we can actually circumvent the tedious task of constructing the metric for which the curvature is strictly negative and bounded away from 0 in the proof of Picard's theorem 2.27.

We saw in chapter 2 that any holomorphic map $f : (\mathbb{D}, \rho) \to (\mathbb{D}, \rho)$ is distance decreasing. Now that we have constructed the Poincare metric for any hyperbolic domain of $\mathbb{C}$, we prove a similar theorem for any holomorphic map between hyperbolic domains of $\mathbb{C}$.

Theorem 5.7. *Suppose $f : (S_1, \mu_1) \to (S_2, \mu_2)$ be a holomorphic map between hyperbolic domains S_1 and S_2 of $\mathbb{C}$, where μ_1 and μ_2 are Poincare metrics of S_1 and S_2 respectively. Then f is distance decreasing.*

Proof. From lemma 5.2 it follows that the map f can be lifted to $\tilde{f} : \mathbb{D} \to \mathbb{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} y \in (\mathbb{D}, \rho) & \xrightarrow{\tilde{f}} & (\mathbb{D}, \rho) \ni \tilde{f}(y) \\ \downarrow p_1 & & \downarrow p_2 \\ z \in (S_1, \mu_1) & \xrightarrow{f} & (S_2, \mu_2) \ni f(z) \end{array}$$

In order to prove that f is distance decreasing, we need to prove that $\mu_1(z) \geq \mu_2(f(z))$ for any $z \in S_1$. We note that as seen in theorem 5.5, p_1 and p_2 are local isometries. Let z be an arbitrary element in S_1 and let $y \in \mathbb{D}$ be such that $p_1(y) = z$. Since $p_2 \circ \tilde{f} = f \circ p_1$, we have $(p_2 \circ \tilde{f})(y) = (f \circ p_1)(y) = f(z)$. We thus have

$$\begin{aligned} (p_2 \circ \tilde{f})(y) &= f(z) \\ \therefore \mu_2(p_2 \circ \tilde{f})(y) &= \mu_2(f(z)) \end{aligned}$$

$$\therefore \rho(\tilde{f}(y)) = \mu_2(f(z)) \tag{5.3.1}$$

But we know that $\tilde{f}$ is distance decreasing and hence $\rho(y) \geq \rho(\tilde{f}(y))$. Now p_1 being a local isometry, we have $\rho(y) = \mu_1(p_1(y)) = \mu_1(z)$. Hence we have

$$\mu_1(z) \geq \rho(\tilde{f}(y)) \tag{5.3.2}$$

Thus from (5.3.1) and (5.3.2) above we get $\mu_1(z) \geq \mu_2(f(z))$ as required. □

5.4. Great Picard's Theorem

We have already given a proof of Great Picard's theorem in chapter 3 using normal families. In this section we will give an alternate proof using the theory developed so far in the present chapter.

Lemma 5.8. *Suppose that μ is the Poincare metric of $\mathbb{D} \setminus \{0\}$. The length of the curve $\gamma(t) = re^{2\pi i t}$, $0 \leq t < 1$ in $(\mathbb{D} \setminus \{0\}, \mu)$ is $|\frac{2\pi}{logr}|$.*

PROOF. Let $\mathbb{H}$ denote the upper half plane and $\tilde{\rho}$ denote the poincare metric on it. This is given by $\tilde{\rho}(x+iy) = \frac{1}{y}$. The universal covering map $p : (\mathbb{H}, \tilde{\rho}) \to (\mathbb{D} \setminus \{0\}, \mu)$ given by $p(z) = e^{iz}$ is a local isometry. We use this to actually determine the metric μ. Suppose $z = (x+iy) \in \mathbb{H}$ is such that $|p(z)| = |e^{iz}| = r$. Then $e^{-y} = r$ and hence $y = -logr$. According to theorem 5.5, the metric μ is given by:

$$\begin{aligned} \mu(e^{iz}) &= \frac{\tilde{\rho}(z)}{|e^{iz}|} \\ \therefore \mu(e^{iz}) &= \frac{1}{-rlogr} \end{aligned}$$

Thus the length of curve γ is $l_\mu(\gamma) = \frac{2\pi r}{-rlogr} = \frac{2\pi}{|logr|}$. □

Lemma 5.9. *Suppose $f : \mathbb{D} \setminus \{0\} \to \hat{\mathbb{C}} \setminus \{a, b, c\}$ is a holomorphic function, then f can be extended to a holomorphic function on the whole of $\mathbb{D}$.*

PROOF. We consider the following three cases:

(1) Suppose $\lim_{z \to 0} f(z)$ exists and is one of $\{a, b, c\}$. Then 0 is either a removable singularity or a pole of f and hence in this case f extends to a holomorphic function on the whole of $\mathbb{D}$.

(2) For $0 < r < 1$, let $A_r = D(0, r) \setminus \{0\}$. Assume that for any sequence in $\mathbb{D} \setminus \{0\}$ such that $(z_n) \to 0$, it holds that limit points of $\{f(z_n)\}$ is contained in $\{a, b, c\}$. Then $\forall \epsilon > 0$, $\exists \delta > 0$ such that $f(A_\delta) \subset D(a, \epsilon) \cup D(b, \epsilon) \cup D(c, \epsilon)$. Hence f thought of as a complex valued function (after coordinate change if necessary) is bounded. Thus 0 is again a pole or a removable singularity and hence f can be holomorphically extended to the whole of $\mathbb{D}$.

(3) Suppose the situations discussed in the above 2 cases donot arise. Then $\hat{\mathbb{C}}$ being compact, $\exists (z_n) \to 0$ and $f(z_n) \to m \in \hat{\mathbb{C}} \setminus \{a, b, c\}$. The space $\hat{\mathbb{C}} \setminus \{a, b, c\}$ is hyperbolic and let $p : \mathbb{D} \to \hat{\mathbb{C}} \setminus \{a, b, c\}$ be its universal covering map. Let μ be the Poincare metric on $\hat{\mathbb{C}} \setminus \{a, b, c\}$. Let $D_\mu(m, r)$ be an open disc which is also an evenly covered neighborhood of m.

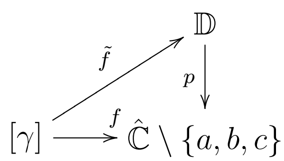
Let $z \in (z_n)$ be such that $|\frac{2\pi}{log|z|}| < \frac{r}{2}$ and $d_\mu(f(z), m) < \frac{r}{2}$. Let $\delta = |z|$. We will show that the image of the curve $\gamma(t) = |z|e^{2\pi i t}, 0 \leq t \leq 1$ under f is contained in $D_\mu(m, r)$. Let $q \in f([\gamma])$. By the distance decreasing property of holomorphic maps between hyperbolic domains (theorem 5.7) the following inequality holds:

$$d_\mu(f(z), q) \leq l_\mu(f \circ \gamma) < \frac{r}{2}$$

By our very choice of z we have the inequality $d_\mu(m, f(z)) < \frac{r}{2}$.

$$\begin{aligned} \therefore d(m,q) &< d(m, f(z)) + d_\mu(f(z), q) \\ &= r \end{aligned}$$

We have shown that for any $q \in f([\gamma])$, $q \in D_\mu(m,r)$. Hence $f([\gamma]) \subset D_\mu(m,r)$. Hence we have:



There exists a lift $\tilde{f}$ of $f|_{[\gamma]}$. The lift is given by $\tilde{f} = f \circ p^{-1}|_{D_\mu(m,r)}$. $\mathbb{D}$ being simply connected, by lemma 4.3, this is possible iff $f_*(\pi_1([\gamma], z))$ is the identity element in $\pi_1(\gamma, z)$. But since any loop in $\mathbb{D} \setminus \{0\}$ itself is equivalent to either the trivial loop or γ, it follows that $f_*(\pi_1(\mathbb{D} \setminus \{0\}))$ is the identity element in $\pi_1(\gamma, z)$. Thus by lemma 4.3, f lifts to a map $\tilde{f} : \mathbb{D} \setminus \{0\} \to \mathbb{D}$. Thus $\tilde{f}$ and hence f can be extended to a holomorphic function on the whole of $\mathbb{D}$.

□

Theorem 5.10 (Great Picard's Theorem)**.** *Suppose $f : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ is a holomorphic function and 0 is an essential singularity of f, then for every $0 < r < 1$, $f(D(0,r) \setminus \{0\})$ misses atmost one point in $\mathbb{C}$.*

Proof. Let f be as in the statement of the theorem. Assume that $\mathrm{Img}(f) \subset \mathbb{C} \setminus \{a,b\}$, then f can be considered as a holomorphic map taking values in $\hat{\mathbb{C}}$. Then $f : \mathbb{D} \setminus \{0\} \to \hat{\mathbb{C}} \setminus \{a, b, \infty\}$. By the above lemma 5.9, it follows that f can be extended to a holomorphic function on the whole of $\mathbb{D}$. This implies that 0 is either a removable singularity or a pole. This is a contradiction. Hence our assumption that $\mathrm{Img}(f) \subset \mathbb{C} \setminus \{a,b\}$ is false.

Thus f can miss atmost one point in $\mathbb{C}$. □

Part 2

The Uniformization Theorem

CHAPTER 6

Solution of the Dirichlet problem

In this chapter we will introduce and later discuss properties of harmonic and subharmonic functions. We will elucidate Perron's method for the solution of the Dirichlet's problem. The discussion in this chapter is mainly based on the article of Kumaresan [**9**]. We also refer the reader to the books of Narasimhan [**10**] and Gamelin [**8**] for a wonderful exposition on the material covered in this chapter.

6.1. Harmonic Functions

6.1.1. Basic properties of Harmonic functions. Let U be a non empty open subset of $\mathbb{C}$ or $\mathbb{R}^2$.

Definition 6.1 (Harmonic function)**.** *A C^2 function $f : U \to \mathbb{R}$ is said to be harmonic if $\Delta f \equiv 0$ on U, where the operator $\Delta = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}$.*

Lemma 6.2. *Let U and W be non empty subsets of $\mathbb{C}$. Let $F : W \to U$ be a holomorphic function and let $f : U \to \mathbb{R}$ be a harmonic function. Then the composite function $f \circ F : W \to \mathbb{R}$ is also harmonic.*

Proof. If F is constant, then so is $f \circ F$. So we will assume that F is nonconstant. Let $w \in W$ and let $r, R > 0$ be such that $D(w, r) \subset W$, $D(F(w), R) \subset U$ and $f(D(w, r)) \subset D(F(w), R)$. f being harmonic on $D(F(w), R)$, there exists a holomorphic function G on $D(F(w), R)$ such that $Re(G) = f$. Then $f \circ F = Re(G \circ F)$ on $D(w, r)$. $G \circ F$ is holomorphic and hence $f \circ F$ is harmonic in a neighbourhood of w and hence on the whole of W. □

Remark 6.3. In the above theorem, instead of $F = f_1 + if_2$ being a holomorphic function, we assume simply that f_1 and f_2 are harmonic functions. Then the conclusion of the theorem does not hold. That is, there exists a harmonic function h such that $h \circ F$ is *not* harmonic. Consider $F : \mathbb{C} \to \mathbb{C}$ defined by $F(x + iy) = (2x+3y)+i(5x+4y)$ and $h(x+iy) = x^2-y^2$, then $(h\circ F)(x+iy) = 21x^2+7y^2+28xy$ which is not harmonic.

It follows from the Cauchy-Riemann equations that the real and imaginary parts of holomorphic functions are harmonic. This fact helps us to derive some properties of harmonic functions which are analogous to those of holomorphic functions.

Proposition 6.4 (Maximum-Minimum principle for harmonic functions)**.** *Let $U \subset \mathbb{C}$ be open and let $f : U \to \mathbb{R}$ be a nonconstant harmonic function. Then f does not have both maximum and minimum in U.*

In particular if U is such that $\overline{U}$ is compact then the maximum and minimum of f are not attained anywhere in U and hence they are attained on ∂U.

PROOF. This is the analogue of the Maximum and Minimum modulus principle of holomorphic functions. Let $b \in U$. Let the disc $D(b, r) \subset U$. Then there exists a holomorphic function F on $D(b, r)$ such that $Re(F) = f$. Since f is nonconstant, so is F. Hence F is an open map. In particular, there exists a disc $D(F(b), R) \subset F(D(b, r))$. Hence the open interval $(f(b) - R, f(b) + R) \subset f(D(b, r)) \subset f(U)$.

Thus $f(b)$ is neither the supremum nor the infimum of f in U as required. $\square$

Proposition 6.5 (Mean value property)**.** *Let $f : U \to \mathbb{R}$ be a harmonic function. Suppose that for some $b \in U$, $\overline{D(b, r)} \subset U$. Then*

$$f(b) = \frac{1}{2\pi} \int_0^{2\pi} f(b + re^{i\theta}) d\theta \tag{6.1.1}$$

PROOF. This is the analogue of the Cauchy integral formula and is often called the *Mean value property* of harmonic functions. Since $\overline{D(b, r)} \subset U$, we can find $s > r$ such that $D(b, s) \subset U$. We can find a holomorphic function F such that $Re(F) = f$ on $D(b, s)$. Let $\gamma(\theta) = b + re^{i\theta}$. By Cauchy integral formula we have:

$$\begin{aligned} F(b) &= \frac{1}{2\pi i} \oint_\gamma \frac{F(\zeta)}{\zeta - b} d\zeta \\ &= \frac{1}{2\pi} \int_0^{2\pi} F(b + re^{i\theta}) d\theta \\ \therefore ReF(b) &= \frac{1}{2\pi} \int_0^{2\pi} Re(F(b + re^{i\theta})) d\theta \\ \therefore f(b) &= \frac{1}{2\pi} \int_0^{2\pi} f(b + re^{i\theta}) d\theta \end{aligned}$$

$\square$

Remark 6.6. (§1.6, Chapter 1, [**11**]) Harmonic functions also satisfy a *volume mean value property.* If f is harmonic in some neighborhood of $\overline{D(a, R)}$ then the following holds:

$$f(a) = \frac{1}{\pi r^2} \int_0^R \int_0^{2\pi} f(r,\theta) r dr d\theta$$

It follows from equation 6.1.1 that it is possible to know the value of $f(b)$ by knowing the values of f on the circle $\partial D(b,r)$. The following discussion ensures that we can get a similar such formula for $f(z)$, $\forall z \in D(b,r)$, in terms of values of f on the circle $\partial D(b,r)$. For the time being let us assume $b = 0$ and $r = 1$. Let $a \in D(0,1)$. Let ϕ_a denote the conformal self map of $D(0,1)$ which is given by $\phi_a(z) = \frac{z+a}{1+\bar{a}z}$. By composing f with ϕ_a, we still get a harmonic function, and using the mean value property for $f \circ \phi_a$ gives the desired result which is summarized in the following proposition.

Proposition 6.7. *Let f be a harmonic function defined on U, where $\overline{D(0,1)} \subset U$. For any $a \in D(0,1)$ we have*

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1-|a|^2}{|e^{i\theta} - a|^2} d\theta$$

PROOF. By lemma 6.2 the function $f \circ \phi_a : D(0,1) \to \mathbb{R}$ is harmonic. By the mean value property we have

$$\begin{aligned}(f \circ \phi_a)(0) &= \frac{1}{2\pi} \int_0^{2\pi} (f \circ \phi_a)(e^{i\theta}) d\theta \\ f(a) &= \frac{1}{2\pi} \int_0^{2\pi} f\left(\frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}\right) d\theta\end{aligned}$$

In the above equation, substituting $\left(\frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}\right) = e^{i\alpha}$, we get:

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\alpha}) \frac{1-|a|^2}{|e^{i\alpha} - a|^2} d\alpha$$

□

Corollary 6.8. *For any $z \in \mathbb{D}$, the value of the following integral is 1.*

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta} - z|^2} d\theta = 1$$

PROOF. Apply proposition 6.7 to the harmonic function $f \equiv 1$. □

For a general b and r, it can be easily worked out (by composing the harmonic function $f : D(b,r) \to \mathbb{R}$ with the function $g(z) = rz + b$, $f \circ g$ and applying the result of the above proposition) that:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(b+re^{i\theta})\frac{r^2-|z-b|^2}{|re^{i\theta}-(z-b)|^2}d\theta \tag{6.1.2}$$

6.1.2. Dirichlet problem for the disc. Throughout this section we shall denote $D(0,1)$ by $\mathbb{D}$.

In this section we shall answer the question: Given a continuous function $f : \partial\mathbb{D} \to \mathbb{R}$, is it possible to extend f to the whole of $\mathbb{D}$ such that it is harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$? This is the Dirichlet problem for a disc.

Let us assume for the moment that such a function exists, call it $\tilde{f}$. Then by proposition 6.7 and by the maximum and minimum principle for harmonic functions (proposition 6.4), it follows that:

$$\tilde{f}(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|z|^2}{|e^{i\theta}-z|^2}d\theta \ , \forall z \in \mathbb{D}$$

This motivates our proof for the solution of the Dirichlet problem for $\mathbb{D}$.

Theorem 6.9. *Let $f : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function. Then the function $\tilde{f}$ defined below is harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$*

$$\tilde{f}(z) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|z|^2}{|e^{i\theta}-z|^2}d\theta & \text{for } z \in \mathbb{D} \\ f(z) & \text{for } z \in \partial\mathbb{D} \end{cases}$$

Proof. We will first prove that $\tilde{f}$ is harmonic on $\mathbb{D}$. Let $\gamma(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$. Consider the function g on $\mathbb{D}$ defined by:

$$\begin{aligned} g(z) &= \frac{1}{2\pi i}\oint_\gamma f(\zeta)\frac{\zeta+z}{\zeta(\zeta-z)}d\zeta \\ &= \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta-z}d\zeta + z\frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)/\zeta}{\zeta-z}d\zeta \end{aligned}$$

f being continuous on $\mathbb{D}$, the two factors in the above equation define a holomorphic function and hence g is also holomorphic on $\mathbb{D}$. Expanding the integral on the RHS of the above equation:

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)d\theta$$

Since $Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) = \frac{1-|z|^2}{|e^{i\theta}-z|^2}$, it follows that on $\mathbb{D}$, $\tilde{f} = Re(g)$. Since g is holomorphic, $\tilde{f}$ is a harmonic function on $\mathbb{D}$.

We will now prove that $\tilde{f}$ is continuous on $\overline{\mathbb{D}}$. For this it is sufficient to prove that for every $b \in \partial\mathbb{D}$, $\lim_{z\to b} \tilde{f}(z) = \tilde{f}(b) = f(b)$.

Given $\epsilon > 0$, we wish to find $\delta > 0$ such that $\forall z \in D(b,\delta) \cap \mathbb{D}$, $|\tilde{f}(z) - f(b)| < \epsilon$. For $z \in \mathbb{D}$ we have:

$$\tilde{f}(z) - f(b) = \frac{1}{2\pi}\int_0^{2\pi} [f(e^{i\theta}) - f(b)] \left(\frac{1-|z|^2}{|e^{i\theta} - z|^2}\right) d\theta \tag{6.1.3}$$

First choose $R > 0$ such that whenever $\zeta \in D(b, 2R) \cap \partial\mathbb{D}$

$$|f(\zeta) - f(b)| < \frac{\epsilon}{2}$$

Let $\overline{D(b,2R)} \cap \partial\mathbb{D} = \{e^{i\theta_1}, e^{i\theta_2}\}$ where $\theta_1 < \theta_2$. The integral in equation 6.1.3 can be split into sum of two integrals as:

$$\begin{aligned}\tilde{f}(z) - f(b) &= \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} [f(e^{i\theta}) - f(b)] \left(\frac{1-|z|^2}{|e^{i\theta} - z|^2}\right) d\theta + \\ &\quad \frac{1}{2\pi}\int_{\theta_2}^{2\pi+\theta_1} [f(e^{i\theta}) - f(b)] \left(\frac{1-|z|^2}{|e^{i\theta} - z|^2}\right) d\theta\end{aligned}$$

We will call the first of these integrals I_1 and the second I_2. By our very choice of R, we have the following inequality for any $z \in \mathbb{D}$:

$$\begin{aligned}|I_1| &\leq \frac{\epsilon}{2}\frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \left(\frac{1-|z|^2}{|e^{i\theta} - z|^2}\right) d\theta \\ &\leq \frac{\epsilon}{2}\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1-|z|^2}{|e^{i\theta} - z|^2}\right) d\theta \\ &= \frac{\epsilon}{2}\end{aligned}$$

Let $M > 0$ be such that $|f| \leq M$ on $\partial\mathbb{D}$. Now we choose $\delta > 0$ such that $\delta < R$ and $1 - |z|^2 \leq \frac{\epsilon R^2}{4M}$, whenever $z \in D(b,\delta)$. Let $\tilde{\gamma}(\theta) = e^{i\theta}$ for $\theta_2 \leq \theta \leq 2\pi + \theta_1$. Then for any $\zeta \in \tilde{\gamma}$ and for any $z \in D(b,\delta)$ we have the following inequalities:

$$\begin{aligned}|z - \zeta| &\geq |\zeta - b| - |z - b| \\ &\geq 2R - \delta \\ &\geq 2R - R = R\end{aligned}$$

Thus we can now conclude that, for z, R and δ as above:

$$\begin{aligned}I_2 &\leq \frac{2M}{R^2}(1 - |z|^2) \\ &\leq \frac{\epsilon}{2}\end{aligned}$$

Thus this is the required δ and hence $\tilde{f}$ is continuous on $\overline{\mathbb{D}}$. $\square$

6.1.3. Mean value property. We saw in proposition 6.5 that harmonic functions satisfy the mean value property. We will prove in this section that the mean value property by itself characterizes harmonic functions, i.e., a function satisfying the mean value property is necessarily harmonic.

Proposition 6.10. *Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ be a continuous function satisfying: $\forall p \in U$, $\exists\ r > 0$ such that $\overline{D(p,r)} \subset U$ and $\forall\ 0 < s \leq r$ the following holds*

$$f(p) = \frac{1}{2\pi}\int_0^{2\pi} f(p + se^{i\theta})d\theta$$

Then f is harmonic on U.

Proof. We will first prove that f satisfies the maximum/minimum principle, i.e., f does not attain maximum and minimum in U unless f is a constant. We prove this by contradiction. Assume that f is nonconstant and that there exists $q \in U$ such that $f(q) = max\{f(z)|z \in U\}$. Let $r > 0$ be such that $\overline{D(q,r)} \subset U$ and $\forall\ 0 < s \leq r$ the following holds:

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} f(q + se^{i\theta})d\theta &= f(q) \\ \therefore \frac{1}{2\pi}\int_0^{2\pi} [f(q) - f(q + se^{i\theta})]d\theta &= 0 \end{aligned}$$

But in the above equation $[f(q) - f(q + se^{i\theta})] \geq 0$, hence for the integral to be zero, $f(q) = f(q + se^{i\theta})$, $\forall\ 0 \leq \theta \leq 2\pi$. Hence on $D(q,r)$, $f(z) \equiv f(q)$. Hence the set $V = \{z \in U|\ f(z) = f(q)\}$ is a nonempty set that is both open and closed. Hence $V = U$ contradicting our assumption that f is nonconstant.

A similar proof works for the case when we assume f attains minimum at a point in U as well. Now we are all set to prove that f is harmonic. Let $p \in U$ and $r > 0$ be such that $\forall\ 0 < s \leq r$, the mean value property holds. Now consider $f|_{\partial D(p,r)}$, this is a continuous function. Let $\tilde{f}$ be the continuous, harmonic extension of f to the disc $D(p,r)$. Now consider $\tilde{f} - f$ on $D(p,r)$. $\tilde{f}$ and f both satisfy the mean value property and hence so does $\tilde{f} - f$. Hence $\tilde{f} - f$ satisfies the maximum/minimum principle as well. Hence both the maximum and minimum of $\tilde{f} - f$ occurs on $\partial D(p,r)$. But $\tilde{f} - f \equiv 0$ on $\partial D(p,r)$. Hence $\tilde{f} \equiv f$ on $D(p,r)$ too. Thus f is a harmonic function on the whole of U. $\square$

Lemma 6.11. *Suppose $\{f_n\}$ is a sequence of harmonic functions on an open set U which converges uniformly on compact subsets of U to the function f. Then f is also a harmonic function on U.*

PROOF. Let $b \in U$ and let $\overline{D(b,r)} \subset U$. Since each f_n is harmonic, by the mean value property we have:

$$f_n(b) = \frac{1}{2\pi}\int_0^{2\pi} f_n(b + re^{i\theta})d\theta$$

Since $f_n \to f$ normally on U we have:

$$\begin{aligned} \lim_{n\to\infty} f_n(b) &= \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f_n(b + re^{i\theta})d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \lim_{n\to\infty}(f_n(b + re^{i\theta}))d\theta \\ \therefore f(b) &= \frac{1}{2\pi}\int_0^{2\pi} f(b + re^{i\theta})d\theta \end{aligned}$$

Thus f satisfies the mean value property on U and hence by proposition 6.10, f is harmonic. □

6.1.4. Harnack's Principle. We shall derive an interesting property of harmonic functions called the *Harnack's Principle* by making use of equation 6.1.2.

Proposition 6.12 (Harnack's Inequality)**.** *Let $f : U \to \mathbb{R}$ be a nonnegative harmonic function. Let $\overline{D(b,r)} \subset U$. For $0 < \epsilon < 1$ let D_ϵ denote the disc $D_\epsilon = \{z|\ |z - b| < r\epsilon\}$. Then $\forall z \in D_\epsilon$ we have:*

$$f(b)\left(\frac{1-\epsilon}{1+\epsilon}\right) \le f(z) \le f(b)\left(\frac{1+\epsilon}{1-\epsilon}\right)$$

PROOF. From equation 6.1.2 it follows that:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(b + re^{i\theta})\frac{r^2 - |z-b|^2}{|re^{i\theta} - (z-b)|^2}d\theta$$

In the above equation let $w(\theta) = re^{i\theta}$ and $\zeta = (z-b)$, then $\frac{r^2-|z-b|^2}{|re^{i\theta}-(z-b)|^2} = \frac{|w(\theta)|^2-|\zeta|^2}{|w(\theta)-\zeta|^2}$. By triangle inequality we have:

$$\begin{aligned} |w(\theta)| - |\zeta| \quad &\leq |w(\theta) - \zeta| \leq \quad |w(\theta)| + |\zeta| \\ \therefore \left(\frac{|w(\theta)| - |\zeta|}{|w(\theta)| + |\zeta|}\right) \quad &\leq \left(\tfrac{r^2-|z-b|^2}{|re^{i\theta}-(z-b)|^2}\right) \leq \quad \left(\frac{|w(\theta)| + |\zeta|}{|w(\theta)| - |\zeta|}\right) \\ \therefore \left(\frac{r - |\zeta|}{r + |\zeta|}\right) \quad &\leq \left(\tfrac{r^2-|z-b|^2}{|re^{i\theta}-(z-b)|^2}\right) \leq \quad \left(\frac{r + |\zeta|}{r - |\zeta|}\right) \end{aligned}$$

From the above inequalities and the fact that f is nonnegative, we have $\forall z \in D(b,r)$:

$$\begin{aligned} \left(\frac{r-|z-b|}{r+|z-b|}\right)\frac{1}{2\pi}\int_0^{2\pi} f(b+re^{i\theta})d\theta \quad &\leq f(z) \leq \quad \left(\frac{r+|z-b|}{r-|z-b|}\right)\frac{1}{2\pi}\int_0^{2\pi} f(b+re^{i\theta})d\theta \\ \therefore \left(\frac{r-|z-b|}{r+|z-b|}\right) f(b) \quad &\leq f(z) \leq \quad \left(\frac{r+|z-b|}{r-|z-b|}\right) f(b) \end{aligned}$$

Thus for all $z \in \overline{D_\epsilon}$ we have the desired result:

$$f(b)\left(\frac{1-\epsilon}{1+\epsilon}\right) \leq f(z) \leq f(b)\left(\frac{1+\epsilon}{1-\epsilon}\right)$$

□

Theorem 6.13 (Harnack's Principle)**.** *Suppose that $f_1 \leq f_2 \leq f_3 \leq \cdots$ is an increasing sequence of harmonic functions on a domain $U \subset \mathbb{C}$. Then one of the following occurs:*

(1) *f_n converges uniformly on compact subsets of U to a harmonic function*
(2) *$f_n \to \infty$ uniformly on compact subsets of U.*

Proof. It suffices to prove that if there exists some $p \in U$ such that the sequence $f_n(p)$ converges, then f_n converges uniformly (to a necessarily harmonic function) on U. So assume that there exists such a $p \in U$. Let $\overline{D(p,r)} \subset U$. Then for every $\zeta \in D(p, \frac{r}{2})$ and $\forall n \in \mathbb{N}$, it follows from Harnack's inequality (proposition 6.12) that

$$\frac{1}{3}[f_n(p) - f_1(p)] \leq f_n(\zeta) - f_1(\zeta) \leq 3[f_n(p) - f_1(p)]$$

Hence for every $\zeta \in D(p, \frac{r}{2})$, $(f_n(\zeta))$ is an increasing sequence which is also bounded and so convergent. We shall now prove that f_n converges uniformly on $D(p, \frac{r}{2})$. Similarly for any $n, m \in \mathbb{N}$ with $n \geq m$ and for every $\zeta \in D(p, \frac{r}{2})$

$$f_n(\zeta) - f_m(\zeta) \leq 3[f_n(p) - f_m(p)]$$

Thus for any $n, m \in \mathbb{N}$ and for every $\zeta \in D(p, \frac{r}{2})$ the following holds:

$$|f_n(\zeta) - f_m(\zeta)| \leq 3|f_n(p) - f_m(p)|$$

Hence f_n converges uniformly on $D(p, \frac{r}{2})$.

Let $V = \{z \in U |$ the sequence $(f_n(z))$ converges$\}$. The above discussion shows that V is open. We will show below that V is closed as well and this along with the above discussion proves that f_n converges normally on U.

Let (p_n) be a sequence in V such that $p_n \to p \in U$. Assume that $\overline{D(p,r)} \subset U$. Choose k large enough such that $|p_k - p| < r/4$. Then $\overline{D(p_k, r/2)} \subset \overline{D(p,r)} \subset U$. By the discussion above since $f_n(p_k)$ converges, so does $f_n(\zeta)$, $\forall \zeta \in D(p_k, r/4)$. Hence we have shown that $p \in V$. Thus V is closed as well.

□

6.1.5. Singularities. Analogous to the Riemann removable singularity theorem for a holomorphic function, we will prove a theorem for harmonic functions in this section.

Proposition 6.14. *Let $f : U \to \mathbb{R}$ be a harmonic function and let $\overline{D(p,r)} \setminus \{p\} \subset U$. If f is bounded on $\overline{D(p,r)} \setminus \{p\}$, then f can be extended to a harmonic function on the whole of $\overline{D(p,r)}$, i.e., we can assign a value for f at p so that the extended function is harmonic.*

Proof. Let h be the harmonic function on $D(p,r)$ which is continuous on $\overline{D(p,r)}$ such that $h = f$ on $\partial D(p,r)$. For $\epsilon > 0$ consider the harmonic function g_ϵ on $D(p,r) \setminus \{p\}$ defined as:

$$g_\epsilon(z) = (h(z) - f(z)) + \epsilon log \left|\frac{z-p}{r}\right|$$

Since both h and f are bounded in $D(p,r) \setminus \{p\}$, so is $h - f$. So the limit, $\lim_{z \to p} g_\epsilon(z) = -\infty$. So we can choose a small enough δ such that $\forall |\zeta - p| < \delta$, $g_\epsilon(\zeta) \leq 0$. By the maximum/minimum principle of harmonic functions it follows that $g_\epsilon \leq 0$ on $D(p,r) \setminus \{p\}$. Letting $\epsilon \to 0$, we find that $h \leq f$ on $D(p,r) \setminus \{p\}$.

By going through the same arguments as above for the harmonic function $k_\epsilon(z) = (f(z) - h(z)) + \epsilon log \left|\frac{z-p}{r}\right|$, we can conclude that $f \leq h$ on $D(p,r) \setminus \{p\}$. Hence $h = f$ on $D(p,r) \setminus \{p\}$. Thus by defining $f(p) = h(p)$, we get the desired result.

□

6.2. Subharmonic Functions

Definition 6.15 (Subharmonic function)**.** *Let $U \subset \mathbb{C}$ be a nonempty open set. A continuous function $f : U \to \mathbb{R}$ is said to be subharmonic if for every $\overline{D(p,r)} \subset U$ and a harmonic function h defined in a neighborhood of $\overline{D(p,r)}$ satisfying $h \geq f$ on $\partial D(p,r)$, it holds that $h \geq f$ on whole of $D(p,r)$.*

Remark 6.16. It is easy to see that harmonic functions are also subharmonic. Subharmonic functions are the $\mathbb{C}$-analogue of $\mathbb{R}$-convex functions: The analogue of the Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in one dimension is $\frac{\partial^2}{\partial x^2}$ and the analogue of harmonic functions in the class of C^2 functions on a domain of $\mathbb{R}$ are the linear functions ($\frac{\partial^2 f}{\partial x^2} \equiv 0 \Rightarrow f$ is a linear function). What is a convex function on (a,b)? A continuous function $g : (a,b) \to \mathbb{R}$ is said to be convex if it holds that whenever $c, d \in (a,b)$ and $c < d$ we have:

$$g(\zeta) \leq h_{cd}(\zeta) \ , \ \forall \ \zeta \in (c,d)$$

where h_{cd} is the linear function on $[c,d]$ such that $h_{cd}(c) = g(c)$ and $h_{cd}(d) = g(d)$. This definition of convex function is equivalent to saying that:

$$g(\zeta) \leq h(\zeta) \ , \ \forall \ \zeta \in (c,d)$$

where h is any linear function on $[c,d]$ such that $h(c) \geq g(c)$ and $h(d) \geq g(d)$ (Here c and d are to thought of as $\{c,d\} = \partial[c,d]$).

We know that a C^2 function f on $\mathbb{R}$ is convex iff $\frac{\partial^2 f}{\partial x^2} \geq 0$. Similarly a C^2 function g on a domain in $\mathbb{C}$ is subharmonic iff $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})g \geq 0$ (Sec. 2, Chapter XV, [**8**]).

The following proposition is the analogue of the mean value property of harmonic functions.

Proposition 6.17. *Let $U \subset \mathbb{C}$ be a nonempty open set and let $f : U \to \mathbb{R}$ be a continuous function. Then f is subharmonic on U iff for any $p \in U$ and $r > 0$ such that $\overline{D(p,r)} \subset U$ the following holds:*

$$f(p) \leq \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta}) d\theta$$

Proof. ($\Rightarrow$) We first assume that f is subharmonic. Let $\overline{D(p,r)} \subset U$. Let $\tilde{f}$ be the harmonic function on $D(p,r)$ which is continuous on $\overline{D(p,r)}$ and such that $f = \tilde{f}$ on $\partial D(p,r)$. Then $f(z) \leq \tilde{f}(z)$ on $D(p,r)$ and in particular $f(p) \leq \tilde{f}(p)$. By applying the mean value property of harmonic functions to $\tilde{f}$, we have

$$\tilde{f}(p) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(p + re^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta}) d\theta$$

It thus follows that $f(p) \leq \frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})d\theta$.

($\Leftarrow$) We now prove the converse. For this let us assume that f is a continuous function with the property mentioned in the theorem. We will now prove that f satisfies a maximum principle, in particular if f is nonconstant then f does not attain its maximum at any interior point of U. For any point $p \in U$ with $\overline{D(p,r)} \subset U$, we have the inequality:

$$\begin{aligned} f(p) &\leq \frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})d\theta \\ \therefore 0 &\leq \frac{1}{2\pi}\int_0^{2\pi} [f(p+re^{i\theta}) - f(p)]d\theta \end{aligned}$$

Thus the above inequality shows that p cannot be a maximum for f. We initially started with an arbitrary $p \in U$, hence f cannot attain maximum in U.

Now let $\overline{D(q,s)} \subset U$ and let h be a function which is harmonic on $D(q,s)$ and continuous on $\overline{D(q,s)}$ satisfying $h \geq f$ on $\partial D(q,s)$. The function $f-h$ on $\overline{D(q,s)}$ is a continuous function satisfying the inequality mentioned in the statement of the theorem. Hence from the above discussion it follows that $f-h$ also satisfies the maximum principle. Hence the maximum of $f-h$ on $\overline{D(q,s)}$ is attained on $\partial D(q,s)$. But on $\partial D(q,s)$, $f-h \leq 0$, hence on $\overline{D(q,s)}$ too $f-h \leq 0$. We have thus proved that f is subharmonic. $\square$

In the proof of the preceding proposition we have also proved the following lemma:

Lemma 6.18 (Maximum Principle)**.** *Let $U \subset \mathbb{C}$ be a domain and let $f : U \to \mathbb{R}$ be a subharmonic function. Then f does not attain its maximum anywhere on U. In particular if $V \subset U$ is an open set with compact closure, then the maximum of f on $\overline{V}$ is attained on ∂V and nowhere in V.*

Subharmonic functions satisfy *only* the maximum principle. They do not satisfy any analogous minimum principle.

From the characterization of subharmonic functions proved in proposition 6.17 it easily follows that:

Corollary 6.19. *If u and v are two subharmonic functions on U, then the function f defined by*

$$f(z) = max\{u(z), v(z)\},\ \forall z \in U$$

is also subharmonic on U.

Definition 6.20 (Poisson modification)**.** *Let $f : U \to \mathbb{R}$ be a subharmonic function on a domain U and let $\overline{D(p,r)} \subset U$. The Poisson modification of f for $D(p,r)$ is the function $\tilde{f}$ defined as:*

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \in U \setminus D(p,r) \\ \frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\frac{r^2-|z-p|^2}{|re^{i\theta}-(z-p)|^2}d\theta & \text{if } z \in D(p,r) \end{cases}$$

Lemma 6.21. *Let $f : U \to \mathbb{R}$ be a subharmonic function and let $\overline{D(p,r)} \subset U$. Then the Poisson modification of f for $D(p,r)$, $\tilde{f}$, is subharmonic on U.*

PROOF. Note that the Poisson modification of f for $D(p,r)$ is harmonic in the disc $D(p,r)$ and it is subharmonic on $U \setminus D(p,r)$. Also note that $\tilde{f} = f$ on $\partial D(p,r)$. Hence it suffices to prove that for every $\zeta \in \partial D(p,r)$ and $R > 0$ such that $\overline{D(\zeta,R)} \subset U$ it holds that $\tilde{f}(\zeta) \leq \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(\zeta + Re^{i\theta})d\theta$.

If $R \geq 2r$ then $\forall\ 0 \leq \theta \leq 2\pi$, $\tilde{f}(\zeta + Re^{i\theta}) = f(\zeta + Re^{i\theta})$. Then in this case

$$\tilde{f}(\zeta) = f(\zeta) \leq \frac{1}{2\pi}\int_0^{2\pi} f(\zeta + Re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(\zeta + Re^{i\theta})d\theta$$

Now assume that $R < 2r$. Let $\theta_1 < \theta_2$ be such that $\overline{D(\zeta,R)} \cap \overline{D(p,r)} = \{\zeta + Re^{i\theta_1}, \zeta + Re^{i\theta_2}\}$. Then we have the following:

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(\zeta + Re^{i\theta})d\theta &= \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \tilde{f}(\zeta + Re^{i\theta})d\theta + \frac{1}{2\pi}\int_{\theta_2}^{2\pi+\theta_2} \tilde{f}(\zeta + Re^{i\theta})d\theta \\ &= \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \tilde{f}(\zeta + Re^{i\theta})d\theta + \frac{1}{2\pi}\int_{\theta_2}^{2\pi+\theta_2} f(\zeta + Re^{i\theta})d\theta \\ &\geq \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} f(\zeta + Re^{i\theta})d\theta + \frac{1}{2\pi}\int_{\theta_2}^{2\pi+\theta_2} f(\zeta + Re^{i\theta})d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\zeta + Re^{i\theta})d\theta \geq f(\zeta) = \tilde{f}(\zeta) \end{aligned}$$

It thus follows that $\tilde{f}$ is subharmonic on U. □

Definition 6.22 (Perron family)**.** *A family $\mathfrak{F}$ of subharmonic functions on U is said to be a Perron family if it satisfies the following conditions:*

(1) *$max\{u,v\} \in \mathfrak{F}$ whenever $u, v \in \mathfrak{F}$.*
(2) *$\tilde{u} \in \mathfrak{F}$ whenever $u \in \mathfrak{F}$, where $\tilde{u}$ is a Poisson modification of u.*
(3) *$u \equiv 0$ outside a compact subset of U.*

Theorem 6.23. *Let $\mathfrak{F}$ be a Perron family on a domain $U \subset \mathbb{C}$. Then the function f defined by*

$$f(z) = sup\{u(z)|\ u \in \mathfrak{F}\}$$

is either harmonic on U or $f \equiv \infty$.

PROOF. It suffices to prove that if there exists $p \in U$ such that $f(p) \in \mathbb{R}$, then the function f is harmonic on U. So assume that such a p exists. Let $\overline{D(p,r)} \subset U$. Let $u_1, u_2, u_3, \ldots$ be a sequence of functions in $\mathfrak{F}$ such that $\lim_{n\to\infty} u_n(p) = f(p)$. Let $\tilde{u_n}$ denote the Poisson modification of u_n for $D(p,r)$. Then $\tilde{u_n} \geq u_n$ and $\tilde{u_n} \in \mathfrak{F}$. It thus follows that $\lim_{n\to\infty} \tilde{u_n}(p) = f(p)$. Now consider the sequence $v_1, v_2, v_3, \ldots$ of $\mathfrak{F}$ defined by

$$\begin{aligned} v_1 &= u_1 \\ v_2 &= \max(u_1, u_2) \\ \vdots & \quad \vdots \\ v_n &= \max(u_1, u_2, \ldots, u_n) \\ \vdots & \quad \vdots \end{aligned}$$

$\mathfrak{F}$ being a Perron family, each $v_n \in \mathfrak{F}$ and $v_1 \leq v_2 \leq v_3 \leq \cdots$

Let $\tilde{v_n}$ denote the Poisson modification of v_n for $D(p,r)$, then each $\tilde{v_n} \in \mathfrak{F}$ and also $\tilde{v_1} \leq \tilde{v_2} \leq \tilde{v_3} \leq \cdots$ is an increasing sequence of harmonic functions on $D(p,r)$. We also have $\lim_{n\to\infty} \tilde{v_n}(p) = f(p)$. Thus by Harnack's principle the function $h(z) = \lim_{n\to\infty} \tilde{v_n}(z)$ is harmonic on $D(p,r)$.

We will show that $h = f$ on $D(p,r)$. Suppose that there exists $q \in D(p,r)$ such that $h(q) \neq f(q)$. Let $w_1, w_2, \ldots$ be a sequence of functions in $\mathfrak{F}$ such that $\lim_{n\to\infty} w_n(q) = f(q)$. Now consider the function x_n defined by $x_n = \max\{w_n, \tilde{v_n}\}$. Then each $x_n \in \mathfrak{F}$. Now consider $y_n = \max(x_1, x_2, \ldots, x_n)$. The sequence of functions $\tilde{y_1}, \tilde{y_2}, \tilde{y_3}, \ldots$ is an increasing sequence of harmonic functions in $D(p,r)$ and hence by Harnack's principle it follows that $g(z) = \lim_{n\to\infty} \tilde{y_n}(z)$ is a harmonic function which also satisfies $g(p) = f(p)$ and $g(q) = f(q)$.

By our very construction, g and h satisfy $g \geq h$ on $\overline{D(p,r)}$. But since $g(p) = h(p)$, it follows that $g = h$ on $D(p,r)$ and in particular $g(q) = h(q)$. Hence $f(q) = h(q)$, thus contradicting our assumption. Thus $h = f$ on $D(p,r)$ and hence f is harmonic on $D(p,r)$.

The above discussion shows that the set $V = \{z \in U | f(z) \in \mathbb{R}\}$ is open in U. It also says something more: Whenever $\overline{D(\zeta, R)} \subset U$ and $f(\zeta) \in \mathbb{R}$, it also holds that $f(z) \in \mathbb{R}$, $\forall z \in D(\zeta, R)$.

This is used to show that V is also closed in U so that $V = U$ as required. Let $(\zeta_n) \subset V$ be such that $\zeta_n \to \zeta \in U$. Suppose that $\overline{D(\zeta, 2r)} \subset U$. Choose k large enough so that $|\zeta_k - \zeta| < r$. Then clearly $\overline{D(\zeta_k, r)} \subset U$. Hence by the remark above it follows that $f(z) \in \mathbb{R}$, $\forall z \in D(\zeta_k, r)$ and in particular $f(\zeta) \in \mathbb{R}$. □

6.3. Dirichlet Problem and its solution by Perron method

In this section we will be concerned about the question: Given a domain $U \subset \mathbb{C}$ and a continuous function $f : \partial U \to \mathbb{R}$, can we extend it to a continuous function $\tilde{f}$ on U such that it is harmonic on U? We shall devise a necessary and sufficient condition on the boundary ∂U so that this is always possible.

We shall first consider an example when this is not possible.

Example 6.24. Consider the domain $U = \mathbb{D} \setminus \{0\}$. Then $\partial U = \partial\mathbb{D} \cup \{0\}$. Consider f on ∂U which is

$$f(z) = \begin{cases} 1 & \text{if } z \in \partial\mathbb{D} \\ 0 & \text{if } z = 0 \end{cases}$$

Let us assume that there exists a continuous extension of f, call it $\tilde{f}$, to the whole of $\overline{\mathbb{D}}$ which is harmonic on $\mathbb{D} \setminus \{0\}$. Then by the maximum/minimum principle of harmonic functions, it follows that $0 \leq \tilde{f} \leq 1$ on $\overline{\mathbb{D}}$. Now since $\tilde{f}$ is a bounded harmonic function in a neighborhood of 0, it follows from proposition 6.1.4 that $\tilde{f}$ is harmonic on $\mathbb{D}$. But again by maximum/minimum principle, since $\tilde{f}|_{\partial\mathbb{D}} \equiv 1$, $\tilde{f} \equiv 1$ on $\overline{\mathbb{D}}$. This is a contradiction since $\tilde{f}(0) = 0$. Hence such an extension is not possible in the present case.

In the preceding example, the boundary point 0 posed the problem. In order for the solution to exist we require the boundary to satisfy certain conditions. The following notions will be required to frame the afore mentioned conditions.

Definition 6.25 (Peak point)**.** *Let $U \subset \mathbb{C}$ be an open set and let $p \in \partial U$. Then p is called a peak point if there exists a continuous function f_p on $N \cap \overline{U}$, where N is a neighborhood of p such that $\overline{N}$ is compact and satisfies the following properties:*

(1) *f_p is subharmonic on $N \cap U$*
(2) *$f_p \leq 0$ on $N \cap \overline{U}$*
(3) *$\{z \in N \cap \overline{U} | f_p(z) = 0\} = \{p\}$*

The function f_p is called a peaking function or a barrier at p

Before we proceed towards formulating conditions on the boundary when the Dirichlet problem has a solution, we look at a few examples when a point on the boundary is a peak point.

Example 6.26. Consider $U = \mathbb{D}$. And consider the point $p = 1$. The function $f_p(z) = 1 - Re(z)$ is a barrier for U at p. Since composition of a subharmonic function by a conformal map remains a subharmonic function, it follows that every point $q \in \partial\mathbb{D}$ is a peak point.

Example 6.27. Suppose $p \in U$ is such that there exists $r > 0$ such that $\overline{D(q,r)} \cap \overline{U} = \{p\}$. Then p is a peak point. Consider the function $f(z) = r/(z-q)$. Then $f(U) \subset \mathbb{D}$. Suppose that $r/p - q = e^{i\theta}$ and consider the function $g(z) = e^{-i\theta}z$. Then the function $g \circ f$ is a barrier for U at p.

It follows from the very definition that for a domain U, $p \in \partial U$ being a peak point is a local property. In the following proposition we shall see how this local property can be used to get a desired global subharmonic function on U.

Proposition 6.28. *Let $U \subset \mathbb{C}$ be a domain and let $p \in \partial U$ be a peak point. Let $m < M$ be given. Then there exists a continuous function f on $\overline{U}$ which is subharmonic on U and satisfies the following properties:*

(1) $f \leq M$ *on* $D(p,r) \cap \overline{U}$, *for some* $r > 0$
(2) $f \equiv m$ *on* $\overline{U} \setminus D(p,r)$
(3) $f(p) = M$

Proof. Let $f_p : D(p,2r) \cap \overline{U} \to \mathbb{R}$ be a barrier for U at p. Consider f_p on $\overline{D(p,r)} \cap \overline{U}$. Let K be the maximum of f_p on $\partial D(p,r) \cap \overline{U}$. Since $f_p \leq 0$, we can choose $N \in \mathbb{N}$ large enough so that $M + NK < m$. Now the function h defined below is subharmonic and satisfies the three properties mentioned above:

$$h(z) = \begin{cases} m & \text{for } z \in \overline{U} \setminus D(p,r) \\ max\{m, M + Nf_p(z)\} & \text{for } z \in \overline{D(p,r)} \cap \overline{U} \end{cases}$$

□

Theorem 6.29 (Dirichlet Problem, Theorem 34 & 37, §7, [**9**])**.** *Suppose that $U \subset \mathbb{C}$ is a bounded domain. Then for any continuous function $f : \partial U \to \mathbb{R}$, there exists a continuous extension of f, call it $\tilde{f}$, on $\overline{U}$ such that it is harmonic on U iff every point $p \in \partial U$ is a peak point.*

Proof. ($\Rightarrow$) Consider a continuous function $g : \partial U \to \mathbb{R}$ such that $g < 0$ on $\partial U \setminus \{p\}$ and $g(p) = 0$. Let $\tilde{g}$ be a continuous extension of g to U such that it is harmonic on U. This $\tilde{g}$ will serve as the barrier for U at p.

($\Leftarrow$) To prove the other way equivalence, we will construct a Perron family on U and show that the associated Perron function is the solution of the Dirichlet problem. We start with a continuous function $f : \partial U \to \mathbb{R}$. Since ∂U is compact, f is bounded. Let $|f| \leq M$ on ∂U. Consider the family of functions $\mathfrak{F}$ on U satisfying the following properties:

(1) u is subharmonic on U and continuous on $\overline{U}$.
(2) $|u| \leq M$ on U.
(3) For every $p \in \partial U$, $u(p) \leq f(p)$.

The family $\mathfrak{F}$ is a Perron family. Since $\mathfrak{F}$ is uniformly bounded, it follows from theorem 6.23 that the associated Perron function, call it h, is harmonic on U. We will show below that whenever $p \in \partial U$ is a peak point, $\lim_{U \ni z \to p} h(z) = f(p)$. This will show that h is the Dirichlet solution for the present case.

Given $\epsilon > 0$, we wish to find $\delta > 0$ such that whenever $z \in D(p,\delta) \cap \overline{U}$ the following inequality holds:

$$|h(z) - f(p)| \leq \epsilon$$

$$f(p) - \epsilon \leq h(z) \leq f(p) + \epsilon$$

We will first find a δ_1 satisfying the inequality $f(p) - \epsilon \leq h(z)$. Since h is the associated Perron function of the family $\mathfrak{F}$, it suffices to prove that there exists $u \in \mathfrak{F}$ such that on $D(p,\delta) \cap \overline{U}$, $u(z) \geq f(p) - \epsilon$.

It follows from proposition 6.28 that it is possible to find a continuous function u on $\overline{U}$ which is subharmonic on U and satisfies the following conditions:

(1) $u(p) = f(p)$
(2) $u(z) \leq f(p)$ on $D(p,R) \cap \overline{U}$ for some $R > 0$
(3) $u(z) \equiv -M$ on $\overline{U} \setminus D(p,R)$

So choose $\delta_1 > 0$ such that $u(z) \geq f(p) - \epsilon$ on $D(p,\delta_1)$ and $|f(z) - f(p)| \leq \epsilon/2$ on $\partial U \cap D(p,\delta_1)$.

Now we will find $\delta < \delta_1$ such that it satisfies the inequality $h(z) \leq f(p) + \epsilon$. It is clear that is the required δ. It follows from proposition 6.28 that there exists a subharmonic function w on U which is continuous on $\overline{U}$ and satisfies the following properties:

(1) $w(p) = -f(p)$
(2) $w(z) \leq -f(p)$ on $D(p,s) \cap \overline{U}$ for some $\delta_1 > s > 0$.

(3) $w(z) \equiv -M$ on $\overline{U} \setminus D(p,s)$

Now for any $u \in \mathfrak{F}$ we have $\forall z \in \ D(p,s) \cap \partial U$:

$$u(z) \leq f(z) \leq f(p) + \epsilon/2$$

$$u(z) + w(z) \leq -f(p) + f(p) = \epsilon/2$$

Now for any $\zeta \in \partial D(p,s) \cap U$ we have

$$w(\zeta) + u(\zeta) \leq -M + M = 0$$

Now by the maximum property of subharmonic functions, it follows that on $D(p,s) \cap U$

$$w(z) + u(z) \leq \epsilon/2$$

It thus follows that $w(z) + h(z) \leq \epsilon/2$ on $D(p,s) \cap U$. By continuity of w at p, we can choose $\delta < s$ such that on $D(p,\delta) \cap \overline{U}$, it holds that $|w(z) - f(z)| \leq \epsilon/2$. Hence on $D(p,\delta) \cap \overline{U}$, $h(z) \leq f(p) + \epsilon$.

□

The following theorem gives a weaker criterion, than that of existence of a peaking function, for a boundary point to be a peak point.

Theorem 6.30 (Bouligand, Theorem 5.6, Chapter 11, [**10**])**.** *Let $U \subset \mathbb{C}$ be an open set and let $p \in \partial U$. Suppose that there exists a continuous function f_p on $N \cap \overline{U}$, where N is a neighborhood of p such that $\overline{N}$ is compact and satisfies the following properties:*

(1) *f_p is subharmonic on $N \cap U$*
(2) *$f_p \leq 0$ on $N \cap \overline{U}$*

Then the point p is a peak point.

The function f_p in the above theorem is *not* a peaking function, but a weaker version of the peaking function. Let us call the function f_p in the above theorem as weak peaking function. In general it is much easier to construct a weak peaking function than a peaking function for a boundary point.

Lemma 6.31. *Suppose $U \subset \mathbb{C}$ is a open. Let $a, b \in \partial U$ be such that both of them are contained in the same connected component of ∂U. Consider the holomorphic function $\phi(z) = \frac{z-a}{z-b}$ defined on U. Then there exists a holomorphic function g on U which satisfies:*

$$e^{g(z)} = \phi(z), \ \forall z \in U$$

That is there exists a holomorphic branch of log $\left(\frac{z-a}{z-b}\right)$ defined on U.

PROOF. Consider the holomorphic function h defined on U by:

$$\begin{aligned} h(z) &= \frac{\phi'(z)}{\phi(z)} \\ &= \frac{1}{z-a} - \frac{1}{z-b} \end{aligned}$$

It suffices to prove that h has a holomorphic antiderivative g on U. Then g (modulo an additive constant) would be the required function. Let $\gamma \subset U$ be any simple closed curve. Then it follows from the Jordan curve theorem that both a and b lie in the same component of $[\gamma]^c$. Hence we have the following:

$$\begin{aligned} \oint_\gamma \frac{\phi'(\zeta)}{\phi(\zeta)} d\zeta &= \oint_\gamma \frac{1}{\zeta - a} d\zeta - \oint_\gamma \frac{1}{\zeta - b} d\zeta \\ &= 0 \end{aligned}$$

It thus follows from Cauchy-Goursat theorem that h has a holomorphic antiderivative. This completes the proof. □

Proposition 6.32. *Let $U \subset \mathbb{C}$ be a domain. Suppose $p \in \partial U$ is such that the connected component of ∂U containing p is not $\{p\}$, then p is a peak point point.*

PROOF. Let $q \neq p$ be a point of the connected component of ∂U containing p. It follows from the preceding lemma that there exists a holomorphic function g on U such that:

$$e^{g(z)} = \frac{z-p}{z-q}, \ \forall \ z \in U$$

The function g can be thought of as $g(z) = \log\left(\frac{z-p}{z-q}\right)$. For $0 < r < \frac{|p-q|}{2}$, consider the function $H(z) = \frac{1}{g(z)}$ which is holomorphic on $D(p,r) \cap U$. Then we have the following:

$$h(z) = Re(H(z)) = Re\left(\frac{1}{g(z)}\right) = \frac{log(|\frac{z-p}{z-q}|)}{|log(\frac{z-p}{z-q})|^2}$$

On $D(p,r) \cap U$, we have the following inequality:

$$|h(z)| = \frac{log(|\frac{z-p}{z-q}|)}{|log(\frac{z-p}{z-q})|^2} \leq \frac{log|\frac{z-p}{z-q}|}{log|\frac{z-p}{z-q}|^2} = \frac{1}{log|\frac{z-p}{z-q}|}$$

Thus on $D(p,r) \cap U$, h is harmonic, negative and satisfies $\lim_{U \ni \zeta \to p} h(z) = 0$. Thus p is a peak point* and hence is a peak point. This proves our claim. □

The above proposition gives a lot more examples of situations when a boundary point is a peak point than those considered at the beginning of this section.

6.4. Generalization to Riemann surfaces

In this section we will define the notions of harmonic functions, subharmonic functions on a Riemann surface. Throughout this section, R will denote a Riemann surface, $(U_\alpha, \phi_\alpha)_{\alpha\in\mathfrak{A}}$ the corresponding conformal structure on R.

Definition 6.33 (Harmonic function)**.** *A real valued function $f : R \to \mathbb{R}$ is said to be harmonic if for every $p \in R$, there exists some $U_\alpha \ni p$ such that the function f_α given by:*

$$f_\alpha(z) = f \circ \phi_\alpha^{-1}(z) \ , \ \forall z \in \phi_\alpha(U_\alpha)$$

is harmonic on $\phi_\alpha(U_\alpha)$.

Definition 6.34 (Subharmonic function)**.** *A real valued function $f : R \to \mathbb{R}$ is said to be subharmonic if for every $p \in R$, there exists some $U_\alpha \ni p$ such that the function f_α given by:*

$$f_\alpha(z) = f \circ \phi_\alpha^{-1}(z) \ , \ \forall z \in \phi_\alpha(U_\alpha)$$

is subharmonic on $\phi_\alpha(U_\alpha)$.

Remark 6.35. Let U and W be domains in $\mathbb{C}$. Suppose that $f : U \to \mathbb{R}$ is a harmonic (resp. subharmonic) function and $g : W \to U$ is a conformal map. Then the composite function $f \circ g : W \to \mathbb{R}$ is also harmonic (resp. subharmonic). Thus if $f : R \to \mathbb{R}$ is a harmonic (resp. subharmonic) function, then the function defined by:

$$f_\alpha(z) = f \circ \phi_\alpha^{-1}(z) \ , \ \forall z \in \phi_\alpha(U_\alpha)$$

is harmonic (resp. subharmonic) on $\phi_\alpha(U_\alpha)$, for every $\alpha \in \mathfrak{A}$.

Having made this remark, everything we have considered so far about harmonic and subharmonic functions on domains in $\mathbb{C}$ carries over to harmonic and subharmonic functions on a Riemann surface.

CHAPTER 7

Uniformization theorem

The Uniformization theorem that was introduced in chapter 5 is a special case (dimension one) of the 22^{nd} problem in the Hilbert's list of problems and one of the ten he presented at the International Congress of Mathematicians, Paris in 1900. H. Poincaré and P.Koebe solved this special case in 1907. In this chapter we will give a proof of the Uniformization theorem following the proof in the book of Gamelin [**8**]. The idea of the proof is based on analytic continuation on the Riemann surface.

7.1. Preliminaries

Lemma 7.1. *If f is a non zero holomorphic function on any open subset $U \subset \mathbb{C}$, then $log|f|$ is a harmonic function on the whole of U.*

PROOF. Let $p \in U$ and $D(p,r) \subset U$ for some $r > 0$. f is a non zero holomorphic function on $D(p,r)$ which is simply connected. Hence there exists a holomorphic function F on $D(p,r)$ such that $f = e^F$. Hence $log|f| = Re(F)$ which is harmonic.

Harmonicity is a local property and we have proved above that f is harmonic in some neighborhood of every point in U. Hence f is harmonic on whole of U. □

Definition 7.2 (Logarithmic pole)**.** *Suppose that $h : D(p,r) \setminus \{p\} \to \mathbb{R}$ is a harmonic function. h is said to have a logarithmic pole at p if the function $f(z) = h(z) + log|z-p|$ is harmonic on $D(p,r)$.*

The following discussion justifies the term *logarithmic pole* in the above definition.

(1) Suppose that $g : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ is a holomorphic function with a simple pole at 0. Then we can write $g(z) = \frac{f(z)}{z}$, where f is a holomorphic function on $\mathbb{D}$ which is non zero in a neighborhood of 0 (For simplicity we will assume that f is non zero on $\mathbb{D}$). It follows from the above lemma that $log|g|$ is harmonic on $\mathbb{D} \setminus \{0\}$.

$$\begin{aligned} log|g(z)| &= log|f(z)| - log|z| \\ log|g(z)| + log|z| &= log|f(z)| \end{aligned}$$

Again by the preceding lemma it follows that $log|f|$ is harmonic on $\mathbb{D}$. Thus $log|g|$ is a harmonic function on $\mathbb{D} \setminus \{0\}$ with a logarithmic pole at 0.

(2) Now suppose that $h : \mathbb{D} \setminus \{0\} \to \mathbb{R}$ is a harmonic function with a logarithmic pole at 0. So the function f defined by

$$f(z) = h(z) + log|z|$$

is harmonic on $\mathbb{D}$. Now f is a harmonic function on $\mathbb{D}$. Hence there exists a holomorphic function F such that $f = Re(F)$. Now consider the holomorphic function $g(z) = e^{F(z)}$ on $\mathbb{D}$. Thus $f = log|g|$, where g is a nonzero holomorphic function on $\mathbb{D}$. Hence we have:

$$f(z) = log|g| = h(z) + log|z|$$

$$\therefore h(z) = log\left|\frac{g(z)}{z}\right|$$

Hence we conclude that a harmonic function h on $\mathbb{D} \setminus \{0\}$ has a logarithmic pole at 0 iff $h = log|H|$, where H is a holomorphic function on $\mathbb{D} \setminus \{0\}$ with a simple pole at 0.

7.2. Green's function

7.2.1. Definition. Throughout this section we will assume R to be a Riemann surface and $q \in R$ to be an arbitrary point in R.

Consider the family of subharmonic functions on $R \setminus \{q\}$, $\mathcal{F}_q$ defined by:

$$\mathcal{F}_q = \{u : R \setminus \{q\} \to \mathbb{R} |\ u \text{ satisfies the following conditions}\}$$

(1) u is subharmonic on $\mathbb{R} \setminus \{q\}$
(2) u is zero outside a compact subset of R
(3) $u(p) + log|z(p)|$ is subharmonic on $\mathcal{D}_q$, where $\mathcal{D}_q$ is a coordinate disc centered at q and $z : \mathcal{D}_q \to \mathbb{D}$ is a coordinate map such that $z(q) = 0$.

The family $\mathcal{F}_q$ of subharmonic functions on $R \setminus \{q\}$ is a Perron family. The associated Perron function is either harmonic or is identically infinity on $R \setminus \{q\}$.

Definition 7.3 (Green's function)**.** *If the associated Perron function for the family $\mathcal{F}_q$ exists, then we say that the Green's function for R with a logarithmic pole at q exists and the function is denoted by $g(p, q)$ and if it does not exists we say that the Green's function does not exist.*

Remark 7.4. We shall now show that the definition of $\mathcal{F}_q$ above is independent of the local coordinate chart $(\mathcal{D}_q, z)$ at q. Let $(\mathcal{D}'_q, \phi)$ be another coordinate chart for R at q, where $\mathcal{D}'_q$ is a coordinate disc such that $\phi(q) = 0$.

Assume that u is a subharmonic function on $\mathbb{R} \setminus \{q\}$ which satisfies the first two conditions and the third condition in the definition of $\mathcal{F}_q$ with respect to the coordinate chart $(\mathcal{D}_q, z)$. We will show that u satisfies the third condition with respect to $(\mathcal{D}'_q, \phi)$ as well. For any $\zeta \in \mathcal{D}'_q$ wherever the following makes sense, we have:

$$\begin{aligned} log|\phi(\zeta)| &= log|\phi \circ z^{-1}(z(\zeta))| \\ &= log|f(z(\zeta))| \end{aligned}$$

where f is a conformal map on a neighborhood of 0 and such that $f(0) = 0$. Hence $f(x) = xg(x)$ in a neighborhood of 0, where $g \neq 0$. We therefore have:

$$\begin{aligned} log|\phi(\zeta)| &= log|f(z(\zeta))| \\ &= log|z(\zeta)| + log|g(z(\zeta))| \\ \therefore u(\zeta) + log|\phi(\zeta)| &= [u(\zeta) + log|z(\zeta)|] + log|g(z(\zeta))| \end{aligned}$$

The first of the two terms in the above equation is subharmonic by assumption and since $g \circ z$ is conformal and nonzero, the second term is harmonic by lemma 7.1. Hence the sum and therefore $u(\zeta) + log|\phi(\zeta)|$ is subharmonic as claimed.

7.2.2. Properties of Green's function. Assuming that the green's function for R with a logarithmic pole at q exits, we shall derive some properties of the green's function $g(p, q)$.

Since $u \equiv 0$ is a member of $\mathcal{F}_q$ and since $g(p, q) = sup\{u(p) \mid u \in \mathcal{F}_q\}$, it follows that:

Lemma 7.5. *The green's function for R with a logarithmic pole at q satisfies $g(p, q) \geq 0$, for every $p \in R$.*

Lemma 7.6. *The function $h(p) = g(p, q) + log|z(p)|$ is harmonic on the the coordinate disc $\mathcal{D}_q$, where $(\mathcal{D}_q, z)$ is a coordinate chart at q.*

Proof. It is clear that h is harmonic on $\mathcal{D}_q \setminus \{q\}$. It suffices to prove that the function h is bounded in a neighborhood of q, in which case q will be a removable singularity of h. Let $\mathcal{D}_q = \{\zeta \in R||z(\zeta)| \leq r\}$ and let $\mathcal{D} = \{\zeta \in R||z(\zeta)| \leq \frac{r}{2}\}$. Consider the set $S = \partial\mathcal{D}$. g is bounded on S. Let $M > 0$ be such that $g \leq M$ on S. We then have the following inequality:

$$g(p, q) + log|z(p)| \leq M + log\left|\frac{r}{2}\right|, \ \forall\, p \in S$$

For any $u \in \mathcal{F}_q$, we have the following inequality:

$$u(p) + log|z(p)| \leq M + log\left|\frac{r}{2}\right|, \ \forall \ p \in S$$

By maximum principle of subharmonic functions, it follows that:

$$u(p) + log|z(p)| \leq M + log\left|\frac{r}{2}\right|, \ \forall \ p \in \mathcal{D}$$

In the above inequality, taking the supremum over all $u \in \mathcal{F}_q$ we get:

$$g(p,q) + log|z(p)| \leq M + log\left|\frac{r}{2}\right|, \ \forall \ p \in \mathcal{D}$$

Hence the function h is harmonic in a neighborhood of q and hence on all of $\mathcal{D}_q$. It thus follows by the analogue of proposition 6.14 for harmonic functions on Riemann surfaces that h is harmonic on $\mathcal{D}_q$. □

Definition 7.7 (Logarithmic pole)**.** *A harmonic function $f : R \setminus \{q\} \to \mathbb{R}$ is said to have a logarithmic pole at q if the function h defined by:*

$$h(p) = f(p) + log|z(p)|$$

is harmonic on $\mathcal{D}_q$, where $(\mathcal{D}_q, z)$ is a coordinate chart, $\mathcal{D}_q$ is a coordinate disc and $z(q) = 0$.

As was shown in Remark 7.4, it can be shown that the above definition is independent of the coordinate chart $(\mathcal{D}_q, z)$.

Lemma 7.8. *Suppose that h is a positive harmonic function on $R \setminus \{q\}$ which has a logarithmic pole at q, then $h \geq g$ on $R \setminus \{q\}$.*

Proof. In order to prove the lemma, it suffices to show that $u(p) \leq h(p)$, $\forall p \in R\setminus\{q\}$ and $\forall u \in \mathcal{F}_q$. So consider an arbitrary $u \in \mathcal{F}_q$. There exists a coordinate disc $\mathcal{D}$ (corresponding coordinate chart $(\mathcal{D}, z)$) on which $u(p) + log|z(p)|$ is subharmonic and $h(p) + log|z(p)|$ is harmonic. Hence $u - h$ is subharmonic on $\mathcal{D}$. It is also subharmonic on $\mathbb{R} \setminus \{q\}$ and hence it is subharmonic on the whole of R. Since $u \equiv 0$ outside a compact subset of R and h is positive, it follows from the maximum principle for subharmonic functions that $u \leq h$ on the whole of R. □

Lemma 7.9. *Suppose that the green's function for R with a logarithmic pole at q exists. Then inf $\{g(p,q)|p \in R \setminus \{q\}\} = 0$.*

Proof. Suppose that inf $\{g(p,q)|p \in R \setminus \{q\}\} = \delta > 0$. Then the function $f(p) = g(p,q) - \delta$ is a positive harmonic function on $R \setminus \{q\}$ with a logarithmic pole at q. By the preceding lemma, $f(p) \geq g(p,q)$ which is a contradiction. Hence the infimum of g is 0 on $R \setminus \{q\}$. □

Remark 7.10. Green's functions were extensively studied in Physics. They play an important role in solving *boundary value problems* in electrostatics. In a domain R that is free of charge the electrostatic potential ϕ is a harmonic function, i.e., ϕ satisfies $\Delta\phi \equiv 0$, where Δ is the Laplacian operator. In general (in the presence of charges in R) the electric potential ϕ satisfies $\Delta\phi = \rho$, where ρ is the charge density in R. Suppose that R is a domain with charge density ρ and the boundary of R, ∂R satisfies certain smoothness conditions. A common boundary value problem encountered in electrostatics is the following:

Calculate the potential ϕ in R under the condition: $\phi|_{\partial R} = f$, for some potential on the boundary.

The solution ϕ of the above problem with $\rho(p_0) = \infty$ and $\rho \equiv 0$ on $R \setminus \{p_0\}$ and $\phi|_{\partial R} \equiv 0$ is by our definition Green's function on R with a logarithmic pole at p_0. Physically the green's function on R with a logarithmic pole at p_0 is the electric potential in R when a unit charge is placed at p_0 and the boundary is grounded.

7.2.3. Harmonic Measure and Green's function. In this section we shall do the following:

(1) Define another harmonic function (on $R\backslash\mathcal{D}$, where $\mathcal{D}$ is some open coordinate disc) called the harmonic measure.
(2) Establish the connection between existence/non existence of green's function and the existence/non existence of harmonic measure.
(3) Show that the existence or non existence of green's function for a surface R with a logarithmic pole at q is not really dependent on the point of q. That is if $g(\zeta, q)$ exists for some $q \in R$, then $g(\zeta, p)$ exists for every $p \in R$.
(4) Show that the existence of harmonic measure on $R \setminus \mathcal{D}$ is a property of the "boundary" of R so that the existence/non existence of green's function depends on the "boundary" of R.

Hence the existence/non existence of green's function serves as a tool for identifying different surfaces on the basis of their "boundary". This can be thought of as the first essential step towards proving the Uniformization theorem, in that it singles out surfaces whose "boundary" is an arc. We shall then use the green's function to construct a conformal map between the (simply connected) surface and the unit disc.

Let $\mathcal{D}$ be an open coordinate disc in R. Consider the family $\mathcal{M}_{\mathcal{D}}$ of subharmonic functions defined on $R \setminus \mathcal{D}$ by:

$$\mathcal{M}_{\mathcal{D}} = \{u : R \setminus \mathcal{D} \to \mathbb{R} \mid u \text{ satisfies the following properties}\}$$

(1) u is subharmonic on $R \setminus \mathcal{D}$
(2) $u \leq 1$ on $R \setminus \mathcal{D}$
(3) $u \equiv 0$ outside a compact subset of $R \setminus \mathcal{D}$

The family $\mathcal{M}_{\mathcal{D}}$ is a Perron family on $R \setminus \mathcal{D}$. Since the constant function $f \equiv 0$ is a member of $\mathcal{M}_{\mathcal{D}}$, the associated Perron function w is a harmonic function which satisfies $0 \leq w \leq 1$.

$\partial\mathcal{D}$ is a compact subset of R. Let $\{\mathcal{D}_n\}_{n=1}^{N}$ be a collection of open coordinate discs each centered at a point of $\partial\mathcal{D}$ such that it covers $\partial\mathcal{D}$ and also satisfies $\overline{\mathcal{D}_n} \subset R$ for every $1 \leq n \leq N$. Now let $S = (R \setminus \overline{\mathcal{D}}) \cap (\cup \mathcal{D}_n)$. $S \subset R$ is a connected open subset and hence is a Riemann surface. S is an open region of R that is bounded by two disjoint arcs, one of which is $\partial\mathcal{D}$ and the other can be thought of as being made up of pieces of $\partial\mathcal{D}_n$ joined together. Every point of ∂S is a peak point and the Dirichlet problem has a solution for every continuous function on ∂S, in particular for the function $f \equiv 1$ on $\partial\mathcal{D}$ and $f \equiv 0$ on $\partial S \setminus \partial\mathcal{D}$.

Consider the subfamily of $\mathcal{M}_{\mathcal{D}}$ which consists of functions $u \in \mathcal{M}_{\mathcal{D}}$ which are identically zero outside $\overline{S}$. This is a Perron family and the associated Perron function, $\tilde{f}$, is the Dirichlet solution for the domain S and the corresponding boundary function f. Clearly $\tilde{f} < w|_S$. Since $\tilde{f}(\zeta) \to 1$ as $\zeta \to \zeta_0 \in \partial\mathcal{D}$, it follows that $w(\zeta) \to 1$. Hence there are two possibilities for the harmonic function w: $0 < w < 1$ or $w \equiv 1$.

Definition 7.11 (Harmonic measure)**.** *If $0 < w < 1$ we say that the harmonic measure for $R \setminus \mathcal{D}$ exists and the function w is called its harmonic measure. If $w \equiv 1$, then we say that the harmonic measure for $R \setminus \mathcal{D}$ does not exist.*

Theorem 7.12. *Let $\overline{\mathcal{D}} \subset R$, where $\mathcal{D}$ is an open coordinate disc. Then for any $q \in \mathcal{D}$, the green's function for R with a pole at q exists iff the harmonic measure for $R \setminus \mathcal{D}$ exists.*

Proof. Let us first assume that the green's function with a logarithmic pole at q exists for some $q \in \mathcal{D}$. We shall now prove the existence of harmonic measure for $R \setminus \mathcal{D}$. Let M be the maximum of g on $\partial\mathcal{D}$. Then $\frac{g}{M} \geq 1$ on $\partial\mathcal{D}$. We thus have $\frac{g}{M} \geq u$ on $\partial\mathcal{D}$ for any $u \in \mathcal{M}_{\mathcal{D}}$. Since g is positive and $u \equiv 0$ outside a compact subset of R, it follows by the maximum principle of subharmonic functions that the subharmonic function $u - \frac{g}{M} \leq 0$ on the whole of $R \setminus \mathcal{D}$. Hence $w(p) \leq \frac{g(p,q)}{M}$, for any $p \in R \setminus \mathcal{D}$. It follows by lemma 7.9 that $\inf_{p \in R \setminus \mathcal{D}} w(p) = 0$ and hence the case $w \equiv 1$ is not possible. Thus $0 < w < 1$ and hence the harmonic measure for $R \setminus \mathcal{D}$ exists.

We will now prove the converse: assume existence of harmonic measure for $R \setminus \mathcal{D}$ and prove the existence of green's function with a logarithmic pole at q for every $q \in \mathcal{D}$. Let $q \in \mathcal{D}$. In order to prove that green's function $g(\cdot, q)$ exists it suffices to prove that every $u \in \mathcal{F}_q$ and every $p \in R \setminus \{q\}$ satisfies $u(p) < M_p$, for some $M_p \in \mathbb{R}$.

Suppose that $\mathcal{D} = z^{-1}(D(0,r))$ and let $z(q_0) = 0$. Choose $s > r$ such that the closed coordinate disc $\mathcal{D}_s = z^{-1}(D(0,s))$ is defined. Let $q \in \mathcal{D}$ and let $u \in \mathcal{F}_q$. The function $h(p) = u(p) + log|z(p) - z(q)|$ is subharmonic on $\mathcal{D}$. Let M, κ, C_1 and C_2 be as below:

$$\begin{aligned} M &= \sup\{u(p) : p \in \partial\mathcal{D}_s\} \\ C_1 &= \sup\{|log|z(p) - z(q)|| : p \in \partial\mathcal{D}\} \\ C_2 &= \sup\{|log|z(p) - z(q)|| : p \in \partial\mathcal{D}_s\} \\ C &= C_1 + C_2 \\ \kappa &= \sup\{w(p) : p \in \partial\mathcal{D}_s\} \end{aligned}$$

Then the function $h \leq M + C_2$ on $\partial\mathcal{D}_s$. Hence by the maximum principle of subharmonic functions, it follows that $h \leq M + C_2$ on whole of $\mathcal{D}_s$ and in particular on $\partial\mathcal{D}$. Also note that $w \equiv 1$ on $\partial\mathcal{D}$ and $u \equiv 0$ outside a compact subset. Hence again by the maximum principle we have:

$$u(p) \leq (M + C)w(p), \forall\ p \in R \setminus \mathcal{D}$$

In the above equation, taking supremum of u and w over the set $\partial\mathcal{D}_s$ we have:

$$M \leq (M + C)\kappa$$

$$\therefore M \leq \frac{C\kappa}{1 - \kappa}$$

Note that M in the above estimate is independent of u! Hence we have $g(p, q) \leq M$, $\forall p \in \partial\mathcal{D}_s$. Hence the associated Perron function of the family $\mathcal{F}_q$ is a finite harmonic function. Hence the green's function $g(\cdot, q)$ exists.

□

Corollary 7.13. *If the green's function exists for some $q_0 \in R$, then it exists for all $q \in R$.*

Proof. It follows from the preceding theorem that the subset $S \subset R$ on which green's function exists is both open and closed. And in the present case S is non empty, hence it is the whole of R as claimed. □

Having proved the above corollary, we can simply talk about the existence/non existence of green's function for a surface.

Before proceeding further, we shall consider three basic examples $\mathbb{D}$, $\mathbb{C}$ and $\hat{\mathbb{C}}$. We shall find out in which of the three cases green's function exists and calculate it explicitly in the case when it does. The equivalence of the existence of green's function and the existence of harmonic measure (theorem 7.12) comes in very handy in many situations when we want to determine if green's function exists for a surface.

The unit disc, $\mathbb{D}$: The harmonic measure for $\mathbb{D} \setminus \mathbb{D}(p, r)$, where $D(p, r) \subset \mathbb{D}$ exists because the function w is the solution of the Dirichlet problem for the domain $\mathbb{D} \setminus D(p, r)$ and the function $f \equiv 1$ on $\partial D(p, r)$ and $f \equiv 0$ on $\partial\mathbb{D}$. Hence green's function exists for $\mathbb{D}$.

Consider the function $h(z) = log\left(\frac{1}{|z|}\right)$. This is a positive harmonic function on $\mathbb{D}$ with a logarithmic pole at 0. It follows by lemma 7.8 that $h(z) \geq g(z, 0),\ \forall\ z \in \mathbb{D}$. For every $0 < r < 1$ consider the subharmonic function on $\mathbb{D} \setminus \{0\}$ given by:

$$\begin{aligned} u_r(z) &= log\left(\frac{r}{|z|}\right), \forall z \in D(0, r) \setminus \{0\} \\ &= 0, \text{ otherwise} \end{aligned}$$

Clearly each $u_r \in \mathcal{F}_0$ and $u_r(z) \to log\left(\frac{1}{|z|}\right)$ as $r \to 1$. Thus $h(z) \leq g(z, 0)$. Hence $g(z, 0) = log\left(\frac{1}{|z|}\right)$.

The complex plane, $\mathbb{C}$: For $n \in \mathbb{N}$, consider the subharmonic function on $\mathbb{C} \setminus \{0\}$, u_n defined by:

$$\begin{aligned} u_n(z) &= log\left(\frac{n}{|z|}\right),\ \forall\ z \in D(0, n) \setminus \{0\} \\ &= 0,\ \forall\ z \in \mathbb{C} \setminus D(0, n) \end{aligned}$$

Then each $u_n \in \mathcal{F}_0$ and for any $z \in \mathbb{C} \setminus \{0\}$, $u_n(z) \to \infty$. Hence the green's function for $\mathbb{C}$ does not exist.

The Riemann Sphere, $\hat{\mathbb{C}}$: In a similar way as we did for the $\mathbb{C}$ case, it can be shown that green's function for $\hat{\mathbb{C}}$ does not exist.

Lemma 7.14. *Suppose that $f : R \to S$ is a conformal map and assume that the green's function for one of R or S exists. Then green's function for other surface also exists and $g_S(f(p), f(q)) = g_R(p, q)$.*

PROOF. Without loss of generality, assume that green's function for S exists. Let $u \in \mathcal{F}_{f(q)}$. Since f is a conformal map, $u \circ f \in \mathcal{F}_q$. Also for any $v \in \mathcal{F}_q$, $v \circ f^{-1} \in \mathcal{F}_{f(q)}$. Thus every member of $\mathcal{F}_q$ is of the form $u \circ f$ for some $u \in \mathcal{F}_{f(q)}$. Let $p \in R \setminus \{q\}$. Then for any $u \in \mathcal{F}_{f(q)}$ we have the following:

$$(u \circ f)(p) = u(f(p)) \leq g_S(f(p), f(q))$$
$$\therefore g_R(p,q) \leq g_S(f(p), f(q))$$

Similarly $g_S(f(p), f(q)) \leq g_R(p,q)$ and hence $g_S(f(p), f(q)) = g_R(p,q)$ as claimed. □

7.3. Uniformization Theorem - Part 1

The Uniformization theorem states that the only simply connected Riemann surfaces are $\mathbb{D}$, $\mathbb{C}$ and $\hat{\mathbb{C}}$. In this section we will see how the existence of green's function acts an indicator for the surface to be conformal to the disc.

7.3.1. Proof of Uniformization Theorem - Part 1.

Theorem 7.15 (The Uniformization Theorem). *Let R be a simply connected Riemann surface. Then R is conformally equivalent to either the unit disc $\mathbb{D}$, the complex plane $\mathbb{C}$ or the Riemann sphere $\hat{\mathbb{C}}$.*

We divide the proof into two parts: in the first part we shall consider simply connected Riemann surfaces for which Green's function exists and in the second part we shall consider simply connected Riemann surfaces for which Green's function does not exist.

PROOF. [Part 1] We shall show that if R is a simply connected Riemann surface for which Green's function exists, then R is conformally equivalent to $\mathbb{D}$. Let $q_0 \in R$ and let $g(\cdot, q_0)$ be the corresponding green's function for R with a logarithmic pole at q_0. As seen in section 7.1, there exists a holomorphic function $\tilde{f}$ in a deleted neighborhood of q_0 with a simple pole at q_0 or equivalently there exists a holomorphic function f in a neighborhood of q_0 with a simple zero at q_0 satisfying:

$$\begin{aligned} log|\tilde{f}(p)| &= g(p, q_0) \\ log|f(p)| &= -g(p, q_0) \\ \text{Equivalently, } |f(p)| &= e^{-g(p,q_0)} \end{aligned}$$

in a neighborhood of q_0. Let $\mathcal{D}$ be any coordinate disc in $R \setminus \{q\}$. Since $g(\cdot, q_0)$ is a harmonic function on $\mathcal{D}$, there exists a holomorphic function G on $\mathcal{D}$ such that $\mathrm{Re}(G) = -g$. The holomorphic function e^G satisfies $|e^G| = e^{-g}$. Suppose that $\mathcal{D}'$ is another coordinate disc in $R \setminus \{q\}$ and G' is the corresponding holomorphic function on $\mathcal{D}'$ satisfying $\mathrm{Re}(G') = -g$ on $\mathcal{D}'$. If $\mathcal{D} \cap \mathcal{D}' \neq \phi$, then on the intersection the holomorphic function e^G and $e^{G'}$ satisfy $|\frac{e^G}{e^{G'}}| = 1$ and hence $e^G = e^{i\alpha}e^{G'}$ for some $\alpha \in \mathbb{R}$. The function $e^{i\alpha}e^{G'}$ on $\mathcal{D}'$ satisfies $|e^{i\alpha}e^{G'}| = e^{-g}$ and also agrees with e^G on $\mathcal{D} \cap \mathcal{D}'$. Thus the function f can be continued analytically along any curve in R. R being simply connected, this implies that there exists an analytic function φ on the whole of R which equals f in a neighborhood of q_0 and also satisfies:

$$|\varphi(p)| = e^{-g(p,q_0)},\ \forall\, p \in R$$

ϕ is a non constant holomorphic function on R. Since $g \geq 0$, $|\varphi| \leq 1$ on R. Thus $\varphi : R \to \mathbb{D}$ is a non constant holomorphic function. If we can establish φ to be an injective map, then since image of φ is a simply connected subset of $\mathbb{D}$ it will follow by Riemann mapping theorem that R is conformally equivalent to $\mathbb{D}$. So we will now prove that φ is an injection.

Let $q_1 \in R \setminus \{q\}$. For any $a \in \mathbb{D}$, let ϕ_a denote the conformal self map of $\mathbb{D}$ given by $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$. Now consider the composition $\psi = \phi_{\varphi(q_1)} \circ \varphi$:

$$\psi(p) = \frac{\varphi(p) - \varphi(q_1)}{1 - \overline{\varphi(q_1)}\varphi(p)}$$

ψ is a coordinate map on a neighborhood of q_1 and $\psi(q_1) = 0$. Thus for any $u \in \mathcal{F}_{q_1}$, the Perron family which defines the green's function at q_1, the following function

$$u(p) + \log|\psi(p)|$$

is subharmonic on the whole of R and since $u(p) + \log|\psi(p)| \leq 0$ outside a compact subset of R the following holds:

$$u(p) + \log|\psi(p)| \leq 0\ ,\ \forall p \in R \setminus \{q_1\}$$

$$\therefore g(p, q_1) + \log|\psi(p)| \leq 0\ ,\ \forall p \in R \setminus \{q_1\} \tag{7.3.1}$$

The above inequality for $p = q_0$ yields:

$$g(q_0, q_1) \leq g(q_1, q_0)$$

Since q_0 and q_1 are arbitrary, we also have:

$$g(q_0, q_1) \geq g(q_1, q_0)$$

$$\therefore g(q_0, q_1) = g(q_1, q_0)$$

Thus equality is attained in (7.3.1) for q_0 which is an interior point. Hence by the maximum principle we have:

$$g(p, q_1) + \log|\psi(p)| = 0 \ , \ \forall p \in R \setminus \{q_1\}$$

$$\therefore |\psi(p)| = e^{-g(p,q_1)}$$

The above equation implies that ψ attains the value 0 only for $p = q_1$. This implies that φ takes the value $\varphi(q_1)$ only at q_1. Since we started with an arbitrary q_1, it follows that φ is an injection.

□

7.4. Uniformization Theorem - Part 2

In the previous section we considered Riemann surfaces R for which Green's function exists and making use of the green's function we constructed a conformal map from R to $\mathbb{D}$. Analogous to the green's function, in this section we will construct a bipolar green's function and then later use it construct an analytic map from R to the Riemann sphere $\mathbb{C}^*$.

7.4.1. Bipolar Green's Function.

Definition 7.16. *Let R be a Riemann surface and let q_1, $q_2 \in R$ be distinct points. Let Δ_1 and Δ_2 be disjoint coordinate discs centered at q_1 and q_2 with corresponsing coordinate maps z_1 and z_2 respectively such that $z_1(q_1) = 0$ and $z_2(q_2) = 0$. A harmonic function $G(\cdot, q_1, q_2)$ on $R \setminus \{q_1, q_2\}$ is called a bipolar green's function on R with poles at q_1 and q_2 if it satisfies the following conditions:*

1. *$G(p, q_1, q_2) + log|z_1(p)|$ is harmonic on Δ_1.*
2. *$G(p, q_1, q_2) - log|z_2(p)|$ is harmonic on Δ_2*
3. *$G(p, q_1, q_2)$ is bounded on $R \setminus (\Delta_1 \cup \Delta_2)$.*

We make the following observations about $G(p, q_1, q_2)$:

1. $G(p, q_1, q_2)$ has a logarithmic pole at q_1 and $-G(p, q_1, q_2)$ has a logarithmic pole at q_2. From the discussion in section 7.1, it follows that there exists a holomorphic function f_1 on Δ_1 with a simple zero at q_1 and a holomorphic function f_2 on $\Delta_2 \setminus \{q_2\}$ with a simple pole at q_2 which satisfy:

$$\begin{aligned} |f_1(p)| &= e^{-G(p,q_1,q_2)} \text{ on } \Delta_1 \\ |f_2(p)| &= e^{-G(p,q_1,q_2)} \text{ on } \Delta_2 \end{aligned}$$

(2) If $G(p, q_1, q_2)$ is a bipolar green's function on R with poles at q_1 and q_2 and if h is any bounded harmonic function on R, then $G(p, q_1, q_2) + h(p)$ is also a bipolar green's function on R with poles at q_1 and q_2. Conversely, if G_1 and G_2 are both bipolar green's function on R with poles at q_1 and q_2 then $G_1 - G_2$ is a bounded harmonic function on R.

Let us explicitly calculate a bipolar green's function for a few cases:

(1) Suppose R is a Riemann surface for which green's function exists. Then a bipolar green's function with a pole at q_1 and q_2 is $G(p, q_1, q_2) = g(p, q_1) - g(p, q_2)$.
(2) Consider the Riemann sphere $\mathbb{C}^*$ and points $q_1 = 0$ and $q_2 = \infty$. Then $G(p, 0, \infty) = -\log|z|$. Since any bounded harmonic function on $\mathbb{C}^*$ is a constant, any bipolar green's function for $\mathbb{C}^*$ with a pole at 0 and ∞ is of the form $G(p, 0, \infty) + c$, for some constant c.
(3) Consider the complex plane $\mathbb{C}$. For any distinct points q_1, $q_2 \in \mathbb{C}$, a bipolar green's function with poles at q_1 and q_2 is $G(p, q_1, q_2) = \log\left(\frac{z-q_2}{z-q_1}\right)$.

We will now prove that for any Riemann surface R and points $q_1, q_2 \in R$, there exists a bipolar green's function for R with poles at q_1 and q_2.

Lemma 7.17. *Let S be a finite bordered Riemann surface. Let q_1, $q_2 \in R$ be distinct points. Then there exists $C > 0$ such that for every Riemann surface R for which green's function exists and which is such that $R \supset S \cup \partial S$, the following holds:*

$$|g_R(p, q_1) - g_R(p, q_2)| < C \ , \ \forall p \in R \setminus \{\Delta_1 \cup \Delta_2\}$$

where $\Delta_1 = \{p : |z_1(p)| < \sigma\}$ and $\Delta_2 = \{p : |z_2(p)| < \sigma\}$ are disjoint coordinate discs centered at q_1 and q_2 respectively and z_1 and z_2 are coordinate maps on a neighborhood of q_1 and q_2 respectively with $z_1(q_1) = 0$ and $z_2(q_2) = 0$.

Proof. Let R be any Riemann surface for which green's function exists and which is such that $R \supset S \cup \partial S$. Let $0 < \rho < \sigma$. For $i = 1, 2$ let A_i denote the coordinate disc $A_i = \{p : |z_i(p)| < \rho\}$. Let M_i and N_i be the maximum of $g_R(p, q_i)$ on ∂A_i and $\partial \Delta_i$ respectively. The function $g_R(p, q_i) + \log|z_i(p)|$ is harmonic on Δ_i and for $p \in \partial \Delta_i$ (and hence for every $p \in \Delta_i$), the following inequality holds:

$$g_R(p, q_i) + \log|z_i(p)| < N_i + \log(\sigma)$$

$$\therefore M_i + \log(\rho) < N_i + \log(\sigma)$$

Thus there exists $p_i \in \partial \Delta_i$ such that:

$$M_i + \log(\rho) \le g_R(p_i, q_i) + \log(\sigma)$$

$$\therefore M_i - g_R(p_i, q_i) \leq \log\left(\frac{\sigma}{\rho}\right)$$

The function $M_i - g_R(p, q_i)$ is a positive harmonic function on $S \setminus \{A_1 \cup A_2\}$. Thus by the Harnack's estimate for function $M_i - g_R(p, q_i)$ on the compact set $\partial\Delta_1 \cup \partial\Delta_2$ we have:

$$\frac{1}{K} \leq \frac{M_i - g_R(p, q_i)}{M_i - g_R(q, q_i)} \leq K \ , \ \forall p, q \in \partial\Delta_1 \cup \partial\Delta_2$$

The above inequality with $q = p_i$ becomes:

$$M_i - g_R(p, q_i) \leq K_1 \ , \ \text{for some } K_1 > 0 \text{ and } \forall p \in \partial\Delta_1 \cup \partial\Delta_2$$

Note that the constant K_1 in the above inequality is independent of R. We thus have for every $p \in \partial\Delta_1 \cup \partial\Delta_2$ (and hence on $R \setminus \{\Delta_1 \cup \Delta_2\}$):

$$M_i - K_1 \leq g_R(p, q_i) \leq M_i \tag{7.4.1}$$

The function $g_R(p, q_1)$ is harmonic on Δ_2 and hence by the maximum principle, the above inequality holds on Δ_2 as well, i.e.,

$$M_1 - K_1 \leq g_R(p, q_1) \leq M_1 \ , \ \forall p \in \Delta_2$$

$$\therefore M_1 - K_1 \leq g_R(q_2, q_1) \leq M_1$$

Similarly we also have the following inequality:

$$M_2 - K_1 \leq g_R(q_1, q_2) \leq M_2$$

But since $g_R(q_1, q_2) = g_R(q_2, q_1)$ as seen in the preceding section we obtain:

$$|M_1 - M_2| \leq K_1$$

Thus it follows from equation (7.4.1) that:

$$\therefore |g_R(p, q_1) - g_R(p, q_2)| \leq 2K_1 \ , \ \forall p \in R \setminus \{\Delta_1 \cup \Delta_2\}$$

□

Theorem 7.18. *Let R be any Riemann surface and let q_1, $q_2 \in R$ be any two distinct points. Then there exists a bipolar green's function for R with poles at q_1 and q_2.*

Proof. Let $p_0 \in R \setminus \{q_1, q_2\}$ be an arbitrary point. Consider the Riemann surface $R_n = R \setminus \Delta_n$, where Δ_n is a coordinate disc (for a particular coordinate map z) of radius $1/n$ each of which is centered at p_0 and is contained in $R \setminus \{q_1, q_2\}$. Green's function exists for each R_n. So consider the function $f_n = g_n(p, q_1) - g_n(p, q_2)$, where g_n is the green's function for the surface R_n. Each of the f_n is a bipolar green's function for R_n with poles at q_1 and q_2. The preceding lemma shows that the family $\{f_n\}$ is uniformly bounded and hence by an analogous result of Montel's theorem for harmonic functions, it follows that there exists a convergent subsequence in $\{f_n\}$. The limit function will be a bipolar green's function for R with poles at q_1 and q_2. □

7.4.2. Proof of Uniformization Theorem - Part 2.

Lemma 7.19. *Let R be a Riemann surface and let $f : R \to \mathbb{C}$ be a nonconstant, bounded analytic function. Then the green's function for R exists.*

Proof. Since f is a nonconstant function, there exists $q \in R$ such that the order of zero of the function $h(p) = f(p) - f(q)$ at q is one. Let $M > 0$ be such that $|h| \leq M$. So h defines a coordinate map on a neighborhood of q. We will now show that green's function for R with a logarithmic pole at q exists. For any $u \in \mathcal{F}_q$, the function

$$u(p) + \log|h(p)|$$

is subharmonic on R. Since u vanishes outside a compact subset of R, it follows by the maximum principle for subharmonic functions that:

$$u(p) + \log|h(p)| \leq \log M \ , \ \forall p \in R$$

$$\therefore u(p) \leq \log M - \log|h(p)|, \ \forall p \in R$$

The above equation proves that the green's function for R with a logarithmic pole at q exists.

□

Theorem 7.20 (The Uniformization Theorem)**.**

Proof. [Part-2] We adopt a proof very similar to that of Part-1. Assume that for the Riemann surface R, green's function does not exist. Let q_1, $q_2 \in R$ be distinct points. Let $G(p, q_1, q_2)$ be a bipolar green's function on R with poles at q_1 and q_2. Let f be a holomorphic function on $\Delta_2 \setminus \{q_2\}$ with a simple pole at q_2 and which satisfies:

$$|f(p)| = e^{-G(p,q_1,q_2)} \text{ on } \Delta_2 \tag{7.4.2}$$

Along any curve $\gamma \in R$, there exists an analytic continuation of the function element (f, Δ_2) which satisfies a similar equation as (7.4.2) on the corresponding coordinate disc. Since R is simply connected there exists an analytic function on the whole of R, call it φ, satisfying:

$$|\varphi(p)| = e^{-G(p,q_1,q_2)}, \ \forall p \in R$$

The function φ satisfies the following properties:

(1) $\varphi : R \to \mathbb{C}^*$ is an analytic function.
(2) $\varphi(p) = 0$ iff $p = q_1$.
(3) $\varphi(p) = \infty$ iff $p = q_2$.
(4) There exists $m, M > 0$ such that $m \leq |\varphi| \leq M$ on $R \setminus \{\Delta_1 \cup \Delta_2\}$.

We will now show that φ is an injection. Let $q_0 \in R \setminus \{q_1, q_2\}$ and let φ_0 denote the corresponding analytic function on R which satisfies $|\varphi_0(p)| = e^{-G(p,q_0,q_2)}$. Consider the function ψ defined by:

$$\psi(p) = \frac{\varphi(p) - \varphi(q_0)}{\varphi_0(p)}$$

The function ψ is analytic on R and is bounded in a neighborhood of q_0 and q_2. Since φ is also bounded in a neighborhood of q_1, it follows that ψ itself is bounded on the whole of R. Hence by lemma 7.19, it follows that ψ is non zero constant, K.

$$\therefore \varphi(p) = K\varphi_0(p) + \varphi(q_0)$$

Since $\varphi_0(p) = 0$ iff $p = q_0$, it follows that $\varphi(p) = \varphi(q_0)$ iff $p = q_0$. This holds for all $q_0 \in R \setminus \{q_1, q_2\}$. Hence the function φ is an injection.

$\varphi(R) \subset \mathbb{C}^*$ is a simply connected subset. If $\mathbb{C}^* \setminus \varphi(R)$ contains more than one point, then by Riemann mapping theorem $\varphi(R)$ is conformally equivalent to $\mathbb{D}$. This implies that green's function for R exists, which is a contradiction. Hence $\mathbb{C}^* \setminus \varphi(R)$ contains at the most one element. If it is an empty set, then R is conformally equivalent to $\mathbb{C}^*$. Otherwise if $\mathbb{C}^* \setminus \varphi(R)$ is a singleton then R is conformally equivalent to $\mathbb{C}$. Thus we have proved the Uniformization theorem. $\square$

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