AN INTRODUCTION TO NORMAL NUMBERS

A thesis submitted towards partial fulfillment of
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by

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Certificate

This is to certify that this dissertation entitled "**An Introduction to Normal Numbers**" towards the partial fulfillment of the BS-MS Dual Degree at Indian Institute of Science Education and Research Pune, represents original research carried out by **Rajesh Kumar Yadav** at Harish Chandra Research Institute Allahabad under the supervision of **Dr. Ravindranathan Thangadurai** during the academic year 2010-2011.

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Abstract

This study is about normality of real numbers. In this study we will mainly look at the expansion of real numbers to any integer base $b (b \geq 2)$ and depending on that we will introduce the concept of normality. We will look at frequency of digit strings in the expansion of any real number to an integer base and if all possible digit strings of length k are equally frequent for each k in the former expansion, then we say the number is normal to the base b. While it is generally believed that many familiar irrational constants and algebraic irrationals are normal, normality has been proven only for numbers which are purposefully invented to be normal. In this study we will see different criteria for proving normality and also give an overview of the main results till the date. We will also give the complete proof of Borel's theorem i.e. Almost all real numbers are absolutely normal. Subsequently we will see some examples of normal numbers.

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Chapter 1

Introduction

Émile Borel introduced the concept of normality in 1909 in a paper addressing the question of probability on a countably infinite sequence of trails[2]. Borel proved that almost all real numbers are normal in the sense that the set of real numbers which are not normal is of measure zero. If all possible digit strings of length k are equally frequent for each k in the decimal expansion of the number to the integral base $b(b \geq 2)$, then we say the number is normal to the base b. Below we will give precise definition of normality. Although many worked on normal numbers but there has been very little progress beyond Borel's original work.

Progress has been limited to the discovery or rather the invention of new classes of normal numbers[1, 4, 16]. The very first of which was given by Champernowne and many people tried to give the proof of normality of this number but normality of this number has been proven only to the base 10. Also this number is proved to be transcendental. Below we will discuss about this number elaborately.

But nothing is known about the normality of algebraic irrational number nor of any well-known irrational constant. The only numbers known to be normal have been constructed purposefully for proving their normality. Some people have also tried to discover ways of constructing new normal numbers.

Although experimental evidences strongly suggest that many, if not all familiar irrationals are indeed normal, for example people plotted first 10^5 digits of π as a random walk and by looking at these plots people tried to predict the normality of π. But unless we have a rigorous proof of these type of predictions we can't really say anything.

In this sense, this topic is completely open.

Chapter 2

b-ary expansion of a real number

Let us define the b-ary expansion of a real number.

Definition 2.0.1. A positive b-ary expansion is a series of the form

$$a_0 + \frac{a_1}{b} + \frac{a_2}{b^2} + ... + \frac{a_n}{b^n} + ...$$

denoted by $a_0.a_1a_2...a_n....$, where $b, a_0 \in \mathbb{Z}_+$, $b \geq 2$ and $a_n \in \{0, 1, 2, ..., b-1\}$ for each $n \in \mathbb{N}$.

Theorem 2.0.1. *Every positive b-ary expansion converges to a positive real number and we then say that the b-ary expansion represents the real number.*

Proof. Consider the positive b-ary form $a_0.a_1a_2...a_n....$ Now

$$\begin{aligned} s_1 &= a_0 + \frac{a_1}{b} \leq a_0 + \frac{b-1}{b} \\ s_2 &= a_0 + \frac{a_1}{b} + \frac{a_2}{b^2} \\ &\leq a_0 + \frac{b-1}{b} + \frac{b-1}{b^2} \\ . \quad & \quad . \\ . \quad & \quad . \end{aligned}$$

By induction we have

$$\begin{aligned} s_n &\leq a_0 + \frac{b-1}{b}\left(1 + \frac{1}{b} + ... + \frac{1}{b^{n-1}}\right) \\ &\leq a_0 + \frac{b-1}{b}\left(\frac{1}{1-\frac{1}{b}}\right) = a_0 + 1 \end{aligned}$$

Then $\{s_n\}$ is an increasing sequence of positive reals and is bounded above by $a_0 + 1$. Hence $\{s_n\}$ is convergent to a real number $a \in \mathbb{R}$. (We then say $a_0.a_1a_2...a_n....$ represents a.) $\square$

Now we will prove some theorems about b-ary expansion of positive real numbers and the main idea behind the proofs is based on Nested Interval Theorem. So first let us prove this theorem and subsequently we shall proceed on the same line.

Theorem 2.0.2. (***Nested Interval Theorem***) *Let $I_k \subset \mathbb{R}$ be finite intervals with endpoints a_k and b_k such that*

- $I_{k+1} \subset I_k$ *for each* $k \in \mathbb{N}$
- $lim_{k\to\infty} l(I_k) = lim_{k\to\infty} b_k - a_k \to 0$

Then there exists a unique $c \in \mathbb{R}$ such that $a_k \leq c \leq b_k$ for all k.

Proof. Consider the set $A := \{a_k : k \in \mathbb{N}\}$. This set is nonempty, bounded above by each of b_n. Hence by the least upper bound property of $\mathbb{R}$, there exists $c \in \mathbb{R}$ such that $c = \sup A$. Then $c \leq b_n$, since c is the least upper bound of A and each b_n is an upper bound for A. Also, since c is an upper bound for A, $a_n \leq c$ for all n. Thus, $a_n \leq c \leq b_n$.

Now we will check the uniqueness of c. If d is also such that $a_n \leq d \leq b_n$ for each n, then, $c, d \in [a_n, b_n]$ for each n. From this we conclude that $|c-d| \leq b_n - a_n$ for all n. As $b_n - a_n \to 0$, it follows that $|c-d| = 0$. Hence $c = d$ and thus c is unique. □

The main question here is how can we be sure that there is always a b-ary representation corresponding every positive real number. The next theorem will ensure that in fact we have a b-ary representation for every positive real number.

Theorem 2.0.3. *Every positive real number has a positive b-ary representation.*

Proof. Given a real number $a \geq 0$, we construct a series as follows: Let a_0 be the greatest integer less than or equal to a. Let

a_1 be the greatest integer such that $a_0 + \frac{a_1}{b} \leq a < a_0 + 1$

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a_n be the greatest integer such that $a_0 + \dfrac{a_1}{b} + ... + \dfrac{a_n}{b^n} \leq a < a_0 + \dfrac{a_1}{b} + ... + \dfrac{a_n + 1}{b^n}$

We claim that $a_0.a_1a_2...a_n...$ represents a. For this we need to show that

1. $a_0.a_1a_2...a_n...$ is a b-ary form, i.e., $a_n \in \{0, 1, ..., b-1\}$ for all $n \in \mathbb{N}$

2. $a = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{b^n}$

1. Now $a_0 + \frac{a_1}{b} \leq a$. So $a_1 \geq 0$. Also, $\frac{a_1}{b} \leq a - a_0 < 1$, so $a_1 < b$ or $0 \leq a_1 \leq b-1$. By induction, we find that $0 \leq a_n \leq b-1$.

We prove by induction that we can find integers a_n, $0 \leq a_n \leq b-1$ such that

$$\sum_{i=0}^{n-1} \frac{a_i}{b^i} + \frac{a_n}{b^n} \leq a < \sum_{i=0}^{n-1} \frac{a_i}{b^i} + \frac{a_n + 1}{b^n}$$

This integer is denoted by a_n.

Assume $a_1, ..., a_{n-1}$ are chosen for $n \geq 1$. Let a_n be the greatest integer k such that

$$\sum_{i=0}^{n-1} \frac{a_i}{b^i} + \frac{k}{b^n} \leq a < \sum_{i=0}^{n-1} \frac{a_i}{b^i} + \frac{k+1}{b^n}$$

holds. Then $a_n \geq 0$. Also $a_n < b$. For, otherwise, $a_n \geq b$, so that we can write $a_n = b + {a_n}'$, with ${a_n}' \geq 0$. But then

$$\sum_{i=0}^{n-1} \frac{a_i}{b^i} = a_0 + \frac{a_1}{b} + ... + \frac{a_{n-1}}{b^{n-1}} + \frac{1}{b^{n-1}} + \frac{{a_n}'}{b^n} \leq a.$$

In particular,

$$a_0 + \frac{a_1}{b} + ... + \frac{a_{n-1}+1}{b^{n-1}} \leq a,$$

which contradicts our induction hypothesis on a_{n-1}.

2. $|a - \left(a_0 + \frac{a_1}{b} + ... + \frac{a_n}{b^n}\right)| < \frac{1}{b^n}$ and hence it follows that $a = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{b^n}$. Thus the theorem follows. $\square$

Now we are sure that we have a b-ary expansion for every positive real number. Now we will prove the uniqueness of this b-ary expansion in the next theorem.

Theorem 2.0.4. *Two distinct positive b-ary expansion*

$$a_0 + \sum_{k=1}^{\infty} \frac{a_k}{b^k}$$

and

$$\alpha_0 + \sum_{k=1}^{\infty} \frac{\alpha_k}{b^k}$$

represent the same positive real number if and only if there exists $N \in \mathbb{N}$ such that

1. $a_k = \alpha_k$ *for* $0 \leq k \leq N$.

2. $a_{N+1} = \alpha_{N+1} - 1$ (*or* $\alpha_{N+1} = a_{N+1} - 1$).

3. $a_k = b-1$ *and* $\alpha_k = 0$ *for* $k > N+1$ (*or* $\alpha_k = b-1$ *and* $a_k = 0$ *for every* $k > N+1$).

Proof. Suppose that the two b-ary expansions have the specified properties. Then if

$$s_n := a_0 + \sum_{k=1}^{n} \frac{a_k}{b^k} \quad \text{and} \quad \sigma_n := \alpha_0 + \sum_{k=1}^{n} \frac{\alpha_k}{b^k}$$

then $\sigma_n - s_n \leq \frac{1}{b^n}$ for all n. Let $\sigma_n \to \alpha$. Then $\alpha - \sigma_n \leq \frac{1}{b^n}$ for all n and hence

$$\alpha_n - s_n = \alpha_n - \sigma_n + \sigma_n - s_n \leq \frac{2}{b^n}, \quad \text{for all} \quad n.$$

Hence $s_n \to \alpha$, and so the two b-ary expansions represent the same real number. Conversely, assume that the two b-ary expansions represent the same real number a. We have $a - s_n \leq \frac{1}{b^n}$ and $a - \sigma_n \leq \frac{1}{b^n}$ for all n. This implies $|\sigma_n - s_n| \leq \frac{1}{b^n}$ for all n.

Let $a_k = \alpha_k$ for $0 \leq k \leq n$ for some $n \in \mathbb{Z}_+$. Since $|\sigma_{n+1} - s_{n+1}| \leq \frac{1}{b^{n+1}}$, we have $|\alpha_{n+1} - a_{n+1}| = 0$ or 1. Suppose $a_{n+1} = \alpha_{n+1} - 1$. Then $\sigma_{n+1} = s_{n+1} + \frac{1}{b^{n+1}}$. But $a \leq s_{n+1} + \frac{1}{b^{n+1}} = \sigma_{n+1}$. Therefore $a = \sigma_{n+1}$ and hence $\alpha_k = 0$ for $k > n+1$. Thus $a = s_{n+1} + \frac{1}{b^{n+1}} = \sigma_{n+1}$. To have $a - s_{n+2} \leq \frac{1}{b^{n+2}}$, we require $a_{n+2} = b-1$. For,

$$\begin{aligned}
\frac{1}{b^{n+2}} &\geq a - s_{n+2} \\
&= s_{n+1} + \frac{1}{b^{n+1}} - \left(s_{n+1} + \frac{a_{n+2}}{b^{n+2}}\right) \\
&= \frac{b}{b^{n+2}} - \frac{a_{n+2}}{b^{n+2}} = \frac{b - a_{n+2}}{b^{n+2}}
\end{aligned}$$

Thus, we must have $b - a_{n+2} \leq 1$ or $b-1 \geq a_{n+2} \geq b-1$. Hence $a_{n+2} = b-1$. Also, $a = s_{n+1} + \frac{1}{b^{n+1}} = s_{n+2} + \frac{a_{n+2}}{b^{n+2}} + \epsilon = s_{n+2} + \frac{b-1}{b^{n+2}} + \epsilon$. So, $\epsilon = \frac{1}{b^{n+2}}$. By induction, we have $a_k = b-1$ for $k > n+1$. $\square$

Definition 2.0.2. A sequence $a_1, a_2, ..., a_k, ...$ in integers is called **eventually priodic** if there is an N_0 and $p \in \mathbb{N}$ such that $a_{k+p} = a_k \quad \forall k > N_0$. The smallest such p is called **period**.

Theorem 2.0.5. *A real number $r \in \mathbb{R}$ has an eventually periodic if and only if $r \in \mathbb{Q}$.*

Proof. Suppose $r \in \mathbb{R}$ has an eventually periodic b-ary expansion for some $b \geq 2$.

$$r = b_0 b_1 ... b_k . a_1 a_2 ... a_N .... \quad \text{where } b_i, a_j \in \{0, 1, ..., b-1\}$$

Now by our assumption, $r = b_0 b_1 ... b_{k_1} . a_0 a_1 ... a_{k_2} \overline{c_1 c_2 ... c_{k_3}}$. Now we can expand this as

$$\begin{aligned}
r &= b_0 . b^{k_1} + b_1 . b^{k_1 - 1} + ... + b_{k_1} + \frac{a_0}{b} + \frac{a_1}{b^2} + ... + \frac{a_{k_2}}{b^{k_2}} \\
&+ \frac{1}{b^{k_2}} \left(\frac{c_1}{b} + \frac{c_2}{b^2} + ... + \frac{c_{k_3}}{b^{k_3}} + \frac{c_1}{b^{k_3+1}} + ... + \frac{c_{k_3}}{b^{2k_3}} + ... \right) \\
&= \frac{p}{q} + \frac{1}{b^{k_2}} \left(\left(\frac{c_1}{b} + \frac{c_2}{b^2} + ... \frac{c_{k_3}}{b^{k_3}} \right) + \frac{1}{b^{k_3}} \left(\frac{c_1}{b} + \frac{c_2}{b^2} + ... \right) + ... \right) \\
&= \frac{p}{q} + \frac{p_1}{q_1} \left(1 + \frac{1}{b^{k_3}} + \frac{1}{b^{2k_3}} + ... \right) \\
&= \frac{p}{q} + \frac{p_1}{q_1} . \frac{1}{1 - \frac{1}{b^{k_3}}} \\
&= \frac{p}{q} + \frac{p_1}{q_1} . \frac{b^{k_3}}{b^{k_3} - 1} \in \mathbb{Q}
\end{aligned}$$

where $p, q, p_1, q_1 \in \mathbb{Z}_+$. So we are done with one part.

Now conversely suppose that $r = \frac{p}{q}$, where $p, q \in \mathbb{Z}_+$ and $(p, q) = 1$.

Without loss of generality assume that $p < q$ or $r = \text{integer} + \frac{p}{q}$, where $p < q$.

If $(q, b) \neq 1$, then $q = {q_1}^{r_1} {q_2}^{r_2} ... {q_n}^{r_n} q'$ where q_i's are prime factors of q, r_i's are positive integers and $(q', b) = 1$. Let $t = \max\{r_1, r_2, ..., r_n\}$. Note that $\frac{p}{q}$ is eventually periodic if and only if $b^t \frac{p}{q}$ is eventually periodic.

So without loss of generality we can assume that $(q, b) = 1$, By Euler's theorem, there exists an integer $k \geq 1$ such that

$$b^k \equiv 1 (\text{mod} q)$$

So we can write

$$b^k = 1 + qd \qquad \text{for some} \qquad d \in \mathbb{Z}_+$$

So notice that

$$\begin{aligned} \frac{p}{q} = \frac{pd}{qd} = \frac{pd}{b^k - 1} = \frac{pd}{b^k\left(1 - \frac{1}{b^k}\right)} \\ = \frac{pd}{b^k}\left(1 + \frac{1}{b^k} + \frac{1}{b^{2k}} + ... + ...\right) \end{aligned}$$

Since $pd < qd < b^k$, the number of non-zero digits in the b-ary expansion of $\frac{pd}{b^k}$ is at most k. So $\frac{p}{q}$ has b-ary expansion which is periodic as $(q, b) = 1$ and eventually periodic if $(q, b) \neq 1$. $\square$

Chapter 3

Normal Numbers and some properties

3.1 Interesting Numbers

In the view of describing the theory of normal numbers it will be very useful to introduce the concept of interesting numbers. We can define interesting numbers as follows:

Definition 3.1.1. (**Interesting Number**) A real number α is called *interesting* if for any natural base $b \geq 2$, the base-b representation of α contains every finite pattern of digits $0, 1, 2, ..., b-1$ infinitely many times.

In the previous chapter we defined b-ary expansion of real numbers. We also proved the result that any rational number α in any natural base b has an infinitely repeating pattern of some or all digits $0, 1, ..., b-1$ in their b-ary expansions. No rational number is interesting because it misses many patterns.

Definition 3.1.2. (**Lebesgue measure**) The Lebesgue measure on $\mathbb{R}$ is a set function $\mu : \mathbb{R} \to [0, \infty)$ such that for any interval $I = [a, b]$ in $\mathbb{R}$, we have $\mu(I) = b - a$.

We know that the set of rational numbers is countable, so its Lebesgue measure is zero. So in this sense we can say that all real numbers are irrationals. From the previous chapter we know that any irrational number represented in any natural base $b(b \geq 2)$ has a nonterminating and non-repeating b-ary expansion. Some irrational numbers contains only a finite number of some of the digits in their b-ary expansion. This has been proven that these irrational numbers make a set of Lebesgue measure zero[11]. Hence, almost all real numbers when written in base b have an infinite number of every digit $0, 1, \ldots, b-1$. Therefore almost all real numbers are interesting numbers.

Remark: We stated the above result about interesting numbers since this is analogous to the result proved by Émile Borel in 1909 about normal numbers. We will give complete proof of Borel's result in the coming chapters.

3.2 Definition of Normality and some properties

Let b be an integer and $b \geq 2$. For a given real number α, there is a unique b-ary expansion of the form

$$\alpha = [\alpha] + \sum_{n=1}^{\infty} a_n b^{-n} \tag{3.2.1}$$

where $[x]$ denotes integer part of x, $0 \leq a_n < b$, and $a_n \neq 0$ infinitely often. This second condition on a_n i.e., $a_n \neq 0$ is to ensure the unique representation of certain rational numbers.

Notation: For a fixed real number α, we write $A(d, b, N)$ to denote the number of occurrences of the integer d in the set $\{a_1, a_2, .., a_N\}$ with the a_n given by (3.2.1).

Now we will introduce some basic definitions based on the above notation,

Definition 3.2.1. (Simply Normal) We call a real number α to be simply normal in the base b if

$$\lim_{N\to\infty}\frac{A(d,b,N)}{N}=\frac{1}{b}$$

for every d with $0 \leq d \leq b-1$.

Finally we give the definition of normality which is as follows;

Definition 3.2.2. (Absolutely Normal) A real number is called absolutely normal if it is simply normal to all the bases b^n where $n = 1, 2, 3, \ldots.$ for every base b greater than 1.

Definition 3.2.3. (Entirely Normal) A real number is called *entirely normal to base b* if it is simply normal to all bases $b^n, n = 1, 2, \ldots.$

We notice that it is very easy to produce a simply normal number to any given base. For example, 0.0123456789 in the base 10 is simply normal. Further we will see that almost all real numbers are absolutely normal, which will be the main focus of this thesis. This result was first proved by Borel in 1909.

Here we would give a criterion to decide **Non-normality** of a number, which will give us the motivation to prove some further results.

By the definition of simply normal number we see that if α is not a simply normal number, then there is some $\epsilon > 0$ for which

$$\left|\frac{A(d,b,N)}{N}-\frac{1}{b}\right|>\epsilon$$

for infinitely many N. We will first prove some elementary properties of normality to the given base. Let us start with a theorem.

Theorem 3.2.1. *[10] Let b and n both be integers ≥ 2. Then, if α is simply normal to base b^n, it is simply normal to the base b.*

Proof. Write $c = b^n$. Since α is simply normal to the base b^n,

$$\frac{A(d, c, N)}{N} = \frac{1}{c} + o(1) \tag{3.2.2}$$

for all d with $0 \leq d \leq c - 1$. We notice that every integer d between 0 and $c - 1$ can be written in the form

$$\sum_{j=0}^{n-1} c_j(d) b^j \quad \text{with} \quad 0 \leq c_j(d) \leq b - 1. \tag{3.2.3}$$

Now assume that

$$\{\alpha\} = \sum_{m=1}^{\infty} a_m b^{-m} = \sum_{r=1}^{\infty} d_r c^{-r}.$$

where $\{x\}$ denotes the fractional part of x.

Now we can rewrite the above equation using equation (3.2.3) as follows:

$$\{\alpha\} = \sum_{m=1}^{\infty} a_m b^{-m} = \sum_{r=1}^{\infty} \left(\sum_{j=0}^{n-1} c_j(d_r) b^j \right) b^{-nr}$$

By the above expression we can conclude that if a is given, with $0 \leq a \leq b-1$, then the integer a appears exactly k times among the numbers a_m with $tn+1 \leq m \leq (t+1)n$ if and only if the equation

$$c_j(d_t) = a$$

has exactly k solutions. Now we write

$$D(k) = \{d : 0 \leq d \leq c - 1, c_j(d) = a \ \text{ has } k \text{ solutions}\}.$$

Clearly $D(k)$ has $\binom{n}{k}(b - 1)^{n-k}$ members since we have chosen k values out of n and for each of the remaining $n - k$ positions there are $b - 1$ possibilities. Let N be a positive integer. We then have

$$\begin{aligned}\frac{A(a,b,Nn)}{Nn} &= \sum_{k=0}^{n}\frac{k}{n}\sum_{d\in D(k)}\frac{A(d,c,N)}{N}\\ &= \sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}(b-1)^{n-k}(c^{-1}+o(1))\\ &= \sum_{r=0}^{n-1}\binom{n-1}{r}(b-1)^{n-1-r}(c^{-1}+o(1))\\ &= \frac{1}{b}+o(1)\end{aligned}$$

by equation (3.2.2) and binomial theorem. Because

$$A(a,b,N+r)-A(a,b,N)<r,$$

moreover, let $N=nM+r$ where $r<n$, then

$$\begin{aligned}\frac{A(a,b,N)}{N} &= \frac{A(a,b,nM+r)}{nM+r}<\frac{A(a,b,nM+r)}{nM}\\ &< \frac{A(a,b,nM)}{nM}+\frac{r}{nM}<\frac{A(a,b,nM)}{nM}+\frac{1}{M}\end{aligned}$$

as $N\to\infty$, $nM\to\infty$ and $M\to\infty$. So we get

$$\lim_{N\to\infty}\frac{A(a,b,N)}{N}=\frac{1}{b},$$

Since this holds for each a, we conclude that α is simply normal to the base b. □

Corollary 3.2.2. *The statements 'α is simply normal to all bases b^n, $n=1,2,\ldots.$' and 'α is simply normal to all base b^{nt}, $t=1,2,\ldots.$ and $n\in\mathbb{Z}_+$' are equivalent.*

Proof. It is very much clear from the statement itself. Both the statements imply each other clearly. □

Remark.. We notice that the theorem (3.2.1) is not true in the reverse direction. As an example, 0.0123456789 is simply normal to base 10 but not simply normal in base 100. Borel calls a number **entirément normal** if each of $\alpha, \alpha b, \alpha b^2, ...$ is simply normal to every base $b, b^2, b^3, ...$ From the definition of entirely normal number we can see that entirement normal and entirely normal are equivalent. The next theorem which we will prove, was first demonstrated by S.S.Pillai[14, 15].

Theorem 3.2.3. *[10] If α is simply normal to all bases b^n, $n = 1, 2, \ldots,$ then each of $\alpha, \alpha b, \alpha b^2, ...$ is simply normal to every base $b^n, b^{2n}, b^{3n}, ...$*

Proof. We only need to show that αb is simply normal to all bases b^n,where $n = 1, 2, ...$ because once we show that αb is simply normal to all bases b^n, from the corollary it will be simply normal to every base $b^n, b^{2n}, b^{3n}, ..$ and rest will follow from the theorem itself. We shall write $A^*(d, c, N)$ to denote the number of occurrences of digit d among the numbers $a_1, a_2, ..., a_N$ where

$$b\alpha = [b\alpha] + \sum_{j=1}^{\infty} a_j c^{-j}$$

and $A(d, c, N)$ for the corresponding function for α. Since multiplying by b just shifts one digit of b-ary expansion of α to the right. We have $|A^*(d, b, N) - A(d, b, N)| \leq 1$. Since α is simply normal to all bases b^n, $n = 1, 2, ...$, αb is simply normal to base b. Now we shall prove that αb is simply normal to every base $b^j (j \geq 2)$ by using the fact that α is simply normal to every base b^{jr} with r arbitrary large.
Now let a be an integer between 0 and $b^j - 1$, and suppose that r is a positive integer. Given an integer d with $0 \leq d \leq b^{jr} - 1$, define $g_m = g_m(d)$ by

$$bd - [db^{1-jr}]b^{jr} = \sum_{m=0}^{r-1} b^{mj} g_m, \quad 0 \leq g_m \leq b^j \tag{3.2.4}$$

Here we can notice that when integer d is between 0 and $b^{jr-1}-1$, we can write

$$db = \sum_{m=0}^{r-1} g_m b^{mj}, \quad 0 \le g_m \le b^{j-1}$$

and when $b^{jr-1} \le d \le b^{jr}$, calculate x such that $db - xb^{jr} < b^{jr}$, then we can see that this x is nothing but $[bd^{1-jr}]$ which verifies the significance of equation (3.2.4). We define $D(k)$ to be the set

$$D(k) = \{d : 0 \le d \le b^{jr}-1, g_m = a \ \text{ has exactly } k \text{ solutions for } \ m \ge 1\}$$

We note that the cardinality of $D(k)$ is

$$\binom{r-1}{k} b^j (b^j-1)^{r-k-1}.$$

It follows that the definition of $D(k)$ that

$$A^*(a, b^j, Nr) \ge \sum_{k=0}^{r-1} k \sum_{d \in D(k)} A(d, b^{jr}, N)$$

Hence, since α is simply normal to base b^{jr},

$$\begin{aligned} \frac{A^*(a, b^j, Nr)}{Nr} &\ge \sum_{k=0}^{r-1} \frac{k}{r} \binom{r-1}{k} b^j (b^j-1)^{r-k-1} (b^{-jr} + o(1)) \\ &= (1-1/r) b^{-j} (1+o(1)). \end{aligned}$$

Thus for any integer M we have

$$\frac{A^*(a, b^j, M)}{M} \ge (1-1/r) b^{-j} (1+o(1)).$$

It follows that

$$\operatorname{liminf}_{N\to\infty} \frac{A^*(a, b^j, N)}{N} \ge b^{-j}(1-1/r). \tag{3.2.5}$$

Since this holds for every a and

$$\sum_{a=0}^{b^j-1} \frac{A^*(a, b^j, N)}{N} = 1$$

we must also have

$$\text{limsup}_{N\to\infty}\frac{A^*(a,b^j,N)}{N} \le b^{-j}(1+1/r) \quad \text{for all} \;\; a. \tag{3.2.6}$$

For otherwise, suppose for some a we have

$$\text{limsup}_{N\to\infty}\frac{A^*(a,b^j,N)}{N} > \frac{1}{b^j}\left(1+\frac{1}{r}\right)$$

Then

$$\begin{aligned}\text{limsup}_{N\to\infty}\sum_{a=0}^{b^j-1}\frac{A^*(a,b^j,N)}{N} &> \sum_{a=0}^{b^j-1}\frac{1}{b^j}\left(1+\frac{1}{r}\right) \\ &> \frac{b^j}{b^j}\left(1+\frac{1}{r}\right) = \left(1+\frac{1}{r}\right)\end{aligned}$$

which is a contradiction.

Since (3.2.5) and (3.2.6) hold for arbitrary large r, we get

$$\text{liminf}_{N\to\infty}\frac{A^*(a,b^j,N)}{N} \ge b^{-j} \quad \text{and} \quad \text{limsup}_{N\to\infty}\frac{A^*(a,b^j,N)}{N} \le b^{-j}$$

Since

$$\text{liminf}_{N\to\infty}\frac{A^*(a,b^j,N)}{N} < \text{limsup}_{N\to\infty}\frac{A^*(a,b^j,N)}{N}$$

We get

$$\text{liminf}_{N\to\infty}\frac{A^*(a,b^j,N)}{N} = \text{limsup}_{N\to\infty}\frac{A^*(a,b^j,N)}{N} = \text{lim}_{N\to\infty}\frac{A^*(a,b^j,N)}{N} = b^{-j}$$

we may deduce that

$$\text{lim}_{N\to\infty}\frac{A^*(a,b^j,N)}{N} = b^{-j}, \quad \text{for} \;\; 0 \le a \le b^j - 1.$$

We thus completes the proof of the fact that αb is simply normal to every base b^j. $\square$

Now we will introduce another definition of normality, given for example by Hardy and Wright[9], Niven and Zuckerman[12]. which is as follows:

Definition 3.2.4. (Normality of a number) Given a block B_k of k digits in base b, we then write $A(B_k, b, N)$ for the number of occurrences of B_k in the block of digits $a_1, a_2, .., a_N$. We call a number *normal to base b* if

$$\lim_{N\to\infty}\frac{A(B_k, b, N)}{N} = b^{-k} \quad \text{for all} \quad k \geq 1, \quad \text{and all} \quad B_k.$$

The next theorem shows that this is equivalent to the definition we chosen for an entirely normal number. Before proving the theorem we will prove one very important lemma which will give us a criterion to decide non-normality. If α is not a normal number, then there is some $\epsilon > 0$ for which

$$\left|\frac{A(d, b, N)}{N} - \frac{1}{b}\right| > \epsilon$$

On the line of the above equation we will proceed towards a lemma.

Let $b \geq 2$ be an integer. Let k be an integer such that $k \geq b + 2$. Let B_k denote the block of k-digits, say, $B_k = a_n a_{n+1} ... a_{n+k-1}$ where $0 \leq a_i \leq b - 1$. Let d be a given digit in the base b such that $0 \leq d \leq b - 1$.

Let n_d denote the number of times the given digit d occurs in B_k. Since the total number of digits in B_k is k, the expected number of times d occurs in B_k is $\frac{k}{b}$ i.e. the average. The following lemma computes the number of different B_k's such that if the difference between the expected number and actual number is $\geq \epsilon k$, then the number of different such blocks $B_k \leq \epsilon b^k$. Note that the number of distinct blocks B_k is b^k.

Lemma 3.2.4. *The number of distinct blocks B_k for which $|n_d - \frac{k}{b}| \geq \epsilon k$ holds is at most ϵb^k for every $k > k_0(\epsilon)$ or $k \gg 0$.*

Proof. First consider those blocks B_k for which

$$\begin{aligned} \frac{k}{b} - n_d &\geq \epsilon k \\ \Rightarrow \quad n_d &\leq \frac{k}{b} - \epsilon k = \left(\frac{1}{b} - \epsilon\right) k \end{aligned}$$

The number of such blocks is

$$\sum_{0 \leq j < k(\frac{1}{b} - \epsilon)} \binom{k}{j} (b-1)^{k-j} = Y \text{ (say)}$$

and we know

$$\sum_{j=0}^{k} \binom{k}{j} (b-1)^{k-j} = b^k$$

But we can notice that

$$\begin{aligned} \frac{\binom{k}{j}(b-1)^{k-j}}{\binom{k}{j+1}(b-1)^{k-j-1}} &= \frac{\frac{k!}{j!(k-j)!}}{\frac{k!}{(j+1)!(k-j-1)!}} \\ &= \frac{(j+1)(b-1)}{k-j-1} \end{aligned}$$

Put $j = \frac{k}{b} - \theta$. Then we get

$$\frac{(j+1)(b-1)}{k-j-1} = \frac{(\frac{k}{b} - \theta + 1)(b-1)}{k - \frac{k}{b} + \theta - 1}$$

Since $j = \frac{k}{b} - \theta \in \mathbb{Z}$, we see that $\theta \in \mathbb{Q}$.

Now we notice that

$$\begin{aligned} \left(\frac{k}{b} - \theta + 1\right)(b-1) &= k - \theta b + b - \left(\frac{k}{b} - \theta + 1\right) \\ &= k + b - \theta - \left(\frac{k}{b} - \theta + 1\right) \end{aligned}$$

So we have

$$\begin{aligned} \frac{(\frac{k}{b} - \theta + 1)(b-1)}{k - (\frac{k}{b} - \theta + 1)} &= \frac{k - \theta b + b - (\frac{k}{b} - \theta + 1)}{k - (\frac{k}{b} - \theta + 1)} \\ &= \frac{b(1-\theta)}{k - (\frac{k}{b} - \theta + 1)} + 1 \end{aligned}$$

Now we want to prove that

$$\begin{aligned}
\frac{b(1-\theta)}{k-(\frac{k}{b}-\theta+1}+1 &< 1-\frac{b(\theta-1)}{k} \\
\Leftrightarrow\ kb\theta+k+(\theta-1)b(\theta-1) &< 2kb
\end{aligned}$$

Since $\theta < \frac{k}{b}$ and $k\epsilon < \theta$; we see that $k\epsilon - 1 < \theta - 1$ and L.H.S. is much smaller than R.H.S.

Moreover,

$$\begin{aligned}
1-\frac{b(\theta-1)}{k} &< 1-\frac{b(k\epsilon-1)}{k} < 1-\epsilon b+\frac{b}{k} \\
\Rightarrow 1-\frac{b(\theta-1)}{k} &< 1-\frac{\epsilon b}{3}
\end{aligned}$$

After getting the above inequality we can write

$$\binom{k}{j}(b-1)^{k-j} < \left(1-\frac{\epsilon b}{3}\right)\binom{k}{j+1}(b-1)^{k-j-1} \quad \forall j \tag{3.2.7}$$

We know that

$$Y = \sum_{0\le j<\frac{k}{b}-k\epsilon}\binom{k}{j}(b-1)^{k-j} \quad \text{and} \quad b^k = \sum_{0\le j\le k}\binom{k}{j}(b-1)^{k-j}$$

Other than the terms in Y, we have $k-\frac{k}{b}+\epsilon k$ terms in b^k.

Moreover

$$b^k \ge \sum_{j=k-\frac{\epsilon k}{2}}^{k-\frac{k}{b}+\frac{\epsilon k}{2}}\binom{k}{j}(b-1)^{k-j}$$

So we have

$$\begin{aligned}
\left(1-\frac{\epsilon b}{3}\right)^{\frac{\epsilon k}{2}} b^k &\geq \left(1-\frac{\epsilon b}{3}\right)^{\frac{\epsilon k}{2}} \sum_{j=k-\frac{\epsilon k}{2}}^{k-\frac{k}{b}+\frac{\epsilon k}{2}} \binom{k}{j}(b-1)^{k-j} \\
&= \sum_{j=k-\frac{\epsilon k}{2}}^{k-\frac{k}{b}+\frac{\epsilon k}{2}} \left(1-\frac{\epsilon b}{3}\right)^{\frac{\epsilon k}{2}} \binom{k}{j}(b-1)^{k-j} \\
&\geq \sum_{j=0}^{\frac{k}{b}-k\epsilon} \binom{k}{j}(b-1)^{k-j} \quad \text{By} \quad (3.2.7) \\
&= Y
\end{aligned}$$

The estimate in equation (3.2.7) is valid for large k's.

The number of blocks B_k satisfying $\frac{k}{b} - n_d \geq \epsilon k$ is less than or equal to $(1-\frac{\epsilon b}{3})^{\frac{\epsilon k}{2}} b^k$.

Now we have to prove that $(1-\frac{\epsilon b}{3})^{\frac{\epsilon k}{2}} < \epsilon \;\; \forall \;\; k > k_0(\epsilon)$.

Since $(1-\frac{\epsilon b}{3})^{\frac{\epsilon k}{2}} \to 0$ as $k \to \infty$, this is indeed true for all $k > k_0(\epsilon)$.

So finally we can conclude that the number of blocks B_k satisfying $\frac{k}{b} - n_d \geq \epsilon k$ is at most ϵb^k.

Analogously one can do the same calculations for the other case which is $n_d - \frac{k}{b} > \epsilon k$.

Thus the lemma is proved. $\square$

Now we will prove a theorem which shows the equivalence of the definition (3.2.4) of normality and our definition of entirely normal numbers. We will use the above lemma to prove this theorem.

Suppose E and F are two blocks of digits to base b and F is not shorter than E, say,

$$F = a_1 a_2 \ldots a_k a_{k+1} \ldots a_{k+s} \quad \text{and} \quad E = b_1 b_2 \ldots b_k$$

then $R_j(E, F)$, where $1 \leq j \leq k$, denotes the number of times E occurs in F with the first digit of E appears in the position $h \equiv j(\text{mod}k)$. For instance,

$$\begin{aligned} F &= a_1a_2...a_ka_{k+1}...a_{k+h}...a_{k+s} \\ &= a_1a_2...a_ka_{k+1}...a_{2k}a_{2k+1}...a_{3k}....a_{lk+m} \text{ where } s = lk+m \text{ and } m < k \end{aligned}$$

then

$$F = a_1...a_ka_{k+1}...a_{k+j-1}b_1b_2...b_k.....$$

if E appears in F with the first digit at $k+j \equiv j(\text{mod}k)$.

Theorem 3.2.5. *[10] A real number α is entirely normal to the base b if and only if*

$$lim_{N\to\infty}\frac{A(B_k,b,N)}{N} = b^{-k} \quad \textit{for all} \quad k \geq 1, \quad \textit{and all} \quad B_k.$$

Proof. Suppose that α is entirely normal to base b and let B_k be given block of k digits. Therefore from the definition we know that each of $\alpha, b\alpha, ..., b^{k-1}\alpha$ is simply normal to base b^k. Now we can write B_k as a single digit, say d, to base b^k. The occurrence of the block B_k as

$$a_{kr+j+1}...a_{kr+k+j},$$

for some j with $0 \leq j \leq k-1$, is then equivalent to the digit d being in the $(r-1)$th position in the expansion of $b^j\alpha$ to base b^k. Write $A_j(d,b^k,N)$ for the number of occurrences of d among the first N digits in the expansion of $b^j\alpha$ to base b^k. Then

$$\begin{aligned} \frac{A(B_k,b,Nk)}{Nk} &= \frac{1}{k}\sum_{j=0}^{k-1}\frac{A_j(d,b^k,N)}{N} \\ &= b^{-k}(1+o(1)), \end{aligned}$$

since $b^j\alpha$ is simply normal to base b^k. Thus

$$\lim_{N\to\infty}\frac{A(B_k,b,N)}{N} = b^{-k}$$

and this establish the only if part of the proof.

If B_k denotes a block of k digits of α in base b and $A(B_k, b, N)$ denotes the number of times B_k appears in the first N digits of α. Assume that

$$\lim_{N\to\infty}\frac{A(B_k,b,N)}{N} = \frac{1}{b^k} \ \ \forall \ \ k \geq 1 \ \text{ and } \ \forall \ \ B_k.$$

We need to prove that

$$\lim_{N\to\infty}\frac{A(d,b^r,N)}{N} = \frac{1}{b^r} \ \ \forall \ \ r \geq 1 \ \ \forall \ \ d \ \text{ digit in base } \ b^r.$$

since r is arbitrary, it will follow that α is entirely normal to base b. Notice that when $r = 1$, the assertion is trivial by our assumption, take $k = 1$ and $B_1 = d$ is a given digit in base b.

$$\frac{1}{b} = \lim_{N\to\infty}\frac{A(B_1,b,N)}{N} = \lim_{N\to\infty}\frac{A(d,b,N)}{N}$$

So assume that $r > 1$. Let d be a digit in base b^r. Then we can write d as a block of r digits to base b as $B = b_1b_2...b_r$. Let $\mathbb{B} = \mathbb{B}(s,\epsilon)$ be the set of all blocks H of $s(s > r)$ digits to the base b such that

$$\max_j|R_j(B,H) - \frac{\frac{s}{r}}{b^r}| \geq \epsilon s.$$

By Lemma (3.2.4), with given d and $\epsilon > 0$, the number of different blocks H of $\frac{s}{r}$ digits to the base b^r(or in other words H of s digits to the base b) for which

$$\left|n_d - \frac{s}{rb^r}\right| \geq \epsilon s$$

is less than or equal to $\epsilon b^{\frac{rs}{r}} = \epsilon b^s$ for every $s > s_0(\epsilon)$, here n_d denotes the number of times d occurs in H. That is, we have $|\mathbb{B}| \leq \epsilon b^s \forall s > s_0(\epsilon)$

Let $s = \max(s_0, \frac{r}{\epsilon})$ and let $H \in \mathbb{B}$ of s digit block. Then $A(H, b, N)$ denotes the number of times the block H appears in first N digits of α. By our assumption,

$$\lim_{N\to\infty} \frac{A(H, b, N)}{N} = \frac{1}{b^s}$$

which is equivalent to say that

$$A(H, b, N) = \frac{N}{b^s} + o(N) \qquad \text{for all sufficiently large N's}$$

So we can write

$$\begin{aligned}
\sum_{H\in\mathbb{B}} A(H, b, N) &= \sum_{H\in\mathbb{B}} \left(\frac{N}{b^s} + o(N)\right) \\
&= \frac{N}{b^s}|\mathbb{B}| + o(N|\mathbb{B}|) \\
&\leq \frac{N}{b^s}\epsilon b^s + o(N) \\
&= \epsilon N + o(N) \leq 2\epsilon N \quad \text{for all sufficiently large N's.}
\end{aligned}$$

We now write D_t for the block of digits $a_t \dots a_{t+s-1}$ in the expansion of α in base b. Let

$$S(N) = \{t : D_t \in \mathbb{B}, t \leq N - s + 1\}$$

Then from above we can see that $|S(N)| \leq 2\epsilon N$ for all large N. Suppose that N is a multiple of r. For a given j such that $1 \leq j \leq r$, suppose $R_j(B, D_t) \geq 1$ i.e.

$$b_1 = a_t, b_2 = a_{t+1}, \dots, b_r = a_{t+r-1}$$

has solution. Then

$$R_j(B, D_{t+a}) \geq 1 \quad \text{for all} \quad a = 0, -1, -2, \dots, -(s-r),$$

this is because of the reason that,

$$\begin{aligned} D_t &= b_1 b_2 \ldots . b_r a_{t+r+1} \ldots . a_{t+s-1} \\ D_{t-1} &= a_{t-1} b_1 b_2 \ldots . b_r a_{t+r+1} \ldots . a_{t+s-2} \\ D_{t-2} &= a_{t-2} a_{t-1} b_1 b_2 \ldots . b_r a_{t+r+1} \ldots . a_{t+s-3} \\ &. \\ &. \\ D_{t-(s-r)} &= a_{t-s+r} a_{t-s+r+1} \ldots . b_1 b_2 \ldots . b_r \end{aligned}$$

which implies that

$$R_j(B, D_t) \geq 1, \ldots, R_j(B, D_{t-s+r}) \geq 1.$$

For a fixed j, $1 \leq j \leq r$, one solution to

$$b_1 = a_t, b_2 = a_{t+1}, ..., b_r = a_{t+r-1}$$

give rise to $(s-r+1)$ counting to $\sum_{t=1}^{N-s+1} R_j(B, D_t)$.

If $R_j(B, D_t) \geq 1$, then $A(d, b^r, N/r) \geq 1$ as $B = b_1 b_2 \ldots . b_r = d$ in the base b^r (by assumption).

So $(s-r+1)A(d, b^r, N/r)$ counts d at least $(s-r+1)$ times. Then

$$|(s-r+1)A(d, b^r, N/r) - \sum_{t=1}^{N-s+1} R_j(B, D_t)| \leq 2s^2. \tag{3.2.8}$$

We already know that $S(N) \leq 2\epsilon N$ and for any $t \in S(N)$, we have

$$o \leq R_j(B, D_t) \leq s-r+1 \;\; \forall \;\; 1 \leq j \leq r$$

For $t \notin S(N)$ we have

$$|R_j(B, D_t) - \frac{s/r}{b^r}| \leq \epsilon s.$$

since for $t \in S(N)$ we had inequality otherwise. That is to say that

$$R_j(B, D_t) \leq \epsilon s + \frac{s}{rb^r} = s\left(\epsilon + \frac{1}{rb^r}\right) \tag{3.2.9}$$

and $|t \leq N - s + 1/t \notin S(N)| = N - s + 1 - |S(N)| \geq N - s + 1 - 2\epsilon N$.

We need to prove that

$$\left|\frac{A(d, b^r, N/r)}{N/r} - \frac{1}{b^r}\right| < \delta \quad \text{for any} \quad \delta > 0$$

which is equivalent to prove

$$\left|A(d, b^r, N/r) - \frac{N/r}{b^r}\right| < \frac{N\delta}{r} \quad \text{for any} \quad \delta > 0$$

So consider

$$\begin{aligned}
\left|A(d, b^r, N/r) - \frac{N/r}{b^r}\right| &= \left|A(d, b^r, N/r) - \sum_{j=1}^{r}\sum_{t=1}^{N-s+1}\frac{R_j(B, D_t)}{s-r+1} + \sum_{j=1}^{r}\sum_{t=1}^{N-s+1}\frac{R_j(B, D_t)}{s-r+1} - \frac{N}{rb^r}\right| \\
&\leq \left|A(d, b^r, N/r) - \sum_{j=1}^{r}\sum_{t=1}^{N-s+1}\frac{R_j(B, D_t)}{s-r+1}\right| + \left|\sum_{j=1}^{r}\sum_{t=1}^{N-s+1}\frac{R_j(B, D_t)}{s-r+1} - \frac{N}{rb^r}\right| \\
&\leq \frac{2s^2}{s-r+1} + \left|\sum_{j=1}^{r}\left(\sum_{t\in S(N)}\frac{R_j(B, D_t)}{s-r+1} + \sum_{t\notin S(N)}\frac{R_j(B, D_t)}{s-r+1}\right) - \frac{N}{rb^r}\right| \\
&\leq \frac{2s^2}{s-r+1} + \sum_{j=1}^{r}\sum_{t\in S(N)}\frac{R_j(B, D_t)}{s-r+1} + \left|\sum_{j=1}^{r}\sum_{t\notin S(N)}\frac{R_j(B, D_t)}{s-r+1} - \frac{N}{rb^r}\right| \\
&\leq \frac{2s^2}{s-r+1} + \frac{s-r+1}{s-r+1}\sum_{j=1}^{r}|S(N)| + \frac{1}{s-r+1}\sum_{j=1}^{r}\left|\sum_{t\notin S(N)} R_j(B, D_t) - \frac{N}{rb^r}\right|
\end{aligned}$$

Form above calculation and equation (3.2.9) we have

$$\left|A(d, b^r, N/r) - \frac{N/r}{b^r}\right| \leq \frac{2s^2}{s-r+1} + 2\epsilon rN + \frac{N/r}{b^r}\left(\frac{s}{s-r+1} - 1\right)$$

for all large N. Moreover,

$$\begin{aligned}
\left|\frac{A(d, b^r, N/r)}{N/r} - \frac{1}{b^r}\right| &\leq 4\epsilon r + \frac{2s^2}{s-r+1}.\frac{r}{N} + \frac{1}{b^r}.\frac{r-1}{s-r+1} \\
&\leq 1 - \epsilon \;\; \forall \;\; N \gg 0
\end{aligned}$$

Since ϵ was arbitrary, this gives us

$$\lim_{N\to\infty} \frac{A(d, b^r, N)}{N} = \frac{1}{b^r}.$$

This establishes our proof. $\square$

Chapter 4

The first Borel-Cantelli lemma and Borel's Theorem

In this chapter we will prove Borel's theorem which is one of the most important theorem in the theory of normal numbers. In 1909 Borel proved that almost all real numbers are absolutely normal. We will prove this theorem using first Borel-Cantelli lemma and results proven in the earlier section.

Now let us first prove Borel-cantelli lemma.

Lemma 4.0.6. ***(The first Borel-Cantelli lemma)****[10] Let X be a measurable space with measure μ. Let $A_j (j = 1, 2, ...)$ be a collection of measurable subsets of X. Then, if*

$$\sum_{j=1}^{\infty} \mu(A_j) < \infty,$$

then almost all elements of X (with respect to μ) belong to only finitely many of the A_j.

Proof. Notice that any subset of X, say ζ, which belongs to infinitely many of the A_j

may be written as

$$\zeta = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j$$

which is equivalent to write

$$\zeta = \text{limsup } A_j.$$

Since a countable intersection of a countable union of measurable sets is measurable, ζ is a measurable set. Therefore we have

$$\mu(\zeta) \leq \mu(\bigcup_{j=n}^{\infty} A_j) \text{ for every } n \geq 1.$$

Hence

$$\mu(\zeta) \leq \sum_{j=n}^{\infty} \mu(A_j) \text{ for every } n \geq 1.$$

Since $\sum_{j=1}^{\infty} \mu(A_j) < \infty$, for a given $\epsilon > 0, \exists$ N such that $\sum_{j=n}^{\infty} \mu(A_j) < \epsilon \; \forall \quad n \geq N$. Thus $\mu(\zeta) = 0$, as required. □

Now we will prove our main theorem i.e. Borel's Theorem.

Theorem 4.0.7. *[10]Almost all real numbers are absolutely normal.*

It is enough to show that almost all real numbers are simply normal to any natural base $b \geq 2$. Let $\zeta(b)$ denote the set of real numbers which are not simply normal to the base b. Then the set of real numbers which are not absolutely normal will be

$$\bigcup_{b=2}^{\infty} \zeta(b)$$

If we prove $\mu(\zeta(b)) = 0$ for every $b \geq 2$, then

$$\mu\left(\bigcup_{b=2}^{\infty} \zeta(b)\right) = 0$$

as required to establish the theorem.

Proof. Let d be an integer between 0 and $b-1$, and, for $\epsilon > 0$, write $\mathbb{B}(N, \epsilon)$ for the set of all blocks B_N of N digits such that

$$\left| n_d - \frac{N}{b} \right| \geq \epsilon N \Leftrightarrow \left| \frac{n_d}{N} - \frac{1}{b} \right| \geq \epsilon$$

where n_d denotes the number of times the given digit d occurs in B_N.

From lemma (3.2.4) we have

$$|\mathbb{B}(N, \epsilon)| \leq \epsilon b^N$$

since ϵ is given, $\exists \quad \rho = \rho(\epsilon) > 0$ such that $e^{-\rho N} < \epsilon \quad \forall \quad N \geq N_0$. So we have

$$|\mathbb{B}(N, \epsilon)| \ll b^N e^{-\rho N}$$

Now let

$$X(N, \epsilon) = \bigcup_{H \in \mathbb{B}(N, \epsilon)} \left[\frac{H}{b^N}, \frac{H+1}{b^N} \right)$$

where H is here regarded as a number written in scale b. Therefore

$$\begin{aligned} \mu(X(N, \epsilon)) &\leq \sum_{H \in \mathbb{B}(N, \epsilon)} \mu \left(\left[\frac{H}{b^N}, \frac{H+1}{b^N} \right) \right) \\ &= \frac{1}{b^N} |\mathbb{B}(N, \epsilon)| \ll e^{-\rho N} \end{aligned}$$

We then have

$$\sum_{N \geq N_0} \mu(X(N, \epsilon)) \ll \sum_{N \geq N_0} e^{-\rho N} << \infty$$

Let $E(\epsilon)$ be the subset of $[0, 1)$ belonging to $X(N, \epsilon)$ for infinitely many $N \geq N_0$. Then, by Lemma (4.0.6), we have $\mu(E(\epsilon)) = 0$.

Therefore by choosing $\epsilon = \frac{1}{2^r}$ for $r = 1, 2, ...$, we see that $\mu\left(E\left(\frac{1}{2^r}\right)\right) = 0 \quad \forall \quad r = 1, 2, ...,$. Hence the set

$$\zeta = \bigcup_{r=1}^{\infty} E(2^{-r})$$

has measure zero. Now, if α is not a normal number, then there is some $\epsilon > 0$ for which

$$\left|\frac{A(d, b, N)}{N} - \frac{1}{b}\right| > \epsilon$$

for infinitely many N. Hence α belongs to $E(2^{-r})$ for some r, and thus to ζ. Since ζ has measure zero, we conclude that almost all $\alpha \in [0, 1)$ and hence almost all real α are normal. This concludes the proof of Borel's theorem. $\square$

Chapter 5

Non-normal numbers are uncountable

Borel proved that almost all real numbers are absolutely normal i.e. the set of non-normal numbers is of measure zero. Wall[17] noted that even if the set of non normal numbers is of measure zero, it is uncountably infinite set. Wall considered Liouville numbers $\alpha = \sum_{j=0}^{\infty} a_j b^{-j!}$(where a_j can be taken from any finite set of integers, not necessarily $\{0, 1, ..., b-1\}$) as an example of a set of numbers which are uncountable and can be proven not normal. Now we will give a direct proof of the fact that set of non normal numbers is uncountable. Our proof is based on Cantor diagonalisation argument, so lets first introduce this concept.

5.1 Cantor diagonalisation technique

This technique was introduced by George Cantor to prove that there exists infinite sets which can not be put into one to one correspondence with the set of natural numbers. Such sets are called uncountable sets. Consider an infinite sequence of the form $(t_1, t_2, t_3, \ldots.)$, where each t_i is an infinite sequence of 0's and 1's. Now we consider an uncountable listing of t_i's, for instance take it as

$$\begin{aligned} t_1 &= (0,0,0,0,0,0,0....) \\ t_2 &= (1,1,1,1,1,1,0,...) \\ t_3 &= (0,1,0,1,0,1,0,...) \\ t_4 &= (1,0,1,0,1,0,1,...) \\ .... \end{aligned}$$

For each m and n, let $t_{n,m}$ be the mth element of the nth sequence on the list, so for each n,

$$t_n = (t_{n,1}, t_{n,2}, t_{n,3}, ...).$$

Now we build a sequence t_0 in such a way that its first element is different from the first element of t_1, its second element is different from the second element of t_2 and in general, its nth element is different from the nth element of t_n. So if $t_{m,m}$ is 1, then $t_{0,m}$ is 0, otherwise $t_{0,m}$ is 1. So for instance:

$$t_0 = (1, 0, 1, 1, ...)$$

We can see that this sequence t_0 is distinct from all the sequences in the above list. This is because of the fact that, if t_0 is same as t_n, then $t_{0,n} = t_{n,n}$ which is contradiction due to construction of t_0.

Therefore any set T consisting all the sequences of zeros and ones can not be put into a countable listing. In other words, T is uncountable.

Now we will prove our main theorem.

Theorem 5.1.1. *In any base b, there are uncountably many non-normal numbers.*

Proof. Let T be any countably infinite set of numbers which are not normal in the base b. Then we can index T by integers, so T is a sequence α_j with

$$\alpha_j = .\alpha_{1,j}\alpha_{2,j}\alpha_{3,j}...(\text{mod } 1).$$

We write α_j with a tale of zeros for simplicity.

We construct the number $\beta = .\beta_1\beta_2\beta_3....$ by setting

$$\begin{aligned} \beta_{2j-1}\beta_{2j} &= 01 \quad \text{if } \alpha_{2j-1,j}\alpha_{2j,j} \neq 00 \\ \beta_{2j-1}\beta_{2j} &= 00 \quad \text{otherwise} \end{aligned}$$

Here ab denotes a 2-string, not a product.

The number β is not normal to the base b, since it contains no 2-strings other than 01 and 00. But we can guarantee by Cantor Diagonalisation technique that β is different from each element of T. So no countable set contains all the members of T. Therefore the set of non normal numbers is uncountable. □

Chapter 6

Example of Normal and Non-normal numbers

6.1 Numbers Proven to be not Normal

We will give some examples of numbers whose non-normality can be proved. The Liouville numbers of the form $\alpha = \sum_{j=0}^{\infty} a_j b^{-j!}$, are not normal in the base b because there are too many zeros in the b-ary expansion of α.

It is very easy to construct non normal numbers. In the b-ary expansion, write down digits in such a way that either there are too many or too few of some specific digit. As an example, we can look at the following number

$$\alpha = .10110111011110.....$$

It is very easy to check that above number is not normal in any base as it contains too many 1's in its expansion.

6.2 Numbers Proven to be Normal

Champernowne[4] produced an example of a normal number in 1933 which was proven to be normal to base 10. The Chamernowne number is

$$\alpha = .123456789101112......$$

α is written in base 10, and we can construct this number by concatenating the natural numbers written in the base 10. This number is the very first and probably the best known example of normal number. Now we will give a complete proof of Champernowne Number's normality to the base 10.

Theorem 6.2.1. *[13] The Chamernowne number i.e.* $C = 0.1234567...$ *is normal to the base* 10.

Proof. Let B_n denote the block consisting of first n-digits of C. We will think of B_n as partitioned into blocks corresponding to the natural numbers. So we will write

$$B_n = 1, 2, 3, 4, ..., 10, 11, ..., d_1 d_2 ... d_m, ... \tag{6.2.1}$$

The last complete natural number in this partitioning of B_n is assumed to have digits $d_1 d_2 ... d_m$ and if we denote this number by w, we have

$$w = d_1 10^{m-1} + d_2 10^{m-2} + ... + d_m 10^0 \quad ; d_1 \neq 0$$

Notice that in equation (6.2.1) there are at most m digits after the last comma, and, since there are at most $w + 1$ partitions, we can see that

$$n \leq m(w + 1) \tag{6.2.2}$$

It will be convenient to define the numbers

$$w_j = [w10^{-j}] = d_1 10^{m-1-j} + d_2 10^{m-2-j} + ... + d_{m-j}; \quad \text{where } j = 1, 2, .., m-1.$$

Our main concern is to estimate the number of occurrences of B_k in B_n, where $B_k = b_1 b_2 ... b_k$. In making this estimate we count only the occurrences of B_k inside the partitions which make up, ignoring occurrences that straddle the commas therein. We look for blocks of digits in equation (6.2.1) having the form

$$y_1 y_2 ... y_s b_1 b_2 ... b_k z_1 z_2 ... z_t = Y_s B_k Z_t \tag{6.2.3}$$

there being at most m digits in any partition in equation (6.2.1), we require that $s + k + t \leq m$. Moreover, the natural number corresponding to the block of digits equation (6.2.3) must not exceed w, and this will guarantee if we require that

$$y_1 10^{s-1} + y_2 10^{s-2} + ... + y_s < d_1 10^{m-k-t-1} + d_2 10^{m-k-t-2} + ... + d_{m-k-t} = w_{k+t}$$

in view of this, we have $w_{k+t} - 1$ possible values for the block Y_s in equation (6.2.3), and 10^t possible values for the t digits of Z_t. Thus the number of blocks of digits of the form equation (6.2.3) to be found in equation (6.2.1) is at least

$$\sum_{t=0}^{m-k-1} 10^t (w_{k+t} - 1)$$

where we get the maximum value $m - k - 1$ for t by setting $s = 1$ in the inequality $s + k + t \leq m$.

Now let $N(B_k, B_n)$ denote the total number of occurrences of B_k in equation (6.2.1) and so we have

$$N(B_k, B_n) \geq \sum_{t=0}^{m-k-1} 10^t (w_{k+t} - 1)$$

By definition of w_j we have $w_j > w10^{-j} - 1$, and so

$$\begin{aligned} N(B_k, B_n) &> \sum_{t=0}^{m-k-1} 10^t (w10^{-k-t} - 2) \\ &= \sum_{t=0}^{m-k-1} w.10^{-k} - \sum_{t=0}^{m-k-1} 2.10^t \\ &> (m-k)w.10^{-k} - 10^{m-k}. \end{aligned}$$

Dividing the left hand side by n and the right hand side by $m(w+1)$ we see by equation (6.2.2) that

$$\begin{aligned} \frac{1}{n} N(B_k, B_n) &> \frac{m-k}{m}.\frac{w}{w+1}.10^{-k} - \frac{10^{m-k}}{w+1}.\frac{1}{m} \\ &= 10^{-k} - 10^{-k} \left\{ \frac{1}{w+1} + \frac{w}{w+1}.\frac{k}{m} \right\} - \frac{10^{m-k}}{w+1}.\frac{1}{m} \end{aligned}$$

Notice that

$$\frac{10^{m-k}}{w+1} \leq \frac{10^{m-1}}{w+1} \leq \frac{w}{w+1} < 1$$

Also we see that, as $n \to \infty$, $w \to \infty$, $m \to \infty$, $\frac{k}{m} \to 0$.

Hence for any given $\epsilon > 0$ we can choose n sufficiently large so that

$$\frac{1}{n} N(B_k, B_n) > 10^{-k} - \epsilon.$$

and we have

$$\text{liminf}_{n\to\infty} \frac{1}{n} N(B_k, B_n) \geq 10^{-k}$$

This holds for all blocks B_k of k digits. So we can conclude that

$$\lim_{n\to\infty} \frac{1}{n} N(B_k, B_n) = 10^{-k}$$

Thus our proof is completed. $\square$

Remark: We can also very easily prove that Champernowne number is irrational. For, suppose its rational, then its expansion to base 10 will be periodic. Let $a_1 a_2 ... a_k$ be the repeating sequence in the expansion. We will have 10^{2k} somewhere in the expansion which is obvious from the way it is constructed. So now in 10^{2k} we will have consecutive $2k$ zeros and we would not have the repeating sequence among these zeros, which is a contradiction to our assumption of rationality. Hence **Champernowne number is irrational**.

Champernowne made the following conjecture:

Conjecture.(Champernowne) The number which is constructed by concatenation of prime numbers i.e.,

$$\alpha = .23571113.....$$

is normal to the base 10.

This conjecture was later proved by Copeland and Erdös[6] in 1946 as a corollary of a more general result, which is as follows:

Theorem 6.2.2. *Let $\{a_j\}$ be an increasing sequence of integers such that for any $\theta < 1$ we have, for N sufficiently large, that the number of a_j less than N is greater than N^θ. Then if the $\{a_j\}$ are written in the base b, the number $.a_1 a_2 a_3 ...$ is normal in the base b.*

The proof of this theorem is based on the concept of (k, ϵ) normality, which we will introduce in the later section.

The corollary follows from the prime number theorem. Prime number theorem implies that for any $c < 1$, if $\pi(N)$ is the number of primes not greater than N, $\pi(N) > \frac{cN}{logN}$ if N is large enough. Since $\frac{cN}{logN} > N^\theta$ for any $\theta < 1$, for sufficiently large values of N, the sequence of prime numbers satisfies the condition of the Copeland Erdös theorem.

With this theorem in hand, the normality of Champernowne's number follows as a trivial corollary.

In the paper on the concatenated primes, Copeland and Erd$\ddot{o}$s made the following conjecture:

Conjecture. If $q(x)$ is a polynomial in x which takes positive values when x takes a positive integer value, then the number

$$.q(1)q(2)q(3)...$$

formed by concatenating the values of the polynomial at $x = 1, 2, 3....$ to the base 10 is normal in base 10.

This conjecture was later proved by Davenport and Erd$\ddot{o}$s[7] in 1952.

6.3 Numbers expected to be Normal

Borel introduced the concept of normality in 1909. There is an explicit relationship between Borel's concept of normality and randomness in the digits of a number. In fact, Borel was thinking of numbers arising from random sequencing of digits. let us see how one can build such a number in base 10, by drawing one of the 10 balls from 0 to 9 out of a box, recording the digit on the ball. Then replace the ball and continue forever. Then with probability 1 such a number is normal to the base 10, Borel proved this fact.

A number like Champernowne number or number formed by concatenating primes in base 10 are highly patterned.

On the other side, till today nobody has been able to find out any pattern in the expansions of irrational constants like π, e, and $\sqrt{2}$ in whatever base we take. Although some people performed statistical tests which were consistent with the random behavior. But this is no evidence to say anything about the normality of these numbers. So normality of such irrational constants is still an open question.

Chapter 7

Uniform distribution and Normal numbers

In this chapter we will introduce the concept of uniform distribution and later we will prove Weyl's theorem which gives us a criterion to decide whether an infinite sequence of real numbers is uniformly distributed.

7.1 Uniform distribution in the unit interval

Let S be a finite set of real numbers $\alpha_1, \alpha_2, ..., \alpha_Q$ contained in the unit interval $[0, 1)$, that is

$$0 \leq \alpha_j \leq 1, \qquad 1 \leq j \leq Q$$

Definition 7.1.1. Given any pair of real numbers a, b such that $0 \leq a < b \leq 1$, we define an *Interval function* $\phi(a, b)$ which is equal to the number of α's which are contained in the interval $[a, b)$, that is those numbers α_j for which we have

$$a \leq \alpha_j < b \quad , 1 \leq j \leq Q$$

Definition 7.1.2. The *Discrepency* of the set S denoted by D is defined as

$$D = \sup_{a,b} \left| \frac{\phi(a,b)}{Q} - (b-a) \right| \tag{7.1.1}$$

Clearly, $0 < D \leq 1$. If we denote the interval $[a, b)$ by I and its length by $|I|$ and write $\phi(I)$ for $\phi(a, b)$, then the above definition can be rewritten as

$$D = \sup_{I \subset [0,1)} \left| \frac{\phi(I)}{Q} - |I| \right|$$

Given an infinite sequence of real numbers $\alpha_1, \alpha_2, ...$ in the interval $[0, 1)$, we denote D_n the discrepancy of the first n terms of the sequence. We say a sequence (α_j) is **Uniformly distribution**, if $D_n \to 0$ as $n \to \infty$.

Let $\phi_n(a, b) = \phi(I)$ be the number of α_j's with $a \leq \alpha_j < b$ and $1 \leq j \leq n$. It follows from the definition that if sequence (α_j) is uniformly distributed in $[0, 1)$, then clearly

$$\frac{\phi_n(a,b)}{n} \to (b-a) \tag{7.1.2}$$

as $n \to \infty$, for each pair of real numbers a, b such that $0 \leq a < b \leq 1$. But the converse is also true i.e. if equation (7.1.2) holds for each such interval $[a, b)$, then the sequence (α_j) is uniformly distributed.

Theorem 7.1.1. *[5] An infinite sequence of real numbers (α_i), $i = 1, 2, ...$, such that $0 \leq \alpha_i < 1$, is uniformly distributed, if and only if*

$$\frac{\phi_n(a,b)}{n} \to (b-a),$$

as $n \to \infty$, for each pair of real numbers a and b, such that $0 \leq a < b \leq 1$. Here $\phi_n(a, b)$ equals the number of α_j, such that $a \leq \alpha_j \leq b$, and $1 \leq j \leq n$.

Proof. For the interval $[0,1)$ can be split up into a finite number of subintervals (I_k), say each of length δ, $0<\delta<1$. Now given any interval $[c,d)$, where $0\le c<d\le 1$, let r denote the number of intervals (I_k), each of length δ, which lie in the interior of $[c,d)$. Their total length is $r\delta$ and we have $r\delta > (d-c)-2\delta$. If r' denotes the number of intervals I_k which intersect $[c,d)$, then $r'\delta < (d-c)+2\delta$.
Since (7.1.2) holds for each interval $[a,b)$, it holds, in particular, for an interval I_k of length δ. Thus given $\epsilon>0$, there exists a number $N(\epsilon)$, such that

$$\delta-\epsilon \le \frac{\phi_n(I_k)}{n} \le \delta+\epsilon,$$

for all $n>N(\epsilon)$ and all k. If we choose $\epsilon=\delta^2$, we get

$$(1-\delta)\delta \le \frac{\phi_n(I_k)}{n} \le (1+\delta)\delta,$$

for all $n>N'(\delta)$. Hence

$$r\delta(1-\delta) \le \frac{1}{n}\sum_{I_k\subset[c,d)} \phi_n(I_k) \le \frac{\phi_n(c,d)}{n} \le \frac{1}{n}\sum_{I_k\cap[c,d)\neq\phi} \phi_n(I_k) \le r'\delta(1+\delta),$$

for all $n>N'(\delta)$, which implies that

$$((d-c)-2\delta)(1-\delta) \le \frac{\phi_n(c,d)}{n} \le ((d-c)+2\delta)(1+\delta),$$

since $d-c\le 1$, it follows that

$$\left|\frac{\phi_n(c,d)}{n}-(d-c)\right| \le 3\delta+2\delta^2,$$

for $n>N'(\delta)$, for any interval $[c,d)\subset[0,1)$, with δ independent of the interval. This implies that $D_n\to 0$ as $n\to\infty$. $\square$

7.2 Uniform distribution modulo 1

An infinite sequence of real numbers (α_j), not necessarily contained in the unit interval, is said to be *uniformly distributed modulo 1*, if the corresponding sequence of fractional parts $(\{\alpha_j\})$ is uniformly distributed in the sense already defined as above. Thus, if D_n is the discrepancy,as defined in the previous section,of the first n terms of the sequence $(\{\alpha_j\})$, then $D_n \to 0$ as $n \to \infty$. We shall see this condition has an alternative, but equivalent, formulation in terms of a new notion of *discrepancy modulo 1.* Given any set S of real numbers $\alpha_1, \alpha_2, ..., \alpha_Q$, let T denote the set of real numbers $(\alpha_k + t)$, where $1 \leq k \leq Q$ and t runs through all integers. Given any pair of real numbers a and b, such that $b \geq a$, let $\phi^*(a, b)$ denote the number of elements of T, which are contained in the interval $[a, b)$. Then

$$\phi^*(a+t, b+t) = \phi^*(a, b) \tag{7.2.1}$$

for any integer t. Further

$$\phi^*(a, b) = \phi(a, b), \qquad \text{if} \quad 0 \leq a < b \leq 1 \tag{7.2.2}$$

where $\phi(a, b)$ is defined, as in the previous section, for $(\{\alpha_k\})$, $1 \leq k \leq Q$.

Definition 7.2.1. (**Discrepancy modulo 1**) The *discrepancy modulo* 1 of the set S is defined to be D^*, where

$$D^* = \sup_{0 \leq b-a \leq 1} \left| \frac{\phi^*(a, b)}{Q} - (b - a) \right| \tag{7.2.3}$$

Here a runs through all real numbers, but in the view of equation (7.2.1), we may assume that $0 \leq a < 1$.

Theorem 7.2.1. *[5] An infinite sequence of real numbers (α_j) is uniformly distributed modulo 1, if and only if $D_n^* \to 0$ as $n \to \infty$, where D_n^* is the discrepancy modulo 1 of the first n terms of the sequence (α_i).*

Proof. If D is the discrepancy of the fractional parts of the numbers in S, we have trivially $D \leq D^*$, because of equation (7.1.1), (7.2.2) and (7.2.3). On the other hand, we also have $D^* \leq 2D$, since any interval $[a, b)$, where $0 \leq a < 1$ and $b - a \leq 1$, is disjoint union of at most two intervals each of which is of the form $[a', b')$, where either $0 \leq a' < b' \leq 1$, or $1 \leq a' < b' \leq 2$. Thus

$$\phi^*(a,b) = \sum \phi^*(a',b'), \qquad b - a = \sum (b' - a'),$$

where the sum $\sum$ extends over at most two terms. Hence

$$\left| \frac{\phi^*(a,b)}{Q} - (b-a) \right| \leq \left| \frac{\phi^*(a',b')}{Q} - (b'-a') \right| \leq 2D,$$

because of equation (7.1.1), (7.2.1) and (7.2.2) and of the fact there are at most two terms in $\sum$. Therefore $D^* \leq 2D$.

Thus given a set S of real numbers (α_j), $1 \leq j \leq Q$, we have defined first the discrepancy D of their fractional parts, and secondly D^*, their discrepancy modulo 1, and the two are connected by the inequalities

$$D \leq D^* \leq 2D \tag{7.2.4}$$

If (α_j) is an infinite sequence of real numbers, not necessarily contained in the unit interval, let D_n denote the discrepancy of the first n terms of the corresponding sequence of the fractional parts $(\{\alpha_i\})$, while D_n^* denotes the discrepancy modulo 1. It follows from equation (7.2.4) that if $D_n \to 0$ as $n \to \infty$, then $D_n^* \to 0$ as $n \to \infty$, and conversely. $\square$

Now we will prove the Weyl's theorem which is as follows:

Theorem 7.2.2. *(**Weyl's Theorem**)[5] If (α_j) is an infinite sequence of real numbers, such that $0 \leq \alpha_< 1$, for $j = 1, 2, ...,$ a necessary and sufficient condition for (α_j) to be uniformly distributed is that*

$$lim_{n\to\infty}\frac{1}{n}\sum_{h=1}^{n} f(\alpha_h) = \int_0^1 f(x)dx, \tag{7.2.5}$$

for every function f which is Riemann integrable in $0 \leq x \leq 1$.

Proof. We may assume f to be real valued for otherwise we can consider the real and imaginary parts separately.

The sufficiency of condition (7.2.5) for the sequence (α_j) to be uniformly distributed is easy to prove. Given any interval $[a, b)$ such that $0 \leq a < b \leq 1$, we take f to be the characteristic function of $[a, b)$ i.e. $f(x) = 1$ if $a \leq x < b$, while $f(x) = 0$ otherwise. Then

$$\frac{1}{n}\sum_{h=1}^{n} f(\alpha_h) = \frac{\phi_n(a,b)}{n} \tag{7.2.6}$$

while $\int_0^1 f(x)dx = b - a$. Condition (7.2.5) therefore implies that

$$\lim_{n\to\infty}\frac{\phi_n(a,b)}{n} = b - a, \tag{7.2.7}$$

which by theorem (7.1.1), implies that the sequence (α_j) is uniformly distributed.

Conversely, if (α_j) is uniformly distributed, then equation (7.2.7) holds, so that equation (7.2.5) holds for the characteristic function f on any interval $[a, b)$ contained in $[0, 1]$, and because of linearity, condition (7.2.5) holds also for any step function in $[0, 1]$. If f is Riemann integrable in $[0, 1]$, then given $\epsilon > 0$, one can find two step functions f_1, f_2 such that $f_1 \leq f \leq f_2$, and $\int_0^1 (f_2(x) - f_1(x))dx < \epsilon$. Since condition

(7.2.5) holds for f_1, we have

$$\lim_{n\to\infty} \sum_{h=1}^{n} f_1(\alpha_h) = \int_0^1 f_1(x)dx \geq \int_0^1 f(x)dx - \epsilon,$$

so that, if n is sufficiently large,

$$\frac{1}{n}\sum_{h=1}^{n} f_1(\alpha_h) > \int_0^1 f(x)dx - 2\epsilon.$$

Since $f \geq f_1$, it follows that

$$\frac{1}{n}\sum_{h=1}^{n} f(\alpha_h) > \int_0^1 f(x)dx - 2\epsilon,$$

for sufficiently large n. Similarly we get

$$\frac{1}{n}\sum_{h=1}^{n} f(\alpha_h) < \int_0^1 f(x)dx + 2\epsilon,$$

for sufficiently large n. Thus

$$\left|\frac{1}{n}\sum_{h=1}^{n} f(\alpha_h) - \int_0^1 f(x)dx\right| < 2\epsilon,$$

for sufficiently large n, which proves (7.2.5) for every Riemann integrable function in $[0, 1]$. $\square$

Theorem 7.2.3. *[5] If (β_j) is an infinite sequence of real numbers, not necessarily contained in the unit interval, a necessary and sufficient condition for (β_j) to be uniformly distributed modulo 1 is that*

$$lim_{n\to\infty}\frac{1}{n}\sum_{h=1}^{n} e^{2\pi i m\beta_h} = 0, \tag{7.2.8}$$

for every integer $m \neq 0$, where $i^2 = -1$.

Proof. Let (β_j) be uniformly distributed modulo 1, and let α_j denote the fractional part of β_j. Then (α_j) is uniformly distributed in the unit interval. If in theorem (7.2.2) we take $f(x) = e^{2\pi imx}$, where m is an integer, and $m \neq 0$, it follows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{h=1}^{n} e^{2\pi im\alpha_h} = \int_0^1 e^{2\pi imx}dx = 0,$$

which is the same as equation (7.2.8), since α_h differs from β_h by an integer. Conversely, if equation (7.2.8) holds for every integer $m \neq 0$, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{h=1}^{n} e^{2\pi im\alpha_h} = 0,$$

and we shall show that condition (7.2.5) is satisfied for every Riemann integrable function in $[0,1]$. Obviously (7.2.5) holds for $f(x) = 1$, and it holds, by our hypothesis, for $f(x) = e^{2\pi imx}$, where m is an integer and $m \neq 0$. Hence it holds also for any trigonometric polynomial of the form

$$a_0 + (a_1\cos 2\pi x + b_1\sin 2\pi x) + ... + (a_m\cos 2\pi mx + b_m\sin 2\pi mx),$$

where the a's and b's are constants. Now any continuous periodic function f, of period 1, can be approximated by a trigonometric polynomial of that kind. That is, given $\epsilon > 0$, there exists a trigonometric polynomial f_ϵ, such that

$$|f - f_\epsilon| < \epsilon.$$

Set $f_1 = f_\epsilon - \epsilon$, and $f_2 = f_\epsilon + \epsilon$, so that $f_1 \leq f \leq f_2$, and $\int_0^1 (f_2(x) - f_1(x))dx = 2\epsilon$. As in the proof of theorem (7.1.1), it follows that (7.2.5) holds for any continuous periodic function of period 1. Confining attention to the basic interval $[0,1]$, for any step function f in $[0,1]$ we can find two continuous periodic functions f_1, f_2 such that $f_1 \leq f \leq f_2$, and $\int_0^1 (f_2(x) - f_1(x))dx < \epsilon$. Hence (7.2.5) holds for a step function f

in $[0,1]$, which implies, as before, that it holds for any Riemann integrable function in $[0,1]$. This completes the proof of theorem. $\square$

As an application of above theorem we have

Theorem 7.2.4. *[5] If ζ is any irrational number, the the infinite sequence $(n\zeta)$, $n = 1, 2, ...$ is uniformly distributed modulo 1.*

Proof. Let m be an integer different from zero. Set $m\zeta = \eta$. We wish to show that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{h=1}^{n} e^{2\pi i h\eta} = 0.$$

As η is real, but not integral, since ζ is irrational, we have

$$\left|\sum_{h=1}^{n} e^{2\pi i h\eta}\right| = \left|\frac{e^{2\pi i(n+1)\eta - 2\pi i\eta}}{2\pi i\eta - 1}\right| \leq \frac{2}{|2\pi i\eta - 1|} = \frac{1}{|\sin\pi\eta|},$$

so that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{h=1}^{n} e^{2\pi i h\eta} = 0.$$

$\square$

There is a connection between normal numbers and the concept of uniform distribution modulo 1, which can be given by the following theorem.

Theorem 7.2.5. *[13] The number α is normal to the base b if and only if the number $\alpha, b\alpha, b^2\alpha, ...$ are are uniformly distributed modulo 1.*

Proof. Suppose first that $\alpha, b\alpha, b^2\alpha, ...$ are uniformly distributed modulo 1. Let the b-ary expansion of the fractional part of α be $.\alpha_1\alpha_2\alpha_3...$ to the base b, and let $B_k = b_1b_2...b_k$ be any block of k digits. We must prove that B_k occurs in $.\alpha_1\alpha_2\alpha_3...$ with frequency b^{-k}. Let I denote the open interval of y satisfying

$$.b_1b_2...b_k < y < b^{-k} + .b_1b_2...b_k,$$

so that I has length b^{-k}. The decimal expansion to base b of every such number y begins with the digits $b_1b_2...b_k$ i.e. B_k.

Next, if the fractional part $(b^m\alpha)$ of any number $b^m\alpha$ has b-ary expansion beginning with $.b_1b_2...b_k$, then $(b^m\alpha)$ belongs to the interval I. For the only other possibility are that $(b^m\alpha)$ is one of the end points of the closure of the interval, such as $(b^m\alpha) = .b_1b_2...b_k$. But these two possibilities are clearly ruled out by the hypothesis. It follows from the definition of uniform distribution that

$$\lim_{n\to\infty}\frac{1}{n}N(B_k, X_n) = \lim_{n\to\infty}\frac{n(I)}{n} = b^{-k},$$

where $n(I)$ denotes the number of those points $(\alpha), (b\alpha), ..., (b^{n-1}\alpha)$ which lie in the interval I.

Conversely, suppose that α is normal to base b. For any positive integer m, divide the unit interval into b^m closed subintervals,

$$(0, b^{-m}), (b^{-m}, 2b^{-m}), (2b^{-m}, 3b^{-m}), ..., (1 - b^{-m}, 1). \tag{7.2.9}$$

Note that $(b^j\alpha)$, being irrational for every integer j, is not the end point of any of these intervals. The normality of the number α implies that the points $(\alpha), (b\alpha), ...$ are distributed with equal frequency in these intervals. That is to say, if R denotes any one of the subintervals (7.2.9), and if $n(R)$ denotes the number of those points $(\alpha), (b\alpha), ..., (b^{n-1}\alpha)$ which lie in R, then

$$\lim_{n\to\infty}\frac{n(R)}{n} = b^{-m}. \tag{7.2.10}$$

Next, let I be any subinterval of the unit interval, say $I = (c, d)$ with $0 \leq c < d \leq 1$. We are not assuming anything about the closure properties of I, so that c and d may or may not belong to I. Let R_1 be the collection of those intervals (7.2.9) that lies entirely inside I. Thus the length of R_1 is at least $d - c - 2b^{-m}$. Hence, given any $\epsilon > 0$, we see by (7.2.10) that, for all n sufficiently large,

$$\frac{n(I)}{n} \geq \frac{n(R_1)}{n} \geq d - c - 2b^{-m} - \frac{\epsilon}{2}.$$

If we choose m large enough so that $b^{-m} < \frac{\epsilon}{4}$, then we can conclude that

$$\frac{n(I)}{n} \geq d - c - \epsilon$$

for n sufficiently large.

Similarly, if we write R_2 for the collection of those intervals (7.2.9) that have one or more points in common with I, we see that the length of R_2 is less that $d - c - 2b^{-m}$. Also we conclude that

$$\frac{n(I)}{n} \leq \frac{n(R_2)}{n} \leq d - c + 2b^{-m} + \frac{\epsilon}{2} \leq d - c + \epsilon$$

for all n sufficiently large. Since ϵ can be made arbitrary small, we conclude that

$$\lim_{n \to \infty} \frac{n(I)}{n} = d - c,$$

and theorem is proved. $\square$

The above theorem can be used to prove such a proposition as: if α is normal to base b, so is $j\alpha$ for any non-zero integer j. This result holds for any rational $j \neq 0$, but the proof of this stronger statement is more difficult and require some work.

Chapter 8

Normality in case of integers

8.1 Integer analogue of normality

The original concept of normality introduced by Borel deals with the b-ary expansion of the real numbers and for checking the normality we look at the limiting frequency of the digits in the fractional part of the real numbers.

Besicovitch studied the same digit pattern in case of integers. Any integer can have only finitely many digits in its expansion to any base, so it is absurd to talk about limiting frequency of the digits in case of integers. But we can certainly look at the relative frequencies of the digits and digit strings in the base b expansion of integers. Besicovitch introduced the following definition for a natural number n expressed in base b.

$$n = a_\lambda a_{\lambda-1} ... a_1 a_0$$

so the a_j are the digits of n in the base b. Now we will define some new type of normalities.

Definition 8.1.1. (**ϵ- Normal**) The number n is ϵ-normal if the frequency of each digit $\{0, 1, 2..., b-1\}$in the expansion of n differs from $1/b$ by less than ϵ.

Definition 8.1.2. (**(k,ϵ)- Normal**) The number n is (k, ϵ) normal if the frequency of each k digit string in the expansion of n differs from $1/b^k$ by less than ϵ.

The definition of ϵ-normality of integers with respect to some base b, is incisively analogous to the definition of simple normality of the real numbers and the concept of (k, ϵ)-normality is loosely analogous to simple normality of real numbers with respect to base b^k.
We can further extend the ideas to the next level with the following definition.

Definition 8.1.3. A natural number n is (k, ϵ)-normal upto N if it is (k, ϵ)-normal for every positive integer $k \leq N$.

We will not develop any further theory in case of integers but we can notice that this is presumptively the closest analogy of normality one can give in case of integers. Besicovitch[3] proved that almost all integers are ϵ-normal and (k, ϵ)-normal for any choice of k and ϵ. He also proved that almost all square of integers are ϵ-normal. As we can see this is analogous to Borel's result that almost all real numbers are absolutely normal.

8.2 Hanson's construction of normal number

Hanson made an interesting bridge between the concepts of normality and (k, ϵ)-normality.
Hanson's result gives us a way to construct normal numbers. We will give the complete proof of Hanson's result here.

Theorem 8.2.1. *[8] Let $\{a_n\}$ be an increasing sequence of positive integers having the property that, for any given k and any given $\epsilon > 0$, almost all a_n are (k, ϵ)-normal* *in the scale b. Let ν_i denote the number of digits in $a_i (i = 1, 2, 3, ..)$ and let*

$$S_n = \sum_{i=1}^{n} \nu_i$$

Then a sufficient condition that the number $x = .a_1a_2a_3...$ be normal in the base b is that

$$n\nu_n = O(S_n). \tag{8.2.1}$$

Proof. Let $b_1b_2...b_k$ be any sequence of k digits of the base b. Let m be an integer and let n be such that $S_n \leq m \leq S_{n+1}$. Let $N_m(x, b_1b_2...b_k)$ denote the number of occurrences of $.b_1b_2...b_k$ in the first m digits of x. Then for a given $\epsilon > 0$,

$$N_m(x, b_1b_2...b_k) \geq (b^{-k} - \epsilon) \sum_{\lambda}{}^{\prime} (\nu_\lambda - k + 1),$$

where $\sum'$ is taken over the values of $\lambda \geq n$ for which a_λ is (k, ϵ)-normal.

Let the number of integers among $a_1, a_2, ..., a_n$ which are not (k, ϵ)-normal be denoted by ω_n. By hypothesis

$$\omega_n = o(n) \qquad \text{as} \qquad n \to \infty$$

Also, $\nu_\lambda \leq \nu_n$ for every $\lambda < n$. Hence

$$\begin{aligned} N_m(x, b_1b_2...b_k) &\geq (b^{-k} - \epsilon) \left\{ \sum_{\lambda}{}^{\prime} \nu_\lambda - (n - \omega_n)(k-1) \right\} \\ &> (b^{-k} - \epsilon)(S_n - \omega_n\nu_n - nk) \end{aligned}$$

and

$$\begin{aligned} \frac{N_m(x, b_1b_2...b_k)}{m - k + 1} &> (b^{-k} - \epsilon) \frac{S_n - \omega_n\nu_n - nk}{S_{n+1}} \\ &= (b^{-k} - \epsilon) \left\{ 1 - \frac{1}{n+1} . \frac{(n+1)\nu_{n+1}}{S_{n+1}} \right\} . \left(1 - \frac{\omega_n}{n} . \frac{n\nu_n}{S_n} - \frac{k}{\nu_n} . \frac{n\nu_n}{S_n} \right) \end{aligned}$$

Which by (8.2.1) approaches to $b^{-k} - \epsilon$ as $n \to \infty$. Hence

$$\text{lininf}_{m\to\infty} \frac{N_m(x, b_1 b_2 ... b_k)}{m - k + 1} \geq (b^{-k} - \epsilon),$$

and since ϵ is arbitrary,

$$\text{lininf}_{m\to\infty} \frac{N_m(x, b_1 b_2 ... b_k)}{m - k + 1} \geq b^{-k}$$

Since it is true for every k-digit sequence, we have

$$\lim_{m\to\infty} \frac{N_m(x, b_1 b_2 ... b_k)}{m - k + 1} = b^{-k}.$$

and x is normal in the base b. $\square$

Chapter 9

Questions for further research

Borel conjectured that all real algebraic irrational constants are normal to every base. So the great open question is the normality or lack of normality of all well-known irrational constants. The research in this field provides enough motivation to work towards normality of numbers like π, e, $log2$, $\zeta(3)$, $\zeta(5)$.

In 1909, Borel introduced the concept of normality and proved that almost all real numbers are normal. But till today, there are very few real numbers provably normal and creating a new normal number or proving the normality of well-known algebraic irrationals or any other real number is one of the most outstanding question facing mathematicians.

Normality of the well known algebraic constant is a great question but even the question of simple normality of any real number is a big question in itself.

We defined the concept of Interesting numbers in chapter 2, the definition of interesting number is much weaker than the normality condition. But proving that some particular number is interesting or producing an interesting number is also an outstanding problem in this field.

Now we will state few open problems.

Main Problem. Let $\alpha \in [0,1]\backslash\mathbb{Q}$ be a real number and let $b \in \mathbb{N}, b \geq 2$. Then prove or disprove that α is simply normal to the base b.

We have considered α in the unit interval as the integer part of a number has no impact on the limiting distribution of the digits. We can see that the above problem is quite general, but even some very specific cases are open till today. Now we will consider a weaker case of the above set up.

Problem 1. Let $\alpha \in [0,1]\backslash\mathbb{Q}$ be a real number. Prove or disprove that at least three digits occur infinitely many times in the decimal expansion of α.

Remark: There is nothing special about three digits pattern. In fact, we can consider any finite digit patterns.

Problem 2. Prove or disprove that $\sqrt{2}$ is an interesting number.

The problems stated above are just a few but we can state a number of similar problems in this field. There has been very little progress beyond Borel's original work, so this field is very much open for mathematicians.

Bibliography

[1] *A general arithmetic construction of transcendental non-liouville normal numbers from rational fractions*, Acta Arith. **16** (1969/70), 239–253.

[2] Émile Borel, *Les probabilitiés dénombrables et leurs applications arithmétiques*, Supplemento ai rendicontidel Circolo Matematico di Palermo **27** (1909), 247–271.

[3] A. S. Besicovitch, *The asymptotic distribution of the numerals in the decimal representation of the squares of the natural numbers*, Mathematische Zeitschrift **39** (1934), 146–156.

[4] D. G. Champernowne, *The construction of decimals normal in the scale of ten*, Journal of the London Mathematical Society **3** (1933), 254–260.

[5] K. chandrasekharan, *Introduction to analytical number theory*, Springer-Verlag, Berlin-Heidelberg, New York, 1968.

[6] A. H. Copeland and P. Erdös, *Note on normal numbers*, Bulletin of the American Mathematical Society **52** (1946), no. 1, 857–860.

[7] H. Davenport and P. Erdös, *Note on normal decimals*, Canadian J. Math. **4** (1952), 58–63.

[8] H. A. Hanson, *Some relations between various types of normality of numbers*, Canadian Journal of Mathematics **6** (1954), 477–485.

[9] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Clarendon Press, Oxford, 1979.

[10] Glyn Harman, *Metric number theory*, London Mathematical Society Monographs New Series 18, pp. 1-10, London.

[11] Saied Hosseini Khayat, *Almost all real numbers are interesting*, Electrical Engineering Department Ferdowsi University of Mashhad, Iran, August 2007.

[12] I. Niven and H. S. Zuckerman, *On the definition of normal numbers*, Pacific J. Math. **1** (1951), 103–109.

[13] Ivan Niven, *Irrational numbers*, The Carus Mathematical Monographs, vol. 11, pp. 94-114, The Mathematical Association of America, Distributed by John Wiley and Sons, Inc., New York, N. Y., 1956.

[14] S. S. Pillai, *On normal numbers*, Proc. Indian Acad. Sci. Sect. A **10** (1939), 13–15.

[15] S. S. Pillai, *On normal numbers*, Proc. Indian Acad. Sci. Sect. A **12** (1940), 179–184.

[16] R. G. Stoneham, *On absolute (j, ϵ)-normality in the rational fractions with applications to normal numbers*, Acta Arith. **22** (1972/73), 277–286.

[17] D. D. Wall, *Normal numbers*, Ph.D. thesis, Berkely, Department of Mathematics, 1940.