Combinatorial Surface

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This is to certify that this thesis entitled ”Combinatorial Surface” submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents work carried out by Sandeep Suman under the supervision of Ashish Kumar Upadhyay.

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Abstract

Combinatorial Surface

by Sandeep Suman

The contents of this thesis are basic notions in combinatorial topology and simplicial homology theory. Some existing techniques of how we can compute the homology groups are also presented. The results on classification of all closed combinatorial surfaces has been discussed. Methods of generating triangulation of surfaces by using computers has been discussed. It is based on the project work which I undertook in IIT Patna as reading project followed by programming to generate triangulations.

Contents

Chapter 1

Introduction

Topology usually refers to point-set topology developed from the general theory of sets by Georg Cantor and first book appeared in 1912 by Hausdorff. The study of combinatorial topology now algebraic topology was started by Henri Poincaré during 1895-1901. It is not based on set theory but geometric problem of curves, surfaces and the geometry of Euclidean spaces. One of the problem, point set topology and algebraic topology tries to answer is to classify topological spaces up to homeomorphism.

In this report we will see the examples of two of main working techniques in algebraic topology first is the idea of invariants(here topological invariants) and other is known classification theorem. Topological invariant is a property of a topological space which is invariant under homeomorphism. For example connectedness and compactness are topological invariants. In general, classification theorem tries to distinguish topological spaces on the basis of some equivalence relation and each topological space belongs to exactly one class. We want to find a set of topological invariants, if they are identical for any two topological spaces then they belong to the same equivalence class. For example a closed surface is represented by a polygon with even number of edges. Polygon has directed edges appears exactly twice. We can use signed symbols to distinguish two direction of an edge. Then Euler characteristic($V - E + F$) and orientability will be able to classify all closed surfaces.

We associate groups with topological spaces which are topological invariant, *i.e.*, homeomorphic space has isomorphic group. Once we define a method to associate a group to the topological space, we can find some results about the topological space using algebraic arguments. If two space have non-isomorphic group then we clearly say that they are not homeomorphic. But usually it is not other way round, *i.e.*, if two groups are isomorphic then we can not say the underlying space is homeomorphic. Two most common group we study are homotopy and homology.

First homotopy group or fundamental group is based on loops and curves on topological space. If we are able to continuously deform one loop to other then they belong to the same class. A trivial loop is the one which can be continuously deformed into a point. A simply connected space is the one in which any loop can be deformed into a point and hence the fundamental group of a simply connected space is trivial. $\mathbb{R}^2$ and $\mathbb{S}^2$ are examples of simply connected space. In circle($\mathbb{S}^1$), a loop which encloses the hole can not be deformed into point thus it is a non-trivial element of the fundamental group of the circle. Two loops belong to same equivalence class iff number of time it will go around the hole and their direction will be the same. It's fundamental group is a infinite group generated by one element. $\mathbb{R}^2$ and circle has different fundamental group hence they are not homeomorphic. But we can see that $\mathbb{R}^2$ and $\mathbb{S}^2$ has same fundamental group but they are not homeomorphic, $\mathbb{R}^2$ is not compact while $\mathbb{S}^2$ is compact. A loop is a continuous map from $\mathbb{S}^1$ to the topological space X. Then there is a generalization of higher dimensional homotopy group in which n-th dimensional loop can be thought as continuous map from $\mathbb{S}^n$ to the topological space X.

Like homotopy, homology group also look for holes in the topological space. Consider a simple configuration below:

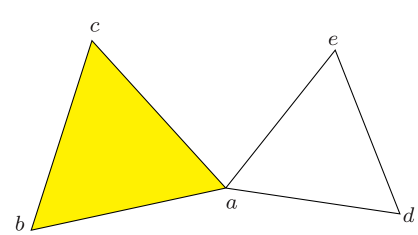

Figure 1.1: Polyhedron

It consist of the triangle $< abc >$ and it's boundary and also edges $< ad >, < de >$ and $< ae >$. Such kind of space are called polyhedron. A 2-chain will be a linear combination of triangles while a 1-chain will be linear combination of edges. For simplicity take the coefficient modulo 2, *i.e.*, zero or one. A 2-chain is $< abc >$ because our polyhedron has only one triangle, while a 1-chain is any combination of 6 edges in the polyhedron. For example, a 1-chain is

$$< ab > + < ac > + < de >,$$

Other edges can be thought as has coefficient zero.
Now we define a boundary operator which is actually the boundary of these closed sets as follows

$$\begin{aligned} \partial < abc > &= < ab > + < bc > + < ac > \\ \partial < ab > &= < a > + < b > \end{aligned}$$

Extending this linearly we can find the boundary of any chain. For If any p-chain is the boundary of some $(p+1)$-chain, then it is called a p-boundary. Also a p-cycle is a p-chain such that it's boundary is zero.
Now the 1-cycle $< ab > + < bc > + < ac >$ encloses a 2-chain $< abc >$, while 1-cycle $< ad > + < ae > + < de >$ bounds a hole. Thus we see that a non-trivial cycle can be the boundary of a higher dimensional chain or it encloses a hole. This is the idea of homology group to find holes in the polyhedron by finding cycles which are not boundaries.

Chapter 2

Basic Combinatorial Topology

In this section our aim is to define polyhedra. We recall the basic definitions from Croom [1].

Definition 2.0.1. *A set of $k+1$ points are geometrically independent if it is not contained in the hyperplane of dimension less than k.*

If $\{a_0, a_1, \cdots, a_k\}$ are the geometrically independent points than $\{a_1 - a_0, \cdots, a_k - a_0\}$ will span a vector space of dimension k.

Definition 2.0.2 (Simplex)**.** *Let $A = \{a_0, a_1, \cdots, a_k\}$ be a set of geometrically independent points in $\mathbb{R}^n$. Then a k-dimensional geometric simplex or k-simplex, σ^k, spanned by A is the set of points in $\mathbb{R}^n$ such that,*

$$\sigma^k = \left\{ x = \sum_{i=0}^{k} \lambda_i a_i \mid, \lambda_i \in \mathbb{R}, \lambda_i \geq 0, \sum \lambda_i = 1 \right\}$$

Here the numbers $\lambda_0, \cdots, \lambda_k$ are called the barycentric coordinates of the points x and $a_0, \cdots, a_i$ are called the vertices of the simplex, σ^k. We can represent $\sigma^k =< a_0 a_1 \cdots a_k >$.
If all the λ_i are strictly positive then the simplex spanned by the $\{a_0, a_1, \cdots, a_k\}$ will be an open subset of $\mathbb{R}^n$ and called open geometric k-simplex.

Definition 2.0.3 (Faces)**.** *A simplex σ^k is a face of a simplex $\sigma^n, k \leq n$ if each vertex of σ^k is a vertex of σ^n. All the faces of the σ^n other than σ^n itself are called proper faces.*

Example 2.0.4. Let $< a_0 a_1 a_2 >$ be a 2-simplex. Then the 2-simplex will be itself and 1-simplexes will be $< a_0 a_1 >$, $< a_1 a_2 >$ and $< a_0 a_2 >$, and 0-simplex will be $< a_0 >$, $< a_1 >$ and $< a_2 >$.

Definition 2.0.5. *Two simplexes are called properly joined if they do not intersect or their intersection will be a face of both.*

Once we have our bricks and the way to combine them we can now look at the structure which can actually represent some topological spaces.

Definition 2.0.6 (Geometric Complex)**.** *A geometric complex or complex, K, is a finite collection of simplexes with two condition.*

- *Each face of a simplex in K is also a member of K.*
- *Any two simplex in K must be properly joined.*

The dimension of K is the largest positive integer n, such that $\sigma^n \in K$. The union of members of K with the Euclidean subspace topology is denoted by $\mid K \mid$ and called geometric carrier of K or the polyhedron associated with K.

Definition 2.0.7. *Let X be a topological space such that, there exist a geometrical complex K whose geometric carrier $\mid K \mid$ is homeomorphic to the X, then X is said to be a triangulable space, and K is called the triangulation of X.*

Thus K represent the topological space X in a simple way and we can expect some of the properties can be shared by both.

Definition 2.0.8. *The closure of a k-simplex σ^k, $Cl(\sigma^k)$, is the complex consisting of σ^k and all its faces.*

Definition 2.0.9. *If K is a geometric complex. For a positive integer r, the r-skeleton is the complex consist of all simplexes of K of dimension less than or equal to r.*

Example 2.0.10. A three simplex $\sigma^3 =< a_0a_1a_2a_3 >$ is a tetrahedron which 2-skeleton will be it's boundary and homeomorphic to the sphere(S^2). Consider K be the collection of all proper faces of the σ^3, Then K will be the triangulation of S^2.

Example 2.0.11. Möbius strip is obtained by identifying two opposite ends of a rectangle with a flip.

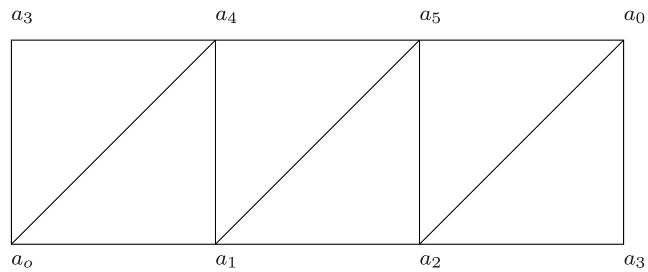

Figure 2.1: A Triangulation of Möbius strip

Definition 2.0.12 (Orientation of Geometric Complexes)**.** *An oriented n-simplex, is obtained by choosing an ordering of it's vertices, The equivalence class of even permutation of the chosen ordering is positively oriented while other is called negatively oriented.*
An oriented geometric complex is obtained by assigning an orientation to each of its simplexes.

We can assign orientation to each of it's geometric simplex separately or we can give an ordering of all it's vertices, which will induce ordering of each simplex by removing all the vertices which are not in the simplex from the ordered list of all vertices.

Definition 2.0.13. *Let K be an oriented geometric complex with simplexes σ^{p+1} and σ^p whose dimensions differ by 1. Then we can associate an incidence number $[\sigma^{p+1}, \sigma^p]$ defined as follows:*

- $[\sigma^{p+1}, \sigma^p] = 0$ *If σ^p is not a face of σ^{p+1}.*
- $[\sigma^{p+1}, \sigma^p] = 1$ *If $\sigma^p =< a_0 \cdots a_p >$ is a face of σ^{p+1} and v be a vertex of σ^{p+1} doesn't belongs to σ^p and $< va_0 \cdots a_p >$ belongs to the even permutation of the σ^{p+1}.*
- $[\sigma^{p+1}, \sigma^p] = -1$ *If $\sigma^p =< a_0 \cdots a_p >$ is a face of σ^{p+1} and v be a vertex of σ^{p+1} doesn't belongs to σ^p and $< va_0 \cdots a_p >$ belongs to the odd permutation of the σ^{p+1}.*

Example 2.0.14. Let $+\sigma^1 = + < a_0 a_1 >$, then $[\sigma^1, < a_0 >] = -1$ and $[\sigma^1, < a_1 >] = 1$

Theorem 2.0.15. *Let K be an oriented complex, σ^p is an oriented p-simplex of K and σ^{p-2} is a face of σ^p. Then,*

$$\sum[\sigma^p, \sigma^{p-1}][\sigma^{p-1}, \sigma^{p-2}] = 0, \sigma^{p-1} \in K$$

Proof. Let $+\sigma^{p-2} =< v_0 \cdots v_{p-2} >$, Then σ^p will have two more vertex a and b, We can assume $+\sigma^p =< abv_0 \cdots v_{p-2} >$ All non-zero terms occur in the summation for two values of σ^{p-1},

$$\sigma_1^{p-1} =< av_0 \cdots v_{p-2} >, \qquad \sigma_2^{p-1} =< bv_0 \cdots v_{p-2} >$$

Now we have to solve each combination of the orientation of these σ^{p-1}.

Case I. Suppose that,

$$+\sigma_1^{p-1} = + < av_0 \cdots v_{p-2} >, \qquad +\sigma_2^{p-1} = + < bv_0 \cdots v_{p-2} >$$

Then,

$$[\sigma^p, \sigma_1^{p-1}] = -1, [\sigma_1^{p-1}, \sigma^{p-2}] = +1,$$
$$[\sigma^p, \sigma_2^{p-1}] = +1, [\sigma_2^{p-1}, \sigma^{p-2}] = +1,$$

So the summation in the product is zero.

Case II. Suppose that,

$$+\sigma_1^{p-1} = + < av_0 \cdots v_{p-2} >, \qquad +\sigma_2^{p-1} = - < bv_0 \cdots v_{p-2} >$$

Then,

$$[\sigma^p, \sigma_1^{p-1}] = -1, [\sigma_1^{p-1}, \sigma^{p-2}] = +1,$$
$$[\sigma^p, \sigma_2^{p-1}] = -1, [\sigma_2^{p-1}, \sigma^{p-2}] = -1,$$

So the summation in the product is again zero.

Similarly,

Case III. Suppose that,

$$+\sigma_1^{p-1} = - < av_0 \cdots v_{p-2} >, \qquad +\sigma_2^{p-1} = + < bv_0 \cdots v_{p-2} >$$

Then,

$$[\sigma^p, \sigma_1^{p-1}] = +1, [\sigma_1^{p-1}, \sigma^{p-2}] = -1,$$

$$[\sigma^p, \sigma_2^{p-1}] = +1, [\sigma_2^{p-1}, \sigma^{p-2}] = +1,$$

So the summation in the product is again zero.

Case IV. Suppose that,

$$+\sigma_1^{p-1} = - < av_0 \cdots v_{p-2} >, \qquad +\sigma_2^{p-1} = - < bv_0 \cdots v_{p-2} >$$

Then,

$$[\sigma^p, \sigma_1^{p-1}] = +1, [\sigma_1^{p-1}, \sigma^{p-2}] = -1,$$

$$[\sigma^p, \sigma_2^{p-1}] = -1, [\sigma_2^{p-1}, \sigma^{p-2}] = -1,$$

So the summation in the product is again zero. Hence, the expression in theorem is zero. $\square$

2.1 Simplicial Homology Group

Definition 2.1.1. *Let K be a oriented geometric complex. For a given positive integer, a p-chain is a function c_p from the family of the oriented p-simplexes of K to the integers such that, for each p-simplex σ^p, $c_p(-\sigma^p) = -c_p(\sigma^p)$. The group structure of $\mathbb{Z}$ will induce a group structure on the family of p-chains. This is called a p-dimensional chain group of K and the group is denoted by $C_p(K)$.*
An elementry p-chain is a p-chain c_p for which there is a p-simplex σ^p, such that $c_p(\tau^p) = 0, \forall \tau^p \neq \sigma^p$.

An arbitary p-chain, d_p can be written as finite sum of of elementary p-chains as follows

$$d_p = \sum g_i \cdot \sigma_i^p$$

Where i runs over all p-simplexes.

Now if $c_p = \sum f_i \cdot \sigma_i^p$ and $d_p = \sum g_i \cdot \sigma_i^p$ are two p-chain on K, Then

(i) $c_p + d_p = \sum(f_i + g_i) \cdot \sigma_i^p$

(ii) The additive inverse of the chain c_p in the group will be $-c_p = \sum -f_i \cdot \sigma_i^p$

(iii) The chain group $C_p(K)$ is isomorphic to the direct sum of finite number of $\mathbb{Z}$. Suppose there are n number of p-simplexes in K. Then $C_p(K)$ is isomorphic to direct sum of n number of copies of $\mathbb{Z}$. One isomorphism is given by

$$\sum_{i=1}^{n} g_i \cdot \sigma_i^p \leftrightarrow (g_1, g_2, \cdots g_n).$$

Definition 2.1.2. *Let $g \cdot \sigma^p$ is an elementary p-chain with $p \geq 1$, the boundary of $g \cdot \sigma^p$, $\partial(g \cdot \sigma^p)$ is defined by*

$$\partial(g \cdot \sigma^p) = \sum [\sigma^p, \sigma_i^{p-1}] g \cdot \sigma_i^{p-1}, \qquad \sigma^{p-1} \in K$$

If $c_p = \sum f_i \cdot \sigma_i^p$ is an arbitrary p-chain, Then

$$\partial(c_p) = \sum \partial(f_i \cdot \sigma_i^p)$$

The boundary of the 0-chain is defined to be zero.

Thus boundary ∂ is homomorphism of the groups $C_p(K)$ and $C_{p-1}(K)$.

Theorem 2.1.3. *If K is an oriented complex and $p \geq 2$, then the composition $\partial\partial : C_p(K) \to C_{p-2}(K)$ in the diagram*

$$C_p(K) \xrightarrow{\partial} C_{p-1}(K) \xrightarrow{\partial} C_{p-2}(K)$$

is a the trivial homomorphism.

Proof. We have to show that $\partial\partial(c_p) = 0, \forall\ c_p \in C_p(K)$. Due to linearity of ∂, it is sufficient to show this for an arbitrary elementary p-chain $g \cdot \sigma^p$.

$$\begin{aligned}
\partial\partial(g \cdot \sigma^p) &= \partial\Big(\sum_{\sigma_i^{p-1} \in K} [\sigma^p, \sigma_i^{p-1}] g \cdot \sigma_i^{p-1}\Big) \\
&= \sum_{\sigma_i^{p-1} \in K} \partial([\sigma^p, \sigma_i^{p-1}] g \cdot \sigma_i^{p-1}) \\
&= \sum_{\sigma_i^{p-1} \in K} \sum_{\sigma_j^{p-2} \in K} [\sigma^p, \sigma_i^{p-1}][\sigma_i^{p-1}, \sigma_j^{p-2}] g \cdot \sigma_j^{p-2}
\end{aligned}$$

Reversing the order of summation and applying theorem 2.0.15, we get

$$\begin{aligned}
\partial\partial(g \cdot \sigma^p) &= \sum_{\sigma_j^{p-2} \in K} \sum_{\sigma_i^{p-1} \in K} [\sigma^p, \sigma_i^{p-1}][\sigma_i^{p-1}, \sigma_j^{p-2}] g \cdot \sigma_j^{p-2} \\
&= 0
\end{aligned}$$

□

Definition 2.1.4 (Cycle and Boundary)**.** *Let K be an oriented geometric complex. A p-dimensional cycle or p-cycle for $p \geq 1$, on K is a p-chain z_p such* *that* $\partial(z_p) = 0$. *Thus the family of p-cycle is the kernel of the homomorphism* $\partial : C_p(K) \to C_{p-1}(K)$ *and subgroup of* $C_p(K)$. *This subgroup is denoted by* $Z_p(K)$, *is called the p-dimensional boundary group of* K.

A p-chain b_p *is a p-dimensional boundary on* K, *or p-boundary, if there is a* $(p+1)$*-chain* c_{p+1} *such that* $\partial(c_{p+1}) = b_p$. *The family of p-dimensional boundary is* $\partial(C_{p+1}(K))$ *will be a subgroup of* $C_p(K)$. *This subgroup is denoted by* $B_p(K)$.

If the dimension of K *is* n. *Then there is no p-chain for* $p > n$, *Hence* $C_p(K)$ *is zero* $\forall\ p > n$ *and thus* $B_n(K) = 0$.

Theorem 2.1.5. *If* K *is an oriented complex, then* $B_p(K) \subset Z_p(K)\ \forall\ p$ *such that* $0 \leq p \leq n$, *where* n *is the dimension of* K.

Proof. From the definition p-dimensional boundary $B_p(K) \cong \partial(C_{p+1}(K))$. Let $c_{p+1} \in C_{p+1}(K)$, using theorem 2.1.3, we know that $\partial\partial(c_{p+1}) = 0$. Hence, $\partial(c_{p+1}) \in Z_p(K) \Rightarrow B_p(K) \subset Z_p(K) \forall p$ □

Definition 2.1.6. *Two p-cycle* w_p *and* c_p *on a complex* K *are homologous,* $w_p \sim c_p$, *If there is a* $(p+1)$*-chain* c_{p+1} *such that*

$$\partial(c_{p+1}) = w_p - c_p.$$

Then if a p-cycle t_p is the boundary of a $(p+1)$-chain, then t_p is homologous to zero, *i.e.*, $t_p \sim 0$.

This is a equivalence relation which leads to the partition of the $Z_p(K)$,

$$[z_p] = \{w_p \in Z_p(K) \mid w_p \sim z_p\}.$$

Definition 2.1.7. *If* K *is an oriented complex, the p-dimensional homology group of* K *is the quotient group*

$$H_p(K) = Z_p(K)/B_p(K).$$

Example 2.1.8. Let K be the closure of a 2-simplex $< a_0a_1a_2 >$ with orientation induced by the ordering of the vertex $a_0 < a_1 < a_2$.

Then K contain positively oriented 2-simplex $< a_0a_1a_2 >$, 1-simplex $< a_0a_1 >, < a_1a_2 >$ and $< a_0a_2 >$ and 0-simplex $< a_0 >, < a_1 >$ and $< a_2 >$.

An arbitary 0-chain will be of the form

$$c_0 = g_0 < a_0 > + g_1 < a_1 > + g_2 < a_2 >;\ g_i \in \mathbb{Z}.$$

Then $C_0(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Also $C_0(K) \cong Z_0(K)$, Hence $Z_0(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.
An arbitrary 1-chain will be of the form

$$c_1 = h_0 < a_0a_1 > +h_1 < a_1a_2 > +h_2 < a_0a_2 >;\ h_i \in \mathbb{Z}.$$

Then $C_1(K) \cong c_0 = g_0 < a_0 > +g_1 < a_1 > +g_2 < a_2 >;\ g_i \in \mathbb{Z}$. Now

$$\begin{aligned} \partial(c_1) &= h_0(< a_1 > - < a_0 >) + h_1(< a_2 > - < a_1 >) + h_2(< a_2 > - < a_0 >) \\ &= (-h_0 - h_2) < a_0 > +(h_0 - h_1) < a_1 > +(h_0 + h_2) < a_2 > \end{aligned}$$

Now c_1 will be a 1-cycle, if $\partial(c_1) = 0$ hence,

$$-h_0 - h_2 = 0, h_0 - h_1 = 0, h_0 + h_2 = 0$$

$$\Rightarrow h_0 = h_1 = -h_2 = h.$$

The form of 1-cycle will be

$$h < a_0a_1 > +h < a_1a_2 > -h < a_0a_2 > .$$

Since there is only one independent variable in 1-cycle. Hence $C_1(K) \cong \mathbb{Z}$.
There is only on 2-simplex in K. Hence 2-chain will be of the form

$$c_2 = h < a_0a_1a_2 >; h \in \mathbb{Z}.$$

Here, $C_2(K) \cong \mathbb{Z}$. Also,

$$\partial(c_2) = h(< a_0a_1 > + < a_1a_2 > - < a_0a_2 >).$$

The 2-cycle has one independent variable, Hence $Z_2(K) \cong \mathbb{Z}$. Also the form of 2-cycle is the same the 2-dimensional boundary, Hence $B_2(K) \cong Z_2(K)$.
Now $H_2(K) \cong 0$
1-cycle and 1-boundaries has same form, Hence

$$Z_1(K) \cong B_1(K) \Rightarrow H_1(K) = 0.$$

To find $H_1(K)$ we observe that any 1-cycle has the form

$$g_0 < a_0 > +g_1 < a_1 > +g_2 < a_2 >= \partial(g_1 < a_0a_1 > +g_2 < a_0a_2 >)+(g_0+g_1+g_2) < a_0 >$$

Hence every 1-cycle is homologous to $h < a_0 >$ for h is an integer, hence $H_1(K) = \mathbb{Z}$. Overall we get $H_0(K) = \mathbb{Z}, H_1(K) = 0$ and $H_2(K) = 0$.

Theorem 2.1.9. *Let K_1 and K_2 be the oriented geometric complex of same complex K with different orientation. Then groups $H_p(K_1) \cong H_p(K_2)$ for each dimension p.*

Proof. To prove this we will find a isomorphism of the $H_p(K_1) \longrightarrow H_p(K_2)$.
Let ${}^1\sigma^p$ and ${}^2\sigma^p$ be the positively oriented simplex in the complex K_1 and K_2 for the simplex $\sigma^p \in K$. We can have a function α defined on the simlexes of K which takes the value ± 1, Then,

$${}^1\sigma^p = \alpha(\sigma^p)^2\sigma^p.$$

Define a sequence of homomorphism $\phi = \{\phi_p\}$

$$\phi_p : C_p(K_1) \longrightarrow C_p(K_2)$$

such that,

$$\phi_p(\sum g_i \cdot {}^1\sigma_i^p) = \sum \alpha(\sigma_i^p) g_i \cdot {}^2\sigma_i^p$$

Where $\sum g_i \cdot {}^1\sigma_i^p$ is arbitrary p-chain on K_1.
For an elementary p-chain $g \cdot {}^1\sigma^p$ on $K_1, p \geq 1$, We have

$$\begin{aligned}
\phi_{p-1}\partial(g \cdot {}^1\sigma^p) &= \phi_{p-1}\Big(\sum_{\sigma^{p-1} \in K} [{}^1\sigma^p, {}^1\sigma^{p-1}] g \cdot {}^1\sigma^p \Big) \\
&= \sum_{\sigma^{p-1} \in K} [{}^1\sigma^p, {}^1\sigma^{p-1}] \alpha(\sigma^{p-1}) g \cdot {}^2\sigma^p \\
&= \sum_{\sigma^{p-1} \in K} \alpha(\sigma^p)\alpha(\sigma^{p-1})[{}^2\sigma^p, {}^2\sigma^{p-1}] \alpha(\sigma^{p-1}) g \cdot {}^2\sigma^p \\
&= \sum_{\sigma^{p-1} \in K} [{}^2\sigma^p, {}^2\sigma^{p-1}] (\alpha(\sigma^p) g) \cdot {}^2\sigma^p \\
&= \partial(\alpha(\sigma^p) g \cdot {}^2\sigma^p) \\
&= \partial\phi_p(g \cdot {}^1\sigma^p)
\end{aligned}$$

Thus we got the relation $\phi_{p-1}\partial = \partial\phi_p$, or we can say the following diagram commutes:

$$\begin{tikzcd} C_p(K_1) \arrow[r, "\phi_p"] \arrow[d, "\partial"'] & C_p(K_2) \arrow[d, "\partial"] \\ C_{p-1}(K_1) \arrow[r, "\phi_{p-1}"'] & C_{p-1}(K_2) \end{tikzcd}$$

Now, $z_p \in Z_p(K_1)$, then using above relation

$$\partial\phi_p(z_p) = \phi_{p-1}\partial(z_p) = \phi_{p-1}(0) = 0,$$

Hence $\phi_p(z_p) \in Z_p(K_2) \Rightarrow \phi_p(Z_p(K_1)) \subset Z_p(K_2)$.
Also $\phi_p\partial(c_{p+1}) \in B_p(K_1)$, then

$$\phi_p\partial(c_{p+1}) = \partial\phi_{p+1}(c_{p+1}),$$

hence $\phi_p\partial(c_{p+1}) \in B_p(K_2) \Rightarrow \phi_p(B_p(K_1)) \subset B_p(K_2)$.
Then ϕ_p^* will induce an homomorphism between $H_p(K_1))$ and $H_p(K_2)$ defined as follows

$$\phi_p^*([z_p]) = [\phi_p(z_p)]$$

for each class $[z_p] \in H_p(K_1)$.
Reversing the role of K_1 and K_2 we will get an homomorphism from $H_p(K_2)$ to $H_p(K_1)$. Which tells us that both are isomorphic.

□

Definition 2.1.10. *Let K be a complex. Two simplexes s_1 and s_2 are connected if either of the following condition is satisfied:*
(a) $s_1 \cap s_2 = \phi$; and
(b) There is sequence $\sigma_1, \cdots \sigma_n$ of 1-simplexes of K such that $s_1 \cap \sigma_1$ a vertex of s_1 and $s_2 \cap \sigma_n$ is a vertex of s_2, and $si_i \cap \sigma_{i+1}$ is a common vertex of σ_i and σ_{i+1} for $1 \leq i < p$.
It means that is a polygonal path exist between these two disjoint simplexes.

This is an equivalence relation on K. It will partition the set K into different combinatorial component. If the complex contain only one combinatorial component then it is said to be connected.

Theorem 2.1.11. *Let K be a complex with r combinatorial components, then $H_0(K)$ is isomorphic to the direct sum of r copies of the group $\mathbb{Z}$.*

Proof. Let K' be a combinatorial component of K. Then any $< a >, < b > \in K'$ we have a sequence of 1-simplexes

$$< aa_0 >, < a_0a_1 >, \cdots , < a_nb >$$

such that any two consecutive 1-simplex will have a common vertex. We can define a 1-chain c_1 on the sequence of this 1-simplex by assigning either g or $-g$ to each simplex so that $\partial(c_1)$ is either $g\cdot < b > -g\cdot < a >$ or $g\cdot < b > +g\cdot < a >$. Hence any 0-chain $g\cdot < b >$ on K' is homologous to either $g\cdot < a >$ or $-g\cdot < a >$. Hence any 0-chain on K' is homologous to an elementary 0-chain $h\cdot < a >$ for some integer h. Applying the result for each combinatorial component of $K_1, \cdots K_r$ of K, for vertex $a_i \in K_i$ any 0-chain K_i is homologous to the 0-chain of the form $h_i\cdot < a_i >$ for some integer h_i. Then any 0-chain c_0 on K will be

$$c_0 \sim \sum_{i=1}^{r} h_i\cdot < a_i > .$$

Suppose that two 0-chain $\sum h_i\cdot < a_i >$ and $\sum g_i\cdot < a_i >$ belongs to the same homology class. Then there exist a 1-chain such that,

$$\partial(c_1) = \sum (g_i - h_i)\cdot < a_i >$$

Now each a_i belongs to the different combinatorial components of K, then the equation holds only if $g_i = h_i \forall i$. Hence each homology class $c_0 \in H_o(K)$ has a unique representative of the form $h_i\cdot < a_i >$. The natural bijection

$$\sum h_i\cdot < a_i > \longrightarrow (h_1, \cdots , h_r)$$

will the isomorphism between $H_0(K)$ to the direct sum of r copies of $\mathbb{Z}$. □

Definition 2.1.12 (n-pseudomanifold). *An n-pseudomanifold is a complex K with the following properties:*
(i) Each simplex of K is a face of some n-simplex or K.
(ii) Each $(n-1)$-simplex is a face is exactly the face of two n-simplex.
(iii) If σ_1^n and σ_2^n are any two n-simplex in K, there exist a sequence of n-simplex beginning with σ_1^n and ending with σ_2^n such that any two consecutive simplex has a common $(n-1)$-simplex.

We can expect triangulation of a manifold is a pseudomanifold. But not all manifold are triangulable. Hence if a manifold is triangulable then it's triangulation will be a pseudomanifold.

Example 2.1.13. The boundary of the 3-simplex is homeomorphic to $\mathbb{S}^2$. Thus the family of all proper faces of a 3-simplex is a 2-pseudomanifold as well as triangulation of $\mathbb{S}^2$.

Euler characteristic plays an important role in theory of surface. Euler characteristic can be defined for surfaces made of properly joined convex polygons by Kinsey [3].

Definition 2.1.14. *Let K be a surface made of properly joined convex polygons, Then the Euler characteristic denoted by $\chi(K)$ is defined as*

$$\chi(K) = V - E + F$$

Where V is number of vertices, E is number of edges and F is number of 2-dimensional faces.
If we triangulate each polygons then we get 2-pseudomanifold. Hence the Euler characteristic of a 2-pseudomanifold, K, will be

$$\chi(K) = \alpha_0 - \alpha_1 + \alpha_2$$

Where α_0 is number of vertices, α_1 is number of 1-simplexes and α_2 is number of 2-simplexes.

Definition 2.1.15. *A rectilinear polyhedron in $\mathbb{R}^3$ is a solid bounded by* **properly joined** *convex polygons. The bounding polygon are faces, then the intersection of faces are called edges and the intersection of edges are vertices. A simple polyhedron is a rectilinear polyhedron whose boundary is homeomorphic to $\mathbb{S}^2$.*

Theorem 2.1.16 (Euler's Theorem)**.** *If S is a simple polyhedron with V vertices, E edges, and F faces, then $V - E + F = 2$.*

Proof. S is a simple polyhedron. It's boundary consist of polygons which may not be triangular. But we can make it triangular. Triangulation doesn't change the value of $V - E + F = 2$. Now remove one of the face, which reduce the number of faces by one, other will remain the same. Hence we will have to show

$$V - E + F = 1.$$

Rest part will be a plane made of triangles. Remove one triangle from boundary by either removing an edge or a vertex the value $V-E+F$ will remain constant. Repeat the process till we get only one triangle. For a triangle the value of $V - E + F = 3 - 3 + 1 = 1$. ☐

Theorem 2.1.17. *Let K be a 2-pseudomanifold with α_0 number of vertices, α_1 edges and α_2 faces(2-simplexes). Then,*
(i) $3\alpha_2 = 2\alpha_1$
(ii) $\alpha_1 = 3(\alpha_0 - \chi(K))$.
(iii) $\alpha_0 \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi(K)})$

Proof. A face consist of 3-edges and each edge is belongs to exactly two faces. Since there are α_2 number of faces then total number of edges will be $3\alpha_2$ in which each edges counted twice, Hence $3\alpha_2 = 2\alpha_1$.
Also,

$$\begin{aligned} \chi(K) &= \alpha_0 - \alpha_1 + \alpha_2 \\ &= \alpha_0 - \alpha_1 + \frac{2}{3}\alpha_1 \\ &= \alpha_0 - \frac{1}{3}\alpha_1 \end{aligned}$$

Then,

$$\alpha_1 = 3(\alpha_0 - \chi(K)).$$

To prove the third inequality, we have

$$\alpha_1 \leq \binom{\alpha_0}{2} = \frac{1}{2}\alpha_0(\alpha_0 - 1)$$

Now,

$$\begin{aligned}
\alpha_1 &\le \frac{1}{2}\alpha_0(\alpha_0-1) \\
3(\alpha_0-\chi(K)) &\le \frac{1}{2}\alpha_0(\alpha_0-1) \\
-3\chi(K) &\le \frac{1}{2}\alpha_0(\alpha_0-1)-3\alpha_0 \\
&\le \frac{1}{2}\alpha_0(\alpha_0-7) \\
-6\chi(K) &\le \alpha_0^2-7\alpha_0 \\
49-24\chi(K) &\le (2\alpha_0-7)^2
\end{aligned}$$

Hence,

$$\alpha_0 \ge \frac{1}{2}(7+\sqrt{49-24\chi(K)})$$

□

Definition 2.1.18. *Let K be an n-pseudomanifold. If for all $\sigma^{n-1} \in K$, a common face of two n-simplex σ_1^n and σ_2^n. We have orientation of K such that*

$$[\sigma_1^n, \sigma^{n-1}] = -[\sigma_2^n, \sigma^{n-1}]$$

Then, K has an coherent orientation. A n-pseudomanifold is orientable if it can be given an coherent orientation. Otherwise it is non-orientable.
In case of orientable n-pseudomanifold each $(n-1)$-simplex should get incidence number with opposite sign with respect to two n-simplex of which it is a common face.

Theorem 2.1.19. *An n-pseudomanifold K is orientable if and only if the nth homology group $H_n(K)$ is not trivial.*

Proof. ($\Rightarrow$) Let K is orientable. Then there is an orientation such that, if σ^{n-1} is $(n-1)$-face of σ_1^n and σ_2^n, with

$$[\sigma_1^n, \sigma^{n-1}] = -[\sigma_2^n, \sigma^{n-1}]$$

We will explicitly find a non-trivial element of $H_n(K)$. Let any n-chain with fixed integer g of the form

$$\begin{aligned} c &= \sum_{\sigma_i^n \in K} g \cdot \sigma_i^n \\ \partial(c) &= \sum_{\sigma_i^n \in K} \sum_{\sigma_j^{n-1} \in K} [\sigma_i^n, \sigma_j^{n-1}] g \cdot \sigma_j^{n-1} \\ \partial(c) &= \sum_{\sigma_j^{n-1} \in K} \sum_{\sigma_i^n \in K} [\sigma_i^n, \sigma_j^{n-1}] g \cdot \sigma_j^{n-1} \end{aligned}$$

Since σ_j^{n-1} will be the face of exactly two n-simplex and using the condition of orientability we get,

$$\begin{aligned} \partial(c) &= \sum_{\sigma_j^{n-1} \in K} 0 \\ &= 0 \end{aligned}$$

Thus c is a n-cycle, *i.e.*, $Z_n(K) \neq 0$ but, $B_n(K) = 0$. Hence we can conclude that $H_n(K) \neq 0$.

($\Leftarrow$) Let $H_n(K) \neq 0$ and

$$z = \sum_{\sigma_i^n \in K} g_i \cdot \sigma_i^n$$

is a non-zero n-cycle. Then,

$$\begin{aligned} \partial(z) &= \sum_{\sigma_i^n \in K} \sum_{\sigma_j^{n-1} \in K} [\sigma_i^n, \sigma_j^{n-1}] g \cdot \sigma_j^{n-1} \\ \partial(z) &= \sum_{\sigma_j^{n-1} \in K} \sum_{\sigma_i^n \in K} [\sigma_i^n, \sigma_j^{n-1}] g \cdot \sigma_j^{n-1} \end{aligned}$$

From the definition of n-pseudomanifold we know that any two n dimensional simplex is connected by a series of a sequence of n dimensional simplexes where intersection of consecutive term will have a common $(n-1)$ dimensional face.

$$\partial(z) = \sum_{\sigma_j^{n-1} \in K} ([\sigma_{j1}^n, \sigma_j^{n-1}] g_{j1} + [\sigma_{j2}^n, \sigma_j^{n-1}] g_{j2}) \cdot \sigma_j^{n-1}$$

Where σ_{j1}^n and σ_{j2}^n are two faces with common face σ_j^{n-1}.

Since $\partial(z) = 0$ hence g_i should have a same absolute value for all i with different sign, *i.e.*, $g_i = \pm g$.

Now reverse the orientation of σ_i^n for $g_i = -g$. Then the above cycle will be

$$z = \sum_{\sigma_i^n \in K} g \cdot \sigma_i^n = g\Big(\sum_{\sigma_i^n \in K} 1 \cdot \sigma_i^n\Big)$$

Hence $\sum_{\sigma_i^n \in K} 1 \cdot \sigma_i^n$ is an n-cycle. This assures us that each common $(n-1)$-face will have positive incidence number with respect to one simplex while negative incidence number with respect to the other simplex, *i.e.*, K is orientable. □

2.2 Classification of Combinatorial Surface

A combinatorial surface is 2-pseudomanifold. Here we will present a classification theorem for closed combinatorial surface as given by Dehn and Heegard(1907). I will follow this section from Stillwell [2].

Definition 2.2.1 (Schemata). *A closed combinatorial surface can be build from finite set of polygons with oriented edges labelled by letters, and every letter appears twice. Such a system is called schema.*

Triangulation of such surface can be obtained by triangulating each polygons.

Example 2.2.2. Let's have a simple example which can convey the main idea. Suppose we have a schema,

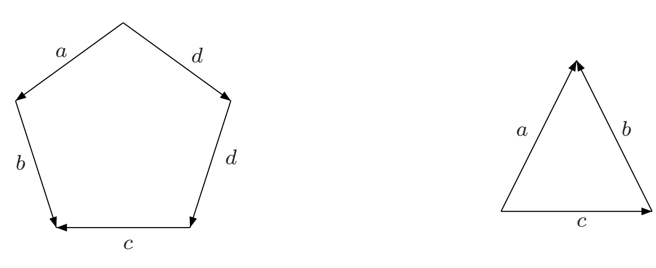

Figure 2.2: A Schema

To understand this surface we have to know how the corner of the polygons fits together. Start with any vertex (say A). We will go around A in a small circular path identifying each till we get a complete disc. All the vertex which identified with A gets the same label A. Now take a vertex which is not labelled, label it and find

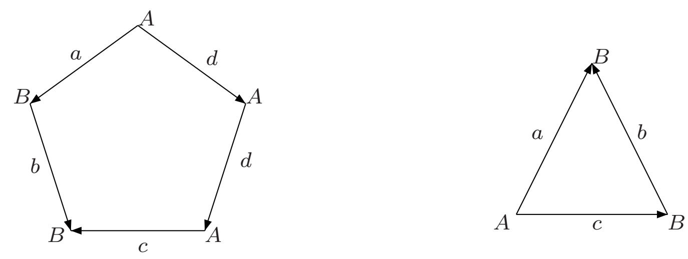

Figure 2.3: A Schema After Labelling the Vertex

all the vertices which will identified with this. After labelling the vertices we get the following picture:

Thus we got two vertices and the Euler characteristic $\chi = V - E + F = 2 - 4 + 2 = 0$. Hence it can be either Torus or Klein bottle.

Combinatorial surface have polygons with directed edges. Each polygon can be represented by a word made of edges. In the above example the two polygons are $abc^{-1}d^{-1}d^{-1}$ and cba^{-1}.
Now if some edge is on two different polygon of the schema then we can identify and get one polygon. Thus any connected combinatorial surface can be represented by only one polygon with directed edges as follows:

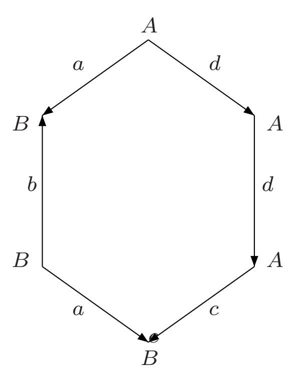

Figure 2.4: Surface with one Polygon

Thus any compact connected combinatorial surface can be represented by single word of edges in which each edge will come twice.
A portion of the boundary of a polygon of the form $aba^{-1}b^{-1}$ and aa is called *handle* and *crosscap* respectively.

Now the classification theorem in terms of handle and crosscap can be stated as follows:

Theorem 2.2.3. *Any connected combinatorial surface can be one of the following three types:*

- *Sphere* (aa^{-1}).
- *Sphere with n-handle*$(a_1a_2a_1^{-1}a_2^{-1}\cdots a_{2n-1}a_na_{2n-1}^{-1}a_n^{-1})$.
- *Sphere with k-crosscaps*$(a_1a_1\cdots a_na_n)$.

Proof. The main tool for the proof is cut and paste. We can cut a polygon into two by forming two identified edges. Paste is the method of attaching two polygon on identified edges.

We follow a step by step method to reduce any surface to a standard normal form of type one of the above in the theorem.

Step I. Reduction to a single polygon with single vertex.

Since the surface is connected we can get a single polygon by pasting the identified edges of different polygon. Before doing anything **collapse** the pair like aa^{-1}. Now partition the vertices of this polygon into equivalence classes of vertices which are identified together. If number of classes is ≥ 2. We can always reduce it to one by the following method. Let A and B are two different class then number of points in B is decreased by one by the following construction:

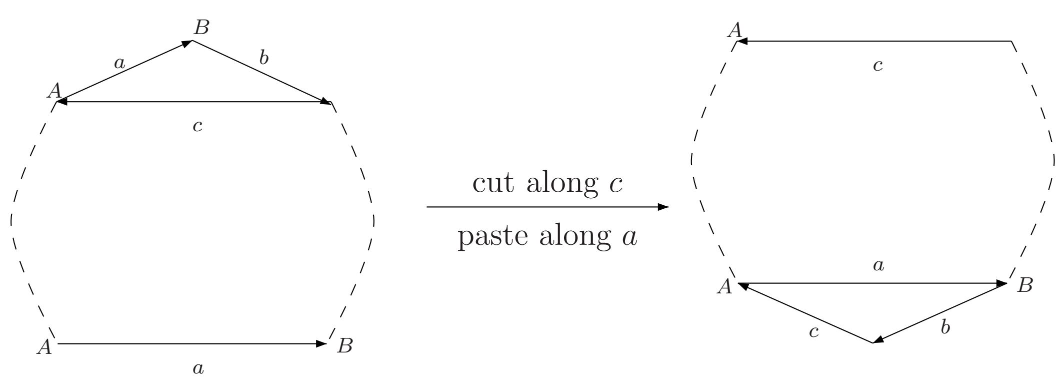

Figure 2.5: Reducing one Vertex of Type B

We will continue this process till we get only one edge in the equivalence class B. But then the edges incident on B will be are of the type a and a^{-1} hence collapse which leads to elimination of all vertices of type B. Thus the number equivalence class of vertices is reduced by one. We will continue this process till we get all the vertices of same equivalence class. The advantage of this construction is that if we now do the cut and paste operation all the vertices will remain in the same class.

Step II. Normal form for same direction edges.

After the above process the surface will have same direction edges and opposite direction. The aim of this step is to bring all same direction edges to form crosscap, *i.e.*, of type aa. The following process will make a crosscap corresponding to each pair of same direction edges.

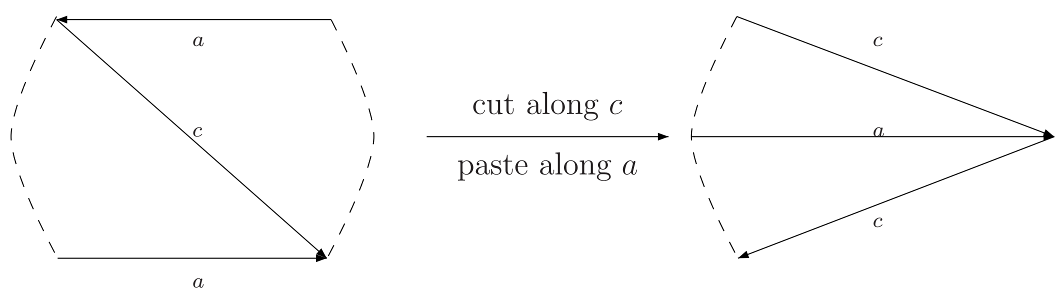

Figure 2.6: Making Same Direction Edges Adjacent

This process will be repeated till we get crosscaps for every pair of same direction edges. Also note that it does not effect ordering of other edges.

Step III. Normal form for opposite direction edges.

Now we want to get a normal form for opposite direction edges. If they are adjacent then either it will be a sphere or it will collapse. Since all the vertices identified to the same point hence oppositely directed edges will occur in crossed pairs as follows

$$\cdots a \cdots b \cdots a^{-1} \cdots b^{-1} \cdots$$

The following process will form a handle corresponding to each crossed pair of the above type:

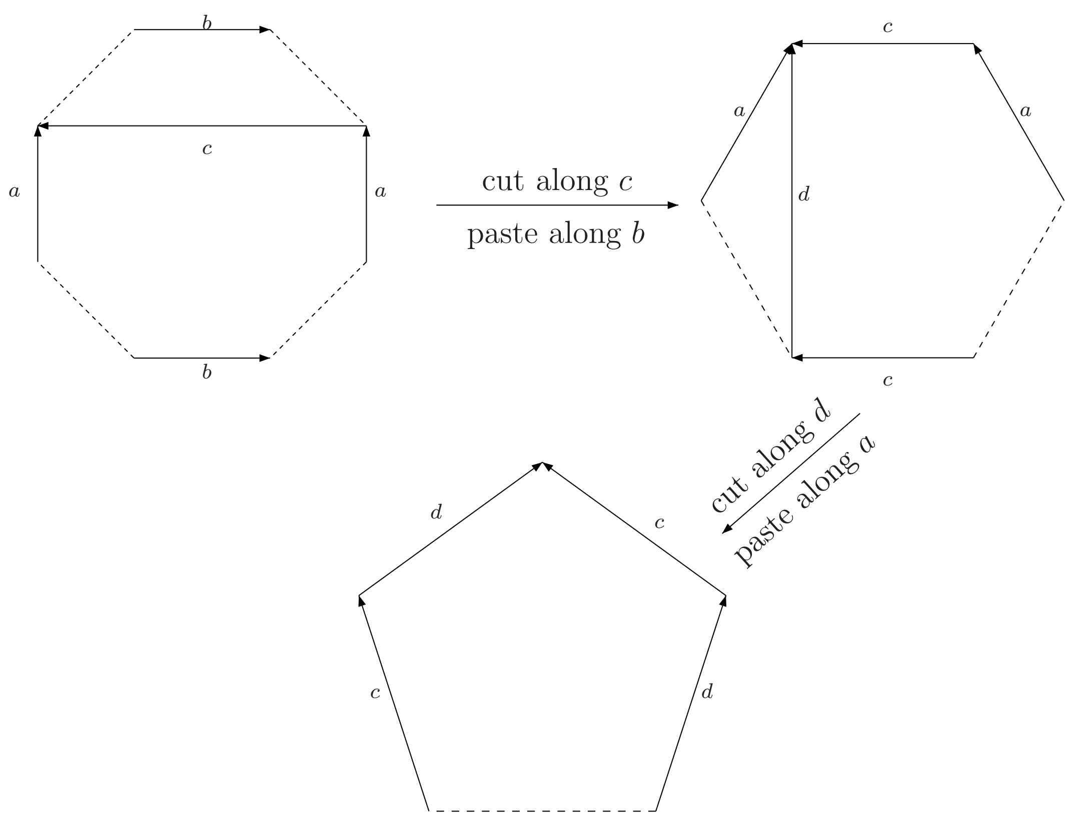

Figure 2.7: Getting Handle from a Pair of Opposite directed Edges

Repeating the process a finite number of times it will exhaust all such crossed pair.

Step IV. Conversion of handle into crosscap.
The resulting surface from all above steps will have either handles or a mixture of crosscaps and handle. But our standard form can have only one. Standard normal form is achieved by the property that one handle with one crosscap lead to three crosscaps, Below the construction has been shown:

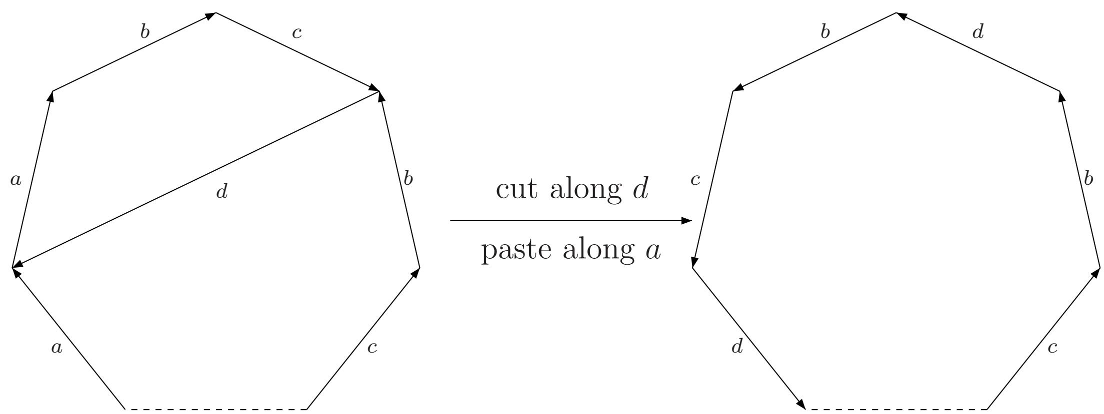

Figure 2.8: Transforming Handle into Crosscaps

We will use same process in step II to convert this into crosscaps. Thus all compact combinatorial surface can be brought to one of the above type. ☐

Remark 2.2.4. If any combinatorial surface will have a single pair of same direction edges then it will be non-orientable. Thus in example 2.2.2 we had a non-orientable surface and hence Klien bottle.

T. Rado in 1925 proved that every compact surface is triangulable. If we consider this result, then we actually get a classification of all compact surfaces.
In the proof of classification of combinatorial surface we used "cut and paste" as only tools to carry out all the process. In both the process the quantity Euler characteristic remain constant.

Theorem 2.2.5. *Any combinatorial surface can be identified with it's Euler Characteristic and orientability.*

Proof. Compute the Euler characteristic for all three type of surface as follows:

- Sphere $\chi = 2 - 1 + 1 = 2$.
- Sphere with n-handle $\chi = 1 - 2n + 1 = 2 - 2n$
- Sphere with k-crosscaps $\chi = 1 - k + 1 = 2 - k$

We know from theorem 2.1.14 that sphere has Euler characteristic 2 and we can see that only sphere can have Euler characteristic 2. Orientable surfaces are represented by sphere with n-handle while non-orientable surface are represented by sphere with k-crosscaps. All orientable surface are distinguished by Euler characteristic and similarly for non-orientable surface. If we take orientation of the surface into account we can find a surface uniquely. Thus orientation and Euler characteristic is able to identify a combinatorial surface uniquely.

$\square$

Chapter 3

Searching a Triangulation

Every compact surface is triangulable. Let K be triangulation of a connected, compact surface without boundary. Then all its edges will be incident on exactly two faces and each face will meet three edges such that.

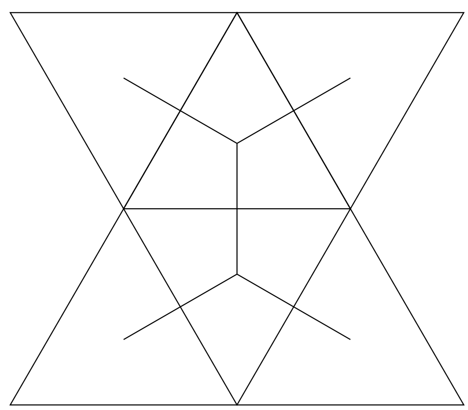

Figure 3.1: Triangulation as a Graph

We can think the situation as a graph such that each triangle represents a vertex and two vertices are connected if there is a common face between them. Thus a triangulation will be a graph such that each vertex has degree exactly three. It is sufficient to write the triangle with sides to represent a triangulation, thus the minimal spanning tree of this graph will be a desired triangulation. We can use Depth-first search or Breadth-first search algorithms to find a minimal spanning tree of the triangulation.

An important issue will be to justify whether the surface is orientable or not. In orientable surface each edge incident on two triangles with opposite incidence numbers, so we can start with any arbitrary node and give it a positive orientation. Then it's neighbour will get an orientation such that the common edge will get opposite orientation from both the face. If we are able to orient the whole K then it is orientable else non-orientable.

Triangulation has finite number of elements and we can use computer to do different type of computation related to them. The theory applied effectively on combinatorial surfaces mainly due to classification theorem. The classification theorem requires only Euler characteristic and orientabiliy. Euler characteristic has a simple expression in terms of number of edges, vertices and faces, $V - E + F$, hence it is easy to compute. So we will focus on the way we can handle orientability from Edelsbrunner and Harer [6].

Ordered Triangle

The most fundamental piece of data structure will consist of triangle and a function which keeps track of orientation and direction it is connected to other such triangles. An ordered triangle can be represented by permutation on three letters hence, **Set of all ordered triangle will form a group isomorphic to symmetric group of three elements,** *i.e.*, S_3. Each element of this group represent an ordered triangle. But since we want to distinguish on the basis of orientability we will have two class and each class has three elements of S^3. We represent a triangle by 123, then cyclic shift retains the orientation while swapping any two element will change the orientation. It will better to define two function to handle such thing as follows:

$SWAP$ swaps two of the leading edge of triangle and change the orientation. Thus the function $SHIFT$ and $SWAP$ are able to navigate us from a standard triangle labelled 123 with leading edge 12 to other ordered triangles.
We store a triangle in a node. We will have a pointer, μ to the node and an integer, i, which represent the ordered version of the triangle 123. $i = 0, 1, 2, 4, 5, 6$ for $123, 312, 231, 213, 132, 321$. Now the functions $SHIFT$ and $SWAP$ can be defined as follows:

function SHIFT((μ, i))
  if $i \leq 2$ **then return** $(\mu, (i + 1)\ mod\ 3)$
  elsereturn $(\mu, (i + 1)\ mod\ 3 + 4)$

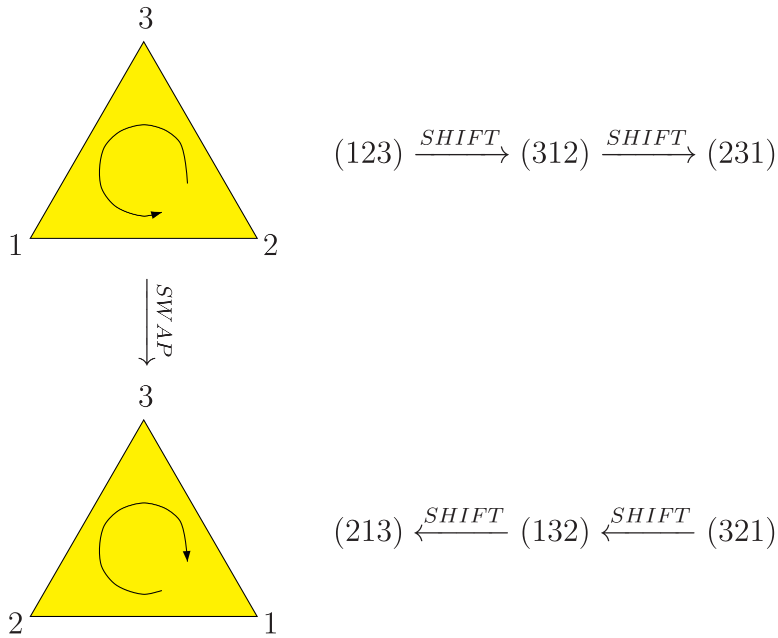

Figure 3.2: Getting all ordered triangle from one using $SHIFT$ and $SWAP$ function

end if
end function

and

function SWAP((μ, i)) **return** $(\mu, (i+4)\ mod\ 8)$
end function

Data Structure

Here we will see the data structure for representing the triangulation of a connected, compact, 2-pseudomanifold without boundary. We have an array $V[n]$ to store the vertex. To store the graph of triangulation as discussed above we need pointers which refers to other nodes of neighboring triangles. The degree of each node will be three hence each node has to store pointers of these three neighbors and the vertices of the triangle itself from V.

Here abc be a triangle and x, y, z the respective third vertices of the neighbour triangles. Each ordered version of triangle points to it's leading edge and shares the leading edge with on of it's neighbour. Let μ, μ_x, μ_y, μ_z points towards the four triangle with ordering given by $i = 0$ corresponding to the triangle abc, abx, ayc, zbc in the above figure. Let a is stored in position i in V and ab is the leading edge of abx, the ordered triangle stores pointers $(\mu, 0).org = 1$ and $(\mu, 0).fnext = (\mu_x, 0)$. Similarly if

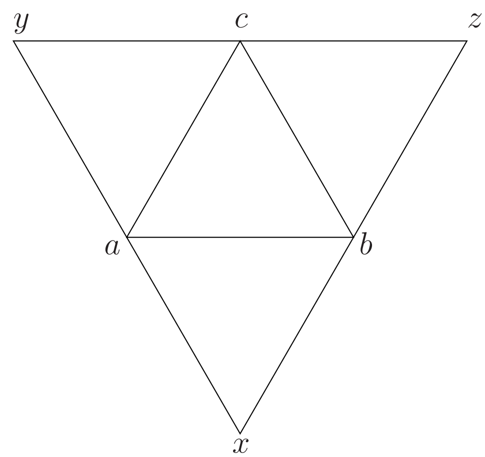

Figure 3.3: A Triangle with Neighbours

j and k be the position of b and c in V, then other five ordered triangle will store the pointer j, k, j, k, i and to the ordered triangles $(\mu, 1)(\mu_x, 1), (\mu_y, 1), (\mu_z, 1)$. Now with the following function we can move around triangulation.

Searching the Tree. Now we have stored whole triangulation in our data structure. To find the triangulation we have to write all the triangles, *i.e.*, find the spanning tree of the graph and write all the nodes of this tree. This can be done by two basic search algorithms Breadth-first search and Depth first search. Here we use Depth first search. Let all the nodes are unmarked in the beginning, we can start the search with an arbitrary triangle μ_0.

function VISIT((μ))
  if μ is marked **then**
    mark μ;P1
    for all neighbour ν of μ **do** visit (ν)
    end for;P2
  else;P3
  end if
end function

Where $P1, P2$ and $P3$ are the conditions we can impose to the different stage to customize our search. Now we will put appropriate condition on statement $P1, P2$ and $P3$. We will try to orient each triangle such that shared edges between any two neighboring triangles should get opposite direction. If we are able to find such orientation for all triangulation, then we will get orientable surface otherwise non-orientable surface.

Suppose we will start with triangle (μ_0, i_0).

function ISORIENTABLE((μ, i))

if μ is unmarked **then**

mark μ and choose orientation containing i;

$b_x \leftarrow IsOrinetable(FNEXT(SWAP(\mu, i)))$

$b_y \leftarrow IsOrinetable(FNEXT(SWAP(\mu, i)))$

$b_z \leftarrow IsOrinetable(SWAP^2(SYM(\mu, i)))$ **return** b_x, b_y and b_z

elsereturn [orientation of μ contains i]

end if

end function

3.1 Generating a lexicographic triangulation

Lexicographical ordering is the ordering similar to the alphabetical ordering. The algorithm following by Lutz [4] will be able to generate a triangulation of a surface on fixed number of vertices in lexicographical order.

Let $\{1, \cdots, n\}$ be a set of n-vertices. We have to find a connected 2-dimensional geometric complex such that every edge is contained in exactly two triangles. Since we are interested in finding triangulation of surfaces up to combinatorial equivalence, *i.e.*, relabelling of vertices, hence 123 will always be present in the triangulation.

In lexicographically minimal triangulation, the collection $B_{deg(1)}$ of triangles containing the vertex 1 is of the form

$$123, 124, 146, 157, \cdots, 1deg(1)(deg(1)+1)$$

, where $deg(1)$ is the degree of the vertex 1 and $3 \leq deg(1) \leq n-1$.

Now, in a lexicographically sorted list of triangulated surface the beginning segment of B_k will come before B_{k+1}. So, we will start searching with B_3 and then by B_4 by the method of *backtracking* as in Lutz [4]:

Start with some triangle and add further triangles as long as no edge is contained in more than two triangles. If this condition is violated, then backtrack. A set of triangles is closed if every of its edges is contained in exactly two triangles. If the link of every closed set of triangles is a circle, then this set of triangles gives a triangulated surface.

We can implement the above idea using sparse matrix. Say we have fixed number

of vertices n. The number of edges will be ${}^nC_2$ and number of faces will be ${}^nC_3$. Store the triangle in rows of a triangle-edge incidence matrix. As we go along the triangulation we will add the corresponding rows. If any entries become 3 it means that corresponding edges belongs to three triangles, which is not allowed. Then we will backtrack till we get all the entries consist of 2 and 0. It will then be a candidate for a triangulated surface.
We have to check whether a candidate is actually a triangulated surface. For this we have to check that neighbourhood of every vertex is a disc.

Example 3.1.1. Similar method had been used in higher dimension to compute all combinatorial 3-manifold on 8 vertex. A. Altshuler in his paper "Combinatorial 3-manifold with few vertices" computes all possible combinatorial 3-manifold with 8 vertices. He uses an algorithm which can be extended to find a combinatorial manifold on given set of vertices satisfying a set of conditions.
The method is based on exhaustive search, *e.g.*, searching all possible combinatorial 3-manifolds on given number of vertices. This algorithm is a search tree which grows in size while simultaneously satisfying the condition of a 3-pseudomanifold. Searching the tree for a given set of conditions either we get a desired geometric complex or we are in a position to say that no geometric complex satisfies the given set of condition. The basic steps involved are as follows:
Suppose our aim is to find a combinatorial 3-manifold on n vertices.

- Take any 3-simplex say $(1, 2, 3, 4)$ that will be root of the search tree.
- A 2-simplex will be precisely the face of two three simplices and this will be the most prominent condition in this process. A two simplex will be covered if it is the face of two 3-simplices. Consider any non-covered face of the 3-simplex say $(1, 2, 3)$.
- Now consider all possible new simplex which can contain the above 2-simplex. This will form the children of the above node if none of its face is already covered.
- Since the number of vertex is finite this process will end up in finite steps and we can search for a particular triangulation along this tree.

3.2 Programs/Language useful for computation

SAGE. SAGE is a Mathematics open-source software. It is actually an interface which brings many open source Mathematics program at one place. It is based on python. It can also be used as python compiler. It has many inbuilt Mathematical functions. Latest version, SAGE 4.8 has many function which helps us do calculation based on finite geometric complex. Using these functions specific calculation can be done very easily.

GAP. "GAP-Groups, Algorithms, programming" is a software for computational discrete algebra with an emphasis on computational group theory. It has a programming language and many inbuilt functions. It also has large data libraries of groups. Different type of symmetric properties of geometric complexes can be studied by making different groups act on these objects as group of automorphism.

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