Some Radiation problems in Electrodynamics and Gravity

A thesis submitted towards partial fulfilment of
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by

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Certificate

This is to certify that this thesis entitled "Some Radiation problems in Electrodynamics and Gravity" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by "Darshan Gajanan Joshi" at "Tata Institute of Fundamental Research, Mumbai", under the supervision of "Prof. C. S. Unnikrishnan" during the academic year 2011-2012.

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Abstract

An accelerated charged particle radiates according to the Classical Electrodynamics. In standard literature, the dynamics of an accelerated charged particle is described by introducing a 'radiation reaction force' i.e. by resorting to the Lorentz-Abraham-Dirac (LAD) equation of motion. However, it is clear that the LAD equation of motion give pathological solutions even in the simplest case. Many alternate possibilities are discussed in literature. However, these remedies are not successful in eradicating these issues. Here we approach this issue from energy conservation point of view. Using simple energy conservation relation we derive the equation of motion to describe the dynamics of accelerating charged particle. To the point we have explored it in few cases, we find our equation of motion free of any unphysical solutions. A similar situation arises in Gravity due to emission of gravitational waves. We extend our idea to write the equation of motion in this case. Further, we study the 'Unruh Effect' which states that an accelerating detector in flat Minkowski spacetime vacuum would find itself in a thermal bath which has a temperature proportional to its acceleration. Our aim is to consolidate various ideas presented in literature on this issue so that it can serve as a good starting point for further research on the connection between these phenomena.

Contents

Chapter 1

Introduction

Radiation has played a pivotal role in development of Physics. It is by the virtue of radiation that we have made immense advances in high speed communication, medical sciences etc. Even at the fundamental level it has been crucial on deciding upon the fate of a theory. One remarkable example is that of discarding the Rutherford's model of atom on the argument based on radiation from accelerated electrons. More recent examples are the 'Cosmic Microwave Background Radiation' and 'Gravitational Waves', which have helped the development of Cosmology.

But in the course of time since the establishment of Electromagnetic radiation some of the most foundational issues related to it have been sidelined at the cost of immense applicability of this phenomenon. The most noted one is the equation of motion for an accelerating charged particle. The first attempt to solve this was made by Lorentz and Abraham around 1900. Later, Dirac generalised their idea to the relativistic case. Since then people have identified various problems with this approach and still continue to search for a consistent way to tackle it. The so called 'radiation reaction force' or 'radiation damping' is usually introduced very superficially in standard textbooks (eg. [1]). In the absence of any concrete equation of motion even the physical picture is unnecessarily made complicated and unappealing. It is pretended that there are no issues and that literature has several remedies for it. However, we will see later that these remedies are only case specific and sometimes only give rise to other problems. So our aim is to convince the reader that the existing equation of motion even in the simplest case has unphysical implications and that the existing remedies have no satisfactory answer to it. Hence we take a route, which has been partially taken by some earlier attempts, to tackle this problem in a very simple way. Particularly, in our analysis the energy lost as radiation is not identified with a 'force' at any stage. We apply our method to a few cases and find that at least in the simple cases that we consider, our equation of motion gives physically correct solutions.

Analogous to an accelerating charged particle giving electromagnetic radiation, we have gravitational waves emitted by accelerating massive systems. Here also we have the 'back reaction' issue. Using our idea from electrodynamics, we simply write an equation of motion for this case. However, we do not attempt to solve it in this work. Rather we take new look on the Gravitational waves. It is stated in many standard texts that the reason for quadrapole nature of gravitational waves is that in a gravitating system the center of mass is not accelerating (since no external force) and hence we do not have a second derivative of dipole term in the expression of power as in the case of Electrodynamics. There is however physically more transparent reason for the absence of second derivative of dipole term in power expression. Motivated from a rough sketch in [2] we try to calculate the power emitted by a system of two masses assuming dipolar gravitational radiation from the individual mass. We compare our result to the power obtained from the usual quadrapole formula for gravitational radiation and find that they do not match exactly. However, our picture gives a nice way to perceive quadrapole nature of the gravitational waves.

As a further step in exploration of accelerated systems, we investigate the 'Unruh Effect'. It was shown by Unruh [3] that an accelerating detector in flat Minkowski spacetime vacuum would not detect it as a vacuum state. Rather it would detect it as a many particle state in equilibrium at a temperature ($T = \hbar a/2\pi k_B c$) proportional to its acceleration. The important issue is whether there is any experimental evidence of 'Unruh Effect'. We argue how is this issue in contrast to some assertions based on spin depolarization in accelerator storage rings. This allows us to assert that circular acceleration is not the right setting to expect Unruh Effect in accelerated frames. Apart from this there is a controversial aspect to this effect as to whether or not such a system would radiate/detect radiation. In this section, we present a brief literature survey on this topic.

Chapter 2

Radiation reaction in Electrodynamics

2.1 Lorentz-Abraham-Dirac equation of motion

Soon after the advent of Classical Electrodynamics, Lienard and Wiechert calculated potentials due to moving charges. It was quickly realised that an accelerated charged particle radiates since its field has a component which goes as $1/r$ (r is the distance between the source and the observer) and hence the Poynting vector has non-zero value as $r \to \infty$. For the case of non-relativistic (speed of particle is much less than that of speed of light in vacuum) point charge the power radiated away due to its acceleration is given by the Larmor's formula:

$$P = \frac{q^2 \mathbf{a}^2}{6\pi\epsilon_0 c^3} \tag{2.1}$$

where P is the power radiated, $\mathbf{a}$ is the acceleration of the particle, c is the speed of light in vacuum and q is the charge of the particle. Now by the virtue of the phenomenon of radiation any charged particle undergoing acceleration would loose its energy in the form of radiation. To describe this phenomenon Lorentz was the first to give an equation of motion for a charged particle taking into consideration the radiation emitted by the particle. The analysis is based on the fact that there exists a recoil force which accounts for the energy lost due to the radiation. In the following steps we sketch a heuristic derivation of equation of motion given by Lorentz as found in most of the standard textbooks [1]. According to the standard arguments, the average work done by this radiation reaction force F_{rad} must be equal to the power lost in the form of radiation. So,

$$\int_{t_1}^{t_2} \mathbf{F}_{rad} \cdot \mathbf{v} dt = -\frac{q^2}{6\pi\epsilon_0 c^3} \int_{t_1}^{t_2} \mathbf{a}^2 dt \tag{2.2}$$

Integrating the RHS by parts, we get

$$\int_{t_1}^{t_2} \mathbf{a}^2 dt = \int_{t_1}^{t_2} (\frac{d\mathbf{v}}{dt}) \cdot (\frac{d\mathbf{v}}{dt}) dt = (\mathbf{v} \cdot \frac{d\mathbf{v}}{dt})|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d^2\mathbf{v}}{dt^2} \cdot \mathbf{v} dt \tag{2.3}$$

If we consider periodic motion then at $t = t_2$ both the velocity and acceleration have the same values as they did at $t = t_1$. Hence we get the standard expression of the so called 'radiation reaction force' as follows:

$$\mathbf{F}_{rad} = \frac{q^2 \dot{\mathbf{a}}}{6\pi\epsilon_0 c^3} \tag{2.4}$$

So the equation of motion for a charged particle taking into account the radiation emitted by it in the non-relativistic limit is:

$$m\mathbf{a} = \mathbf{F}_{external} + \mathbf{F}_{rad} \tag{2.5}$$

This was first obtained by Lorentz and then the relativistic version was derived by Abraham. Later Dirac derived it in a covariant form. Thus the Lorentz-Abraham-Dirac (LAD) equation of motion looks as follows (for a simple derivation see [4]):

$$m\dot{\nu}^\mu = F^\mu_{ext} + F^\mu_{self} \tag{2.6}$$

where

$$F^\mu_{self} = \frac{q^2}{6\pi\epsilon_0 c^3} (\ddot{\nu}^\mu - \nu^\mu \dot{\nu}^\alpha \dot{\nu}_\alpha) \tag{2.7}$$

and ν is the four velocity. In this limit F^μ_{self} is further classified as addition of F_{schott} (the first term) and F_{rad} (the second term) [5]. However these details are not required for our further analysis.

2.2 Problems with the Lorentz-Abraham-Dirac equation

Lorentz-Abraham-Dirac equation of motion involves the third derivative of the position or one can say it the 'jerk'. This makes it difficult to obtain solution in many cases. However solving the LAD equation is not a major issue compared to tackling its many unphysical solutions. The major worries are the runaway solutions and solutions predicting acausal preacceleration (for a detailed discussion see [6]). For clarity, we demonstrate below one such obvious inconsistency in the non-relativistic limit. In the absence of any external force the LAD equation of motion takes the form:

$$m\mathbf{a} = \mathbf{F}_{rad} \tag{2.8}$$

So we have the solution,

$$a = a_0 e^{6\pi m\epsilon_0 c^3 t/q^2} \tag{2.9}$$

This means that the acceleration of the particle would keep increasing! This is absurd because we expect the particle to slowly loose away all its energy and come to rest (since there is no mechanism to increase particle's energy). There are many kinds of tricks in the literature to somehow get away from these issues (for eg. setting $a_0 = 0$). However these difficulties persist not only for this non-relativistic case but also in the relativistic regime.

2.3 Remedies in literature to overcome unphysical solutions of LAD equation

In the course of time many interpretations and conclusions have appeared in the literature ([6] gives a detailed account) to address the above problem with the LAD equation of motion. Some have the opinion that the absurd solutions imply the breakdown of point particle limit [7, 8] and one has to resort to calculations taking into account finite size. Due to divergences of self field at the centre of charged particle, renormalization techniques were also introduced [8].While others have taken a view to modify the equation of motion [9]. Recently it has been claimed [6] to settle all these issues with the advent of modified equation of motion in the following form:

$$m\mathbf{a} = \mathbf{F}_{ext} + \frac{q^2}{6\pi m\epsilon_0 c^3}\dot{\mathbf{F}}_{ext} \tag{2.10}$$

Although this is derived in a much rigorous way in [9] still it is quite absurd because in the absence of time varying external force the radiation term apparently goes away! Actually the above equation of motion is also derived in [10] as an iterative correction. In any case the physical motivation of relating the radiation to a force (radiation reaction force) is not very convincing. Even from the radiation pattern one knows that there is no radiation in the direction of the motion (in case of linear motion). Then there exists no physical quantity in this direction (except for the external force) which might affect the particles motion in this direction. In fact the maximum radiation is in the lateral direction. So associating this phenomenon with a 'force' in the direction of motion clearly seems wrong.

2.4 Our approach to Radiation reaction

Here we approach the problem simply from the energy point of view. There were earlier attempts which start with a similar view but involves force at some stage in the discussion. In contrast, in our view energy lost as radiation is never treated as a force of back action. We would be working essentially in the Newtonian limit. Considering the conservation of total energy we argue that the rate of loss of kinetic energy (E) is equal to the sum of rate of work done by external field and power radiated. In the absence of any other energy mechanism this relation seems the most correct.

$$\frac{\partial E}{\partial t} = \mathbf{F}.\mathbf{v} - \frac{q^2 a^2}{6\pi\epsilon_0 c^3} \tag{2.11}$$

Here a is the acceleration of the particle and F is the external force. The last term on the RHS is the standard formula of power radiated by a charged particle (in SI units). Putting the expression for kinetic energy in the above expression we get the following equation:

$$m\mathbf{v}.\frac{d\mathbf{v}}{dt} - \mathbf{F}.\mathbf{v} + \frac{q^2 a^2}{6\pi\epsilon_0 c^3} = 0 \tag{2.12}$$

Let us investigate Eqn. 2.12 in the absence of any external field. So the equation of motion now takes the form,

$$m\mathbf{v}.\frac{d\mathbf{v}}{dt} + \frac{q^2 a^2}{6\pi\epsilon_0 c^3} = 0 \tag{2.13}$$

A trivial solution to the above equation is $\dot{\mathbf{v}} = 0$, which is quite obvious. The other solution is found by solving,

$$m\mathbf{v} + \frac{q^2 a}{6\pi\epsilon_0 c^3} = 0 \tag{2.14}$$

We first differentiate this equation to solve an initial problem in acceleration (because physically that is what governs the dynamics). Here we get the solution,

$$\mathbf{a} = \mathbf{a}_0 e^{-t/\tau} \tag{2.15}$$

$$\tau = \frac{q^2}{6\pi m\epsilon_0 c^3} \tag{2.16}$$

where a_0 is the initial acceleration. Clearly we see that the acceleration decreases exponentially as expected. The velocity follows the following trend:

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}_0\tau - \mathbf{a}_0\tau e^{-t/\tau} \tag{2.17}$$

where v_0 is the initial velocity.

2.4.1 An illustration in Non-relativistic regime

Another case we investigated is that in the presence of constant electric field. Here we proceed by first solving the quadratic in $\dot{v}$ from the equation of motion 2.12.

$$\dot{v} = \frac{-mv \pm \sqrt{m^2v^2 + 4AFv}}{2A} \tag{2.18}$$

where,

$$A = \frac{q^2}{6\pi\epsilon_0 c^3} \tag{2.19}$$

We consider the positive root of Eqn. 2.19 and after making a series of substitutions during integration we get the following expression (considering $F > 0$):

$$\begin{aligned} \frac{t}{\tau} &= 2[\frac{\sqrt{m^2v^2 + 4AFv}}{\sqrt{m^2v^2 + 4AFv} - mv} - \frac{\sqrt{m^2v_0^2 + 4AFv_0}}{\sqrt{m^2v_0^2 + 4AFv_0} - mv_0}] + \\ & \ln[\frac{(\sqrt{m^2v^2 + 4AFv} + mv)(\sqrt{m^2v_0^2 + 4AFv_0} - mv_0)}{(\sqrt{m^2v^2 + 4AFv} - mv)(\sqrt{m^2v_0^2 + 4AFv_0} + mv_0)}] \end{aligned} \tag{2.20}$$

Where v_0 is the initial velocity of the particle and τ is as defined in Eqn. 2.16. Here we see that explicit expression in v seems impossible. So we explore its behavior numerically. For various values of initial conditions and value of electric field we find that the velocity profile is almost similar to the case when radiation is neglected for very low charged objects (even with high q/m ratio). We have plotted one such profile comparison in Fig. 2.1.
The distinction between velocity profile with radiation and without radiation is more evident in particles with higher charge. As an example we have plotted (see Fig. 2.2) a comparison figure for an object of mass $1gm$ and $q/m \approx 10^7$. In both these cases we see that the velocity of object is lower when radiation is considered. This is to be expected because we had the initial velocity in the direction of the external force.
Next we consider the initial velocity to be in the direction opposite to that of the external force. Here we see that the velocity goes to zero much quicker

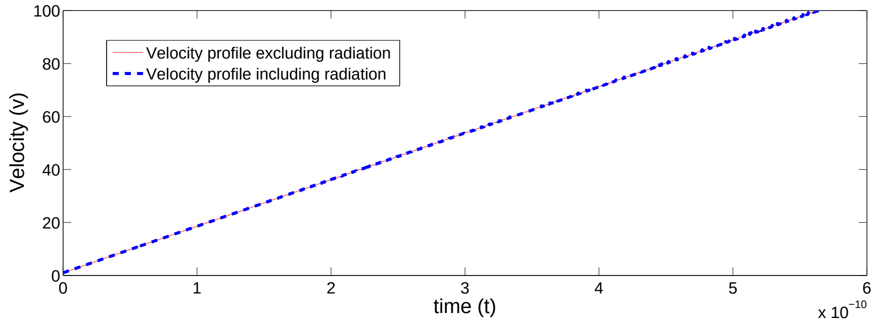

Figure 2.1: We consider a proton with initial velocity $v_0 = 1m/s$ in a constant electric field $E = 1V/m$. The blue curve shows velocity profile corresponding to Eqn. 2.21. The red curve is plotted for the case when radiation is neglected.

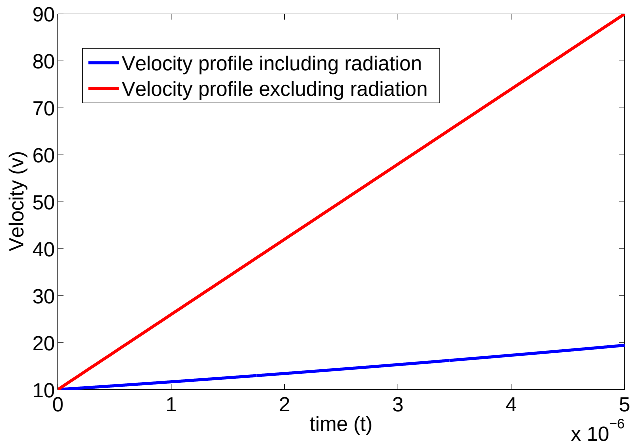

Figure 2.2: We consider an object with mass $1gm$ and $q/m \approx 10^7$ having an initial velocity $v_0 = 10m/s$ in a constant electric field $E = 1V/m$. The blue curve shows velocity profile corresponding to Eqn. 2.21. The red curve is plotted for the case when radiation is neglected.

than the case when radiation is neglected and after that the rise in velocity is again less compared to the case when radiation is neglected (see Fig. 2.3). Even this is expected because when the velocity is in the direction opposite to that of the external force, both radiation and external force are decreasing the kinetic energy of the particle. Whereas once the velocity becomes zero and starts going in direction of the force, external force is increasing the kinetic energy while the radiation is trying to decrease it.

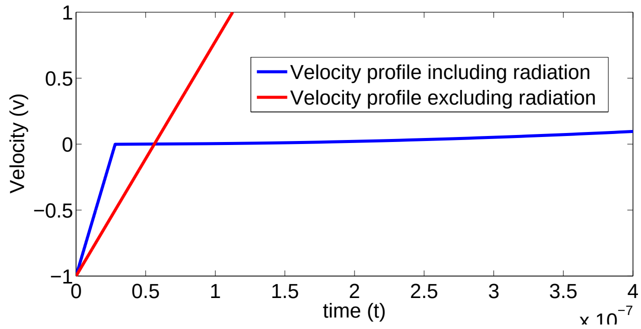

Figure 2.3: We consider an object with mass $1gm$ and $q/m \approx 10^7$ having an initial velocity $v_0 = 10m/s$ in a constant electric field $E = 1V/m$. The blue curve shows velocity profile corresponding to Eqn. 2.21. The red curve is plotted for the case when radiation is neglected. In contrast to above two plots, we see that blue curve is above red curve for negative values of velocity.

2.4.2 An illustration in Relativistic case

All the ideas discussed above can be easily extended to the relativistic limit. According to Lienard's generalization of Larmor's formula,

$$P = \frac{\gamma^6 q^2 \mathbf{a}^2}{6\pi\epsilon_0 c^3} \tag{2.21}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2.22}$$

For the relativistic case, energy of the particle in the presence of an electric field would be:

$$E = \sqrt{p^2c^2 + m^2c^4} + q\phi \tag{2.23}$$

$$\frac{dE}{dt} = \frac{p}{\sqrt{p^2c^2 + m^2c^4}}\frac{dp}{dt} + q\frac{d\phi}{dt} \tag{2.24}$$

$$p = \gamma m v \tag{2.25}$$

$$\frac{p}{\sqrt{p^2c^2 + m^2c^4}} = v \tag{2.26}$$

$$\frac{d\gamma}{dt} = \frac{\gamma^3 v\dot{v}}{c^2} \tag{2.27}$$

where q is the charge and ϕ is the electric potential. Using above relations and demanding conservation of energy once again we equate the rate of change of energy of the particle to the power lost in radiation. Thus we get the following equation of motion:

$$\gamma m v\dot{v} + m\gamma^3\frac{v^3\dot{v}}{c^2} - q\mathbf{E}.\mathbf{v} + \frac{\gamma^6 q^2\dot{\mathbf{v}}^2}{6\pi\epsilon_0 c^3} = 0 \tag{2.28}$$

In general in the presence of a force we may write the EOM as:

$$\gamma m v\dot{v} + m\gamma^3\frac{v^3\dot{v}}{c^2} - \mathbf{F}.\mathbf{v} + \frac{\gamma^6 q^2\dot{\mathbf{v}}^2}{6\pi\epsilon_0 c^3} = 0 \tag{2.29}$$

As we did in the Non-relativistic case, let us investigate this EOM in the absence of any force. In this case one solution is that the velocity remains constant (this is obvious) and the other solution is derived by solving the following equation:

$$\dot{v} = -\frac{mv + \frac{m\gamma^2 v^3}{c^2}}{A\gamma^5} \tag{2.30}$$

$$A = \frac{q^2}{6\pi\epsilon_0 c^3} \tag{2.31}$$

$$\tag{2.32}$$

Upon integration we get the following expression:

$$\frac{c\sqrt{c^2 - v^2} - (c^2 - v^2)\ln[\frac{2(c+\sqrt{c^2-v^2})}{v}]}{(c^2 - v^2)} = -\frac{mt}{A} \tag{2.33}$$

which can be simplified as follows ($A = m\tau$):

$$\frac{t}{\tau} = \ln[\frac{2c}{v}(1 + \frac{1}{\gamma})] - \gamma \tag{2.34}$$

This is a non-algebraic equation. So it seems very difficult to get an explicit expression for v. But we can surely comment on the long time behavior of v by investigating its behavior in the limit of $t \to \infty$. We can easily see that $v \to 0$ as $t \to \infty$. This is again in accordance with our intuition that the velocity must approach zero in the long time limit. Even here we see that the relevant time scale is again τ (as in equation (11)).

2.5 Discussion

Using the basic principle of Energy conservation, we have successfully derived the equation of motion (which are free from any unphysical solutions) for an accelerated charged particle in 1D. The same principle can be easily extended in higher dimensions as well. To base our theory even more concretely, we tried to derive the above equation of motion from a suitable Lagrangian. But since a Lagrangian for a dissipative system is not guaranteed this job was not successful. Of course as a next step we would like to explore the quantum regime. The main motivation would be to address the age old question of 'why an electron in an atom is stable?' (although some claim that Quantum Electrodynamics has answer for this!) Apart from the above discussed problem there is another major debatable issue of 'whether or not a uniformly accelerated charged particle radiates'. Several articles have been written on this due to its deep connection to 'Equivalence Principle' [11]. However all the analysis is based upon consideration that a constant force leads to a constant acceleration. We do not attempt to solve this issue but we would like to point out that in the light of Eqn. 2.19, it is difficult to obtain constant acceleration. A closer analysis of this issue taking into consideration the correct equation of motion is essential.

Chapter 3

A few radiation aspects in Gravity

3.1 Radiation reaction in Gravity

Similar to the radiation reaction in the case of electrodynamics there is analogous effect in the presence of gravity. Due to the emission of gravitational waves a particle looses energy, whose standard expression [12] in terms of the reduced Quadrapole moment tensor is as follows:

$$\frac{dE}{dt} = -\frac{G}{5c^5}\dddot{Q}^2 \tag{3.1}$$

where $\dddot{Q}^2 = \dddot{Q}_{\alpha\beta}\dddot{Q}^{\alpha\beta}$

$$Q_{\alpha\beta} = I_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}I \tag{3.2}$$

$$I^{\alpha\beta}(t) = \int y^\alpha y^\beta T^{00}(t, \mathbf{y})d^3y \tag{3.3}$$

$I^{\alpha\beta}$ is the Quadrapole moment tensor and T^{ab} is the stress-energy tensor. Now we will consider the simplest case of constant gravitational field. This means that all the components of the metric tensor are independent of the coordinate x^0, also called the world time. In this case [10] the energy of the particle is given by:

$$E = -c\frac{\partial S}{\partial x^0} \tag{3.4}$$

$$E = \gamma mc^2\sqrt{g_{00}} \tag{3.5}$$

Where $S = -mc\int ds$ is the action and $ds = g_{ab}dx^a dx^b$ is the invariant interval. Using our earlier ideas we equate the rate of change of this energy (note that taking time derivative is a bit subtle issue in GR - for eg. refer chap. 10 of [10]) to the energy radiated away to get the following relation:

$$\gamma^3 \sqrt{g_{00}} m v \dot{v} + \frac{G}{5c^5} \dddot{Q}^2 = 0 \tag{3.6}$$

3.2 Power emitted by Gravitational Waves: An attempt to write it as a power from two individual dipoles

Power from dipoles

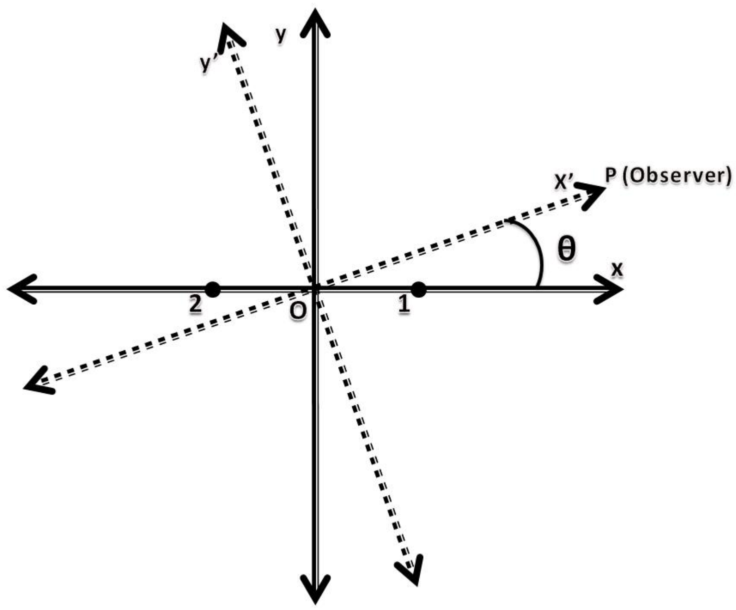

Figure 3.1: Two particles of equal mass undergoing harmonic oscillation. The dotted lines represent the rotated co-ordinate system at an angle θ w.r.t the original co-ordinate

MTW [2] considers a system of two particles attached to the opposite ends of a rotating rod and notes that roughly dipole power from two particles (acquired by adding the individual amplitudes with appropriate phase difference) is proportional to the standard quadrapole formula. Here we consider a simple system where two particles of equal mass are executing Harmonic oscillation along x-axis and their motion is out of phase (because of momentum conservation). The two masses are placed along the x-axis such that the mean position of particle 1 is $x = x_0/2$ and that of particle 2 is $x = -x_0/2$. We can describe their motion as follows:

$$x_1 = A\cos(\omega t) + \frac{x_0}{2} \tag{3.7}$$

$$x_2 = -A\cos(\omega t) - \frac{x_0}{2} \tag{3.8}$$

x_1 is the position of particle 1 and x_2 is that of particle 2.Here A is the Amplitude of the oscillation such that $A < x_0/2$.

Now let us consider an observer positioned at a distance R along a vector which makes an angle θ with the x-axis. Our aim is to calculate the power received by this observer as dipole radiation from particle 1 and that from particle 2. Throughout the calculation we will assume non-relativistic motion.

We know that along the direction of acceleration of a dipole there is no radiation. Hence it is convenient to consider a co-ordinate system in which the corresponding x-axis (we call it x'-axis) points toward the observer. Now we see that the motion along the x'-axis would not contribute anything. Only the oscillations along y'-axis would play a significant role. As we proceed, let us first write the position of the two particles in this new co-ordinate system.

$$x'_1 = (A\cos(\omega t) + \frac{x_0}{2})\cos\theta \tag{3.9}$$

$$x'_2 = -(A\cos(\omega t) - \frac{x_0}{2})\cos\theta \tag{3.10}$$

$$y'_1 = -(A\cos(\omega t) + \frac{x_0}{2})\sin\theta \tag{3.11}$$

$$y'_2 = (A\cos(\omega t) - \frac{x_0}{2})\sin\theta \tag{3.12}$$

Now for the dipole we need the second derivate of the y co-ordinates. Let us denote the individual dipole field amplitudes for particles 1 and 2 at the point P (where the observer is located) by g_1 and g_2 respectively. Apart from proportionality constants (which we have set to unity) and a factor of $1/R$, we can now write:

$$g_1 = m\ddot{y}'_1 \tag{3.13}$$

$$g_2 = m\ddot{y}'_2 \tag{3.14}$$

Now let the final field amplitude at point P be G_f. Then we can write,

$$|G_f|^2 = |g_1|^2 + |g_2|^2 + 2g_1 \cdot g_2 \cos(\Delta\phi) \tag{3.15}$$

where $\Delta\phi$ is the phase difference between the two waves arriving at P. Using simple geometry we can see that at any instant of time the path difference between the two waves is $\Delta x = |x_1 - x_2| \cos\theta$. So the corresponding phase difference becomes $\Delta\phi = \omega L \cos\theta$, where $L = |x_1 - x_2|$. Now substituting all the known relations in Eqn. 3.15, we get

$$|G_f|^2 = 2m^2 A^2 \omega^4 \cos^2(\omega t) \sin^2\theta (1 - \cos(\Delta\phi)) \tag{3.16}$$

we can expand $\cos(\Delta\phi) = 1 - (\Delta\phi)^2/2$. Analogous to ED dipole radiation formula, we can write an expression for the Gravitational dipole power which is proportional to the square of the field ($P = -G\,|G_f|^2/4\pi c^3$) and hence we get the gravitational wave power emitted by this source in terms of the individual dipole contributions to be:

$$P = -\frac{G}{4\pi c^5} m^2 A^2 \omega^6 L^2 \cos^2\omega t \sin^2\theta \cos^2\theta \tag{3.17}$$

This is the power emitted per solid angle. To get the total power emitted by the source we need to integrate over the solid angle (this integration gives a factor of $8\pi/15$).A quick glance to the above expression shows that $P = 0$ at $\theta = 0, \pi/2, \pi, 3\pi/2$. This is very much the nature of a Quadrapole radiation! However we should consider a time-average of the above quantity as power. Upon averaging we get:

$$P = -\frac{Gm^2 A^2 \omega^6}{15c^5}[3A^2 + x_0^2] \tag{3.18}$$

Power calculated using usual GW Quadrapole formula

The power of Gravitational waves as calculated using General Theory of Relativity depends on the third derivative of the reduced Quadrapole moment tensor Q.

$$\frac{dE}{dt} = -\frac{G}{5c^5}\dddot{Q}^2 \tag{3.19}$$

where $\dddot{Q}^2 = \dddot{Q}_{\alpha\beta}\dddot{Q}^{\alpha\beta}$

$$Q_{\alpha\beta} = I_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta} I \tag{3.20}$$

$$I^{\alpha\beta}(t) = \int y^\alpha y^\beta T^{00}(t, \mathbf{y}) d^3y \tag{3.21}$$

$I^{\alpha\beta}$ is the Quadrapole moment tensor and T^{ab} is the stress-energy tensor. Basically I is the moment of Inertia. For our system the various components of I are as follows:

$$I = m({x'_1}^2 + {y'_1}^2) + m({x'_2}^2 + {y'_2}^2) \tag{3.22}$$
$$I_{x'x'} = m({x'_1}^2 + {x'_2}^2) \tag{3.23}$$
$$I_{y'y'} = m({y'_1}^2 + {y'_2}^2) \tag{3.24}$$
$$I_{x'y'} = m(x'_1 y'_1 + x'_2 y'_2) \tag{3.25}$$

For clarity, we also write the third derivatives of the corresponding components.

$$\dddot{I} = \dddot{I}_{x'x'} + \dddot{I}_{y'y'} \tag{3.26}$$
$$\dddot{I}_{x'x'} = 4mA\omega^3 \cos^2\theta \sin\omega t[4A\cos\omega t + \frac{x_0}{2}] \tag{3.27}$$
$$\dddot{I}_{y'y'} = 4mA\omega^3 \sin^2\theta \sin\omega t[4A\cos\omega t + \frac{x_0}{2}] \tag{3.28}$$
$$\dddot{I}_{x'y'} = -4mA\omega^3 \sin\theta\cos\theta \sin\omega t[4A\cos\omega t + \frac{x_0}{2}] \tag{3.29}$$
$$\dddot{Q}^2 = \dddot{I}^2_{x'x'} + \dddot{I}^2_{y'y'} + 2\dddot{I}^2_{x'y'} - \frac{\dddot{I}^2}{3} \tag{3.30}$$
$$\dddot{Q}^2 = \frac{128}{3} m^2 A^2 \omega^6 (L - \frac{3x_0}{4})^2 \sin^2\omega t \tag{3.31}$$

Using the above expressions we can now write down the expression for the power as follows:

$$\frac{dE}{dt} = -\frac{128G}{15c^5} m^2 A^2 \omega^6 (L - \frac{3x_0}{4})^2 \sin^2\omega t \tag{3.32}$$

Upon time-averaging the above expression we arrive at the following expression:

$$P = -\frac{4Gm^2A^2\omega^6}{15c^5}[16A^2 + x_0^2] \tag{3.33}$$

Discussion

It is claimed in many textbooks (for eg. [12], [2]) that in a system of masses emitting Gravitational radiation since the center of mass is not accelerating (in the absence of any other force on this system), we can not have dipole radiation. Although this true in a gross sense, the real physical effect is that of individual dipoles radiating with a phase correlation that effectively becomes quadrapolar. In the simple case of two equal masses, we can easily see that the individual dipole contribution cancels only along the line of motion and along the perpendicular direction to it. In other directions one has to take into account proper phase factors and hence the individual dipoles need not cancel out! In fact we have shown that the quadrapole nature (at least qualitatively) can be viewed as an outcome of the interference term of the two individual dipole radiation. However comparing Eqn. 3.18 and 3.33, we see that our approach does not match exactly with the existing formula for power from Gravitational waves.

Chapter 4

Exploration of the Unruh Effect

It has been a dream of physicists since quite some years to unify Gravity and Quantum Mechanics. Many believe that the first step to wards this is the prediction of Hawking radiation. Shortly after that Davies and Unruh [3] predicted a similar result for accelerating particles in flat space time, which is now famously known as the 'Unruh Effect'. In this section we will discuss our study about the Unruh Effect. This would be basically a brief survey rather than any significant original piece of work. The idea is to put together various approaches to this problem that we came across so that the material serves as a good starting point for students to study this problem. However this survey is based on our understanding and interpretation of the discussed problem. So most of the analogies and ideas discussed are from our view point.
Consider the zero temperature vacuum state in flat Minkowski space time. A uniformly accelerated observer (with acceleration a) would not perceive it as a vacuum state, rather the observer would detect that the system is in a thermal bath of temperature

$$T = \frac{\hbar a}{2\pi k_B c} \tag{4.1}$$

where $\hbar$ is the reduced planck's constant, k_B is the Boltzmann constant and c is the value of speed of light in vacuum. This is the basic essence of the Unruh Effect. The effect arises mainly because the notion of vacuum is in general different for observers in different frame of references. This can be understood in the light of Quantum Field Theory [13]. Consider a complete set of orthonormal solutions to the Klein - Gordon equation:

$$\Box f_\alpha(x) = 0 \tag{4.2}$$

Being a complete set, $\{f_\alpha(x)\}$ can be used to expand any real field operator $\hat{\Phi}(x)$ as:

$$\hat{\Phi}(x) = \sum_\alpha \hat{a}_\alpha f_\alpha(x) + \hat{a}^\dagger_\alpha f^\dagger_\alpha(x) \tag{4.3}$$

where $\hat{a}_\alpha$ and $\hat{a}^\dagger_\alpha$ are annihilation and creation operators such that the vacuum state $|0\rangle^f$ satisfies the relation $\hat{a}_\alpha |0\rangle^f = 0$ for all α. Also,

$$[\hat{a}_\alpha, \hat{a}_\beta] = \left[\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta\right] = 0 \tag{4.4}$$

$$\left[\hat{a}_\alpha, \hat{a}^\dagger_\beta\right] = \delta_{\alpha\beta} \tag{4.5}$$

Annihilation and creation operators or one can say the mode functions $f_\alpha(x)$ can change if one makes a transformation to another frame of reference. This would then result in general in change of states too, particularly the vacuum state. As an illustration one can consider a new set of mode functions constructed through Bogoliubov transformation:

$$F_\alpha(x) \equiv \sum_\beta M_{\alpha\beta} f_\beta + N_{\alpha\beta} f^\dagger_\beta \tag{4.6}$$

Now,

$$\hat{\Phi}(x) = \sum_\alpha \hat{a}_\alpha f_\alpha(x) + \hat{a}^\dagger_\alpha f^\dagger_\alpha(x) = \sum_\alpha \hat{A}_\alpha F_\alpha(x) + \hat{A}^\dagger_\alpha F^\dagger_\alpha(x) \tag{4.7}$$

such that

$$\hat{a}_\alpha = \sum_\beta (M_{\alpha\beta}\hat{A}_\beta + N^*_{\beta\alpha}\hat{A}^\dagger_\beta) \tag{4.8}$$

$$\hat{a}^\dagger_\alpha = \sum_\beta (M^*_{\beta\alpha}\hat{A}^\dagger_\beta + N_{\beta\alpha}\hat{A}_\beta) \tag{4.9}$$

The vacuum state corresponding to this new set of mode functions is defined by the relation $\hat{A}_\alpha |0\rangle^F = 0$. The two vacua state $|0\rangle^F$ and $|0\rangle^f$ are entirely different. In fact, it turns out that the expectation value of the particle number in the state $|0\rangle^f$ calculated using the new mode functions is

$$\langle n\rangle =^f \langle 0|\, \hat{A}^\dagger_\alpha \hat{A}_\alpha \,|0\rangle^f = \sum_\beta |N_{\alpha\beta}|^2 \tag{4.10}$$

It means that the state which was vacuum for a choice of mode functions (in this case $\{f_\alpha\}$) can be a many particle state (for non-zero value of $N_{\alpha\beta}$) for another choice of mode functions (in this case $\{F_\alpha\}$)!

4.1 Derivation of Unruh effect

All inertial frames are related to each other via Lorentz transformation which is essentially a linear transformation and hence the form of the solution to Klein-Gordon equation remains same. So all inertial observers share the same vacuum state. However, in the case of a uniformly accelerated frame, the transformation for mode functions is non-trivial. To discuss the dynamics of uniformly accelerated system one needs to resort to Rindler co-ordinate system. The region with $|t| < z$ and $|t| < -z$ in the flat space time (t, z) is called Right and Left Rindler wedge respectively. The co-ordinate transformation to the right rindler wedge is given by (for simplicity, throughout the derivation we will stick to (1+1) dimension):

$$t = \frac{e^{a\xi}}{a} sinh(a\tau) \tag{4.11}$$

$$z = \frac{e^{a\xi}}{a} cosh(a\tau) \tag{4.12}$$

and that for the left rindler wedge is:

$$t = \frac{e^{a\bar{\xi}}}{a} sinh(a\bar{\tau}) \tag{4.13}$$

$$z = -\frac{e^{a\bar{\xi}}}{a} cosh(a\bar{\tau}) \tag{4.14}$$

where a is a positive constant. The physical motivation to consider this co-ordinate system is that $\xi = 0$ corresponds to the world line of a particle with uniform acceleration a and τ is the proper time in this frame of reference.

The massless scalar field $\hat{\Phi}(t, z)$ in flat minkowski space time can be expanded in terms of the solutions to the K-G equation as discussed above (we proceed with the derivation as discussed in [14]).

$$\hat{\Phi}(t, z) = \int_0^\infty \frac{dk}{\sqrt{4\pi k}} (\hat{b}_{-k} e^{-\iota k(t-z)} + \hat{b}_k e^{-\iota k(t+z)} + \hat{b}_{-k} e^{\iota k(t-z)} + \hat{b}_k e^{\iota k(t+z)}) \tag{4.15}$$

The creation and annihilation operators satisfy the commutation relation:

$$\left[\hat{b}_{\pm k}, \hat{b}^\dagger_{\pm k'}\right] = \delta(k - k') \tag{4.16}$$

Since the right and left moving solutions are independent we can group their respective terms by defining $U = t - z$ and $V = t + z$. Thus we can write $\hat{\Phi}(t, z) = \hat{\Phi}(U) + \hat{\Phi}(V)$, where

$$\hat{\Phi}(V) = \int_0^\infty dk [\hat{b}_k f_k(V) + \hat{b}_k f_k^*(V)] \tag{4.17}$$

$$f_k(V) = \frac{1}{\sqrt{4\pi k}} e^{-\iota k V} \tag{4.18}$$

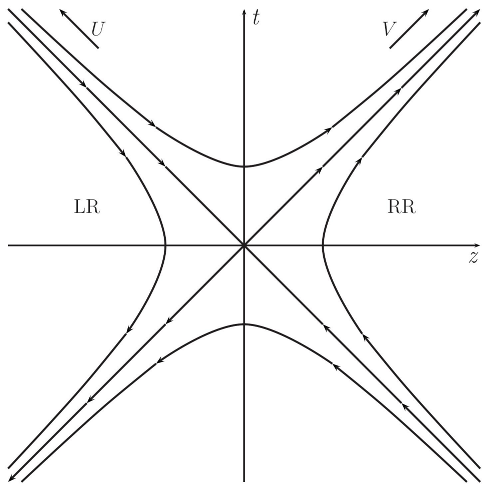

Figure 4.1: The region $|t| < z$ is the Right Rindler wedge and the region $|t| < -z$ is the Left Rindler wedge

Also we will now concentrate only on the left moving sector of the field since the treatment for the right moving part is essentially the same. The minkowski vacuum state $|0_M\rangle$ is defined by the relation $\hat{b}_k |0_M\rangle = 0$ for all k. Now in the right rindler wedge, we have a similar K-G equation:

$$(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\xi^2})g_\omega = 0 \tag{4.19}$$

So the solutions are again similar to that we discussed above in the case of minkowski. Here also we can separate out the left and right moving sectors by defining $u = \tau - \xi$ and $v = \tau + \xi$. Note that,

$$U = -a^{-1}e^{-au} \tag{4.20}$$
$$V = a^{-1}e^{av} \tag{4.21}$$

As discussed earlier we can now expand the scalar field $\hat{\Phi}$ in terms of these new mode functions such that,

$$\hat{\Phi}(V) = \int_0^\infty d\omega[\hat{a}^R_\omega g_\omega(v) + \hat{a}^{R\dagger}_\omega g^*_\omega(v)] \tag{4.22}$$
$$g_\omega(v) = (4\pi\omega)^{-1/2}e^{-\iota\omega v} \tag{4.23}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators for this new choice. These operators satisfy the usual commutation relations discussed above.

Similar procedure is to be followed in the left rindler wedge. Here we define $\bar{u} = \tau - \xi$ and $\bar{v} = \tau + \xi$ for the same reason discussed above. All the relations written for the right rindler wedge hold in this case with v replaced by $\bar{v}$ (such that $V = -a^{-1}e^{-a\bar{v}}$) and the creation/annihilation operators of right rindler wedge replaced by those in the left rindler wedge ($\hat{a_\omega}^L$ and $\hat{a_\omega}^{L\dagger}$). The rindler vacuum state $|0_R\rangle$ is defined by the relation $\hat{a}^R_\omega |0_R\rangle = \hat{a}^L_\omega |0_R\rangle = 0$.
Since both g_ω and f_k form complete set and are used to expand the scalar field, g_ω can be written as a linear combination of f_k and f_k^*. Thus we have,

$$\theta(V)g_\omega(v) = \int_0^\infty \frac{dk}{\sqrt{4\pi k}}(\alpha^R_{\omega k}e^{-\iota kV} + \beta^R_{\omega k}e^{\iota kV}) \tag{4.24}$$

$$\theta(-V)g_\omega(\bar{v}) = \int_0^\infty \frac{dk}{\sqrt{4\pi k}}(\alpha^L_{\omega k}e^{-\iota kV} + \beta^L_{\omega k}e^{\iota kV}) \tag{4.25}$$

where $\theta(V)$ is the Heaviside function such that $\theta(V) = 0$ if $V < 0$ and $\theta(V) = 1$ if $V > 0$. The α and β in the above relation are called the Bogoliubov coefficients. These are crucial in the derivation of the Unruh effect.
Let us now proceed to calculate the Bogoliubov coefficients. In order to calculate $\alpha^R_{\omega k}$, we multiply Eqn. 4.24 by $e^{\iota kV}/2\pi$ and integrate over V. This gives us:

$$\alpha^R_{\omega k} = \sqrt{\frac{k}{\omega}}\int_0^\infty \frac{dV}{2\pi}(aV)^{-\iota\omega/a}e^{\iota kV} \tag{4.26}$$

Thus we get,

$$\alpha^R_{\omega k} = \frac{\iota e^{\pi\omega c/2a}}{2\pi\sqrt{\omega k}}(\frac{a}{k})^{-\iota/\omega a}\Gamma(1 - \iota\frac{\omega}{a}) \tag{4.27}$$

After similar procedure we get rest of the bogoliubov coefficients as follows:

$$\beta^R_{\omega k} = -\frac{\iota e^{-\pi\omega c/2a}}{2\pi\sqrt{\omega k}}(\frac{a}{k})^{-\iota/\omega a}\Gamma(1 - \iota\frac{\omega}{a}) \tag{4.28}$$

$$\alpha^L_{\omega k} = -\frac{\iota e^{\pi\omega c/2a}}{2\pi\sqrt{\omega k}}(\frac{a}{k})^{\iota/\omega a}\Gamma(1 + \iota\frac{\omega}{a}) \tag{4.29}$$

$$\beta^L_{\omega k} = \frac{\iota e^{-\pi\omega c/2a}}{2\pi\sqrt{\omega k}}(\frac{a}{k})^{\iota/\omega a}\Gamma(1 + \iota\frac{\omega}{a}) \tag{4.30}$$

Note that

$$\beta^L_{\omega k} = -e^{-\pi\omega/a}\alpha^{R*}_{\omega k} \tag{4.31}$$

$$\beta^R_{\omega k} = -e^{-\pi\omega/a}\alpha^{L*}_{\omega k} \tag{4.32}$$

Now the most crucial part is to construct functions which are linear combination of positive frequency modes $e^{-\iota kV}$ in Minkowski space time. This is done by substituting above relations in Eqn. 4.24 and 4.25.

$$G_\omega(V) = \theta(V)g_\omega(v) + e^{-\pi\omega/a}\theta(-V)g^*_\omega(\bar{v}) \tag{4.33}$$
$$\bar{G}_\omega(V) = \theta(-V)g_\omega(\bar{v}) + e^{-\pi\omega/a}\theta(V)g^*_\omega(v) \tag{4.34}$$

Now inverting these two relations and substituting into

$$\hat{\Phi}(V) = \int_0^\infty d\omega[\theta(V)(\hat{a}^R_\omega g_\omega(v) + \hat{a}^{R\dagger}_\omega g^*_\omega(v)) + \theta(-V)(\hat{a}^L_\omega g_\omega(\bar{v}) + \hat{a}^{L\dagger}_\omega g^*_\omega(\bar{v}))] \tag{4.35}$$

we get

$$\hat{\Phi}(V) \propto \int_0^\infty d\omega[G_\omega(V)(\hat{a}^R_\omega - e^{-\pi\omega/a}\hat{a}^{L\dagger}_\omega) + \bar{G}_\omega(V)(\hat{a}^L_\omega - e^{-\pi\omega/a}\hat{a}^{R\dagger}_\omega) + H.c.] \tag{4.36}$$

At this point it is important to stress the need of doing all this mathematical jugglery. Our aim is to find a relation between the old and new annihilation/creation operators. For this we proceed by expressing the scalar field in terms of the two sets of complete mode functions (which we did). Then we write one of the modes in terms of the other. We did this by expressing g_ω in terms of f_k. Now we must substitute these relations in the scalar field expansion and make a comparison to the earlier expansion to get the desired relations between annihilation and creation operators. Simply putting, the process is similar to substituting Eqn. 4.6 in Eqn. 4.7 to get Eqn. 4.8 and 4.9. Now the problem here is that the two Rindler wedges are causally disconnected and one can see that easily even from Eqn. 4.35. So we need another function in terms of f_k which is continuous and hence we made an effort to construct the functions G_ω and $\bar{G}_\omega$ which connects the two rindler wedges.

Now upon comparing Eqn. 4.36 to Eqn. 4.15, we find that $(\hat{a}^R_\omega - e^{-\pi\omega/a}\hat{a}^{L\dagger}_\omega)$ and $(\hat{a}^L_\omega - e^{-\pi\omega/a}\hat{a}^{R\dagger}_\omega)$ take the role of $\hat{b}_k$ and hence must annihilate the Minkowski vacuum state.

$$(\hat{a}^R_\omega - e^{-\pi\omega/a}\hat{a}^{L\dagger}_\omega)\left|0_M\right\rangle = 0 \tag{4.37}$$
$$(\hat{a}^L_\omega - e^{-\pi\omega/a}\hat{a}^{R\dagger}_\omega)\left|0_M\right\rangle = 0 \tag{4.38}$$

To make our further calculations simpler and easier to understand we discretize the Rindler energy levels ω. Thus as a modification in notations we will replace ω with ω_i. So using the discrete version of the above relations and the commutators of the creation and annihilation operators, we find that

$$\left\langle 0_M\right| \hat{a}^{R\dagger}_{\omega_i}\hat{a}^R_{\omega_i}\left|0_M\right\rangle = e^{-2\pi\omega_i/a}\left\langle 0_M\right| \hat{a}^{L\dagger}_{\omega_i}\hat{a}^L_{\omega_i}\left|0_M\right\rangle + e^{-2\pi\omega_i/a} \tag{4.39}$$
$$\left\langle 0_M\right| \hat{a}^{L\dagger}_{\omega_i}\hat{a}^L_{\omega_i}\left|0_M\right\rangle = e^{-2\pi\omega_i/a}\left\langle 0_M\right| \hat{a}^{R\dagger}_{\omega_i}\hat{a}^R_{\omega_i}\left|0_M\right\rangle + e^{-2\pi\omega_i/a} \tag{4.40}$$

Now solving these two relations we find

$$\langle 0_M | \hat{a}^{R\dagger}_{\omega_i} \hat{a}^{R}_{\omega_i} |0_M\rangle = \langle 0_M | \hat{a}^{L\dagger}_{\omega_i} \hat{a}^{L}_{\omega_i} |0_M\rangle = \frac{1}{e^{2\pi\omega_i/a} - 1} \tag{4.41}$$

This gives the expectation value for the particle number in the minkowski vacuum state as seen by the accelerated observer. We see that this is identical to the Bose-Einstein distribution with an associated temperature T_u (Unruh Temperature) given by

$$T_u = \frac{\hbar a}{2\pi k_B c} \tag{4.42}$$

In short this means that a system undergoing uniform acceleration with respect to zero-temperature vacuum will come to equilibrium at an effective temperature given by above expression.
Initially it was thought that the presence of a horizon (due to acceleration of system from infinite past to infinite future) is essential for the Unruh effect. However, it has been shown that even for finite time acceleration, the effect remains valid [15, 16].

4.2 Alternate derivation of the Unruh Effect

The derivation of the Unruh effect given above is technically very challenging. There exists another way to derive this result in a more physically intuitive way [17]. We all are accustomed to the phenomenon of Doppler effect. Any observer moving at a constant velocity with respect to a wave source would experience a constant shift in the frequency of emitted way. Well, what happens if instead of constant velocity we consider constant acceleration? Intuitively we expect a time dependent Doppler shift i.e. the shift in frequency observed would change with time. More precisely speaking, the phase of the wave would be time dependent.
The trajectory of a uniformly accelerated particle in a (1+1) flat Minkowski spacetime (t,z) is given by:

$$t(\tau) = \frac{c}{a} sinh(\frac{a\tau}{c}) \tag{4.43}$$

$$z(\tau) = \frac{c^2}{a} cosh(\frac{a\tau}{c}) \tag{4.44}$$

where τ is the proper time of the accelerated observer and a is the acceleration. Standard Minkowski plane wave with frequency ω_k has the form proportional to $e^{\iota\phi_\pm} \equiv e^{\iota(kz \pm \omega_k t)}$, $k = \omega_k/c$. Now if we substitute above relations in the expression for ϕ, we get:

$$\phi_{\pm} = kz \pm \omega_k t = \frac{\omega_k c}{a} e^{\pm a\tau/c} \tag{4.45}$$

One quickly see the meaning of this relation if one identifies that

$$\phi_{\pm} \equiv \int^{\tau} \omega'_k(\tau')d\tau' \tag{4.46}$$

This means that

$$\omega'_k(\tau) = \omega_k e^{a\tau/c} \tag{4.47}$$

which is consequence of time dependent Doppler shift. From this one calculate the frequency spectrum $S(\Omega)$.

$$S(\Omega) \equiv \left| \int_{-\infty}^{\infty} d\tau e^{\iota\Omega\tau} e^{\iota(\omega_k c/a)e^{a\tau/c}} \right|^2 = \frac{2\pi c}{\Omega a} \frac{1}{e^{2\pi\Omega c/a} - 1} \tag{4.48}$$

We see the planck factor appearing in the above expression which hints towards Unruh effect. But this is not quite the complete derivation. The reason is the following. We have considered only a single frequency of the Minkowski mode and hence we would detect only a single frequency (though Doppler shifted) at a time in an accelerated frame too. However, quantum field in vacuum has components at all frequencies. So, if we consider all these frequencies only then can we realize the Unruh effect. Consider again a massless scalar field in (1+1) dimension quantized in a volume V:

$$\hat{\Phi} = \sum_k \sqrt{\frac{2\pi\hbar c^2}{\omega_k V}} (\hat{a}_k e^{-\iota\omega_k t} + \hat{a}_k^{\dagger} e^{\iota\omega_k t}) \tag{4.49}$$

Consider the fourier transform operator,

$$\hat{g}(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \hat{\Phi} e^{\iota\Omega t} \tag{4.50}$$

Since for a thermal state, $\hat{a}_k^{\dagger}\hat{a}_k$ has the expectation value $(e^{\hbar\omega_k/k_B T} - 1)^{-1}$, the expectation value $\left\langle \hat{g}^{\dagger}(\Omega)\hat{g}(\Omega') \right\rangle$ in thermal equilibrium is

$$\left\langle \hat{g}^{\dagger}(\Omega)\hat{g}(\Omega') \right\rangle = \sum_k \frac{2\pi\hbar c^2}{\omega_k V} \left\langle \hat{a}_k^{\dagger}\hat{a}_k \right\rangle \delta(\Omega - \Omega')\delta(\omega_k - \Omega) \tag{4.51}$$

In the limit of $V \to \infty$ we have

$$\left\langle \hat{g}^{\dagger}(\Omega)\hat{g}(\Omega') \right\rangle = \hbar c^2 \int_{-\infty}^{\infty} \frac{dk}{\omega_k} \frac{\delta(|k|\, c - \Omega)\delta(\Omega - \Omega')}{(e^{\hbar\Omega/k_B T} - 1)} = \frac{2\hbar c/\Omega}{(e^{\hbar\Omega/k_B T} - 1)} \delta(\Omega - \Omega') \tag{4.52}$$

Now let us do a similar calculation for an observer in uniform acceleration in the vacuum field of the accelerated frame. Here we have to use appropriate Doppler shifted frequencies.

$$\hat{g}(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau e^{\iota\Omega\tau} \sum_k \sqrt{\frac{2\pi\hbar c^2}{\omega_k V}}[\hat{a}_k e^{\iota(\epsilon_k \omega_k c/a)e^{-\epsilon_k a\tau/c}} + \hat{a}_k^\dagger e^{-\iota(\epsilon_k \omega_k c/a)e^{-\epsilon_k a\tau/c}}] \tag{4.53}$$

where ϵ_k is $|k|/k$. By using the fact that $\left\langle \hat{a}_k \hat{a}_{k'}^\dagger \right\rangle = \delta_{kk'}$ and performing the integral over τ as above, we get the expectation value for the correlation function to be

$$\left\langle \hat{g}^\dagger(\Omega)\hat{g}(\Omega')\right\rangle = \frac{\hbar c^2}{\pi a}\left|\Gamma(\frac{\iota\Omega c}{a})\right|^2 e^{-\pi\Omega c/a}\delta(\Omega-\Omega') = \frac{2\hbar c/\Omega}{e^{2\pi\Omega c/a}-1}\delta(\Omega-\Omega') \tag{4.54}$$

Thus we get the usual expression describing the Unruh Effect.

4.3 Experimental approaches to detect Unruh Effect

Although Unruh effect has sound mathematical basis and also a physically intuitive understanding, it has a controversial aspect to it and it is whether or not such an accelerated system emits radiation (we discuss this issue in the next section). This has stimulated a great interest in devicing an experiment to settle this issue. At the same time it is easy to see how difficult it is to proceed with an experiment involving purely mechanical motion. The value of acceleration corresponding to a Unruh temperature of 1 K is $a = 2.4 \times 10^{20} m/s^2$. Any body linearly accelerated at this value is likely to be distorted and deformed [18]. So alternate methods are to be deployed.
Although high linear accelerations are difficult to achieve, one can reach very high centripetal accelerations. Accelerations as high as of the order of $10^{23} m/s^2$ (this gives corresponding unruh temperature of the order of $1000K$) can be achieved in accelerator rings. It was well known theoretically and experimentally that circulating electrons in such storage rings become polarized to a very high degree. But the slight depolarization observed appealed Bell and Leinaas [18, 19] to investigate if this was an evidence for the Unruh effect. However the temperature they found for the distribution of residual depolarized electrons was about 1.4 times higher than that predicted by the expression for the Unruh temperature. Another similar scenario arises within an atom due to centripetal acceleration of the electrons. This suggests that

a study of distribution of electrons in energy levels very close to that of the ground state at very low ambient temperatures could reveal some connection to Unruh effect. There is a recent proposal to study this effect in larger atoms like oxygen and fluorine [20].
If we consider a Hydrogen atom, the acceleration ($a = \gamma^2 c^2/r$) in the rest frame of an electron in its ground state (i.e. at a distance of first bohr orbit $r_0 = 5.3 \times 10^{-11} m$ such that $\gamma = 1$) is $a = 1.69 \times 10^{27} m/s^2$. This gives tremendous amount of corresponding Unruh temperature ($T_u = 48364262K$). Now the energy corresponding to the hyperfine splitting of hydrogen atom at $21cm$ wavelength is about $E_{hy} = 9.4 \times 10^{-27} J$. It is easy to see that $k_B T_u >> E_{hy}$. Hence if there existed Unruh radiation, we must expect that the probability distribution in the hyperfine states to be 1/2. However, most of the Hydrogen in the Universe is in the ground hyperfine state! As a possible resolution we conjecture that 'Unruh Effect', if it exists, is only applicable for linear acceleration and not for circular case with nearly constant angular velocity.
Another recent proposal [21] suggests using 'Berry's phase' to observe the Unruh effect. According to the investigation, the berry's phase for an inertial detector is different from that of an accelerated detector. This difference if observed, according to them can be viewed as an outcome of the Unruh effect. There are also proposals to construct superconducting circuits to realize this effect (on the similar lines to the way Dynamical Casimir Effect is being claimed to have been observed). Also looking at electrons in microwave cavity is another approach (for more details see [14]). However, most of these experimental proposals aim at observing the Unruh effect and are not aimed at proving or disproving the existence of the corresponding Unruh radiation. So the radiation aspect to this effect still remains controversial.

4.4 Radiation corresponding to the Unruh Effect

Many feel it obvious for the accelerated system to radiate since it is at an elevated temperature w.r.t. minkowski zero temperature [15]. However, Ford and O'Connell [22] have shown using detailed balance calculation that the uniformly accelerated system would not radiate. There are few others who have the opinion that there is no Unruh radiation.
We find the phenomenon of 'Unruh Effect' very similar to the picture of 'pseudo force' in Newtonian mechanics. Consider a ball attached to string hanging from the ceiling of a closed car with an observer inside it. In an

inertial frame this setup has its natural ground state in the vertical position. However if the closed car mentioned above is uniformly accelerating then the ball's equilibrium position is shifted from vertical position to a position at an angle which depends on the acceleration of the car. Now the observer accustomed to being in an inertial frame would obviously conclude that the ball is in an excited state in the presence of a force (which he can not detect!). However, since this is the new equilibrium position there is no question of the ball coming back to the vertical position to minimize its energy. The situation in case of Unruh effect is somewhat similar. In the accelerated frame the system is in its new vacuum state. Then why should it radiate? As long as the system is accelerating there should not be any radiation emitted by the system. However, if such an accelerating system comes to a halt suddenly, then we can expect it to radiate because the state in which it is freezed is a many particle state w.r.t. Minkowski spacetime! However, even if such radiation does exist it would not be appropriate to call it Unruh radiation since this would be an outcome of a transient phenomenon. But surely it can serve as a good proof to Unruh effect because the existence of such radiation after stopping (i.e. after removal of external forcing agent) must necessarily imply that the system was in an excited state. This is a bit difficult to work out and requires some time. So it was not possible to make progress in this direction.

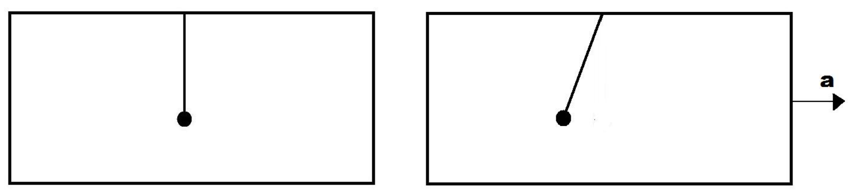

Figure 4.2: Schematic of string-ball system in a stationary car and an accelerating car

Also in general for many atoms the corresponding Unruh temperature due to centripetal acceleration is very high. However we do not see any spontaneous excitations to excited state in such atoms [23]. So it is very hard to believe that radiation corresponding to Unruh effect does exist.

4.5 Further Work

We have conjectured above the absence of 'Unruh Effect' in the circular case. However we would like to make some progress in the experimental determination of the 'Unruh Effect' in future. Apart from this we mainly aim at working on three important issues:

1. Study of radiation phenomenon in the context of relation between acceleration and Gravity. This includes study of 'Equivalence Principle' in the presence of radiating charged particle.

2. Relation between Unruh Effect, Hawking radiation and Dynamical Casimir Effect. Since all these phenomena are related to the 'Quantum vacuum', it is important to understand their relation to each other. Already many attempts have been made in this direction. We would like to revisit these due to its importance from experimental determination point of view.

3. The puzzle regarding the absence of radiation from stationary orbits in atoms in the context of rest of the radiation phenomena.

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