Introduction to Affine and Projective Varieties

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This is to certify that this thesis entitled ”Introduction to Affine and Projective Varieties” submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents work carried out by Ayesha Fatima under the supervision of Prof. Nitin Nitsure.

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To Baba...
For everything

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Abstract

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by Ayesha Fatima

In this thesis we give detailed solutions of the exercises in the first chapter of the textbook 'Algebraic Geometry' by Robin Hartshorne. We have followed this with an essay in which we have proved two theorems which bring out some relationships between the algebro-geometric notions and those coming from complex manifolds.

Contents

Chapter 1

Affine Varieties

Exercise 1.0.1. *(a) Let Y be the plane curve $y = x^2$ i.e., Y is the zero set of the polynomial $f = y - x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.*

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

(c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to either $A(Y)$ or $A(Z)$. Which one is it when?

Solution:

(a) We have $Y = \mathcal{Z}(y - x^2)$. We prove that $y - x^2$ is an irreducible polynomial. Consider $y - x^2$ as a polynomial in x with coefficients in $k[y]$. Suppose $y - x^2$ is reducible. Then it has two factors each of degree 1. Let $ax + b$ and $cx + d$ be the two linear factors where a, b, $c,$ and d are elements of $k[y]$. Therefore we have $ac = -1$. Therefore a and c are elements of k such that $a = -1/c$. We also have $ad + bc = 0$. Putting $c = -1/a$ in this, we get $a^2d - b = 0$ i.e., $b - a^2d = 0$. But $bd = y$ i.e., $(ad)^2 = y$. This means that y is a square of a polynomial in y which is not true. Therefore $y - x^2$ is irreducible. Since $k[x, y]$ is a Unique Factorization Domain, $(y - x^2)$ is a prime ideal of $k[x, y]$ and $\sqrt{(y - x^2)} = (y - x^2)$. Therefore $I(Y) = \sqrt{(y - x^2)} = (y - x^2)$.

$A(Y) = k[x, y]/I(Y) = k[x, y]/(y - x^2)$. We claim that $k[x, y]/(y - x^2)$ is isomorphic to a polynomial ring in one variable $k[t]$. Define a map $\phi : k[x, y] \longrightarrow k[t]$ by $f(x, y) \mapsto f(t, t^2)$. This map is clearly a ring homomorphism. Also, any polynomial $f(t) \in k[t]$ is the image of the polynomial $f(x) \in k[x, y]$. Therefore this is a surjective ring homomorphism.

Let $f(x, y)$ be an element in the ideal generated by $y - x^2$. Therefore $f(x, y) = (y - x^2)g(x, y)$ for some polynomial $g(x, y)$ and thus $f(t, t^2) = 0$. Therefore $(y - x^2) \subsetneq ker\phi$.
Let $f(x, y)$ be an element of $ker\phi$. Consider $f(x, y)$ as a polynomial in y with coefficients in $k[x]$. If we divide $f(x, y)$ by the polynomial $y - x^2$, which is a linear polynomial in y, then we have $f(x, y) = (x^2 - y)g(x, y) + h(x, y)$. Since $deg\, h(x, y) < deg\,(y - x^2) = 1$, $h(x, y)$ is polynomial in y of degree 0 i.e., a polynomial in x. Since $f(t, t^2) = 0$, we have $h(t) = 0$. Therefore $h(x)$ is the zero polynomial and hence $f(x, y) \in (y - x^2)$. Therefore $ker\,\phi = (y - x^2)$. Therefore we have $A(Y) = k[x, y]/(y - x^2) \cong k[t]$.

(b) $Z = \mathcal{Z}(xy - 1)$. We claim that $xy - 1$ is irreducible. Consider $xy - 1$ as a polynomial in x with coefficients in $k[y]$. Suppose it is reducible. Then it has two linear factors. Suppose $ax + b$ and $cx + d$ are the two linear factors of $xy - 1$. Then $ac = 0$ and $bd = 1$. Therefore both b and d are elements of k. also either a or c is equal to 0. Suppose $a = 0$. Then $ax + b \in k$. This contradicts the fact that $ax + b$ is a polynomial of degree 1 in x. Therefore $xy - 1$ is an irreducible polynomial and $(xy - 1)$ is a prime ideal of $k[x, y]$ and thus $\sqrt{(xy - 1)} = (xy - 1)$. Therefore $I(Z) = (xy - 1)$ and $A(Z) = k[x, y]/I(Z) = k[x, y]/(xy - 1)$.

We claim that $k[x, y]/(xy - 1)$ is isomorphic to the Laurent polynomial ring in x, $k[x, \frac{1}{x}]$. Define a map $\phi : k[x, y] \longrightarrow k[x, \frac{1}{x}]$ by sending the polynomial $f(x, y)$ to the Laurent polynomial $f(x, \frac{1}{x})$. This map is clearly a ring homomorphism. Also, ϕ is surjective because any Laurent polynomial $f(x, \frac{1}{x})$ is the image of the polynomial $f(x, y)$.

Suppose $f(x, y)$ is a polynomial in $(xy - 1)$. Then $f(x, y) = (xy - 1)g(x, y)$ for some polynomial $g(x, y)$ in $k[x, y]$. Therefore $f(x, \frac{1}{x}) = 0$ and $f(x, y) \in ker\phi$. Therefore $(xy - 1) \subset ker\phi$.

Let $f(x, y)$ be an element of $ker\phi$. Consider $f(x, y)$ as a polynomial in y with coefficients in $k[x]$. If we divide $f(x, y)$ by the polynomial $xy - 1$, which is a linear polynomial in y, then we have $f(x, y) = (xy - 1)g(x, y) + h(x, y)$. Since $deg\, h(x, y) < deg\,(xy - 1) = 1$, $h(x, y)$ is polynomial in y of degree 0 i.e., a polynomial in x. Since $f(x, \frac{1}{x}) = 0$, we have $h(x) = 0$. Therefore $f(x, y) \in (xy - 1)$ and thus $ker\phi = (xy - 1)$. Thus $A(Y) = k[x, y]/(xy - 1) \cong k[x, \frac{1}{x}]$.

Suppose $k[x, \frac{1}{x}] \cong k[t]$, polynomial ring in the variable t. Suppose φ is an isomorphism from $k[x, \frac{1}{x}]$ to $k[t]$. Since φ maps an invertible element to an invertible element, $\varphi(k^{\times}) \subset k^{\times}$. Also since x is an invertible element of $k[x, \frac{1}{x}]$, $\varphi(x)$ has to be an invertible element of $k[t]$ and therefore an element of $k^{\times}$. Therefore $\varphi(k[x, \frac{1}{x}]) \subset k$. This is a contradiction. Therefore $k[x, \frac{1}{x}]$ is not isomorphic to a polynomial ring in one variable.

(c) Suppose $f = ax^2 + bxy + cy^2 + dx + ey + f$ be any irreducible quadratic polynomial. We let $x = ucos\,\theta - v\,sin\,\theta$ and $y = u\,sin\,\theta + v\,cos\,\theta$ for a some angle θ (This amounts to a rotation of axes by an angle θ). Substituting these equations in f and letting the coefficient of uv be 0, we get $tan\,\theta = b/a - c$. The equation is now of the form $Au^2 + Cv^2 + Du + Ev + F = 0$. Completing the squares and by a change of coordinates, we can convert the irreducible equation to one of the following standard forms:

$Y = X^2$ (parabola) $\longrightarrow$ $(eq.\,1)$ (when $AC = 0$)

$\dfrac{X^2}{A_1^2} + \dfrac{Y^2}{B_1^2} = 1$ (ellipse) $\longrightarrow$ $(eq.\,2)$ (when $AC > 0$)

$\dfrac{X^2}{A_1^2} - \dfrac{Y^2}{B_1^2} = 1$ (hyperbola) $\longrightarrow$ $(eq.\,3)$ (when $AC < 0$)

Equation (1) is the case considered in part (a). Putting $X_1 = X/A_1$ and $Y_1 = iY/B_1$ in the equation (2) converts it to the equation $X_1^2 - Y_1^2 = 1$. Putting $X_1 = X/A_1$ and $Y_1 = Y/B_1$ in the equation (2) converts it to the equation $X_1^2 - Y_1^2 = 1$. Putting $X_1 = (U - V)/\sqrt{2}$ and $Y_1 = U + V/\sqrt{2}$, the equation $X_1^2 - Y_1^2 = 1$ gets converted to $UV = 1$ which is the same as the case considered in part (b).

Exercise 1.0.2 (The twisted cubic curve)**.** *Let $Y \subsetneq \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find the generators of the ideal*

$I(Y)$. Show that $A(Y)$ is isomorphic to the polynomial ring in one variable over k. We say that Y is given by the parametric representation $x = t$, $y = t^2$, $z = t^3$.

Solution:
$Y = \mathcal{Z}(y - x^2, z - x^3)$. We claim that $I = (y - x^2, z - x^3)$ is a radical ideal. Consider the map $\phi : k[x, y, z]/I \longrightarrow k[x]$ which sends the element $f(x, y, z)$ to the element $f(x, x^2, x^3)$ and the map $\psi : k[x] \longleftrightarrow k[x, y, z]/I$ sending the element $f(x)$ to itself. Then ϕ and ψ are ring homomorphisms. Also $\phi(\psi(f)) = f$ for any polynomial $f \in k[x]$. If f is an element of $k[x, y, z]/I$, then $\psi(\phi(f))$ $= \psi(f(x, x^2, x^3)) = f(x, x^2, x^3)$. Since $x^2 \equiv y$ and $x^3 \equiv z$ in $k[x, y, z]/I$, $f(x, x^2, x^3) \equiv f(x, y, z)$ in $k[x, y, z]/I$. Therefore the ring homomorphisms ϕ and ψ are inverses of each other and thus $k[x, y, z]/(y - x^2, z - x^3) \cong k[x]$. Since $k[x]$ is an integral domain, $(y - x^2, z - x^3)$ is a prime ideal and thus a radical ideal. Therefore $I(Y) = \sqrt{(y - x^2, z - x^3)} = (y - x^2, z - x^3)$.

Therefore $A(Y) = k[x, y, z]/I(Y) \cong k[x]$ and $dim\, Y = dim\, A(Y) = dim\, k[x] = 1$. Therefore $A(Y)$ is an affine variety of dimension 1.

Exercise 1.0.3. *Let Y be the algebraic set in $\mathbb{A}^3$ defined by two polynomials $x^2 - yz$ and $xz - x$. Show that Y is the union of three irreducible components. Describe them and find their prime ideals.*

Solution:
$Y = \mathcal{Z}(x^2 - yz, xz - x)$. Therefore for any point $(x, y, z) \in Y$ we have $x^2 - yz = 0$ and $x(z - 1) = 0$. If $x = 0$, we have $yz = 0$ and therefore either $y = 0$ or $z = 0$. Therefore any point of the form $(0, t, 0)$ or $(0, 0, t)$ in $\mathbb{A}^3$ belongs to Y where $t \in k$.

If $z = 1$, we have $x^2 = y$. Therefore any point of the form $(t, t^2, 1)$ belongs to Y for all $t \in k$.
Therefore $\mathcal{Z}(x^2 - yz, xz - x) = \mathcal{Z}(x, z) \cup \mathcal{Z}(x, y) \cup \mathcal{Z}(x^2 - y, z - 1)$.

Let $I_1 = (x, z)$, $I_2 = (x, y)$ and $I_3 = (x^2 - y, z - 1)$. We claim that we have $k[x, y, z]/I_i \cong k[t]$ for $i = 1, 2, 3$.
To prove for $i = 1$:
Let $\phi : k[x, y\, z] \longrightarrow k[y]$ be the map defined by sending the element $f(x, y, z)$ to the element $f(0, y, 0)$. This map is clearly a ring homomorphism. Also, any

$f(y) \in k[y]$ is the image of the element $f(y) \in k[x, y\, z]$. Therefore ϕ is a surjective ring homomorphism. Let $f(x, y, z) \in (x, z)$. Then
$f(x, y, z) = xg(x, y, z) + zh(x, y, z)$ for some polynomials $g(x, y, z)$,
$h(x, y, z) \in k[x, y, z]$. Since $\phi(f(x, y, z)) = 0$, $f(x, y, z) \in ker\, \phi$. Therefore $(x, z) \subset ker\, \phi$. Conversely, let $f(x, y, z)$ be an element of $ker\, \phi$. Therefore $f(0, y, 0) = 0$. Write $f(x, y, z)$ as $xg(x, y, z) +\ h(y, z)$ where g and h are polynomials. Therefore $h(y, 0) = 0$. Write $h(y, z)$ as $zp(y) + q(y)$ where p and q are polynomials. Therefore $q(y) = 0$. Thus $f(x, y, z) = xg(x, y, z) +\ zp(y)$ i.e., $f(x, y, z) \in (x, z)$. Therefore $k[x, y, z]/(x, z) \cong k[y] \cong k[t]$.

To prove for $i = 2$:
Let $\psi : k[x, y\, z] \longrightarrow k[z]$ be the map defined by sending the element $f(x, y, z)$ to the element $f(0, 0, z)$. This map is clearly a ring homomorphism. Also, any $f(z) \in k[z]$ is the image of the element $f(z) \in k[x, y\, z]$. Therefore ψ is a surjective ring homomorphism. Let $f(x, y, z) \in (x, y)$. Then
$f(x, y, z) = xg(x, y, z) + yh(x, y, z)$ for some polynomials $g(x, y, z)$,
$h(x, y, z) \in k[x, y, z]$. Since $\psi(f(x, y, z)) = 0$, $f(x, y, z) \in ker\, \psi$. Therefore $(x, y) \subset ker\, \psi$. Conversely, let $f(x, y, z)$ be an element of $ker\, \psi$. Therefore $f(0, 0, z) = 0$. Write $f(x, y, z)$ as $xg(x, y, z) +\ h(y, z)$ where g and h are polynomials. Therefore $h(0, z) = 0$. Write $h(y, z)$ as $yp(z) + q(z)$ where p and q are polynomials. Therefore $q(z) = 0$. Thus $f(x, y, z) = xg(x, y, z) +\ yp(z)$ i.e., $f(x, y, z) \in (x, y)$. Therefore $k[x, y, z]/(x, y) \cong k[z] \cong k[t]$.

To prove for $i = 3$: Let $\varphi : k[x, y, z]/I_3 \longrightarrow k[x]$ be the map defined by sending the element $f(x, y, z)$ to the element $f(x, x^2, 1)$. This map is clearly a ring homomorphism. Also, any $f(x) \in k[x]$ is the image of the element $f(x) \in k[x, y, z]$. Therefore φ is a surjective ring homomorphism. Let $\phi : k[x] \longrightarrow k[x, y, z]/I_3$ be the map defined by sending the element $f(x)$ to itself. ϕ is clearly a ring homomorphism. Also $\varphi(\phi(f(x))) = f(x)$ for any element $f(x) \in k[x]$. For any element $f(x, y, z) \in k[x, y, z]/I_3$, $\phi(\varphi(f(x, y, z))) = f(x, x^2, 1)$. Since $x^2 \equiv y$ and $z \equiv 1$ in $k[x, y, z]/I_3$, we have $f(x, x^2, 1) \equiv f(x, y, z)$ in $k[x, y, z]/I_3$. Therefore the ring homomorphisms ϕ and φ are inverses of each other and thus $k[x, y, z]/I_3 \cong k[x] \cong k[t]$.

Since $k[t]$ is an integral domain, I_i is a prime ideal and thus $I(\mathcal{Z}(I_i)) = \sqrt{I_i} = I_i$ for each $i = 1, 2, 3$. Therefore we have $\mathcal{Z}(I_1)$, $\mathcal{Z}(I_2)$ and $\mathcal{Z}(I_3)$ as the irreducible components of Y with I_1, I_2 and I_3 as their respective prime ideals.

Exercise 1.0.4. *If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.*

Solution:
Proper closed subsets of $\mathbb{A}^1$ are finite subsets of $\mathbb{A}^1$. Closed sets of $\mathbb{A}^1 \times \mathbb{A}^1$ in product topology are finite union of basic open sets which are of the form $X \times Y$ where X and Y are closed in $\mathbb{A}^1$. If X (or Y) is equal to $\mathbb{A}^1$, then $X \times Y$ looks like a finite union of horizontal (or vertical) lines in $\mathbb{A}^1 \times \mathbb{A}^1$. If both X and Y are proper subsets of $\mathbb{A}^1$, then $X \times Y$ is finite set of points in $\mathbb{A}^1 \times \mathbb{A}^1$. If both X and Y are equal to $\mathbb{A}^1$ (or $\emptyset$), then $X \times Y$ is equal to $\mathbb{A}^1 \times \mathbb{A}^1$ (or $\emptyset$).
Consider the closed set $\mathcal{Z}(y - x)$ in $\mathbb{A}^2$. It is an infinite subset of $\mathbb{A}^2$ because it is equal to the set $\{(t, t) | \, t \in k\}$. We claim that it is also not equal to union of finite number of vertical and horizontal lines. Suppose that it is equal to a union of finite number of vertical and horizontal lines H_i and V_j where $i = 1, \dots, n$ and $j = 1, \dots, m$ for some non-negative integers m and n. Each H_i is of the form $\{(t, h_i) \, | \, t \in k\}$ for a fixed element $h_i \in k$ and each V_j is of the form $\{(v_j, t) \, | \, t \in k\}$ for a fixed element $v_j \in k$. Each H_i has only one point of the form (t, t) i.e., the point (h_i, h_i) and each V_j has only one point of the form (t, t) i.e., (v_j, v_j). Therefore there are only finitely many points of the form (t, t) in $(\bigcup_{i=1}^{n} H_i) \cup (\bigcup_{j=1}^{m} V_j)$. This is a contradiction since $\mathcal{Z}(y - x) = \{(t, t) | \, t \in k\}$ has infinitely many points (because k is an algebraically closed field). Therefore $\mathcal{Z}(y - x)$ is not a closed set of $\mathbb{A}^1 \times \mathbb{A}^1$. Therefore the product topology and the Zariski topology on $\mathbb{A}^2$ is not the same.

Exercise 1.0.5. *Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.*

Solution:
Suppose B is a k-algebra such that $B \cong k[x_1, x_2, \dots, x_n]/I(Y)$ where Y is an algebraic set in $\mathbb{A}^n$. Clearly B is a finitely generated k-algebra. Let $\overline{x}$ be nilpotent element of B. Then $\overline{x}^m = 0$ for some m. Therefore $x^m \in I(Y)$ i.e., $x \in \sqrt{I(Y)} = I(Y)$ (because $I(Y)$ is a radical ideal).

Conversely, suppose B is a finitely generated k-algebra with no nilpotent elements. Then B is of the form $k[x_1, x_2, \ldots, x_n]/I$ for some n. We claim that I is a radical ideal. If $x \in \sqrt{I}$, then $x^m \in I$ for some m and $\overline{x}$ is a nilpotent element of B. But B has no nilpotent elements; therefore, $\overline{x} = 0$ i.e., $x \in I$. Therefore I is a radical ideal and $I(\mathcal{Z}(I)) = I$. Thus B is the affine coordinate ring of $\mathcal{Z}(I)$.

Exercise 1.0.6. *Any non empty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.*

Solution:

Let U be a non-empty open subset of the irreducible topological space X. Suppose closure of U, $\overline{U}$, is a proper subset of X. Then X can be written as union of two non empty proper closed subsets U^c and $\overline{U}$ which is a contradiction. Therefore $\overline{U} = X$ i.e., U is dense.

Suppose U is reducible. Let $U = A_1 \cup A_2$ where A_1 and A_2 are proper closed subsets of U.

Then $A_1 = B_1 \cap U$ and $A_2 = B_2 \cap U$ for some proper closed subsets B_1 and B_2 of X.

Then $X = (B_1 \cup U^c) \cup B_2$. Since $B_1 \cup U^c$ and B_2 are proper closed subsets of X, this is a contradiction. Therefore U is irreducible.

Suppose $\overline{Y}$ is reducible. Let $\overline{Y} = A_1 \cup A_2$ where A_1 and A_2 are proper closed subsets of $\overline{Y}$. Since $\overline{Y}$ is the smallest closed subset containing Y, Y is not properly contained in either A_1 or A_2 and $A_1 \cap Y$ and $A_2 \cap Y$ are proper closed subsets of Y. But then $Y = (A_1 \cap Y) \cup (A_2 \cap Y)$ which is a contradiction. Therefore $\overline{Y}$ is irreducible.

Exercise 1.0.7. *(a) Show that the following conditions are equivalent for a topological space X:*

(i) X is noetherian; (ii) every non-empty family of closed subsets has a minimal element;

(iii) X satisfies the ascending chain condition for open sets (iv) every non-empty family of open subsets has a maximal element.

(b) *A noetherian topological space is quasi compact, i.e., every open cover has a finite sub-cover.*

(c) *Any subset of a noetherian topological space is noetherian in its induced topology.*

(d) *A noetherian space which is also Hausdorff must be a finite set with the discrete topology.*

Solution:

(a) 1) $\implies$ 2): Suppose there exists a non empty family Σ of closed subsets of X which has no minimal element. Let C_1 be any closed set in Σ. Since C_1 is not a minimal element of Σ, there exists a closed subset C_2 such that $C_2 \subsetneq C_1$. Since C_2 is not minimal, there exists a closed subset C_3 in Σ such that $C_3 \subsetneq C_2$. Proceeding in this way, we can produce by the axiom of choice an infinite strictly decreasing chain of closed sets of Σ. Therefore X is not noetherian.
2) $\implies$ 3): Consider an ascending chain of open subsets $U_1 \subsetneq U_2 \subsetneq \ldots \subsetneq U_n \subsetneq \ldots$. Let $C_i = U_i^c$. Then $C_1 \supset C_2 \supset \ldots C_n \supset \ldots$ terminates because the collection of closed subsets $\{C_i\}$ has a minimal element. Therefore the chain $U_1\ U_2 \subsetneq \ldots \subsetneq U_n \subsetneq \ldots$ terminates.
3) $\implies$ 4): Suppose there exists a non empty family Σ of open subsets which has no maximal element. Let U_1 be any open set of Σ. Since U_1 is not maximal, there exists an open set U_2 in Σ such that $U_1 \subsetneq U_2$. Since U_2 is not maximal, there exists an open set U_3 in Σ such that $U_2 \subsetneq U_3$. proceeding in this way, we can construct by axiom of choice an infinite strictly increasing chain of open sets in Σ.
4) $\implies$ 1): Consider any descending chain $C_1 \supset C_2 \supset \ldots C_n \supset \ldots$ of X. Let $U_i = C_i$. Then $\{U_i\}$ is a family of open subsets of X. Let U_n be the maximal element of this family. Then $U_m = U_{m+1}$ for all $m \geq n$. Therefore $C_m = C_{m+1}$ for all $m \geq n$ and the descending chain of closed subsets terminates.

(b) Let $\{U_i\}_{i \in I}$ be an open cover of the noetherian topological space X. Consider the family Σ consisting of all open sets which are finite unions of open sets in the open cover. Then Σ has a maximal element. Suppose the maximal element is $U_1 \cup U_2 \cup \ldots U_n$. For any open set U_i in the open cover, $U_1 \cup U_2 \cup \ldots U_n \cup U_i = U_1 \cup U_2 \cup \ldots U_n$, i.e., $U_i \subset U_1 \cup U_2 \cup \ldots U_n$. Therefore $U_1 \cup U_2 \cup \ldots U_n$ forms a finite sub-cover of the open cover $\{U_i\}_{i \in I}$. Therefore there exists a finite sub-cover of every open cover of X and X is quasi compact.

(c) Let Y be a subset of the noetherian topological space X. Let $A_0 \subsetneq A_1 \subsetneq A_2 \ldots$ be an ascending chain of open subsets of Y. Then $A_i = B_i \cap Y$ for some open subset B_i of X. We have an ascending chain of open subsets $B_0 \subsetneq B_0 \cup B_1 \subsetneq B_0 \cup B_1 \cup B_2 \subsetneq \ldots$. Therefore for some n we have $B_0 \cup B_1 \cup \ldots \cup B_n = B_0 \cup B_1 \cup \ldots \cup B_n \cup B_{n+1}$ i.e., $B_{n+1} \subsetneq B_0 \cup B_1 \cup \ldots B_n$. Thus $B_{n+1} \cap Y \subsetneq (B_0 \cup B_1 \cup \ldots B_n) \cap Y = (B_0 \cap Y) \cup (B_1 \cap Y) \cup \ldots \cup (B_n \cap Y) = A_0 \cup A_1 \cup \ldots \cup A_n = A_n$ Therefore $A_{n+1} = A_n$ and any ascending chain of open subsets of Y terminates and Y is noetherian.

(d) Let Y be any subset of a Hausdorff, noetherian topological space X. From part (c) we have that Y is noetherian. Therefore Y is quasi compact. We claim that any quasi-compact subset of a Hausdorff space is closed. To prove this let $x \in X \setminus Y$ be a point. Since X is Hausdorff, for any $y \neq x$ in X we can find two disjoint open subsets U_y and V_y of X such that $y \in U_y$ and $x \in V_y$. Therefore $\{U_y\}_{y \in Y}$ is an open cover of Y. Since Y is quasi-compact, there exists a finite sub-cover of $\{U_y\}_{y \in Y}$. Let $U_{y_1}, U_{y_2}, \ldots U_{y_n}$ be the finite sub-cover. $A = \bigcap_{i=1}^{n} V_{y_i}$ is an open subset of X containing x such that $A \cap U_{y_i} = \emptyset$ for $i = 1, 2, \ldots n$. Therefore $A \cap Y = \emptyset$. Therefore for every $x \in X \setminus Y$ we can find an open subset A of X such that $x \in A$ and $A \subsetneq X \setminus Y$ i.e., $X \setminus Y$ is open. Therefore Y is closed. Therefore any subset of a Hausdorff, noetherian topological space X is closed and thus X has discrete topology. So, $X \setminus \{x\}$ is closed for any point $x \in X$ and thus $\{x\}$ is open. Consider the open cover $\bigcup_{x \in X} \{x\}$ of X. This has a finite sub-cover and thus X has a finite number of points.

Exercise 1.0.8. *Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \nsubseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.*

Solution:
Let $H = \mathcal{Z}(f)$ for some irreducible polynomial $f \in k[x_1, x_2, \ldots x_n]$. Let X be any closed irreducible component of $H \cap Y$. Since $X \subsetneq H \cap Y$ in Y, $I(H \cap Y) \subsetneq I(X)$ in $A(Y)$. Since X is an irreducible component i.e., a maximal irreducible subset, $I(X)$ is a minimal prime ideal containing $I(Y \cap H)$. Therefore $I(X)$ is a minimal prime ideal of $A(Y)$ containing f. Since $Y \nsubseteq H$, f does not belong to $I(Y)$ and therefore is not zero or nilpotent in $A(Y)$. From Krull's Haupidealsatz we have $ht(I(X)) = 1$ in $A(Y)$. We have $dim\, A(Y) = dim\, Y = r$. Therefore $dim\, A(Y)/I(X) = dim\, A(Y) - ht(I(X)) = r - 1$. $dim\, A(X) = dim\, k[x_1, x_2, \ldots x_n]/I(X) = dim\, A(Y)/I(X)$ because $I(Y) \subsetneq I(X)$. Therefore $dim\, X = dim\, A(X) = r - 1$.

Exercise 1.0.9. *Let $\mathfrak{a} \subsetneq A = k[x_1, x_2, \ldots x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $\mathcal{Z}(\mathfrak{a})$ has dimension $\geq n - r$.*

Solution:
Let Y be an irreducible component of $\mathcal{Z}(\mathfrak{a})$. Therefore $I(Y)$ is a minimal prime over $I(\mathcal{Z}(\mathfrak{a})) = \sqrt{(f_1, \ldots, f_r)}$. We claim that $I(Y)$ is minimal prime ideal over $\mathfrak{a}$. Suppose Q is a prime ideal such that $\mathfrak{a} \subset Q \subset I(Y)$. Since $\sqrt{\mathfrak{a}}$ is the intersection of all prime ideals that contain $\mathfrak{a}$, $\sqrt{\mathfrak{a}} \subset Q \subset I(Y)$. This contradicts the minimality of $I(Y)$ over $\sqrt{\mathfrak{a}}$. Therefore $I(Y)$ is a minimal prime ideal containing $\mathfrak{a}$. Krull's dimension theorem states that in a noetherian ring any prime ideal which is minimal over an ideal generated by r elements has height $\leq r$. Therefore $height\, I(Y) \leq r$. $Dim\, Y = dim\, A(Y) = dim\, A - height\, I(Y) \geq n - r$.

Exercise 1.0.10. *(a) If Y is any subset of a topological space X, then $dim\, Y \leq dim\, X$.*

(b) If X is a topological space which is covered by a family open subsets $\{U_i\}$, then $dim\, X = sup\, dim\, U_i$.

(c) Give an example of a topological space X and a dense open subset U with $dim\, U < dim\, X$.

(d) If Y is a closed subset of an irreducible finite dimensional topological space X, and if $dim\, Y = dim\, X$, then $Y = X$.

(e) Give an example of a noetherian topological space of infinite dimension.

Solution:

(a) Let $Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_t$ be any chain of irreducible closed subsets of Y. Consider the chain of irreducible closed subsets of X, $\overline{Y_0} \subsetneq \overline{Y_1} \subsetneq \ldots \subsetneq \overline{Y_t}$. We claim that $\overline{Y_i}$ are distinct. Assume the contrary. Suppose $\overline{Y_i} = \overline{Y_{i+1}}$. Since $Y_{i+1} \subsetneq \overline{Y_{i+1}} = \overline{Y_i}$, we have $Y \cap Y_{i+1} \subsetneq Y \cap \overline{Y_i} = Y_i$ which is a contradiction. This proves the claim. Thus $dim\, Y \leq dim\, X$.

(b) Let $X = \bigcup U_i$. Since $dim\, X \geq dim\, U_i$, $dim\, X \geq sup\, dim\, U_i$. Consider any chain of distinct irreducible closed subsets $Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_t$ of X. For any open set U of X, $Z_i \cap U$ is an open subset of the irreducible subset Z_i and is therefore irreducible. Also, closure of $U \cap Z_i$ in Z_i is equal to Z_i. Since Z_i is closed in X, closure of $U \cap Z_i$ in X is also equal to Z_i. Let x be any point in Z_0. Since $\{U_i\}$ forms an open cover of X, $x \in U_i$ for some i. Then $Z_0 \cap U_i \subset \ldots \subset Z_t \cap U_i$ is a chain of irreducible closed subsets of U_i. Also, each of the subsets in the chain is distinct. If $Z_j \cap U_i = Z_k \cap U_i$, then $\overline{Z_j \cap U_i} = Z_j = \overline{Z_k} \cap U_i = Z_k$ which is not true. Therefore we get a chain of distinct irreducible subsets of U_i. Therefore $dim\, U_i \geq dim\, X$ and hence $sup\, dim\, U_i \geq X$. Therefore, $dim\, X = sup\, dim\, U_i$.

(c) Let $X = \{a, b, c\}$. Define topology on X by letting X, $\emptyset$, $\{a, b\}$ and $\{a\}$ to be the closed subsets. $U = \{c\}$ is an open subset. The smallest closed subset containing U i.e., closure of U is X itself. Therefore U is a dense open subset of X. Since no non empty closed subset is contained in U, $dim\, U = 0$. But $\{a, b\} \subsetneq X$ is a chain of irreducible subsets of X. Therefore $dim\, X \geq 1$.

(d) Suppose $dim\, X = dim\, Y = n$. Let $Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n$ be a chain of closed irreducible subsets of Y. Since Y is closed in X, each of the Y_i is closed in X. Therefore we have a chain $Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subsetneq X$ of closed irreducible subsets of X. If $Y_n \subsetneq X$, we have $dim\, X \geq n + 1$ which is a contradiction. Therefore $Y_n = X$. But $Y_n \subsetneq Y$. Therefore $X = Y$.

(e) Let $X = \{a_1, a_2, \ldots a_n, \ldots\}$. Let the closed sets of X be X, $\emptyset$ and all sets of the form $\{a_i\}_{i=1}^n$. This space is noetherian. But it has infinite dimension because we have an infinite chain $\{a_1\} \subsetneq \{a_1, a_2\} \subsetneq \{a_1, a_2, a_3\} \ldots$ of irreducible closed subsets of X.

Exercise 1.0.11. *Let $Y \subsetneq \mathbb{A}^3$ be the curve parametrically given by $x = t^3$, $y = t^4$, $z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by two elements. We say that Y is not a local complete intersection.*

Solution:
For a monomial $f = x^\alpha y^\beta z^\gamma \in k[x, y, z]$ we define $deg_w(f)$ (weighted degree) to be $3\alpha + 4\beta + 5\gamma$. For any polynomial $f \in k[x, y, z]$ we define $deg_w(f)$ to be the maximum of the weighted degrees of the monomial terms of f. Therefore, the minimum weighted degree that a non- zero polynomial can have is 3. We call a polynomial f weighted homogeneous if all its monomial have the same weighted degrees.

Suppose $f =$ be a polynomial in $I(Y)$. We can write f as $g_1 + g_2 + \ldots g_r$ where $g_i = a_{i1} x^{\alpha_{i1}} y^{\beta_{i1}} z^{\gamma_{i1}} + a_{i2} x^{\alpha_{i2}} y^{\beta_{i2}} z^{\gamma_{i2}} + \ldots + a_{in} x^{\alpha_{in}} y^{\beta_{in}} z^{\gamma_{in}}$ is a weighted homogeneous polynomial of weighted degree d_i. Therefore $3\alpha_{ij} + 4\beta_{ij} + 5\gamma_i = d_i$ for $j = i, \ldots, n$. $f(t^3, t^4, t^5) = 0$ for all t. Therefore, $t^{d_1}(a_{11} + \ldots + a_{1n}) + t^{d_2}(a_{21} + \ldots + a_{2n}) + \ldots t^{d_r}(a_{r1} + \ldots + a_{rn}) = 0$. Therefore, $a_{i1} + \ldots + a_{in} = 0$ for each $i = 1, \ldots, r$. Therefore f belongs to the ideal generated by the set of weighted homogeneous polynomials with sum of coefficients equal to 0.

Conversely, assume that $f = a_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} + a_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + \ldots + a_n x^{\alpha_n} y^{\beta_n} z^{\gamma_n}$ is weighted homogeneous with sum of coefficients equal to 0. Then $f(t^3, t^4, t^5) = 0$. Therefore we have that $I(Y)$ is the ideal generated by the set of weighted homogeneous polynomials with sum of coefficients equal to 0.

We claim that (1, 0, 1) and (0, 2, 0) are the only non-negative integer solutions of $3\alpha + 4\beta + 5\gamma = 8$. For any positive integers α and γ, $3\alpha + 5\gamma \geq 8$. Therefore $8 - 4\beta \geq 8$ which implies that $\beta = 0$. Then $3\alpha + 5\gamma = 8$. For $\gamma \geq 2$, $3\alpha + 5\gamma \geq 10$. Therefore $\gamma = 1$ which implies $\alpha = 1$. Therefore (1, 0, 1) is the only non-negative solution of $3\alpha + 4\beta + 5\gamma = 8$ such that both $\alpha, \gamma \neq 0$. Suppose $\gamma = 0$ and $\alpha \neq 0$. Then $3\alpha = 8 - 4\beta$. Therefore $8 - 4\alpha \geq 3$ which implies that $\beta \leq 5/4$ i.e., $\beta = 1$. Therefore $3\alpha = 4$ which is clearly a contradiction. Therefore there exist no solution for which $\gamma = 0$ and $\alpha \neq 0$.

Suppose $\alpha = 0$ and $\gamma \neq 0$. Therefore $5\gamma = 8 - 4\beta$. Therefore $8 - 4\alpha \geq 5$ which implies that $\beta \leq 3/4$ i.e., $\beta = 0$. Therefore $5\alpha = 8$ which is clearly a contradiction. Therefore there exist no solutions for which $\alpha = 0$ and $\gamma \neq 0$.

Therefore $\alpha = 0$ and $\gamma = 0$ which implies that $\beta = 2$. Therefore (0, 2, 0) is the only solution which allows $\gamma = 0$ and $\alpha = 0$.

Thus (0, 2, 0) and (1, 0, 1) are the only solutions and therefore $f_1 = x\,z - y^2$ is the only weighted homogeneous polynomial (upto multiplication by an element of k) of weighted degree 8 in $I(Y)$.

We claim that (3, 0, 0) and (0, 1, 1) are the only non-negative integer solutions of $3\alpha + 4\beta + 5\gamma = 9$. For any positive integers β and γ, $4\beta + 5\gamma \geq 9$. Therefore $9 - 3\alpha \geq 9$ which implies that $\alpha = 0$. Then $4\beta + 5\gamma = 9$. For $\gamma \geq 2$, $4\beta + 5\gamma \geq 10$. Therefore $\gamma = 1$ which implies $\beta = 1$. Therefore (0, 1, 1) is the only solution of $3\alpha + 4\beta + 5\gamma = 9$ such that both $\beta, \gamma \neq 0$.

Suppose $\beta = 0$ and $\gamma \neq 0$. Then $3\alpha + 5\gamma = 9$. Therefore $5\gamma = 9 - 3\alpha \geq 5$ which implies $\alpha \leq 4/3$. Therefore $\alpha = 0$ or 1. When $\alpha = 0$, we get $5\gamma = 9$ which is not possible. When $\alpha = 1$, we get $5\alpha = 6$ which is also not possible. Therefore there exists no solution for which $\beta = 0$ and $\gamma \neq 0$.

Suppose $\beta \neq 0$ and $\gamma = 0$. Then $3\alpha + 4\beta = 9$. Therefore $4\beta = 9 - 3\alpha \geq 4$ which implies $\alpha \leq 5/3$. Therefore $\alpha = 1$ or 0. When $\alpha = 0$, we get $4\beta = 9$ which is not possible. When $\alpha = 1$, we get $4\beta = 6$, which is also not possible. Therefore there exists no solution for which $\beta \neq 0$ and $\gamma = 0$.

When $\beta = \gamma = 0$, we get $3\alpha = 9$ or $\alpha = 9$. Therefore (3, 0, 0) is the only solution which allows both β and γ to be equal to 0.

Therefore (3, 0 0) and (0, 1, 1) are the only solutions and therefore $f_2 = x^3 - y\,z$ is the only weighted homogeneous polynomial (upto multiplication by an element of k) of weighted degree 9 in $I(Y)$.

We claim that $3\alpha + 4\beta + 5\gamma = 7$ has only one non-negative integer solution (1, 1 0). If both β and γ are non zero, then $3\alpha + 4\beta + 5\gamma \geq 9$. Therefore either β or γ has to be 0. If $\beta = 0$, then $3\alpha + 5\gamma = 7$. $\alpha = 0$ (or $\gamma = 0$) is not possible because 7 is not a multiple of 5 (or 3). But for $\gamma, \alpha \geq 1$, $3\alpha + 5\gamma \geq 8$. Therefore there is no solution with $\beta = 0$. If $\gamma = 0$, then $3\alpha + 4\beta = 7$. $\alpha = 0$ (or $\beta = 0$) is not possible because 7 is not a multiple of 4 (or 3). For $\alpha \neq 0$ and $\beta \neq 0$, there is only one solution (1, 1, 0) which is thus the only solution.

By similar arguments we can show that there is only one non-negative integer solution to $3\,\alpha\, + 4\,\beta\, +\, 5\,\gamma\, =\, n$ for $n\, =\, 3,\, 4,\, 5,\, 6$ and no non-negative integer solution for $n\, =\, 2,\, 3$.

Since any weighted homogeneous polynomial of $I(Y)$ is such that the sum of coefficients is 0, $f_1\, =\, x\,z\, -\, y^2$ and $f_2\, =\, x^3\, -\, y\,z$ are the two weighted homogeneous non-zero polynomials of least weighted degree that belong to $I(Y)$ (upto multiplication by an element of k).

Suppose $I(Y)$ is generated by two elements. We then claim that the generators are f_1 and f_2. Let g_1 and g_2 be the two generators of $I(Y)$. Then, for $i\, =\, 1,\, 2$ g_i is a weighted homogeneous polynomial whose sum of coefficients is zero. Since $f_1\, \in\, I(Y)$, $f_1\, =\, g_1 h_1\, +\, g_2 h_2$ for some polynomials $h_1,\, h_2\, \in\, k[x,\, y,\, z]$. Therefore $deg_w(f_1)\, =\, max\{deg_w(g_1)\, +\, deg_w(h_1),\, deg_w(g_2)\, +\, deg_w(h_2)\}$. Suppose $deg_w(g_1)\, +\, deg_w(h_1)\, \geq\, deg_w(g_2)\, +\, deg_w(h_2)$. Then, $deg_w(f_1)\, =\, deg_w(g_1)\, + deg_w(h_1)$. Since f_1 is the homogeneous polynomial of least weighted degree that belongs to $I(Y)$, $deg_w(h_1)\, =\, 0$, i.e., $h_1\, =\, a\, \in\, k$. Therefore $g_1\, =\, a\, f_1$ for some scalar a.
Since we are assuming $g_1\, \neq\, g_2$, $deg_w(g_2)\, \geq\, 9$. Since $f_2\, \in\, I(Y)$, $f_2\, =\, af_1 h_1\, +\, g_2 h_2$ for some polynomials $h_1,\, h_2\, \in\, k[x,\, y,\, z]$. Suppose $deg_w(g_2)\, >\, 9$. Then $h_2\, =\, 0$ which implies that $deg_w(h_1)\, =\, 1$ which is not possible. Therefore $deg_w(g_2)\, =\, 9$. Since $x^3\, -\, y\,z$ is the only weighted homogeneous polynomial in $I(Y)$ of weighted degree 9 (upto multiplication by an element of k), we have $g_2\, =\, bf_2$ for some $b\, \in\, k$.

Consider the weighted degree 10 polynomial $f\, =\, x^2\,y\, -\, z^2$ in $I(Y)$. Then it cannot be written as an element of the ideal generated by f_1 and f_2. Because if $f\, =\, f_1 h_1\, +\, f_2 h_2$ for some polynomials $h_1,\, h_2\, \in\, k[x,\, y,\, z]$, then either $deg_w(h_1)\, =\, 2$ or $deg_w(h_2)\, =\, 2$ both of which are not possible. Therefore $I(Y)$ cannot be generated 2 elements.

To prove that $I(Y)$ is a prime ideal:
Suppose $f\, =\, a_1\, x^{\alpha_1}\, y^{\beta_1}\, z^{\gamma_1}\, + a_2\, x^{\alpha_2}\, y^{\beta_2}\, z^{\gamma_2}\, +\, \ldots\, +\, a_n\, x^{\alpha_n}\, y^{\beta_n}\, z^{\gamma_n}$ and $g\, =\, b_1\, x^{\lambda_1}\, y^{\mu_1}\, z^{\nu_1}\, + b_2\, x^{\lambda_2}\, y^{\mu_2}\, z^{\nu_2}\, +\, \ldots\, +\, b_m\, x^{\lambda_m}\, y^{\mu_m}\, z^{\nu_m}$ be two polynomials such that $fg\, \in\, I(Y)$. Then the sum of coefficients of fg, $(a_1\, +\, a_2\, +\, \ldots\, + a_n)\,(b_1\, +\, b_2\, + \ldots\, +\, b_m)$ is equal to zero. Therefore either $(a_1\, +\, a_2\, +\, \ldots\, + a_n)\, =\, 0$ or $(b_1\, +\, b_2\, + \ldots\, +\, b_m)\, =\, 0$. Also replacing x with X^3,

y with Y^4 and z with Z^5, we have fg is a homogeneous polynomial (in the usual sense) in X, Y and Z. Therefore either f or g is a homogeneous polynomial (in the usual sense) in X, Y and Z. Therefore either f or g is weighted homogeneous in x, y and z and thus $I(Y)$ is a prime ideal.

Exercise 1.0.12. *Give an example of an irreducible polynomial $f \in \mathbb{R}[x, y]$ whose zero set $\mathcal{Z}(f)$ in $\mathbb{A}^2_{\mathbb{R}}$ is not irreducible.*

Solution:
Let $f = x^2(x^2 - 1) + y^2 = x^4 - 2x^3 + x^2 + y^2$. We claim that f is irreducible in $\mathbb{R}[x, y]$. Consider f as a polynomial in y with coefficients in $\mathbb{R}[x]$. If f were reducible then it has two factors $ay + b$ and $cy + d$, each of degree 1 in y. Here $a, b, c, d \in \mathbb{R}[x]$. Then $ac = 1$ which implies that $a, c \in \mathbb{R}$. Also, $ad + bc = 0$. Putting $c = 1/a$ in this we get $b = -a^2d$. Therefore $bd = x^4 - 2x^2 + x^2 = (x^2 - x)^2 = -(ad)^2$. Since $a \in \mathbb{R}$, this implies that $a^2 = -1$ which is not possible in $\mathbb{R}$. Therefore f is irreducible. But $\mathcal{Z}(f) = \{ (0, 0), (1, 0)\}$ which is not an irreducible subset of $\mathbb{A}^2_{\mathbb{R}}$ because it can be written as a union of two closed proper subsets $\mathcal{Z}(x^2 + y^2) = \{(0, 0)\}$ and $\mathcal{Z}((x - 1)^2 + y^2) = \{(1, 0)\}$.

Chapter 2

Projective Varieties

Exercise 2.0.13 (Homogeneous Nullstellenstaz)**.** *Prove the homogeneous nullstellenstaz which states that if $\mathfrak{a} \subset S$ is a homogeneous ideal, and if f is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in \mathcal{Z}(\mathfrak{a})$ in $\mathbb{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$.*

Solution:

Let $\mathfrak{a}$ be a proper ideal of S. Let $\hat{\mathcal{Z}}(\mathfrak{a})$ denote the set $\{P \in \mathbb{A}^{n+1} \,|\, f(P) = 0 \ \forall\, f \in \mathfrak{a}\}$. For any point $P = (a_0 : \dots : a_n) \in \mathbb{P}^n$, let $\hat{P}$ denote the subset $\{(ta_0, ta_1, \dots ta_n) \in \mathbb{A}^{n+1} \,|\, t \in k\}$. Since $\mathfrak{a}$ is a homogeneous ideal of S, if $(a_0, a_1, \dots, a_n) \in \hat{\mathcal{Z}}(\mathfrak{a})$ then $\hat{P} \subset \hat{\mathcal{Z}}(\mathfrak{a})$. Therefore if $P \in \mathcal{Z}(\mathfrak{a})$, then $\hat{P} \subset \hat{\mathcal{Z}}(\mathfrak{a})$. Moreover, $\hat{\mathcal{Z}}(\mathfrak{a})$ is exactly equal to these points and 0.

If a non constant homogeneous polynomial f vanishes at all points of $\mathcal{Z}(\mathfrak{a})$, then it vanishes at all points of $\hat{\mathcal{Z}}(\mathfrak{a})$. From the usual nullstellensatz, we have $f^q \in \mathfrak{a}$ for some q.

Exercise 2.0.14. *For any homogeneous ideal $\mathfrak{a} \subset S$ show that the following conditions are equivalent:*
(i) $\mathcal{Z}(\mathfrak{a}) = \emptyset$;
(ii) $\sqrt{\mathfrak{a}}$ *is either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;*
(iii) $\mathfrak{a} \supset S_d$ *for some $d > 0$.*

Solution:

i) $\Longrightarrow$ ii):

Case i :

If $\mathfrak{a}$ is a proper ideal of S, then from the solution to problem 1, we know that if $\mathcal{Z}(\mathfrak{a}) = \emptyset$ then $\hat{\mathcal{Z}}(\mathfrak{a}) = \{0\}$. Let $I(\hat{\mathcal{Z}}(\mathfrak{a}))$ be the ideal of $\hat{\mathcal{Z}}(\mathfrak{a})$ when considered as a subset of $\mathbb{A}^{n+1}$. Then we have $\sqrt{\mathfrak{a}} = I(\hat{\mathcal{Z}}(\mathfrak{a})) = I(0) = S_+$ i.e., all polynomials with constant term equal to 0.

Case ii:

If $\mathfrak{a} = S$, then $\hat{\mathcal{Z}}(\mathfrak{a}) = \emptyset$ and therefore $\sqrt{\mathfrak{a}} = I(Z_a(\mathfrak{a})) = I(\emptyset) = S$.

ii) $\Longrightarrow$ iii): Consider the ideal I generated by the elements $x_0, x_1, \ldots, x_n$. Then $I \subset \sqrt{\mathfrak{a}}$. Since I is finitely generated $I^d \subset \mathfrak{a}$ for some $d > 0$. But any element of S_d belongs to I^d. Therefore $S_d \subset \mathfrak{a}$ for some $d > 0$.

iii) $\Longrightarrow$ i): Suppose $P = (a_0 : a_1; \ldots : a_n) \in \mathcal{Z}(\mathfrak{a})$. Then $f(P) = 0$ for any homogeneous polynomial $f \in \mathfrak{a}$. But since $S_d \subset \mathfrak{a}$, $x_i^d \in \mathfrak{a}$ for $i = 0$ to n. Therefore $a_i^d = 0$ for $i = 0$ to n. Since S is an integral domain, $a_i = 0$ for $i = 0$ to n. This is not possible. Therefore $\mathcal{Z}(\mathfrak{a}) = \emptyset$.

Exercise 2.0.15. *(a) If $T_1 \subset T_2$ are subsets of S^h, then $\mathcal{Z}(T_1) \supset \mathcal{Z}(T_2)$.*

(b) If $Y_1 \subset Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supset I(Y_2)$.

(c) For any two subsets Y_1, Y_2 of $\mathbb{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(d) If $\mathfrak{a} \subset S$ is a homogeneous ideal with $\mathcal{Z}(\mathfrak{a}) \neq \emptyset$, then $I(\mathcal{Z}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(e) For any subset $Y \subset \mathbb{P}^n$, $\mathcal{Z}(I(Y)) = \overline{Y}$.

Solution:

(a) Let $P \in \mathcal{Z}(T_2)$. Then $f(P) = 0$ for all $f \in T_2$ and hence for all f in T_1. Therefore $P \in \mathcal{Z}(T_1)$.

(b) Let $A = \{f \in S \,|\, f \textit{ is homogeneous and } f(P) = 0 \textit{ for all } P \in Y_2\}$ and $B = \{f \in S \,|\, f \textit{ is homogeneous and } f(P) = 0 \textit{ for all } P \in Y_1\}$. We have $A \subset B$ and therefore $(A) \subset (B)$. But $(A) = I(Y_2)$ and $(B) = I(Y_1)$. Therefore $I(Y_1) \supset I(Y_2)$.

(c) Let A and B be as above. Then $I(Y_1 \cup Y_2)$ is the ideal generated by the homogeneous polynomials f that vanish at points of $Y_1 \cup Y_2$. Clearly, if any homogeneous polynomial f vanishes on Y_1 and on Y_2, then f vanishes at $Y_1 \cup Y_2$. Therefore $I(Y_1) \cap I(Y_2) \subset I(Y_1 \cup Y_2)$. Conversely, consider any polynomial f in the generating set of $I(Y_1 \cup Y_2)$. Then f is a homogeneous polynomial that vanishes on all points of $Y_1 \cup Y_2$ and therefore f vanishes on all points of Y_1 and on all points of Y_2. Therefore $f \in A$ and $f \in B$. Therefore $f \in I(Y_1) \cap I(Y_2)$.

(d) From problem 1 we have that $I(\mathcal{Z}(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$. Since $\mathfrak{a}$ is a homogeneous ideal, $\sqrt{\mathfrak{a}}$ is homogeneous. Let f be any one of the generating elements of $\sqrt{\mathfrak{a}}$. Then $f^q \in \mathfrak{a}$ for some $q > 0$. Also, f^q is a homogeneous element. Since for any homogeneous polynomial $g \in \mathfrak{a}$, $g(P) = 0$ for all points $P \in \mathcal{Z}(\mathfrak{a})$, we have $f^q(P) = 0$ for all points $P \in \mathcal{Z}(\mathfrak{a})$. Therefore $f(P) = 0$ for all points $P \in \mathcal{Z}(\mathfrak{a})$. Therefore $f \in I(\mathcal{Z}(\mathfrak{a}))$ and $I(\mathcal{Z}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(e) Suppose $P \notin \mathcal{Z}(I(Y))$. Then there is a homogeneous polynomial $f \in I(Y)$ such that $f(P) \neq 0$. But $f(Q) = 0$ for any point $Q \in Y$. Therefore $P \notin Y$. Therefore $Y \subset \mathcal{Z}(I(Y))$. Since $\overline{Y}$ is the smallest closed subset containing Y, $\overline{Y} \subset \mathcal{Z}(I(Y))$. To prove the converse, assume that $P \notin \overline{Y}$. Therefore there is a homogeneous polynomial $f \in I(\overline{Y})$ such that $f(P) \neq 0$. Since $Y \subset \overline{Y}$, $I(\overline{Y}) \subset I(Y)$. Therefore $f \in I(Y)$. Since $f(P) \neq 0$, $P \notin \mathcal{Z}(I(Y))$. Therefore $\mathcal{Z}(I(Y)) \subset \overline{Y}$. Hence $\mathcal{Z}(I(Y)) = \overline{Y}$.

Exercise 2.0.16. *(a) There is a one-one inclusion reversing correspondence between algebraic sets in $\mathbb{P}^n$, and the homogeneous ideals of S not equal to S_+, given by $Y \longmapsto I(Y)$ and $\mathfrak{a} \longmapsto \mathcal{Z}(\mathfrak{a})$. Note: Since S_+ does not occur in this correspondence it is sometimes called the irrelevant maximal ideal of S.*

(b) *An algebraic set $Y \subset \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.*

(c) *Show that $\mathbb{P}^n$ itself irreducible.*

Solution:

(a) For any subset $Y \subset \mathbb{P}^n$, $I(Y)$ is clearly a homogeneous ideal. Also, when Y is a closed subset i.e., $Y = \mathcal{Z}(\mathfrak{a})$, we have from part (d) of Exercise 2.3 $\sqrt{I(Y)} = I(\mathcal{Z}(I(Y)))$. From part (e) of the same Exercise, $\mathcal{Z}(I(Y)) = \overline{Y} = Y$. Therefore $\sqrt{I(Y)} = I(Y)$. Therefore the map I is from the set of algebraic sets to the set of homogeneous radical ideals. Also, if $I(Y_1) = I(Y_2)$, then $\mathcal{Z}(I(Y_1)) = \mathcal{Z}(I(Y_2))$ i.e., $Y_1 = Y_2$. Therefore the map is 1-1. If I_1 and I_2 are radical ideals none equal to S_+ or S such that $\mathcal{Z}(I_1) = \mathcal{Z}(I_2)$, then $I(\mathcal{Z}(I_1)) = I(\mathcal{Z}(I_2))$ .i.e., $I_1 = I_2$. Also, when $\mathfrak{a} = S$ $\mathcal{Z}(\mathfrak{a}) = \emptyset$. When $\mathcal{Z}(\mathfrak{a}) \neq \emptyset$ from exercise 2.3 we get that $I(\mathcal{Z}(\mathfrak{a})) = \mathfrak{a}$. When $\mathcal{Z}(\mathfrak{a}) = \emptyset$, we have $\sqrt{\mathfrak{a}} = S$ or S_+. But since we are not considering S_+ in the image of the map I, $\sqrt{\mathfrak{a}} = S$. Therefore these two maps are inverses of each other. Also, from part (a) and (b) of Exercise 2.3, these maps are inclusion reversing.

(b) Suppose $I(Y)$ is not a prime ideal. Then there exist homogeneous elements f and g such that $fg \in I(Y)$ but $f \notin I(Y)$ and $g \notin I(Y)$. Let $Y_1 = \mathcal{Z}(f) \cap Y$ and $Y_2 = \mathcal{Z}(g) \cap Y$. Then Y_1 and Y_2 are proper closed subsets of Y Then $Y_1 \cup Y_2 \subset Y$. Since $fg \in I(Y)$, $\mathcal{Z}(I(Y)) = Y \subset \mathcal{Z}(fg) = \mathcal{Z}(f) \cup \mathcal{Z}(g)$. Therefore $Y = Y_1 \cup Y_2$ and Y is reducible.

Conversely, assume that Y is reducible. Let $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are proper closed subsets of Y. Since $I(Y) \subset I(Y_1)$ and $I(Y) \subset I(Y_2)$, there exist polynomials $f_1 \in I(Y_1) \setminus I(Y)$ and $f_2 \in I(Y_2) \setminus I(Y)$. But $f_1 f_2 \in I(Y_1) \cap I(Y_2) = I(Y_1 \cup Y_2) = I(Y)$. Therefore $I(Y)$ is not a prime ideal.

(c) $I(\mathbb{P}^n) = \{0\}$. Since the zero ideal is a prime ideal (because S is an integral domain), $\mathbb{P}^n$ is irreducible.

Exercise 2.0.17. *(a) $\mathbb{P}^n$ is a noetherian topological space.*

(b) Every algebraic set in $\mathbb{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing the another. These are called the irreducible components.

Solution:

(a) Let $Y_1 \supset Y_2 \supset \ldots \supset Y_n \supset \ldots$ be a decreasing chain of closed subsets of $\mathbb{P}^n$. Then $I(Y_1) \subset I(Y_2) \subset \ldots \subset I(Y_n) \subset \ldots$ is a chain of homogeneous radical ideals in S. Since S is homogeneous, there exists an N such that $I(Y_N) = I(Y_i)$ for all $i > N$. Therefore $\mathcal{Z}(I(Y_i)) = \mathcal{Z}(I(Y_N))$ for all $i > N$. Therefore every descending chain of closed subsets in $\mathbb{P}^n$ terminates and thus $\mathbb{P}^n$ is a noetherian topological space.

(b) The statement is true because of the result which states that every closed subset of a noetherian topological space can be written as a finite union of irreducible closed subsets, no one containing another.

Exercise 2.0.18. *If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $dim\, S(Y) = dim\, Y + 1$.*

Solution:
Let U_i be the open set of $\mathbb{P}^n$ defined by $a_i \neq 0$. Let $\varphi_i : U_0 \longrightarrow \mathbb{A}^n$ be the homeomorphism defined by sending the point $(a_0, a_1, \ldots, a_n)$ to the point with the affine coordinate $\left(\frac{a_0}{a_i}, \ldots, \frac{x_n}{a_i}\right)$ with $\frac{a_i}{a_i}$ omitted. We may assume, for notational convenience, that $i = 0$. Let $Y_0 = \varphi(Y \cap U_0)$ and let $A(Y_0)$ be the affine coordinate ring of Y_0. Assume that $Y_0 \neq \emptyset$. We note that localization is exact i.e., for any ring S and any ideal I of S, $D^{-1}(S/I) = D^{-1}S/D^{-1}I$ where D is a any multiplicatively closed subset of S. Define a map $\theta : k[y_1, y_2, \ldots, y_n] \longrightarrow S(Y)_{x_0}$ by sending the polynomial $f(y_1, \ldots, y_n)$ to the element $f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)\, mod\, I(Y)_{x_0}$.

We claim that $ker\, \theta = I(Y_0)$. To prove the claim first suppose that $f \in ker\theta$. Therefore $f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in I(Y)_{x_0}$. Suppose $deg\, f = e$. Then $x_0^e f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in I(Y)$. Therefore for any point $a = (a_0 : a_1 : \ldots : a_n) \in Y$,

$a_0^e f\left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right) = 0$. If $a \in Y \cap U_0$, then $a_0 \neq 0$ and therefore $f\left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right) = 0$. Therefore $f \in I(Y_0)$.

Conversely if $f \in I(Y_0)$, then for any point $a = \left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right) \in Y_0$, $f\left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right) = 0$. Suppose $deg\, f = e$. Then $a_0^e f\left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right) = 0$. Therefore $x_0^e f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in I(Y)$ and thus $f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in I(Y)_{x_0}$. This proves the claim that $ker\,\theta = I(Y_0)$.

Therefore $A(Y_0)$ is isomorphic to a subring of $S(Y)_{x_0}$. We identify $A(Y_0)$ with the subring of $S(Y)_{x_0}$ that is the isomorphic image of $A(Y_0)$. Any element of $A(Y_0)$ is a homogeneous element of degree 0. Also, if some element $f(x_0, \ldots, x_n)/x_0^e \in S(Y)_{x_0}$ is homogeneous of degree 0, then $deg\, f = e$. Therefore, $f(x_0, \ldots, x_n)/x_0^e = f\left(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = \theta(f(1, y_1, \ldots, y_n)$. Therefore, image of φ is equal to the set of all homogeneous elements of degree 0 in $S(Y)_{x_0}$ and thus $A(Y_0)$ is isomorphic to the subring of homogeneous elements of degree 0 of the localized ring $S(Y)_{x_0}$. We identify $A(Y_0)$ with this subring of $S(Y)_{x_0}$.

We claim that $S(Y)_{x_0} \cong A(Y_0)[x_0, x_0^{-1}]$. To prove this, consider an element $f(x_0, \ldots, x_n)/x_0^e \in S(Y)_{x_0}$. Suppose degree of $f = d$. Then f can be written as $g_0 + g_1 + \ldots + g_d$ where g_i is homogeneous of degree i. Then, $f(x_0, \ldots, x_n)/x_0^e = \frac{g_0}{x_0^e} + \ldots + \frac{g_d}{x_0}$. We have $\frac{g_i}{x_0^e} = \frac{g_i}{x_0^i} x_0^{i-e}$. Since $\frac{g_i}{x_0^i}$ is a homogeneous element of degree 0, it is an element of $A(Y_0)$. Therefore, $\frac{g_i}{x_0^e}$ is an element of $A(Y_0)[x_0, x_0^{-1}]$ and hence $f(x_0, \ldots, x_n)/x_0^e$ is an element of $A(Y_0)[x_0, x_0^{-1}]$. Therefore $S(Y)_{x_0} \cong A(Y_0)[x_0, x_0^{-1}]$ and thus $dim\, S(Y)_{x_0} = dim\, A(Y_0)[x_0, x_0^{-1}]$.

This result is independent of the assumption that $i = 0$ and can be deduced for any i for which $Y_i \neq \emptyset$. Therefore $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$ and $dim\, S(Y)_{x_i} = dim\, A(Y_i)[x_i, x_i^{-1}]$ for any i for which $Y_i \neq \emptyset$.

Let $A(Y_i)[x_i, x_i^{-1}]$ be denoted by B_i. When $Y_i \neq \emptyset$, we know that B_i is an integral domain which is finitely generated as a $k-$algebra. Therefore $dim\, B_i$ is equal to the transcendence degree of the quotient field $K(B_i)$ of B_i over k. But $K(B_i) = K\,(A(Y_i))(x_i)$. Therefore the transcendence degree over k of $K(B_i)$ is equal to the transcendence degree of $K(A(Y_i)) + 1$. Since transcendence degree of $K(A(Y_i)) = dim\, A(Y_i) = dim\, Y_i$, we have $dim\, S(Y_i)_{x_i} = dim\, Y_i + 1$.

We have $dim\, S(Y_i)_{x_i}$ is equal to the transcendence degree of the quotient field $K(S(Y_i)_{x_i})$. But $K(S(Y_i)_{x_i}) = K(S(Y))$. Therefore $dim\, S(Y_i)_{x_i} = dim\, S(Y)$ and thus $dim\, Y_i = dim\, S(Y) - 1$ whenever $Y_i \neq \emptyset$.

Since φ is a homeomorphism, $dim\, Y_i = dim\, (Y \cap U_i)$. But $\{Y \cap U_i\}$ forms an open cover of Y. Therefore, from exercise 1.10, $dim\, Y = sup\, dim\, (Y \cap U_i) = sup$ dim Y_i. But whenever $Y_i \neq \emptyset$, $dim\, Y_i = dim\, S(Y) - 1$ and is equal to -1 whenever $Y_i = \emptyset$. Therefore $dim\, Y = dim\, S(Y) - 1$.

Exercise 2.0.19. *(a)* $dim\, \mathbb{P}^n = n$.

(b) If $Y \subset \mathbb{P}^n$ *is a quasi projective variety, then* $dim\, Y = dim\, \overline{Y}$.

Solution:

(a) From the exercise 2.6, we know that $dim\, \mathbb{P}^n = dim\, k[x_0, \ldots x_n] - 1$. From Theorem 1.8A. we know that $dim\, k[x_0, \ldots, x_n]$ is equal to the transcendence degree of the quotient field of $k[x_0, \ldots, x_n]$ over k which is equal to $n + 1$. Therefore $dim\, \mathbb{P}^n = n + 1 - 1 = n$.

(b) If $Y \subset \mathbb{P}^n$ is a quasi-projective variety, then $Y \cap U_i$ is a quasi-affine variety when U_i is identified with $\mathbb{A}^n$ using the homeomorphism of Theorem 2.2. Also, since U_i is an open cover of $\mathbb{P}^n$, there exist at least one i for which $Y \cap U_i \neq \emptyset$.

We know that closure of $Y \cap U_i$ in $U_i \cong \mathbb{A}^n$ is equal to $\overline{Y} \cap U_i$. Let Y_0 denote the closure of $Y \cap U_i$ in U_i. Then the closure $\overline{Y_0}$ of Y_0 in $\mathbb{A}^n$ is equal to $\overline{Y}$. Since Y_0 is closed in U_i which is open in $\mathbb{A}^n$, we get $Y_0 = \overline{Y} \cap U_i$.

The family of curves $\{Y \cap U_i\}$ is an open cover of Y. Therefore from Exercise 1.10, $dim\, Y = sup\, dim\, Y \cap U_i$

From proposition 1.10, we have $dim\, (Y \cap U_i) = dim\, (\overline{Y} \cap U_i)$. From the solution to Exercise 2.6, $dim\, (\overline{Y} \cap U_i) = dim\, \overline{Y}$ whenever $\overline{Y} \cap U_i \neq \emptyset$. When $\overline{Y} \cap U_i = \emptyset$, $dim\, \overline{Y} \cap U_i = -1$. Therefore $dim\, Y = sup\{dim\, \overline{Y}, -1\} = dim\, \overline{Y}$.

Exercise 2.0.20. *A projective variety* $Y \subset \mathbb{P}^n$ *has dimension* $n - 1$ *if and only if it is the zero set of a single irreducible homogeneous polynomial* f *of positive degree.* Y *is called a hypersurface in* $\mathbb{P}^n$.

Solution:
Suppose $dim\, Y = n - 1$. We have $dim\, S(Y) + height\, I(Y) = dim\, S$. Since $dim\, S(Y) = dim\, Y + 1$, we have height of $I(Y) = 1$. Since S is a noetherian integral domain which is a unique factorization domain, from theorem 1.12A, $I(Y)$ is a principal ideal. Therefore $I(Y) = (f)$ for some irreducible polynomial $f \in S$. Therefore $Y = I(\mathcal{Z}(f)) = \sqrt{(f)} = (f)$. Therefore Y is the zero set of a single irreducible homogeneous polynomial.
Conversely, let $Y = \mathcal{Z}(f)$ where f is an irreducible homogeneous polynomial of positive degree. Therefore $I(Y) = I(\mathcal{Z}(f)) = \sqrt{(f)} = (f)$. Since S is a unique factorization domain, $I(Y)$ is a prime ideal and height of $I(Y) = 1$. Therefore $dim\, S(Y) = n$ and thus $dim\, Y = n - 1$.

Exercise 2.0.21 (Projective closure of an affine variety)**.** *If $Y \subset \mathbb{A}^n$ is an affine variety, we identify $\mathbb{A}^n$ with an open set $U_0 \subset \mathbb{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbb{P}^n$, which is called the projective closure of Y.*

(a) Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, where β is as in the proof of proposition 2.2 .

(b) Let $Y \subset \mathbb{A}^3$ be the twisted cubic curve (as in problem 1.2). Its projective closure $\overline{Y}$ is called the twisted cubic curve in $\mathbb{P}^3$. Find the generators of $I(Y)$ and $I(\overline{Y})$ and use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ does not necessarily generate $I(\overline{Y})$.

Solution:

(a) We recall that α is the map from the set S^h of homogeneous elements of $S = k[x_0, \ldots, x_n]$ to $k[y_1, \ldots, y_n]$ defined by sending the homogeneous element f to the element $f(1, y_1, \ldots, y_n)$. We also recall that β is the map from $k[y_1, \ldots, y_n]$ to the set S^h which sends a polynomial g of degree e to the element $x_0^e g(x_1/x_0, \ldots, x_n/x_0)$.

Let $A = \beta(I(Y))$ and let $J = (A)$. Any element of A, will be of the form $x_0^e g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$ where g is an element of degree e in $I(Y)$. If we identify Y with $Y_0 = \varphi(Y \cap U_0)$, then $Y \subset \mathcal{Z}(J)$. Therefore $\overline{Y} \subset \mathcal{Z}(J)$. Therefore

$I(\mathcal{Z}(J)) \subset I(\overline{Y})$, i.e., $\sqrt{J} \subset I(\overline{Y})$. Therefore $J \subset I(\overline{Y})$. Conversely, let $f \in I(\overline{Y})$. If we identify any point $(a_1, \ldots, a_n)$ of Y with the point $(1, a_1, \ldots, a_n)$ of $Y_0 \subset \overline{Y}$, then $f(1, a_1, \ldots, a_n) = 0$. Since $\alpha(f) = f(1, y_1, \ldots, y_n)$, we have $\alpha(f)(a_1, \ldots, a_n) = 0$ for any point $(a_1, \ldots, a_n) \in Y$. Therefore $\alpha(f) \in I(Y)$ and hence $f = \beta(\alpha(f)) \in \beta(I(Y)) = A$. Therefore $f \in J$.

(b) Let $X = \{(p^3 : p^2t : pt^2 : t^3) \,|\, p, t, \in k, \textit{not both } 0\} \subset \mathbb{P}^3$ be a subset. Consider the subset $X_0 = \{(p^3 : p^2t : pt^2 : t^3) \,|\, p, t, \in k\,;\, p \neq 0\}$ of X. When $p \neq 0$, then $(p^3 : p^2t : pt^2 : t^3) = \left(1 : \frac{t}{p} : \frac{t^2}{p^2} : \frac{t^3}{p^3}\right)$. Therefore X_0 is equal to the subset $\varphi_0^{-1}(Y)$ of U_0 where φ_0 is the homeomorphism of proposition 2.2. Henceforth we identify X_0 with its homeomorphic image in U_0 and call it Y. Also $X_1 = X \setminus Y$ is the set consisting of the single point $(0, 0, 0, t^3) = (0, 0, 0, 1) \in \mathbb{P}^3$.

Let $I \subset k[u, x, y, z]$ be the ideal $(xy - uz, uy - x^2, xz - y^2)$. We claim that $\mathcal{Z}(I) = X$. Clearly, any point of X belongs to $\mathcal{Z}(I)$. To prove the converse, let $P = (a_0 : a_1 : a_2 : a_3)$ be a point in $\mathcal{Z}(I)$. If $a_0 = 0$, then $a_0a_2 = a_1^2 = 0$ which implies that $a_1 = 0$. Then $a_2^2 = 0$ which implies that $a_2 = 0$. Therefore P is a point of the form $(0 : 0 : 0 : t) \in \mathbb{P}^3$ which is the point corresponding to X_1. Now suppose that $a_0 \neq 0$. We can put $a_0 = 1$. Let $a_1 = h$ for some $h \in k$. Then $a_0a_2 = a_1^2$ implies that $a_2 = h^2$. Then $a_1a_2 = a_3$ implies that $a_3 = h^3$. Therefore $P = (1 : h : h^2 : h^3)$ for some $h \in k$. Let $h = \frac{t}{p}$ for some $t, p \in k$ and $p \neq 0$. Then $P = (p^3 : p^2t : pt^2 : t^3)$. Thus P is a point of A. This proves the claim that $\mathcal{Z}(I) = X$.

We now claim that $\overline{Y} = X = A \cup X_1$. Since we have that $X = Y \cup \{(0 : 0 : 0 : 1)\}$, it is enough to show that $P = (0 : 0 : 0 : 1) \in \overline{Y} = \mathcal{Z}(I(Y))$. Consider any polynomial f in $I(Y)$. Any point of $Y \cap U_3$ is of the form $\left(\dfrac{p^3}{t^3} : \dfrac{p^2}{t^2} : \dfrac{p}{t} : 1\right)$ for some $t, p \in k^\times$. Therefore $f\left(\dfrac{p^3}{t^3}, \dfrac{p^2}{t^2}, \dfrac{p}{t} : 1\right) = 0$. Let $f(x_0, x_1, x_2, 1) = g(y_0, y_1, y_2) \in k[y_0, y_1, y_2]$. Then $g\left(\dfrac{p^3}{t^3}, \dfrac{p^2}{t^2}, \dfrac{p}{t}\right) = 0$ for all $p, t \in k^\times$. Therefore $g(s^3, s^2, s) = 0$ for all $s \in k^\times$ which implies that $g(x^3, x^2, x) \in k[x]$ is the zero polynomial. Therefore $g(0, 0, 0) = f(0, 0, 0, 1) = 0$ which proves that $(0; 0 : 0 : 1) \in \overline{Y}$.

We have $\beta(f_1) = uy - x^2$ and $\beta(f_2) = zu^2 - x^3$. Consider the generator $G = xz - y^2$ of I. Suppose $G = h_1(uy - x^2) + h_2(zu^2 - x^3)$. Now, any monomial of $h_1(uy - x^2) + h_2(zu^2 - x^3)$ will be a multiple of either uy or x^2 or zu^2 or x^3. But neither xz nor y^2 is a multiple of any of these terms. Therefore $G \notin (\beta(f_1), \beta(f_2)$. But $G \in I \subset \sqrt{I} = I(\overline{Y})$. Therefore $\beta(f_1)$, $\beta(f_2)$ do not generate $I(\overline{Y})$.

Exercise 2.0.22 (The cone over a projective variety)**.** *Let $Y \subset \mathbb{P}^n$ be a non empty algebraic set, and let $\theta : \mathbb{A}^{n+1} \setminus \{(0, \dots, 0)\} \longrightarrow \mathbb{P}^n$ be the map which sends the point with affine coordinates $(a_0, \dots, a_n)$ to the point with homogeneous coordinates $(a_0, \dots, a_n)$. We define the affine cone over Y to be*

$$C(Y) = \theta^{-1}(Y) \cup \{(0, \dots 0)\}.$$

(a) Show that $C(Y)$ is an algebraic set in $\mathbb{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \dots, x_n]$.

(b) $C(Y)$ is irreducible if and only if Y is.

(c) $\dim C(Y) = \dim Y + 1$.
Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbb{P}^{n+1}$. This is called the projective cone over Y.

Solution:

(a) For any point $P = (a_0 : \dots : a_n) \in Y \subset \mathbb{P}^n$, we get $\theta^{-1}(P) = \{(ta_0, \dots, ta_n) \mid t \in k\}$. Suppose $Y = \mathcal{Z}(f_1, \dots, f_r)$ for some homogeneous polynomials $f_i \in k[x_0, \dots, x_n]$. We claim that $C(Y) = \hat{\mathcal{Z}}(f_1, \dots, f_r) \subset \mathbb{A}^{n+1}$. Suppose $P = (a_0, \dots a_n) \in C(Y)$. Then either $P = (0, \dots, 0)$ or $P \in \theta^{-1}(Y)$. Since f_i are homogeneous polynomials, $(0, \dots, 0) \in \mathcal{Z}(f_1, \dots, f_r)$. If $P \in \theta^{-1}(Y)$, then $P = (tb_0, \dots, tb_n)$ for some point $(b_0 : \dots : b_n) \in Y$. Since f_i is homogeneous $f_i(b_0, \dots b_n) = 0$ implies that $f_i(tb_0, \ldots, tb_n) = 0$. Therefore $P \in \mathcal{Z}(f_1, \ldots, f_r)$. Conversely, suppose $P = (a_0, \ldots a_n) \in \mathcal{Z}(f_1, \ldots, f_r)$. Then either $P = (0, \ldots, 0)$ or $P \neq (0, \ldots, 0)$. In the latter case, $P \in \theta^{-1}(a_0, \ldots, a_n)$ for $(a_0 : \ldots : a_n) \in \mathbb{P}^n$. But since $P \in \mathcal{Z}(f_1, \ldots, f_r) = Y \subset \mathbb{P}^n$, $P \in \theta^{-1}(Y)$. Therefore $C(Y)$ is an affine algebraic subset of $\mathbb{A}^{n+1}$.

To prove that $I(C(Y)) = I(Y)$, let f be one of the elements in the generating set of $I(Y)$. Since f is a homogeneous element $f(0, \ldots, 0) = 0$. Also, any point in $\theta^{-1}(Y)$ is of the form $(ta_0, \ldots, ta_n)$ for some point $(a_0 : \ldots : a_n) \in Y$. Since f is homogeneous, $f(a_0, \ldots, a_n) = 0$ implies $f(ta_0, \ldots, ta_n) = 0$ for all $t \in k^{\times}$. Therefore $f \in I(C(Y))$. Conversely, assume $f \in I(C(Y))$. Therefore, $f(ta_0, \ldots, ta_n) = 0$ for any point $P = (a_0 : \ldots : a_n) \in Y$ and any $t \in k$. Suppose $deg\, f = d$. Then $f = f_0 + f_1 + \ldots + f_d$ where f_i is a homogeneous polynomial of degree i. Therefore, $f_0(P) + tf_1(P) + \ldots + t^d f_d(P) = 0$ for all $t \in k$. This implies that the polynomial $F = f_0(P) + xf_1(P) + \ldots + x^d f_d(P) \in k[x]$ has all elements of k as its roots. Since k is an algebraically closed field, this is possible only if F is the zero polynomial. Therefore $f_i(P) = 0$ for $i = 0, \ldots, d$. Therefore for each $i = 0, \ldots, d$, f_i is a homogeneous polynomial such that $f_i(P) = 0$ for any point $P \in Y$. Therefore $f \in I(Y)$.

(b) From Corollary 1.4, $C(Y)$ is irreducible if and only if $I(C(Y))$ is a prime ideal. From Exercise 2.4, $Y \subset \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal. Therefore Y is irreducible if and only if $C(Y)$ is irreducible.

(c) From proposition 1.7, $dim\, C(Y) = dim\, k[x_0, \ldots, x_n]/I(C(Y)) = S(Y)$. From Exercise 2.6, $dim\, S(Y) = dim\, Y + 1$. Therefore $dim\, C(Y) = dim\, Y + 1$.

Exercise 2.0.23 (Linear Varieties in $\mathbb{P}^n$)**.** *A hypersurface defined by a linear polynomial is called a hyperplane.*

(a) Show that the following two conditions are equivalent for a variety Y in $\mathbb{P}^n$:

(i) $I(Y)$ can be generated by linear polynomials.

(ii) Y can be written as an intersection of hyperplanes.

In this case we say that Y is a linear variety in $\mathbb{P}^n$.

(b) If Y is a linear variety of dimension r in $\mathbb{P}^n$, show that $I(Y)$ is minimally generated by $n - r$ linear polynomials.

(c) Let Y and Z be linear varieties in $\mathbb{P}^n$, with $\dim Y = r$ and $\dim Z = s$. If $r + s - n \geq 0$, then $Y \cap Z \neq \emptyset$. Furthermore, if $Y \cap Z \neq \emptyset$, then $Y \cap Z$ is a linear variety of dimension $\geq r + s - n$.

Solution:

(a) i) $\Rightarrow$ ii): Suppose $I(Y) = (f_1), f_2, \ldots, f_n)$ where f_i are linear polynomials. Then $Y = \mathcal{Z}(I(Y)) = Y = \mathcal{Z}(f_1) \cap \ldots \mathcal{Z}(f_n)$.

ii) $\Rightarrow$ ii): Suppose $Y\,\mathcal{Z}(l_1, \ldots, l_r)$ for some linear homogeneous polynomials l_i. We know that by a change of coordinates we can assume $l_i = x_i$. Therefore $Y = \mathcal{Z}(x_1, \ldots, x_r)$. Let $I = (x_1, \ldots, x_r)$. We know that $k[x_1, \ldots, x_n]/I = k[x_{r+1}, \ldots, x_n]$ which is an integral domain. Therefore I is a prime ideal and thus a radical ideal. Therefore $I(Y) = \sqrt{I} = I$ which gives that $I(Y)$ is generated by linear polynomials.

(b) Let $Y = \mathcal{Z}(f_1, \ldots, f_t)$ be a linear variety. Then any point of Y is the non-trivial solution of the system of t linear equations $\{f_i\}_{i=1}^t$. Therefore Y is the solution set of t linear equations and thus is a subspace of $\mathbb{A}^n$. Let $dim_V(Y)$ denote the dimension of Y as a subspace of $\mathbb{A}^n$. Assume that $dim_V(Y) = r$. We claim that Y can be written as $\mathcal{Z}(l_1, \ldots, l_{n-r})$ for some linear homogeneous equations l_i. Let A be the coefficient matrix of the system of equations $\{f_i\}_{i=1}^t$. Then A is an $t \times n$ matrix. Let $T : V^n \longrightarrow V^t$ be the linear transformation corresponding to A where V^n, V^t are vector spaces over k of dimension n and t respectively. Then Y is the null space of T. Now nullity of $T = dim_V(Y) = r$. Therefore rank of T, which is equal to the dimension of the range of T, is equal to $n - r$. Therefore T can be considered as a surjective linear transformation from V^n to V^{n-r}. Let $T^{'}$ be the map T with the co-domain restricted to V^{n-r}. Then Y is the null space of $T^{'}$ and thus Y is the solution space of $n - r$ equations. Therefore Y can be written as $\mathcal{Z}(l_1, \ldots, l_{n-r})$.

Let $dim\,(Y)$ denote the dimension of Y as a topological space. We claim that $dim_V(Y) = dim\,(Y)$, which will solve the exercise. Suppose $dim_V(Y) = r$.

Then Y can be written as an intersection of $n - r$ hyper-planes. Suppose $Y = \mathcal{Z}(l_0, \ldots, l_{n-r-1})$ for some linear homogeneous polynomials l_i. Then from part (a) we know that $I(Y) = (l_0, \ldots, l_{n-r-1})$. By a linear change of coordinates we may assume that $I(Y) = (x_0, \ldots, x_{n-r-1})$. We have that $S(Y) = k[x_0, \ldots, x_n]/I(Y)$ which can be shown to be isomorphic to $k[x_{n-r}, \ldots, x_n]$. From theorem 1.8A, we know that $dim\, S(Y) =$ transcendence degree of $k(x_{n-r}, \ldots, x_n)$ over k, which is equal to $r + 1$. Therefore $dim\,(Y) = dim\, S(Y) - 1 = r = dim_V\,(Y)$.

We know that $dim\, Y = r$ implies that $dim\, S(Y) = r + 1$. Since $ht\, I(Y) + dim\, S(Y) = n + 1$, we have height $I(Y) = n - r$. Suppose $I(Y) = (l_1, l_2, \ldots, l_m)$. Since Y is a variety $I(Y)$ is a prime ideal and therefore is the minimal prime over $(l_1, l_2, \ldots, l_m)$. Krull's dimension theorem states that in a noetherian ring the height of a prime ideal P which is minimal over an ideal $I = (a_1, a_2, \ldots, a_m)$ is $\leq m$. Therefore we have $ht\, I(Y) \leq m$. Therefore, $m \geq n - r$.

(c) Suppose $Y = \mathcal{Z}(f_1) \cap \ldots \mathcal{Z}(f_2) \cap \ldots \cap \mathcal{Z}(f_l)$ and $Z = \mathcal{Z}(g_1) \cap \mathcal{Z}(g_2) \cap \ldots \cap \mathcal{Z}(g_m)$ where f_i and g_j are linear equations in the variables $x_0, x_1, \ldots, x_n$. From part (b) we know that $l \geq n - r$ and $m \geq n - s$. Any point in $Y \cap Z$ is a non trivial solution of the system of $l + m$ linear equations $\{f_i\}_{i=1}^{l} \cup \{g_j\}_{j=1}^{m}$. Suppose $Y \cap Z = \emptyset$ (i.e., the system of equations has only trivial solution). Then $n + 1 \geq l + m$. But $l + m \geq (n - r) + (n - s)$. Therefore $r + s - n < 0$. Therefore $Y \cap Z \neq \emptyset$ if $r + s - n \geq 0$.

Exercise 2.0.24 (The d-Uple Embedding)**.** *For given n, $d > 0$, let $M_0, \ldots, M_N$ be all monomials of degree d in the variables $x_0, \ldots, x_n$ where $N = \binom{n+d}{n} - 1$. We define the mapping $\rho_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ by sending the point $P = (a_0, a_1, \ldots a_n)$ to the point $\rho_d(P) = (M_0(a), M_1(a), \ldots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example if $n = 1$ $d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic.*

(a) Let $\theta : k[y_0, \ldots, y_N] \longrightarrow k[x_0, \ldots, x_n]$ be the homomorphism defined by sending

y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $\mathcal{Z}(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

(b) *Show that the image of ρ_d is exactly $\mathcal{Z}(\mathfrak{a})$.*

(c) *Show that ρ_d is a homeomorphism of $\mathbb{P}^n$ onto the projective variety $\mathcal{Z}(\mathfrak{a})$.*

(d) *Show that the twisted cubic curve in $\mathbb{P}^3$ is equal to the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.*

Solution:

(a) We know that $k[y_0, \ldots, y_N]/\mathfrak{a}$ is isomorphic to a subring of $k[x_0, \ldots, x_n]$. Since $k[x_0, \ldots, x_n]$ is an integral domain and since subring of an integral domain is an integral domain, $k[y_0, \ldots, y_N]/\mathfrak{a}$ is an integral domain and thus $\mathfrak{a}$ is a prime ideal.

Let f be an element of $ker\,\theta$. Suppose $f = f_0 + f_1 + \ldots + f_r$ where f_i is a homogeneous polynomial of degree i. We have $\theta(f) = f(M_0, \ldots, M_n) = \theta(f_0) + \ldots + \theta(f_r) = 0$. But $\theta(f_i)$ is a homogeneous polynomial of degree di. Therefore each $\theta(f_i) = 0$ and thus each $f_i \in ker\,\theta$. Therefore for any $f \in ker\,\theta$, each of the homogeneous component of f belongs to $ker\,\theta$. Therefore $ker\,\theta$ is an homogeneous ideal.

(b) We first exhibit a set of generators for $\mathfrak{a}$. We know that each M_i is of the form $x_0^{\alpha_{i0}} x_1^{\alpha_{i1}} \ldots x_n^{\alpha_{in}}$ where α_{ik} are non negative integers such that $\sum_{k=0}^{n} \alpha_{ik} = d$. Let $\mathcal{I}$ be the ideal generated by the set of all polynomials of the form $\prod_{i \in I} y_i^{d_i} - \prod_{j \in J} y_j^{d_j}$ for which $\sum_{i \in I} d_i \alpha_{ik} = \sum_{j \in J} d_j \alpha_{jk}$, for some subsets I, J of $\{0, \ldots, N\}$, for each $k = 0, 1, \ldots, n$. We claim that $\mathfrak{a} = \mathcal{I}$. For any element $f = \prod_{i \in I} y_i^{d_i} - \prod_{j \in J} y_j^{d_j}$ of the generating set of the polynomials $\theta(f) = \prod_{i \in I} M_i^{d_i} - \prod_{j \in J} M_j^{d_j}$. For a fixed k, the exponent of x_k in $\prod_{i \in I} M_i^{d_i}$ is equal to $\sum_{i \in I} d_i \alpha_{ik}$ and the exponent of x_k in $\prod_{j \in J} M_j^{d_j}$ is equal to $\sum_{j \in J} d_j \alpha_{jk}$. Since for each k, $\sum_{i \in I} d_i \alpha_{ik} = \sum_{j \in J} d_j \alpha_{jk}$, we get that $\theta(f) = 0$. Therefore $\mathcal{I} \subset ker\,\theta$.

To prove the converse, let $f \in \mathfrak{a}$. We can group together the monomials $a_m \prod_{i=0}^{N} y_i^{d_{mi}}$ of f for which $\theta(\prod_{i=0}^{N} y_i^{d_{mi}}) = \prod_{i=0}^{N} M_i^{\beta_{mi}}$ is same. Suppose after a permutation of the terms of f that $\prod_{i=0}^{N} M_i^{d_{mi}} = \prod_{i=0}^{N} M_i^{d_{ni}} = M$ for all $m, n \in \{1, \ldots, l\}$. This means that for each $m, n \in \{1, \ldots, l\}$, $\sum_{i \in I} d_i \alpha_{im} = \sum_{n \in J} d_n \alpha_{nm}$. Let Y_m denote $\prod_{i=0}^{N} y_i^{d_{mi}}$. Therefore $Y_m - Y_n$ is an element of the generating set of $\mathcal{I}$. Also, $F_M = (a_1 + a_2 + \ldots + a_l)M = 0$ and therefore $a_1 + a_2 + \ldots + a_l = 0$. Then $f_M = \theta^{-1}(F_M) = a_1 Y_1 + \ldots + a_l Y_l$.

Using the property that $\sum_{i=1}^{l} a_i = 0$ we get that $f_M = a_1(Y_1 - Y_2) + (a_1 + a_2)(Y_2 - Y_3) + \ldots\ldots + (a_1 + \ldots + a_{l-1})(Y_{l-1} - Y_l)$. Therefore $f_M \in \mathcal{I}$. But f is a sum of such f_M. Therefore $f \in \mathcal{I}$ which proves that $\mathfrak{a} \subset \mathcal{I}$. Therefore $\mathfrak{a} = \mathcal{I}$.

Suppose $P = (M_0(a), M_1(a), \ldots, M_N(a))$ for some point $a \in \mathbb{P}^n$. Consider any one of the generating elements of $\mathfrak{a}$: $\prod_{i \in I} y_i^{d_i} - \prod_{j \in J} y_j^{d_j}$ for which $\sum_{i \in I} d_i \alpha_{ik} = \sum_{j \in J} d_j \alpha_{jk}$ for each $k = 0, 1, \ldots, n$. Then $F(P) = \prod_{i \in I} M_i^{d_i}(a) - \prod_{j \in J} M_j^{d_j}(a)$. Since $F \in \mathfrak{a}$, we get that $\prod_{i \in I} M_i - \prod_{j \in J} M_j = 0$ and therefore $F(P) = 0$. Therefore $P \in \mathcal{Z}(\mathfrak{a})$ proving that image of ρ_d is contained in $\mathcal{Z}(\mathfrak{a})$.

To prove the converse, consider $P = (b_0, \ldots, b_N) \in \mathcal{Z}(\mathfrak{a})$. We can label the coordinates of $\mathbb{P}^N$ using $n+1$ tuples $a_0 a_1 \ldots a_n$ such that $\sum_{i=0}^{n} a_i = d$. Consider, after relabelling the coordinates, the $a_0 a_1 \ldots a_n$-th coordinate $b_{a_0 \ldots a_n}$ of P. Since $P \in \mathcal{Z}(\mathfrak{a})$, $\prod_{i \in I} b_{\alpha_{i1} \ldots \alpha_{in}} = \prod_{j \in J} b_{\beta_{j1} \ldots \beta_{jn}}$ whenever for each $k = 1, \ldots, n$, $\sum_{i \in I} \alpha_{ik} = \sum_{j \in J} \alpha_{jk}$. Using these conditions we derive that $(b_{a_0 \ldots a_n})^d = (b_{d00\ldots0})^{a_0} (b_{0d0\ldots0})^{a_1} \ldots\ldots (b_{00\ldots0d})^{a_n}$. If each coordinate of the form $b_{00\ldots d\ldots 00} = 0$, then $b_{a_0 a_1 \ldots a_n} = 0$. Since $b_{a_0 a_1 \ldots a_n} = 0$ was any general coordinate of P, this would imply that all the coordinates of P are zero which is false. This implies that at least one of the coordinates of the form $b_{00\ldots d\ldots 00} \neq 0$.

Suppose, after a permutation of the coordinates, $b_{d00\ldots 00} \neq 0$. Let $u = (u_0, u_1, \ldots un) \in \mathbb{P}^n$ be such that

$$u_0 = b_{d00\ldots\ldots 00}$$

$$u_1 = b_{d-1,100\ldots 0}$$

$$u_2 = b_{d-1,010\ldots0}$$
$$\vdots$$
$$u_n = b_{d-100\ldots01}$$

We claim that $\rho_d(u) = P$. We have $\rho_d(u) = \rho_d\left(1, \frac{u_1}{u_0}, \ldots, \frac{u_n}{u_0}\right)$. The $(a_0, \ldots, a_n)$-th coordinate of the image is

$$= \left(\frac{u_0}{u_0}\right)^{a_0}\left(\frac{u_1}{u_0}\right)^{a_1}\cdots\left(\frac{u_n}{u_0}\right)^{a_n}$$

$$= u_0^{-d}u_0^{a_0}u_1^{a_1}\ldots\ldots\ldots u_n^{a_n}$$

Using the fact that $P \in \mathcal{Z}(\mathfrak{a})$, we derive that $(b_{d00\ldots00})(u_0^{-d}u_0^{a_0}u_1^{a_1}\ldots u_n^{a_n}) = b_{a_0a_1\ldots a_n}$. Therefore the $(a_0, \ldots, a_n)$-th coordinate of the $\rho_d(u)$ is $\frac{b_{a_0a_1\ldots a_n}}{b_{d00\ldots00}}$. Since $b_{d00\ldots00}$ is a constant which is same for each coordinate of $\rho_d(u)$, we get that $\rho_d(u) = P$.

(c) To show that ρ_d is a continuous mapping, consider a closed subset $\mathcal{Z}(\beta)$ of $\mathcal{Z}(\mathfrak{a})$. Suppose $\mathcal{Z}(\beta) = (f_1, \ldots, f_r)$ where f_i are homogeneous polynomials in $k[x_0, \ldots, x_N]$. We claim that $\rho_d^{-1}(\mathcal{Z}(\beta)) = (\theta(f_1), \ldots, \theta(f_r))$. Suppose $P = (a_0, \ldots, a_n) \in \rho_d^{-1}(\mathcal{Z}(\beta))$ i.e, $\rho_d(P) \in \mathcal{Z}(\beta)$. Since $\rho_d(P) = (M_0(a), \ldots, M_N(a))$, $f_i(M_0(a), \ldots, M_N) = 0$ for all $i = 1$ to l. Therefore $\theta(f_i)(P) = 0$ for all $i = 1$ to r. Therefore $P \in \mathcal{Z}(\theta(f_1), \ldots, \theta(f_r))$.

To prove the converse, consider a point $P = (a_0, \ldots, a_n) \in \mathcal{Z}(\theta(f_1), \ldots, \theta(f_r))$. Therefore $\theta(f_i)(a_0, \ldots, a_n) = 0$ which implies that $f_i(M_0(a), \ldots, M_N(a)) = 0$. Therefore $P \in \rho_d^{-1}(\mathcal{Z}(\beta))$. This proves that ρ_d is a continuous map onto $\mathcal{Z}(\mathfrak{a})$.

To prove that ρ_d is a closed map, consider a closed subset $\mathcal{Z}(\mathcal{J}_0)$ of $\mathbb{P}^n$. Suppose $\mathcal{J}_0 = (f_1, f_2, \ldots, f_l)$. Consider the ideal $\mathcal{J} = (f_1^d, \ldots f_l^d)$. Since $\mathcal{J} \subset \mathcal{J}_0$, $\mathcal{Z}(\mathcal{J}_0) \subset \mathcal{Z}(\mathcal{J})$. Also, if $f^d(P) = 0$, then $f(P) = 0$. Therefore $\mathcal{Z}(\mathcal{J}) \subset \mathcal{Z}(\mathcal{J}_0)$. Therefore $\mathcal{Z}(\mathcal{J}_0) = \mathcal{Z}(\mathcal{J})$.

Consider any monomial $F = x_0^{\alpha_0}x_1^{\alpha_1}\ldots x_n^{\alpha_n} \in k[x_0, \ldots, x_n]$. $F^d = x_0^{d\alpha_0}x_1^{d\alpha_1}\ldots x_n^{d\alpha_n} = (x_0^d)^{\alpha_0}(x_1^d)^{\alpha_1}\ldots(x_n^d)^{\alpha_n}$. This is clearly an element in the image of θ. Since any f_i^d is a sum of elements of the form F^d, $f_i^d \in Im(\theta)$ and thus $\mathcal{J} \subset Im(\theta)$.

Let I be the ideal $\theta^{-1}(\mathcal{J})$. Then I is generated by the elements $\theta^{-1}(f_i^d)$ for $i = 1$ to l. Let $\theta^{-1}(f_i^d)$ be denoted by g_i. We claim that $\mathcal{Z}(I) = \rho_d(\mathcal{Z}(\mathcal{J}))$. Suppose $P = (a_0, \ldots, a_n) \in \rho_d^{-1}(\mathcal{Z}(I))$. Then $\rho_d(P) = (M_0(a), \ldots, M_N(a)) \in \mathcal{Z}(I)$. Therefore $g_i(\rho_d(P)) = g_i(M_0(a), \ldots, M_N(a)) = 0$ which implies that $\theta(g_i)(P) = 0$. Since $\theta(g_i) = f_i$, $P \in \mathcal{Z}(\mathcal{J})$. Now suppose $P = (a_0, \ldots, a_n) \in \mathcal{Z}(\mathcal{J})$. Therefore $f^d(P) = \theta(g_i)(P) = 0$ which implies that $g_i(M_0(a), \ldots, M_N(a)) = g_i(\rho_d(P)) = 0$. Therefore $\rho_d(P) \in \mathcal{Z}(\mathcal{J})$ and hence $P \in \rho_d^{-1}(\mathcal{Z}(\mathcal{J}))$. This proves that ρ_d^{-1} is continuous. Therefore ρ_d is a homeomorphism of $\mathbb{P}^n$ onto the projective variety $\mathcal{Z}(\mathfrak{a})$.

(d) Let $M_0 = x^3$, $M_1 = x^2y$, $M_2 = xy^2$ and $M_3 = y^3$. Then the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$ maps a point $P = (a, b)$ to the point (a^3, a^2b, ab^2, b^3). Therefore $Im(\rho_d) = \{a^3, a^2b, ab^2, b^3) \mid a, b \in k\}$ which is the twisted cubic curve in $\mathbb{P}^3$.

Exercise 2.0.25. *Let Y be the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. This is the Veronese surface. If $Z \subset Y$ is a closed curve (a curve is a variety of dimension 1), show that there exist a hypersurface $V \subset \mathbb{P}^5$ such that $V \cap Y = Z$.*

Solution:

Since ρ_2 is a homeomorphism of $\mathbb{P}^2$ onto the image $Y \subset \mathbb{P}^5$, $\rho_2^{-1}(Z)$ is a closed curve of $\mathbb{P}^2$ whenever Z is a closed curve of Y. A closed curve of $\mathbb{P}^2$ is of the form $\mathcal{Z}(F)$ for some homogeneous polynomial in $k[x_0, x_1, x_1]$. Let $M_0 = x_0^2$, $M_1 = x_1^2$, $M_2 = x_2^2$, $M_3 = x_0x_1$, $M_4 = x_1x_2$ and $M_5 = x_2x_0$. $F(x_0, x_1, x_2)^2$ is a homogeneous polynomial of degree $2m$ where m is the degree of F. Consider any monomial term $x^{2\alpha}y^{2\beta}z^{2\gamma}$ of F^2. Without loss of generality, we can assume that $\alpha \geq \beta \geq \gamma$. Therefore $x^{2\alpha}y^{2\beta}z^{2\gamma} = (xy)^{2\beta}(xz)^{2\alpha - 2\beta}(z)^{2\gamma - 2\alpha - 2\beta}$. Therefore each monomial of F^2 is a product of some powers of M_i's. Therefore F^2 is a polynomial in M_i's. We substitute y_i in place of M_i in F^2 to get a polynomial G in $k[y_0, y_1, y_2, y_3, y_4, y_5]$.

We claim that $\mathcal{Z}(G) \cap Y = \rho_2(\mathcal{Z}(F))$. Consider a point $P \in \mathcal{Z}(G) \cap Y$. Let $a = (a_0, a_1, a_2) = \rho_2^{-1}(P)$. Hence $P = (M_0(a), M_1(a), \ldots, M_5(a))$. Therefore $G(M_0(a), M_1(a), \ldots, M_5(a)) = F^2(a_0^2, a_1^2, a_2^2, a_0a_1, a_1a_2, a_2a_0) = 0$. Therefore

$F^2(a_0, a_1, a_2) = 0$ and thus $F(a_0, a_1, a_2) = 0$ which implies that $\rho_2^{-1}(P) \in \mathcal{Z}(F)$. Therefore $P \in \rho_2(\mathcal{Z}(F))$ and thus $\mathcal{Z}(G) \cap Y \subset \rho_2(\mathcal{Z}(F))$

To prove that $\mathcal{Z}(G) \cap Y \supset \rho_2(\mathcal{Z}(F))$, assume that $P = (b_0, \ldots, b_N) \in \rho_2(\mathcal{Z}(F))$. So $P = \rho_2(a)$ for some $a = (a_0, a_1, a_2) \in \mathcal{Z}(F)$ i.e.,
$P = (M_0(a), M_1(a), \ldots, M_5(a))$ Hence $F(a_0, a_1, a_2) = 0$ and therefore
$F^2(a_0, a_1, a_2) = F^2(a_0^2, a_1^2, a_2^2, a_0a_1, a_1a_2, a_2a_0) = 0$. Therefore
$G(P) = G(M_0(a), M_1(a), \ldots, M_5(a)) = F^2(a_0^2, a_1^2, a_2^2, a_0a_1, a_1a_2, a_2a_0) = 0$.
Therefore $P \in \mathcal{Z}(G)$ and thus $\mathcal{Z}(G) \cap Y \supset \rho_2(\mathcal{Z}(F))$. This proves the claim.

Exercise 2.0.26 (The Segre Embedding). *Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \longrightarrow \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_ib_j, \ldots)$ in lexicographic order where $N = rs + r + s$. Note that ψ is well defined and injective. It is called the Segre embedding. show that the image of ψ is a subvariety of $\mathbb{P}^N$.*

Solution:
Any element of $\mathbb{P}^N$, where $N = rs + r + s$, can be considered as an $(r + 1) \times (s + 1)$ matrix. Let $r_0 = r + 1$ and $s_0 = s + 1$. Consider any matrix $M = \psi(P \times Q)$ in the image of ψ where $P = (a_0 : \ldots : a_r) \in \mathbb{P}^r$ and $Q = (b_0 : \ldots : b_s) \in \mathbb{P}^s$. We can consider P as a $1 \times r_0$ matrix over k and Q as a $1 \times s_0$ matrix over k. Then $M = {}^tPQ$.

We claim that a $r_0 \times s_0$ matrix M is of the form ${}^tPQ$ for some $1 \times r_0$ matrix P and $1 \times s_0$ matrix Q if and only if rank of M is 1. Suppose $M = {}^tPQ$. Let $T_1 : k^{s_0} \to k$ be the linear transformation corresponding to Q where k^{s_0} be the vector space over k of dimension s_0. Let $T_2 : k \to k^{r_0}$ be the linear transformation corresponding to ${}^tP$. Then the linear transformation corresponding to the matrix M is $T_0 \circ T_1$. Since the range of T_1 is a subspace of k, the rank of T_1 is either 1 or 0. But Q is not the zero matrix. Therefore the rank of T_1 is 1. Also, the rank of T_2 is either 1 or 0. But since P is not the zero matrix, the rank of T_2 is 1. Therefore the rank of $T_2 \circ T_1$ is 1 and therefore rank of M is 1.

Conversely, suppose rank of M is 1. Let $T : k^{s_0} \longrightarrow k^{r_0}$ be the linear transformation corresponding to M. Then the rank of T, which is the dimension of the image of T, is 1. Therefore we can write T as a composition $T_2 \circ T_1$ where $T_1 : k^{s_0} \longrightarrow k$ is the surjective linear transformation which is equal to T but with the co-domain restricted to the range of T and $T_2 : k \longrightarrow k^{r_0}$ is the natural inclusion of k in k^{r_0}. Let P' be the $r_0 \times 1$ matrix corresponding to T_2 and let Q be the $1 \times s_0$ matrix corresponding to T_1. Then $M = {}^tPQ$ where $P = {}^tP'$. This proves the claim.

Since the rank of M is 1, the determinant of any 2×2 minor of M is 0. Therefore the image of ψ is the set of all $r_0 \times s_0$ matrices for which the determinant of any 2×2 minor is 0. The determinant of any 2×2 minor is an homogeneous expression of degree 2. Therefore the image of ψ is a closed subset of $\mathbb{P}^N$.

Exercise 2.0.27 (The Quadric Surface in $\mathbb{P}^3$)**.** *Consider the surface Q (a surface is a variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw = 0$.*

(a) *Show that Q is equal to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.*

(b) *Show that Q contains two families of lines (a line is a linear variety of dimension 1) $\{ L_t \}$, $\{ M_t \}$, each parametrized by $t \in \mathbb{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \emptyset$; if $M_t \neq M_u$, $M_t \cap M_u = \emptyset$, and for all t, u, $L_t \cap M_u =$ one point.*

(c) *Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ (where each $\mathbb{P}^1$ has its zariski topology).*

Solution:

(a) We can relabel the coordinates of $\mathbb{P}^3$ to let $x = z_{11}$, $y = z_{00}$, $z = z_{01}$ and $w = z_{10}$. Then from the solution to Exercise 14, the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ is equal to $\mathcal{Z}(z_{11}z_{00} - z_{10}z_{01})$ which is nothing but Q.

(b) Consider a fixed point $t = (t_0, t_1) \in \mathbb{P}^1$. Then for any point $b = (b_0, b_1) \in \mathbb{P}^3$, $\psi(t \times b) = (t_0b_0, t_0b_1, t_1b_0, t_1b_1)$. Therefore $\psi(t \times b) = \mathcal{Z}(t_0z - t_1x, t_1y - t_0w)$. This is a linear variety. Let this be called L_t. As t varies L_t defines a family of curves in Q.

Let $I = (t_0z - t_1x, t_1y - t_0w)$. For any fixed $t = (t_0, t_1)$, either $t_0 \neq 0$ or $t_1 \neq 0$. Without loss of generality we may assume that $t_0 \neq 0$. Let $l = t_1/t_0$. Then $I = (z - lx, y - ly)$. Let $\varphi_1 : k[x, y, z, w]/I \longrightarrow k[x, y]$ be the map sending the $f(x, y, z, w) \, mod \, I$ to $f(x, y, lx, lw)$. Let $\varphi_2 : k[x, y] \longleftrightarrow k[x, y, z, w]/I$ be the map sending $f(x, y)$ to $f(x, y) \, mod \, I$. Then it is clear that φ_1 and φ_2 are ring homomorphisms. Also, $\varphi_1(\varphi_2(f(x, y))) = \varphi_1(f(x, y) \, mod \, I) = f(x, y)$. Also, for any element $f(x, y, z, w) \in k[x, y, z, w] \, mod \, I$, $\varphi_2(\varphi_1(f)) = \varphi_2(f(x, y, lx, lw))$. Since $lx \equiv z$ and $lw \equiv z$ in $k[x, y, z, w]/I$, $f(x, y, z, w) = f(x, y, lx, ly)$ in $k[x, y, z, w]/I$. Hence the ring homomorphisms φ_1 and φ_2 are inverses of each other which gives us that $k[x, y, z, w]/I \cong k[x, y]$. Therefore I is a prime ideal and therefore a radical ideal. Since $I(\mathcal{Z}(I)) = \sqrt{I} = I$ is a prime ideal, $\mathcal{Z}(I) = L_t$ is irreducible.

Also, since it is a subset of Q, $dim \, L_t \leq dim \, Q = 2$. Since Q is defined by an irreducible polynomial, Q is irreducible. From Exercise 1.10(d), if $dim \, L_t = 2$, then $Q = L_t$ which is not true. Also, since L_t does not consists of a single point, $P \subset L_t$ is a chain of distinct irreducible closed subsets of L_t for any point $P \in L_t$. Therefore $dim \, L_t \neq 0$. Therefore $dim \, L_t = 1$.

Consider a fixed point $t = (t_0, t_1) \in \mathbb{P}^1$. Then for any point $b = (a_0, a_1) \in \mathbb{P}^3$, $\psi(a \times t) = (a_0t_0, a_0t_1, a_1t_1, a_1t_1)$. Therefore $\psi(a \times t) = \mathcal{Z}(t_1x - t_0y, t_1z - t_0w)$. This is a linear variety. Let this be called M_t. As t varies M_t defines a family of curves in Q. Using arguments similar to the ones used in calculation of $dim \, L_t$, it can be calculated that $dim \, M_t = 1$.

Suppose that $L_t \neq L_u$ i.e., $u \neq t$. Suppose for some $b = (b_0, b_1)$ and $c = (c_0, c_1)$, $t \times b = u \times c$. Therefore $(t_0b_0, t_0b_1, t_1b_0, t_1b_1) = (u_0c_0, u_0c_1, u_1c_0, u_1c_1)$. Therefore $t_0b_0 = \lambda u_0c_0$, $t_0b_1 = \lambda u_0c_1$, $t_1b_0 = \lambda u_1c_0$ and $t_1b_1 = \lambda u_1c_1$ for some $\lambda \in k^{\times}$. We have that either $c_0 \neq 0$ or $c_1 \neq 0$. If $c_0 \neq 0$, then $\lambda t_1u_0c_0 = t_0t_1b_0 = \lambda t_0u_1c_0$. Since $c_0, \neq 0$; $\lambda \neq 0$, $t_1u_0 = t_0u_1$ which implies that $t = u$. But this is a contradiction. Therefore $L_t \cap L_u = \emptyset$. If $c_1 \neq 0$, $\lambda u_0c_1t_1 = t_1t_0b_1 = \lambda u_1c_1t_0$ implies that $t_0u_1 = t_1u_0$ i.e., $t = u$. This is a contradiction. Therefore $L_t \cap L_u = \emptyset$. Similarly it can be proved that if $M_t \neq M_u$, then $M_t \cap M_u = \emptyset$. Also, $M_t \cap L_u = \{t \times u\}$.

(c) Consider the twisted cubic curve in $\mathbb{P}^3$. It is equal to the subset $X = \{(p^2t,\, pt^2,\, p^3,\, t^3)\,|\, p,\, t, \in k\}$. Then clearly any point of twisted cubic curve lies on Q. We claim that $X \neq L_t$ or M_u for any $t,\, u \in \mathbb{P}^1$. Any point of the twisted cubic curve satisfies the equation $y^2 = wx$. Any point of M_t, for a fixed $t \in \mathbb{P}^1$, is of the form $(a_0t_0,\, a_0t_1,\, a_1t_0,\, a_1t_1)$ for some $(a_0,\, a_1) \in \mathbb{P}^1$. This point does not satisfy the given equation. Therefore $M_t \neq X$ for any $t \in \mathbb{P}^1$. Also. any point of L_u, for a fixed $u \in \mathbb{P}^1$, is of the form $(t_0a_0,\, t_0a_1,\, t_1a_0,\, t_1a_1)$ for some $(a_0,\, a_1) \in \mathbb{P}^1$. This point also does not satisfy the given equation. Therefore $X \neq L_u$ for any $u \in \mathbb{P}^1$.

Exercise 2.0.28. *(a) The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in $\mathbb{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.*

(b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the two ideals. For example, let C be the conic in $\mathbb{P}^2$ given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of on point P, but $I(C) + I(L) \neq I(P)$.

Solution:

(a) Any point $P = (x,\, y,\, z,\, w)$ in $Q_1 \cap Q_2$ satisfies the equations $x^2 - yw$ and $xy - zw$. When $w = 0$, $x^2 = 0$ which implies that $x = 0$. Therefore, $(0,\, y,\, z,\, 0) \in Q_1 \cap Q_2$ for any $y,\, z \in k$. These points are given by $L = \mathcal{Z}(x,\, w)$. We claim that $dim\, L = 1$. Let $I = (x,\, w)$. It can be proved that $k[x,\, y,\, z,\, w]/I \cong k[y,\, z]$ which is an integral domain. Therefore I is prime and hence a radical ideal. Since $L = \mathcal{Z}(I)$, $I(L) = \sqrt{I} = I$. Now, $dim\, S(L) = dim\, k[x,\, y] = 2$. Therefore $dim\, L = 1$. Hence L is a linear variety of dimension 1 and hence a line.

When $w \neq 0$, we may assume that $w = 1$. Then we have $x^2 = y$ and $xy = z$ which implies that $z = x^3$. Therefore any point of $Q_1 \cap Q_2$ for which $w \neq 0$ is of the form $(x,\, x^2,\, x^3,\, 1)$ which is the twisted cubic curve.

(b) Let C be the conic $\mathcal{Z}(x^2 - yz)$ and let L be the line $\mathcal{Z}(y)$. Any point (x, y, z) in $C \cap L$ has $y = 0$ and therefore $x^2 = 0$. Therefore $C \cap L$ consists of one point $(0, 0, 1)$. Therefore $C \cap L = \mathcal{Z}(x, y)$. Let $J = (x, y)$ It can be shown that $k[x, y, z]/J$ is isomorphic to $k[z]$ which is a PID and hence J is a prime ideal. Therefore $I(C \cap L) = \sqrt{J} = J$. Also, it can be shown that $k[x, y, z]/(y) \cong k[x, z]$ which is an integral domain. Therefore $I_1 = (y)$ is a prime ideal. Also, it can be shown that $x^2 - yz$ is an irreducible element of $k[x, y, z]$. Since $k[x, y, z]$ is a UFD, $I_2 = (x^2 - yz)$ is a prime ideal. Since $L = \mathcal{Z}(I_1)$ and $C = \mathcal{Z}(I_2)$, we have $I(L) = \sqrt{I_1} = I_1$ and $I(C) = \sqrt{I_2} = I_2$.

$I(C) + I(L) = \{(x^2 - yz)f_1 + yf_2 \,|\, f_1, f_2 \in k[x, y, z]\}$. We claim that $x \in I(C) + I(L)$. Any term of $(x^2 - yz)f_1$ is a multiple of either x^2 or yz and any term yf_2 is a multiple of y. Since x is not a multiple of either x^2, yz or y, $x \notin I(C) + I(L)$. Therefore $I(P) \neq I(C) + I(L)$.

Exercise 2.0.29 (Complete Intersections)**.** *A variety Y of dimension r in $\mathbb{P}^n$ is a strict complete intersection if $I(Y)$ can be generated by $n - r$ elements. Y is a set theoretic complete intersection if Y can be written as the intersection of $n - r$ hyperplanes.*

(a) Let Y be a variety in $\mathbb{P}^n$, let $Y = \mathcal{Z}(\mathfrak{a})$; and suppose that $\mathfrak{a}$ can be generated q elements. Then show that $\dim Y \geq n - q$.

(b) Show that a strict complete intersection is a set theoretic complete intersection.

(c) The converse of (b) is false. For example let Y be the twisted cubic curve in $\mathbb{P}^3$. Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1 and H_2 of degree 2 and 3 respectively, such that $Y = H_1 \cap H_2$.

(d) It is an unsolved problem whether every closed irreducible curve in $\mathbb{P}^3$ is a set theoretic intersection of two surfaces.

Solution:

(a) Let $\mathfrak{a} = \mathcal{Z}(f_1 \ldots, f_q)$. Since $Y = \mathcal{Z}(\mathfrak{a})$, $I(Y) = \sqrt{\mathfrak{a}}$. Since Y is a variety, $I(Y)$ is a prime ideal. We know that for any ideal I, $\sqrt{I}$ is the intersection of all prime ideals containing I. Therefore $I(Y)$ is the minimal prime ideal over $\mathfrak{a}$. From Krull's dimension theorem we get that $htI(Y) \leq q$. From Theorem 1.8A(b) we get that $dim\, S(Y) \geq n + 1 - q$. Since $dim\, Y = dim\, S(Y) - 1$, we get that $dim\, Y \geq n - q$.

(b) We assume that the variety Y of dimension r is a strict complete intersection, i.e., $I(Y)$ can be generated by $n - r$ elements. Let $I(Y) = (f_1, \ldots, f_{n-r})$. Then $Y = \mathcal{Z}(I(Y))\, \mathcal{Z}(f_1, \ldots, f_{n-r}) = \bigcap_{i=1}^{n-r} \mathcal{Z}(f_i)$. Since each $\mathcal{Z}(f_i)$ is a hypersurface, Y can be written as an intersection of $n - r$ hypersurfaces and is thus a set theoretic complete intersection.

(c) We know that the twisted cubic curve in $\mathbb{P}^3$ consists of points of the form $(p^3 : p^2t : pt^2 : t^3)$ for some $p, t \in k$ such that $p^2 + t^2 \neq 0$. It is clear that no homogeneous polynomial of degree 1 in $k[u, x, y, z]$ belongs to $I(Y)$ because any homogeneous polynomial of degree 1 is of the form $x_i - ax_j$ for some $a \in k^\times$ and some $x_i\, x_j \in \{u, x, y, z\}$. Also, any polynomial in $I(Y)$ can be written as $u^{d_1}x^{d_1}y^{d_2}z^{d_3}$ such that $3d_0 + 2d_1 + d_2 = 0$ and $d_1 + 2d_2 + 3d_3 = 0$. Using these equations we can deduce that the only homogeneous polynomials of degree 2 that belong to $I(Y)$ are $f_1 = uz - xy$, $f_2 = uy - x^2$ and $f_3 = xz - y^2$.

Suppose $I(Y) = (g_1, g_2)$ for some homogeneous polynomials $g_1, g_2 \in k[u, x, y, z]$. Then $f_1 = g_1h_1 + g_2h_2$ for some $h_1, h_2 \in k[u, x, y, z]$. Since f_1 is homogeneous ,we may assume that h_1 and h_2 are homogeneous polynomials. Now, we have that g_1h_1 is homogeneous of degree $deg\, g_1 + deg\, h_1$ and g_2h_2 is homogeneous of degree $deg\, g_2 + deg\, h_2$. Therefore $deg\, g_1 + deg\, h_1 = deg\, g_2 + deg\, h_2 = 2$. Therefore $deg\, g_i \leq 2$ for $i = 1, 2$. But $I(Y)$ has no homogeneous polynomials of degree 1. Therefore $deg\, g_i = 2$ for $i = 1, 2$ and thus $\{g_1, g_2\} \subset \{f_1, f_2, f_3\}$.

Suppose $I(Y) = (f_1, f_2)$. Then $f_3 = h_1f_1 + h_2f_2$ for some homogeneous polynomials h_i. But using arguments similar to the above, we get that $h_1, h_2 \in k$. Say $h_i = a_i \in k$ for $i = 1, 2$. But clearly $xz - y^2$ cannot be written as $a_1(uz - xy) + a_2(uy - x^2)$ for any constants $a_1, a_2 \in k$. Therefore $I(Y) \neq (f_1, f_2)$. Similarly, if $I(Y) = (f_2, f_3)$, then $f_1 = a_2 f_2 + a_3 f_3$ for some constants $a_2, a_3 \in k$. But clearly this is not possible. Therefore $I(Y) \neq (f_2, f_3)$. Similarly, it can be shown that $I(Y) \neq (f_1, f_3)$. Therefore $I(Y)$ is not generated by two elements and is thus not a strict complete intersection.

Let $H_1 = \mathcal{Z}(y^2 - xz)$ and $H_2 = \mathcal{Z}(x^3 + u^2 z - 2xyu)$. We claim that $Y = H_1 \cap H_2$. Clearly any point $(p^3 : p^2 t : pt^2 : t^3)$ of Y lies on both H_1 and H_2. Consider a point $P = (a : b : c : d) \in H_1 \cap H_2$. Then $c^2 = bd$ and thus $b^2c^2 = b^3 d$. But $b^3 = 2abc - a^2 d$. Therefore $b^2c^2 = 2abcd - a^2d^2$ which implies that $(bc - ad)^2 = 0$ i.e., $bc = ad$.

Since $c^2 = bd$, we get $a^2c^2 - a^2bd = 0$ which gives that $a^2c^2 + b^4 - b^4 - a^2bd = 0$. But $b^3 + a^2 d = 2abc$. Therefore $b^4 + a^2c^2 - 2b^2ca = (b^2 - ac)^2 = 0$. Therefore $b^2 = ac$. Clearly any point of this form is a point in Y. Therefore $H_1 \cap H_2 \subset Y$ proving that $Y = H_1 \cap H_2$. This proves that Y is a set theoretic complete intersection. Therefore every set theoretic complete intersection need not be a strict complete intersection.

Chapter 3

Morphisms

We fix some notation which will be used throughout this section. For any variety X, we denote the open set $X \setminus \mathcal{Z}(f)$ by $D(f)$.

We state and prove a lemma which will be then used in solving the exercises.

Lemma (3 E)**.** *Let* $U = D(f)$ *be a Zariski open subset of an affine variety* $X \subset \mathbb{A}^n$ *where* f *is some polynomial in* $k[x_1, \ldots, x_n]$*. Then the ring of regular functions on* U *is the localization* $A(X)[\frac{1}{f}]$*.*

Proof. By definition, a regular function on U is a function g such that for any point $P \in U$ we can write $g = h/l$ for some polynomials h, l in the neighbourhood $U_l = U \cap D(l)$ of P. Hence U is covered by the open subsets U_l. Since the Zariski topology is noetherian, we can find a finite open sub-cover $\{U_\alpha\}$ of U. Here each U_α is of the form $U \cap U_{l_\alpha}$ for some polynomial l_α. We have that $U \subset \bigcup U_\alpha$ which implies that $X \setminus \bigcup U_\alpha \subset X \setminus U$. But $X \setminus U = \mathcal{Z}(f)$. Therefore we have that $\bigcap C_\alpha \subset \mathcal{Z}(f)$ where $C_\alpha = X \setminus U_\alpha$. But $\bigcap C_\alpha$ is the set of common zeroes of $\{l_\alpha\}$. By the Nullstellensatz, we have that there exists a positive integer m such that $f^m \in I$ where I is the ideal generated by the l_α. Therefore we can write $f^m = \sum f_\alpha l_\alpha$ for some polynomials f_α. Therefore $f^m g = \sum (f_\alpha l_\alpha)(h_\alpha/l_\alpha) = \sum f_\alpha h_\alpha$ which implies that $g = \frac{\sum f_\alpha h_\alpha}{f^m}$ which is an element of $A(X)[1/f]$. This proves the lemma. □

Exercise 3.0.30. *Show that*

(a) *Any conic in* $\mathbb{A}^2$ *is isomorphic to either* $\mathbb{A}^1$ *or* $\mathbb{A}^1 - \{0\}$*.*

(b) $\mathbb{A}^1$ *is not isomorphic to any proper open subset of itself.*

(c) *Any conic in* $\mathbb{P}^2$ *is isomorphic to* $\mathbb{P}^1$*.*

(d) $\mathbb{A}^2$ *is not homeomorphic to* $\mathbb{P}^2$*.*

(e) *If an affine variety is isomorphic to a projective variety then it consists of only one point.*

Solution:

(a) From exercise 1.1 we know that a conic Z in $\mathbb{A}^2$ is isomorphic to either the parabola $y = x^2$ or the hyperbola $xy = 1$.
Case i: Let Z be isomorphic to the parabola $y = x^2$. Then $A(Z) \cong k[x]$. But we know that $A(\mathbb{A}^1) = k[x]$. Therefore from proposition 3.5, we get that Z is isomorphic to $\mathbb{A}^1$.
Case ii: Let Z be isomorphic to the hyperbola $xy = 1$. We can define a map $\varphi : \mathbb{A}^1 \setminus \{0\} \longrightarrow Z$ by $x \mapsto (x, x^{-1})$. From Lemma 3.6, we get that this map is a morphism. It is also bijective. To see that φ is an isomorphism we have to check that the inverse map of φ is a morphism. We can check that $\varphi^{-1} : Z \longrightarrow \mathbb{A}^1 \setminus \{0\}$ is given by $(x, y) \mapsto x$. This is the restriction of the projection map which is clearly a morphism.
Therefore any conic Z in $\mathbb{A}^2$ is isomorphic to either $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$.

(b) Suppose $U \subsetneq \mathbb{A}^1$ is isomorphic to $\mathbb{A}^1$. Then $A(U) \cong A(\mathbb{A}^1) = k[x]$. But we know that U is of the form $\mathbb{A}^1 \setminus \{p_1, \ldots, p_n\}$ for some finite number of points $p_i \in \mathbb{A}^1$. Let f be the polynomial having precisely $p_1, \ldots, p_n$ as its roots. Then we know from Lemma 3E that $A(U)$ is of the form $k[x][\frac{1}{f}]$.

Now, suppose $\mathbb{A}^1$ is isomorphic to U. Then from Corollary 3.7 we know that the the coordinate ring of U is isomorphic to the coordinate ring of $\mathbb{A}^1$. Therefore $k[x, \frac{1}{f}]$ is isomorphic to $k[y]$ for some indeterminate y. Let $\phi : k[x, \frac{1}{f}] \longrightarrow k[y]$ be an isomorphism. Let $\phi(x) = p(y) \in k[y]$. Let $f = a_0 + a_1x + \ldots + a_nx^n$. Since ϕ is an isomorphism $\phi(f) = b_0 + b_1p(y) + \ldots + b_np(y)^n$ where $b_i = \phi(a_i) \in k$. Since f is a unit in $k[x, \frac{1}{f}]$ we know that $\phi(f) \in k^\times$. Therefore $p(x) \in k$. This implies that ϕ is not surjective which is a contradiction. Therefore ϕ is not an isomorphism.

(c) We claim that any two conics in $\mathbb{P}^2$ are isomorphic. Consider any conic Y in $\mathbb{P}^2$. Then $Y = \mathcal{Z}(axy + byz + cxz + dx^2 + ey^2 + fz^2)$. Let P, Q, R be any three non collinear points in $\mathbb{P}^2$. Let $P = (p_1 : p_2 : p_3)$, $Q = (q_1 : q_2 : q_3)$, $R = (r_1 : r_2 : r_3)$ be any representations of these points in $\mathbb{P}^2$. Let $v_P = (p_1, p_2, p_3)$, $v_Q = (q_1, q_2, q_3)$, $v_R = (r_1, r_2, r_3)$ be the vectors in k^3 corresponding to these representations of P, Q, R. Then clearly these vectors are linearly independent vectors in the vector space k^3. Now, given any two sets of non collinear points in $\mathbb{P}^2$, we know that they give two sets of basis vectors of the vector space k^3. Hence there exists a linear transformation $T : k^3 \longrightarrow k^3$ which is the change of basis transformation corresponding to these two bases.

But a linear transformation corresponds to a linear change of coordinates which is an isomorphism. Therefore we can assume that the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) lie on Y. Therefore $Y = \mathcal{Z}(axy + byz + cxz)$ for some $a, b, c \in k$, none of them zero. Now, we scale x, y, z by a factor of $\lambda_1 = \sqrt{\frac{c}{ab}}$, $\lambda_2 = \sqrt{\frac{b}{ac}}$ and $\lambda_3 = \sqrt{\frac{a}{bc}}$ respectively i.e., we put $X = \lambda_1 x$, $Y = \lambda_2 y$ and $Z = \lambda_3 z$. Then $Y = \mathcal{Z}(XY + YZ + ZX)$. This proves that any curve in $\mathbb{P}^2$ is isomorphic to the curve $Y = (xy + yz + zx)$ and therefore any two curves in $\mathbb{P}^2$ are isomorphic. Therefore it is enough to prove that the conic $Y = \mathcal{Z}(z^2 - xy)$ is isomorphic to $\mathbb{P}^1$. We can check that $Y = \{(t^2, u^2, tu) \in \mathbb{P}^2 \mid t, u, \in k;\ both\ \ not\ \ 0\}$. Define a map $\rho : \mathbb{P}^1 \longrightarrow Y$ by $(t, u) \mapsto (t^2, u^2, tu)$. Then clearly, ρ is a morphism. Also, the inverse map $\rho^{-1} ; Y \longrightarrow \mathbb{P}^1$ can be given by $(x, y, z) \mapsto (1, \frac{z}{x})$ when $x \neq 0$ and by $(x, y, z) \mapsto (\frac{z}{y}, 1)$ when $y \neq 0$. This map is clearly a morphism. Therefore Y is isomorphic to $\mathbb{P}^1$ and hence any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

(d) Let $\mathcal{C} \subset \mathbb{A}^2$ be irreducible closed of dimension 1. If $\mathbb{A}^2$ is homeomorphic to $\mathbb{P}^2$, then the dimension of the homeomorphic image of $\mathcal{C}$ is of dimension 1 in $\mathbb{P}^2$ and hence is a curve in $\mathbb{P}^2$. If we take two curves $\mathcal{Z}(f)$, $\mathcal{Z}(g)$ in $\mathbb{A}^2$ which have an empty intersection, say two parallel lines, then the homeomorphic image of these curves in $\mathbb{P}^2$ should have an empty intersection. To prove this let $\rho : \mathbb{A}^2 \longrightarrow \mathbb{P}^2$ be the homeomorphism and $\rho(\mathcal{Z}(f)) = \mathcal{C}_1$ and $\rho(\mathcal{Z}(g)) = \mathcal{C}_2$. If $P \in \mathcal{C}_1 \cap \mathcal{C}_2$, then $\rho^{-1}(P) \in \mathcal{Z}(f) \cap \mathcal{Z}(g)$ which is a contradiction. Therefore $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. We show in exercise 3.7 that any two curves in $\mathbb{P}^2$ have a non-empty intersection. Therefore there exists no homeomorphism between $\mathbb{A}^2$ and $\mathbb{P}^2$.

(e) Suppose $Y \subset \mathbb{A}^n$ is an affine variety. Suppose $Y : \varphi \longrightarrow Z$ is an isomorphism where Z is a projective variety. Then from the proof of Proposition 3.5, φ induces an isomorphism of the ring of regular functions $\mathcal{O}(Y)$ and $\mathcal{O}(Z)$. We know, from Theorem 3.2 and Theorem 3.4, that $\mathcal{O}(Y) = A(Y)$ and $\mathcal{O}(Z) = k$. Therefore $A(Y) = k[x_1, \ldots, x_n]/I(Y) \cong k$. This implies that $I(Y)$ is a maximal ideal of $k[x_1, \ldots, x_n]$ which in turn implies that Y consists of only a single point.

Exercise 3.0.31. *A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.*

(a) For example, let $\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^2$ be defined by $t \longmapsto (t^2, t^3)$. Show that φ defines a bijective bi-continuous morphism of $\mathbb{A}^1$ onto the curve $y^2 - x^3$, but that φ is not an isomorphism.

(b) For another example, let the characteristic of the base field be $p > 0$, and define a map $\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ by $t \mapsto t^p$. Show that φ is bijective and bi-continuous but not an isomorphism. This is called the Frobenius morphism.

Solution:

(a) Let H denote the curve $\mathcal{Z}(y^2 - x^3)$. Let $\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^2$ be the map defined by $t \mapsto (t^2, t^3)$. Let x_1 and x_2 be the coordinate functions on $\mathbb{A}^2$. Then the map $x_1 \circ \varphi : X \longrightarrow k$ is given by $t \mapsto t^2$ and the map $x_2 \circ \varphi : X \longrightarrow k$ is given by $t \mapsto t^3$. Clearly, these maps are regular functions. Therefore from Lemma 3.6, we get that $\varphi : X \longrightarrow Y$ is a morphism. This map is clearly a bijection onto the image, which is the curve H. Now consider an closed subset $X \subset \mathbb{A}^1$. Then X consists of a finite number of points. Since φ is a bijective map, we get that $\varphi(X)$ is a finite set and thus is a closed subset of $\mathbb{A}^2$ and hence φ is a closed map. If we let μ denote the inverse map of φ, then for any closed subset $X \subset \mathbb{A}^1$ we have that $\mu^{-1}(X) = \varphi(X)$ which is a closed subset of $\mathbb{A}^2$. Therefore φ is a bicontinuous bijection.

Now,, suppose that φ is an isomorphism of $\mathbb{A}^1$ onto H. Then from proposition 3.5 we know that the morphism $\psi = \varphi : H \longrightarrow \mathbb{A}^1$ induces a k-algebra isomorphism $\psi^{\#}$ from $A(\mathbb{A}^1) = k[t]$ onto $A(H) = k[x, y]/(y^2 - x^3)$ given by $f(t) \mapsto f(t) \circ \psi = f(\frac{y}{x})$. But this map is clearly not surjective and hence is not an isomorphism. Therefore φ is not an isomorphism.

We can further show that H is not isomorphic to $\mathbb{A}^1$. Suppose that H is isomorphic to $\mathbb{A}^1$. Then from proposition 3.5, $A(H) \cong A(\mathbb{A}^1) = k[x]$. We now show that $A(H) = k[x, y]/(y^2 - x^3)$ is not a Unique Factorization Domain (UFD). But since $k[x]$ is a UFD, we get a contradiction to the assumption that $A(H) \cong A(\mathbb{A}^2)$ which proves that φ is not an isomorphism.

Let us denote $k[x, y]/(y^2 - x^3)$ by R. We claim that R is isomorphic to the $k[t^2, t^3]$ for some indeterminate t. We define a map $\varphi : k[x, y] \longrightarrow k[t^2, t^3]$ by $a \mapsto a$ for all $a \in k$, $x \mapsto t^2$ and $y \mapsto t^3$. Then clearly this map is a k-algebra homomorphism. Also, it can be easily checked that $ker\,\varphi = (y^2 - x^3)$. This proves the claim. We now prove that $k[t^2, t^3]$ is not integrally closed in its field of fractions which will prove that R is not a unique factorization domain. We know that the field of fractions of R is equal to the field $k(t)$. Consider the monic polynomial $x^3 - t^3$ with coefficients in $k[t^2, t^3]$. Then clearly t is a root of this monic polynomial and hence t is integral over $k[t^2, t^3]$. But $t \notin k[t^2, t^3]$ which proves that $k[t^2, t^3]$ is not integrally closed.

(b) Suppose $char\,k = p > 0$. Then the morphism $\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ defined by $t \mapsto t^p$ is an injective. Indeed, for any $t, s \in k$, we have $0 = t^p - s^p = (t - s)^p$ which implies that $t = s$. Also, since k is algebraically closed field, φ is surjective. We know that the proper closed subsets of $\mathbb{A}^1$ are precisely the finite subsets of $\mathbb{A}^1$. Therefore to prove that φ is a continuous map, it is enough to prove that the inverse image of a single point is a closed subset of $\mathbb{A}^1$. But since φ is a bijection, the inverse image of the singleton set is a singleton set, which is a closed subset of $\mathbb{A}^1$. Therefore φ is a continuous map. The same argument proves that $\varphi^{-1} : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$, which is a bijection, is a continuous map. Therefore φ is a bicontinuous, bijection.

Consider $x_1 \circ \varphi : \mathbb{A}^1 \longrightarrow k$, where x_1 is the coordinate function on $\mathbb{A}^1$, which is nothing but the identity function. Then $x_1 \circ \varphi$ is defined by $t \mapsto t^p$ and hence is a regular function. Therefore from Lemma 3.6, we get that φ is a morphism.

Suppose $\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ is an isomorphism. Then the map $h : k[x] \longrightarrow k[x]$ defined by $f(x) \mapsto f(x) \circ \varphi$ is an isomorphism. But $f(x) \circ \varphi = f(x^p)$. This map is clearly not surjective and hence is not an isomorphism. Therefore φ is not an isomorphism.

Exercise 3.0.32. *(a) Let $\varphi : X \longrightarrow Y$ be a morphism. Then for each $P \in X$, φ induces a homomorphism of local rings $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \longrightarrow \mathcal{O}_{P,X}$.*

(b) Show that a morphism φ is an isomorphism if and only if φ is a homeomorphism, and the induced map φ_P^* on the local rings is an isomorphism, for all $P \in X$.

(c) Show that if $\varphi(X)$ is dense in Y, then the map φ_P^* is injective for all $P \in X$.

Solution:

(a) Let $\varphi : X \longrightarrow Y$ be a morphism. Consider the induced map $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} :\longrightarrow \mathcal{O}_{P,X}$ defined by $(V, f) \mapsto (\varphi^{-1}(V), f \circ \varphi)$. Since φ is a morphism and $f : V \longrightarrow k$ is a regular map, we know that $f \circ \varphi : \varphi^{-1}(V) \longrightarrow k$ is a regular map. To check that this map is well defined, suppose that $(V_1, f_1) = (V_2, f_2)$ i.e., $f_1 = f_2$ on $V_1 \cap V_2$. Then we want to prove that $f_1 \circ \varphi = f_2 \circ \varphi$ on $\varphi^{-1}(V_1) \cap \varphi^{-1}(V_2)$. Consider $x \in \varphi^{-1}(V_1) \cap \varphi^{-1}(V_2)$. Then $\varphi(x) \in V_1 \cap V_2$. Therefore $f_1(\varphi(x)) = f_2(\varphi(x))$ which proves that $f_1 \circ \varphi = f_2 \circ \varphi$ on $\varphi^{-1}(V_1) \cap \varphi^{-1}(V_2)$.

To check that the map is a ring homomorphism. Consider $\varphi_P^*((V_1, f_1) + (V_2, f_2)) = \varphi_P^*(V_1 \cap V_2, f_1 + f_2) = (\varphi^{-1}(V_1 \cap V_2), (f_1 + f_2) \circ \varphi) = (\varphi^{-1}(V_1) \cap \varphi^{-1}(V_2), f_1 \circ \varphi + f_2 \circ \varphi) = (\varphi^{-1}(V_1), f_1 \circ \varphi) + (\varphi^{-1}(V_2), f_2 \circ \varphi) = \varphi_P^*(V_1, f_1) + \varphi_P^*(V_2, f_2)$. This proves that φ_P^* is a ring homomorphism.

(b) If φ is an isomorphism, then it is a homeomorphism. Also, for any $P \in X$ the map $(\varphi^{-1})_P^*$ defines an inverse homomorphism for the map φ_P^*.

Conversely, assume that φ is an homeomorphism and that φ_P^* is an isomorphism for each $P \in X$. We have to prove that $\varphi^{-1} : Y \longrightarrow X$ is a morphism. Let $U \subset X$ be any open subset of X and $f : U \longrightarrow k$ be any regular map. Let $P \in U$ be any point. Then $(U, f) \in \mathcal{O}_{P,X}$. Since φ_P^* is an isomorphism, it has an inverse morphism. Let $\mu_P^* : \mathcal{O}_{P,X} \longrightarrow \mathcal{O}_{\varphi(P),Y}$ be the inverse map of φ_P^*. Then $\mu_P^*(U, f) \in \mathcal{O}_{\varphi(P),Y}$. Suppose $\mu_P^*(U, f) = (V, g)$ where V is an open subset of Y and $g : V \longrightarrow k$ is a regular function. But $(U, f) = \varphi_P^*(\mu_P^*(U, f)) = \varphi_P^*(V, g) = (\varphi^{-1}(V), g \circ \varphi)$. Therefore, $f = g \circ \varphi$ on $U \cap \varphi^{-1}(V)$ which in turn implies that $f \circ \varphi^{-1} = g$ on $V \cap \varphi(U)$.

Therefore, we have that $f \circ \varphi^{-1}$ is a regular function in a open neighbourhood $V \cap \varphi(U)$ of $\varphi(P)$, i.e, $f \circ \varphi^{-1}$ is regular at $\varphi(P)$. Since P is any general of point of U we have that $f \circ \varphi^{-1}$ is regular on $\varphi(U)$. Therefore $\varphi^{-1} : Y \longrightarrow X$ is a morphism.

(c) Consider the morphism $\varphi : X \longrightarrow Y$ and the induced morphism $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \longrightarrow \mathcal{O}_{P,X}$. Suppose $\varphi(X)$ is dense in Y. Consider an element of $(U, f) \in \mathcal{O}_{\varphi(P),Y}$ such that $\varphi_P^*(U, f) = 0$. We now prove that $(U, f) = 0$ which proves that φ_P^* is injective. Now, $\varphi_P^*(U, f) = (f \circ \varphi, \varphi^{-1}(U))$. Therefore $f(\varphi(x)) = 0$ for all $x \in \varphi^{-1}(U)$. Therefore $f = 0$ on $U \cap \varphi(X)$ which implies that $U \cap \varphi(X) \subset \mathcal{Z}(f)$. If we prove that $U \subset \mathcal{Z}(f)$, then $f(x) = 0$ for all $x \in U$ and therefore $(U, f) = 0$ which will prove the claim. We have $\varphi(X) \subset \mathcal{Z}(f) \cup U^c$ which is a closed subset of Y. Also, since U is non-empty, U^c is a proper subset of Y. Now, suppose $U \not\subset \mathcal{Z}(f)$. Therefore $\mathcal{Z}(f) \cup U^c \neq Y$. Therefore $\mathcal{Z}(f) \cup U^c$ is a proper closed subset of Y which contains $\varphi(X)$, which contradicts the property that $\varphi(X)$ is dense in Y. Therefore $U \subset \mathcal{Z}(f)$ as required.

Exercise 3.0.33. *Show that the $d-$uple embedding of $\mathbb{P}^n$ is an isomorphism onto its image.*

Solution:

Let $\rho_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ be the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ where $N = \binom{n+d}{n} - 1$. It is defined by sending the point $P = (a_0 : a_1 : \ldots : a_n)$ to the point $\rho_d(P) = (M_0(a) : \ldots : M_N(a))$ where M_i are the monomials of degree d in $n + 1$ variables. Since the map is defined by polynomial functions, it is clearly a morphism. Also, from the solution to the exercise 2.12, we know that ρ_d is a homeomorphism onto the image $\rho_d(\mathbb{P}^n)$.

We label the coordinates of $\mathbb{P}^N$ using $n + 1$ tuples $a_0a_1 \ldots a_n$ such that $\sum_{i=0}^n a_i = d$. Then from Exercise 2.12 we know that for any point in $\rho_d(\mathbb{P}^n)$ at least one of the coordinates of the form $b_{00\ldots00d\ldots00}$ is on zero. Suppose after permutation of the coordinates we assume that $b_{d00\ldots00} \neq 0$, then we know from the same exercise that $\rho^{-1}(P) = (u_1, \ldots, u_n)$ where $u_0 = b_{d00\ldots00}$, $u_1 = b_{d-1,1\ldots00}$, $u_2 = b_{d-1,01\ldots00}, \ldots\ldots, u_n = b_{d-1,00\ldots01}$. Therefore ρ_d^{-1} is a morphism.

Exercise 3.0.34. *By the abuse of language, we will say that a variety 'is affine' if it is isomorphic to an affine variety. Let* $H \subset \mathbb{P}^n$ *be any hypersurface, show that* $\mathbb{P}^n - H$ *is affine.*

Solution:
Let $H \subset \mathbb{P}^n$ be a surface of degree d. Suppose $H = \mathcal{Z}(\sum a_{i_0 \ldots i_n} x_0^{i_0} \ldots x_n^{i_n})$. Let L be the hyperplane $\mathcal{Z}(\sum a_{i_0 \ldots a_{i_n}} x_{i_0 i_1 \ldots x_{i_n}})$. Consider the d-uple embedding of $\mathbb{P}^n$. Then $\rho_d(H) = L \cap \rho_d(\mathbb{P}^n)$. We know that $\mathbb{P}^N \setminus \mathcal{Z}(Y_j)$ is isomorphic to $\mathbb{A}^N$. Also, any two hyperplanes are isomorphic (by a linear change of coordinates). Therefore $\mathbb{P}^N \setminus L$ is affine. Since ρ_d is an isomorphism of $\mathbb{P}^n$ onto its image in $\mathbb{P}^N$, we have that $\mathbb{P}^n \setminus H$ is isomorphic to $\rho_d(\mathbb{P}^n \setminus H)$. But $\rho_d(\mathbb{P}^n \setminus H) = (\mathbb{P}^N \setminus L) \cap \rho_d(\mathbb{P}^n)$ which is a closed subset of $\mathbb{P}^N \setminus L$ and is therefore affine.

Exercise 3.0.35. *There are quasi-affine varieties which are not affine. For example, show that* $X = \mathbb{A}^2 - \{(0, 0)\}$ *is not affine.*

Solution:
We know that $X = U_1 \cup U_2$ where $U_1 = \{(x, y) \in X \mid x \neq 0\}$ and $U_2 = \{(x, y) \in X \mid y \neq 0\}$ are open subset of X. Now, consider an element $f \in \mathcal{O}(X)$. Then f is regular at every point of U_1 and hence from the above lemma we get that f is of the form g_1/x^n on U_1 for some positive integer n. We may assume that g_1 is not divisible by x^n. Similarly, we get that f is of the form g_2/y^m on U_2 for some positive integer m such that g_2 is not divisible by y^m. On $U_1 \cap U_2$ we have $g_1/x^n = g_2/y^m$. Therefore $g_1 y^m = g_2 x^n$. But $x^n \nmid g_1$ and $y^m \nmid g_2$ and $k[x, y]$ is a unique factorization domain. Therefore we have that $m = n = 0$ and hence $f = g_1 = g_2$. Therefore $\mathcal{O}(X) \cong k[x, y]$.

Suppose that X is affine. Suppose X is isomorphic to the affine variety $V \subset \mathbb{A}^n$. Then from Proposition 3.5 we get that $A(V) \cong k[x, y]$. But $A(\mathbb{A}^2) \cong k[x, y]$. Now from Corollary 3.7 we get that V is isomorphic to $\mathbb{A}^2$. Therefore we get that X is isomorphic to $\mathbb{A}^2$. Let $A(\mathbb{A}^2) = k[u, t]$. Now from Proposition 3.5 we get that the identity isomorphism h from $A(\mathbb{A}^2) = [u, t]$ to $A(X) = k[x, y]$ induces an isomorphism from X to $\mathbb{A}^2$. But the morphism induced by the identity isomorphism $h : k[u, t] \longrightarrow k[x, y]$ induces the inclusion morphism $i : X \longrightarrow \mathbb{A}^2$. But i is not surjective and hence is not an isomorphism. This proves that X is not affine.

Exercise 3.0.36. *(a) Show that any two curves in $\mathbb{P}^2$ have a non-empty intersection.*

(b) More generally, show that if $Y \subset \mathbb{P}^n$ is a projective variety of dimension ≥ 1, and if H is a hypersurface, then $Y \cap H \neq \emptyset$.

Solution:

(a) Let $Y_1 = \mathcal{Z}(f_1)$ and $Y_2 = \mathcal{Z}(f_2)$ be two curves in $\mathbb{P}^2$, where $f_1,\, f_2$ are homogeneous polynomials in $k[x,\, y,\, z]$. Let $Z = Y_1 \cap Y_2 = \mathcal{Z}(f_1,\, f_2) \subset \mathbb{P}^2$. Let $C(Y) = \mathcal{Z}(f_1,\, f_2) \subset \mathbb{A}^3$. Since $f_1,\, f_2$ are homogeneous polynomials in $k[x,\, y,\, z]$, $O = (0,\, 0,\, 0) \in C(Z)$. Now, Z is non-empty if and only if $C(Z)$ has points other than O. From Proposition 1.13 we know that the dimension of $C(Y_1)$ is 2, where $C(Y_1)$ denotes the cone over Y_1. Also, from exercise 1.8, dimension of every irreducible component of $C(Z)$ is 1. If $C(Z) = \{(0,\, 0,\, 0)\}$, then $dim\, C(Z) = 0$ which is a contradiction. Therefore $C(Z)$ contains points other than $(0,\, 0,\, 0)$ and therefore $Y_1 \cap Y_2 \neq \emptyset$.

(b) Suppose $dim\, Y = r \geq 1$. Then we know that $C(Y)$ is an affine variety of dimension $r + 1$. Now, $Y \cap H \neq \emptyset$ if and only if $C(Y) \cap C(H)$ has some point other than $(0,\, 0,\, 0)$. From exercise 1.8, dimension of every irreducible component of $C(Y) \cap C(H)$ is equal to $r \geq 1$. If $C(Y) \cap C(H) = \{(0,\, 0,\, 0)\}$, then $dim\, (C(Y) \cap C(H)) = 0$, which is a contradiction. Therefore $C(Y) \cap C(H)$ has some point other than $(0,\, 0,\, 0)$ and thus $Y \cap H \neq \emptyset$.

Exercise 3.0.37. *Let H_i and H_j be the hyperplanes in $\mathbb{P}^n$ defined by $x_i = 0$ and $x_j = 0$, with $i \neq j$. Show that any regular function on $\mathbb{P}^n - (H_i \cap H_j)$ is constant.*

Solution:

Let $X = \mathbb{P}^n \setminus (H_i \cap H_j)$. Therefore X consists of points in $\mathbb{P}^n$ where either $x_i \neq 0$ or $x_j \neq 0$. Therefore $X = U_i \cup U_j$ where $U_i = \mathcal{Z}(x_i)^c$ and $U_j = \mathcal{Z}(x_j)^c$. We have that $f \in A(U_i) = k\left[\frac{x_0}{x_i},\, \ldots,\, \frac{x_n}{x_i}\right]$ which implies that $f = g/x_i^r$ on U_i where $g \in k[x_0,\, \ldots,\, x_n]$ and $r = deg\,(g)$. Similarly we get that $f = h/x_j^s$ on U_j where $h \in k[x_0,\, \ldots,\, x_n]$ and $s = deg\,(h)$. Also, on $U_i \cap U_j$ we get that $x_j^s g = x_i^r h$. Using the fact that $k[x_0,\, \ldots,\, x_n]$ is a unique factorization domain we get that $r = s = 0$. Therefore $f = g = h$ which are degree 0 polynomials and hence constants. Therefore $f \in k$.

Exercise 3.0.38. *The homogeneous coordinate ring of a projective variety is not invariant under isomorphism. For example, let $X = \mathbb{P}^1$ and let Y be the 2−uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$. Then $X \cong Y$. But show that $S(X) \ncong S(Y)$.*

Solution:
Let $\rho_2 : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ be the 2-uple embedding map. Then ρ_2 is given by $(a, b) \mapsto (a^2, ab, b^2)$. Also, $\rho_2(\mathbb{P}^1) = \mathcal{Z}(y^2 - xz)$. Let $X = \mathbb{P}^1$ and $Y = \rho_2(\mathbb{P}^1)$. From exercise 3.4, we know that $Y \cong X$. We know that $S(X) = k[t, u]$ and $S(Y) = k[x, y, z]/(y^2 - xz)$. Let R denote $k[x, y, z]/(y^2 - xz)$. We have to show that $k[x, y] \ncong R$. We know from Hilbert Nullstellensatz that every maximal ideal of $k[t, u]$ is generated by two elements. We construct a maximal ideal in R which is not generated by two elements which will prove that R is not isomorphic to $k[u, t]$. Let us denote the polynomial $y^2 - xz$ by f. Consider the maximal ideal $M = (x, y, z)$ of the polynomial ring $k[x, y, z]$. Then clearly $f \in M^2$. We have that $M/(f)$ is a maximal ideal of R. Let this ideal be denoted by $\mathfrak{m}$. Then we claim that $\mathfrak{m}$ is not generated by two elements. Assume the contrary. Let $g, h\, k[x, y, z]$ be polynomials such that $\mathfrak{m} = (\overline{g}, \overline{h})$. This implies that $M = (f, g, h)$. But we know that M/M^2 is a three dimensional vector space over k. We also know that if $M = (f, g, h)$, then M/M^2 is generated by $\overline{f}, \overline{g}, \overline{h}$ as a k vector space. But since $\overline{f} = 0$, we have that M/M^2 is generated by $\overline{g}, \overline{h}$ which is a contradiction since M/M^2 is a three dimensional vector space. Therefore $\mathfrak{m}$ is not generated by two elements. This proves that $k[x, y, z]/(y^2 - xz) \ncong k[u, t]$.

Exercise 3.0.39 (Subvarieties)**.** *A subset of a topological space is called locally closed if its is an open subset of its closure, or, equivalently, if it is the intersection of an open set with a closed set.*

If X is a quasi affine (or a quasi projective) variety and if Y is an irreducible locally closed subset, then Y is also quasi affine(respectively, quasi projective) variety, by virtue of being a locally closed subset of the same affine or projective space. We call this the induced structure on Y and we call Y a subvariety of X.

Now let $\varphi : X \longrightarrow Y$ be a morphism, let $X' \subset X$ and $Y' \subset Y$ be irreducible locally closed subsets such that $\varphi(X') \subset Y'$. Show that $\varphi|_{X'} : X' \longrightarrow Y'$ is a morphism.

Solution:

Consider the map $\varphi \circ i : X' \longrightarrow Y$ where i is the inclusion morphism $X' \hookrightarrow X$. Since the composition of morphisms is a morphism, we get that $\varphi \circ i$ is a morphism. Let $\varphi \circ i$ be denoted by φ'. Since $\varphi(X') \subset Y'$, we know that $Im\,\varphi' \subset Y'$. Now consider the map $\varphi|_{X'} : X' \longrightarrow Y'$ defined by restricting the co-domain of φ' to Y'. This map is also clearly a morphism.

Exercise 3.0.40. *Let X be any variety and let $P \in X$. Show there is a 1-1 correspondence between the prime ideals of the local ring $\mathcal{O}_P$ and the closed sub-varieties of X containing P.*

Solution:

Case 1: Suppose X is a quasi-affine variety. Suppose X is an open subset of the affine variety $Z \subset \mathbb{A}^n$. From theorem 3.2, we know that the prime ideals of $\mathcal{O}_P$ are in 1-1 correspondence with the prime ideals $A(Z)$ contained in $\mathfrak{m}_P$. But the prime ideals $\mathfrak{p}$ of $A(Z)$, contained in $\mathfrak{m}_P$, correspond to the varieties Y of Z containing P. Therefore there is a 1-1 correspondence between the prime ideals of $\mathcal{O}_P$ and the varieties of Z containing P.

But if Y is a variety of Z, then $Y \cap X$ is an irreducible locally closed subset of X. Also, if Y_1 and Y_2 are two varieties of Z such that $Y_1 \cap X = Y_2 \cap X$ then $Y_1 = Y_2$. Because if $Y_1 \neq Y_2$, then $A = Y_1 \cap Y_2$ and $B = (Y_1 \cap X)^c$ are two proper closed subsets of Y_1 such that $Y_1 = A \cup B$ which contradicts the irreducibility of Y_1. Therefore, there is a 1-1 correspondence between the varieties of Z containing P and the closed sub-varieties of X containing P. This implies that there is a 1-1 correspondence between the prime ideals of $\mathcal{O}_P$ and the closed sub-varieties of X containing P.

Case 2: Suppose X is a quasi-projective variety. Suppose X is an open subset of the projective variety $Z \subset \mathbb{P}^n$. From theorem 3.4, we know that the prime ideals of $\mathcal{O}_P$ are in 1-1 correspondence with the homogeneous prime ideals of $S(Y)$ contained in $\mathfrak{m}_P$. But the homogeneous prime ideals $\mathfrak{p}$ of $A(Z)$, contained in $\mathfrak{m}_P$, correspond to the varieties Y of Z containing P. Therefore there is a 1-1 correspondence between the prime ideals of $\mathcal{O}_P$ and the varieties of Z containing P.

Now, arguing as in the quasi-affine case, we get that there is a 1-1 correspondence between the prime ideals of $\mathcal{O}_P$ and the varieties of Z containing P.

Exercise 3.0.41. *If P is a point on the variety X, then the $dim\, \mathcal{O}_P = dim\, X$.*

Solution:

If X is an affine variety, then the result is the same as proposition 3.2(c). Suppose $X \subset \mathbb{P}^n$ is a projective variety, then it is covered by a finite number of affine open subsets $\{U_i\}_{i-=1}^{n+1}$ of $\mathbb{P}^n$. Since $P \in X$, $P \in U_i$ for some $i = 1, \ldots, n+1$. Without loss of generality, we may assume that $P \in U_1$. Then we know that $\mathcal{O}_{P, U_1} = \mathcal{O}_{P, X}$. Since U_1 is an affine variety, we get from proposition 3.2(c) that $dim\, U_1 = dim\, \mathcal{O}_{P, U_1} = dim\, \mathcal{O}_{P, X}$. From the exercise 1.10 we know that if $\{U_i\}$ forms an open cover of a irreducible noetherian space, then $U_i \cap X \neq \emptyset$ implies that $dim\, U_i = dim\, X$. Therefore $dim\, \mathcal{O}_{P, X} = dim\, X$.

Exercise 3.0.42 (The Local Ring of a Subvariety)**.** *Let $Y \subset X$ be a subvariety. Let $\mathcal{O}_{Y, X}$ be the set of equivalence classes (U, f) where $U \subset X$ is open, $U \cap Y \neq \emptyset$, and f is a regular function on U. We say (U, f) is equivalent to (V, g), if $f = g$ on $U \cap V$ Show that $\mathcal{O}_{Y, X}$ is a local ring with residue field $K(Y)$ and dimension $= dim\, X - dim\, Y$. It is the local ring of Y on X. Note if $Y = P$ is a point we get $\mathcal{O}_P$, and if $Y = X$ we get $K(X)$. Note also that if Y is not a point, then $K(Y)$ is not algebraically closed, so in this way we get local rings whose residue fields are not algebraically closed.*

Solution:

Let $\mathfrak{m}_Y$ denote the ideal $\{(U, f) \in \mathcal{O}_{Y, X} \mid f(x) = 0 \,\forall\, x \in U \cap Y\}$. We prove that any element of $\mathcal{O}_{Y, X}$ not in $\mathfrak{m}_Y$ is a unit which proves that $\mathcal{O}_{Y, X}$ is a local ring with maximal ideal $\mathfrak{m}_Y$. Suppose $(U, f) \notin \mathfrak{m}_Y$. Then $f \neq 0$ on $U \cap Y$. Therefore $\exists\, P \in Y \cap U$ such that $f(P) \neq 0$. Since f is regular on U, there exists a neighbourhood V of P in U such that $f = \frac{f_1}{f_2}$ on V. Let $g = \frac{f_2}{f_1}$. Consider $(g, V)(f, U) = (fg, V \cap U) = (I_d, V \cap U)$. Therefore (f, U) is a unit in $\mathcal{O}_{Y, X}$.

We now claim that $\mathcal{O}_{Y, X}/\mathfrak{m}_Y \cong K(Y)$. Consider an element $(U, f) \in \mathcal{O}_{Y, X}$. Therefore f is a regular function on U. Therefore f is regular on $U \cap Y$ which is an open subset of Y. We define a map $\varphi : \mathcal{O}_{Y, X} \longrightarrow K(Y)$ by $\varphi(U, f) = (U \cap Y, f)$. It can be checked that this ring homomorphism. Now consider an element in $K(Y)$. It is an equivalence class of the form (V, f) where V is a non empty open subset of Y and f is a regular function on V and where two pairs (V_1, f_1) and (V_2, f_2) are identified if $f_1 = f_2$ on $V_1 \cap V_2$. Since f is regular on V for any point $P \in V$, we can find a open neighbourhood V_0, open in V, of P such that $f = h/g$ on V_0. Since $P \in V \cap D(g)$, we have that $V \cap D(g) \neq \emptyset$. Also, $(V, f) = (V \cap D(g), f)$. Also, $D(g) \cap Y \neq \emptyset$ and $(D(g) \cap Y, h/g) = (V, f)$. Therefore we have that $\varphi(D(g), h/g) = (D(g) \cap Y, h/g) = (V, f)$ proving that φ is surjective.

Consider any element $(U, f) \in \mathfrak{m}_Y$. Then $\varphi(U, f) = (U \cap Y, f) = 0$. Therefore $\mathfrak{m}_Y \subset ker\, \varphi$. Since $\mathfrak{m}_Y$ is a maximal ideal, we get that $\mathfrak{m}_Y = ker\, \varphi$. Therefore $K(Y) \cong \mathcal{O}_{Y,X}/\mathfrak{m}_Y$ and hence $K(Y)$ is the residue field of $\mathcal{O}_{Y,X}$.

Suppose X is a projective variety. Let $\{X_i\}$ be the open cover of X by affine open subsets where $X_i = U_i \cap X$ and U_i is as defined before proposition 2.2. Then $\{Y \cap X_i\}$ is a cover of Y by affine open subsets. Also, $Y \cap X_i = Y \cap U_i$. Let $Y \cap X_i$ be denoted by Y_i. Now $Y_i \neq \emptyset$ implies that $X_i \neq \emptyset$. Also, by definition of the local ring $\mathcal{O}_{Y,X}$ we know that $\mathcal{O}_{Y,X} \cong \mathcal{O}_{Y_i, X_i}$. Also, from the solution to the exercise 2.6 we know that $dim\, X = dim\, X_i$ and $dim\, Y = dim\, Y_i$. Therefore it is enough if we prove that dimension of $\mathcal{O}_{Y,X}$ is equal to $dim\, X - dim\, Y$ in the case when X is a affine variety.

Let us assume that X is an affine variety. For this we make the claim that $\mathcal{O}_{Y,X} \cong A(X)_{\mathcal{I}}$ where $\mathcal{I}$ is the prime ideal of regular functions on X vanishing on Y. Any element of $A(X)_{\mathcal{I}}$ is of the form f/g where g is a polynomial which does not vanish on Y. Let $U = D(g)$. Then $U \cap Y \neq \emptyset$. Define $\varphi : A(X)_{\mathcal{I}} \longrightarrow \mathcal{O}_{Y,X}$ by sending the f/g to the equivalence class $(U, f/g)$. Suppose f_1/g_1 and f_2/g_2 are two elements of $A(X)_{\mathcal{I}}$ such that $(f_1/g_1, U_1) = (f_2/g_2, U_2)$ where $U_1 = D(g_1)$ and $U_2 = D(g_2)$. Then $f_1/g_1 = f_2/g_2$ on $U_1 \cap U_2$. Now $X \setminus (U_1 \cap U_2)$ is a closed set and hence is of the form $\mathcal{Z}(h_1, \ldots, h_l)$ for some polynomials h_i. Therefore for any $i = 1, \ldots, l$ we have $h_i\,(f_1 g_2 - f_2 g_1) = 0$. Since $Y \cap (U_1 \cap U_2) \neq \emptyset$, $h_i \notin \mathcal{I}$ for any $i = 1, \ldots, l$. Therefore $f_1/g_1 = f_2/g_2$ in $A(X)_{\mathcal{I}}$. Therefore the map is injective.

Consider any element (U, f) where f is a regular function on U. Then for any point $P \in U$, there exists a neighbourhood $V \subset U$ containing P such that $f = f_1/g_1$ on V and $g \not\equiv 0$ on V. Since $g_1(P) \neq 0$, we have that $g \notin \mathcal{I}$. We claim that (U, f) is the image of f_1/g_1. But we know that the image of f_1/g_1 is $(U_0, f_1/g_1)$ where $U_0 = D(g_1)$. Therefore it is enough to prove that $f = f_1/g_1$ on $U \cap U_0$. But we can take $V = U_0 \cap U$ which proves the claim. Therefore $dim\, \mathcal{O}_{Y,X} = ht(\mathcal{I})$. Recall that $A(X)/\mathcal{I} = A(Y)$. Then it follows that $dim\, \mathcal{O}_{Y,X} = ht\,(\mathcal{I}) =$

$dim\, A(X) - dim\, A(Y) = dim\, X - dim\, Y$.

Exercise 3.0.43 (Projection from a point)**.** *Let $\mathbb{P}^n$ be a hyperplane in $\mathbb{P}^{n+1}$ and let $P \in \mathbb{P}^{n+1} - \mathbb{P}^n$. Define a mapping $\varphi : \mathbb{P}^{n+1} - \{P\} \longrightarrow \mathbb{P}^n$ by $\varphi(Q) =$ the intersection of the unique line containing P and Q with $\mathbb{P}^n$.*

(a) Show that φ is a morphism.

(b) Let $Y \subset \mathbb{P}^3$ be the twisted cubic curve which is the image of the 3−uple embedding of $\mathbb{P}^1$. If t, u are the homogeneous coordinates of $\mathbb{P}^1$, we say that Y is the curve given parametrically by $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$. Let $P = (0, 0, 1, 0)$, and let $\mathbb{P}^2$ be the hyperplane $z = 0$. Show that the projection of Y from P is the cuspidal cubic curve and find its equation in the plane.

Solution:

(a) We are given that $\mathbb{P}^n$ is a hyperplane in $\mathbb{P}^{n+1}$ and $P \in \mathbb{P}^{n+1} \smallsetminus \mathbb{P}^n$. Suppose the coordinates of $\mathbb{P}^{n+1}$ are given by $x_0, \ldots, x_n$. By a linear change of coordinates, we may assume that $\mathbb{P}^n$ is given by $x_n = 0$ and that $P = (0 : 0 : \ldots : 1)$. Suppose $Q \in \mathbb{P}^{n+1} \smallsetminus \{P\}$ is given by $(z_0 : \ldots : z_n)$. Then $\varphi(Q) = (z_0 : \ldots : z_{n-1})$. We now prove that the map φ is continuous. Any closed subset of $\mathbb{P}^n$ is of the form $\mathcal{Z}(\mathfrak{a})$ for some ideal $\mathfrak{a} = (f_1, \ldots, f_r)$ of $k[x_0, \ldots, x_n]$. We claim that $\varphi^{-1}(\mathcal{Z}(\mathfrak{a})) = \mathfrak{b}$ where $\mathfrak{b}$ is the ideal of $k[x_0, \ldots, x_{n+1}]$ generated by the elements $f_1, \ldots, f_r$ when considered as polynomials of $k[x_0, \ldots, x_{n+1}]$. Suppose $T = (t_0 : \ldots : t_{n+1}) \in \mathcal{Z}(\mathfrak{b})$. Then for each $i = 1, \ldots, r$, $f_i(T) = 0$ i.e, $f(t_0, \ldots, t_n) = 0$. Therefore $\varphi(T) \in \mathcal{Z}(\mathfrak{a})$ which implies that $T \in \varphi^{-1}(\mathcal{Z}(\mathfrak{a}))$. Conversely, assume that $T = (t_0 : \ldots : t_{n+1}) \in \varphi^{-1}(\mathcal{Z}(\mathfrak{a}))$. Let $S = \varphi(T)$. Then $S = (t_0 : \ldots : t_n)$ and $S \in \mathcal{Z}(\mathfrak{a})$. Therefore for each $i = 1, \ldots, r$, we have $f_i(S) = 0$ and therefore $T \in \mathcal{Z}(\mathfrak{b})$.

To prove that φ is a morphism consider any open subset U of $\mathbb{P}^n$ and a regular function $f : U \longrightarrow k$. Let $P \in U$ be any point. Then there exists an open neighbourhood of $W \subset U$ of P such that $f = \dfrac{f_1}{f_2}$ where $f_1, f_2 \in k[x_0, \ldots, x_n]$. We have $\dfrac{f_1}{f_2} \circ \varphi = \dfrac{f_1}{f_2}$ where f_1, f_2 are considered as polynomials in $k[x_0, \ldots, x_{n+1}]$. Therefore for any point $\varphi^{-1}(P) \in \varphi^{-1}(U)$, there exists a neighbourhood $\varphi^{-1}(W)$ such that $f \circ \varphi = \dfrac{f_1}{f_2}$ for some polynomials $f_1,\, f_2 \in k[x_0, \ldots, x_{n+1}]$. Therefore $f \circ \varphi : \varphi^{-1}(U) \longrightarrow k$ is a regular function and therefore φ is a morphism.

(b) The map $\varphi : \mathbb{P}^3 \setminus P : \longrightarrow \mathbb{P}^2$ where $P = (0, 0, 1, 0)$ is given by $(x : y : z : w) \mapsto (x : y : w)$. If Y is the twisted cubic curve, parametrized by $(x : y : z : w) = (t^3 : t^2u : tu^2 : u^3)$, then the projection of Y from P is parametrized by $(x : y : z) = (t^3 : t^2u : u^3)$. Let Z denote the projection of Y from P. Then clearly $Z \subset \mathcal{Z}(y^3 - x^2w)$. Now consider any point $Q = (x : y : w) \in \mathcal{Z}(y^3 - x^2w)$. When $x = 0$, the point $Q = (0 : 0 : w)$ for arbitrary values of w. Consider the case when $x \neq 0$. We can assume that $x = 1$. Therefore the point is of the form $(1 : y : y^3)$ for arbitrary values of y. Put $y = \frac{u}{t}$. Then $(1 : y : y^3) = (t^3 : t^2u : u^3)$. In the case when $t = 0$, this is equal to the point $(0 : 0 : w)$. Therefore we have that $Z \subset \mathcal{Z}(y^3 - x^2w)$. Therefore projection from P is equal to $\mathcal{Z}(y^3 - x^2w)$ which is the twisted cubic curve.

Exercise 3.0.44 (Product of Affine Varieties)**.** *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties.*

(a) Show that $X \times Y \subset \mathbb{A}^{n+m}$ with its induced topology is irreducible.

(b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

(c) Show that $X \times Y$ is a product in the category of varieties, i.e., show (i) the projections $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ are morphisms, and (ii) given a variety Z, and the morphisms $\varphi_1 : Z \longrightarrow X$, $\varphi_2 : Z \longrightarrow Y$, there is a unique morphism $\varphi : Z \longrightarrow X \times Y$ such that $p_i \circ \varphi = \varphi_i$ for $i = 1, 2$.

(d) Show that $\dim X \times Y = \dim X + \dim Y$.

Solution:

(a) Suppose $X \times Y$ is the union of two closed sets $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subset Z_i\}$, $i = 1, 2$. We claim that $X = X_1 \cup X_2$. It is clear that $X_1 \cup X_2 \subset X$. Conversely, consider a point $x \in X$. Then $(x \times Y) \cap Z_i$ is a closed subset of $X \times Y$ because it is the intersection of two closed subsets Z_i and $x \times Y$. Also, it is clear that $x \times Y = ((x \times Y) \cap Z_1) \cup ((x \times Y) \cap Z_2)$. Since Y is irreducible and $x \times Y$ is the homeomorphic image of Y, $x \times Y$ is irreducible. Therefore $x \times Y = (x \times Y) \cap Z_i$ for either $i = 1, 2$. Therefore $x \times Y \subset Z_i$ for either $i = 1, 2$. Therefore $x \in X_1 \cup X_2$. Hence $X = X_1 \cup X_2$.

We now prove that X_i are closed subsets of X. Let $U_i = X \subset X_i$ and $V_i = X \times Y \subset Z_i$. We now claim that $P(V_i) = U_i$. Since V_i is open subset of $X \times Y$ and since P is an open map, this proves that U_i is an open subset of X and hence that X_i is an closed subset of X. Consider the case when $i = 1$. Consider a point $a \in P(V_1)$. Therefore there exists a point $b \in Y$ such that $a \times b \in V_1$. Suppose that $a \in X_1$. Then $a \times Y \subset Z_1$ which implies that $a \times b \in Z_1$ which is a contradiction. Therefore $a \notin X_1$ and hence $P(V_1) \subset U_1$.

Conversely let $a \in U_1$. Since $a \notin X_1$, we have that $a \times Y \not\subset Z_1$. Therefore there exists a point $b \in Y$ such that $a \times b \notin Z_1$. Therefore $a \times b \in V_1$ and hence $a \in P(V_1)$. This proves that $P(V_1) = U_1$. Similarly we can prove that $P(V_2) = U_2$.

Since X is irreducible $X = X_i$ for either $i = 1, 2$. Therefore $X \times Y = Z_i$ for either $i = 1, 2$ and hence $X \times Y$ is irreducible.

(b) To prove that $A(X \times Y) \cong A(X) \otimes_k A(Y)$, we first define a map $F : A(X) \times A(Y) \longrightarrow A(X \times Y)$ given by sending the ordered pair (f, g) to the element $f \times g \in A(X \times Y)$ where $f \times g$ is the defined by $f \times g(x, y) = f(x)g(y)$. Clearly this map is a bilinear map and hence by the universal property of tensor product there exists a unique homomorphism $\tilde{F} : A(X) \otimes_k A(Y) \longrightarrow A(X \times Y)$ given by $f \otimes g \mapsto f \times g$.

Now $A(X \times Y)$ is generated as a k-algebra by the elements $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ where $x_i = X_i \, mod I(X \times Y)$ and $y_j = Y_j \, mod I(X \times Y)$ where X_i and Y_j are the coordinate functions of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively. Clearly $\tilde{F}(x_i \times 1) = x_i$ for $i = 1, \ldots, n$ and $\tilde{F}(1 \times y_j)\, y_j$ for $j = 1, \ldots, m$. Therefore the map $\tilde{F}$ is surjective.

To prove that $\tilde{F}$ is injective consider an element $f \otimes g \in A(X) \otimes A(Y)$ such that $\tilde{F}(f \otimes g) = 0$. Therefore $f \times g = 0$. Therefore $f(x)g(y) = 0$ for all $x \in X$ and $y \in Y$. Therefore either $f = 0$ or $g = 0$. Therefore $f \otimes g = 0$.

(c) From lemma 3.6, we get that the projection maps are morphisms. Suppose φ, ϕ are two morphism from Z to $X \times Y$ such that $p_i \circ \varphi = \varphi_i$ and $p_i \circ \phi = \varphi_i$ for $i = 1, 2$. Since for any $z \in Z$, $p_i(\varphi(z)) = p_i(\phi(z))$ for $i = 1, 2$, we get that $\varphi = \phi$. Therefore if such a map exists, it is unique.

Define a map $\varphi = \varphi_1 \times \varphi_2 : Z \longrightarrow X \times Y$ by $\varphi(z) = (\varphi_1(z), \varphi_2(z))$. This map clearly satisfies the property $p_i \circ \varphi = \varphi_i$. Since X is a affine variety, we know from Lemma 3.6 that there exist n regular functions $f_1, \ldots, f_n \in A(X)$ such that $\varphi_1 = (f_1, \ldots, f_n)$. Similarly there exist m regular functions $g_1, \ldots, g_m \in A(Y)$ such that $\varphi_2 = (g_1, \ldots, g_m)$. Therefore $\varphi = (f_1, \ldots, f_n, g_1, \ldots, g_m)$ and hence is a morphism.

(d) We have that $A(X \times Y) = A(X) \otimes_k A(Y)$. Therefore we have to prove that $dim\, A(X) \otimes_k A(Y) = dim\, A(X) + dim\, A(Y)$. Let $dim\, A(X) = r$ and $dim\, A(Y) = s$. We know that $A(X)$ and $A(Y)$ are finitely generated k-algebras. Suppose $A(X)$ is generated by $x_1, x_2, \ldots, x_m$ over k as an algebra such that $x_1, \ldots, x_r$ are algebraically independent. Similarly assume that $A(Y)$ is generated by $y_1, \ldots, y_n$ over k as an algebra such that $y_1, \ldots, y_s$ are algebraically independent.

We claim that $A(X) \otimes A(Y)$ is generated by $x_1 \otimes 1, \ldots, x_m \otimes 1$ $1 \otimes y_1, \ldots, 1 \otimes y_n$ over k as an algebra. We know that $A(X) \otimes A(Y)$ is generated by elements of the form $a \otimes b$ as a k-module (i.e, k-vector space) where $a \in A(X)$ and $b \in A(Y)$. But $a \otimes b = (a \otimes 1)(1 \otimes b)$. We know that $a \otimes 1$ is given by a polynomial in $x_i \otimes 1$ and $1 \otimes b$ is given by a polynomial in $1 \otimes y_j$. Therefore $a \otimes b$ is given by a polynomial in $x_i \otimes 1$ and $1 \otimes y_j$. Therefore $A(X) \otimes_k A(Y)$ is generated by $x_1 \otimes 1, \ldots, x_m \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_n$ as a k-algebra.

We now claim that $x_1 \otimes 1, \ldots, x_r \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_s$ are algebraically independent.
Suppose $\sum a_{\alpha 1 \ldots \alpha r \beta 1 \ldots \beta s} (x_1 \otimes 1)^{\alpha 1} \ldots (x_r \otimes 1)^{\alpha r} (1 \otimes y_1)^{\beta 1} \ldots (1 \otimes y_s)^{\beta s} = 0$ for some $a_{\alpha 1 \ldots \alpha r \beta 1 \ldots \beta s} \in k$. Then for any point $(u_1, \ldots, u_m) \in X$ we have that

$\sum a_{\alpha 1 \ldots \alpha r \beta 1 \ldots \beta s} (u_1^{\alpha 1} \ldots u_r^{\alpha r})(1 \otimes y_1^{\beta 1} \ldots y_s^{\beta s}) = 0$. We know that in the tensor product of k-algebras, $1 \otimes y = 0$ if and only if $y = 0$. We use this fact along with the fact that the y_i are algebraically independent to conclude that in the above summation each of the coefficients $a_i(u_1, \ldots, u_m)$, which is a polynomial in u_i with coefficients from among the $a_{\alpha 1 \ldots \alpha r \beta 1 \ldots \beta s}$, is 0. But this is true for each $(u_1, \ldots, u_m) \in X$. Hence each of the polynomials $a_i(x_1, \ldots, x_m) = 0$. We now use the fact that x_i are algebraically independent to conclude that each of the $a_{\alpha 1 \ldots \alpha r \beta 1 \ldots \beta s} = 0$. Therefore we get that $x_1 \otimes 1, \ldots, x_r \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_s,$ are algebraically independent. This proves that the $dim\, X \times Y \geq dim\, X + dim\, Y$.

Suppose after some relabelling of the x_i we get that $x_1 \otimes 1, \ldots, x_t \otimes 1$, $1 \otimes y_1, \ldots 1 \otimes y_s$ are algebraically independent for some $t < r$. We claim that this implies that $x_1, \ldots, x_t$ are algebraically independent which is not true since the dimension of $A(X)$ is $r > t$. Suppose $x_1, \ldots, x_t$ are algebraically dependent. Then there exist $\{a_{\alpha 1 \ldots \alpha t}\}$ not all zero, such that $\sum a_{\alpha 1 \ldots \alpha t}\, x_1^{\alpha 1} \ldots x_t^{\alpha t} = 0$. Therefore $1 \otimes \sum a_{\alpha 1 \ldots \alpha 2}\, x_1^{\alpha 1} \ldots x_t^{\alpha t} = 0$ which implies that $\sum a_{\alpha 1 \ldots \alpha t}(1 \otimes x_1)^{\alpha 1} \ldots (1 \otimes x_t)^{\alpha t} = 0$. But this implies that $x_1 \otimes 1, \ldots, x_t \otimes 1, 1 \otimes y_1, \ldots 1 \otimes y_s$ are algebraically dependent. Therefore $x_1 \otimes 1, \ldots, x_t \otimes 1, 1 \otimes y_1, \ldots 1 \otimes y_s$ are algebraically dependent for any $t < r$.

Similarly if we suppose that after some relabelling of the y_j that $x_1 \otimes 1, \ldots, x_r \otimes 1, 1 \otimes y_1, \ldots 1 \otimes y_u$ are algebraically independent for some $u < s$. Then by an argument similar to above we get a contradiction. Therefore $x_1 \otimes 1, \ldots, x_r \otimes 1, 1 \otimes y_1, \ldots 1 \otimes y_s$ is the smallest algebraically independent subset from $x_1 \otimes 1, \ldots, x_m \otimes 1, 1 \otimes y_1, \ldots 1 \otimes y_n$. This proves that $dim\, X \times Y = dim\, X + dim\, Y$.

Exercise 3.0.45 (Product of Quasi-projective Varieties)**.** *Use the Segre embedding to identify $\mathbb{P}^n \times \mathbb{P}^m$ with its image and hence give it the structure of a projective variety. Now for any two quasi-projective varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$, consider $X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$.*

(a) Show that $X \times Y$ is a quasi-projective variety.

(b) If X and Y are both projective, show that $X \times Y$ is projective.

(c) Show that $X \times Y$ is a product in the category of varieties.

Solution:

We first prove (b) and then use the proof to give a proof of (a).

(b) It is clear that $X \times Y = (X \times \mathbb{P}^m) \cap (\mathbb{P}^n \times Y)$. We now claim that $X \times \mathbb{P}^m$ and $\mathbb{P}^n \times Y$ are closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ which will prove that $X \times Y$ is a closed subset of $\mathbb{P}^n \times \mathbb{P}^m$. Let $X = \mathcal{Z}(\mathfrak{a})$ and $Y = \mathcal{Z}(\mathfrak{b})$. Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} \,|\, i = 0, \ldots n\,;\, j = 0, \ldots m\}$. Suppose $\mathfrak{a} = \langle f_1, \ldots, f_r \rangle$ for some homogeneous polynomials $f_l \in k[x_0, \ldots, m]$. For each $l = 1, \ldots, r$ and $j = 0, \ldots, m$, define $f_{lj} = f_l(z_{0j}, z_{1j}, \ldots z_{nj}) \in k[\{z_{ij}\}]$. For each $j = 0, \ldots m$ define $\mathfrak{a}_j \subset k[\{z_{ij}\}]$ to be the ideal $\langle f_{1j}, \ldots, f_{rj} \rangle$.

We claim that $X \times \mathbb{P}^m = \bigcap_{j=0}^{m} \mathcal{Z}(\mathfrak{a}_j)$ and hence is a closed subset of $\mathbb{P}^n \times \mathbb{P}^m$. Let $\varphi : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$ be the Segre embedding. Consider a point $\varphi(a, b) \in X \times \mathbb{P}^m$. Then $a \in X$ and $\varphi(a, b) = (a_0 b_0, a_0 b_1, \ldots, a_i b_j, \ldots, a_n b_m)$ in the lexicographic order. Suppose $b_j \neq 0$. To illustrate the point we consider the case when $i = 0$. Then $\varphi(a \times b) = (a_0, a_0 b_1/b_0, \ldots, a_i, \ldots, a_n, \ldots, a_n b_m/b_0)$. Therefore $\varphi(a, b) \in \mathcal{Z}(\mathfrak{a}_0)$. Also when $b_0 = 0$, it is very clear that $\varphi(a, b) \in \mathcal{Z}(\mathfrak{a}_0)$. Therefore $\varphi(a, b) \in \bigcap_{j=0}^{m} \mathcal{Z}(\mathfrak{a}_j)$ and hence $X \times \mathbb{P}^m \subset \bigcap_{j=0}^{m} \mathcal{Z}(\mathfrak{a}_j)$.

Conversely let $\varphi(a, b) \in \bigcap_{j=0}^{m} \mathcal{Z}(\mathfrak{a}_j)$. Suppose $b_j \neq 0$. Since $\varphi(a, b) \in \mathcal{Z}(\mathfrak{a}_j)$, for each $l = 1, \ldots, r$, $f_{lj}(\varphi(a, b)) = 0$. But $f_{lj}(\varphi(a, b)) = f_l(a_0 b_j, a_1 b_j, \ldots, a_n b_j) = b_j f_l(a_0, \ldots, a_n)$ since f_l are homogeneous polynomials. Since $b_j \neq 0$, we get that $f_l(a_0, \ldots, a_n) = 0$ for each $l = 1, \ldots, r$. Therefore $a \in \mathcal{Z}(\mathfrak{a}) = X$ and hence $\varphi(a, b) \in X \times \mathbb{P}^m$ proving that $X \times \mathbb{P}^m = \bigcap_{j=0}^{m} \mathcal{Z}(\mathfrak{a}_j)$. We can similarly prove that $\mathbb{P}^n \times Y$ is a closed subset of $\mathbb{P}^n \times \mathbb{P}^m$.

(a) Suppose X is an open subset of the projective variety X_0 and Y is an open subset of the projective variety Y_0. Let $X_0 \setminus X = C$ and $Y_0 \setminus Y = D$. Since D is a closed subset of Y_0, we know that D is of the form $D_0 \cap Y_0$ for some closed subset D_0 of $\mathbb{P}^m$. Now, $X_0 \times D = (X_0 \times Y_0) \cap (X_0 \times D)$. We know from part (a) that $X_0 \times D$ is a closed subset of $\mathbb{P}^n \times \mathbb{P}^m$ and therefore $X_0 \times D$ is a closed subset of $X_0 \times Y_0$. We can similarly prove that $C \times Y_0$ is a closed subset of $X_0 \times Y_0$.

We now claim that $X_0 \times Y_0 = (X \times Y) \cup (X_0 \times D) \cup (C \times Y_0)$. It is clear that $(X \times Y) \cup (X_0 \times D) \cup (C \times Y_0) \subset X_0 \times Y_0$. Conversely, suppose that $\varphi(a, b) \in X_0 \times Y_0$. If $a \in X$ and $b \in Y$, then $\varphi(a, b) \in X \times Y$. If $b \in Y_0 \setminus Y$, then $\varphi(a, b) \in X_0 \times D$ and if $a \in X_0 \setminus X$, then $\varphi(a, b) \in C \times Y_0$. Therefore $\varphi(a, b) \in (X \times Y) \cup (X_0 \times D) \cup (C \times Y_0)$. This proves the claim.

We now claim that $(X_0 \times D) \cup (C \times Y_0) \cup (X \times Y) = \emptyset$. Suppose $\varphi(a, b) \in (X_0 \times D) \cup (C \times Y_0) \cup (X \times Y)$. Since $\varphi(a, b) \in X \times Y$, we have that $a \in X$ and $b \in Y$. Now $\varphi(a, b)$ also belongs to $(X_0 \times D) \cup (C \times Y_0)$. Consider the case when $\varphi(a, b) \in X_0 \times D$. Therefore $b \in D$ and we get a contradiction. Similarly if $\varphi(a, b) \in C \times Y_0$, we get a contradiction. This proves the claim.

Now, $X \times Y = (X_0 \times Y_0) \setminus (X_0 \times D \cup C \times Y_0)$. Since $X_0 \times D \cup C \times Y_0$ is a closed subset of $X_0 \times Y_0$, we get that $X \times Y$ is an open subset of $X_0 \times Y_0$ and hence is a quasi-projective variety.

(c) Let the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^N$ where $N = nm + n + m$ be denoted by σ. Let $a = (a_0, a_1, \ldots, a_n) \in X$ and $b = (b_0, \ldots, b_m) \in Y$ be any two points. We may assume that $a_0 \neq 0$ and $b_0 \neq 0$. Consider the point $\sigma(a \times b) = (a_0b_0, a_0b_1, \ldots, a_nb_m) \in X \times Y$. Let $P_1 : X \times Y \longrightarrow X$ be the map defined by $(a_0b_0, a_0b_1, \ldots, a_nb_m) \mapsto (a_0b_0, a_1b_0, \ldots a_nb_0)$ and $P_2 : X \times Y \longrightarrow Y$ be the map defined by $(a_0b_0, a_0b_1, \ldots, a_nb_m) \mapsto (a_0b_0, a_0b_1, \ldots a_0b_m)$. Since these maps are defined by polynomials locally they are morphisms. These maps are the projection maps.

Let $\varphi_1 : Z \longrightarrow X$ and $\varphi_2 : Z \longrightarrow Y$ be any two morphisms. Define a map $F : Z \longrightarrow X \times Y$ by $z \mapsto (a_0b_0, a_0b_1, \ldots, a_nb_m)$ where $\varphi_1(z) = (a_0, \ldots, a_n) = a$ and $\varphi_2(z) = (b_0, b_1, \ldots, b_m) = b$. We claim that this map is morphism. It is enough to prove that this map is locally defined by quotients of polynomial functions. We know that for any point $a \in X$ and $b \in Y$ we have affine open neighbourhoods $U \subset X$ and $Y \subset Y$ containing a and b respectively such that the product of affine varieties $U \times V$ is isomorphic to $\sigma(U \times V)$. Let $W = \varphi_1^{-1}(U) \cap \varphi_2^{-1}(V) \subset Z$. Then the restriction of φ_1 to W, $\varphi_1|_W : W \longrightarrow U$ is a morphism where U is an affine variety. Therefore $\varphi_1|_W$ is defined by polynomial functions. Similarly the restriction of φ_2 to W, $\varphi_2|_W : W \longrightarrow V$ is defined by polynomial functions. Now, the restriction of F to W, $F|_W : W \longrightarrow U \times V$ is given by $z \mapsto \sigma_0(\varphi_1^{'}(z), \varphi_2^{'}(z))$ where $\sigma_0 = \sigma|_{U \times V}$, $\varphi_1^{'} = \varphi_1|_W$ and $\varphi_2^{'} = \varphi_2|_W$. Since σ_0, $\varphi_1|_W$ and $\varphi_2|_W$ are morphisms of affine varieties and hence are given by polynomial functions and hence $F|_W$ is given by polynomial functions. Hence we have that F is given by polynomial functions locally and hence F is a morphism.

Also, $P_1 \circ F = \varphi_1$ and $P_2 \circ F = \varphi_2$. Hence $X \times Y$ is a product in the category of varieties.

Exercise 3.0.46 (Normal Variety). *A variety X is said too be Normal at a point $P \in X$ if $\mathcal{O}_P$ is integrally closed. X is normal if it is normal at every point.*

(a) Show that every conic in $\mathbb{P}^2$ is normal.

(b) Show that the quadric surfaces Q_1, Q_2 given by $Q_1 : xy = zw$; Q_2 ; $xy = z^2$ are normal.

(c) Show that the cuspidal cubic $y^2 = x^3$ in $\mathbb{A}^2$ is not normal.

(d) If Y is affine, then Y is normal if and only if $A(Y)$ is integrally closed.

(e) Let Y be an affine variety. Show that there exists a normal affine variety $\tilde{Y}$ and a morphism $\pi : \tilde{Y} \longrightarrow Y$ with the property that whenever Z is a normal variety and $\varphi : Z \longrightarrow Y$ is a dominant morphism (i.e., $\varphi(Z)$ is dense in Y), then there is a unique morphism $\theta : Z \longrightarrow \tilde{Y}$ such that $\varphi = \pi \circ \theta$. $\tilde{Y}$ is called the normalization of Y.

Solution:
(a) From Exercise 3.1 we know that every conic $\mathcal{C}$ in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$ and hence $S(\mathcal{C}) \cong k[x, y]$. If we consider any point $P \in \mathcal{C}$ then from Theorem 3.4 we know that $\mathcal{O}_P \cong k[x, y]_{(\mathfrak{m}_P)}$. But from the proof of the same theorem we know that $k[x, y]_{(\mathfrak{m}_P)} \cong k[x]_{\mathfrak{m}'_P}$ where $\mathfrak{m}'_P$ is the ideal of $A(Y_i) = k[x]$ corresponding to P and Y_i is the affine open subset of $\mathbb{P}^1$ containing P. But $k[x]_{\mathfrak{m}'_P}$ is a discrete valuation ring and hence is integrally closed. This proves that $\mathcal{O}_P$ is integrally closed for $P \in \mathcal{C}$ and hence any conic $\mathcal{C}$ in $\mathbb{P}^2$ is normal.

We first prove (d) and then apply it to prove (c).

(d) From theorem 3.2 we know that for any point $P \in Y \subset \mathbb{A}^n$, $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$ where $\mathfrak{m}_P \subset A(Y)$ is the ideal of functions vanishing at P. Also, there is a 1-1 correspondence between ideals of $A(Y)$ and points of Y. Suppose Y is a normal variety, i.e., Y is normal at every point $P \in Y$. Then $\mathcal{O}_P = A(Y)_{\mathfrak{m}_P}$ is integrally closed for each maximal ideal $\mathfrak{m}_P$ of Y. But we know that an integral domain R is integrally closed if and only if R_M is integrally closed for each maximal ideal M of R. Therefore Y is normal if and only if $A(Y)$ is integrally closed.

(c) In the view of the above, to prove that Y is not normal it is enough to prove that $A(Y)$ is not integrally closed. When $Y = \mathcal{Z}(y^2 - x^3) \subset \mathbb{A}^2$, $A(Y) = k[x, y]/(y^2 - x^3)$. Let $k(Y)$ denote the field of fractions of $A(Y)$. Consider the element $t = y/x \in k(Y)$. Clearly $t^2 = x$ in $A(Y)$. Therefore t is an integral element of $k(Y)$. But $t \in A(Y)$. Therefore $A(Y)$ is not integrally closed and hence Y is not normal.

(e) Let $A(Y)$ denote the coordinate ring of Y and let $k(Y)$ denote the field of fractions of $A(Y)$. Let A denote the integral closure of $A(Y)$ in $k(Y)$. Then we know that A is an integrally closed. We claim that $A = A(X)$ for some affine variety X. To prove this it is enough to rove that A is a finitely generated k-algebra with no nilpotent elements. Since Y is a variety, $\mathcal{I}(Y)$ is a prime ideal and hence the nilradical of $A(Y)$ is the zero ideal From theorem 3.9 A, we know that A is finitely generated as an $A(Y)$ module and hence the nilradical of A is the zero ideal. Therefore A has no nilpotent elements. Also, from theorem 3.9 A, A is a finitely generated k-algebra. Therefore $A = A(X)$ for some affine variety X. Also, since A is integrally closed we have that X is a normal variety. We claim that X satisfies the property stated in the exercise.

We know that when $\varphi : Z \longrightarrow Y$ is any morphism of affine varieties, then the induced homomorphism of the affine algebras $\tilde{\varphi} : A(Y) \longrightarrow A(Z)$ is injective if and only if $\varphi(Z)$ is a dense subset of Y. Therefore, in the view of theorem 3.5, the property of the variety $\tilde{Y}$ stated in the exercise is the same as saying this: There exists a homomorphism $f_\pi : A(Y) \longrightarrow A(\tilde{Y})$ with the property that whenever $A(Z)$ is a integrally closed ring and $f_\varphi : A(\tilde{Y}) \longrightarrow A(Z)$ is an injective homomorphism, then there exists a unique homomorphism $f_\theta : A(Y) \longrightarrow A(Z)$ such that $f_\varphi \circ f_\pi = f_\theta$. Let f_π be the inclusion map. Since f_φ is an injective morphism, we have that the homomorphic image of $A(Y)$ is isomorphic to $A(\tilde{Y})$. Let $B = f_\varphi(A(Y))$. Then $B \subset A(Z)$ where $A(Z)$ is integrally closed. Therefore $A(Z) \subset \tilde{B}$, where $\tilde{B}$ represents the algebraic closure of B. But $\tilde{B}$ is isomorphic to $A(\tilde{Y})$. Let f_θ be the homomorphism. Then we have that $f_\theta \circ f_\pi = f_\varphi$.

Exercise 3.0.47 (Projectively Normal Varieties)**.** *A projective variety $Y \subset \mathbb{P}^n$ is projectively normal (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed. If Y is projectively normal, then Y is normal.*

Solution:
We want to prove that $\mathcal{O}_P$ is normal for any point $P \in Y$. We know that $S(Y) = k[X_0, \dots, X_n]/\mathcal{I}(Y)$ has a graded structure. Let $S(Y) = \oplus \sum_{i=0}^{\infty} S_i$ where $S_0 = k$. Also, $S(Y)$ is a finitely generated as an algebra over k by $x_0, \dots, x_n$ where $x_i \equiv X_i \, mod \, \mathcal{I}(Y)$. Let $\mathfrak{m}_P$ be the ideal generated by the set of homogeneous polynomials $f \in S(Y)$ such that $f(P) = 0$. Let T be the set of homogeneous elements of $S(Y)$ not in $\mathfrak{m}_P$. Then $T^{-1}S(Y)$ has graded structure. Let $T^{-1}S(Y) = \oplus \sum_{i=-\infty}^{\infty} R_i$. Using the notation fixed before Theorem 3.4 we denote R_0 by $S(Y)_{(\mathfrak{m}_P)}$. Then from Theorem 3.4 we know that $\mathcal{O}_P = S(Y)_{\mathfrak{m}_P}$. Now, $\mathfrak{m}_P$ is a prime ideal of $S(Y)$ such that $\mathfrak{m}_P \subsetneq (x_0, \dots, x_n)$. Therefore there exists an i such that $x_i \notin \mathfrak{m}_P$. Without loss of generality, we may assume that $x_0 \notin \mathfrak{m}_P$. Let U be the multiplicatively closed subset $\{1, x_0, x_0^2, \dots\}$. Then clearly $U^{-1}S(Y)$ has a graded structure. Let the $U^{-1}S(Y) = \oplus \sum_{i=-\infty}^{\infty} R_i'$. Then clearly $U^{-1}S(Y) = R_0'[x_0, x_0^{-1}]$.

We now make two claims.
Claim 1: x_0 is transcendental over R_0' which implies that $U^{-1}S$ is the Laurent polynomial ring over R_0'.
Claim 2: R_0 is a localization of R_0'.

Suppose we prove these two claims. Then the proof proceeds as follows. Since $S(Y)$ is integrally closed, $U^{-1}S(Y)$ is integrally closed. But $U^{-1}S(Y) = R_0'[x_0, x_0^{-1}]$. Therefore R_0' is integrally closed. Since R_0 is a localization of R_0' we get that R_0 is integrally closed.

To prove the first claim: Suppose x_0 is the root of a polynomial with coefficients in R_0'. Suppose $\frac{f_n}{x_0^{d_n}} x_0^n + \frac{f_{n-1}}{x_0^{d_{n-1}}} x_0^{n-1} + \dots + \frac{f_0}{x_0^{d_0}} = 0 \in U^{-1}S(Y)$ where f_i are homogeneous polynomials of degree d_i in $S(Y)$. Let $d = max\{d_n, d_{n-1}, \dots, d_0\}$. Then $\frac{f_n x_0^{n+d-d_n} + f_{n-1} x_0^{n-1+d-d_n} \dots + f_0 x_0^{d-d_0}}{x_0^d} = 0$ in $U^{-1}S(Y)$. Therefore $\exists\, r \geq 0$ such that $x_0^r(f_n x_0^{n+d-d_n} + f_{n-1} x_0^{n-1+d-d_n} \dots + f_0 x_0^{d-d_0}) = 0$ in $S(Y)$. Now, $x_0^r = 0 \implies x_0 = 0 \implies X_0 \in I(Y)$. But we know that $I(Y) \subset M_P$ where M_P is the contraction of the ideal $\mathfrak{m}_P$ in $S(Y)$. This implies that $X_0 \in M_P \implies x_0 \in \mathfrak{m}_P$

which is a contradiction. Therefore we have that $f_n x_0^{n+d-d_n} + f_{n-1} x_0^{n-1+d-d_n} \ldots + f_0 x_0^{d-d_0} = 0$. But $f_{n-i} x_0^{n-i+d-d_n}$ is a homogeneous polynomial of degree $n - i + d$. Therefore each $f_{n-i} x_0^{n-i+d-d_n} = 0$. But $x_0 \neq 0$. Therefore $f_{n-i} = 0$. Therefore x_0 is transcendental over R_0'.

To prove the second claim: Let f/g be an element in R_0. Then f, g are homogeneous polynomials in $S(Y)$ of the same degree such that $g \notin \mathfrak{m}_P$. Consider the ideal Q_0 of R_0' generated by the elements of the form $\frac{h}{x_0^{deg\, h}}$ where $h \in \mathfrak{m}_P$. Then clearly Q_0 is a prime ideal of R_0'. Define a map $\varphi : R_0 \longrightarrow R'_{0Q_0}$ by $f/g \mapsto \frac{f}{x_0^\alpha} / \frac{g}{x_0^\alpha}$ where $\alpha = deg\, f = deg\, g$. It can be checked that this map is a ring homomorphism. Also, it can be checked that this map is surjective. Suppose $\varphi(f_1/g_1) = \varphi(f_2/g_2)$. This implies that there exists an element $g_3/x_0^{\alpha_3} \notin Q_0$ such that $\frac{g_3}{x_0^{\alpha_3}} \left(\frac{f_1}{x_0^{\alpha_1}} \frac{g_2}{x_0^{\alpha_2}} - \frac{f_2}{x_0^{\alpha_2}} \frac{g_1}{x_0^{\alpha_1}} \right) = 0$. This implies that $\frac{g_3 f_1 g_2 - g_3 f_2 g_1}{x_0^{\alpha_1 + \alpha_2 + \alpha_3}} = 0$. This implies that $g_3(f_1 g_2 - g_1 f_2) = 0$. Now, since $g_3/x_0^{\alpha_3} \notin Q_0$, we have that $g_3 \neq 0$. This implies that $f_1 g_2 - g_1 f_2 = 0$ which in turn implies that $f_1/g_1 = f_2/g_2$ proving that the map is injective. Therefore φ is an isomorphism and hence that R_0 is localization of R_0'.

Exercise 3.0.48 (Automorphism of $\mathbb{A}^n$)**.** *Let $\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ be a morphism of $\mathbb{A}^n$ to $\mathbb{A}^n$ given by n polynomials $f_1, \ldots, f_n$ of n variables $x_1, \ldots, x_n$. Let $J_\varphi = det\,|\partial f_i/\partial x_j|$ be the Jacobian polynomial of φ.*

If φ is an isomorphism (in which case we call φ an automorphism of $\mathbb{A}^n$) show that J_φ is a non-zero constant polynomial. The converse of (a) is an unsolved problem, even for $n = 2$.

Solution:

Let $\phi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ be the inverse morphism of the morphism φ. Suppose ϕ is given by the polynomials $g_1, \ldots, g_n$ of n variables. Let J_ϕ denote the Jacobian polynomial of ϕ and let $J_{\varphi \circ \phi}$ denote the Jacobian polynomial of $\varphi \circ \phi$. But we know that $\varphi \circ \phi = I_d$ where I_d denotes the identity morphism which is given by the polynomials $x_1, \ldots, x_n$ and hence $J_{I_d} = 1$. But $J_{\varphi \circ \phi}(X) = J_\varphi(\phi(X)) J_\phi(X)$. Therefore J_φ is a unit in $k[x_1, \ldots, x_n]$ and hence J_φ is a non-zero constant polynomial.

Exercise 3.0.49 (Group Varieties)**.** *A group variety consists of a variety Y together with the morphism $\mu : Y \times Y \longrightarrow Y$, such that the set of points of Y with the operation given by μ is a group, and such that the inverse map $y \mapsto y^{-1}$ is also a morphism of $Y \longrightarrow Y$.*

(a) The additive group $\mathbf{G}_a$ is given by the variety $\mathbb{A}^1$ and the morphism $\mu : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$ defined by $\mu(a, b) = a + b$. Show that it is a group variety.

(b) The multiplicative group $\mathbf{G}_m$ is given by the variety $\mathbb{A}^1 - \{(0)\}$ and the morphism $\mu(a, b)\, ab$. Show that it is a group variety.

(c) If G is a group variety, and X is any variety, show that the set $Hom(X, G)$ has a natural group structure.

(d) For any variety X, show that $Hom(X, \mathbf{G}_a)$ is isomorphic to $\mathcal{O}(X)$ as a group under addition.

(e) For any variety X, show that $Hom(X, \mathbf{G}_m)$ is isomorphic to the group of units in $\mathcal{O}(X)$, under multiplication.

Solution:

(a) Clearly, the operation given by μ is an associative binary operation. Also, for any $a \in \mathbb{A}^1$, $-a \in \mathbb{A}^1$ is the inverse of a under the operation defined by μ. The element $0 \in \mathbb{A}^1$ is the identity element. Therefore $\mathbf{G}_a$ is a group. The inverse map $x \mapsto -x$ is a morphism of $\mathbb{A}^1 \to \mathbb{A}^1$ from lemma 3.6. Therefore $\mathbf{G}_a$ is a group variety.

(b) Let $X = \mathbb{A}^1 \setminus \{(0)\}$. Then we can check that operation defined by μ makes X into group. Consider the inverse map $x \mapsto \frac{1}{x}$. Then clearly this map is a morphism of X and hence $\mathbf{G}_m$ is a group variety.

(c) Let φ_1 and φ_2 be any two morphism from X to G. Define the operation $*$ on $Hom(X, G)$ by $\varphi_1 * \varphi_2(x) = \varphi_1(x)\varphi_x$ where the multiplication on the right is in G. We have that $\varphi^{-1}(x) = \varphi(x)^{-1}$. Since in G, the inverse map is a morphism, the map ψ defined by $\psi(x) = \varphi(x)^{-1}$ is a morphism. The associativity property of this operation follows from the associativity property in the group variety G. Therefore $Hom\,(X, G)$ has a natural group structure.

(d) Consider any element $f \in Hom(X, \mathbf{G}_a)$. Then f is a morphism from X to $\mathbf{G}_a = \mathbb{A}^1 = k$. Therefore f is a regular function on X and hence an element of $\mathcal{O}(X)$. This defines a bijection between $\mathcal{O}(X)$ and $Hom(X, \mathbf{G}_a)$. Let this bijection be called F. For any two elements $\varphi_1, \varphi_2 \in Hom(X, \mathbf{G}_a)$, we have $\varphi_1 + \varphi_2(x) = \varphi_1(x) + \varphi_2(x)$. Therefore $F(\varphi_1 + \varphi_2) = F(\varphi_1) + F(\varphi_2)$ and hence F is a group isomorphism.

(e) We know that $G_{\mathbf{m}} = \mathbb{A}^1 \setminus \{(0)\} = D(t)$. Therefore we know from Lemma 3E that $\mathcal{O}(G_{\mathbf{m}}) = k[t][\frac{1}{t}] = k(t)$. Therefore there is a bijection μ between the set $Hom(k(t), \mathcal{O}(X))$ and the set $\mathcal{O}(X)^*$. We claim that this bijection is a group homomorphism. Suppose $h_1 : k(t) \longrightarrow \mathcal{O}(X)^*$ and $h_2 : k(t) \longrightarrow \mathcal{O}(X)^*$ be two k-algebra homomorphisms. Then $\mu(h_1) = h_1(t) \in \mathcal{O}(X)^*$ and $\mu(h_2) = h_2(t) \in \mathcal{O}(X)^*$. We have that $h_1h_2 : k(t) \longrightarrow \mathcal{O}(X)^*$ is the k-algebra homomorphism defined by $h_1h_2(t) = h_1(t)h_2(t) \in \mathcal{O}(X)^*$. Therefore $\mu(h_1h_2) = h_1h_2(t) = h_1(t)h_2(t)$ and hence μ is a group homomorphism. Since it is a bijection we get that $Hom(k(t), \mathcal{O}(X))$ is isomorphic to $\mathcal{O}(X)^*$.

We know from Proposition 3.5 that there exists a bijection $\beta : Hom(k(t), \mathcal{O}(X)) \longrightarrow Hom(X, G_{\mathbf{m}})$. We know from the proof of the same proposition that if $h : k(t) \longrightarrow \mathcal{O}(X)$ is k-algebra homomorphism then $\beta(h)$ is the morphism $\varphi_h : X \longrightarrow G_{\mathbf{m}}$ given by $P \mapsto h(t)(P)$. If $h^{'} : k(t)\, \mathcal{O}(X)$ is another k-algebra homomorphism, then $hh^{'}$ is the k-algebra homomorphism given by $t \mapsto h(t)h^{'}(t)$. Then $\beta(hh^{'}) = \varphi_{hh'}$ which is given by $P \mapsto h(t)h^{'}(t)(P)\, h(t)(P)h^{'}(t)(P)$. Therefore β is a bijective group homomorphism and hence is an isomorphism. Therefore we get that $Hom(X, G_{\mathbf{m}}) \cong Hom(k(t), \mathcal{O}(X)) \cong \mathcal{O}(X)^*$.

Chapter 4

Rational Maps

Exercise 4.0.50. *If f and g are regular functions on open subsets U and V of a variety X, and if $f = g$ on $U \cap V$, show that the function which is f on U and g on V is a regular function on $U \cup V$. Conclude that if f is a rational function on X, then there is a largest open subset U of X on which f is represented by a regular function. We say that f is defined at the points of U.*

Solution:

Let F be the function which is equal to f on U and g on V. Consider any point $P \in U \cup V$. Then either $P \in U$ or $P \in V$. Without loss of generality we may assume that $P \in U$. Since f is regular on U, there exists an open neighbourhood W of P such that $f = f_1/f_2$ on W. Also, $W \subset U$ is open in the variety X because U is open in X. Therefore for any point $P \in U \cup V$, there exists an open neighbourhood $W \subset X$ of P such that $f = f_1/f_2$ on W. Therefore the function F is regular on all points $P \in U \cap V$ and hence is regular on $U \cup V$.

Any rational function on X is an equivalence class (U, f) where f is a regular function on U. Let $\mathcal{F} = \{(U_i, f_i)\}_{i \in I}$ be the family of all pairs that occur in the equivalence class of (U, f). This implies that for any two $i, j \in I$, $f_i = f_j$ on $U_i = U_j$. Let $V = \bigcup_{i \in I} U_i$ and let F be the function on V which is equal to f_i on U_i. Then F is regular on V. Let U_0 be any open subset such that f is represented as a regular function f_0 on U_0. Then $(U_0, f_0) \in \mathcal{F}$ and hence $U_0 \subset V$. Therefore V is the largest subset on which the rational function f is represented as a regular function.

Exercise 4.0.51. *Same problem for rational maps. If φ is a rational map of X to Y, show that there is a largest open set on which φ is represented by a morphism. We say that the rational map is defined on the points of that open set.*

Solution:
Let X and Y be any two varieties. Let U and V be any two non-empty open subsets of X. Let φ_U be a morphism of U to Y and let φ_V be a morphism of V to Y. Suppose that $\varphi_U = \varphi_V$ on $U \cap V$. Let F be the function which is equal to φ_U on U and φ_V on V. Then we claim that F is a morphism of $U \cup V$ to Y. Clearly F is a continuous map. Let W be any open subset of Y and let $f : W \longrightarrow k$ be any regular function. We have to prove that $f \circ F : F^{-1}(W) \longrightarrow k$ is a regular function. Now, $F^{-1}(W) = \varphi_U^{-1}(W) \cup \varphi_V^{-1}(W)$. Consider any $P \in F^{-1}(W)$. Then either $P \in \varphi_U^{-1}(W)$ or $P \in \varphi_V^{-1}(W)$. We may assume that $P \in \varphi_U^{-1}(W)$. Since φ_U is regular, there exists an open subset $U_0 \subset \varphi_U^{-1}(W)$ containing P such that $f \circ \varphi_U$ is of the form f_1/f_2 on U_0. But $F = \varphi_U$ on U_0. Also, U_0 is an open subset of $\varphi_U^{-1}(W)$ which is an open subset of $F^{-1}(W)$. Therefore U_0 is an open subset of $F^{-1}(W)$. Therefore for any point $P \in F^{-1}(W)$ we have an open subset U_0 containing P such that $f \circ F$ is of the form f_1/f_2 on U_0. Therefore $f \circ F : F^{-1}(W) \longrightarrow k$ is a regular function which proves that F is a morphism.
Any rational map $\varphi : X \longrightarrow Y$ is an equivalence class of pairs (U, φ_U) where U is a non empty open subset of X and φ_U is a morphism of U to Y. Let $\mathcal{F} = \{(U_i, \varphi_{U_i})\}$ be the family of all pairs that occur in the equivalence class of (U, φ_U). This implies that for any $i, j \in I$, $\varphi_{U_i} = \varphi_{U_j}$ on $U_i \cap U_j$. Let $V = \bigcup_{i \in I} U_i$ and let F be the function which is equal to φ_{U_i} on U_i. Then F is a morphism on V. Let U_0 be any open subset such that φ is represented by a morphism φ_{U_0} on U_0. Then $(U_0, \varphi_{U_0}) \in \mathcal{F}$ and hence $U_0 \subset V$. Therefore V is the largest open subset on which φ is represented as a morphism.

Exercise 4.0.52. *(a) Let f be a rational map on $\mathbb{P}^2$ given by $f = x_1/x_0$. Find the set of points where f is defined and the corresponding regular function.*

(b) Now think of this function as a rational map from $\mathbb{P}^2$ to $\mathbb{A}^1$. Embed $\mathbb{A}^1$ in $\mathbb{P}^1$, and let $\varphi : \mathbb{P}^2 \longrightarrow \mathbb{P}^1$ be the resulting rational map. Find the set of points where φ is defined, and describe the corresponding morphism.

Solution:

(a) The rational map $f = x_1/x_0$ on $\mathbb{P}^2$ can be represented by the equivalence class $(U_{x_0}, x_1/x_0)$ where U_{x_0} represents the open set $\mathcal{Z}(x_0)^c$. We claim that f is defined at U_0. Suppose $(U_{x_0}, x_1/x_0) = (V, g)$ and that $V \not\subset U_{x_0}$. Let $P = (a_0, a_1, a_2) \in V \setminus U_{x_0}$. There exists a neighbourhood W of P such that g has the form g_1/g_2 on W where g_1 and g_2 are homogeneous polynomials of same degree. Also, since $g_2(P) \neq 0$, $x_0 \nmid g_2$. Also, $g_1/g_2 = x_1/x_0$ on $W \cap U_{x_0}$. We can homeomorphically identify U_{x_0} with $\mathbb{A}^2$ and consider $W \cap U_{x_0}$ as an open subset of $\mathbb{A}^2$. Let this open subset be denoted by W_0. Therefore we have that $g_1(1, x_1, x_2) = x_1 g_2(1, x_1, x_2)$ on $W_0 \subset \mathbb{A}^2$. This implies that $W_0 \subset \mathcal{Z}\left(g_1(1, x_1, x_2) - x_1 g_2(1, x_1, x_2)\right) = Z$. If $Z \neq \mathbb{A}^2$, then Z^c is a non empty subset of $\mathbb{A}^2$ which is disjoint from W_0. But any two non empty open subsets of $\mathbb{A}^2$ intersect. This implies that $Z = \mathcal{Z}\left(g_1(1, x_1, x_2) - x_1 g_2(1, x_1, x_2)\right) = \mathbb{A}^2$. Hence $g_1(1, x_1, x_2) = x_1 g_2(1, x_1, x_2)$ on $\mathbb{A}^2$. This implies that degree of $g_1(1, x_1, x_2) =$ (degree of $g_2(1, x_1, x_2) + 1$). Now, since $x_0 \nmid g_2$, the degree of $g_2(1, x_1, x_2)$ is the same as the degree of $g_2(x_0, x_1, x_2)$ which in turn is the same as the degree of g_1. Now, degree of $g_1(x_0, x_1, x_2)$ is greater than or equal to the degree of $g_1(1, x_1, x_2)$. But degree of $g_1(1, x_1, x_2)$ is strictly greater than the degree of $g_2(x_0, x_1, x_2)$. But this gives a contradiction to the fact that degree of $g_1(x_0, x_1, x_2) =$ degree of $g_2(x_0, x_1, x_2)$. Hence $V \subset U_{x_0}$ and hence the rational map is defined on U_{x_0}.

(b) Let U_0 denote the open subset $\mathcal{Z}(x_0)^c$ of $\mathbb{P}^2$. Let $\varphi_0 : U_0 \longrightarrow \mathbb{P}^1$ be the morphism given by $(x_0, x_1, x_2) \mapsto (1, x_1/x_0)$. Let U_1 denote the open subset $\mathcal{Z}(x_1)^c$ of $\mathbb{P}^2$. Let $\varphi_1 : U_1 \longrightarrow \mathbb{P}^1$ be the morphism given by $(x_0, x_1, x_2) \mapsto (x_0/x_1, 1)$. Then $\varphi_0 = \varphi_1$ on $U_0 \cap U_1$. Therefore $(U_0, \varphi_0) = (U_1, \varphi_1)$ and the given rational map can be represented by the equivalence class $(U_0, \varphi_0) = (U_1, \varphi_1)$. Clearly, the rational map is defined at the points of the open set $U = U_0 \cup U_1$ and is represented by the morphism $\varphi : U \longrightarrow \mathbb{P}^1$ given by $(x_0, x_1, x_2) \mapsto (1, x_1/x_0)$ when $x_0 \neq 0$ and by $(x_0, x_1, x_2) \mapsto (x_0/x_1, 1)$ when $x_1 \neq 0$. But $U = \mathbb{P}^2 \setminus \{P\}$ where P is the point $(0, 0, 1)$.

We now claim that the morphism φ cannot be extended to the point P. Assume the contrary. Therefore there exists a neighbourhood W of P and a morphism $\mu : W \longrightarrow \mathbb{P}^1$ such that $\mu = \varphi$ on $W \cap U$. Therefore $\mu(x_0, x_1, x_1) = (1, x_1/x_0)$ when $x_0 \neq 0$ and $\mu(x_0, x_1, x_2) = (x_0/x_1, 1)$ when $x_1 \neq 0$. Let $W_0 = W \cap U_0$. We can identify U_0 by $\mathbb{A}^2$ and thus consider W_0 as an open subset of $\mathbb{A}^2$. Let $W_0^{'} = W_0 \cup \{P\}$. Consider the restriction of μ to $W_0^{'}$. It is a morphism and hence the map $F = \pi_2 \circ \mu : W_0^{'} \longrightarrow k$ is a regular map, where π_2 is the projection on the second coordinate. Therefore there exists a neighbourhood $V \subset W_0^{'}$ of P where F has the form f_1/f_2 for some homogeneous polynomials f_1, f_2 of the same degree. But on $V \setminus \{P\}$, $F = x_1/x_0$. But $V \setminus \{P\}$ is an open subset of $W_0^{'} \setminus \{P\} = W_0$ which is an open subset of $\mathbb{A}^2$. Therefore $f_1(1, x_1, x_2) = x_1 f_2(1, x_1, x_2)$ on the open subset $V \setminus \{P\}$. Using the same arguments as in part (a), $f_1(1, x_1, x_2) = x_1 f_2(1, x_1, x_2)$ on the whole of $\mathbb{A}^2$. Since $f_2(0, 0, 1) \neq 0$, we have that $x_0 \nmid f_2(x_0, x_1, x_2)$ and hence the degree of $f_2(1, x_1, x_2) =$ degree of $f_2(x_0, x_1, x_2)$. Let d_1 denote the degree of $f_i(x_0, x_1, x_2)$ for $i = 1, 2$. Therefore we have that the degree of $f_1(1, x_1, x_2) = 1 + d_2$. But $d_1 \geq$ degree of $f_1(1, x_1, x_2) = 1 + d_2 > d_2$ which implies that $d_1 > d_2$ which is a contradiction since $d_1 = d_2$. Therefore the morphism φ cannot be extended to the point P.

Exercise 4.0.53. *A variety Y is rational if it is birationally equivalent to $\mathbb{P}^n$ for some n (or equivalently by (4.5), if $K(Y)$ is a pure transcendental extension of k).*

(a) Any cubic in $\mathbb{P}^2$ is a rational curve.

(b) The cuspidal cubic $y^2 - x^3$ is a rational curve.

(c) Let Y be the nodal curve $y^2 z = x^2(x + z)$ in $\mathbb{P}^2$. Show that the projection φ from the point $P = (0, 0, 1)$ to the line $z = 0$ induces a birational map from Y to $\mathbb{P}^1$. Thus Y is a rational curve.

Solution:

(a) From exercise 3.1 (c), we know that any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$ and hence is birationally isomorphic to $\mathbb{P}^1$. Therefore any conic in $\mathbb{P}^2$ is a rational.

(b) Let $Y = \mathcal{Z}(y^2 - x^3)$. Consider the morphism $\varphi : \mathbb{A}^1 \longrightarrow Y$ given by $t \mapsto (t^2, t^3)$. Let $U = \mathbb{A}^1 \setminus \{0\}$. Then U is open in $\mathbb{A}^1$ which is open in $\mathbb{P}^1$. Let

$Y^{'} = Y \setminus \{(0, 0)\}$. Then Y' is an open subset of Y. Now the map $\varphi^{'} : U \longrightarrow Y^{'}$ given by $t \mapsto (t^2, t^3)$ is a morphism. Also, the map $\phi : Y^{'} \longrightarrow U$ given by $(x, y) \mapsto y/x$ is an inverse morphism to φ. Hence the open subsets $U \subset \mathbb{P}^1$ and $Y^{'} \subset Y$ are isomorphic and hence Y is birationally equivalent to $\mathbb{P}^1$ and hence is rational.

(c) Let $Y = \mathcal{Z}(y^2z - x^3\, x^2z) \subset \mathbb{P}^2$. Let $P = (0, 0, 1)$ and let $\mathbb{P}^1 \subset \mathbb{P}^2$ be given by $z = 0$. Then the projection $\varphi : \mathbb{P}^2 \setminus \{P\} \longrightarrow \mathbb{P}^1$ is given by $(x, y, z) \mapsto (x, y)$. Let φ again denote the restriction of this morphism to $Y \setminus \{P\}$. Let $\mu : \mathbb{P}^1 \longrightarrow Y \setminus \{P\}$ be the map defined by $(x, y) \mapsto (x, y, \frac{x^3}{y^2 - x^2})$. This map is well defined because if $y^2 = x^2$ in Y, then $y^3 = 0$ which implies that $y = 0$ and hence $x = 0$. Also, since μ is defined by quotients of polynomials it is a morphism. It is also clear that μ is the inverse morphism to φ. Hence the open subset $Y \setminus \{P\}$ is isomorphic to $\mathbb{P}^1$ and hence Y is birationally equivalent to $\mathbb{P}^1$. Therefore Y is a rational curve.

Exercise 4.0.54. *Show that the quadric surface $Q : xy = zw$ in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$, but not isomorphic to $\mathbb{P}^2$.*

Solution:
Let $W = Q \cap U_w$ where U_w denotes the open subset $\mathcal{Z}(w)^c$. Therefore W is an open subset of Q. We define a morphism $\varphi : W \longrightarrow \mathbb{A}^2$ by $(w, x, y, z) \mapsto (x/w, z/w)$. We define a morphism $\mu : \mathbb{A}^2 \longrightarrow W$ by $(x, y) \mapsto (1, x, y, xy)$. Clearly φ and μ are inverses of each other and thus W is isomorphic to $\mathbb{A}^2$ which is an open subset of $\mathbb{P}^2$. Therefore $\mathbb{P}^2$ and Q are birationally equivalent.
If Q were isomorphic to $\mathbb{P}^2$, then Q would be homeomorphic to $\mathbb{P}^2$. We know from exercise 2.15, that Q contains a family of lines $\{L_t\}$ with the property that if $L_t \neq L_u$, then $L_t \cap L_u = \emptyset$. But we have proved in exercise 3.7 that any two curves in $\mathbb{P}^2$ have a non-empty intersection. Therefore Q is not homeomorphic (and hence not isomorphic) to $\mathbb{P}^2$.

Exercise 4.0.55 (Plane Cremona Transformations)**.** *A birational map of $\mathbb{P}^2$ into itself is called a plane Cremona transformation. We give an example called the Quadratic Transformation. It is a rational map $\varphi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ given by $(a_0, a_1, a_2) \mapsto (a_1a_2, a_0a_2, a_0a_1)$ when no two of a_0, a_1, a_2 are 0.*

(a) Show that φ is birational and is its own inverse.

(b) Find open sets $U, V \subset \mathbb{P}^2$ such that $\varphi : U \longrightarrow V$ is an isomorphism.

(c) Find the open sets where φ and φ^{-1} are defined, describe the corresponding morphisms.

Solution:

(a) Let U be the open neighbourhood where no two of a_0, a_1, a_2 are 0. Let $\varphi : U \longrightarrow \mathbb{P}^2$ be the morphism defined by $(a_0, a_1, a_2) \mapsto (a_1a_2, a_0a_2, a_0a_1)$. We claim that the rational map defined by the equivalence class (U, φ) is its own inverse. Consider the composition $(U, \varphi) \circ (U, \varphi)$. It is given by the equivalence class $(\varphi^{-1}(U), \varphi \circ \varphi)$.

We have that $U = \mathbb{P}^2 \setminus \{A, B, C\}$ where $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. Therefore $\varphi^{-1}(U)$ will be equal to $\mathbb{P}^2 \setminus (\varphi^{-1}(A) \cup \varphi^{-1}(B) \cup \varphi^{-1}(C))$. We now claim that $\varphi^{-1}(A) = \mathcal{Z}(x_0)$. Clearly for any point $P = (0, t, u) \in \mathcal{Z}(x_0)$, $\varphi(P) = (1, 0, 0)$. Therefore $\mathcal{Z}(x_0) \subset \varphi^{-1}(A)$. Now, suppose $(a_0, a_1, a_2) \in \varphi^{-1}(A)$. Therefore $\varphi(a_0, a_1, a_2) = (a_1a_2, a_0a_2, a_0a_1) = (1, 0, 0)$. Hence $a_0a_2 = 0$ and $a_0a_1 = 0$. But since $a_1a_2 = 1$, we have that $a_1, a_2 \neq 0$. Therefore $a_0 = 0$ and hence $(a_0, a_1, a_2) \in \mathcal{Z}(x_0)$. Similarly we can prove that $\varphi^{-1}(B) = \mathcal{Z}(x_1)$ and $\varphi^{-1}(C) = \mathcal{Z}(x_2)$. Therefore $\varphi^{-1}(U) = \mathbb{P}^2 \setminus \mathcal{Z}(x_0x_1x_2)$. Let this open subset be denoted by V. Then it can be easily checked that $\varphi \circ \varphi = I_d$ on V. Therefore (U, φ) is birational and is its own inverse.

(b) From the proof of the corollary 4.5, we know that φ gives an isomorphism of the open subset $\varphi^{-1}(\varphi^{-1}(U))$ to itself. Now, $\varphi^{-1}(\varphi^{-1}(U)) = \varphi^{-1}(V)$ where V is as in part (a). We claim that $\varphi^{-1}(V) = V$. Clearly, $V \subset U$ which implies that $\varphi^{-1}(V) \subset \varphi^{-1}(U) = V$. Consider any $(a_0, a_1, a_2) \in V$. Now, $\varphi(a_0, a_1, a_2) = (a_1a_2, a_0a_2, a_0a_1)$. Since $a_0, a_1, a_2 \neq 0$, $\varphi(a_0, a_1, a_2) \in V$. This proves that $\varphi^{-1}(V) = V$. Hence φ induces an isomorphism of the open subset V to itself.

(c) It is clear from the definition of the rational map φ that it is defined at all the points of the open subset $\mathbb{P}^2 \setminus \{A, B, C\}$ where $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$. Let this open subset be denoted by U. We now claim that the morphism $\varphi : U \longrightarrow \mathbb{P}^2$ defined by $(a_0, a_1, a_2) \mapsto (a_1a_2, a_0a_2, a_0a_1)$ cannot be extended to any of the points A, B, C.

Suppose that φ can be extended to the point A. Then there exists an open subset W of A and a morphism $\mu : W \longrightarrow \mathbb{P}^2$ such that $\varphi = \mu$ on $W \cap U$. Let $W_1 = W \cap U_1$ where $U_1 = \mathcal{Z}(x_1)^c$. Since U_1 can be identified with $\mathbb{A}^2$, we can consider W_1 as an open subset of $\mathbb{A}^2$. Clearly $A \notin W_1$. Let $W_1^{'} = W_1 \cup \{A\}$. Consider the restriction of the map μ to $W_1^{'}$. It is a morphism and hence the map $F = \pi \circ \mu : W_1^{'} \longrightarrow k$ is a regular map where π represents the projection onto the first coordinate. Therefore there exists a neighbourhood $V \subset W_1^{'}$ of A where F has the form f_1/f_2 for some homogeneous polynomials f_1, f_2 of same degree. But on $V \setminus \{A\}$, we know that F has the form x_1x_2. Now $V \setminus \{A\}$ is an open subset of $W_1^{'} \setminus \{A\} = W_1$ which is an open subset of $\mathbb{A}^2$. Therefore $f_1(x_0, 1, x_2) = x_2f_2(x_0, 1, x_2)$ on the open subset $V \setminus \{A\}$. Using the same argument as in 4.3 (a), we get that $f_1(x_0, 1, x_2) = x_2f_2(x_0, 1, x_2)$ on the whole of $\mathbb{A}^2$. Let $\overline{f_i} = f_i(x_0, 1, x_2)$ for $i = 1, 2$. Therefore $deg(\overline{f_1}) = deg(\overline{f_2}) + 1$. Since $f_2(1, 0, 0) \neq 0$, we have that $x_1 \nmid f_2$ and hence $deg(f_2) = deg(\overline{f_2})$. Therefore we have that $deg(\overline{f_1}) = deg(f_2) + 1$. Since $deg(\overline{f_1}) \leq deg(f_1)$, we get that $1 + deg(f_2) \leq deg(f_1)$ which implies that $deg(f_2) < deg(f_1)$ which is a contradiction. Therefore the morphism φ cannot be extended to the point A. Using similar arguments we can prove that φ cannot be extended to any of the points A, B or C. Therefore U is the largest open set on which the rational map φ can be expressed as a morphism.

Exercise 4.0.56. *Let X and Y be two varieties. Suppose there are points $P \in X$ and $Q \in Y$ such that the local rings $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$ are isomorphic as k-algebras. Then show that there are open sets $P \in U \subset X$ and $Q \in V \subset Y$ and an isomorphism of U to V which sends P to Q.*

Solution:
Let $X^{'}$ and $Y^{'}$ be any affine open sets in X and Y respectively such that $P \in X^{'}$ and $Q \in Y^{'}$. Then we know that $\mathcal{O}_{P,X} \cong \mathcal{O}_{P,X^{'}}$ and $\mathcal{O}_{Q,Y} \cong \mathcal{O}_{Q,Y^{'}}$. Also, if we get an isomorphism of U and V such that $\varphi(P) = Q$ where $U \subset X^{'}$ and $V \subset Y^{'}$ are open neighbourhoods of P, Q respectively, then we are done with the problem. So we may assume that X and Y are affine varieties.

From theorem 3.2 we know that $\mathcal{O}_{P,X} \cong A(X)_{\mathfrak{m}_P}$ and $\mathcal{O}_{Q,Y} \cong A(Y)_{\mathfrak{m}_Q}$. Therefore we have inclusions $A(X) \hookrightarrow \mathcal{O}_{P,X}$ and $A(Y) \hookrightarrow \mathcal{O}_{Q,Y}$. Let $\theta : \mathcal{O}_{Q,Y} \longrightarrow \mathcal{O}_{P,X}$ be the isomorphism. Suppose $A(Y)$ is generated by $y_1, y_2, \ldots y_m$ as a k-algebra. Then $\theta(y_i) \in \mathcal{O}_{P,X}$ for $i = 1, \ldots m$. Therefore for each $i = 1, \ldots, m$, we have $\theta(y_i) = (U_i, f_i)$ where U_i is an open subset of X and $f_i ; U_i \longrightarrow k$ is a regular function. Let $U = \bigcap_{i=1}^m U_i$. Then $\theta(y_i)$ is a regular function on U for each $i = 1, \ldots, m$. Therefore θ defines a homomorphism from $A(Y)$ to $\mathcal{O}(U) = A(U)$ and hence from Proposition 3.5 we have a morphism $\varphi : U \longrightarrow Y$.
Let $\mathfrak{m}_P \subset A(U)$ be the ideal corresponding to the point P. We claim that $\theta^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{\varphi(P)}$. Suppose $f \in \theta^{-1}(\mathfrak{m}_P)$. Then $f = \theta^{-1}(g)$ for some $g \in \mathfrak{m}_P$. Therefore $\theta(f) = f \circ \varphi = g$. Since $g(P) = 0$, we have that $f(\varphi(P)) = 0$. Therefore $f \in \mathfrak{m}_{\varphi(P)}$. Conversely let $f \in \mathfrak{m}_{\varphi(P)}$. Therefore $f(\varphi(P)) = 0$ which implies that $f \circ \varphi(P) = 0$. Therefore $f \circ \varphi \in \mathfrak{m}_P$. But $f \circ \varphi = \theta(f)$. Therefore $\theta(f) \in \mathfrak{m}_P$ and hence $f \in \theta^{-1}(\mathfrak{m}_P)$. This proves the claim.
Let $\mathfrak{m}_P^{'}$ be the unique maximal ideal of $\mathcal{O}_{P,X}$ and let $\mathfrak{m}_Q^{'}$ be the unique maximal ideal of $\mathcal{O}_{Q,Y}$. Since θ is an isomorphism of $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$, we have $\theta^{-1}(\mathfrak{m}_P^{'}) = \mathfrak{m}_Q^{'}$. Since $\mathcal{O}_{P,X} \cong A(X)_{\mathfrak{m}_P}$, we know that $\mathfrak{m}_P^{'} = \mathfrak{m}_P\mathcal{O}_{P,X}$ and similarly $\mathfrak{m}_Q^{'} = \mathfrak{m}_Q\mathcal{O}_{Q,Y}$. Therefore $\theta^{-1}(\mathfrak{m}_P) \subset \mathfrak{m}_Q$. But we have proved that $\theta^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{\varphi(P)}$. Therefore $\mathfrak{m}_Q = \mathfrak{m}_{\varphi(P)}$. From theorem 3.2 we know that there is a 1-1 correspondence between the points of Y and the maximal ideals of $A(Y)$. Therefore we get that $\varphi(P) = Q$.
Let $\eta = \theta^{-1}$. Then using the same arguments as above we get a morphism $\mu : V \longrightarrow X$ such that $\mu(Q) = P$ where V is an open subset containing Q. Now, $\varphi^* = \theta$ and $\mu^* = \eta$. Hence $(\varphi \circ \mu)^* = \mu^* \circ \varphi^* = I_d$ where I_d is the identity homomorphism of the ring $\mathcal{O}_{Q,Y}$. Therefore $\varphi \circ \mu = I_d$ on $\mu^{-1}(U)$ where I_d is the identity morphism of the open subset $\mu^{-1}(U)$. Similarly we get that $\mu \circ \varphi = I_d$ on $\varphi^{-1}(V)$ where I_d is the identity morphism of the set $\varphi^{-1}(V)$. Now restricting the open sets as in the proof of corollary 4.5, we get that an isomorphism of open subsets of P and Q which maps P to Q.

Exercise 4.0.57. *Let Y be the cuspidal cubic curve $y^2 - x^3$ in $\mathbb{A}^2$. Blow up the point $(0, 0)$, let E be the exceptional curve, and let $\tilde{Y}$ be the strict transform of Y. Show that E meets $\tilde{Y}$ in one point, and that $\tilde{Y} \cong \mathbb{A}^1$. In this case the morphism $\varphi : \tilde{Y} \longrightarrow Y$ is bijective and bicontinuous, but it is not an isomorphism.*

Solution:
Let t, u be the homogeneous coordinates for $\mathbb{P}^1$. Let X denote the blowing-up of $\mathbb{A}^2$ at O. It is defined by the equation $xu = ty$ inside $\mathbb{A}^2 \times \mathbb{P}^1$. The total inverse image of Y in X is obtained by considering the equations $y^2 = x^3$ and $xu = ty$ in $\mathbb{A}^2 \times \mathbb{P}^1$. We know that $\mathbb{P}^1$ is covered by two open sets $t \neq 0$ and $u \neq 0$. When $t \neq 0$, we set $t = 1$ and then obtain the equations $y^2 = x^3$ and $xu = ty$ which gives the the reducible equation $x^3 = x^2u^2$. The first irreducible component of this is given by $x = 0$, $y = 0$ and u arbitrary. This corresponds to the exceptional curve E. The other irreducible component is given $x = u^3$, $y = u^3$. This is $\tilde{Y}$. Clearly $\tilde{Y}$ meets E only in one point which is $P = (0, 0) \times (1, u) \in \mathbb{A}^1 \times \mathbb{P}^1$. Clearly the map $\phi : \tilde{Y} \longrightarrow \mathbb{A}^1$ defined by $(u^2, u^3) \times (1, u) \mapsto u$ is a morphism. Also, the map $\mu : \mathbb{A}^1 \longrightarrow \tilde{Y}$ given by $u \mapsto (u^2, u^3) \times (1, u)$ is the inverse morphism to ϕ. Therefore $\mathbb{A}^1$ is isomorphic to $\tilde{Y}$.
Since φ induces an isomorphism of $\tilde{Y} \setminus \varphi^{-1}(O)$ to $Y \setminus O$, we know that φ is a bijective bicontinuous map of these two sets. Now since $\varphi^{-1}(O) \cap \tilde{Y}$ is the singleton set, φ is a bijective map from Y to $\tilde{Y}$. Also, since $\varphi^{-1}(O) = P$, the inverse image of the closed subset $\{O\}$ is the closed set $\{P\}$. Hence the map φ is bicontinuous.

Chapter 5

Nonsingular Varieties

Exercise 5.0.58 (Multiplicities)**.** *Let $Y \subset \mathbb{A}^2$ be the curve defined by the equation $f(x, y) = 0$. Let $P = (a, b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0, 0)$. Then write f as a sum $f = f_0 + \ldots, + f_d$ where f_i is a homogeneous polynomial of degree i in x and y. Then we define the multiplicity of P on Y, denoted by $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. The linear factors of f_r are called the tangent directions at P. Show that $\mu_P(Y) = 1 \Leftrightarrow P$ is a nonsingular point of Y*

Solution:
We know from Theorem 5.1 that a variety $Y \subset \mathbb{A}^n$ is non singular at a point $P \in Y$ if and only if the local ring $\mathcal{O}_{P,Y}$ is a regular local ring. A linear change of coordinates will change $\mathcal{O}_{P,Y}$ only upto an isomorphism. Therefore the nonsingularity property of a point on a variety remains unchanged under a linear change of coordinates. Then after a linear change of coordinates such that $P = (0, 0)$ we know that f has the form $f_0 + f_1 + \ldots + f_d$ where f_i is a homogeneous polynomial of degree i in x and y. Since $P = (0, 0) \in Y$, we know that $f_0 = 0$ and since $\mu_P(Y) = 1$ we have that $f_1 \neq 0$. Suppose the linear term is $\alpha x + \beta y$. Then both α and β cant be zero simultaneously. Now, $\partial f/\partial x(P) = \alpha$ and $\partial f/\partial y(P) = \beta$ and hence both are not simultaneously zero and hence P is non singular.
To prove the converse also we make a linear change of coordinates such that $P = (0, 0)$. Suppose now that $f = f_1 + \ldots + f_d$. Suppose the linear term is $\alpha x + \beta y$. Then $\partial f/\partial x(P) = \alpha$ and $\partial f/\partial y(P) = \beta$. Since f is non singular at P, both α and β cant be simultaneously zero and hence $f_1 \neq 0$. Therefore $\mu_P(Y) = 1$.

Exercise 5.0.59. *For every degree $d > 0$ and every $p = 0$ or a prime number, give the equation of a non singular curve of degree d in $\mathbb{P}^2$ over a field k of characteristic p.*

Solution:
For characteristic 0 we consider the curve Y given by $f = x^d + +y^d + z^d$. Then $\partial f/\partial x = dx^{d-1}$, $\partial f/\partial y = dy^{d-1}$ and $\partial f/\partial z = dz^{d-1}$. Hence the Jacobian of this curve at any point $P \in \mathbb{P}^2$ is a non zero row matrix. Now we know from Exercise 5.8 we know that this implies that Y is a non singular curve at every point of $\mathbb{P}^2$ and hence is a non singular curve. Also, when the field has a positive characteristic p such that p does not divide d, then f still satisfies the Jacobian condition and hence is a non singular curve.
When the positive characteristic p is such that p divides d, then we can consider the curve Y given by $f = xy^{d-1} + yz^{d-1} + zx^{d-1}$. Then $\partial f/\partial x = (d-1)zx^{d-2} + y^{d-1}$, $\partial f/\partial y = (d-1)xy^{d-2} + z^{d-1}$ and $\partial f/\partial z = (d-1)yz^{d-2} + x^{d-1}$. We can check for the solutions in each of the affine open subsets of $\mathbb{P}^2$, $\{x \neq 0\}$, $\{y \neq 0\}$ and $\{z \neq 0\}$. For example to check for a solution in $\{x \neq 0\}$, we put $x = 1$. Then it can be checked that the three equations have no solution in this affine open subset. Similarly it can be checked that there are no solutions in each of the other two affine open subsets. Hence the Jacobian of Y at any point $P \in \mathbb{P}^2$ is a non zero row matrix. Therefore, from Exercise 5.8, we get that Y is a non singular curve.

Exercise 5.0.60 (Blowing Up Curve Singularities)**.**

(a) Let Y be the cusp $x^6 + y^6 = xy$ or the node $y^2 + x^4 + y^4 - x^3$. Show that the curve $\tilde{Y}$ obtained by blowing up Y at $(0, 0)$ is non singular.

(b) We define a node(also called ordinary double point) to be a point of multiplicity 2 of a plane curve with distinct tangent directions. If P is a node on a plane curve Y, show that $\varphi^{-1}(P)$ consists of two distinct non singular points on the blown up curve $\tilde{Y}$. We say that 'blowing up P resolves the singularity at P'.

(c) Let $P \in Y$ be the tacnode of $x^2 = x^4 + y^4$. If $\varphi : \tilde{Y} \longrightarrow Y$ be the blowing up at P, show that $\varphi^{-1}(P)$ is a node. Using (b) we can see that a tacnode can be resolved using two successive blowings-up.

(d) *Let Y be the plane curve $y^3 = x^5$ which has a 'higher order cusp' at O. Show that O is a triple point; that blowing up O gives rise to a double point and that one further blowing up resolves the singularity.*

Solution:

(a) We first consider the cusp $x^6 + y^6 = xy$. We know that $\tilde{Y} - \varphi^{-1}(O)$ is isomorphic to $Y - O$ where φ is the blowing up map of $\mathbb{A}^2$ at O. It can be checked that all the points of $Y - O$ are non singular and thus all the points of $\tilde{Y} - \varphi^{-1}(O)$ are non singular. Therefore we need to only check the singularity of the points on $\tilde{Y} \cap \varphi^{-1}(O)$.

Let the homogeneous coordinates of $\mathbb{P}^1$ be t, u. Let X denote the blowing up of $\mathbb{A}^2$ at the origin. We get the total inverse image of Y in X by considering the equations $xu = ty$ and $x^6 + y^6 = xy$. We first consider the affine open subset U_t of $\mathbb{P}^1$ given by $t \neq 0$. To determine $\tilde{Y} \cap U_t$, we set $t = 1$ and then obtain the equations $xu = y$ and $x^6 + y^6 = xy$ which gives the equation $x^2(x^4 + x^4u^6 - u) = 0$. The first irreducible component of this is given by $x = 0$, $y = 0$ and u arbitrary. This corresponds to $\varphi^{-1}(O) \cap U_t$.

The second irreducible component $x^4 + x^4u^2 - u = 0$ along with the equation $y = xu$ defines $\tilde{Y} \cap U_t$. Now $\tilde{Y} \cap U_t$ meets $\varphi^{-1}(O) \cap U_t$ in the point $(0, 0, 0)$. Now the Jacobian matrix of $\tilde{Y} \cap U_t$ at the point (0, 0, 0) is $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which clearly has rank 2. Now, since $\tilde{Y}$ is birationally equivalent to Y we get, from Theorem 3.2 and Corollary 4.5, that the dimension of $\tilde{Y}$ is the same as the dimension of Y which is equal to 1. Hence the dimension of $\tilde{Y} \cap U_t$ is also 1. Therefore we get that $\tilde{Y} \cap U_t$ is non singular at the point O.

We now consider the affine open subset U_u of $\mathbb{P}^2$ given by $u \neq 0$. For this we set $u = 1$. Then we get the equation $y^2(t^6y^4 + y^4 - t) = 0$. The first irreducible component of this, given by $x = 0$, $y = 0$ and t arbitrary is $\varphi^{-1}(O) \cap U_u$. The second irreducible component $t^6y^4 + y^4 - t = 0$ along with $x = ty$ defines $\tilde{Y} \cap U_u$. Now, $\tilde{Y} \cap U_u$ meets $\varphi^{-1}(O) \cap U_u$ in the point (0, 0, 0). Now the Jacobian matrix of $\tilde{Y} \cap U_u$ at the point (0, 0, 0) is $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which clearly has rank 2. Using the arguments same as above we get that the dimension of $\tilde{Y} \cap U_u$ is 1 and hence that $\tilde{Y} \cap U_u$ is non singular at the point O. Therefore we get that $\tilde{Y}$ is a non singular variety.

We now consider the cusp Z defined by $y^2 + x^4 + y^4 - x^3 = 0$. Let the homogeneous coordinates of $\mathbb{P}^1$ be t, u. Let X be as before. Then the total inverse image of Z in X is obtained by considering the equations $y^2 + x^4 + y^4 - x^3 = 0$ and $xu = ty$. We know that each point of $Z - O$ is non singular. Hence, as noted above, we need to only check the singularity of the points on $\tilde{Z} \cap \varphi^{-1}(O)$. We first consider the affine open subset U_t of $\mathbb{P}^1$ given by $t \neq 0$. To determine $\tilde{Z} \cap U_t$, we set $t = 1$ and then obtain the equations $y = xu$ and $y^2 + x^4 + y^4 - x^3 = 0$ which gives us the equation $x^2(u^2 + x^2 + x^2u^4 - x) = 0$. The first irreducible component of this is given by $x = 0$, $y = 0$ and u arbitrary. This corresponds to $\varphi^{-1}(O) \cap U_t$.

The second irreducible component $u^2 + x^2 + x^2u^4 - x = 0$ along with the equation $y = xu$ corresponds to $\tilde{Z} \cap U_t$. Now, $\tilde{Z} \cap U_t$ meets $\varphi^{-1}(O) \cap U_t$ in the point (0, 0, 0). The Jacobian of $\tilde{Z} \cap U_t$ at the point (0, 0, 0) is $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which clearly has rank 2. Using arguments similar to the ones used in the case of cusp we get that the dimension of $\tilde{Z} \cap U_t$ is 1 and hence that $\tilde{Z} \cap U_t$ is non singular at the point (0, 0, 0).

We can easily check that $(\tilde{Z} \cap \varphi^{-1}(O)) \cap U_u\, \emptyset$. Hence the $\tilde{Z} \cap \varphi^{-1}(O)$ is non singular. Therefore we get that $\tilde{Z}$ is a non singular variety.

(b) By a linear change of coordinates we can assume that the node is $P = (0, 0)$. Let Y be defined by the equation $f = f_1 + f_2 + \ldots + f_d$. Since $\mu_P(Y) = 2$ we have that $f_1 = 0$. Also since there are two distinct tangent directions, we have that $f_2 = (\alpha_1 x + \beta_1 y)(\alpha_2 x + \beta_2 y)$ such that $\alpha_1/\alpha_2 \neq \beta_1/\beta_2$. We can write f as $f_2 + g(x, y)$ where $g(x, y)$ has only terms of degree 3 or more. By another change of coordinates we can assume that $f_2 = xy$.

We let the homogeneous coordinates of $\mathbb{P}^2$ be t, u. Then the total inverse image of Y in the blow up of $\mathbb{A}^2$ at the origin, X, is given by $xu = ty$ and $f(x, y) = 0$. We first consider the affine open subset given by $t \neq 0$. We set $t = 1$ to get the equation $xu = y$. Substituting in f, we get that $f = x^2u + g(x, xu)$. Since $g(x, xu)$ has terms of degree 3 or more, we can write $g(x, xu)$ as $x^3h(x, xu)$ for some polynomial h. Therefore $f = x^2(u + xh(x, xu))$. Now, $\tilde{Y} \cap U_t$ is given by $u + xh(x, xu) = 0$ and $y = xu$ and it meets $\varphi^{-1}(P)$ at the point $(0, 0, 0)$. This point corresponds to the point $(0, 0, 1, 0)$ in $\mathbb{A}^2 \times \mathbb{P}^1$. The Jacobian matrix of this variety at any point $Q = (a, b, c)$ is given by $\begin{bmatrix} -b & -a & 1 \\ 1 + b\dfrac{\partial h}{\partial u}(Q) & b\dfrac{\partial h}{\partial x}(Q) + h(Q) & 0 \end{bmatrix}$. Therefore at the point $(0, 0, 0)$ the Jacobian matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ which clearly has rank 2. Since the dimension of $\tilde{Y} \cap U_t$ is 1 we get that $\tilde{Y} \cap U_t$ is non singular at the point P.

We now consider the affine open subset U_u of $\mathbb{P}^2$ given by $u \neq 0$. We set $u = 1$ in the equations to obtain the equation $x = ty$. Substituting in $f(x, y)$ we get $ty^2 + g(ty, y) = 0$. Since the degree of $g(x, y)$ is ≥ 3, we can write $g(x, y)$ as $y^3h(ty, t)$ for some polynomial h. Hence $f = y^2(t + yh(ty, y))$. Now $\tilde{Y} \cap U_u$ is given by $t + yh(ty, y) = 0$ and $x = ty$ and meets $\varphi^{-1}(P)$ in $(0, 0, 0)$. This point is the point $(0, 0, 0, 1)$ in $\mathbb{A}^2 \times \mathbb{P}^1$. It can be checked that the Jacobian matrix of this variety is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ which clearly has rank 2. Since the dimension of $\tilde{Y} \cap U_u$ is 1 we get that $\tilde{Y} \cap U_u$ is non singular at the point P.

Therefore we get that $\varphi^{-1}(P) \cap \tilde{Y} = \{(0, 0, 0, 1), (0, 0, 1, 0)\}$ both of which are non singular points.

(c) Let the homogeneous coordinates of $\mathbb{P}^1$ be t, u. Then the total inverse image of Y in the blow up of $\mathbb{A}^2$ at the origin is given by $xu = ty$ and $x^2 = x^4 + y^4$. We first consider the affine piece given by $t \neq 0$ by putting $t = 1$. We then get the equations $xu = y$ and $x^2 = x^4 + y^4$ from which we obtain the equation $x^2 - x^4 - x^4u^4 = 0$. This has two irreducible components, $x = 0$ and $x^2 + x^2u^4 - 1 = 0$. The first component corresponds to the exceptional curve, $E = \psi^{-1}(P) \cap U_t$ where $\psi : X \longrightarrow \mathbb{A}^2$ is the blowing up of $\mathbb{A}^2$ at the origin. The second component corresponds to $\tilde{Y} \cap U_t$. We can check that $\tilde{Y} \cap U_t \cap E = \emptyset$.

We now consider the second affine subset U_u given by $u \neq 0$ by setting $u = 1$. We then get the equations $ty = x$ and $x^2 = x^4 + y^4$ from which we get the equation $t^2y^2 - t^4y^4 - y^4$. This has two irreducible components $y = 0$ and $t^2 - t^4y^2 - y^2 = 0$. The first component corresponds to the exceptional curve, $E = \psi^{-1}(P) \cap U_u$ where $\psi : X \longrightarrow \mathbb{A}^2$ is the blowing up of $\mathbb{A}^2$ at the origin.

The second component corresponds to $\tilde{Y} \cap U_u$. This intersects E at the point $(0, 0, 0)$. The lowest degree terms of $t^2 - t^4y^2 - y^2 = 0$ are $t^2 - y^2 = (t + y)(t - y)$ and hence $\varphi^{-1}(P) \cap U_u$ is a node.

(d) Let Y be the variety defined by $y^3 - x^5 = 0$. Then clearly the origin is a triple point of Y. Let the homogeneous coordinates of $\mathbb{P}^1$ be u, t. The total inverse image of Y in the blow up of origin at origin is given by the equations $xu = ty$ and $y^3 - x^5 = 0$. We first consider the affine piece U_u given by $u \neq 0$ by setting $u = 1$. Then we get equations $x = yt$ and $y^3 - x^5 = 0$ from which we obtain the equation $y^5t^5 - y^3 = 0$. This has two irreducible components $y = 0$ and $y^2t^5 - 1 = 0$. The first component corresponds to the exceptional curve $E = \psi^{-1}(P) \cap U_t$ where ψ is the blowing up of $\mathbb{A}^2$ at the origin. The second component corresponds to $\tilde{Y} \cap U_u$. We can check that $\tilde{Y} \cap U_u \cap E = \emptyset$. Therefore this component of $\tilde{Y}$ has no singularity.

We consider the affine piece U_t given by $t \neq 0$ by setting $t = 1$. Then we get the equations $y = xu$ and $y^3 - x^5 = 0$ from which we obtain the equation $x^3u^3 - x^5 = 0$. This has two irreducible components $x = 0$ and $u^3 - x^2 = 0$. The first component corresponds to the exceptional curve $E = \psi^{-1}(P) \cap U_t$ where ψ is the blowing up of $\mathbb{A}^2$ at the origin. The second component corresponds to $\tilde{Y} \cap U_t$. This is the cuspidal cubic curve with a double point at the origin.

Let us denote this cuspidal cubic curve by Z. We now blow up Z at the point $(0, 0)$. Let the homogeneous coordinate of $\mathbb{P}^1$ be w, v. Then the inverse image of Z in the blow up of $\mathbb{A}^2$ at the origin is given by the equations $xw = uv$ and $u^3 - x^2 = 0$. We first consider the affine piece U_w given by $w \neq 0$ by setting $w = 1$. We then get the equations $x = uv$ and $u^3 - x^2 = 0$ from which we get the equation $u^3 - u^2v^2 = 0$. This has two irreducible components $u = 0$ and $u - v^2 = 0$. The first component corresponds to $E = \varphi^{-1}(P) \cap U_w$ where φ is the blow up of $\mathbb{A}^2$ at the origin. The second component corresponds to $\tilde{Z} \cap U_w$. This meets E at the point $(0, 0, 0)$ and it can be checked that $\tilde{Z} \cap U_w$ is non singular at this point.

We now consider the affine piece U_v given by $v \neq 0$ by setting $v = 1$. We then get the equations $xw = u$ and $u^3 - x^2 = 0$ from which we get the equation $x^3w^3 - x^2 = 0$. This has two irreducible components $x = 0$ and $xw^3 - 1 = 0$.

The first component corresponds to $E = \varphi^{-1}(P) \cap U_v$ and the second component corresponds to $\tilde{Z} \cap U_v$. We can check that $\tilde{Z} \cap U_w \cap E = \emptyset$.

Therefore we cusp at the point P obtained by blowing up Y is resolved by a subsequent blowing up.

Chapter 6

Varieties and Submanifolds

6.1 Introduction

Complex analysis deals with holomorphic functions which are defined on open subsets of the euclidean topology on $\mathbb{C}^n$. This leads to the notion of holomorphic submanifolds of $\mathbb{C}^n$ (and $\mathbb{P}^n(\mathbb{C})$) which are loosely speaking subsets of $\mathbb{C}^n$ (and $\mathbb{P}^n(\mathbb{C})$) which are locally given by holomorphic functions.

On the other hand we have defined on $\mathbb{C}$ and $\mathbb{P}^2(\mathbb{C})$ a topology called the Zariski topology in which the closed subsets are the set of common zeroes of polynomials functions. The closed subsets in the Zariski topology are called algebraic varieties. These may or may not be reducible.

In this essay we state and prove two basic yet remarkable theorems which bring out the relationship between these two kinds of subspaces of $\mathbb{P}^2(\mathbb{C})$, namely the algebraic varieties and the analytical submanifolds thus connecting the algebro-geometric notions over abstract fields with the ideas coming from complex manifolds.

6.2 Closed Submanifolds of $\mathbb{C}^n$

In this section we define a closed holomorphic submanifold of $\mathbb{C}^n$. We begin by defining a few preliminaries.

Definition 6.2.1 (Holomorphic Function)**.** *A map $f : U \longrightarrow \mathbb{C}$ from an open subset $U \subset \mathbb{C}^n$ to $\mathbb{C}$ is said to be holomorphic (or complex analytic) if for any point $P = (a_1, \ldots, a_n) \in U$ there exists an neighbourhood of P in which f can be expressed* *as a convergent power series in the n complex variables $x_1 - a_1, \ldots, x_n - a_n$.*

A map $f : U \longrightarrow \mathbb{C}^m$ which is given by $(f_1, \ldots, f_m)$ is said to be holomorphic if each of the $f_i : U \longrightarrow \mathbb{C}$ is holomorphic.

Definition 6.2.2 (Holomorphic Isomorphism). *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open subsets. A map $f : U \longrightarrow V$ is called a holomorphic isomorphism if it is a topological homeomorphism such that f and the inverse map $g : V \longrightarrow U$ are holomorphic when regarded as maps from $U \longrightarrow \mathbb{C}^m$ and $V \longrightarrow C^n$ respectively.*

It can be checked that when $U \subset \mathbb{C}^n$ is non empty and when there is a holomorphic isomorphism $f : U \longrightarrow V$ fro some $V \subset \mathbb{C}^m$, then $m = n$.

Definition 6.2.3 (Holomorphic coordinate chart). *The tuple $(U, u_1, \ldots, u_n)$ where $U \subset \mathbb{C}^n$ is non empty and open and each of the $u_i : U \longrightarrow \mathbb{C}$ is holomorphic, is called a holomorphic coordinate chart if the resulting map $u : U \longrightarrow \mathbb{C}^n$ is a holomorphic isomorphism of U onto an open subset $V \subset \mathbb{C}^n$.*

We now define a special kind of holomorphic coordinate chart which will be used in the definition of a closed submanifold of $\mathbb{C}^n$.

Definition 6.2.4 (Cubical coordinate chart(polydisk)). *Let $(U, u_1, \ldots, u_n)$ be a holomorphic coordinate chart such that the map $u : U \longrightarrow \mathbb{C}^n$ is a holomorphic isomorphism of U onto an open subset $V \subset \mathbb{C}^n$.*

If V is of the form $\{(b_1, \ldots, b_n) \in \mathbb{C}^n \mid |b_i| < a\}$ fro some $a > 0$, then the holomorphic coordinate chart is called a cubical coordinate chart.

The point $P \in U$ for which $u_i(P) = 0$ for all $i = 1, \ldots, n$ is called the centre of the cubical coordinate chart.

Definition 6.2.5 (Locally closed holomorphic submanifold of $\mathbb{C}^n$). *A non empty locally closed subset $X \in \mathbb{C}^n$ is called a locally closed submanifold of $\mathbb{C}^n$ if each $P \in X$ is the centre of a cubical coordinate chart $(U, u_1, \ldots, u_n)$ such that $X \cap U = \{Q \mid u_i(Q) = 0 \, for \, all \, d+1 \leq i \leq n\}$ where $d \leq n$ is a positive integer.*

The positive integer d is called the dimension of the locally closed submanifold.

When X is an empty it vacuously satisfies the criteria of being a locally closed submanifold and therefore we adopt the convention of giving it dimension $-\infty$.

If a locally closed submanifold X of $\mathbb{C}^n$ is closed in an open subset V of $\mathbb{C}^n$, we call it a closed submanifold of V.

6.3 Implicit Function Theorem

In this section we state the implicit function theorem which serves to connect analytical geometry and algebraic geometry specifically by showing that non singular algebraic varieties are closed holomorphic submanifolds. We illustrate this use of the theorem in the later sections after defining the concept of non-singularity.

Theorem 6.3.1 (Implicit Function Theorem). *Let $x_1, \ldots, x_m$ be the linear coordinates on $\mathbb{C}^m$ and let $y_1, \ldots, y_n$ be the linear coordinates on $\mathbb{C}^n$. Therefore we get on $\mathbb{C}^{m+n}$ the linear coordinates $x_1, \ldots, x_m, y_1, \ldots, y_n$. Let $U \subset \mathbb{C}^{m+n}$ be an open neighbourhood of the origin $0 \in \mathbb{C}^{m+n}$. Let $f = (f_1, \ldots, f_n) : U \longrightarrow \mathbb{C}^n$ be a holomorphic map such that $f(0) = 0$. If the $n \times n$ matrix*

$$\left(\frac{\partial f_i}{\partial y_j}\right)_{1 \le i, j \le n}$$

is invertible at the point $0 \in \mathbb{C}^{m+n}$, then there exists open neighbourhoods $V_a \subset \mathbb{C}^m$ and $W_b \subset \mathbb{C}^n$ which are given $|x_i| < a$ and $|y_j| < b$ for some positive real numbers a, b and a holomorphic function $g = (g_1, \ldots, g_n) = V_a \longrightarrow W_b$ such that $V_a \times V_b \subset U$ and

$$f^{-1}(0) \cap (V_a \times W_b) = \{(x, g(x)) | \, x \in V_a\}.$$

Proof. See Proposition 1.1.11, Chapter 1, p 11 of [2] □

This theorem tells us that under suitable conditions the level set of a holomorphic map is locally the graph of a holomorphic function.

Now, if $f : U \longrightarrow \mathbb{C}^n$ be a holomorphic map where $U \subset \mathbb{C}^q$ is an open subset. Suppose for each $P \in U$ with $f(P) = 0$, the rank of the $n \times q$ matrix

$$\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{1 \le i \le n, \, 1 \le j \le q}$$

is n. If $f^{-1}(0)$ is non empty, we get from the Implicit Function Theorem that it is a closed holomorphic submanifold of U of dimension $q - n$.

6.4 Non-singular varieties

In this section we define the concept of a non-singular algebraic variety of $\mathbb{C}^n$ and then use implicit function theorem to connect it to the concept of a holomorphic submanifold of $\mathbb{C}^n$.

Definition 6.4.1 (Non-singular affine variety)**.** *Let $Y = \mathcal{Z}(f_1, \ldots, f_t) \subset \mathbb{C}^n$ be an algebraic variety. Then Y is said to be non-singular at a point $P \in Y$ if the rank of the matrix $\left(\dfrac{\partial f_i}{\partial x_j}(P)\right)$ is $n - r$ where r is the dimension of Y.*

The variety Y is said to be non-singular if it is non-singular at every point of Y.

Similarly we can define the concept of a non singular algebraic subset of $\mathbb{P}^n$.

Definition 6.4.2 (Non singular projective variety)**.** *Let $Y \subset \mathbb{P}^n$ be the set $\mathcal{Z}(f_1, \ldots, f_t)$ where f_i are homogeneous polynomials in $k[x_0, x_n]$. Let $P \in Y$ be the point with the homogeneous coordinates $(a_0, \ldots, a_n)$. The Y is said to be non-singular at the point P if the rank of the matrix $\left(\dfrac{\partial f_i}{\partial x_j}(P)\right)$ is $n - r$ where r is the dimension of Y.*

The matrix $\left(\dfrac{\partial f_i}{\partial x_j}(P)\right)$ is called the Jacobian matrix of Y. If $Y = \mathcal{Z}(f_1, \ldots f_t) = \mathcal{Z}(g_1, \ldots, g_s)$, then it can be shown that the rank of the matrix $\left(\dfrac{\partial f_i}{\partial x_j}(P)\right)$ is the same as the rank of the matrix $\left(\dfrac{\partial g_i}{\partial x_j}(P)\right)$ for any $P \in Y$. Therefore the notion of non-singularity of a variety is independent of the set of generators of the variety. This criteria for non-singularity is called the Jacobian criteria.

We recall a few definitions from algebraic geometry here which will aid us in giving an equivalent condition for son singularity.

Let $Y \subset \mathbb{C}^n$ be an algebraic variety.

Definition 6.4.3 (Regular Functions on $\mathbb{C}^n$)**.** *A function $f : Y \longrightarrow \mathbb{C}$ is said to be regular at a point $P \in Y$ if there is an open neighbourhood $U \subset Y$ with $P \in U$ and two polynomials $g, h \in k[x_1, \ldots, x_n]$ such that h is no where zero on U and $f = g/h$ on U.*

We say that f is regular on Y if it is regular at every point on Y.

Definition 6.4.4 (Regular functions on $\mathbb{P}^n$)**.** *A function $f : Y \longrightarrow \mathbb{C}$ is said to be regular at a point $P \in Y$ if there is an open neighbourhood $U \subset Y$ with $P \in U$ and* *two homogeneous polynomials g, $h \in k[x_o, \ldots, x_n]$ of the same degree such that h is no where zero on U and $f = g/h$ on U.*

We say that f is regular on Y if it is regular at every point of Y.

Definition 6.4.5 (Local ring of a point on a variety)**.** *Suppose Y is an algebraic subset of $\mathbb{C}^n$ (or $\mathbb{P}^n$) and suppose $P \in Y$ is a point. Then the set of all pairs (U, f) where U is an open subset of Y containing P and f is a regular function on U with the equivalent condition that $(U, f) = (V, g)$ if $f = g$ on $U \cap V$ is a ring. This ring is called the local ring of P on Y and is denoted by $\mathcal{O}_{P,Y}$.*

The local ring $\mathcal{O}_{P,Y}$ is basically the ring of germs of regular functions near P. It can be checked that it is a local ring and the set of germs of regular functions which vanish at P is its the maximal ideal. We denote this maximal ideal by $\mathfrak{m}_P$.

We now define the algebraic notion of a regular local ring.

Definition 6.4.6 (Regular Local Ring)**.** *A noetherian local ring $(A, \mathfrak{m})$ with residue field k is called regular local if $dim_k \mathfrak{m}/\mathfrak{m}^2 = dim\, A$.*

We now state a theorem which gives an equivalent condition for non singularity in terms of the local ring.

Theorem 6.4.7. *Let $Y \subset \mathbb{C}^n$ be an algebraic set. Let $P \in Y$ be a point. Then Y is non-singular at P if and only if the local ring $\mathcal{O}_{P,Y}$ is a regular local ring.*

Proof. See Theorem 5.1, Chapter I, p 32 of [1] □

We next state a theorem which tells us that most of the points of an algebraic variety Y are non-singular.

Theorem 6.4.8. *Let Y be a variety. Then the set $SingY$ of singular points of Y is proper closed subset of Y.*

Proof. See Theorem 5.3, Chapter I, p 33 of [1] □

We now use the implicit function theorem to show that non-singular algebraic varieties are closed holomorphic submanifolds. Suppose now that the $Y \subset \mathbb{C}^n$ is an irreducible non singular algebraic set of dimension r. Suppose that $Y = \mathcal{Z}(f_1, \ldots, f_n)$. Since Y is non-singular, at every point P of Y the rank of the matrix $\left(\dfrac{\partial f_i}{\partial x_j}(P)\right)$ is $n - r$. Then from the implicit function theorem we know that Y is a closed submanifold of dimension r in $\mathbb{C}^n$.

6.5 Chow's Theorem

We now state and prove a theorem which serves to show an important connection between algebraic geometry and analytical geometry by showing that the local property of being analytic in $\mathbb{P}^n$ is equivalent to the global property of being algebraic. In this essay we concentrate on the special case of $n = 2$. This result allows us to apply many analytical methods to algebraic geometry.

Theorem 6.5.1. *Any* 1*-dimensional holomorphic closed submanifold of* $\mathbb{P}^2_{\mathbb{C}}$ *is a non-singular irreducible algebraic curve in* $\mathbb{P}^2_{\mathbb{C}}$.

Proof. If M is a straight line, then there is nothing to prove, so assume that M is not a straight line. Note that $\mathbb{P}^2$ has an affine open cover by three copies of $\mathbb{C}^2$, each the complement of one axis. Therefore, $M \cap \mathbb{C}^2$ will be a non-empty closed submanifold for each of these $\mathbb{C}^2$. We first assume that $M \cap \mathbb{C}^2$is connected (this assumption is necessarily satisfied as we will show later).

Therefore, we now begin by considering a 1-dimensional holomorphic closed connected submanifold $M_0 \subset \mathbb{C}^2$.

By definition of a submanifold, around any $Q \in M_0$ there is a rectangular open neighbourhood (polydisk) W such that $M_0 \cap W$ is the graph of a holomorphic function $y = g(x)$ or graph of a holomorphic function $x = h(y)$.

Let $P \in \mathbb{C}^2$ be such that $P \notin M_0$. Suppose that the line $\overline{PQ}$ is tangent to M_0 at Q. Then around Q we can choose new local coordinates (u, v) which form a polydisk U such that u and v are first degree polynomials in x and y, and for any line L passing through P and intersecting U, the set $L \cap U$ is given by $v =$ constant. Hence locally (by shrinking U if necessary) there will be a holomorphic function $f(u)$ such that $M_0 \cap U$ is the graph $v = f(u)$. Suppose that Q is the point $(u, v) = (a, b)$.

Let $T = \{Q \in M_0 | \overline{PQ} \textit{ is tangent to } M_0 \textit{ at } Q\}$. We will show that T is closed and discrete. First we will show that T is discrete, that is, each point of T is isolated. The condition that $Q = (a, b)$ is a point in T is that $\frac{df}{du}(a) = 0$. Since f is holomorphic, we have that $\frac{df}{du}$ is holomorphic. Therefore if $\frac{df}{du}(a) = 0$, then either a is an isolated zero of $\frac{df}{du}$ or $\frac{df}{du} \equiv 0$ in a neighbourhood of a. If former is the case then we are done. Suppose $\frac{df}{du} \equiv 0$ in a neighbourhood of a. Then f is a constant function. Since $f(a) = b$, we have that $f(u) = b$. Let L be the line whose intersection with U is given by $v = b$. Hence $L \cap U \subset M_0$. Let L be given by $ax + by + c = 0$ where x, y are the cartesian coordinates on $\mathbb{C}^2$. Then $ax + by + c$ is a holomorphic function on M_0

which vanishes in the open set $M_0 \cap U$, so vanishes on all of M_0 as M_0 is connected, so $L \subset M_0$, which means $P \in M_0$, a contradiction. This completes the proof that T is discrete. The set T is closed in M for if T has a limit point in M not in T then it is a limit point of zeros of df/du, so df/du is identically zero in a neighbourhood of the point on M. Then the above argument will show again that $P \in M_0$, a contradiction.

Let $M \subset \mathbb{P}^2_{\mathbb{C}}$ be a closed holomorphic submanifold of dimension 1. Let the homogeneous coordinates of $\mathbb{P}^2$ be X, Y, Z. Since $\mathbb{P}^2_{\mathbb{C}}$ is compact, we have that M is compact. We can write $\mathbb{P}^2_{\mathbb{C}}$ as $\mathbb{C}^2 \cup \mathbb{C}^2 \cup \mathbb{C}^2$ and then we get that $M \cap \mathbb{C}^2$ is a closed submanifold of dimension 1 in $\mathbb{C}^2$. We will first assume that $M \cap \mathbb{C}^2$ is connected.

Let $P \in \mathbb{P}^2_{\mathbb{C}}$ be such that $P \notin M$ and let $T = \{Q \in M | \overline{PQ} \textit{ is tangent to } M \textit{ at } Q\}$. Since each $T \cap \mathbb{C}^2$ is discrete and closed, we have that T is discrete and closed. Hence T is finite by compactness of M.

Let $T = \{Q_1, \ldots, Q_d\}$. Let L be a line in $\mathbb{P}^2$ such that $P \notin L$. Let $\pi : M \longrightarrow L$ be the projection from the point P. We can choose linear coordinates on $\mathbb{P}^2$ such that $P = (0,1,0)$ and the line L is given by $Y = 0$ and $(1,0,0) \notin \pi(T)$. Therefore $(1,0,0), (0,0,1) \in L$. Also, the projection is now given by $\pi(a,b,c) = (a,c)$.

Let $U_Z \subset \mathbb{P}^2$ be the affine open subset given by $Z \neq 0$. Then the affine coordinates of U_Z are $x = X/Z$ and $y = Y/Z$. Now, the lines through P will be given by $x = t$ where $t \in \mathbb{C}$ and the projection from P will be given by $\pi(x,y) = x$. Let $M_Z = M \cap U_Z$. Now, for any point $(a,b) \in M_Z$, there exists a holomorphic function h and a positive real number r such that $y = h(x)$ in an open disc D of radius r around a, i.e., $D = \{x \in \mathbb{C} \mid |x - a| < r\}$.

Let $a \notin \pi(T) \subset L$. Suppose there are exactly m points in M over $x = a$, given by $y = b_1, \ldots, b_m$. Now, for each (a, b_i), there exists a holomorphic function h_i and a positive real number r_i such that $y = h_i(x)$ in an open disc D_i of radius r around a. Now, $\Gamma_{h_1}(D_1) \cup \ldots \cup \Gamma_{h_m}(D_m)$ is open in M_Z. Therefore $M_Z \setminus (\Gamma_{h_1}(D_1) \cup \ldots \cup \Gamma_{h_m}(D_m))$ is closed in M_Z and hence compact in M_Z (since M_Z is compact).

Let $M_Z \setminus (\Gamma_{h_1}(D_1) \cup \ldots \cup \Gamma_{h_m}(D_m))$ be denoted by N. Therefore $\pi(N)$ is compact and hence closed. We know that $a \notin \pi(N)$. Therefore there exists a small neighbourhood U of a such that $U \cap \pi(N) = \emptyset$. Therefore $\pi^{-1}(U) \subset \Gamma_{h_1}(D_1) \cup \ldots \cup \Gamma_{h_m}(D_m)$. Therefore for any $a \in U$, any point of $\pi^{-1}(a)$ will be of the form $(a, h_i(a))$ for some $i = 1, \ldots, m$. Therefore $\pi^{-1}(a)$ consists of at most m points. Also, we can choose U so small that $\pi^{-1}(U) \cap \Gamma_{h_i}(D_i) \cap \Gamma_{h_j}(D_j) = \emptyset$. Therefore for any $a \in U$, $\pi^{-1}(a)$ consists of exactly m points. Therefore the cardinality of the fibre of $\pi : M_Z \longrightarrow L$

is locally constant on $L - \pi(T)$. But $L - \pi(T) = \mathbb{P}^1-$ finite set $= S^2-$ finite set and therefore is connected. Therefore the cardinality of the fibre on $L - \pi(T)$ is constant, say m.

Around a small enough disk with center at any $a \in L - \pi(T)$, we therefore have well defined holomorphic functions $s_1(x), s_2(x), \ldots, s_m(x)$, which are elementary symmetric functions of the $h_i(x)$, that is,

$$y^m - s_1(x)y^{m-1} + \ldots + (-1)^m s_m(x) = \prod_{i=1}^{m}(y - h_i(x))$$

As the ordering of the h_i does not matter in the definition of s_i, these are well defined functions on $L - \pi(T)$. Let $F(x, y) = y^m - s_1(x)y^{m-1} + \ldots + (-1)^m s_m(x)$, which is a polynomial in y with coefficients which are holomorphic functions on $L - \pi(T)$.

We will next examine the behaviour of the $s_i(x)$ at points of $\pi(T)$ and at $x = \infty$. Consider a point $a \in \pi(T)$, and let $D - \{a\}$ be a punctured disk around a which does not contain any other point of $\pi(T)$. For any $x \in D - \{a\}$, there are m values of y which satisfy $F(x, y) = 0$. We claim that all these m values are bounded by a constant c, that is, $|y| \leq c$. For if not, there exists a sequence $a_n \to a$ in $D - \{a\}$, and for each a_n a root b_n of $F(a_n, y) = 0$, such that $|b_n| \to \infty$. In $\mathbb{P}^2$, we have $(a_n, b_n, 1) = (a_n/b_n, 1, 1/b_n)$, and these points have $P = (0, 1, 0)$ as their limit in $\mathbb{P}^2$. This contradicts the assumption that $P \notin M$. Therefore each of the m locally defined functions h_j is bounded by c in any small disk in $D - \{a\}$, so their elementary symmetric combinations $s_i(x)$ are bounded around each point a of $\pi(T)$, so by Riemann removable singularity theorem, the s_i extend to entire functions on L.

We next examine the behaviour of s_i at the point $x = \infty$ on the line L. By the choice of coordinates we have that the point $x = \infty$ is the point $Q = (1, 0, 0) \in \mathbb{P}^2$. Also, we have that $(1, 0, 0) \notin \pi(T)$. Consider the affine open set U_X of $\mathbb{P}^2$ given by $X \neq 0$. Let the affine coordinates on U_Z be denoted by $u = \frac{Y}{X}$ and $w = \frac{Z}{X}$. Therefore $u = \frac{y}{x}$ and $w = \frac{1}{x}$. Now, the line joining $(0, 1, 0)$ and $(1, 0, 0)$ is given by $w = \frac{Z}{X} = 0$. Each of the m points in the fibre over Q is of the form $(u, w) = (c_i, 0)$. Let the local description of the manifold around $(c_i, 0)$ be given by the holomorphic function $u = g_i(w)$. Using the same argument as was used in the affine open neighbourhood U_Z, we get that g_i is bounded in a neighbourhood of $w = 0$ because it take the values in a neighbourhood of c_i. Let $c = max\{|g_i(0)|\}$. If we substitute $u = \frac{y}{x}$, we get that $y = xg_i(w)$. Therefore $|y| \sim c|x|$ as $x \to \infty$ $(w \to 0)$. But $y = h_i(x)$. Therefore each

$h_i(x) \sim c|x|$ as $x \to \infty$. Therefore we have that $s_i(x) \sim x^i c'$ as $x \to \infty$ for some constant c'.

We know that if $s(x)$ is an entire function on $\mathbb{C}$ and if there exists a constant c and an integer $r \geq 0$ such that $|s(x)| < c|x|^r$, then $s(x) \in \mathbb{C}[x]$. Therefore we get that each of the $s_i(x)$ are polynomials in x. In particular, $F(x,y) = y^m - s_1(x)y^{m-1} + \ldots + (-1)^m s_m(x) \in \mathbb{C}[x,y]$. Therefore we get that $M_Z = M \cap U_Z = \mathcal{Z}(F(x,y))$ is the affine variety defined by $F(x,y)$.

Let $\tilde{F}(X,Y,Z)$ be the homogenization of F with respect to Z. Then we have that $M \subset \mathcal{Z}(\tilde{F})$. We can factor the highest power of Z out of $\tilde{F}$ and therefore assume that Z does not divide $\tilde{F}$. Then we get that $M = \mathcal{Z}(\tilde{F}) \subset \mathbb{P}^2$.

If $M \cap \mathbb{C}^2$ were disconnected, by the above proof each connected component will give rise to a projective curve. But by Bezout's theorem, these curves will intersect. A point of intersection of two irreducible components will be singular, which contradicts the assumption that M is a holomorphic submanifold of $\mathbb{P}^2$. Hence there is no loss of generality in our earlier assumption that $M \cap \mathbb{C}^2$ is connected. In particular, the above argument shows that $\tilde{F}(X,Y,Z)$ is irreducible.

This completes the proof of the theorem.

We now state and prove a theorem which shows that any zariski closed subset in $\mathbb{P}^2$ is connected in the euclidean topology.

Theorem 6.5.2. *If $F(X,Y,Z) \in \mathbb{C}[X,Y,Z]$ is a homogeneous polynomial, then the corresponding locus $\mathcal{Z}(F(X,Y,Z))$ is connected in $\mathbb{P}^2_{\mathbb{C}}$ in the euclidean topology.*

Proof. Let $\mathcal{Z}(F(X,Y,Z))$ be denoted by V. We begin by proving that there are no isolated points on V. Consider any point $P = (a_0, a_1, a_2) \in V$. Suppose $a_2 \neq 0$. We now consider the affine open subset U_Z of $\mathbb{P}^2$ given by $Z \neq 0$. Let $x = \frac{X}{Z}$ and $y = \frac{y}{Z}$. Let the point $(a_0/a_2, a_1/a_2) \in \mathbb{C}^2$ be denoted by P itself. Let $a_0/a_2 = a$ and $a_1/a_2 = b$. Let the dehomogenization of $F(X,Y,Z)$ be denoted by $f(x,y)$. Therefore we have that $f(a,b) = 0$.

Let ϵ be so chosen that in the open disc $D = \{y \in \mathbb{C} \mid |y-b| < \epsilon\}$ there exists only one root b of $f(a,y)$. Suppose this root occurs with a multiplicity r. Then we have that $\frac{1}{2\pi i}\int_{|y-b|=\epsilon} \frac{f_y(a,y)}{f(a,y)}dy = r$ where $f_y(a,y) = \frac{\partial f}{\partial y}(a,y)$. Now, we know that $\frac{1}{2\pi i}\int_{|y-b|=\epsilon} \frac{f_y(x,y)}{f(x,y)}dy$ is a continuous function of x. But this integral can take only positive integral values. Therefore in a small neighbourhood $|x-a| < \delta$, we get that $\frac{1}{2\pi i}\int_{|y-b|=\epsilon} \frac{f_y(x,y)}{f(x,y)}dy = r$. Therefore we get that for any x in the δ neighbourhood of

a, there exists a root y of $f(x,y)=0$. Hence the point (a,b) is not isolated. We now state and prove a lemma which is a special case of the theorem but will be used in the proof of the theorem.

Lemma 6.5.3. *Suppose $f(x,y) \in \mathbb{C}[x,y]$ is irreducible then $\mathcal{Z}(f) \subset \mathbb{C}^2$ is connected.*

Proof. By a linear change of coordinates (Noether normalization), we can assume that $f(x,y) = y^n + a_1(x)y^{n-1} + \ldots + a_n(x)$ where $a_i(x) \in \mathbb{C}[x]$ are polynomials. Let T be the set of points (x,y) where $f=0$ and $\frac{\partial f}{\partial y} = 0$. We claim that the set T is finite. First we recall some facts about discriminants.

If $f, g \in A[y]$ are two polynomials with coefficients in a ring A, their resultant $Res_y(f,g) \in A$ is an element of A. If $\phi : A \to B$ is a ring homomorphism, and if $\phi : A[y] \to B[y]$ again denotes the induced homomorphism, then $\phi(Res_y(f,g)) = Res_y(\phi(f), \phi(g))$, as the resultant is universally given as the determinant of a certain matrix in the coefficients of f and g. The discriminant of $f \in A[y]$ is the element $\mathcal{D}_y(f) \in A$ defined by $\mathcal{D}_y(f) = Res_y(f, df/dy)$. A monic polynomial $f(y) \in A[y]$, where A is a UFD, has repeated factors in its unique factorization into irreducibles in $A[y]$ if and only if $\mathcal{D}_y(f) = 0 \in A$.

Applying the above to the ring $A = \mathbb{C}[x]$ which is a UFD, we get $A[y] = \mathbb{C}[x,y]$. For $f(x,y) \in \mathbb{C}[x,y]$ as above, we get $\mathcal{D}_y(f(x,y)) \in \mathbb{C}[x]$. Putting $x = a$ defines a homomorphism $\phi : \mathbb{C}[x] \to \mathbb{C}$, and $\mathbb{C}$ is also a UFD. By the above, $(\mathcal{D}_y(f))(a) = \mathcal{D}_y(f(a,y)) \in \mathbb{C}$. So the polynomial f and $\frac{\partial f}{\partial y}$ have a common zero at $x = a$, if and only if $\mathcal{D}_y(f)$ vanishes at $x = a$. Since f is irreducible, the discriminant polynomial $\mathcal{D}_y(f)$ is not the zero polynomial and hence has finitely many roots. Therefore there are only finitely many points a where $f(a,y)$ and $\frac{\partial f}{\partial y}(a,y)$ have a common zero. Also, as f is monic in y of degree n, at each zero a of $\mathcal{D}_y(f)$, $f(a,y)$ is a non zero polynomial having at most n roots. Therefore T is a finite set. Let $T = \{Q_1, \ldots, Q_d\}$.

Let the line $y = 0$ be denoted by L and let $\pi : \mathbb{C}^2 \longrightarrow L$ be the projection onto this line given by $\pi(x,y) = x$. As the roots of $\mathcal{D}_y(f)$ are contained in $\pi(T)$, for any $a \in L - \pi(T)$ there exist exactly n distinct points (a, b_i) of $\mathcal{Z}(f)$ which lie over a. By implicit function theorem, there exists in a small disk U in $L - \pi(T)$ with center a holomorphic functions $h_i(x)$ such that $h_i(a) = b_i$ and $f(x, h_i(x)) = 0$, that is, $\pi^{-1}(U) \cap \mathcal{Z}(f)$ is the disjoint union of the graphs of h_i. Let $W = \mathcal{Z}(f) - \pi^{-1}\pi(T)$, and let $p = \pi|_W : W \to L - \pi(T)$ be the projection. The above shows that the open disk U around a is evenly covered, and $p : W \to L - \pi(T)$ is therefore a covering projection of degree n.

Note that by the definition of the $h_i(x)$, we have $f(x,y) = \prod_i (y - h_i(x))$ for all $x \in L - \pi(T)$.

Suppose that $\mathcal{Z}(f)$ is not connected. Suppose $\mathcal{Z}(f) = M_1 \cup M_2$ where M_1 is a connected component and M_2 is its complement. The projections $p : M_1 \to L - \pi(T)$ and $p : M_2 \to L - \pi(T)$ are again covering projections. As $L - \pi(T)$ is connected, these have constant degrees, say n_1 and n_2, so that $n_1 + n_2 = n$. Around each $a \in L - \pi(T)$, the set of the n holomorphic functions h_i gets partitioned into two subsets: the set of n_1 holomorphic functions h'_j which correspond to points in M_1 and the set of remaining n_2 holomorphic functions h''_2 which correspond to points in M_2. Thus, the products $f'(x,y) = \prod_j (y - h'_j(x))$ and $f''(x,y) = \prod_k (y - h''_k(x))$ are well defined over $x \in L - \pi(T)$, and

$$f(x,y) = f'(x,y)f''(x,y).$$

We now claim that $f'(x,y)$ and $f''(x,y)$ are elements of $\mathbb{C}[x,y]$. Note that f' is a polynomial in y with coefficients $s'_i(x)$ which are elementary symmetric polynomials in the $h'_j(x)$, and similarly, f'' is a polynomial in y with coefficients $s''_i(x)$ which are elementary symmetric polynomials in the $h''_k(x)$. As the $h_i(x)$ are roots of $f(x,y) = y^n + a_1(x)y^{n-1} + \ldots + a_n(x)$, and as the coefficients $a_i(x)$ are polynomials, the $h_i(x)$ are bounded around each point of $\pi(T)$ and have polynomial growth at $x = \infty$. (This follows from the basic estimate that if $y^n + b_1 y^{n-1} + \ldots + b_n \in \mathbb{C}[y]$ is any monic polynomial, then its roots are bounded by $max\{1, \sum_i |b_i|\}$.) It follows that the elementary symmetric polynomials $s'_i(x)$ and $s''_i(x)$ are bounded around each point of $\pi(T)$, and have polynomial growth at $x = \infty$. So by the Riemann removable singularity theorem, these are entire functions.

Any entire function with polynomial growth at ∞ is itself a polynomial. Hence the s'_i and the s''_i are themselves elements of $\mathbb{C}[x]$. So $f;(x,y)$ and $f''(x,y)$ are in $\mathbb{C}[x,y]$, and $f = f'f''$. Thus we get a factorization of $f(x,y)$ in $\mathbb{C}[x,y]$, contradicting its irreducibility.

This shows that the assumption that V is not connected must be false. This proves the lemma that an irreducible affine curve is connected in the euclidean topology on $\mathbb{C}^2$. $\square$

We now return to the projective case.

Suppose that the homogeneous polynomial $F(X, Y, Z)$ is irreducible. Therefore, $f(x, y) = F(\frac{X}{Z}, \frac{y}{Z}, 1)$ is irreducible. Therefore we have that $\mathcal{Z}(F) \cap \mathbb{C}^2$ is connected. We have that $\mathcal{Z}(F) \cap \mathcal{Z}(Z)$ is either finite or equal to $\mathcal{Z}(Z)$. If it is equal to $\mathcal{Z}(Z)$, then since F is irreducible we get that $F(X, Y, Z) = Z$ and hence $\mathcal{Z}(F)$ is connected. Otherwise $\mathcal{Z}(F) \cap \mathcal{Z}(Z)$ is a finite set of points. We know that these points are not isolated and hence $\mathcal{Z}(F)$ is connected.

Now suppose F is not irreducible. Suppose $F = F_1^{r_1} F_2^{r_2} \ldots F_s^{r_s}$ where each F_i is irreducible. Then $\mathcal{Z}(F) = \mathcal{Z}(F_1) \cap \ldots \cap \mathcal{Z}(F_s)$. But we know that $\mathcal{Z}(F_i)$ is connected for each $i = 1, \ldots, s$. Also, for any $i, j \in \{1, \ldots, s\}$, we know from Bezout's Theorem that $\mathcal{Z}(F_i) \cap \mathcal{Z}(F_j) \neq \emptyset$. Therefore we get that $\mathcal{Z}(F)$ is connected. $\square$

Corollary 6.5.4. *If $M \subset \mathbb{P}^2$ is a one dimensional holomorphic submanifold, then M is connected.*

Proof. We know from Chow's theorem that M is of the form $\mathcal{Z}(F)$ for some homogeneous $F \in \mathbb{C}[X, Y, Z]$. But from the above theorem we know that $\mathcal{Z}(F)$ is connected. $\square$

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