Fourier Analysis in Number Fields

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This is to certify that this thesis entitled “Fourier Analysis in Number Fields” submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents work carried out by Akshaa Vatwani under the supervision of Dipendra Prasad.

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To Snoopy and Honey

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Abstract

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by Akshaa Vatwani

In this thesis we give an exposition of John Tate's doctoral dissertation titled 'Fourier Analysis in Number Fields and Hecke's Zeta-Functions'. In this dissertation, Tate proved the analytic continuation and functional equation for Hecke's ζ-function over a number field k using what is now known as harmonic analysis over adéles. In his work he first examines the local ζ-function and then uses adéles and idéles to include in a symmetric way all the completions of the field into a single structure, so as to examine the global ζ-function.

We explain required prerequisites and expand upon ideas used in Tate's thesis to give a comprehensive view of Tate's work.

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Chapter 1

Introduction

1.1 Some Background

Early in the 20th century, Hecke gave an analytic continuation and a simple functional equation for the Dedekind ζ -function over the whole plane. He soon realised that his method would work for a ζ-function formed with a new type of ideal character. He finally proved that these 'Hecke' ζ-functions satisfy the same type of functional equation as the Dedekind ζ-function. Hecke's proof used methods only up to a level of mostly elementary complex analysis , but involved long and drawn out computations. Matchett, a student of Artin's, made an attempt before Tate to use idèles and adèles to redefine classical ζ-functions and interpret Hecke characters. But for proving the functional equation, she followed the method of Hecke.

Tate provided a more elegant proof of the functional equation of the Hecke L-series by using Fourier analysis on the adèles and idèles and employing a reformulation of the Grössencharakter in terms of a character on the idèles. Tate's work can be viewed as a reformulation of Hecke's work. Even though what was done by Tate was not new and he was not the first to work on adèles and idèles, he was one of the first to do what is now called harmonic analysis over the adèles and idèles. Tate's work presently is understood as the $GL(1)$ case of automorphic forms. The techniques used in Tate's thesis have had far reaching consequences. Harmonic analysis is now an important area of number theory. The mathematics in Tate's thesis is a foundation stone today for understanding many more advanced concepts in mathematics. One such example is the Langland's program - a web of conjectures linking together many areas of mathematics. Thus though Tate's result was not new, his novel way of exploring the problem has opened doors to many other areas of mathematics today and is one of the reasons why Tate's thesis is considered an invaluable contribution to number theory.

1.2 Locally compact groups and unitary characters

A topological group is a group G together with a topology such that the group operation and the inversion map given as,

$$\begin{aligned} G \times G &\to G \\ (g,h) &\mapsto gh \end{aligned}$$

and

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1}, \end{aligned}$$

are continuous maps on $G \times G$ (with product topology) and G respectively.

In a similar manner, topological field is a field k together with a topology such that addition and multiplication are continuous functions on $k \times k$.

By a neighbourhood, we shall mean a set with an interior, that is a set that contains an open set inside it. Topological groups have many interesting properties. We will state and prove one such property which is of concern to us.

Proposition 1.2.1. *Every neighbourhood U of the identity contains a neighbourhood V of the identity such that $VV(=\{xy : x, y \in V\}) \subseteq U$.*

Proof. We can assume that U is open. The continuous map $\phi : U \times U \to G$ defined by the group operation is continuous, so $\phi^{-1}(U)$ must be an open set inside $U \times U$ containing (e, e), where e is the identity of G. By definition of product topology, there exist sets V_1 and V_2 such that $(e, e) \in V_1 \times V_2 \subseteq U \times U$. Putting $V = V_1 \cap V_2$ proves the proposition. $\square$

Another interesting property is that in a topological group, the concepts of convergent sequence and Cauchy sequence make sense in a very straightforward way.

Definition 1.2.2. *A sequence* $\{g_n\}_{n\in\mathbb{Z}^+}$ *of elements of* G *is said to converge to* g *if given any open neighbourhood* U *of the identity* e*, we can find an integer* N *such that* $g_n g^{-1} \in U$ *for all* $n \geq N$

Definition 1.2.3. *A sequence* $\{g_n\}_{n\in\mathbb{Z}^+}$ *of elements of* G *is said to be Cauchy if given any open neighbourhood* U *of the identity* e*, we can find an integer* N *such that* $g_n g_m^{-1} \in U$ *for all* $n, m \geq N$*.*

Let us now look at a particular type of topological group.

Definition 1.2.4. *A topological group* G *that is both Hausdorff and locally compact (every point admits a compact neighbourhood) is called a locally compact group.*

Proposition 1.2.5. (proposition 1.6, §1.1, [9]) *Let* G *be a Hausdorff topological group. Then a subgroup* H *of* G *that is locally compact (in the subspace topology) is moreover closed.*

Proof. Let K be a compact neighbourhood of the identity e in H. Since H is Hausdorff, K must be closed in H. Hence there exists a closed neighbourhood U of e in G such that $K = U \cap H$. $U \cap H$ must be closed since $U \cap H$ is compact in H, and there by in G as well. By proposition 1.2.1, there exists a neighbourhood V of e in G such that $VV \subseteq U$. We need to show that $\overline{H} \subseteq H$. As $\overline{H}$ is a subgroup of G, every neighbourhood of x^{-1} must intersect H, if $x \in \overline{H}$. In particular we have some element y contained in $Vx^{-1} \cap H$. If we can prove that yx lies in H, then $y \in H$ will imply that x must also lie in H.

Let us now prove that $yx \in U \cap H$. As $U \cap H$ is closed, it is enough to show that every neighbourhood W of yx intersects $U \cap H$. Now $y^{-1}W \cap xV$ is a neighbourhood of x and x lies in $\overline{H}$, hence there exists an element $z \in y^{-1}W \cap xV \cap H$ giving us: $yz \in W \cap H$, $y \in Vx^{-1}$ and $z \in xV$. This means that $yz \in Vx^{-1}xV = VV \subseteq U$. Therefore $W \cap (U \cap H)$ is non empty, proving the result. □

When we talk of an isomorphism or homomorphism between two topological groups, then we not only want the map to respect algebraic structure but also topological structure. An isomorphism between topological groups is understood to be an algebraic isomorphism which is bi-continuous. When dealing with homomorphisms, we talk about continuous homomorphisms. One important class of continuous homomorphisms is the class of continuous homomorphisms from G to the multiplicative group S^1 (or $\mathbb{C}^\times$). Such maps are called unitary characters (or characters). The word unitary obviously arises because these maps take elements of G to elements of absolute value 1. The set of unitary characters of a group G is denoted by $\hat{G}$. It forms a multiplicative group and is also called the Pontryagin dual of G. We define a topology on $\hat{G}$ as follows. Let B be a compact subset of G, and let U be a neighbourhood in S^1. Then we define the subset $W(B,U)$ of $\hat{G}$ by

$$W(B,U) = \{\chi \in \hat{G} : \chi(B) \subseteq U\}$$

These sets act as a sub-base to determine a topology on $\hat{G}$ known as the compact open topology. Since we are dealing with continuous functions, this topology is the same as the topology of compact convergence i.e. a sequence f_n of functions converges to the function f in this topology if and only if for each compact subspace B of G, the sequence of restricted functions $f_{n|B}$ converges uniformly to $f_{|B}$. Using this, one can prove that $\hat{G}$ is in fact a topological group with respect to the compact open topology.

In order to examine dual groups in more detail, we define some key subsets of S^1.

Definition 1.2.6. *Consider the map $\phi : x \mapsto e^{2\pi i x}$ from the reals to S^1. For ϵ real and contained in $(0,1]$, we define $N(\epsilon) \subseteq S^1$ as the image of the symmetric open set $(-\epsilon/3, \epsilon/3)$ under ϕ.*

We are now in a position to prove the following important results.

Theorem 1.2.7. (proposition 3.2(iv), §3.1, [9]) *If G is compact, then $\hat{G}$ is discrete.*

Proof. Given any non-trivial unitary character χ of G, $\chi(G)$ is a subgroup of S^1. It cannot be contained in any set of the form $N(\epsilon)$, $\epsilon \in (0,1]$, because such a set is never closed under multiplication and inversion. As G is compact, $W(G, N(\epsilon))$ is an open set of the character group, but it cannot contain any unitary character other than the trivial character. Thus the singleton set $\{\chi_{triv}\}$ is open in the group of unitary characters, proving that the $\hat{G}$ is discrete. □

Theorem 1.2.8. (proposition 3.2(iii), §3.1, [9]) *If G is discrete, then $\hat{G}$ is compact.*

Proof. If G is discrete, then any homomorphism taking G into S^1 is continuous and thereby a unitary character, giving $\hat{G} = \mathrm{Hom}(G, S^1)$. Also, the compact open topology on $\hat{G}$ reduces to the topology of pointwise convergence because the compact sets in the discrete group G are precisely the finite sets. However, with respect to this topology, $\mathrm{Hom}(G, S^1)$ is a closed subset of the space of all maps from G to S^1, which itself is compact. Thus $\hat{G}$ is a closed subset of a compact space, which means that it is compact. □

1.3 Fourier transform and Pontryagin Duality

For a topological space, the sets contained in the σ-algebra generated by the open sets of that space are called Borel sets. A positive measure defined on the Borel sets of a locally compact Hausdorff space X is called a Borel measure. Let μ be a Borel measure on a locally compact Hausdorff space X. Let E denote a general Borel set. Then μ is said to be outer regular on E if

$$\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}.$$

μ is said to be inner regular on E if

$$\mu(E) = \sup\{\mu(B) : B \subseteq E, B \text{ compact}\}.$$

A Borel measure that is finite on compact sets, outer regular on Borel sets and inner regular on open sets is called a Radon measure. A non-zero Radon measure μ that is both left and right translation invariant is called a Haar measure. This means $\mu(xE) = \mu(E) = \mu(Ex)$, for all $x \in X$ and all Borel sets E of X. We have the following fundamental theorem which we will accept and use without giving the proof.

Theorem 1.3.1. (theorem 1.8, §1.2, [9]) *Let G be a locally compact group. Then G admits a Haar measure. This measure is unique up to multiplication by a scalar.*

In order to define the Fourier transform and state the Fourier inversion formula, we must first introduce functions of a certain type.

Definition 1.3.2. (definition, pg.92, §3.2, [9]) *Let G be a locally compact topological group. Then a Haar measurable function $\phi : G \to \mathbb{C}$ in $L_\infty(G)$ is said to be of positive type if for any $f \in \mathscr{C}_c(G)$ the following inequality holds:*

$$\int \int \phi(s^{-1}t) f(s) ds \overline{f(t)} dt \geq 0.$$

As f has compact support K and the Haar measure is finite on compact sets, the the above integral is bounded by $\|\phi\|_\infty(sup|f|.\mu(K))^2$.

Definition 1.3.3. (definition, pg.102, §3.3, [9]) *Let $f \in L^1(G)$. We define $\hat{f} : \hat{G} \to \mathbb{C}$, the Fourier transform of f, by the formula*

$$\hat{f}(\chi) = \int_G f(y)\overline{\chi}(y)dy$$

for $\chi \in \hat{G}$.

Let $V(G)$ be defined as the complex span of functions of positive type. Let $\mathfrak{B}_1(G) = V(G) \cap L^1(G)$. Then we can define the Fourier inversion formula as follows.

Theorem 1.3.4. (theorem 3.9, §3.3, [9]) *There exists a Haar measure $d\chi$ on $\hat{G}$ such that for all $f \in V^1(G)$,*

$$f(y) = \int_{\hat{G}} \hat{f}(\chi)\chi(y)d\chi.$$

The measure $d\chi$ of this theorem is a measure on $\hat{G}$ which corresponds to the measure dy on G and is called the dual measure of dy.

Just as we took the dual of G, one can again consider the dual of $\hat{G}$ denoted by $\hat{\hat{G}}$. Then we have the map

$$\alpha : G \to \hat{\hat{G}},$$

where $\alpha(y)(\chi) = \chi(y)$. Thus $\alpha(y)$ is a character of $\hat{G}$.

Theorem 1.3.5. (theorem 3.2,§3.4, [9]) *G and $\hat{G}$ are mutually dual, with the map $\alpha : G \to \hat{\hat{G}}$ as an isomorphism of topological groups.*

Proof. We sketch briefly the main steps of the proof. The first step is to show that the map α is injective, that is, $\hat{G}$ separates points in G. For this it is sufficient to show that for an $z \neq e$ in G, there exists a unitary character χ such that $\chi(z) \neq 1$. Then if we define $L_z f(x) = f(zx)$, we get $\hat{f} = \hat{L_z f}$, for all $f \in L_1(G)$. Using the inversion formula gives $f = L_z f$ for all $f \in \mathfrak{B}_1(G)$. As G is Hausdorff, we can find an open neighbourhood U of the identity such that $U \cap (zU)$ is an empty set. by an application of Urysohn's lemma (see [7]) , there exists a non-zero continuous function of positive type with support contained in U. But due to disjointness of U and zU, it is impossible that $L_z f = f$ for such a function f. This proves that there is indeed a unitary character of the required type and α is injective. Next, one defines a topology on the double dual of G as follows. Let $\hat{K}$ be a compact neighborhood of the identity character in $\hat{G}$ and V be an open neighborhood of the identity in S^1. Consider sets of the form

$$W(\hat{K}, V) = \{\phi \in \hat{\hat{G}} : \phi(\hat{K}) \subseteq V\}.$$

Such subsets and their translates constitute a subbasis for the topology on $\hat{\hat{G}}$. We can consider those elements of $W(\hat{K}, V)$ which arise from the elements of G through the map α. They are given by

$$W(\hat{K}, V) \cap \alpha(G).$$

From this we construct a subset of G given by

$$W_G(\hat{K}, V) = \{y \in G : \alpha(y)(\chi) \in V \text{ for all } \chi \in \hat{K}\}.$$

This construction gives us the identity

$$\alpha(W_G(\hat{K}, V)) = W(\hat{K}, V) \cap \alpha(G).$$

This shows that α is a homeomorphism onto its image. The final step is to prove that the image of α is closed and moreover dense in the double dual of G. To show that it is closed, first note that a locally compact and dense subset of a Hausdorff space must be open. Now $\alpha(G)$ is locally compact as it is the homeomorphic image of the locally compact group G, and is dense in its closure in the double dual. Hence it is an open subgroup of its closure. But since every open subgroup of a topological group is also closed, $\alpha(G)$ is closed in the dual of $\hat{G}$. Now if $\alpha(G)$ is not dense in the dual of $\hat{G}$, then it is a proper closed subgroup and there exists a non-zero function $\hat{\phi} \in L_1(\hat{G})$ which vanishes on $\alpha(G)$. For $\chi_0 \in \hat{\hat{G}}$,

$$\hat{\phi}(\hat{\chi}_0) = \int \phi(\chi)\hat{\chi}_0(\chi^{-1})d\chi.$$

Since $\hat{\phi}$ vanishes on $\alpha(G)$, $\int \phi(\chi)\chi(y^{-1})d\chi = 0$, for all $y \in G$. This means that $\phi = 0$ almost everywhere and hence $\hat{\phi} = 0$. This is a contradiction as $\hat{\phi}$ was taken to be non-zero. This shows that the map is onto and completes the proof. $\square$

1.4 Fields and valuations

Let $k_{\mathfrak{p}}$ denote the completion of an algebraic number field at a place $\mathfrak{p}$. A place is an equivalence class of valuations on the field. Ostrowski's well known theorem tells us that if $\mathfrak{p}$ is archimedean then $k_{\mathfrak{p}}$ is either $\mathbb{R}$ or $\mathbb{C}$. Then we are left with the case $\mathfrak{p}$ discrete, which gives us $k_{\mathfrak{p}}$ $\mathfrak{p}$-adic. In this case $k_{\mathfrak{p}} = (\pi)$ contains a ring of integers $\mathcal{O}_{\mathfrak{p}}$ having a single prime ideal $\mathfrak{p}$, this is the set of all non-units of $\mathcal{O}_{\mathfrak{p}}$ and hence the maximal ideal. The field $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}$ is finite, having say $\mathcal{N}\mathfrak{p}$ elements. We have

Lemma 1.4.1. (lemma, §2.7, [2]) *The ring of integers $\mathcal{O}_{\mathfrak{p}}$ can be written as precisely the set of*

$$\alpha = \sum_{j=0}^{\infty} a_j \pi^j,$$

where the a_j run independently through some set Σ of representatives of $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}$ in $\mathcal{O}_{\mathfrak{p}}$.

Theorem 1.4.2. (theorem, §2.7, [2]) *For $\mathfrak{p}$ discrete, $\mathcal{O}_{\mathfrak{p}}$ is compact with respect to the topology arising from the absolute value.*

Proof. Let O_λ, $\lambda \in \Lambda$ ba a family of open sets covering $\mathcal{O}_{\mathfrak{p}}$. To show compactness, we need a finite subcover. Suppose there is no finite subcover. As Σ is the set of representatives of $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}$, $\mathcal{O}_{\mathfrak{p}}$ is the union of the finitely many cosets $a + \mathfrak{p} = a + \pi\mathcal{O}_{\mathfrak{p}}$, with $a \in \Sigma$. As there is no finite subcover by assumption, there exists some $a_0 \in \Sigma$ such that $a_0 + \pi\mathcal{O}_{\mathfrak{p}}$ is not covered by finitely many of the O_λ. Repeat this argument for $a_0 + \pi\mathcal{O}_{\mathfrak{p}}$ in place of $\mathcal{O}_{\mathfrak{p}}$. We get that there is some a_1 such that $a_0 + a_1\pi + \pi^2\mathcal{O}_{\mathfrak{p}}$ does not have a finite subcover. Continuing this process, we get $\alpha = a_0 + a_\pi + a_2\pi^2 + \ldots$. Then $\alpha \in O_{\lambda_0}$ for some $\lambda_0 \in \Sigma$. As O_{λ_0} is open, we have $\alpha + \pi^J\mathcal{O}_{\mathfrak{p}} \subseteq O_{\lambda_0}$ for some J. this gives us a contradiction, thereby proving the theorem. $\square$

There are infinitely many equivalent valuations belonging to the place $\mathfrak{p}$, out of which we choose to work with the following:

$|\alpha|$ = ordinary absolute value if $k_{\mathfrak{p}}$ is real

$|\alpha|$ = square of ordinary absolute value if $k_{\mathfrak{p}}$ is complex

$|\alpha| = (\mathcal{N}\mathfrak{p})^{-\nu}$, where ν is the valuation (ordinal number) of α for $k_{\mathfrak{p}}$ $\mathfrak{p}$-adic.

With the topology associated with this absolute value metric, $k_{\mathfrak{p}}$ is a complete topological field.

For each of these cases, the field $k_{\mathfrak{p}}$ is locally compact. This will be evident if we prove the following stronger statement.

Theorem 1.4.3. *A subset B of $K_{\mathfrak{p}}$ is relatively compact if and only if it is bounded in absolute value.*

Proof. This statement is trivial for the case $k_{\mathfrak{p}}$ real or complex, as it is just the Heine-Borel theorem for these cases. We prove it for the case $\mathfrak{p}$ discrete. Let $B \subseteq k_{\mathfrak{p}}$ be bounded in absolute value. Then there is a large enough integer d such that all elements of B have absolute value less than $(\mathcal{N}\mathfrak{p})^d$ and are hence contained in the set $\pi^{-d}\mathcal{O}_{\mathfrak{p}}$. As $\mathcal{O}_{\mathfrak{p}}$ is compact by the previous theorem and multiplication is a homeomorphism for the topological field $k_{\mathfrak{p}}$, $\pi^{-d}\mathcal{O}_{\mathfrak{p}}$ is also compact. B is thus contained in a compact set and must be relatively compact. Conversely, if B is relatively compact, then its closure $\overline{B}$ is compact. Consider the absolute value map taking $k_{\mathfrak{p}}$ to the reals. As this map is continuous, the image of $k_{\mathfrak{p}}$ under the absolute value map must be compact, hence bounded in $\mathbb{R}$. $\square$

Chapter 2

The Local Field $k_{\mathfrak{p}}$

In this chapter, we examine the local field $k_{\mathfrak{p}}$ which is the completion of the algebraic number field k at a place $\mathfrak{p}$. There are two aspects of this field that we take into account - the additive aspect and the multiplicative aspect. For both, we try to give a definite form to the characters of the field and construct a convenient measure on the field. This chapter lays the foundation for the proof of the main theorem in the local case.

2.1 The unitary characters of $k_{\mathfrak{p}}^{+}$

Let $k_{\mathfrak{p}}^{+}$ denote the additive group of the field $k_{\mathfrak{p}}$, with ξ denoting a general element of $k_{\mathfrak{p}}^{+}$. Let us assume that $k_{\mathfrak{p}}^{+}$ has a non-trivial unitary character χ (we will see that this is always true). Then for $\eta \in k_{\mathfrak{p}}^{+}$, we define the translate of χ by the formula

$$L_{\eta}\chi(\xi) = \chi(\eta\xi).$$

Theorem 2.1.1. (lemma 2.2.1, §2.2, [13]) *The translates of χ are precisely the unitary characters of $k_{\mathfrak{p}}^{+}$. More precisely the following map is an isomorphism of topological groups.*

$$\begin{aligned} \phi : k_{\mathfrak{p}}^{+} &\longrightarrow \hat{k}_{\mathfrak{p}}^{+} \\ \eta &\longmapsto L_{\eta}\chi \end{aligned}$$

Proof. We prove the result by individually proving each of the following assertions:

(i) $L_\eta\chi$ is a unitary character of $k_{\mathfrak{p}}^+$.

$$L_\eta\chi : \xi \longmapsto \eta\xi \longmapsto \chi(\eta\xi)$$

is continuous as it is a composition of multiplication by η and the unitary character χ - both of which are continuous maps.

$$L_\eta\chi(\xi_1 + \xi_2) = \chi(\eta\xi_1 + \eta\xi_2)) = \chi(\eta\xi_1)\chi(\eta\xi_2) = L_\eta\chi(\eta\xi_1)L_\eta\chi(\eta\xi_2),$$

shows that $L_\eta\chi$ is a homomorphism.

(ii)ϕ is a homomorphism
Coming to the map ϕ, we have

$$L_{\eta_1+\eta_2}\chi(\xi) = \chi(\eta_1\xi + \eta_2\xi) = \chi(\eta_1\xi)\chi(\eta_2\xi) = L_\eta\chi(\eta_1\xi)L_\eta\chi(\eta_2\xi).$$

This gives us $\phi(\eta_1 + \eta_2) = \phi(\eta_1)\phi(\eta_2)$, proving that ϕ is a homomorphism from the additive group $k_{\mathfrak{p}}^+$ to the multiplicative group $\hat{k}_{\mathfrak{p}}^+$.

(iii)ϕ is injective
If η is contained in the kernel of ϕ then $L_\eta\chi$ must be the trivial unitary character. This means that

$$\chi(\eta\xi) = 1, \text{ for all } \xi \in k_{\mathfrak{p}}^+.$$

However as multiplication by non-zero η is an automorphism of $k_{\mathfrak{p}}^+$ and χ was assumed to be a non-trivial unitary character, η must be zero if it is contained in the kernel of ϕ. ϕ is thus an algebraic isomorphism onto a subgroup of $\hat{k}_{\mathfrak{p}}^+$

(iv)ϕ is bicontinuous
As the domain and range of ϕ are both topological groups, it is enough to consider continuity of ϕ at zero. Consider a sequence of elements η_i in the topological group $k_{\mathfrak{p}}^+$, converging to zero. We must prove that the images $L_{\eta_i}\chi$ of the η_i under ϕ converge to the trivial unitary character χ_{triv} in the character group. Recalling our notation for open sets of a character group (section 1.1), mathematically this translates into proving that given an open neighbourhood $W(B, U)$ of χ_{triv} in the character group, there exists an integer N such that $L_{\eta_i}\chi \in W(B, U)$ for $i \geq N$, that is,

$$\chi(\eta_i B) \subseteq U \text{ for } i \geq N.$$

As χ is continuous, $\chi^{-1}(U)$ is an open neighbourhood of zero and contains the open ball around zero $B_\delta(0)$ for some $\delta \in \mathbb{R}$, where

$$B_\delta(0) = \{x \in k_{\mathfrak{p}}^{+} : |x| < \delta\}.$$

Since B is compact, all its elements are bounded in absolute value by some integer M. Choosing N such that $|\eta_i| < \delta/M$ for $i \geq N$, we now find that for $i \geq N$, elements of $\eta_i B$ are bounded in absolute value by δ and hence $\eta_i B \subseteq B_\delta(0) \subseteq \chi^{-1}(U)$. This gives us

$$\chi(\eta_i B) \subseteq U, \text{ for } i \geq N$$

as required.

To prove the continuity of the inverse function of ϕ, it suffices to prove continuity at χ_{triv}. Accordingly, if $\{L_{\eta_i}\chi\}$ is a sequence converging to χ_{triv} in $Im(\phi)$, then we must show that the sequence $\{\eta_i\}$ converges to zero in $k_{\mathfrak{p}}^{+}$. Convergence of the sequence $\{L_{\eta_i}\chi\}$ to χ_{triv} implies that given an open neighbourhood of $W(B, U)$ of χ_{triv}, there exists an integer N such that $L_{\eta_i}\chi \in W(B, U)$ for $i \geq N$. In other words,

$$\text{there exists } N \in \mathbb{Z} \text{ such that } \chi(\eta_i B) \subseteq U \text{ for } i \geq N. \tag{2.1.1}$$

The above statement holds for any compact set B in $k_{\mathfrak{p}}^{+}$ and any open neighbourhood U of 1 in S^1. In order to prove our result, we consider a particular type of compact set: $B_M = \{x \in k_{\mathfrak{p}}^{+} : |x| \leq M\}$, and note that

$$\eta_i B_M = \{x \in k_{\mathfrak{p}}^{+} : |x| \leq |\eta_i| M\}.$$

Next we select the set U conveniently by first choosing $\xi \in k_{\mathfrak{p}}^{+}$ such that $\chi(\xi) \neq 1$ and then choosing an open neighbourhood U of 1 in S^1 such that $\chi(\xi) \notin U$. This is possible, as can easily be seen geometrically.

As $\chi(\xi) \notin U$ by choice and the statement 2.1 holds for the sets B_M and U chosen as above, we see that there exists N such that $\xi \notin \eta_i B_M$ for $i \geq N$. As $\eta_i B_M$ consists of precisely all those elements of $k_{\mathfrak{p}}^{+}$ which are bounded in absolute value by $|\eta_i| M$, we see that there exists N such that

$$|\xi| > |\eta_i| M \text{ for } i \geq N$$

Thus there exists an integer N such that $|\eta_i| < |\xi|/M$ for all $i \geq N$ and this entire argument holds for M arbitrarily large. Hence $\{\eta_i\}$ must converge to zero.
(v)Im(ϕ) $= \hat{k_{\mathfrak{p}}^{+}}$
From the assertions proved so far, it is evident that ϕ is a topological as well as algebraic isomorphism of the locally compact group $k_{\mathfrak{p}}^{+}$ onto a subgroup of the character group. In particular, as $\mathrm{Im}(\phi)$ is a locally compact subgroup of the Hausdorff topological group $\hat{k_{\mathfrak{p}}^{+}}$, we can use proposition 1.2.5 to conclude that $\mathrm{Im}(\phi)$ is closed in the character group. If $\mathrm{Im}(\phi)$ were a proper closed subgroup of the character group then there exists non-zero $\xi \in k_{\mathfrak{p}}^{+}$ such that the image of ξ is trivial under all unitary characters contained in $\mathrm{Im}(\phi)$. This means that

$$\chi(\eta\xi) = 1 \text{ for all } \eta \in k_{\mathfrak{p}}^{+}.$$

Since multiplication by non-zero ξ is an automorphism of $k_{\mathfrak{p}}^{+}$, this implies that χ is trivial on $k_{\mathfrak{p}}^{+}$. This is a contradiction. Hence $\mathrm{Im}(\phi)$ must be the whole of the character group.

This proves that ϕ is indeed an isomorphism between the topological group $k_{\mathfrak{p}}^{+}$ and its character group. □

At the core of the above result is the assumption that $k_{\mathfrak{p}}^{+}$ has a non-trivial unitary character χ. We construct a special non-trivial unitary character to show that this assumption is valid.

Definition 2.1.2. *The field $k_{\mathfrak{p}}$ lies above the completion of $\mathbb{Q}$ at some place p, let us denote this completion by R. R is thus defined to be simply the field of real numbers if p is archimedean and the field of p-adic numbers if p is discrete.*

We define a non-trivial continuous additive map λ of R into the group $\mathbb{R} \pmod 1$ for each case.
For the case p archimedean, $R = \mathbb{R}$, we define

$$\lambda(x) = -x \pmod 1.$$

It can be easily seen that this map is non-trivial continuous and additive .

For the case p discrete, $R = \mathbb{Q}_p$, we define λ using the following steps:

Step 1 Choose $\nu \in \mathbb{Z}$ such that $p^{\nu}x \in \mathbb{Z}_p$

Step 2 Choose $n \in \mathbb{Z}$ such that $n \equiv p^\nu x \pmod{p^\nu \mathbb{Z}_p}$

Now put $\lambda(x) = n/p^\nu \pmod 1$.

Lemma 2.1.3. *The map λ defined above for the case p discrete is a well defined map from $\mathbb{Q}_p$ to the group of real numbers modulo 1.*

Proof. The element x of $\mathbb{Q}_p$ can be written as $p^{-k}(a_0 + a_1 p + a_2 p^2 + \cdots)$ for a unique integer k. For $p^\nu x$ to be integral, we choose $\nu \geq k$. Then

$$p^\nu x = p^{\nu-k}(a_0 + a_1 p + a_2 p^2 + \ldots).$$

Choosing n congruent to $p^\nu x$ modulo $p^\nu \mathbb{Z}_p$ means that

$$n \in a_0 p^{\nu-k} + a_1 p^{\nu-k+1} + \ldots + a_{k-1} p^{\nu-1} + p^\nu \mathbb{Z}_p.$$

Then the value taken by n/p^ν is independent of ν, namely

$$n/p^\nu \in \frac{a_0 + a_1 p + a_2 p^2 + \ldots + a_{k-1} p^{k-1}}{p^k} + \mathbb{Z}.$$

Thus n/p^ν is uniquely defined modulo 1, proving the required result. □

Lemma 2.1.4. (lemma 2.2.2, §2.2, [13]) *The map λ described above for the case p discrete is a non-trivial continuous additive map of $\mathbb{Q}_p$ into the group $\mathbb{R} \pmod 1$ as required.*

Proof. To show that the map λ thus constructed satisfies the required conditions, we first show that λ is in fact determined by just the following two properties:

Property i. $\lambda(x)$ is a rational number with only a power of p in the denominator

Property ii. $\lambda(x) - x \in \mathbb{Z}_p$.

To do this, we first use i. to write $\lambda(x)$ as n/p^ν, for some integers n and ν. Then from ii., we get $\frac{n}{p^\nu} - x \in \mathbb{Z}_p$, that is, $n - p^\nu x \in p^\nu \mathbb{Z}_p$. This means that $p^\nu x = n - (n - p^\nu)$ is integral and $n \equiv p^\nu x \pmod{p^\nu}$ as stated in our initial definition of the map λ. Thus the properties i. and ii. are sufficient to determine λ and the value that satisfies i. and ii. is precisely the value given to $\lambda(x)$ by our initial definition.

From ii. we have $\lambda(x) = 0 \Leftrightarrow x \in \mathbb{Z}_p$; easily showing that λ is not a trivial map, in particular it is non-trivial on all elements outside $\mathbb{Z}_p$. It is an additive map because putting $\lambda(x+y) = \lambda(x) + \lambda(y)$ satisfies the above properties for $x + y$. Namely:

i. $\lambda(x) + \lambda(y)$ is a rational number with only a power of p in the denominator as both $\lambda(x)$ and $\lambda(y)$ are themselves of the form n/p^ν.

ii. $\lambda(x) + \lambda(y) - (x+y) = (\lambda(x) - x) + (\lambda(y) - y) \in \mathbb{Z}_p$.

Convergence of a sequence makes sense in $\mathbb{Q}_p$ under the absolute value derived from the p-adic valuation. For continuity of λ we must prove that if $\{x_n\}$ is a sequence converging to x in $\mathbb{Q}_p$ then the sequence $\lambda(x_n)$ converges to $\lambda(x)$ in the group of reals modulo 1. This means that $(x_n - x) \to 0$ must imply $\lambda(x_n - x) \to 0$. Thus it is sufficient to prove continuity at zero. Let $\{y_n\}$ be a sequence of p-adic numbers converging to zero in $\mathbb{Q}_p$. If y_n converges to 0, then so does $|y_n|$ and $\nu_p(y_n)$ diverges to ∞, where ν_p denotes the p-adic valuation. Thus there exists an integer N such that $\nu_p(y_n) \geq 0$ for all $n \geq N$. That is, $y_n \in \mathbb{Z}_p$ for all $n \geq N$. As $y \in \mathbb{Z}_p \Leftrightarrow \lambda(y) = 0$, we have $\lambda(y_n) = 0$ for all $n \geq N$. This proves that $\lambda(y_n) \to 0$ for $y_n \to 0$. $\square$

Now that we have a non-trivial continuous additive map on R into $\mathbb{R}$ (mod 1), we get a non-trivial unitary character on $k_{\mathfrak{p}}^+$ by manipulating λ using the trace map Tr from the field $k_{\mathfrak{p}}^+$ onto R. Define $\Lambda(\xi) = \lambda(Tr\xi)$. Using the fact that Tr is also a non-trivial additive map, we see that

$$\begin{array}{rcl} \chi : k_{\mathfrak{p}}^+ & \to & \mathbb{C} \\ \xi & \mapsto & e^{2\pi i \Lambda(\xi)} \end{array}$$

is our much needed non-trivial unitary character on $k_{\mathfrak{p}}^+$. We now have a definite form for each unitary character of $k_{\mathfrak{p}}^+$ as follows:

Theorem 2.1.5. (theorem 2.2.1, §2.2, [13]) *$k_{\mathfrak{p}}^+$ is naturally its own character group under the identification of $k_{\mathfrak{p}}^+$ with $\hat{k}_{\mathfrak{p}}^+$ given by $\eta \longleftrightarrow L_\eta \chi$, where $L_\eta \chi : \xi \mapsto e^{2\pi i \Lambda(\eta\xi)}$.*

Let us use the term character to denote a continuous homomorphism into the multiplicative group $\mathbb{C}^*$ - not necessarily into S^1 as is in the case of a unitary character. We now compute characters of some common groups.

Characters of $(\mathbb{R}^+, \cdot)$ **:**

Using the commonly used isomorphism $z \mapsto e^z$ between the the additive group $\mathbb{R}$ and the multiplicative group $\mathbb{R}^+$, the problem comes down to finding the characters of $(\mathbb{R}, +)$. Let $\tilde{c}$ be a character of $(\mathbb{R}, +)$ into $\mathbb{C}^*$. Then the map $c(z) = \tilde{c}(z)/|\tilde{c}(z)|$ is a unitary character of $\mathbb{R}^+$ and we can use theorem 2.1.1 to determine c. If we let the non-trivial unitary character χ of theorem 2.1.1 be given by $\chi : z \mapsto e^{iz}$ for $(\mathbb{R}, +)$, then the unitary character c has the form $c(z) = L_y\chi(z) = e^{izy}$ for some $y \in \mathbb{R}$. Now letting $x = log_e(|\tilde{c}(z)|)/z$ (note that x is a well defined real number), we have

$$\tilde{c}(z) = |\tilde{c}(z)| \cdot c(z) = e^{zx} \cdot e^{izy} = e^{zs},$$

where s is the complex number $x + iy$. Thus the characters of $(\mathbb{R}, +)$ have the form $z \mapsto e^{zs}, s \in \mathbb{C}$. Shifting from $\mathbb{R}$ to $\mathbb{R}^+$ via the isomorphism ψ tells us that the characters of $(\mathbb{R}^+, \cdot)$ have the form $e^z \mapsto e^{zs}$, that is, $t \mapsto t^s, s \in \mathbb{C}$. Thus characters of the multiplicative group $\mathbb{R}^+$ are just maps raising to a complex power!

Characters of $(\mathbb{Z}, +)$:

The additive group $\mathbb{Z}$ is generated by the element 1. If we know the value of $\tilde{c}(1)$ for a character $\tilde{c}$ then $\tilde{c}(m) = \tilde{c}(1 + \ldots + 1) = (\tilde{c}(1))^m$ for any $m \in \mathbb{Z}$. Each character is uniquely determined by its value on 1 and this value can range over the whole of $\mathbb{C}^*$. The characters of $\mathbb{Z}$ thus have the form $m \mapsto z^m, z \in \mathbb{C}^*$.

Using the same argument, one sees that the unitary characters of $\mathbb{Z}$ have the form $m \mapsto z^m, z \in S^1$. The character group of $\mathbb{Z}$ is thus S^1. This serves as a good example to point out that theorem 2.1.1 does not apply to $(\mathbb{Z}, +)$ because the proof of the theorem relies on the fact that multiplication by any non-zero element of $k_{\mathfrak{p}}^{+}$ is an automorphism of $k_{\mathfrak{p}}^{+}$, or equivalently that any non-zero element is invertible.

Unitary characters of $(S^1, \cdot)$:

The map $\sigma : x \mapsto e^{2\pi ix}$ from the reals to S^1 shows that S^1 is isomorphic to the quotient group $\mathbb{R}/\mathbb{Z}$.

Let us represent elements of S^1 by $e^{2\pi ix}$ with $x \in \mathbb{R}$. This is equivalent to representing elements of S^1 by the cosets $x + \mathbb{Z}$ with $x \in \mathbb{R}$. If $\bar{c}$ denotes a unitary character of S^1, then define a map c on $\mathbb{R}$ by $c(x) = \bar{c}(e^{2\pi ix})$. It can be easily checked that this map is a continuous homomorphism of $\mathbb{R}$ into S^1 and hence a unitary character of the additive group $\mathbb{R}$. Moreover, for any integer m,

$$c(x + m) = \bar{c}(e^{2\pi i(x+m)}) = \bar{c}(e^{2\pi ix}) = c(x)$$

This means that c restricted to $\mathbb{Z}$ is trivial. Conversely, any unitary character on $\mathbb{R}$ which is trivial on $\mathbb{Z}$ gives us a well defined unitary character on S^1 given by $\bar{c}(e^{2\pi ix}) := c(x)$. Thus the unitary characters of S^1 correspond precisely to the unitary characters of $\mathbb{R}$ which are trivial over $\mathbb{Z}$.

As computed earlier, the unitary characters of $\mathbb{R}$ are given by $c : x \mapsto e^{ixy}$, $y \in \mathbb{R}$ or equivalently by $c : x \mapsto e^{2\pi ixy}$, $y \in \mathbb{R}$. For c to be trivial over $\mathbb{Z}$, it is necessary and sufficient that $c(1) = 1$, which in turn is equivalent to $e^{2\pi iy} = 1$, that is, $y \in \mathbb{Z}$. Thus the unitary characters of S^1 have the form

$$\bar{c}(e^{2\pi ix}) := c(x) = e^{2\pi ixy},\ y \in \mathbb{Z},$$

or more simply put

$$\bar{c} : t \mapsto t^y,\ y \in \mathbb{Z}.$$

In the preceding example we showed that the character group of $\mathbb{Z}$ is S^1. This example shows that the character group of S^1 is in turn given by $\mathbb{Z}$!

Coming back to our field $k_{\mathfrak{p}}^+$ with unitary characters given by $L_\eta\chi$, $\eta \in k_{\mathfrak{p}}^+$, we conclude this section with a result that will be used later. Before that we define in simple terms what is called the different of the field $k_{\mathfrak{p}}$.

Definition 2.1.6. *Let $\mathcal{O}_{\mathfrak{p}}$ denote the ring of integers of $k_{\mathfrak{p}}$, that is $\mathcal{O}_{\mathfrak{p}} = \{x \in k_{\mathfrak{p}} : |x| \leq 1\}$. For R as given in definition 2.1.2, let $\mathcal{O}_R$ denote its ring of integers. We define the following operations*

$$\begin{aligned} \mathcal{O}_{\mathfrak{p}}^{-1} &= \{x \in k_{\mathfrak{p}} : x\mathcal{O}_{\mathfrak{p}} \subseteq \mathcal{O}_{\mathfrak{p}}\} \\ \mathcal{O}_{\mathfrak{p}}^{\ *} &= \{x \in k_{\mathfrak{p}} : Tr(x\mathcal{O}_{\mathfrak{p}}) \subseteq \mathcal{O}_R\}. \end{aligned}$$

Then the different of $k_{\mathfrak{p}}$ is defined to be $\mathfrak{d} = (\mathcal{O}_{\mathfrak{p}}^*)^{-1}$.

Lemma 2.1.7. (lemma 2.2.3, §2.2, [13]) *For the case $\mathfrak{p}$ discrete, the unitary character $L_\eta\chi : \xi \mapsto e^{2\pi i\Lambda(\eta\xi)}$ corresponding to η is trivial on $\mathcal{O}_{\mathfrak{p}}$ if and only if $\eta \in \mathfrak{d}^{-1}$.*

Proof. Notice that for the case $\mathfrak{p}$ discrete, the field R is simply $\mathbb{Q}_p$ and the ring of integers $\mathcal{O}_R$ is given by the p-adic integers $\mathbb{Z}_p$. The map $L_\eta\chi : \xi \mapsto e^{2\pi i\Lambda(\eta\xi)}$ is trivial on $\mathcal{O}_{\mathfrak{p}}$ if and only if $\Lambda(\eta\xi) = \lambda(Tr(\eta\xi)) = 0$ for all $\xi \in \mathcal{O}_{\mathfrak{p}}$. This is equivalent to the condition $Tr(\eta\mathcal{O}_{\mathfrak{p}}) \subseteq \mathbb{Z}_p$, which is true if and only if $\eta \in \mathcal{O}_{\mathfrak{p}}^* = \mathfrak{d}^{-1}$. □

2.2 Measure on $k_{\mathfrak{p}}^+$

As $k_{\mathfrak{p}}^+$ is a locally compact abelian group, there exists a Haar measure μ on it which is unique up to scalar multiplication.

Lemma 2.2.1. (lemma 2.2.4, §2.2, [13]) *Consider a non-zero element α and a measurable set M of $k_{\mathfrak{p}}^{+}$. If we define $\mu_1(M) = \mu(\alpha M)$, then μ_1 is also a Haar measure.*

Proof. The multiplication map $\xi \mapsto \alpha\xi$ is an algebraic automorphism of $k_{\mathfrak{p}}^{+}$ because it is a homomorphism with $\xi \mapsto \alpha^{-1}\xi$ as the inverse map. Moreover, as $k_{\mathfrak{p}}$ is a topological group, it is also bi-continuous. It is thus a topological as well as algebraic automorphism of $k_{\mathfrak{p}}^{+}$. If we have a compact measurable set M of $k_{\mathfrak{p}}^{+}$, then the image αM under the multiplication map is also compact and the finiteness of $\mu(\alpha M)$ implies the finiteness of $\mu_1(M)$.

If we have an open measurable set M of $k_{\mathfrak{p}}^{+}$, then αM is also open and inner regularity of the Haar measure μ on open sets states that

$$\mu(\alpha M) = \sup\{\mu(K') : K' \subseteq \alpha M, K' \text{ compact}\}.$$

As the multiplication map is a topological automorphism, given a compact subset K of M, the translate αK is a compact subset of αM . Such translates thus constitute a subset of the compact subsets of αM. Conversely, for every compact subset K' of αM, the set K given by $\alpha^{-1}K'$ is a compact subset of M. Thus any compact subset of αM is an α- translate of some compact subset of M. This proves that the compact subsets of a translate of M are precisely the translates of the compact subsets of M. This gives

$$\begin{aligned} \mu(\alpha M) &= \sup\{\mu(\alpha K) : K \subseteq M, K \text{ compact}\} \\ \text{i.e. } \mu_1(M) &= \sup\{\mu_1(K) : K \subseteq M, K \text{ compact}\}, \end{aligned}$$

proving inner regularity of μ_1 on open sets. Similarly one can derive outer regularity of μ_1 on Borel sets from the corresponding property for μ, thereby ensuring that all the topological properties of a Haar measure are satisfied by μ_1. To verify translation invariance, we use the additivity of the multiplication map and the translation invariance of μ to get:

$$\mu_1(M + \xi) = \mu(\alpha M + \alpha\xi) = \mu(\alpha M) = \mu_1(M),$$

for any $\xi \in k_{\mathfrak{p}}^{+}$. μ_1 satisfies the defining properties and is thus indeed a Haar measure on $k_{\mathfrak{p}}^{+}$. □

Lemma 2.2.2. (lemma 2.2.5, §2.2, [13]) $\mu(\alpha M) = |\alpha|\mu(M)$.

Proof. As μ_1 and μ both are Haar measures on $k_{\mathfrak{p}}^+$, μ_1 must be a scalar multiple of μ. Denoting this scalar as $\varphi(\alpha)$ since it may be dependent on α, we have $\mu_1 = \varphi(\alpha)\mu$. In particular we observe that $\varphi(\alpha)$ is independent of the set M taken and that $\mu_1(M) = \varphi(\alpha)\mu(M)$ holds for any measurable set M of $k_{\mathfrak{p}}^+$. Consequently M can be chosen in a way that makes $\varphi(\alpha)$ easy to compute. Depending on $\mathfrak{p}$, we have three possibilities for $k_{\mathfrak{p}}^+$:

$k_{\mathfrak{p}}^+$ real:

Let us choose M as the set $[0, 1]$. Then $\alpha M = [0, \alpha]$ where α is a non-zero real number. The Haar measure μ on $\mathbb{R}$ is upto a scalar just the Lebesgue measure. As the length and hence Lebesgue measure of the interval $[0, \alpha]$ is $|\alpha|$ times that of the interval $[0, 1]$, we see that $\mu(\alpha M) = |\alpha|\mu(M)$. This gives $\varphi(\alpha) = |\alpha|$.

$k_{\mathfrak{p}}^+$ complex:

The Haar measure μ on $\mathbb{C}$ is upto a scalar just the ordinary Lebesgue measure on the plane. If we take M to be the square in the plane having vertices as $(0, 0), (0, i), (1, i), (1, 0)$ then αM is the square with vertices $(0, 0), (0, \alpha i), (\alpha, \alpha i), (\alpha, 0)$. It is easy to see that the scaling factor is now the square of the scaling factor in the real case. However by our convention, the absolute value in $\mathbb{C}$ is taken to be the square of the ordinary absolute value, ensuring that the result holds for the complex case.

$k_{\mathfrak{p}}^+$ $\mathfrak{p}$-adic:

Consider the ring of integers $\mathcal{O}_{\mathfrak{p}}$ of $k_{\mathfrak{p}}^+$. This is an open hence Borel set and is thus measurable. It is also compact which means that $\mu(\mathcal{O}_{\mathfrak{p}})$ is finite.

We first consider the case $\alpha \in \mathcal{O}_{\mathfrak{p}}$, so that $\alpha\mathcal{O}_{\mathfrak{p}} \subseteq \mathcal{O}_{\mathfrak{p}}$. Let $m \in \mathbb{Z}$ be the valuation of α, then α can be written as $\pi^m u$, where π is an element of valuation 1 and $u \in \mathcal{O}_{\mathfrak{p}}^\times$. Then

$$\alpha\mathcal{O}_{\mathfrak{p}} = \pi^m u\mathcal{O}_{\mathfrak{p}} = \pi^m\mathcal{O}_{\mathfrak{p}} = \mathfrak{p}^m,$$

where $\mathfrak{p}$ is the unique prime ideal of $\mathcal{O}_{\mathfrak{p}}$. As $\alpha\mathcal{O}_{\mathfrak{p}}$ is a subring of $\mathcal{O}_{\mathfrak{p}}$, we can consider the quotient ring $\mathcal{O}_{\mathfrak{p}}/\alpha\mathcal{O}_{\mathfrak{p}}$. The cardinality of this quotient ring is the number of cosets of $\alpha\mathcal{O}_{\mathfrak{p}}$ in $\mathcal{O}_{\mathfrak{p}}$. Each coset is simply a translation of $\alpha\mathcal{O}_{\mathfrak{p}}$ having the same measure as $\alpha\mathcal{O}_{\mathfrak{p}}$ because of translation invariance of the Haar measure. Thus each coset can effectively be thought of as a copy of $\alpha\mathcal{O}_{\mathfrak{p}}$ and the number of cosets gives us the number of copies of $\alpha\mathcal{O}_{\mathfrak{p}}$ required to cover $\mathcal{O}_{\mathfrak{p}}$. This number is simply

$$|\mathcal{O}_{\mathfrak{p}}/\alpha\mathcal{O}_{\mathfrak{p}}| = |\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^m| = (\mathcal{N}\mathfrak{p})^m = |\alpha|^{-1}.$$

Thus we need $|\alpha|^{-1}$ copies of $\alpha\mathcal{O}_{\mathfrak{p}}$ to cover $\mathcal{O}_{\mathfrak{p}}$, that is, $\mu(\mathcal{O}_{\mathfrak{p}}) = |\alpha|^{-1}\mu(\alpha\mathcal{O}_{\mathfrak{p}})$. This gives the required result.

If $\alpha \notin \mathcal{O}_{\mathfrak{p}}$ then α^{-1} must be contained in $\mathcal{O}_{\mathfrak{p}}$ and the previous argument that was used for $\alpha \in \mathcal{O}_{\mathfrak{p}}$ can now be used for α^{-1} instead. Moreover now $\mathcal{O}_{\mathfrak{p}}$ is a subring of the ring $\alpha\mathcal{O}_{\mathfrak{p}}$. We note that there is a one-one correspondence between cosets of $\mathcal{O}_{\mathfrak{p}}$ in $\alpha\mathcal{O}_{\mathfrak{p}}$ and cosets of $\alpha^{-1}\mathcal{O}_{\mathfrak{p}}$ in $\mathcal{O}_{\mathfrak{p}}$ given by

$$\mathcal{O}_{\mathfrak{p}} + \alpha\xi \leftrightarrow \alpha^{-1}\mathcal{O}_{\mathfrak{p}} + \xi,$$

where ξ denotes the general element of $\mathcal{O}_{\mathfrak{p}}$. Using this correspondence of cosets and the the previous argument for $\alpha^{-1} \in \mathcal{O}_{\mathfrak{p}}$ gives

$$|\alpha\mathcal{O}_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}| = |\mathcal{O}_{\mathfrak{p}}/\alpha^{-1}\mathcal{O}_{\mathfrak{p}}| = |\alpha|.$$

Hence $\mu(\alpha\mathcal{O}_{\mathfrak{p}}) = |\alpha|\mu(\mathcal{O}_{\mathfrak{p}})$ as needed. □

We have thus recorded for a general Haar measure how the measure of a set changes under the multiplication map. For the purpose of integration, our result can be written as

$$d\mu(\alpha\xi) = |\alpha| d\mu(\xi).$$

Let us now try to select a fixed Haar measure on $k_{\mathfrak{p}}^{+}$. We start with a Haar measure $d\mu_0$ on $k_{\mathfrak{p}}^{+}$. Let its dual measure on the character group be given by $d\chi_0$. As the map ϕ of theorem 2.1.1 is an isomorphism of topological groups, the dual measure $d\chi_0$ on the character group corresponds to a measure $d\mu_1$ on $k_{\mathfrak{p}}^{+}$ as follows

$$\int_M d\mu_1 := \int_{\phi(M)} d\chi_0.$$

Through an argument similar to that used to prove lemma 2.2.1, one can show that $d\mu_1$ is a Haar measure on $k_{\mathfrak{p}}^{+}$ and must hence be a scalar multiple of the measure $d\mu_0$, say $d\mu_1 = \lambda d\mu_0$. How can we ensure that the measure $d\mu_1$ corresponding to the dual measure is the same as the original measure $d\mu_0$?

For this purpose, let us now divert to the question of how the dual measure $d\chi$ changes on replacing a Haar measure $d\mu$ by a scalar multiple $d\mu' = c d\mu$. For a function $f \in \mathfrak{B}_1(k_{\mathfrak{p}}^{+})$, the Fourier transform can be taken with respect to either measure. Let us denote the Fourier transform of f with respect to the measure $d\mu$ as $\hat{f}$ and that with respect to the measure $d\mu'$ as $\tilde{f}$. Then with respect to $d\mu$, the Fourier transform and the Fourier inversion formula give us

$$\begin{aligned}\hat{f}(\chi) &= \int f(y)\bar{\chi}(y)d\mu(y)\\ f(y) &= \int \hat{f}(\chi)\chi(y)d\chi(\chi)\end{aligned}$$

Denoting the dual measure of $d\mu'$ as $d\chi'$, we have,

$$\begin{aligned}\tilde{f}(\chi) &= \int f(y)\bar{\chi}(y)d\mu'(y) = c\int f(y)\bar{\chi}(y)d\mu(y) = c\hat{f}(\chi)\\ f(y) &= \int \tilde{f}(\chi)\chi(y)d\chi'(\chi) = c\int \hat{f}(\chi)\chi(y)d\chi'(\chi)\end{aligned}$$

Equating the expressions for $f(y)$ obtained for both measures gives us

$$\int \hat{f}(\chi)\chi(y)d\chi(\chi) = c\int \hat{f}(\chi)\chi(y)d\chi'(\chi)$$

which means that $d\chi' = 1/c \cdot d\chi$. Take our fixed Haar measure to be $\sqrt{\lambda}$. We have

$$\begin{aligned}k_{\mathfrak{p}}^{+} &\longrightarrow & \hat{k}_{\mathfrak{p}}^{+} &\longrightarrow k_{\mathfrak{p}}^{+}\\ d\mu_0 &\longmapsto & d\chi_0 &\longmapsto \lambda d\mu_0\\ \sqrt{\lambda}d\mu_0 &\longmapsto & \frac{1}{\sqrt{\lambda}}d\chi_0 &\longmapsto \frac{\lambda d\mu_0}{\sqrt{\lambda}}\end{aligned}$$

Thus we have a measure that is its own dual when the character group of $k_{\mathfrak{p}}^{+}$ is identified with $k_{\mathfrak{p}}^{+}$ and it is this measure that we choose as a fixed Haar measure for $k_{\mathfrak{p}}^{+}$ henceforth. This measure is given as follows for the following cases :

ordinary Lebesgue measure on $\mathbb{R}$ if $k_{\mathfrak{p}}^{+}$ is real

twice the ordinary Lebesgue measure in the plane if $k_{\mathfrak{p}}^{+}$ is complex

the measure that gives $\mathcal{O}_{\mathfrak{p}}$ the measure $(\mathcal{N}\mathfrak{d})^{-1/2}$ if $k_{\mathfrak{p}}^{+}$ is $\mathfrak{p}$-adic.

From theorem 2.1.4, a general character of $k_{\mathfrak{p}}^{+}$ has the form $\xi \mapsto e^{2\pi i\Lambda(\eta\xi)}$ and can be identified with the element η of $k_{\mathfrak{p}}^{+}$. Using this along with the self-dual measure defined above on $k_{\mathfrak{p}}^{+}$, the inverse Fourier transform has the form of an integral on $k_{\mathfrak{p}}^{+}$ rather than on its character group. If the self dual measure is denoted as $d\mu$, then writing $d\mu(\xi)$ as $d\xi$ for convenience, the Fourier transform and the inversion formula on $k_{\mathfrak{p}}^{+}$ are then given by

$$\hat{f}(\eta) = \int f(\xi)e^{-2\pi i\Lambda(\eta\xi)}d\xi \tag{2.2.1}$$

$$\begin{aligned} f(\xi) &= \int \hat{f}(\eta)e^{2\pi i\Lambda(\eta\xi)}d\eta \qquad (2.2.2) \\ &= \int \hat{f}(\eta)e^{-2\pi i\Lambda(\eta(-\xi))}d\eta = \hat{\hat{f}}(-\xi). \end{aligned}$$

2.3 The characters of $k_{\mathfrak{p}}^{\times}$

Let $k_{\mathfrak{p}}^{\times}$ denote the multiplicative group of $k_{\mathfrak{p}}$ and α denote a general element of this group. Consider the continuous homomorphism $|\cdot| : \alpha \mapsto |\alpha|$ from $k_{\mathfrak{p}}^{\times}$ onto the multiplicative group $\mathbb{R}^{+}$ or $\mathcal{N}\mathfrak{p}^{\mathbb{Z}}$ according to whether $\mathfrak{p}$ is archimedean or discrete. The kernel of this map is given by $\mathcal{O}_{\mathfrak{p}}^{\times} = \{\alpha \in k_{\mathfrak{p}}^{\times} : |\alpha| = 1\}$. As $\mathcal{O}_{\mathfrak{p}}^{\times}$ is closed as well as bounded in absolute value, it is compact. For $\mathfrak{p}$ discrete, the image set $\mathcal{N}\mathfrak{p}^{\mathbb{Z}}$ has discrete topology. Then $\mathcal{O}_{\mathfrak{p}}^{\times}$ is open as it is the inverse image of the set $\{1\}$ which is open in $\mathcal{N}\mathfrak{p}^{\mathbb{Z}}$.

Consider a character c of $k_{\mathfrak{p}}^{\times}$ which is trivial on $\mathcal{O}_{\mathfrak{p}}^{\times}$. If two elements α_1 and α_2 of $k_{\mathfrak{p}}^{\times}$ have the same absolute value, then $\alpha_1\alpha_2^{-1} \in \mathcal{O}_{\mathfrak{p}}^{\times}$ gives us $c(\alpha_1) = c(\alpha_2)$. The value $c(\alpha)$ thus depends solely on the absolute value of α and this simplifies the form taken by c to some extent. Hence it makes sense to examine such special characters - also termed as unramified characters - before moving on to a general character of $k_{\mathfrak{p}}^{\times}$.

Lemma 2.3.1. (lemma 2.3.1, §2.3, [13]) *The unramified characters of $k_{\mathfrak{p}}^{\times}$ are of the form $c(\alpha) = |\alpha|^{s}$, $s \in \mathbb{C}$. For $\mathfrak{p}$ archimedean, s is determined by the character c whereas for $\mathfrak{p}$ discrete, s is determined modulo $2\pi i/log\mathcal{N}\mathfrak{p}$.*

Proof. Let c be an unramified character of $k_{\mathfrak{p}}^{\times}$. As seen from the above discussion, $c(\alpha)$ depends only on the absolute value of α. The set of all absolute values of elements in $k_{\mathfrak{p}}^{\times}$ is called the value group of $k_{\mathfrak{p}}^{\times}$. Define the following function on the value group: $d(|\alpha|) := c(\alpha)$. The map d is a homomorphism because

$$d(|\alpha_1||\alpha_2|) = d(|\alpha_1\alpha_2|) = c(\alpha_1\alpha_2) = c(\alpha_1)c(\alpha_2) = d(|\alpha_1|)d(|\alpha_2|).$$

Also $c(\alpha) = (d \circ |\cdot|)(\alpha)$ tells us that d is continuous. Thus d is a character on the value group of $k_{\mathfrak{p}}^{\times}$ which is given by $(\mathbb{R}^+, \cdot)$ or $(\mathcal{N}\mathfrak{p}^{\mathbb{Z}}, \cdot)$ according to whether $\mathfrak{p}$ is archimedean or discrete.

For the case $\mathfrak{p}$ archimedean, d is a character of $(\mathbb{R}^+, \cdot)$. From the computation of characters of $(\mathbb{R}^+, \cdot)$ following theorem 2.1.4, $d(|\alpha|)$ involves simply raising to a complex exponent and has the form $|\alpha| \mapsto |\alpha|^s$, $s \in \mathbb{C}$. Hence $c(\alpha) = d(|\alpha|) = |\alpha|^s$, $s \in \mathbb{C}$. It is evident that distinct complex numbers s_1 and s_2 will give distinct characters and thus s is determined by the character c.

For the case $\mathfrak{p}$ discrete, we have the following isomorphism between the value group and the additive group of integers:

$$\begin{aligned} \beta : (\mathcal{N}\mathfrak{p}^{\mathbb{Z}}, \cdot) &\rightarrow (\mathbb{Z}, +) \\ \mathcal{N}\mathfrak{p}^m &\mapsto m \end{aligned}$$

From the computation following theorem 2.1.4, we know that the characters of $\mathbb{Z}$ have the form $m \mapsto z^m$, $z \in \mathbb{C}^*$. The complex number z can be written as $r \cdot e^{i\theta}$ with $r, \theta \in \mathbb{R}$. The non-zero real numbers r and e^{θ} can be written as $\mathcal{N}\mathfrak{p}^x$ and $\mathcal{N}\mathfrak{p}^y$ for some real numbers x and y respectively (this can be done by using the logarithm map appropriately). This gives

$$z = r \cdot e^{i\theta} = \mathcal{N}\mathfrak{p}^x \cdot \mathcal{N}\mathfrak{p}^{iy} = \mathcal{N}\mathfrak{p}^s,$$

where $s = x + iy \in \mathbb{C}$. The character of $(\mathcal{N}\mathfrak{p}^{\mathbb{Z}}, \cdot)$ corresponding to the character $m \mapsto z^m$ of $\mathbb{Z}$ is hence given by $\mathcal{N}\mathfrak{p}^m \mapsto (\mathcal{N}\mathfrak{p}^s)^m$ and all characters of $(\mathcal{N}\mathfrak{p}^{\mathbb{Z}}, \cdot)$ are precisely of this form as a consequence of the isomorphism β. As $\mathcal{N}\mathfrak{p}^m$ represents the absolute value of some element $\alpha \in k_{\mathfrak{p}}^{\times}$ for $\mathfrak{p}$ discrete, a general character d of the value group is simply given by $|\alpha| \mapsto |\alpha|^s$ with $s \in \mathbb{C}$. This gives the required form for $c(\alpha)$. We note that if distinct complex numbers s_1 and s_2 give the same character, the corresponding characters must agree on $\mathcal{N}\mathfrak{p}$ (the multiplicative generator of the value group). This gives $\mathcal{N}\mathfrak{p}^{s_1} = \mathcal{N}\mathfrak{p}^{s_2}$, that is, $\mathcal{N}\mathfrak{p}^{s_1 - s_2} = 1$. Writing $\mathcal{N}\mathfrak{p}$ as $\exp(log_e \mathcal{N}\mathfrak{p})$ gives us

$$e^{(s_1 - s_2) log \mathcal{N}\mathfrak{p}} = 1.$$

Thus $(s_1 - s_2) log \mathcal{N}\mathfrak{p} \equiv 0 \pmod{2\pi i}$, giving

$$s_1 \equiv s_2 \pmod{2\pi i / log \mathcal{N}\mathfrak{p}}.$$

As each of these steps is reversible, we see that s is determined modulo $2\pi i/log\mathcal{N}\mathfrak{p}$.

We have shown that every unramified character of $k_{\mathfrak{p}}^{\times}$ is of the form $\alpha \mapsto |\alpha|^s$ with $s \in \mathbb{C}$. It is easy to see that every such map is indeed an unramified character. Thus the unramified characters of $k_{\mathfrak{p}}^{\times}$ are precisely of the required form. □

Using this result, we attempt to give a definite form to a general character of $k_{\mathfrak{p}}^{\times}$. For this we first try to write a general element of $k_{\mathfrak{p}}^{\times}$ in terms of its 'restriction' on $\mathcal{O}_{\mathfrak{p}}^{\times}$. Let us do this separately for the cases $\mathfrak{p}$ archimedean and discrete.

For $\mathfrak{p}$ archimedean, a given element element α of $k_{\mathfrak{p}}^{\times}$ can be written uniquely as $\tilde{\alpha}\rho$, with $\tilde{\alpha} \in \mathcal{O}_{\mathfrak{p}}^{\times}$ and ρ a positive real number. This can be done by taking $\rho = |\alpha|$ and $\tilde{\alpha} = \alpha/|\alpha|$. To show that this representation is unique, let $\tilde{\alpha}_1\rho_1 = \tilde{\alpha}_2\rho_2$. As $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ have absolute value 1 and ρ_1 and ρ_2 are positive real numbers, taking absolute values on both sides gives us $\rho_1 = \rho_2$. This in turn gives $\tilde{\alpha}_1 = \tilde{\alpha}_2$ and thus the representation is unique.

For $\mathfrak{p}$ discrete we first fix an element π of valuation 1. Then we will see that any element α can be written uniquely as $\tilde{\alpha}\rho$, with $\tilde{\alpha} \in \mathcal{O}_{\mathfrak{p}}^{\times}$ and ρ a power of π. If the absolute value of α is given by $\mathcal{N}\mathfrak{p}^{-m}$, then $\alpha\pi^{-m}$ has absolute value 1. Taking $\tilde{\alpha} = \alpha\pi^{-m}$ and $\rho = \pi^m$ gives us $\alpha = \tilde{\alpha}\rho$, with $\tilde{\alpha}$ and ρ as required. If $\tilde{\alpha}_1\rho_1 = \tilde{\alpha}_2\rho_2$, then since $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are units in $\mathcal{O}_{\mathfrak{p}}^{\times}$, taking absolute values gives us $|\rho_1| = |\rho_2|$. As ρ_1 and ρ_2 are simply powers of π, both must be the same power of π, that is, $\rho_1 = \rho_2$. This gives $\tilde{\alpha}_1 = \tilde{\alpha}_2$, proving that this representation for α is unique.

Consider the map $\alpha \mapsto \tilde{\alpha}$. In the archimedean case, this map is given by $\alpha \mapsto \alpha/|\alpha|$ and is easily seen to be a homomorphism. For the discrete case this map looks like $\alpha \mapsto \alpha\pi^{-\nu_{\mathfrak{p}}(\alpha)}$, where $\nu_{\mathfrak{p}}(\alpha)$ denotes the valuation of α. This is a homomorphism because

$$\widetilde{\alpha_1\alpha_2} = \alpha_1\alpha_2\pi^{-\nu_{\mathfrak{p}}(\alpha_1\alpha_2)} = \alpha_1\pi^{-\nu_{\mathfrak{p}}(\alpha_1)}\alpha_2\pi^{-\nu_{\mathfrak{p}}(\alpha_2)} = \tilde{\alpha_1}\tilde{\alpha_2}.$$

Theorem 2.3.2. (theorem 2.3.1, §2.3, [13]) *If $\tilde{\alpha}$ is as defined above, the characters of $k_{\mathfrak{p}}^{\times}$ are precisely the maps of the form $c(\alpha) = \tilde{c}(\tilde{\alpha})|\alpha|^s$, where $\tilde{c}$ is a unitary character of $\mathcal{O}_{\mathfrak{p}}^{\times}$. While s is determined as in lemma* 2.3.1*, $\tilde{c}$ is uniquely determined by the character c.*

Proof. Let c be a character of $k_{\mathfrak{p}}^{\times}$. Define $\tilde{c}$ to be the restriction of c to $\mathcal{O}_{\mathfrak{p}}^{\times}$. It can be seen that $\tilde{c}$ is a character of $\mathcal{O}_{\mathfrak{p}}^{\times}$ and consequently the image of the compact multiplicative group $\mathcal{O}_{\mathfrak{p}}^{\times}$ under $\tilde{c}$ must be a compact subgroup K of $(\mathbb{C}^*, \cdot)$. If we now take the absolute value map on K then the image of this absolute value map must be a compact subgroup of $(\mathbb{R}^+, \cdot)$. However the only compact - or even bounded - subgroup of the multiplicative group $\mathbb{R}^+$ is $\{1\}$. This can be verified by including an element other than 1 and trying to close the set under inversion and multiplication. Hence the image K of $\mathcal{O}_{\mathfrak{p}}^\times$ under $\tilde{c}$ must be a subgroup of S^1. This proves in general that a character on a compact group is in fact a unitary character and in particular that $\tilde{c}$ is a unitary character of $\mathcal{O}_{\mathfrak{p}}^\times$!

For an element α of $k_{\mathfrak{p}}^\times$, we have $\alpha = \tilde{\alpha}\rho$ from the discussion preceding this theorem. Let us now consider the map

$$c/\tilde{c} : \alpha \mapsto c(\alpha)/\tilde{c}(\tilde{\alpha}).$$

This map is well defined because $\tilde{\alpha}$ is uniquely defined for each α and is continuous as it is the quotient of two continuous maps and the denominator does not vanish. It can be readily checked that it is also a homomorphism. If $u \in \mathcal{O}_{\mathfrak{p}}^\times$, then $\tilde{u} = u$, which gives $c/\tilde{c} : u \mapsto 1$ for all $u \in \mathcal{O}_{\mathfrak{p}}^\times$. The map $c/\tilde{c}$ thus is an unramified character on $k_{\mathfrak{p}}^\times$ and must be of the form $\alpha \mapsto |\alpha|^s$, $s \in \mathbb{C}$ according to lemma 2.3.1. Hence

$$c(\alpha) = \tilde{c}(\tilde{\alpha})|\alpha|^s,$$

where $s \in \mathbb{C}$ and $\tilde{c}$ is a unitary character of $\mathcal{O}_{\mathfrak{p}}^\times$. For the converse, is easy to see that every map of this form is indeed a character of $k_{\mathfrak{p}}^\times$. As $\tilde{c}$ is simply the restriction of c to $\mathcal{O}_{\mathfrak{p}}^\times$, it is uniquely determined by the character c. $\square$

This tells us that a character $c(\alpha)$ of $k_{\mathfrak{p}}^\times$ is upto a factor of $|\alpha|^s$ simply a unitary character $\tilde{c}(\tilde{\alpha})$ of $\mathcal{O}_{\mathfrak{p}}^\times$. This simplifies the determination of characters of $k_{\mathfrak{p}}^\times$ to some extent. For this purpose, we define some terms.

As a character c of $k_{\mathfrak{p}}^\times$ has the form $c(\alpha) = \tilde{c}(\alpha)|\alpha|^s$ according to the above theorem, we see that $|c(\alpha)| = |\alpha|^\sigma$, where $\sigma = Re(s)$. As s is either uniquely determined by c or is uniquely determined modulo $2\pi i/\log \mathcal{N}\mathfrak{p}$, we see that $Re(s)$ is always uniquely determined by the character c. We call σ the exponent of c.

2.4 Equivalence classes of characters of $k_{\mathfrak{p}}^\times$

We define two characters to be equivalent if they agree on $\mathcal{O}_{\mathfrak{p}}^\times$ or equivalently if their quotient is an unramified character. Then an equivalence class C of characters consists of all characters of the form $\tilde{c}(\alpha)|\alpha|^s$, where $\tilde{c}$ is a unitary character of $\mathcal{O}_{\mathfrak{p}}^{\times}$ and is fixed for this class, whereas s varies over the complex numbers. Each equivalence class is thus represented by a unitary character of $\mathcal{O}_{\mathfrak{p}}^{\times}$. Let us now determine for different cases, the unitary characters of $\mathcal{O}_{\mathfrak{p}}^{\times}$ and subsequently the equivalence classes of characters.

Case 1. $k_{\mathfrak{p}}$ real:
For the real field, the ring of integers is simply the ring of ordinary integers. Integers which are units are 1 and -1, hence $\mathcal{O}_{\mathfrak{p}}^{\times} = \{1, -1\}$. Let $\tilde{c}$ denote a unitary character of $\{1, -1\}$. Since $\tilde{c}(-1)^2 = \tilde{c}(1)$, the image of -1 must be a square root of unity. Hence the unitary characters of $\mathcal{O}_{\mathfrak{p}}^{\times}$ can be written as:

$$\tilde{c}(\tilde{\alpha}) = \tilde{\alpha}^n,\ n = 0, 1.$$

Thus, there are two equivalence classes. In the equivalence class corresponding to the trivial map, the characters have the form

$$c(\alpha) = \tilde{\alpha}^0|\alpha|^s = |\alpha|^s.$$

As $\tilde{\alpha} = \alpha/|\alpha|$ in this case, the characters in the other equivalence class look like

$$c(\alpha) = \tilde{\alpha}^1|\alpha|^s = (\alpha/|\alpha|)|\alpha|^s = \pm|\alpha|^s,$$

where there is a plus sign if the non zero real variable α is positive and a minus sign otherwise. In summary, the two equivalence classes of characters can be denoted as $||^s$ and $\pm||^s$.

Case 2. $k_{\mathfrak{p}}$ complex:
For the complex field, $\mathcal{O}_{\mathfrak{p}}^{\times}$ is S^1- the set of elements of absolute value 1. The computation following theorem 2.1.4 tells us that the unitary characters of S^1 are given by

$$\tilde{c}(\tilde{\alpha}) = \tilde{\alpha}^n,\ n \in \mathbb{Z}.$$

Let us denote these unitary characters by c_n where n is an integer and $c_n : \tilde{\alpha} \mapsto \tilde{\alpha}^n$. If we denote a general element α of $\mathbb{C}^{\times}$ as $re^{i\theta}$ with $r > 0$, then $\tilde{\alpha} = \alpha/|\alpha| = e^{i\theta}$, giving $c_n : e^{i\theta} \mapsto e^{in\theta}$. As absolute value in $\mathbb{C}$ was chosen to be the square of the usual absolute value, the characters of the equivalence class represented by c_n have the form

$$c(\alpha) = c_n(\tilde{\alpha})|\alpha|^s = e^{in\theta}r^{2s}.$$

For convenience, the equivalence classes can be denoted by $c_n||^s$.

Case 3. $k_{\mathfrak{p}}$ **$\mathfrak{p}$-adic:**

Let us first discuss sets of the type $N_n = 1 + \mathfrak{p}^n$, with $n > 0$. A typical element of N_n has the form $1 + a\pi^n$, where π is a fixed element of valuation 1 and a is any element in the ring of integers.

As the $\mathfrak{p}$-adic absolute value satisfies the property $|a+b|_{\mathfrak{p}} = \max\{|a|_{\mathfrak{p}}, |b|_{\mathfrak{p}}\}$, if $|a|_{\mathfrak{p}} \neq |b|_{\mathfrak{p}}$, we have $|1 + a\pi^n|_{\mathfrak{p}} = 1$, that is, N_n is a subset of $\mathcal{O}_{\mathfrak{p}}^{\times}$. The identity

$$1 = (1 + x)(1 - x + x^2 - x^3 + \ldots)$$

implies that the inverse in $\mathcal{O}_{\mathfrak{p}}^{\times}$ of $1 + a\pi^n$ is given by the $\mathfrak{p}$-adic element $1 - a\pi^n + (a\pi^n)^2 - (a\pi^n)^3 + \cdots$, which is contained in N_n. There is another way to realize this. We see that an element x of the field is contained in $1 + \mathfrak{p}^n$ if and only if $|x - 1| \leq \mathcal{N}\mathfrak{p}^{-n}$. Now let x be an element of N_n. Then using the fact that x has absolute value 1,

$$|x^{-1} - 1| = |x||x^{-1} - 1| = |1 - x| \leq \mathcal{N}\mathfrak{p}^{-n},$$

which gives $x^{-1} \in N_n$. We have shown that N_n is closed under taking inverses. It can be checked that N_n is also closed under multiplication and is thus a subgroup of $\mathcal{O}_{\mathfrak{p}}^{\times}$.

As $1 + \mathfrak{p}^n$ is precisely the set $\{x : |x - 1|_{\mathfrak{p}} < \mathcal{N}\mathfrak{p}^{-n+1}\}$, we see that N_n is thus an 'open ball' around 1. By definition, given any open set containing 1, we can find some $n > 0$ such that N_n is contained in that open set.

In summary, the subgroups N_n of $\mathcal{O}_{\mathfrak{p}}^{\times}$ form a fundamental system of neighbourhoods of 1.

Coming back to the question of determining unitary characters of $\mathcal{O}_{\mathfrak{p}}^{\times}$, let $\tilde{c}$ denote such a unitary character. Choose an open neighbourhood U of 1 in S^1 which contains no subgroups of S^1 other than the trivial subgroup. One example of such an open neighbourhood is the arc running from i to $-i$ in the clockwise direction on the unit circle, excluding both endpoints. The inverse image of U under $\tilde{c}$ will be an open set containing 1 and hence we can find some $n_0 > 0$ such that $N_{n_0} \subseteq \tilde{c}^{-1}(U)$. Thus the image under $\tilde{c}$ of the subgroup N_{n_0} of $\mathcal{O}_{\mathfrak{p}}^{\times}$ is a subgroup of U and must be the trivial subgroup. $\tilde{c}$ is thus trivial on N_{n_0} and must be trivial on all the nested neighbourhoods N_n, $n > n_0$. The point to note is that for any unitary character $\tilde{c}$, we have $\tilde{c}(1+\mathfrak{p}^n) = 1$ for sufficiently large n. Selecting minimal such non-negative n and calling the ideal $\mathfrak{f} = \mathfrak{p}^n$ the conductor of $\tilde{c}$, we can interpret $\tilde{c}$ as a unitary character of the quotient group $\mathcal{O}_{\mathfrak{p}}^{\times}/1 + \mathfrak{f}$. The advantage of this interpretation is that as this quotient group is finite, each unitary character of $\mathcal{O}_{\mathfrak{p}}^{\times}$ can now be described by only a finite table of values. The equivalence classes of characters of the field are represented by the unitary characters of $\mathcal{O}_{\mathfrak{p}}^{\times}$. This is the best we can say because we cannot give any definite form to the unitary characters of $\mathcal{O}_{\mathfrak{p}}^{\times}$ for this case. Every different unitary character of $\mathcal{O}_{\mathfrak{p}}^{\times}$ gives rise to a different equivalence class. Instead of dealing with unitary characters of $\mathcal{O}_{\mathfrak{p}}^{\times}$, we will prefer to deal with characters of $k_{\mathfrak{p}}^{\times}$ by extending a unitary character $\tilde{c}$ of $\mathcal{O}_{\mathfrak{p}}^{\times}$ to a character c of $k_{\mathfrak{p}}^{\times}$ by fixing an element π of valuation 1 and giving it the value 1 under $\tilde{c}$. Then for any element $\alpha = \tilde{\alpha}\pi^{\nu}$ of $k_{\mathfrak{p}}^{\times}$, c is defined to be

$$c(\alpha) = \tilde{c}(\tilde{\alpha})\tilde{c}(\pi^{\nu}) = \tilde{c}(\tilde{\alpha}).$$

Thus the equivalence classes of characters are now represented by such characters of $k_{\mathfrak{p}}^{\times}$ itself! We give such a character c the subscript n if the corresponding 'restriction' $\tilde{c}$ has conductor $\mathfrak{p}^n$. Then the classes of characters are represented by the characters c_n and can be denoted as $c_n||^s$.

This completes our description of the characters of $k_{\mathfrak{p}}^{\times}$ in terms of the unitary characters of $\mathcal{O}_{\mathfrak{p}}^{\times}$ and our subsequent elaboration of the unitary characters of $\mathcal{O}_{\mathfrak{p}}^{\times}$ for different cases.

2.5 Measure on $k_{\mathfrak{p}}^{\times}$

Let $\mathcal{C}_c(k_{\mathfrak{p}}^{\times})$ denote the space of continuous functions on $k_{\mathfrak{p}}^{\times}$ which vanish outside a compact set. Recall that there is a one to one correspondence between Radon measures on $k_{\mathfrak{p}}^{\times}$ and non-trivial functionals on $\mathcal{C}_c(k_{\mathfrak{p}}^{\times})$, given by:

$$d_1\alpha \longleftrightarrow \Phi(g) = \int_{k_{\mathfrak{p}}^{\times}} g(\alpha) d_1\alpha,$$

where $g(\alpha) \in \mathcal{C}_c(k_{\mathfrak{p}}^{\times})$. We have already constructed a measure $d\xi$ on $k_{\mathfrak{p}}^{+}$ in section 2.2. We now use this measure to define on $\mathcal{C}_c(k_{\mathfrak{p}}^{\times})$ the following functional:

$$\Phi(g) = \int_{k^+-\{0\}} g(\xi)|\xi|^{-1}d\xi.$$

This expression makes sense because $g(\xi)|\xi|^{-1} \in \mathcal{C}_c(k^+ - \{0\})$ for $g(\alpha)$ contained in $\mathcal{C}_c(k_{\mathfrak{p}}^{\times})$. This functional on $\mathcal{C}_c(k_{\mathfrak{p}}^{\times})$ corresponds to some Radon measure $d_1\alpha$ on $k_{\mathfrak{p}}^{\times}$ in the sense that $\Phi(g) = \int_{k_{\mathfrak{p}}^{\times}} g(\alpha)d_1\alpha$. In order to prove that this Radon measure is translation invariant and hence a Haar measure, we must show that for any element β of $k_{\mathfrak{p}}^{\times}$, $d_1(\alpha\beta) = d_1(\alpha)$. For this purpose, let us first fix $\beta \in k_{\mathfrak{p}}^{\times}$ and define $h(\alpha) = g(\alpha\beta^{-1})$. Then,

$$\begin{aligned}
\Phi(h) &= \int_{k_{\mathfrak{p}}^+-\{0\}} g(\xi\beta^{-1})|\xi|^{-1}d\xi = \int_{(k_{\mathfrak{p}}^+-\{0\})\beta^{-1}} g(\xi)|\xi\beta|^{-1}d(\xi\beta) \\
&= \int_{k_{\mathfrak{p}}^+-\{0\}} g(\xi)|\xi|^{-1}d\xi,
\end{aligned}$$

giving $\Phi(g) = \Phi(h)$. The last step depends on the fact that multiplication by β^{-1} is an automorphism of $k^+ - \{0\}$ and that $d(\xi\beta) = |\beta|d\xi$, as stated in lemma 2.2.2. As $\Phi(g) = \Phi(h)$, we must have

$$\int_{k_{\mathfrak{p}}^{\times}} g(\alpha)d_1\alpha = \int_{k_{\mathfrak{p}}^{\times}} h(\alpha)d_1\alpha = \int_{k_{\mathfrak{p}}^{\times}} g(\alpha\beta^{-1})d_1\alpha = \int_{k_{\mathfrak{p}}^{\times}} g(\alpha)d_1(\alpha\beta).$$

This gives $d_1(\alpha) = d_1(\alpha\beta)$ for any $\beta \in k_{\mathfrak{p}}^{\times}$ and thereby shows that the Radon measure $d_1\alpha$ corresponding to the functional $\Phi(g)$ is translation invariant and hence a Haar measure on $k_{\mathfrak{p}}^{\times}$.

We have now constructed the required Haar measure but we cannot yet determine whether a function is integrable, because the integral $\int g(\alpha)d_1\alpha$ does not yet make sense for functions other than those contained in $\mathcal{C}_c(k_{\mathfrak{p}}^{\times})$. Hence, define a function $g(\alpha)$ on $k_{\mathfrak{p}}^{\times}$ to be integrable with respect to the measure $d_1\alpha$ if and only if the function $g(\xi)|\xi|^{-1}$ on $k^+ - \{0\}$ is integrable with respect to the additive measure $d\xi$. Moreover, the correspondence $g(\alpha) \leftrightarrow g(\xi)|\xi|^{-1}$ is a one to one correspondence between $\mathcal{C}_c(k_{\mathfrak{p}}^{\times})$ and $\mathcal{C}_c(k_{\mathfrak{p}}^{+} - \{0\})$ and we can view the integrable functions as limits of functions contained in $\mathcal{C}_c(k_{\mathfrak{p}}^{\times})$ and $\mathcal{C}_c(k_{\mathfrak{p}}^{+}-\{0\})$ respectively. This gives us

$$\int_{k_{\mathfrak{p}}^{\times}} g(\alpha)d_1\alpha = \int_{k_{\mathfrak{p}}^{+}-\{0\}} g(\xi)|\xi|^{-1}d\xi,$$

for $g(\alpha) \in L_1(k_{\mathfrak{p}}^{\times})$ and $g(\xi)|\xi|^{-1} \in L_1(k_{\mathfrak{p}}^{+}-\{0\})$. The concept of integrability of a function on $k_{\mathfrak{p}}^{\times}$ now has meaning with respect to the measure $d_1\alpha$ and we can also determine a well defined value for the corresponding integral of the function. Our construction of a multiplicative measure on $k_{\mathfrak{p}}^{\times}$ is thereby complete and is given in terms of the additive measure by $d_1\alpha = d\alpha/|\alpha|$.

Recall that in the additive case (section 2.2), the self dual measure we obtained for the $\mathfrak{p}$-adic case gave $\mathcal{O}_{\mathfrak{p}}$ the measure $(\mathcal{N}\mathfrak{d})^{-1/2}$. Emulating the additive measure, we will normalize the multiplicative measure so that $\mathcal{O}_{\mathfrak{p}}^{\times}$ has the measure $(\mathcal{N}\mathfrak{d})^{-1/2}$. The 'volume' of $\mathcal{O}_{\mathfrak{p}}^{\times}$ under the measure $d_1\alpha$ is given by

$$\int_{\mathcal{O}_{\mathfrak{p}}^{\times}} d_1\alpha = \int_{\mathcal{O}_{\mathfrak{p}}^{\times}} d\xi/|\xi|_{\mathfrak{p}} = \int_{\mathcal{O}_{\mathfrak{p}}^{\times}} d\xi$$

Thus the multiplicative measure of $\mathcal{O}_{\mathfrak{p}}^{\times}$ is same as its additive measure because elements of $\mathcal{O}_{\mathfrak{p}}^{\times}$ have absolute value 1. From the isomorphism between the quotient groups $\mathcal{O}_{\mathfrak{p}}^{\times}/1+\mathfrak{p}$ and $(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})^{\times}$, we see that the number of cosets of $1+\mathfrak{p}$ in $\mathcal{O}_{\mathfrak{p}}^{\times}$ is $\mathcal{N}\mathfrak{p}-1$. This means that $\mathcal{O}_{\mathfrak{p}}^{\times}$ is a disjoint union of $\mathcal{N}\mathfrak{p}-1$ additive translates of $1+\mathfrak{p}$ and hence its measure is $\mathcal{N}\mathfrak{p}-1$ times the measure of $1+\mathfrak{p}$. Similarly, the measure of $\mathcal{O}_{\mathfrak{p}}$ is $\mathcal{N}\mathfrak{p}$ times the measure of the ideal $\mathfrak{p}$. Using these facts, we have

$$\begin{aligned}\int_{\mathcal{O}_{\mathfrak{p}}^{\times}} d\xi &= \mathcal{N}\mathfrak{p}-1\int_{1+\mathfrak{p}} d\xi = \mathcal{N}\mathfrak{p}-1\int_{\mathfrak{p}} d\xi \\ &= \frac{\mathcal{N}\mathfrak{p}-1}{\mathcal{N}\mathfrak{p}}\int_{\mathcal{O}} d\xi = \frac{\mathcal{N}\mathfrak{p}-1}{\mathcal{N}\mathfrak{p}}(\mathcal{N}\mathfrak{d})^{-1/2}\end{aligned}$$

Definition 2.5.1. *Let $d\alpha$ denote the additive measure defined at the end of section 2.2. We define our normalized multiplicative measure $d'\alpha$ as*

$$d'\alpha = d_1\alpha = d\alpha/|\alpha|,$$

for the case $\mathfrak{p}$ *archimedean and*

$$d'\alpha = \frac{\mathcal{N}\mathfrak{p}}{\mathcal{N}\mathfrak{p}-1} d_1\alpha = \frac{\mathcal{N}\mathfrak{p}}{\mathcal{N}\mathfrak{p}-1} \frac{d\alpha}{|\alpha|},$$

if $\mathfrak{p}$ *is discrete.*

Chapter 3

The local ζ-function

The results so far have given some insight into the structure of local fields. Let us now deal with the crux of the theory of this thesis for the local case. We define the ζ-function for the field $k_\mathfrak{p}$ and state its functional equation. All the results of the previous chapter fall seamlessly into place to develop an involved proof of the functional equation and analytic continuation of the ζ-function. This is an important part of the theory as local results serve as a model and motivate the theory in the large in many ways.

3.1 The functional equation

Let $f(\xi)$ denote a complex valued function on $k_\mathfrak{p}^+$ and let $f(\alpha)$ be its restriction to $k_\mathfrak{p}^\times$. We consider the class $\mathfrak{z}_\mathfrak{p}$ of all the functions which satisfy both the conditions:

i. $f(\xi)$ and $\hat{f}(\xi)$ (the Fourier transform of f) are continuous and belong to $L^1(k_\mathfrak{p}^+)$

ii. $f(\alpha)|\alpha|_\mathfrak{p}^\sigma$ and $\hat{f}(\alpha)|\alpha|_\mathfrak{p}^\sigma \in L^1(k_\mathfrak{p}^\times)$ for $\sigma > 0$.

The first condition ensures that the Fourier inversion formula holds for the functions f and $\hat{f}$.

The ζ-function of $k_\mathfrak{p}$ can be thought of as a generalisation of the Fourier transform, in which we have characters instead of unitary characters in the formula.

Definition 3.1.1. (definition 2.4.1, §2.4, [13]) *For each $f \in \mathfrak{z}_\mathfrak{p}$, we have a function of characters c, defined for all characters of exponent σ greater than 0, given by*

$$\zeta(f, c) = \int f(\alpha)c(\alpha)d'\alpha.$$

This is called the ζ-function of $k_{\mathfrak{p}}$.

Recall that an equivalence class C of characters consists of all characters of the form $\tilde{c}(\alpha)|\alpha|^s$, where $\tilde{c}$ is a fixed representative of the class and s varies over the complex numbers. Because of the complex parameter s varying over the whole of $\mathbb{C}$, each equivalence class can be viewed as a Riemann surface (one dimensional complex manifold). As we have seen in theorem 2.3.2, s is determined by c for archimedean $\mathfrak{p}$ while for discrete $\mathfrak{p}$, it is determined mod $2\pi i/\log \mathcal{N}\mathfrak{p}$. This means that for the archimedean case, each equivalence class will be isomorphic to the complex plane. For the discrete case, the equivalence class has the form of a cylinder in the complex plane, as points that differ by an integral multiple of $2\pi i/\log \mathcal{N}\mathfrak{p}$ are identified. With this interpretation of each equivalence class as a surface, the set of all characters appears to be a collection of Riemann surfaces. Hence it makes sense to discuss local properties such as regularity or singularities of a function of characters. Analytic continuation of such a function also makes sense if it is carried out on each surface separately.

In particular we are concerned with such properties for the ζ-function. As this depends on the interpretation of characters as a collection of Riemann surfaces, the values taken by the exponents of the characters play an important role in the results that follow. Let us first examine the regularity of the ζ-function.

Lemma 3.1.2. (lemma 2.4.1, §2.4, [13]) *The ζ-function is regular in the domain of all characters of exponent greater than* 0

Proof. Consider the integral $\int f(\alpha)c(\alpha)|\alpha|^s d'\alpha$ as a function of the complex variable s. Then at $s=0$, this function gives us the local ζ-function $\zeta(f,c)$. Let σ denote the exponent of c. We have

$$\int |(f(\alpha)c(\alpha)|\alpha|^s)|d'\alpha = \int |f(\alpha)|\alpha|^{\sigma+Re(s)}|d'\alpha,$$

which is finite for s close enough to zero, due to condition ii. stated at the beginning of this section. Thus for s near zero, the function $\int f(\alpha)c(\alpha)|\alpha|^s d'\alpha$ is absolutely convergent and its derivative can in fact be obtained by differentiating under the integral sign. That is, for s near zero, the derivative of the above function is given by

$$\int f(\alpha)c(\alpha)|\alpha|^s \log|\alpha| d'\alpha.$$

This means that $\int f(\alpha)c(\alpha)|\alpha|^s d'\alpha$ is an analytic function of s for s near zero. Putting $s = 0$ gives us the required result. □

Lemma 3.1.3. (lemma 2.4.2, §2.4, [13]) *For any functions $f, g \in \mathfrak{z}$ and characters c with exponent contained in the open interval $(0, 1)$, we have*

$$\zeta(f, c)\zeta(\hat{g}, \hat{c}) = \zeta(\hat{f}, \hat{c})\zeta(g, c),$$

where $\hat{c}(\alpha) = |\alpha|c^{-1}(\alpha)$.

Proof. First note that if the exponent of c is given by s, then the exponent of $\hat{c}(\alpha)$ is given by $1 - s$. Thus for exponent of c in the range $(0, 1)$, the exponent of $\hat{c}$ also lies in the same open interval and all the ζ functions in the statement of the theorem are well defined. Using absolute convergence of the integrals, we have

$$\begin{aligned}\zeta(f, c)\zeta(\hat{g}, \hat{c}) &= \int f(\alpha)c(\alpha)d'\alpha \cdot \int \hat{g}(\beta)c^{-1}(\beta)|\beta|d'\beta \\ &= \int\int f(\alpha)\hat{g}(\beta)c(\alpha\beta^{-1})|\beta|d'\alpha\, d'\beta\end{aligned}$$

The double integral is taken over the direct product $k_{\mathfrak{p}}^{\times} \times k_{\mathfrak{p}}^{\times}$. Using the automorphism $(\alpha, \beta) \to (\alpha, \alpha\beta)$ of $k_{\mathfrak{p}}^{\times} \times k_{\mathfrak{p}}^{\times}$ and invariance of the measure term $d'\alpha\, d'\beta$ under this automorphism, we obtain

$$\iint f(\alpha)\hat{g}(\alpha\beta)c(\beta^{-1})|\alpha\beta|d'\alpha\, d'\beta.$$

Fubini's theorem gives us

$$\int \left(\int f(\alpha)\hat{g}(\alpha\beta)|\alpha|d'\alpha \right) c(\beta^{-1})|\beta|d'\beta. \tag{3.1.1}$$

Consider now the inner integral, as the rest of the expression is independent of the functions f and g. Up to the normalisation factor, this integral can be written as $\int f(\alpha)\hat{g}(\alpha\beta)|\alpha|d_1\alpha$. Moving from the multiplicative measure to additive measure using $|\alpha|d_1\alpha = d\xi$, we get $\int f(\xi)\hat{g}(\xi\beta)d\xi$. Using equation 2.2.1 to write $\hat{g}(\xi\beta)$ as an integral over $k_{\mathfrak{p}}^{+}$ and then using Fubini's theorem, we obtain

$$\iint f(\xi)g(\eta)e^{-2\pi i\Lambda(\xi\beta\eta)}d\xi d\eta,$$

which is obviously symmetric in f and g. The inner integral of equation 3.1.1 is thus symmetric in f and g, which means that the same is true for our initial expression $\zeta(f,c)\zeta(\hat{g},\hat{c})$. □

The above result is crucial for establishing analytic continuity and functional equation for the local ζ-function, as we shall see. The ζ-function was defined for characters of exponent greater than 0. Let us fix an equivalence class C of characters and for this class, define an explicit function $f_C \in \mathfrak{z}$. Then the quotient

$$\zeta(f_C, c)/\zeta(\hat{f}_C, \hat{c})$$

is well defined for all characters in C which have exponents in the range $(0,1)$. From lemma 3.1.2 one can see that using any other function of $\mathfrak{z}$ instead of f_C will give the same quotient. This facilitates choosing of a function f_C which will make computation of this quotient easier. We conclude that this quotient is independent of the function f_C used to compute it and is thus simply a function ρ of the characters c in C having exponents in the range $(0,1)$. We have

$$\rho(c) = \zeta(f_C, c)/\zeta(\hat{f}_C, \hat{c}).$$

The function $\rho(c)$ thus defined will turn out to be a meromorphic function of the parameter s. As the entire equivalence class C is described by the parameter s, we have an analytic continuation of $\rho(c)$ over all of C. The entire argument holds for any equivalence class C and so in effect we get a function $\rho(c)$ which makes sense for all characters. From lemma 3.1.2, we see that for any function $f \in \mathfrak{z}$,

$$\zeta(f,c)\zeta(\hat{f}_C, \hat{c}) = \zeta(\hat{f}, \hat{c})\zeta(f_C, c),$$

hence

$$\zeta(f,c) = \rho(c)\zeta(\hat{f}, \hat{c}).$$

This leads us to the *Main Theorem* of the local theory, whose proof now only depends on the following steps for each equivalence class C of characters: choice of a suitable function f_C, computation of $\rho(c)$ using that function and analytic continuity of the function $\rho(c)$ thus computed.

Theorem 3.1.4. (Main theorem : local case) (theorem 2.4.1, §2.4, [13]) *The ζ-*

function has an analytic continuation to the domain of all characters, given by the functional equation

$$\zeta(f,c)=\rho(c)\zeta(\hat{f},\hat{c}).$$

Before we proceed to a rigorous proof of this theorem in the next section, we prove some properties of $\rho(c)$ that follow from the functional equation and will be useful.

Lemma 3.1.5. (lemma 2.4.3, §2.4, [13]) *Let $\bar{c}$ denote the function which takes α to the complex conjugate of $c(\alpha)$. The function ρ satisfies the following properties:*

i. $\rho(\hat{c})=\frac{c(-1)}{\rho(c)}$

ii. $\rho(\bar{c})=c(-1)\overline{\rho(c)}$.

Proof. Let us first check some simple facts which will arise when we start to prove the two properties stated above.
$(i)\hat{\bar{f}}(\eta)=\bar{\hat{f}}(-\eta)$: Using equation 2.2.1,

$$\hat{\bar{f}}(\eta)=\int\bar{f}(\xi)e^{-2\pi i\Lambda(\xi\eta)}d\xi=\int\bar{f}(\xi)\overline{e^{2\pi i\Lambda(\xi\eta)}}d\xi=\bar{\hat{f}}(-\eta)$$

$(ii)\hat{\bar{c}}(\alpha)=\bar{\hat{c}}(\alpha)$: Using the definition of $\hat{c}$, we have

$$\bar{\hat{c}}(\alpha)=\overline{|\alpha|c(\alpha^{-1})}=|\alpha|\bar{c}(\alpha^{-1})=\hat{\bar{c}}$$

$(iii)\bar{\hat{c}}(-\alpha)=\bar{\hat{c}}(\alpha)/c(-1)$:
The complex conjugate of $c(-1)$ is in fact its inverse because $\bar{c}(-1)c(-1)=|c(-1)|^2=|c(-1)^2|=1$. We have

$$\begin{aligned}\bar{\hat{c}}(-\alpha)&=\overline{|\alpha|c((-\alpha)^{-1})}=|\alpha|\bar{c}(-\alpha^{-1})\\&=|\alpha|\bar{c}(-1)\bar{c}(\alpha^{-1})=\hat{\bar{c}}/c(-1).\end{aligned}$$

In order to prove the first property, we apply the functional equation twice to get $\zeta(f,c)=\rho(c)\zeta(\hat{f},\hat{c})=\rho(c)\rho(\hat{c})\zeta(\hat{\hat{f}},\hat{\hat{c}})$. In order to simplify the term $\zeta(\hat{\hat{f}},\hat{\hat{c}})$, let us first note that $\hat{\hat{f}}(\xi)=f(-\xi)$ (see equation 2.2.2) and $\hat{\hat{c}}(\alpha)=|\alpha|\hat{c}(\alpha^{-1})=|\alpha||\alpha^{-1}|c(\alpha)=c(\alpha)$. Then,

$$\zeta(\hat{\hat{f}},\hat{\hat{c}})=\int f(-\alpha)c(\alpha)d'\alpha=c(-1)\int f(\alpha)c(\alpha)d'\alpha=c(-1)\zeta(f,c).$$

This gives $\rho(\hat{c})\rho(c) = 1/c(-1) = c(-1)$ as needed.

For the second property, we apply the functional equation to $\zeta(\bar{f}, \bar{c})$ and use the facts we stated in the beginning of the proof.

$$\begin{aligned} \zeta(\bar{f}, \bar{c}) &= \rho(\bar{c})\zeta(\hat{\bar{f}}, \hat{\bar{c}}) = \rho(\bar{c}) \int \hat{\bar{f}}(-\alpha)\bar{\hat{c}}(\alpha)d'\alpha \\ &= \rho(\bar{c})c(-1) \int \bar{\hat{f}}(\alpha)\bar{\hat{c}}(\alpha)d'\alpha = \rho(\bar{c})c(-1)\overline{\zeta(\hat{f}, \hat{c})} \end{aligned}$$

On the other hand when $\zeta(\bar{f}, \bar{c})$ is written as an integral using the definition of the ζ-function, we find that

$$\zeta(\bar{f}, \bar{c}) = \overline{\zeta(f, c)} = \overline{\rho(c)}\overline{\zeta(\hat{f}, \hat{c})}.$$

Comparing the two expressions obtained for $\zeta(\bar{f}, \bar{c})$ gives us the required result. $\square$

3.2 Proof of the Main theorem for the local case

As observed earlier, the proof of theorem 3.1.4 is now only a matter of computing $\rho(c)$ on each surface C using a convenient function f_C chosen for the equivalence class C. Though the function $\rho(c)$ so constructed is defined for only a certain set of characters in C, we will see that it can be analytically continued to the entire surface C, thereby resulting in the analytic continuation and functional equation for the ζ-function. The proof is divided into three parts, for the cases $k_\mathfrak{p}$ real, complex and $\mathfrak{p}$-adic respecively.

3.2.1 $k_\mathfrak{p}$ Real

Let us try to first understand the preliminary facts. There are two aspects of the field $k_\mathfrak{p}$ - additive and multiplicative. If the field is real then $\xi \in \mathbb{R}^+$ is a real variable while $\alpha \in \mathbb{R}^\times$ is a non-zero real variable. As R (see definition 2.1.2) is the real field, the trace map Tr from $k_\mathfrak{p}^+$ to R is just the identity map, giving us

$$\Lambda(\xi) = -(Tr\xi) = -\xi.$$

The absolute value in the field in this case is just the ordinary absolute value on the reals, the additive measure $d\xi$ is the ordinary Lebesgue measure (as given at the end of section 2.2) and the multiplicative measure is $d'\alpha = d\alpha/|\alpha|$ (as given in definition 2.5.1).

The functions f_C and the Fourier Transforms

As stated under Case 1. of section 2.4, there are two equivalence classes of characters given by $||^s$ and $\pm||^s$ respectively. We choose the functions

$$f(\xi) = e^{-\pi\xi^2} \text{ and } f_\pm(\xi) = \xi e^{-\pi\xi^2},$$

for the first and the second class respectively. One can check that these functions indeed belong to the class of functions given by $\mathfrak{z}_\mathfrak{p}$. A fundamental factor while choosing a suitable function for an equivalence class is not only easy computation of $\zeta(f, c)$, but also a 'nice' expression for the fourier transform, which may be in terms of the original function! This provides some kind of symmetry and makes it easy to compute $\zeta(\hat{f}, \hat{c})$ and hence the quotient $\rho(c)$.

The fourier transforms for this case are computed below:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(\eta)e^{-2\pi i\Lambda(\xi\eta)}d\eta = \int_{-\infty}^{\infty} e^{-\pi\eta^2+2\pi i\xi\eta}d\eta = e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi(\eta-i\xi)^2}d\eta$$

Using Cauchy's integral theorem to make the substitution $\eta \to \eta + i\xi$, we find that the above expression is equal to

$$e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi\eta^2}d\eta = 2e^{-\pi\xi^2}\int_0^{\infty} e^{-\pi\eta^2}d\eta.$$

Putting $\pi\eta^2 = t$ and noting that $d\eta$ is just the usual lebesgue measure, we get

$$\frac{e^{-\pi\xi^2}}{\sqrt{\pi}}\int_0^{\infty} e^{-t}t^{-1/2}dt, \text{ that is, } \frac{e^{-\pi\xi^2}}{\sqrt{\pi}}\Gamma(1/2).$$

As $\Gamma(1/2) = \sqrt{\pi}$, we get $\hat{f}(\xi) = f(\xi)$. Let us now compute the Fourier transform for the other function.

$$\hat{f}_\pm(\xi) = \int_{-\infty}^{\infty} f_\pm(\eta)e^{-2\pi i\Lambda(\xi\eta)}d\eta = \int_{-\infty}^{\infty} \eta e^{-\pi\eta^2+2\pi i\xi\eta}d\eta.$$

To evaluate the above integral, one observes that the indefinite integral $\int \eta e^{-\pi\eta^2+2\pi i\xi\eta}d\eta$

is a function of ξ that is obtained upon differentiating with respect to ξ, the function

$$\frac{1}{2\pi i}\int e^{-\pi\eta^2+2\pi i\xi\eta}d\eta,$$

which is equal to $e^{-\pi\xi^2}$ (from the computation of $\hat{f}(\xi)$ done prior to this). This gives

$$\int \eta e^{-\pi\eta^2+2\pi i\xi\eta}d\eta = i\xi e^{-\pi\xi^2}.$$

Hence $\hat{f}_\pm(\xi) = if_\pm(\xi)$.

The ζ-functions

Let us first deal with the class of characters denoted by $||^s$. The function chosen in this case for computation is f.

$$\zeta(f,||^s) = \int f(\alpha)|\alpha|^s d'\alpha = \int_{-\infty}^{\infty} e^{-\pi\alpha^2}|\alpha|^s\frac{d\alpha}{|\alpha|} = 2\int_0^\infty e^{-\pi\alpha^2}\alpha^{s-1}d\alpha.$$

Putting $\pi\alpha^2 = t$ gives

$$\pi^{-s/2}\int_0^\infty e^{-t}t^{\frac{s}{2}-1}dt = \zeta(f,c) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right).$$

As $\hat{f}(\alpha) = f(\alpha)$ and $\widehat{|\alpha|}^s = |\alpha||\alpha|^{-s}$, we have

$$\zeta(\hat{f},\widehat{||}^s) = \zeta(f,||^{1-s}) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right).$$

Coming now to the equivalence class denoted by $\pm||^s$ and the corresponding function $f_\pm$ chosen for this case,

$$\begin{aligned}
\zeta(f_\pm,\pm||^s) &= \int f_\pm(\alpha)\pm|\alpha|^s d'\alpha \\
&= \int_{-\infty}^0 \alpha e^{-\pi\alpha^2}(-1)|\alpha|^s\frac{d\alpha}{|\alpha|} + \int_0^\infty \alpha e^{-\pi\alpha^2}|\alpha|^s\frac{d\alpha}{|\alpha|} \\
&= \int_{-\infty}^0 \alpha e^{-\pi\alpha^2}(-1)|\alpha|^s\frac{d\alpha}{(-1)\alpha} + \int_0^\infty \alpha e^{-\pi\alpha^2}|\alpha|^s\frac{d\alpha}{\alpha} \\
&= 2\int_0^\infty e^{-\pi\alpha^2}|\alpha|^s d\alpha = 2\int_0^\infty e^{-\pi\alpha^2}\alpha^s d\alpha
\end{aligned}$$

This last integral can be obtained by replacing s by $s+1$ in the computations done for $\zeta(f, ||^s)$ above. This gives

$$\zeta(f_{\pm}, \pm||^s) = \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right).$$

In a similar way one can compute that

$$\zeta(\hat{f}_{\pm}, \pm\widehat{||}^s) = \zeta(if_{\pm}, \pm||^{1-s}) = i\pi^{-\frac{(1-s)+1}{2}}\Gamma\left(\frac{(1-s)+1}{2}\right).$$

Expressions for $\rho(c)$

Taking quotients of the appropriate expressions above, the required function $\rho(c)$ for each equivalence class of characters is obtained as follows

$$\rho(||^s) = \frac{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)} = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s).$$

and

$$\rho(\pm||^s) = i\frac{\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)}{\pi^{-\frac{(1-s)+1}{2}}\Gamma\left(\frac{(1-s)+1}{2}\right)} = -i2^{1-s}\pi^{-s}\sin\left(\frac{\pi s}{2}\right)\Gamma(s).$$

3.2.2 $k_{\mathfrak{p}}$ Complex

Now $\xi \in \mathbb{C}^+$ is a complex variable, which we will denote by $x+iy$, while $\alpha \in \mathbb{C}^\times$ is a non-zero complex variable which will be denoted by $re^{i\theta}$ with $r > 0$. As R (see definition 2.1.2) is again the real field, the trace map Tr from $k_{\mathfrak{p}}^+$ to R is given by $Tr(\xi) = (x+iy)+(x-iy) = 2x$, giving us $\Lambda(\xi) = -(Tr\xi) = -2Re\xi = -2x$.

The absolute value in the field in this case is the square of the ordinary absolute value on the complex numbers, the additive measure $d\xi = 2dxdy$ is twice the ordinary Lebesgue measure (as given at the end of section 2.2) and the multiplicative measure is $d'\alpha = d\alpha/|\alpha|$ (as given in definition 2.5.1). The ordinary Lebesgue measure on the complex numbers can also be written as $(rd\theta)dr$ if one considers polar co-ordinates instead of cartesian co-ordinates. Hence the multiplicative measure has the form

$$d'\alpha = \frac{d\alpha}{|\alpha|} = \frac{2rdrd\theta}{r^2} = \frac{2}{r}drd\theta.$$

The functions f_C and the Fourier Transforms

As discussed in Case 2. of section 2.4, the equivalence classes are denoted as $c_n||^s$, with $n \in \mathbb{Z}$ and the characters of the nth class have the form $c(re^{i\theta}) = e^{in\theta}r^{2s}$. We put

$$f_n(\xi) = \begin{cases} (x-iy)^{|n|}e^{-2\pi(x^2+y^2)}, & n \geq 0 \\ (x+iy)^{|n|}e^{-2\pi(x^2+y^2)}, & n \leq 0 \end{cases}$$

We claim that the fourier transform is given by $\hat{f}_n(\xi) = i^{|n|}f_{-n}(\xi)$ for all n. For $n = 0$, this claim simply says that f_0 is its own fourier transform. We prove this first. Let $\eta = u + iv$ and $\xi = x + iy$. Then

$$\Lambda(\eta\xi) = -2Re((u+iv)(x+iy)) = -2(ux - vy).$$

We have

$$\begin{aligned} \hat{f}_0(\xi) &= \int f_0(\eta)e^{-2\pi i\Lambda(\xi\eta)}d\eta = \int e^{-2\pi(u^2+v^2)}e^{4\pi i(ux-vy)}dudv \\ &= 2\int_{-\infty}^{\infty} e^{-2\pi u^2+4\pi ixu}du \int_{-\infty}^{\infty} e^{-2\pi v^2-4\pi iyv}dv \\ &= 2e^{-2\pi x^2}e^{-2\pi y^2}\int_{-\infty}^{\infty} e^{-2\pi(u-ix)^2}du \int_{-\infty}^{\infty} e^{-2\pi(v+iy)^2}dv \end{aligned}$$

Using Cauchy's theorem, we can replace the variables as follows: $u \to u + ix$ and $v \to v - iy$. This gives

$$2e^{-2\pi x^2}e^{-2\pi y^2}\int_{-\infty}^{\infty} e^{-2\pi u^2}du \int_{-\infty}^{\infty} e^{-2\pi v^2}dv.$$

Let us evaluate the first integral:

$$\int_{-\infty}^{\infty} e^{-2\pi u^2}du = 2\int_0^{\infty} e^{-2\pi u^2}du = \frac{1}{\sqrt{2\pi}}\int e^{-t}t^{-1/2}dt,$$

putting $t = 2\pi u^2$. This gives $\Gamma(1/2)/\sqrt{2\pi}$, which equals $1/\sqrt{2}$. The value of the second integral is also $1/\sqrt{2}$ by symmetry, giving us

$$\hat{f}_0(\xi) = 2e^{-2\pi x^2}e^{-2\pi y^2}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = e^{-2\pi(x^2+y^2)} = f_0(\xi).$$

If the claim is true for some $n \geq 0$, then this establishes that for that n,

$$\int f_n(\eta)e^{-2\pi i\Lambda(\xi\eta)}d\eta = i^n f_{-n}(\xi),$$

or in other words

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(u-iv)^n e^{-2\pi(u^2+v^2)+4\pi i(ux-vy)}2dudv = i^n(x+iy)^n e^{-2\pi(x^2+y^2)}. \tag{3.2.1}$$

Consider the operator

$$D = \frac{1}{4\pi i}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial x}\right).$$

As z^n is analytic and satisfies the Cauchy-Riemann equations, it can be seen that $D((u+iv)^n)$ is zero. Hence applying the operator D to equation 3.2.1 gives us

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(u-iv)^{n+1} e^{-2\pi(u^2+v^2)+4\pi i(ux-vy)}2dudv = i^{n+1}(x+iy)^{n+1} e^{-2\pi(x^2+y^2)},$$

that is,

$$\int f_{n+1}(\eta)e^{-2\pi i\Lambda(\xi\eta)}d\eta = i^{n+1} f_{-(n+1)}(\xi).$$

Thus the claim for $n+1$ is true if it is true for n. This completes the proof by induction for $n \geq 0$. In order to prove the claim for $n < 0$, we first write the claim for $-n$. Applying the fourier transform on both sides and using equation 2.2.2 gives us

$$i^{|n|}\hat{f}_n(\xi) = f_{-n}(-\xi) = (-x-(-iy))^{|n|}e^{-2\pi(x^2+y^2)} = (-1)^{|n|}f_{-n}(\xi),$$

which proves the claim for $n < 0$ as well.

The ζ-functions

For computing the ζ-function, let us work with the co-ordinates r and θ. Denoting $\alpha = x+iy$ as $re^{i\theta}$, gives us $(x-iy) = re^{-i\theta}$. For $n \geq 0$, $(x-iy)^{|n|} = r^{|n|}e^{-i|n|\theta} = r^{|n|}e^{-in\theta}$, whereas for $n < 0$, $(x+iy)^{|n|} = r^{|n|}e^{i|n|\theta} = r^{|n|}e^{-in\theta}$. Thus for all n, f_n is defined to be the function $r^{|n|}e^{-in\theta}e^{-2\pi r^2}$. As discussed in the preceding paragraphs, the characters corresponding to the class denoted by $c_n||^s$ have the form $e^{in\theta}r^{2s}$. We have,

$$\begin{aligned}\zeta(f_n, c_n||^s) &= \int f_n(\alpha)c_n(\alpha)|\alpha|^s d'\alpha = \int_0^\infty \int_0^{2\pi} r^{2(s-1)+|n|}e^{-2\pi r^2} 2rdrd\theta \\ &= 2\pi \int_0^\infty (r^2)^{s-1+\frac{|n|}{2}} e^{-2\pi r^2} d(r^2) = (2\pi)^{(1-s)+\frac{|n|}{2}} \Gamma\left(s + \frac{|n|}{2}\right).\end{aligned}$$

Using the formula for $\hat{f}$ in terms of f and the fact that $\widehat{c_n||^s} = c_{-n}||^{1-s}$, we can compute in a similar way that

$$\zeta(\hat{f}, \widehat{c_n||^s}) = i^{|n|}(2\pi)^{s+\frac{|n|}{2}} \Gamma\left(1 - s + \frac{|n|}{2}\right).$$

Expression for $\rho(c)$
We get

$$\rho(c_n||^s) = (-i)^{|n|} \frac{(2\pi)^{1-s}\Gamma\left(s + \frac{|n|}{2}\right)}{(2\pi)^s\Gamma\left(1 - s + \frac{|n|}{2}\right)}.$$

3.2.3 $k_\mathfrak{p}$ $\mathfrak{p}$-adic

We review some facts. We know that ξ is a $\mathfrak{p}$-adic variable while α is a non-zero $\mathfrak{p}$-adic variable and can be written as $\tilde{\alpha}\pi^\nu$ with π a fixed element of valuation 1 and ν an integer. As R (see definition 2.1.2) is now the field $\mathbb{Q}_\mathfrak{p}$, $\Lambda(\xi) = -\lambda(Tr\xi)$, where λ is the map defined preceding lemma 2.1.3. The absolute value in the field in this case for an element of valuation ν is given by $|\alpha|_\mathfrak{p} = (\mathcal{N}\mathfrak{p})^{-\nu}$. The additive measure $d\xi$ is such that $\mathcal{O}_\mathfrak{p}$ gets measure $(\mathcal{N}\mathfrak{d})^{-1/2}$, as given at the end of section 2.2. The multiplicative (normalized) measure is as given in definition 2.5.1 for the discrete case.

The functions f_C and the Fourier Transforms
As described in case 3 of section 2.4, the equivalence classes are denoted as $c_n||^s$,with $n \geq 0$, where c_n is a character which has conductor $\mathfrak{p}^n$, or rather whose restriction to $\mathcal{O}_\mathfrak{p}^\times$ has conductor $\mathfrak{p}^n$.

We put

$$f_n(\xi) = \begin{cases} e^{2\pi i\Lambda(\xi)}, & \text{for } \xi \in \mathfrak{d}_\mathfrak{p}^{-1}\mathfrak{p}^{-n} \\ 0, & \text{for } \xi \notin \mathfrak{d}_\mathfrak{p}^{-1}\mathfrak{p}^{-n} \end{cases}$$

As the support of f_n is contained in $\mathfrak{d}_{\mathfrak{p}}^{-1}\mathfrak{p}^{-n}$, we have

$$\hat{f}_n(\xi) = \int f_n(\eta) e^{-2\pi\Lambda(\xi\eta)} d\eta = \int_{\mathfrak{d}_{\mathfrak{p}}^{-1}\mathfrak{p}^{-n}} e^{-2\pi\Lambda((\xi-1)\eta)} d\eta.$$

This is the integral of the additive character $\eta \mapsto e^{-2\pi\Lambda((\xi-1)\eta)}$ over the compact subgroup $\mathfrak{d}_{\mathfrak{p}}^{-1}\mathfrak{p}^{-n}$ of $k_{\mathfrak{p}}^{+}$. This character is trivial over this subgroup if and only if $\Lambda((\xi-1)\mathfrak{d}_{\mathfrak{p}}^{-1}\mathfrak{p}^{-n}) = 0$, that is, $\lambda((\xi-1)\mathfrak{d}_{\mathfrak{p}}^{-1}\mathfrak{p}^{-n}) = 0$. This means

$$Tr((\xi-1)\mathfrak{d}_{\mathfrak{p}}^{-1}\mathfrak{p}^{-n}) \subseteq \mathbb{Z}_{\mathfrak{p}}.$$

By arguing in a similar fashion as was done in lemma 2.1.7, we get $(\xi-1)\mathfrak{d}_{\mathfrak{p}}^{-1}\mathfrak{p}^{-n} \subseteq \mathfrak{d}^{-1}$, that is, $(\xi-1)\mathfrak{p}^{-n} \subseteq \mathcal{O}_{\mathfrak{p}}$. This is equivalent to the condition that $(\xi-1) \in \mathfrak{p}^n\mathcal{O}_{\mathfrak{p}}$ or $\xi \equiv 1 \pmod{\mathfrak{p}^n}$. Thus the integral gives us the measure of $\mathfrak{d}_{\mathfrak{p}}^{-1}\mathfrak{p}^{-n}$ if $\xi \equiv 1 \pmod{\mathfrak{p}^n}$. Else the value of the integral is zero as the additive character is non trivial on the above mentioned subgroup.

To compute the measure of $\mathfrak{d}_{\mathfrak{p}}^{-1}\mathfrak{p}^{-n}$ with respect to the additive measure $d\xi$, notice that

$$d(\mathfrak{d}_{\mathfrak{p}}^{-1}\mathfrak{p}^{-n}\mathcal{O}_{\mathfrak{p}}) = |\mathfrak{d}_{\mathfrak{p}}^{-1}||\mathfrak{p}^{-n}|d(\mathcal{O}_{\mathfrak{p}}) = (\mathcal{N}\mathfrak{d})(\mathcal{N}\mathfrak{p})^n(\mathcal{N}\mathfrak{d})^{-1/2} = (\mathcal{N}\mathfrak{d})^{1/2}(\mathcal{N}\mathfrak{p})^n.$$

We thus have the following formula for the fourier transform:

$$\hat{f}_n(\xi) = \begin{cases} (\mathcal{N}\mathfrak{d})^{1/2}(\mathcal{N}\mathfrak{p})^n, & \text{for } \xi \equiv 1 \pmod{\mathfrak{p}^n} \\ 0, & \text{otherwise} \end{cases}$$

The ζ-functions

We know that the prime ideals occurring in the factorisation of $\mathfrak{d}$ are the prime ideals of $\mathcal{O}$ that are ramified over $\mathbb{Q}$. As for k $\mathfrak{p}$-adic, $\mathfrak{p}$ is the only prime ideal of $\mathcal{O}$, we have $\mathfrak{d} = \mathfrak{p}^d$ for some integer d. Then $\mathfrak{d}^{-1}$ is given by $\mathfrak{p}^{-d}$. Let us denote by A_ν the set of all elements of valuation ν or equivalently of absolute value $|(\mathcal{N}\mathfrak{p})^{-\nu}|$. Then if we fix an element π of valuation 1, then A_ν is in fact the set given by $\pi^\nu\mathcal{O}^\times$, that is, it is a multiplicative translate of $\mathcal{O}^\times$. Therefore, under the multiplicative measure, a_ν has the same measure as $\mathcal{O}^\times$, which is, $(\mathcal{N}\mathfrak{d})^{-1/2}$.

We first treat the case $n = 0$. The character c_0 is simply the trivial character and

f_0 is the characteristic function of $\mathfrak{d}^{-1}$. We have

$$\begin{aligned}\zeta(f_0, ||^s) &= \int_{\mathfrak{p}^{-d}} |\alpha|^s d'\alpha = \sum_{\nu=-d}^{\infty} \int_{A_\nu} |\alpha|^s d'\alpha = \sum_{\nu=-d}^{\infty} (\mathcal{N}\mathfrak{p})^{-\nu s} \int_{A_\nu} d'\alpha \\ &= \left(\sum_{\nu=-d}^{\infty} (\mathcal{N}\mathfrak{p})^{-\nu s} \right) (\mathcal{N}\mathfrak{d})^{-1/2} = \frac{(\mathcal{N}\mathfrak{p})^{ds}}{1-(\mathcal{N}\mathfrak{p})^{-s}} (\mathcal{N}\mathfrak{d})^{-1/2} = \frac{(\mathcal{N}\mathfrak{d})^{s-\frac{1}{2}}}{1-(\mathcal{N}\mathfrak{p})^{-s}}.\end{aligned}$$

As $\hat{f}_0$ is $(\mathcal{N}\mathfrak{d})^{1/2}$ times the characteristic function of $\mathcal{O}$, we repeat the steps used to compute $\zeta(f_0, ||^s)$, to get

$$\zeta(\hat{f}_0, ||^s) = \zeta(\hat{f}_0, ||^{1-s}) = (\mathcal{N}\mathfrak{d})^{1/2} \int_{\mathcal{O}} |\alpha|^{1-s} d'\alpha = \zeta(\hat{f}_0, ||^s) = \frac{1}{1-(\mathcal{N}\mathfrak{p})^{s-1}}.$$

Now let us discuss the case $n > 0$. In this case,

$$\begin{aligned}\zeta(f_n, c_n ||^s) &= \int_{\mathfrak{d}^{-1}\mathfrak{p}^{-n}} e^{2\pi i \Lambda(\alpha)} c_n(\alpha) |\alpha|^s d'\alpha \\ &= \sum_{\nu=-d-n}^{\infty} \left(\mathcal{N}\mathfrak{p}^{-\nu s} \int_{A_\nu} e^{2\pi i \Lambda(\alpha)} c_n(\alpha) d'\alpha \right) \qquad (3.2.2)\end{aligned}$$

Let us look at the terms of this sum corresponding to $\nu \geq -d$. In this case, A_ν consists of terms of valuation ν and must be contained in $\mathfrak{d}^{-1} = \mathfrak{p}^{-d}$, since $\nu \geq -d$. Since $\alpha \in A_\nu \subseteq \mathfrak{d}^{-1}$, we have $Tr(\alpha) \in \mathbb{Z}_\mathfrak{p}$. This leads to $\Lambda(\alpha) = 0$. The integral

$$\begin{aligned}\int_{A_\nu} e^{2\pi i \Lambda(\alpha)} c_n(\alpha) d'\alpha &= \int_{A_\nu} c_n(\alpha) d'\alpha = \int_{\pi^\nu \mathcal{O}^\times} c_n(\alpha) d'\alpha \\ &= \int_{\mathcal{O}^\times} c_n(\alpha \pi^\nu) d'\alpha = \int_{\mathcal{O}^\times} c_n(\alpha) d'\alpha = 0,\end{aligned}$$

since c_n has conductor $\mathfrak{p}^n$ with $n > 0$ and so cannot be trivial on $\mathcal{O}^\times$.

Now other than the first term (corresponding to $\nu = -d-n$), we are left with the terms corresponding to $-d-n < \nu < -d$. This inequality makes sense for $\nu \in \mathbb{Z}$ if and only if $n > 1$. We have already seen that breaking up the elements into sets of the form A_ν, on each of which the absolute value remains constant, is of great help. A further tactic is to break A_ν itself into disjoint sets, on each of which the function Λ is now constant! We take such sets to be of the form

$$\alpha_0 + \mathfrak{d}^{-1} = \alpha_0 + \mathfrak{p}^{-d} = \alpha_0(1 + \mathfrak{p}^{-d-\nu}).$$

On each such set, Λ has the value $\Lambda(\alpha_0)$ as $\Lambda(\mathfrak{d}^{-1}) = 0$. The integral

$$\int_{A_\nu} e^{2\pi i\Lambda(\alpha)} c_n(\alpha) d'\alpha$$

now looks like the sum of integrals on these sets. On each such set, the integral takes the form

$$\begin{aligned}\int_{\alpha_0+\mathfrak{d}^{-1}} e^{2\pi i\Lambda(\alpha)} c_n(\alpha) d'\alpha &= e^{2\pi i\Lambda(\alpha_0)} \int_{\alpha_0+\mathfrak{d}^{-1}} c_n(\alpha) d'\alpha = \int_{\alpha_0(1+\mathfrak{p}^{-d-\nu})} c_n(\alpha) d'\alpha \\ &= \int_{1+\mathfrak{p}^{-d-\nu}} c_n(\alpha\alpha_0) d'\alpha = c_n(\alpha_0) \int_{1+\mathfrak{p}^{-d-\nu}} c_n(\alpha) d'\alpha.\end{aligned}$$

Now as $-d-n < \nu$ implies $n > -d-\nu$, and the conductor of c_n is $\mathfrak{p}^n$, c_n must be non trivial on $1+\mathfrak{p}^{-d-\nu}$. As $1+\mathfrak{p}^{-d-\nu}$ is a subgroup of $k^\times$, the above integral is zero. Thus the sum of all such integrals is also zero, that is for $-d-n < \nu < -d$,

$$\int_{A_\nu} e^{2\pi i\Lambda(\alpha)} c_n(\alpha) d'\alpha = 0.$$

In summary, except the first term corresponding to $\nu = -d-n$, all the terms of equation 3.2.2 vanish. Hence

$$\zeta(f_n, c_n||^s) = \mathcal{N}\mathfrak{p}^{(d+n)s} \int_{A_{-d-n}} e^{2\pi i\Lambda(\alpha)} c_n(\alpha) d'\alpha.$$

In order to simplify this expression, we first observe that A_{-d-n} is the set of all elements of valuation $-d-n$ and can be written as $A_{-d-n} = \mathcal{O}^\times \pi^{-d-n}$. We continue our strategy of dissecting elements into smaller sets and now we try to find smaller sets in A_{-d-n} on which Λ and c_n are both constant. If we choose $\{\epsilon\}$ to be a set of representatives of the quotient group $\mathcal{O}^\times/(1+\mathfrak{p}^n)$, then $\mathcal{O}^\times$ can be written as the disjoint union $\bigcup_\epsilon \epsilon(1+\mathfrak{p}^n)$. Then

$$A_{-d-n} = \bigcup_\epsilon \epsilon(1+\mathfrak{p}^n)\pi^{-d-n} = \bigcup_\epsilon (\epsilon\pi^{-d-n} + \epsilon\mathfrak{p}^{-d})$$

As $\epsilon \in \mathcal{O}^\times$, it is a unit and hence $\epsilon\mathfrak{p}^{-d} = \mathfrak{p}^{-d} = \mathfrak{d}^{-1}$. This gives

$$A_{-d-n} = \bigcup_{\epsilon} \epsilon\pi^{-d-n}(1+\mathfrak{p}^n) \tag{3.2.3}$$
$$= \bigcup_{\epsilon} (\epsilon\pi^{-d-n} + \mathfrak{d}^{-1}) \tag{3.2.4}$$

Consider such a set corresponding to some ϵ. Referring to equation 3.2.3, we see that as c_n is trivial on $(1+\mathfrak{p}^n)$, the constant value taken by c_n on this set is $c_n(\epsilon\pi^{-d-n}) = c_n(\epsilon)$. Referring to equation 3.2.4, we see that Λ takes the fixed value $\Lambda(\epsilon\pi^{-d-n})$ on this set.

Therefore, we get

$$\zeta(f_n, c_n||^s) = (\mathcal{N}\mathfrak{p})^{(d+n)s} \left(\sum_{\epsilon} c_n(\epsilon) e^{2\pi i \Lambda(\epsilon/\pi^{d+n})} \right) \int_{1+\mathfrak{p}^n} d'\alpha$$

The Fourier transform $\hat{f}_n$ is $(\mathcal{N}\mathfrak{d})^{1/2}(\mathcal{N}\mathfrak{p})^n$ times the characteristic function of the set $1+\mathfrak{p}^n$. On this set c_n is trivial as its conductor is $\mathfrak{p}^n$ and absolute value is trivial as this set is contained in $\mathcal{O}^\times$. Hence $\widehat{c_n||^s} = c_n^{-1}||^{1-s}$ is trivial on this set. We have

$$\zeta(\hat{f}_n, \widehat{c_n||^s}) = (\mathcal{N}\mathfrak{d})^{1/2}(\mathcal{N}\mathfrak{p})^n \int_{1+\mathfrak{p}^n} d'\alpha,$$

which is a constant.

Expressions for $\rho(c)$

$$\rho(||^s) = (\mathcal{N}\mathfrak{d})^{s-\frac{1}{2}} \frac{1-(\mathcal{N}\mathfrak{p})^{s-1}}{1-(\mathcal{N}\mathfrak{p})^{-s}},$$

whereas for a non trivial character c,

$$\rho(c||^s) = (\mathcal{N}(\mathfrak{d}\mathfrak{f}))^{s-\frac{1}{2}} \left[(\mathcal{N}\mathfrak{f})^{-1/2} \sum_{\epsilon} c(\epsilon) e^{2\pi i \Lambda(\epsilon/\pi^{ord\,\mathfrak{d}\mathfrak{f}})} \right],$$

where $\mathfrak{f}$ is the conductor of c.$\{\epsilon\}$ is a set of representatives of cosets of $1+\mathfrak{f}$ in $\mathcal{O}^\times$ and is a fixed set for each equivalence class of characeters. The expression in square brackets is called as a root-number.

This completes the proof of the Main theorem for the local case.

Chapter 4

Adelès and Idèles

Adèles and idèles are beautiful structures which arise from a single concept called the restricted direct product. As the name suggests, this is a modification of the usual direct product. In this chapter we explain this concept of restricted direct product in the abstract sense. Then we define adèles and idèles using this and examine their properties.

4.1 The abstract restricted direct product

Let $J = \{\nu\}$ be a given set of indices and let J_∞ be a fixed finite subset of J. Now suppose that for each index ν we are given a locally compact group G_ν. For indices $\nu \notin J_\infty$ we are also given a compact open (hence closed) subgroup H_ν of G_ν.

Definition 4.1.1. (definition, pg. 180, §5.1, [9]) *The restricted direct product of G_ν with respect to H_ν is defined as the set*

$$\prod_{\nu \in J}' G_\nu = \{(x_\nu) : x_\nu \in G_\nu \text{ with } x_v \in H_\nu \text{ for all but finitely many } \nu\}.$$

Let G denote the above restricted direct product. If the group operation on G is defined componentwise then G is a subgroup of the ordinary direct product $\prod G_\nu$. The ordinary direct product however also has a topological structure which makes it into a topological group. To define a topology on G, we define a neighbourhood sub-base around identity consisting of sets of the form $\prod N_\nu$ where N_ν is an open neighbourhood of 1 in G_ν with $N_\nu = H_\nu$ for all but finitely many ν.

Theorem 4.1.2. (proposition 5.1.(i)., §5.1, [9]) *G is a locally compact group.*

Proof. For a given finite set of indices S containing J_∞, consider the subgroup G_S of G given by

$$G_S = \prod_{\nu \in S} G_\nu \times \prod_{\nu \notin S} H_\nu$$

Then every element of G is contained in G_S for some such finite set S. This shows that G can be written as the union of all such subgroups. When the topology of G is restricted to G_S, we get the product topology on G_S. This gives an easier way to visualise the topology on G. As G_S is the product of a finite family of locally compact groups G_ν with a family of compact groups H_ν, G_S is locally compact in the product topology. As each $x \in G$ is contained in G_S for some S, the group G is also locally compact under the given topology.

To see that G along with the corresponding topology is a topological group, we consider a sequence of elements $(x_\nu)_n$ converging to the element (x_ν) in G. We know that $(x_\nu) \in G_S$ for some finite set of indices S containing J_∞. Consider the open neighbourhood of (x_ν) in G_S (and therefore in G) given by

$$U = \prod_{\nu \in S} N_\nu \times \prod_{\nu \notin S} H_\nu.$$

By definition of convergence, there exists some integer N such that for all $n > N$, $(x_\nu)_n \in U \subseteq G_S$. After a point, all terms of the sequence lie in G_S and thus the convergence in G can be regarded as a convergence in G_S on which the induced topology is the product topology. This leads to group multiplication and inversion being continuous operations. $\square$

We can embed G_ν in G as follows:

$$\begin{aligned} G_\nu &\to G \\ x &\mapsto (\ldots, 1, 1, x, 1, 1, \ldots). \end{aligned}$$

Under this embedding, G_ν is a closed subgroup of G.

Lemma 4.1.3. (proposition 5.1.(ii)., §5.1, [9]) *A subset K of G is compact if and only if $K \subseteq \prod K_\nu$ where K_ν is a compact subset of G_ν for all ν, and $K_\nu = H_\nu$ for almost all ν. Moreover, if K is a compact neighbourhood then we have equality instead of containment of K*

Proof. Since subsets of the form G_S cover G and are clearly open in the topology defined, a finite number of G_S must cover the compact set K. But a finite union of G_S is obviously contained in a single subset of the form G_{S_0} for some index S_0. If we project $K \subseteq G_{S_0}$ into the group G_ν corresponding to each index ν, the continuous projection map gives us a compact subset K'_ν of $G_n u$ for each index. Moreover, for all but finitely many indices, K'_ν is a subset of H_ν. Thus $K \subseteq \prod K'_\nu \subseteq \prod K_\nu$ where the sets $K\nu$ are as required by the theorem.

Conversely, it can be seen that any set of the above form is a compact subset of some G_S and hence of G. Moreover if K is a compact neighbourhood then it contains some open set of the form N_ν with $N_\nu = H_\nu$ for almost all ν. Combined with the containment of K in $\prod K_\nu$ shown above, this gives the required result. □

A character of G can be written in terms of characters of the components G_ν and conversely, characters of G_ν define a unique character of G. This is made more precise in the following theorem.

Theorem 4.1.4. (lemmas 5.2, 5.3, §5.1, [9]) *Let $\chi \in \hat{G}$. Then χ is trivial on all but finitely many H_ν. Hence we can write χ in terms of its component functions $\chi_\nu = \chi_{|G_\nu}$ as*

$$\chi(y) = \prod \chi_\nu(y_\nu)$$

and this product is well defined. Conversely, given $\chi_\nu \in \hat{G}_\nu$ with $\chi_{|H_\nu} = 1$ for all but finitely many ν, we get a character of G given by

$$\chi(y) = \prod \chi_\nu(y_\nu).$$

Proof. Let $\chi \in \hat{G}$. It is geometrically obvious that we can choose an open neighbourhood U of 1 in $\mathbb{C}^\times$ such that U contains no subgroups other than the trivial group. As χ is continuous, $\chi^{-1}(U)$ is open and hence we can find an open neighbourhood $N = \prod N_\nu$ of the identity such that $\chi(N) \subseteq U$, with $N_\nu = H_\nu$ for all ν outside some finite set of indices S. For any index $\nu_0 \notin S$, consider the subgroup of N given by

$$H_{\nu_0} = \{(\ldots, 1, x, 1, \ldots) : x \in H_{\nu_0}\}.$$

Then $\chi(H_{\nu_0})$ is a subgroup of U and hence is trivial. This holds for any H_ν with $\nu \notin S$. Thus χ is trivial on all but finitely many H_ν and the product formula follows.

In order to prove the converse statement, let S be a finite set of indices such that $\chi_{|H_\nu} = 1$ for all $\nu \notin S$. It is obvious that the finite product $\prod \chi_\nu(y_\nu)$ is well defined and is a homomorphism of G into $\mathbb{C}^\times$. To prove continuity, it is enough to show that given an open neighbourhood U of 1 in $\mathbb{C}$, one can find an open neighbourhood N of the identity in G such that $\chi(N) \subseteq U$.

To show this, one chooses an open neighbourhood V such that $V^{(m)} \subseteq U$. This is possible as $\mathbb{C}^\times$ is a topological group. For each $\nu \in S$, there is a neighbourhood N_ν of the identity in G_ν such that $\chi_\nu(N_\nu) \subseteq V$. Then the set

$$\prod_{\nu \in S} N_\nu \times \prod_{\nu \notin S} H_\nu$$

is an open neighbourhood of the identity in G that satisfies $\chi(N) \subseteq U$. $\square$

This theorem paves the way for writing the group of characters $\hat{G}$ in terms of the groups $\hat{G}_\nu$. We define H_ν^* to be the subgroup of $\hat{G}_\nu$ consisting of the characters that when restricted to H_ν give the trivial map. The compactness of H_ν implies that $W(H_\nu, U)$ is an open set of $\hat{G}_\nu$ in the compact open topology. If U is chosen to be an open neighbourhood of 1 in $\mathbb{C}^*$ that contains no subgroups other than the trivial group then the open set $W(H_\nu, U)$ consists of precisely those characters of $\hat{G}_\nu$ which are trivial on H_ν. Thus H_ν^* is open in $\hat{G}_\nu$. To show that H_ν^* is compact, we identify the characters in H_ν^* with the character group of G_ν/H_ν in the obvious manner. This gives an isomorphism of the topological groups H_ν^* and $(G_\nu/H_\nu)\hat{}$. Since H_ν is open in G_ν, (G_ν/H_ν) is discrete and hence $(G_\nu/H_\nu)\hat{}$ is compact.

Thus H_ν^* is a compact open subgroup of $\hat{G}_\nu$ and taking the restricted direct product of the $\hat{G}_\nu$ with respect to the H_ν^* makes sense.

Theorem 4.1.5. (theorem 5.4, §5.1, [9]) *Let G be the restricted direct product of the G_ν with respect to the H_ν. Then we have the following isomorphism of topological groups*

$$\hat{G} \cong \prod{}' \hat{G}_\nu$$

where the restricted direct product of the G_ν is taken with respect to the subgroups H_ν^*.

Proof. Theorem 4.1.4 already establishes that this is indeed an algebraic isomorphism. It remains to show bicontinuity of the map ϕ taking the tuple (χ_ν) to the product of the components of this tuple, given by $\prod \chi_\nu$.

$$\begin{aligned} \phi : \prod{}' \hat{G}_\nu &\longrightarrow \hat{G} \\ (\chi_\nu) &\longmapsto \prod \chi_\nu . \end{aligned}$$

To prove that *phi* is continuous at the trivial character, let U be a neighbourhood of 1 in $\mathbb{C}^\times$ and let K be a compact neighbourhood of the identity of G. Then by lemma 4.1.3, $K = \prod K_\nu$ where K_ν is a compact neighbourhood of the identity of G_ν and $K_\nu = H_\nu$ for almost all ν. Let a character χ of G lie in the open neighbourhood $W(K, U)$ of the trivial character. Look at the projections of χ given by χ_ν, they are trivial on G_ν for all but finitely many ν. Say that they are non trivial for a set S of m number of indices. As $\mathbb{C}^\times$ is a topological group, we can find some open set V in U such that the set

$$V^{(m)} = \{x_1 x_2 \ldots x_m : x_i \in V, 1 \leq i \leq m\}$$

is contained in U (see proposition 1.2.1). Take

$$N = \prod_{\nu \in S} W(K_\nu, V) \times \prod_{\nu \notin S} H_\nu^* .$$

This is an open set in the restricted product topology. It follows that $N \subseteq \phi^{-1}(W(K, U))$, showing that ϕ is continuous!

On the other hand any neighbourhood of the identity of the restricted product looks like $N = \prod W(K_\nu, U)$ and the open set $W(\prod K_\nu, U)$ of G is contained in its image under ϕ. This means that ϕ is an open map. Thus ϕ is bicontinuous and the result is proved. □

Just as the group of characters on the restricted direct product G can be characterised in terms of the character groups of the individual groups G_ν, so can the Haar measure on G be constructed by the Haar measures on G_ν.

Theorem 4.1.6. (proposition 5.5, §5.1, [9]) *Let G denote the restricted direct product as above. Let dg_ν denote the corresponding Haar measure on G_ν normalized so that*

$$\int_{H_\nu} dg_\nu = 1$$

for all but finitely many ν. Then there is a unique Haar measure dg on G such that for each finite set of indices S containing J_∞, the restriction dg_S, of dg to

$$G_S = \prod_{\nu \in S} G_\nu \times \prod_{\nu \notin S} H_\nu$$

is the product measure.

Proof. Choose a finite set of indices S containing J_∞. Define a measure dg_S by taking the product of the measures dg_ν. The measure dg_S is simply a product measure on G_S and it can be checked that it is indeed a Haar measure on G_S. One notes that the condition

$$\int_{H_\nu} dg_\nu = 1$$

for all but finitely many ν is of critical importance here for making sense of what might otherwise have been a diverging infinite product of measures. In general, a Haar measure on a group is defined uniquely once we specify how it behaves on any open subgroup. Hence this construction of the measure dg_S on the open subgroup G_S of G defines a unique measure on G. But is this measure independent of S? We want it to be independent as we have not specified in the theorem any way of choosing the set S. If we start with a different set of indices S' and construct the measure as above, then we find that the measures given by S and S' agree on $G_{S \cup S'}$. As both measures agree on an open subgroup of G, they agree on G as well. The measure so constructed is thus independent of the choice of the set S. $\square$

Now that we have defined measure in terms of the component measure, let us examine how integration can be done on the restricted direct product.

Lemma 4.1.7. (lemma 3.3.2, §3.3, [13]) *Given for each ν, a continuous function $f_\nu \in L_1(G_\nu)$ such that $f_\nu(g_\nu) = 1$ on H_ν for almost all ν, we define $f(g) = \prod_\nu f_\nu(g_\nu)$. Then f is continuous on G. Also, if S is a set of indices outside which f_ν is trivial on H_ν and $\int_{H_\nu} dg_\nu = 1$, then*

$$\int_{G_S} f(g)dg = \prod_{\nu \in S} \left[\int_{G_\nu} f_\nu(g_\nu) dg_\nu \right].$$

Proof. Consider the restriction of f to $G_S = \prod_{\nu \in S} G_\nu \times \prod_{\nu \notin S} H_\nu$,

$$f_{|G_S} = \prod_{\nu \in S} f(g_\nu).$$

This is a finite product of continuous functions, so f is continuous on G_S. This is true for any set of indices S satisfying the conditions of the theorem. Consider an element g in G and an open set U in $\mathbb{C}$ containing the image $f(g)$ of this element. As g belongs to G_S for some S and is continuous on this G_S, there is an open set N of g in G_S such that $f(N) \subseteq U$. As G_S is open in G, N is an open neighbourhood of g in G and is contained in $f^{-1}(U)$. This shows that f is continuous on G.

For the second part, note that

$$\begin{aligned}\int_{G_S} f(g)dg &= \int_{G_S} f(g)dg_S = \int_{G_S} \left(\prod_{\nu \in S} f(g_\nu)dg_\nu\right)\left(\prod_{\nu \notin S} f(g_\nu)dg_\nu\right) \\ &= \prod_{\nu \in S} \int_{G_\nu} f(g_\nu)dg_\nu \prod_{\nu \notin S} \int_{H_\nu} f(g_\nu)dg_\nu = \prod_{\nu \in S} \left(\int_{G_\nu} f(g_\nu)dg_\nu\right).\end{aligned}$$

□

Theorem 4.1.8. (theorem 3.3.1, §3.3, [13]) *If $f_\nu(g_\nu)$ and $f(g)$ are the functions of the preceding lemma and moreover*

$$\prod_\nu \left(\int_{G_\nu} |f_\nu(g_\nu)|dg_\nu\right) < \infty,$$

then $f(g) \in L_1(G)$ and

$$\int_G f(g)dg = \prod_\nu \left(\int_{G_\nu} f_\nu(g_\nu)dg_\nu\right).$$

Proof. We have

$$\int_G |f(g)|dg = \lim_S \int_{G_S} |f(g)|dg.$$

Now for any set S of indices outside which f_ν is trivial on H_ν and $\int_{H_\nu} dg_\nu = 1$, using lemma 4.1.7, we have $\int_{G_S} f(g)dg = \prod_{\nu\in S}\left[\int_{G_\nu} f_\nu(g_\nu)dg_\nu\right]$. Hence

$$\lim_S \left(\int_{G_S} f(g)dg\right) = \lim_S \left(\prod_{\nu\in S}\left[\int_{G_\nu} f_\nu(g_\nu)dg_\nu\right]\right).$$

This limit exists and is equal to $\prod_\nu \left(\int_{G_\nu} |f_\nu(g_\nu)|dg_\nu\right)$ since $\prod_\nu \left(\int_{G_\nu} |f_\nu(g_\nu)|dg_\nu\right) < \infty$. This gives

$$\int_G |f(g)|dg = \prod_\nu \left(\int_{G_\nu} |f_\nu(g_\nu)|dg_\nu\right)$$

and thereby $f \in L_1(G)$.

For the second part we notice that

$$\prod_\nu \left(\int_{G_\nu} f_\nu(g_\nu)dg_\nu\right) \le \prod_\nu \left(\int_{G_\nu} |f_\nu(g_\nu)|dg_\nu\right) < \infty.$$

Thus all the steps of the first part can be repeated with $|f|$ and $|f_\nu|$ replaced by f and f_ν respectively, to get

$$\int_G f(g)dg = \prod_\nu \left(\int_{G_\nu} f_\nu(g_\nu)dg_\nu\right).$$

□

We now know how characters, measure and integration appear in a restricted direct product with respect to the corresponding properties of the component groups. We now make some similar observations about Fourier transform in a restricted direct product.

A typical element of $\hat{G}$ is denoted by the character c whose components are given by c_ν. That is c is the tuple $(\cdots, c_\nu, \cdots)$. Let the dual measure of dg_ν in $\hat{G}_\nu$ be denoted as dc_ν. For the time being, we let the function $f_\nu(g_\nu)$ be the characteristic function of H_ν. This is an integrable function and so we can take its Fourier transform:

$$\hat{f}_\nu(c_\nu) = \int_{G_\nu} f_\nu(g_\nu)\bar{c}_\nu(g_\nu)dg_\nu = \int_{H_\nu} \bar{c}_\nu(g_\nu)dg_\nu.$$

Therefore,

$$\hat{f}_\nu(c_\nu) = \begin{cases} 0, & \text{if } c_\nu \text{ is non trivial over } H_\nu \\ \int_{H_\nu} dg_\nu, & \text{if } c_\nu \text{ is non trivial over } H_\nu \end{cases}$$

The Fourier transform $\hat{f}_\nu(c_\nu)$ is in fact $\int_{H_\nu} dg_\nu$ times the characteristic function of H_ν^*. Now applying the Fourier inversion formula,

$$f_\nu(g_\nu) = \int_{\hat{G}_\nu} \hat{f}_\nu(c_\nu) c_\nu(g_\nu) dc_\nu = \int_{H_\nu} dg_\nu \int_{H_\nu^*} c_\nu(g_\nu) dc_\nu.$$

As this equation holds for all $g_\nu \in G_\nu$, let us take g_ν to be an element of H_ν. Then $f_\nu(g_\nu) = 1$ as f is the characteristic function of H_ν and $c_\nu(g_\nu) = 1$ since $c_\nu \in H_\nu^*$. This gives

$$\left(\int_{H_\nu} dg_\nu\right)\left(\int_{H_\nu^*} dc_\nu\right) = 1.$$

hence $\int_{H_\nu^*} dc_\nu = 1$ for all but finitely many ν.

Lemma 4.1.9. (lemma 3.3.3, §3.3, [13]) *If $f_\nu(g_\nu) \in \mathfrak{B}_1(G_\nu)$ for all ν and $f_\nu(g_\nu)$ is the characteristic function of H_ν for all but finitely many ν, then $f(g)$ has Fourier transform $\hat{f}(c) = \prod_\nu \hat{f}_\nu(c_\nu)$ and $f(g) \in \mathfrak{B}_1(G)$.*

Proof. Let S be a set of indices such that $f_\nu(g_\nu)$ is the characteristic function of H_ν for all $\nu \notin S$. Using the facts $f_\nu(g_\nu) \in L_1(G_\nu)$ for all ν, H_ν has finite measure for all ν and H_ν has measure 1 for all but finitely many ν,

$$\begin{aligned}\prod_\nu \left(\int_{G_\nu} |f_\nu(g_\nu)\bar{c}_\nu(g_\nu)| dg_\nu\right) &= \prod_\nu \left(\int_{G_\nu} |f_\nu(g_\nu)| dg_\nu\right) \\ &= \prod_{\nu \in S} \left(\int_{G_\nu} |f_\nu(g_\nu)| dg_\nu\right) \prod_{\nu \notin S} \left(\int_{H_\nu} dg_\nu\right) \\ &< \infty.\end{aligned}$$

This means that the conditions of theorem 4.1.8 are satisfied and we can use it to get

$$\hat{f}(c) = \int_G f(g)\bar{c}(g) dg = \prod_\nu \left(\int_{G_\nu} f_\nu(g_\nu)\bar{c}_\nu(g_\nu) dg_\nu\right) = \prod_\nu \hat{f}_\nu(c_\nu).$$

For the second part of the result, we use the facts $\hat{f}_\nu(c_\nu) \in L_1(\hat{G}_\nu)$ for all ν, H_ν^* has finite measure for all ν and H_ν^* has measure 1 for all but finitely many ν. We also know that $\hat{f}_\nu$ is the characteristic function of H_ν^* for all ν outside some finite set S. This gives

$$\int_{\hat{G}} |\hat{f}(c)| dc = \prod_\nu \left(\int_{\hat{G}_\nu} |\hat{f}_\nu(c_\nu)| dc_\nu\right) = \prod_{\nu \in S} \left(\int_{\hat{G}_\nu} |\hat{f}_\nu(c_\nu)| dc_\nu\right) \prod_{\nu \notin S} \left(\int_{H_\nu^*} dc_\nu\right) < \infty.$$

This gives $\hat{f} \in L_1(\hat{G})$ as required. $\square$

Applying the above lemma to the group $\hat{G}$ with measure dc, we obtain the inversion formula

$$f(g) = \int \hat{f}(c)c(g)dc,$$

from the component wise inversion formulas. This proves that the measure dc is in fact the dual of the measure dg, that is, the product measure obtained from the duals of the component measures is the dual of the original measure!

4.2 Adèles

We let k denote a finite algebraic field and $k_\mathfrak{p}$ denote the completion of k at the place $\mathfrak{p}$.

Definition 4.2.1. (definition 4.1.1, §4.1, [13]) *The additive group $\mathbb{A}_k$ of adèles is defined as the restricted direct product over all the places $\mathfrak{p}$, of the additive groups $k_\mathfrak{p}^+$ with respect to the subgroups $\mathcal{O}_\mathfrak{p}$.*

We denote a general adèle by the tuple $x = (\ldots, x_\mathfrak{p}, \ldots)$.

Theorem 4.2.2. *$\mathbb{A}_k$ is its own character group.*

Proof. From theorem 4.1.5, we know that since $\mathbb{A}_k$ is the restricted direct product of $k_\mathfrak{p}^+$ with respect to $\mathcal{O}_\mathfrak{p}$, $\hat{\mathbb{A}_k}$ must be the restricted direct product of $\hat{k}_\mathfrak{p}^+$ with respect to $\mathcal{O}_\mathfrak{p}^*$. A typical character of $\hat{\mathbb{A}_k}$ looks like a tuple of the component-wise local characters $\xi_\mathfrak{p} \mapsto e^{2\pi i \Lambda(\xi\eta)}$. Identifying $\hat{k_\mathfrak{p}}$ with $k_\mathfrak{p}$ via the identification

$$x_\mathfrak{p} \mapsto e^{2\pi i \Lambda_\mathfrak{p}(x_\mathfrak{p}\eta_\mathfrak{p})} \leftrightarrow \eta_\mathfrak{p},$$

we see that due to lemma 2.1.7, this gives us an identification between $\mathcal{O}_\mathfrak{p}^*$ and $\mathfrak{d}_\mathfrak{p}^{-1}$. Thus $\hat{\mathbb{A}_k}$ is in fact the restricted direct product of $k_\mathfrak{p}^+$ with respect to $\mathfrak{d}_\mathfrak{p}^{-1}$, with a typical element given by $\eta = (\ldots, \eta_\mathfrak{p}, \ldots)$. However, due to the classical result that only finitely many primes get ramified, we have $\mathfrak{d}_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$. This is equivalent to saying that any element of $\hat{\mathbb{A}_k}$ has the form $\eta = (\ldots, \eta_\mathfrak{p}, \ldots)$, with $\eta_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$. We introduce the notation $\Lambda = \sum_\mathfrak{p} \Lambda_\mathfrak{p}(x_\mathfrak{p})$ along with component wise multiplication

$$\eta x = (\ldots, \eta_\mathfrak{p} x_\mathfrak{p}, \ldots).$$

Thus $\hat{\mathbb{A}_k} = \mathbb{A}_k$, with a general element $\eta = (\ldots, \eta_\mathfrak{p}, \ldots)$ being identified with the character

$$x = (\ldots, x_\mathfrak{p}, \ldots) \mapsto \prod_\mathfrak{p} \exp(2\pi i \Lambda_\mathfrak{p}(\eta_\mathfrak{p}\xi_\mathfrak{p})) = \exp\left(2\pi i \sum_\mathfrak{p} \Lambda(\eta_\mathfrak{p}\xi_\mathfrak{p})\right).$$

□

As per the discussion of the previous section, we give the group of adèles the measure dx which is the product of the component-wise measures $dx_\mathfrak{p}$ on the groups $k_\mathfrak{p}^+$. As these local measures are self dual, so is the measure dx on $\mathbb{A}_k$. This gives us the following formulae for the Fourier transform and the inverse Fourier transform :

$$\hat{f}(\eta) = \int f(x) e^{-2\pi i \Lambda(\eta x)} dx \tag{4.2.1}$$

$$f(x) = \int \hat{f}(\eta) e^{2\pi i \Lambda(\eta x)} d\eta \tag{4.2.2}$$

We see that the theory for adèles has quite a few similarities with the previously discussed theory for the additive group $k_\mathfrak{p}^+$. As a step in this direction we have the following lemma.

Lemma 4.2.3. (lemma 4.1.1, §4.1, [13]) *The map $\phi_a : x \mapsto ax$ is an automorphism of the group of adèles if and only if $a \in \mathbb{A}_k$ satisfies the conditions $a_\mathfrak{p} \neq 0$ for all $\mathfrak{p}$ and $|a_\mathfrak{p}|_\mathfrak{p} = 1$ for all but finitely many $\mathfrak{p}$.*

Proof. On each component $\mathfrak{p}$, the map is given by the homomorphism $x_\mathfrak{p} \mapsto a_\mathfrak{p} x_\mathfrak{p}$. As $k_\mathfrak{p}$ is a topological field, this multiplication map is continuous. Assume that this map is an automorphism. Then it must be surjective, so in particular we must have $b \in \mathbb{A}_k$ such that $\phi_a(b) = 1$, that is, $a_\mathfrak{p} b_\mathfrak{p} = 1$ for all $\mathfrak{p}$. For such an adèle b to exist, we must have $a_\mathfrak{p} \neq 0$ for all $\mathfrak{p}$. Also as $(\ldots, b_\mathfrak{p}, \ldots) = (\ldots, a_\mathfrak{p}^{-1}, \ldots)$ is an element of $\mathbb{A}_k$, we must have $a_\mathfrak{p}^{-1} \in \mathcal{O}_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$, which gives, $|a_\mathfrak{p}|_\mathfrak{p} = 1$ for all but finitely many $\mathfrak{p}$.

Conversely if the above two conditions are met, then taking $b = (\ldots, a_\mathfrak{p}^{-1}, \ldots)$, the map ϕ_b is the inverse map for the continuous homomorphism ϕ_a and is a map of the same form. This shows that ϕ_a is an automorphism. □

We shall see that the elements of $\mathbb{A}_k$ that satisfy the conditions of the above theorem are precisely those that we will define as idèles in the next section. For the sake of convenience, we call them as idèles in this section as well, even though we have not defined this term precisely yet. With this result in hand, we can now examine how the measure of a set in $\mathbb{A}_k$ changes under multiplicative translation by an idèle, just as we saw how the measure of a set in $k_\mathfrak{p}^+$ changes under multiplicative translation by an element of $k_\mathfrak{p}^\times$. The following result is one of many which show that idèles are just the multiplicative analogue of adèles!

Lemma 4.2.4. (lemma 4.1.2, §4.1, [13]) *For an idèle a,*

$$d(ax) = |a|dx,$$

where $|a| = \prod_\mathfrak{p} |a_\mathfrak{p}|_\mathfrak{p}$.

Proof. As multiplication by an idèle is an automorphism of $\mathbb{A}_k$, it is obvious that the measure $d_0x = d(ax)$ is a Haar measure on $\mathbb{A}_k$ and must differ from dx by a contant factor. To find this constant factor, we can choose any convenient set and compare the two measures for that set. Let $N = \prod N_\mathfrak{p}$ be a compact neighbourhood of 0 in $\mathbb{A}_k$. Then using theorem 4.1.8,

$$\int_{aN} dx = \prod_\mathfrak{p} \int_{a_\mathfrak{p} N_\mathfrak{p}} dx_\mathfrak{p}.$$

From lemma 2.2.2, measure of $a_\mathfrak{p} N_\mathfrak{p}$ is $|a_\mathfrak{p}|_\mathfrak{p}$ times the measure of $N_\mathfrak{p}$. This gives

$$\prod_\mathfrak{p} \left(|a_\mathfrak{p}|_\mathfrak{p} \int_{N_\mathfrak{p}} dx_\mathfrak{p} \right),$$

which is equal to $\prod_\mathfrak{p} |a_\mathfrak{p}|_\mathfrak{p} \int_N dx$. □

We imbed the field k in $\mathbb{A}_k$ by identifying the element ξ of k with the adèle $(\ldots, \xi, \ldots)$. Then k is a subgroup of $\mathbb{A}_k$ and we have the following approximation theorem. If we denote the set of infinite (archimedean) places of k by S_∞, then we have the subgroup $\mathbb{A}_{S_\infty}$ of $\mathbb{A}_k$ given by

$$\mathbb{A}_{S_\infty} = \prod_{\mathfrak{p} \in S_\infty} k_\mathfrak{p}^+ \times \prod_{\mathfrak{p} \notin S_\infty} \mathcal{O}_\mathfrak{p}.$$

$\mathbb{A}_{S_\infty}$ consists of all adèle elements whose components at finite (discrete) places have absolute value less than or equal to one. Henceforth we denote this open subgroup of

$\mathbb{A}_k$ as $\mathbb{A}_\infty$.

Theorem 4.2.5. (theorem 5.8, §5.2, [9]) (The Approximation Theorem)

$$\mathbb{A}_k = k + \mathbb{A}_\infty.$$

Proof. Consider any element $x = (\ldots, x_\mathfrak{p}, \ldots) \in \mathbb{A}_k$. We must prove that there exists $\xi = (\ldots, \xi, \ldots) \in k$ such that $x_\mathfrak{p} - \xi \in \mathcal{O}_\mathfrak{p}$ for all discrete places $\mathfrak{p}$.

There are only finitely many places $\mathfrak{p}$ such that $x_\mathfrak{p} \notin \mathcal{O}_\mathfrak{p}$. Let the discrete primes $\mathfrak{p}$ such that $x_\mathfrak{p} \notin \mathcal{O}_\mathfrak{p}$ be given by $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\}$ for some integer r. Then for each such component $x_\mathfrak{p}$, there exists some power of $\pi_\mathfrak{p}$ ($\pi_\mathfrak{p}$ is a fixed element of valuation 1), which when multiplied by $x_\mathfrak{p}$ gives an element of $\mathcal{O}_\mathfrak{p}$. Doing this for all the places in S, we can find an element m of k such that $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ are the only primes in the factorisation of the ideal m and multiplication by m makes $x_\mathfrak{p}$ integral for $\mathfrak{p} \in S$. For discrete primes which are not in S, $mx_\mathfrak{p}$ is anyway integral because m is an element of k while $x_\mathfrak{p}$ is integral.

In other words, there exists $m \in k$ such that

$$mx_\mathfrak{p} \in \mathcal{O}_\mathfrak{p},$$

for all discrete places $\mathfrak{p}$.

Now choose $\lambda \in \mathcal{O}$ through Chinese Remainder Theorem in the ring of integers $\mathcal{O}$ of the field k as follows: Solve for $i = 1, 2, \ldots, r$, the system of equations

$$mx_{\mathfrak{p}_i} \equiv \lambda \pmod{\mathfrak{p}_i^{n_i}\mathcal{O}_{\mathfrak{p}_i}}, \tag{4.2.3}$$

where n_i are large integers. Take $\xi = \lambda/m$. We must show that this ξ works. As λ is chosen according to the system of equations given by 4.2.3,

$$x_{\mathfrak{p}_i} - \xi = \frac{mx_{\mathfrak{p}_i} - \lambda}{m} \in \frac{\mathfrak{p}_i^{n_i}\mathcal{O}_{\mathfrak{p}_i}}{m}.$$

Notice that we did not specify any value for the variables n_i appearing in 4.2.3. except for saying that they are large integers. In practice, we just need to choose n_i to be atleast the power of $\mathfrak{p}_i$ occuring in the factorisation of the ideal m. This gives us

$$x_{\mathfrak{p}_i} - \xi \in \mathcal{O}_{\mathfrak{p}_i}.$$

For the discrete primes not occurring in S, m is as good as a unit since none of these primes occur in the factorisation of the ideal m. For such primes, as $m \in k$ and $x_{\mathfrak{p}}$ is already integral, so is $mx_{\mathfrak{p}}$. As λ was chosen in the ring of integers $\mathcal{O}$, we get $mx_{\mathfrak{p}} - \lambda \in \mathcal{O}_{\mathfrak{p}}$ and hence

$$x_{\mathfrak{p}} - \xi = \frac{mx_{\mathfrak{p}} - \lambda}{m} \in \frac{\mathcal{O}_{\mathfrak{p}}}{m} = \mathcal{O}_{\mathfrak{p}}.$$

This proves that for the ξ chosen, $x_{\mathfrak{p}} - \xi \in \mathcal{O}_{\mathfrak{p}}$ for all discrete primes □

Corollary 4.2.6. *For the case $K = \mathbb{Q}$, we have*

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{Q} + (\mathbb{R} \times \prod_{p\, prime} \mathbb{Z}_p).$$

We denote the infinite part of $\mathbb{A}_k$ as $\mathbb{A}^{\infty}$. This is the cartesian product of the archimedean completions of k, that is $\mathbb{A}^{\infty} = \prod_{\mathfrak{p} \in S_{\infty}} k_{\mathfrak{p}}^{+}$. Moreover, for any $x \in \mathbb{A}_k$, we denote by x^{∞} the projection of x on $\mathbb{A}^{\infty}$, that is , $x^{\infty} = (\ldots, x_{\mathfrak{p}}, \ldots)_{\mathfrak{p} \in S_{\infty}}$. If a generating equation for k over $\mathbb{Q}$ has r_1 real roots and r_2 pairs of conjugate complex roots, then $\mathbb{A}$ is a vector space over $\mathbb{R}$ of dimension $n = r_1 + 2r_2$.

Lemma 4.2.7. (lemma 4.1.4, §4.1, [13]) *If $\{\omega_1, \omega_2, \ldots, \omega_n\}$ is a minimal basis for the ring of integers $\mathcal{O}$ of k over the rational integers, then $\{\omega_1^{\infty}, \omega_2^{\infty}, \ldots, \omega_n^{\infty}\}$ is a basis for the vector space $\mathbb{A}^{\infty}$ over the reals. Let*

$$D^{\infty} = \{x^{\infty} = \sum_{v=1}^{\eta} x_v \omega_v^{\infty} : 0 \leq x_v < 1\}.$$

If $d = (\det(\omega_i^{(j)}))^2$ denotes the absolute discriminant of k, then the parallelotope D^{∞} has volume $\sqrt{|d|}$ if measured in the measure $dx^{\infty} = \prod_{\mathfrak{p} \in S_{\infty}} dx_{\mathfrak{p}}$.

Proof. Consider the projection $\xi \mapsto \xi^{\infty} = (\ldots, \xi, \ldots)_{\mathfrak{p} \in S_{\infty}}$. This is just the classical embedding of a number field into n-space. According to the classical theorem (see [5]), $d = (\det(\omega_i^{(j)}))^2$ is non-zero, $\omega_1, \omega_2, \ldots, \omega_n$ are linearly independent and D^{∞} has volume given by $2^{-r_2}\sqrt{|d|}$ in case the standard measure is used. However we have used a measure which is twice the ordinary measure in the complex plane and $\mathbb{A}^{\infty}$can be thought of as a product of r_1 real lines and r_2 complex planes. With our chosen measure, there is thus an additional factor of 2^{r_2} and we find that the volume of the fundamental parallelotope is simply $\sqrt{|d|}$. □

Definition 4.2.8. (definition 4.1.2, §4.1, [13]) *We define the additive fundamental domain D to be the set of all x such that $x \in \mathbb{A}_\infty$ and $x^\infty \in D^\infty$. We can write D as $D^\infty \times \prod_{\mathfrak{p} \notin S_\infty} \mathcal{O}_\mathfrak{p}$.*

Theorem 4.2.9. (theorem 4.1.3(1), §4.1, [13]) *D is indeed an additive fundamental domain because any adèle x is congruent to one and only one element of D modulo the field elements, that is, we have the disjoint union*

$$\mathbb{A} = \bigcup_{\xi \in k} (\xi + D).$$

Proof. We must show that an element x of $\mathbb{A}_k$ can be brought from the adèles to a unique element of the fundamental domain D via the field elements. We break this transition from $\mathbb{A}_k$ to D into two steps:

$$\mathbb{A}_k \to \mathbb{A}_\infty \to D.$$

The Approximation theorem (theorem 4.2.5) tells us that given $x \in \mathbb{A}$, there exists a field element ξ_1 such that $x + \xi_1 \in \mathbb{A}_\infty$.

Let us examine this transition in more detail before moving on to the second. In particular this means that $x_\mathfrak{p} + \xi_1 \in \mathcal{O}_\mathfrak{p}$ for all discrete places $\mathfrak{p}$. If there exists some other field element ξ_2 such that $x + \xi_2 \in \mathbb{A}_\infty$, then $\xi_1 - \xi_2 \in \mathbb{A}_\infty$. In particular this means that the field element $\xi_1 - \xi_2$ belongs to $\mathcal{O}_\mathfrak{p}$ for all discrete places $\mathfrak{p}$ and hence must be integral. Thus if there are two field elements taking the adèle x into $\mathbb{A}_\infty$, then they must be congruent modulo $\mathcal{O}$.

We now turn our attention to the transition from $\mathbb{A}_\infty$ to D. As $\{\omega_1^\infty, \omega_2^\infty, \ldots, \omega_n^\infty\}$ is a basis for $\mathbb{A}^\infty$ over the reals, an element y of $\mathbb{A}_\infty = \mathbb{A}^\infty \times \prod_{\mathfrak{p} \notin S_\infty} \mathcal{O}_\mathfrak{p}$ can be uniquely written as

$$y = \left(\sum_{v=1}^{n} x_v \omega_v^\infty \right) \times \prod_{\mathfrak{p} \notin S_\infty} \mathcal{O}_\mathfrak{p},$$

with $x_v \in \mathbb{R}$. We choose $m_v = [x_v]$ where $[\cdot]$ is the step function giving the greatest integer less than or equal to x_v. Take $\gamma = \sum_{v=1}^{n} m_v \omega_v$. This is an element of $\mathcal{O}$ as $\{\omega_1, \omega_2, \ldots, \omega_n\}$ is a minimal basis for the ring of integers $\mathcal{O}$ of k over the rational integers. Now consider $y - \gamma$. For all $\mathfrak{p} \notin S_\infty$, we have $(y - \gamma)_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$. Thus $y - \gamma \in \mathbb{A}_\infty$.

Moreover

$$(y-\gamma)^\infty = \sum_{v=1}^{n} x_v\omega_v^\infty - \sum_{v=1}^{n} m_v\omega_v^\infty = \sum_{v=1}^{n} t_v\omega_v^\infty,$$

with t_v real and between 0 and 1. This means $(y-\gamma)^\infty \in D^\infty$. But these properties of $y-\gamma$ are precisely those stated in definition 4.2.8. This gives $y-\gamma \in D$ and completes our transition from $\mathbb{A}-\infty$ to D

In this construction of γ itself, γ is a uniquely defined element of $\mathcal{O}$. Hence we have the following steps taking an adèle x to an element of D.

$$x \xrightarrow{\xi} x+\xi \xrightarrow{\gamma} x+\xi-\gamma$$

As ξ is a field element that is unique modulo $\mathcal{O}$ and γ is a unique element of $\mathcal{O}$, the step from an adèle to an element of D is carried out through the element $\kappa = (\xi-\gamma)$ which is unique modulo $\mathcal{O}$. We want to show that given adèle x is congruent to one and only one element of D modulo the field elements. Suppose that for a given adèle x, there exist field elements κ_1 and κ_2 such that x can be taken to two elements of D:

$$\begin{aligned} x+\kappa_1 &= d_1 \in D \\ x+\kappa_2 &= d_2 \in D. \end{aligned}$$

As the step from an adèle to an element of D is unique modulo $\mathcal{O}$, we have $\xi_1-\xi_2 \in \mathcal{O}$. This gives $d_1-d_2 \in \mathcal{O}$, making it possible to write d_1-d_2 as $\sum_{v=1}^{n} l_v\omega_v$ with $l_v \in \mathbb{Z}$. Taking the infinite part gives

$$(d_1-d_2)^\infty = \sum_{v=1}^{n} l_v\omega_v^\infty \text{ with } l_v \in \mathbb{Z}. \tag{4.2.4}$$

As $d_1-d_2 \in D$ means that $(d_1-d_2)^\infty \in D^\infty$, we have

$$(d_1-d_2)^\infty = \sum_{v=1}^{n} x_v\omega_v^\infty \text{ with } x_v \in \mathbb{R} \text{ and } 0 \le x_v < 1. \tag{4.2.5}$$

As $(d_1-d_2)^\infty$ must have a unique expression in terms of the ω_v^∞, we combine equations 4.2.4 and 4.2.5 to obtain $x_v = m_v = 0$ for all v, giving $d_1 = d_2$ □

Theorem 4.2.10. (theorem 4.1.3(2), §4.1, [13]) *D has measure* 1.

Proof. As $D = D^\infty \times \prod_{\mathfrak{p} \notin S_\infty} \mathcal{O}_\mathfrak{p}$,

$$\int_D dx = \int_{D^\infty} dx^\infty \prod_{\mathfrak{p} \notin S_\infty} \left(\int_{\mathcal{O}_\mathfrak{p}} dx_\mathfrak{p} \right) = \sqrt{|d|} \prod_{\mathfrak{p} \notin S_\infty} (\mathcal{N}_\mathfrak{p} \mathfrak{d}_\mathfrak{p})^{-1/2}.$$

The discriminant d is the norm of the different of k, which is in turn the product of the local differents $\mathfrak{d}_\mathfrak{p}$. This gives $|d| = \prod_{\mathfrak{p} \in S_\infty} (\mathcal{N}_\mathfrak{p} \mathfrak{d}_\mathfrak{p})$ and completes the proof. □

Let us pause and examine what the last two results mean in a heuristic sense. The disjoint union of theorem 4.2.9 means that the field elements act as a set of lattice points for the group of adèles. D is the fundamental parallelotope formed by these lattice points, D is the set whose translated copies make up the 'volume' of the space of adèles. Compare this to the real plane which has points with integer co-ordinates as lattice points and the unit square as the fundamental parallelotope. From this perspective, it is important and expected that the volume of the fundamental lattice should be non zero. This fact is confirmed by theorem 4.2.10. This brings us to one of the most important results for adèles, establishing all the more its similarity to the real plane with integer lattice points.

Theorem 4.2.11. (corollary 4.1.1, §4.1, [13]) *k is a discrete subgroup of $\mathbb{A}_k$. D is relatively compact and the factor group $\mathbb{A}_k/k$ is compact.*

Proof. We will make use of the fact that as given in lemma 4.2.7, $\mathbb{A}^\infty$ is a vector space over the reals. This means that for most purposes $\mathbb{A}^\infty$ is as good as $\mathbb{R}^n$ and we can assert that

$$N^\infty := \left\{ \left(\sum_{v=1}^{n} x_v \omega_v^\infty \right) : -\frac{1}{2} < x_v < \frac{1}{2}, x_v \in \mathbb{R} \right\}$$

is an open subset of $\mathbb{A}^\infty$. This means that the set

$$N := \left\{ \left(\sum_{v=1}^{n} x_v \omega_v^\infty \right) \times \prod_{\mathfrak{p} \notin S_\infty} \mathcal{O}_\mathfrak{p} : -\frac{1}{2} < x_v < \frac{1}{2}, x_v \in \mathbb{R} \right\}$$

is an open subset of $\mathbb{A}^\infty \times \prod_{\mathfrak{p} \notin S_\infty} \mathcal{O}_\mathfrak{p}$, that is, $\mathbb{A}_\infty$. As $\mathbb{A}_\infty$ is in turn open in $\mathbb{A}_k$, N is thus an open subset of the adèles, containing zero.

Now consider $N \cap k$. If a field element ξ is contained in N, then it must contained inside $\mathcal{O}_\mathfrak{p}$ for every discrete prime $\mathfrak{p}$ and hence must be an element of $\mathcal{O}$. Using the minimal basis for $\mathcal{O}$ over $\mathbb{Z}$ given in lemma 4.2.7, we have $\xi = \sum_{v=1}^{n} m_v\omega_v$, with m_v as integers. This gives

$$\xi^\infty = \sum_{v=1}^{n} m_v\omega_v^\infty.$$

As $\xi \in N$, ξ^∞ must be contained in N^∞, which means that we must have for each integer m_v, $-\frac{1}{2} < m_v < \frac{1}{2}$. This is impossible unless every m_v is zero, that is, zero is the only field element contained in N. Given any field element ξ_0 of k, we can find an open subset of the adèles, namely $N + \xi_0$, so that

$$(N + \xi_0) \cap k = \{\xi_0\}.$$

This proves that k is discrete in the group of adèles.

In order to prove that D is relatively compact, note that

$$D = D^\infty \times \prod_{\mathfrak{p}\notin S_\infty} \mathcal{O}_\mathfrak{p} \subseteq \overline{D^\infty} \times \prod_{\mathfrak{p}\notin S_\infty} \mathcal{O}_\mathfrak{p},$$

where $\overline{D^\infty}$ denotes the closure of D^∞ in $\mathbb{A}^\infty$. Viewing $\mathbb{A}^\infty$ as a vector space over $\mathbb{R}$ as in lemma 4.2.7, we see that $\overline{D^\infty} \subseteq \mathbb{A}^\infty \cong \mathbb{R}^n$ can be regarded as a closed an bounded set in n-space and must be compact. As $\mathcal{O}_\mathfrak{p}$ is compact for each place $\mathfrak{p}$, the set $\overline{D^\infty} \times \prod_{\mathfrak{p}\notin S_\infty} \mathcal{O}_\mathfrak{p}$ containing D is compact, thereby proving that D is relatively compact.

It now remains to prove that the factor group $\mathbb{A}_k/k$ is compact. We know that set theoretically, the factor group $\mathbb{A}_k/k$ is the same as the set D. There is an identification map between these two sets, obtained through the steps discussed in the proof of theorem 4.2.9. From these steps it can be seen that the identification map is in fact continuous. We can extend this identification map so that we get a map ϕ from the closure of D to the closure of the factor group. As k is a discrete subgroup of the adèle group, the factor group is in fact closed. This gives the surjective, continuous map

$$\phi : \overline{D} \longrightarrow \mathbb{A}_k/k.$$

As D is relatively compact, $\overline{D}$ is compact and so is its image under the continous map ϕ. This proves that $\mathbb{A}_k/k$ is compact. $\square$

We now prove some results for the additive map Λ on the adèles.

Lemma 4.2.12. (lemma 4.1.5, §4.1, [13]) $\Lambda(\xi) = 0$ *for all* $\xi \in k$

Proof. By definition, $\Lambda(\xi) = \sum_{\mathfrak{p}} \Lambda_{\mathfrak{p}}(\xi)$. If the prime $\mathfrak{p}$ lies above the rational prime p, then $\Lambda_{\mathfrak{p}}(\xi) = \lambda_{\mathfrak{p}}(Tr^{k_{\mathfrak{p}}}_{\mathbb{Q}_p}(\xi))$. We have

$$\Lambda(\xi) = \sum_{\mathfrak{p}} \lambda_{\mathfrak{p}}(Tr^{k_{\mathfrak{p}}}_{\mathbb{Q}_p}(\xi)) = \sum_{p} \lambda_p \left(\sum_{\mathfrak{p}|p} Tr^{k_{\mathfrak{p}}}_{\mathbb{Q}_p}(\xi) \right)$$

As the trace $Tr^k_{\mathbb{Q}}$ is given by the sum of the local traces $Tr^{k_{\mathfrak{p}}}_{\mathbb{Q}_p}$, we get

$$\Lambda(\xi) = \sum_{p} \lambda_p(Tr^k_{\mathbb{Q}}(\xi)).$$

As $Tr^k_{\mathbb{Q}}(\xi)$ is a rational number and Λ is a map into the reals modulo 1, it is sufficient to prove that $\sum_p \lambda_p(x) \equiv 0 \pmod 1$ for x a rational number.

Let us fix a rational prime q. We know that the domain of λ_p is the completion of $\mathbb{Q}$ at the p-th place. Then $\sum_p \lambda_p(x)$ can be written as

$$\sum_{p} \lambda_p(x) = \left(\sum_{p \neq q, p_\infty} \lambda_p(x) \right) + \lambda_q(x) + \lambda_{p_\infty}(x), \tag{4.2.6}$$

where p_∞ denotes the archimedean completion of the field $\mathbb{Q}$. From the construction of λ above lemma 2.1.3, $\lambda_p(x)$ has only a power of p in the denominator if p is a discrete prime. Hence if $p \neq q$, $\lambda_p(x)$ has a non negative power of q in its factorisation and in particular, all the terms in parenthesis in equation 4.2.6 are q-adic integers. From the same construction, we also know that $\lambda_{p_\infty}(x) = -x$. Moreover from property *ii* stated in the proof of lemma 2.1.4, we see that $\lambda_q(x) - x$ is a q-adic integer. This means that the second and third term of equation 4.2.6 add to give a q-adic integer.

In other words,$\sum_p \lambda_p(x)$ can be written as a sum of q-adic integers and is thus itself a q-adic integer. This argument can be repeated for any rational prime q, which means that $\sum_p \lambda_p(x)$ is an element of the reals modulo 1 that is integral with respect to every rational prime. It follows that $\sum_p \lambda_p(x) \equiv 0 \pmod 1$. □

Theorem 4.2.13. (theorem 4.1.4, §4.1, [13]) *Let* k^* *denote the set of all unitary characters of the adèle group which are trivial on* k. *Then* $k^* = k$

Proof. We must prove that for the self dual group $\mathbb{A}_k$, the unitary character corresponding to the adèle element x is trivial on k if and only if $x \in k$. More succintly, we must prove that

$$\Lambda(x\xi) = 0 \text{ for all } \xi \in k \iff x \in k.$$

As k^* is the unitary character group of the compact quotient group $\mathbb{A}_k/k$, it must be discrete. Lemma 4.2.12 shows that at least $k \subseteq k^*$ is true. Consider the quotient k^*/k. This being a discrete subgroup of the compact group $\mathbb{A}_k/k$, k^*/k must be a finite group. With the action of k on k^* defined as

$$\xi\chi(x) = \chi(\xi x),$$

where $\xi \in k$, $\chi \in k^*$ and x an adèle, k^* is a vector space over k. As k is not a finite field, the index of k in k^* cannot be finite unless it is 1. This gives $k^* = k$. □

4.3 Idèles

In the last section we introduced the term idèles but did not offer a precise definition. We now do so.

Definition 4.3.1. (definition 4.3.1, §4.3, [13]) *The multiplicative group I_k of idèles is the restricted direct product of the groups $k_\mathfrak{p}^\times$ with respect to the subgroups $\mathcal{O}_\mathfrak{p}^\times$.*

Let us denote a general element of the idèle group as $a = (\ldots, a_\mathfrak{p}, \ldots)$. We shall construct a map from the idèles to the ideal group of the field k, which will play an important role in our understanding of the structure of the idèle group. We can associate with each idèle a, an ideal $\phi(a)$, given by

$$\phi(a) = \prod_{\mathfrak{p} \notin S_\infty} \mathfrak{p}^{\nu_\mathfrak{p}(a_\mathfrak{p})}.$$

The map $a \mapsto \phi(a)$ is a continuous homomorphism onto the discrete group of ideals of k. Let us inspect the kernel of this map. The idèle a maps to the identity $\mathcal{O}$ of the ideal group if and only if $\nu_\mathfrak{p}(a_\mathfrak{p}) = 0$ for all $\mathfrak{p} \notin S_\infty$. Equivalently, $a_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^\times$ for all discrete primes. Thus the kernel of ϕ is simply the subgroup I_{S_∞} of I, given by

$$I_{s_\infty} = \prod_{\mathfrak{p} \in S_\infty} k_\mathfrak{p}^\times \times \prod_{\mathfrak{p} \notin S_\infty} \mathcal{O}_\mathfrak{p}^\times.$$

The notion of an idèle not only encompasses the information obtained from the notion of an ideal, but we also obtain information about the components at the archimedean primes as well as at discrete primes where the component is a unit. Thus for a field k, idèles can be thought of as a refined version of ideals; a version which includes information that is otherwise hidden when we represent an element a as the ideal $\phi(a)$! This also explains to some extent the name 'idèle', which is a contraction of the words 'ideal' and 'element'!

Now that we have defined the map ϕ, what can we say about the image of $k^\times$ under this map? For this we first imbed k in I_k in the obvious manner: identify $\alpha \in k^\times$ with the idèle $(\ldots, \alpha, \ldots)$. Then $\phi(\alpha) = \prod_{\mathfrak{p} \notin S_\infty} \mathfrak{p}^{\nu_\mathfrak{p}(\alpha)}$ gives us

$$\phi(\alpha) = \alpha\mathcal{O}.$$

It can be seen that the image of $k^\times$ under this map is precisely the set of all principal ideals of the field.

Let us now review the characters and measure corresponding to the idèle group. A general character $c(a)$ on the idèle group has the form $c(a) = \prod_\mathfrak{p} c_\mathfrak{p}(a_\mathfrak{p})$, where $c_\mathfrak{p}$ is a character on $k_\mathfrak{p}^\times$ and $c_\mathfrak{p}$ is trivial on $\mathcal{O}_\mathfrak{p}^\times$ for all but finitely many $\mathfrak{p}$. The measure da on the idèles is chosen to be $d'a = \prod_\mathfrak{p} d'a_\mathfrak{p}$, where $d'a_\mathfrak{p}$ is the multiplicative measure defined in section 2.5 on $k_\mathfrak{p}^\times$.

Theorem 4.3.2. (Product formula) (theorem 4.3.1, §4.3, [13]) *$|\alpha| = 1$ for $\alpha \in k_\mathfrak{p}^\times$.*

Proof. From theorem 4.2.9, the adèle group can be written as the disjoint union $\mathbb{A}_k = \bigcup_{\xi \in k}(-\xi + D)$. Using this let us write αD as

$$\alpha D = \mathbb{A}_k \cap \alpha D = \bigcup_{\xi \in k} \left((-\xi + D) \cap \alpha D\right). \tag{4.3.1}$$

As theorem 4.2.9 gives $\mathbb{A}_k = \bigcup_{\xi \in k}(D + \xi)$, we must have

$$\alpha \mathbb{A}_k = \bigcup_{\xi \in k} (\alpha D + \alpha \xi).$$

As $\alpha k = k$ (since $\alpha \in k^\times$), and $\alpha \mathbb{A}_k = \mathbb{A}_k$ (using $\alpha \in I_k$ and lemma 4.2.3), we obtain

$$\mathbb{A}_k = \bigcup_{\xi \in k} (\alpha D + \xi).$$

Again let us write D as

$$D = D \cap \mathbb{A}_k = \bigcup_{\xi \in k} \left(D \cap (\alpha D + \xi) \right). \tag{4.3.2}$$

Consider now the 'piece' $D \cap (\alpha D + \xi)$ of D and the 'piece' $(-\xi + D) \cap \alpha D$ of αD. We have the correspondence:

$$\begin{array}{ccc} D \cap (\alpha D + \xi) & \longleftrightarrow & (-\xi + D) \cap \alpha D \\ x & \longleftrightarrow & x - \xi \end{array}$$

As the correspondence between the two sets involves just an additive operation, both sets have the same measure with respect to the additive Haar measure dx on the adèles! As each of the sets D and αD is made up of a disjoint number of such congruent pieces, they must have the same volume. As lemma 4.2.4 implies that measure of αD is $|\alpha|$ times the measure of D, we must have $|\alpha| = 1$. $\square$

Let us now consider the continuous homomorphism $a \mapsto |a| = \prod_{\mathfrak{p}} |a_{\mathfrak{p}}|_{\mathfrak{p}}$ of the idèles into the multiplicative group of positive reals. We call the kernel of this map as the group of norm one idèles and denote it by J_k. This is a closed subgroup of the idèle group and contains k, as can be seen from the product formula. Let us denote a typical element of J_k by b. This subgroup of the idèles helps us to give a convenient description of the idèles.

In section 2.3 we showed that every element α of $k_{\mathfrak{p}}^{\times}$ can be written uniquely as $\tilde{\alpha}\rho$, with $\alpha \in \mathcal{O}_{\mathfrak{p}}^{\times}$ and ρ a positive real or a power of π (a fixed element of valuation 1), according to whether $\mathfrak{p}$ is achimedean or discrete respectively. We attempt to imitate this strategy for the idèles. First fix an archimedean prime $\mathfrak{p}_0$. A general element a of the idèle group can be written as

$$a = (\ldots, a_{\mathfrak{p}}, \ldots) = \left(\ldots, a_{\mathfrak{p}}, \ldots, \frac{a_{\mathfrak{p}_0}}{|a|}, \ldots \right) \cdot (\ldots, 1, \underbrace{|a|}_{\mathfrak{p}_0 th\, place}, 1, \ldots).$$

The first term has absolute value 1 and is thus an element of J_k. Let us denote it as b. The second term can be denoted by just $|a|$, it is in effect just a way of representing the absolute value of the idèle a. Let T be the subgroup of all idèles for which the $\mathfrak{p}_0$th component is a positive real and all the other components are 1. Then each element of T is uniquely determined by its absolute value. That is, if the absolute value of the element is t then it is either given by $(t, 1, 1, \ldots)$ or $(\sqrt{t}, 1, 1, \ldots)$ according to whether the place $\mathfrak{p}_0$ is real or complex (we have written the $\mathfrak{p}_0$th component first). Thus $t \mapsto |t|$ is an isomorphism of T with $\mathbb{R}^+, \cdot)$.

Representing elements of T by just absolute values, we have : an idèle a can be uniquely written in the form $a = |a|b$, with $|a| \in T$ and $b \in J_k$. It is apparent that I is the direct product $T \times J_k$. As T is isomorphic to the multiplicative group of positive reals, we choose the measure on T to be in accordance with section 2.5, that is, $d't = dt/|t| = dt/t$, where dt is the ordinary Lebesgue measure. We do not have any explicit way to select a measure db on J_k, but as measures $d'a$ and $d't$ on I_k and T are known, it is enough to require that $d'a = d't.db$.

Expressing the idèles in this manner makes integration more convenient because

$$\begin{aligned}\int_{I_k} f(a)da &= \iint f(tb)db\frac{dt}{t} \\ &= \int_0^\infty \left[\int_{J_k} f(tb)db\right]\frac{dt}{t} = \int_{J_k}\left[\int_0^\infty f(tb)\frac{dt}{t}\right]db.\end{aligned}$$

Let $J_{S_\infty} := J_k \cap I_{S_\infty}$. Let S'_∞ be the set of all archimedean primes other than the prime $\mathfrak{p}_0$. Consider the continuous homomorphism

$$l(b) : b \mapsto (\ldots, \log|b_\mathfrak{p}|_\mathfrak{p}, \ldots)_{\mathfrak{p}\in S},$$

of J_{S_∞} onto the additive Euclidean r-space, where $r = r_1 + r_2 - 1$. A point to note is that the map leaves out the place $\mathfrak{p}_0$. With this place included, one has the constraint $\prod_\mathfrak{p} |b_\mathfrak{p}|_\mathfrak{p} = 1$ because b is a norm one idèle. As $b \in J_{S_\infty}$ means that $|b_\mathfrak{p}|_\mathfrak{p} = 1$ for $\mathfrak{p} \notin S_\infty$, we get

$$\prod_{\mathfrak{p}\in S_\infty} |b_\mathfrak{p}|_\mathfrak{p} = \prod_{\mathfrak{p}\in S'_\infty} |b_\mathfrak{p}|_\mathfrak{p}.|b_{\mathfrak{p}_o}|_{\mathfrak{p}_o} = 1.$$

As the map excludes the $\mathfrak{p}_0$th component, whose value can be adjusted as needed, the components in the set S'_∞ are free to take any values. This explains surjectivity.

$k^\times \cap J_{S_\infty}$ is the group of all elements $\epsilon \in k^\times$ which are units at all finite primes and hence units of the ring $\mathcal{O}$. The units ξ for which $l(\xi) = 0$ are the roots of unity in k and form a finite cyclic group. As the images $l(\epsilon)$ form a lattice of the highest dimension in the Euclidean r-space, the group of units ϵ, modulo the group of roots of unity ξ, is a free abelian group on r generators. Let $\{\epsilon_i\}$, $1 \le i \le r$, be a basis for the group of units modulo roots of unity. Then the vectors $l(\epsilon_i)$ are a basis for the r-space over the real numbers. For any $b \in J_{S_\infty}$, we may write $l(b)$ uniquely as as $l(b) = \sum_{v=1}^{r} x_v l(\epsilon_v)$, with $x_v \in \mathbb{R}$.

Definition 4.3.3. *We define P to be the fundamental parallelotope spanned by the vectors $l(\epsilon_i)$ in the r-space, that is,*

$$P = \left\{ \sum_{v=1}^{r} x_v l(\epsilon_v) : 0 \leq x_v < 1 \right\}$$

We define Q to be the usual unit cube in r-space, that is,

$$Q = \left\{ (\ldots, x_{\mathfrak{p}}, \ldots)_{\mathfrak{p} \in S'_\infty} : 0 \leq x_{\mathfrak{p}} < 1 \right\}.$$

Lemma 4.3.4. (lemma 4.3.1, §4.3, [13]) *Let $l^{-1}(P)$ denote the set of all $b \in J_{S_\infty}$ such that $l(b) \in P$. Then*

$$\int_{l^{-1}(P)} db = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}} R,$$

where $R = \pm \det(\log |\epsilon_i|_{\mathfrak{p}})$, with $1 \leq i \leq r$ and $\mathfrak{p} \in S'_\infty$.

Proof. Using the fact that l is a homomorphism and the expression for the volume of the fundamental parallelotope over the unit cube,

$$\frac{\text{measure of } l^{-1}(P)}{\text{measure of } l^{-1}(Q)} = \frac{\text{measure of } P}{\text{measure of } Q} = \pm \det(\log |\epsilon_i|_{\mathfrak{p}}) = R.$$

$l^{-1}(Q)$ consists of all $b \in J_{s_\infty}$ such that $l(b) \in Q$, that is $1 \leq |b|_{\mathfrak{p}} < e$, for $\mathfrak{p} \in S'_\infty$. Let Q^* be the corresponding set in I_{S_∞} consisting of all $a \in I_{S_\infty}$ with $1 \leq |a|_{\mathfrak{p}} < e$ for $\mathfrak{p} \in S_\infty$. Then

$$\int_{Q^*} da = \int_J \left[\int_{tb \in Q^*} \frac{dt}{t} \right] db = \int_{l^{-1}(Q)} \left[\int_{|b|_{\mathfrak{p}_0}^{-1}}^{e|b|_{\mathfrak{p}_0}^{-1}} \frac{dt}{t} \right] db,$$

since $a = tb \in Q^* \Leftrightarrow b \in l^{-1}(Q)$ and $1 \leq |tb_{\mathfrak{p}_0}|_{\mathfrak{p}_0} < e$. As the integral within square brackets gives the value 1, we see that $\int_{Q^*} da$ is the same as $\int_{l^{-1}(Q)} db$.

Let $Q^*_{\mathfrak{p}}$ denote the set of all $a_{\mathfrak{p}} \in k^\times_{\mathfrak{p}}$ such that $1 \leq |a|_{\mathfrak{p}} < e$ for $\mathfrak{p} \in S_\infty$. Then

$Q^* = \prod_{\mathfrak{p}\in S_\infty} Q_{\mathfrak{p}}^* \times \prod_{\mathfrak{p}\notin S_\infty} \mathcal{O}_{\mathfrak{p}}^\times$.

$$\int_{Q^*} da = \prod_{\mathfrak{p}\in S_\infty} \int_{Q_{\mathfrak{p}}^*} da_{\mathfrak{p}} \prod_{\mathfrak{p}\notin S_\infty} \int_{\mathcal{O}_{\mathfrak{p}}^\times} da_{\mathfrak{p}} = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}},$$

since for the r_1 places where $\mathfrak{p}$ is real,

$$\int_{Q^*} da = \int_{-e}^{-1} \frac{dx}{|x|} + \int_1^e \frac{dx}{|x|} = 2\int_1^e \frac{dx}{x} = 2,$$

for the r_2 places where $\mathfrak{p}$ is complex,

$$\int_{Q^*} da = \int_0^{2\pi} \int_1^{\sqrt{e}} \frac{2dr d\theta}{r} = 2\pi,$$

and finally for $\mathfrak{p}$ discrete,

$$\prod_{\mathfrak{p}\notin S_\infty} \int_{\mathcal{O}_{\mathfrak{p}}^\times} da_{\mathfrak{p}} = \prod_{\mathfrak{p}\notin S_\infty} (\mathcal{N}_{\mathfrak{p}}\mathfrak{d}_{\mathfrak{p}})^{-1/2} = \frac{1}{\sqrt{|d|}}.$$

□

In the previous section we defined an additive fundamental domain for the adèles modulo the field elements. We now define a multiplicative fundamental domain E, not for the idèles but for the norm one idèles modulo $k^\times$.

Definition 4.3.5. (definition 4.3.2, §4.3, [13]) *Let h be the class number of k and select norm one idèles $b^{(1)}, \dots, b^{(h)}$ such that the corresponding ideals $\phi(b^{(1)}), \dots, \phi(b^{(h)})$ represent the different ideal classes (this can be done as only the finite components play a role in the map ϕ and the infinite components can be adjusted to give absolute value 1 for the idèle). Let w be the number of roots of unity in k. Let E_0 be the subset of all $b \in l^{-1}(P)$ such that $0 \le arg\ b_{\mathfrak{p}_0} < \frac{2\pi}{w}$. Then the multiplicative fundamental domain E, for J mod $k^\times$ is defined to be*

$$E = E_0 b^{(1)} \cup E_0 b^{(2)} \cup \ldots \cup E_0 b^{(h)}.$$

Theorem 4.3.6. (theorem 4.3.2(1), §4.3, [13]) *We have the disjoint union $J_k = \bigcup_\alpha \alpha E$.*

Proof. We perform a series of steps that will take an element b of J_k into βE for some $\beta \in k^\times$. First consider $\phi(b)$, this is an ideal of the field and must belong to a unique ideal class represented by, say, $\phi(b^{(h)})$. Then $\phi(b/b^{(i)})$ must be a principal ideal, say, $\alpha\mathcal{O}$. Then $\phi(b/(b^{(i)}\alpha)) = \mathcal{O}$, implying that $b/(b^{(i)}\alpha)$ must be contained in the kernel of ϕ, that is I_{S_∞}. As $b \in J_k$, $b^{(i)} \in J_{S_\infty}$ and $\alpha \in k^\times$, we must have $b/(b^{(i)}\alpha) \in I_{S_\infty} \cup_k = J_{S_\infty}$, giving

$$l\left(\frac{b}{b^{(i)}\alpha}\right) = \sum_{v=1}^{r} x_v l(\epsilon_v),$$

with $x_v \in \mathbb{R}$. If $[x_v]$ denotes the step function, then $b' := b/(b^{(i)}\alpha \prod \epsilon_v^{[x_v]})$ has under l the image $\sum_{v=1}^r (x_v - [x_v]) l(\epsilon_v) \in P$. Thus $b' \in l^{-1}(P)$. We are still not in E_0, as the $\mathfrak{p}_0$ component of b' may have any argument. We take ζ a root of unity with the closest argument to that of $b'_{\mathfrak{p}_0}$ and smaller than it. Then $b'/\zeta \in E_0$. Putting $\beta = \alpha\zeta \prod_{v=1}^r \epsilon_v^{[x_v]}$, we get $b \in \beta b^{(i)} E_0$, with $\beta \in k^\times$. □

Theorem 4.3.7. (corollary 4.3.1, §4.3, [13]) *$k^\times$ is a discrete subgroup of J_k and $J/k^\times$ is compact.*

Proof. Let us compute the measure of E. As E is made up of h number of copies of E_0, measure of E is h times the measure of E_0. From the definition of E_0 in definition 4.3.5, we find that the measure of E_0 is $1/w$ times the measure of $l^{-1}(P)$, whose measure is known from lemma 4.3.4. This gives measure of E to be

$$= \frac{h}{w} \text{ measure of } l^{-1}(P) = \frac{2^{r_1}(2\pi)^{r_2} hR}{\sqrt{|d|}w}.$$

As measure of E is non-zero, it has an interior, proving that $k^\times$ is discrete in J_k. As E is also relatively compact, $J/k^\times$ is compact. □

Chapter 5

Towards the Main Theorem

With all the groundwork behind us, we now attempt to prove the main result of this thesis. We start with a number theoretic version of the Riemann Roch theorem and builds toward the main result. We define the ζ-function for the field k and prove the analytic continuation for this function. In doing so we see that the elegant functional equation for the ζ-function is simply a consequence of the analytic continuation!

5.1 Riemann-Roch Theorem

Let $\phi(x)$ be a continuous, periodic function over the adèles. By periodic, we mean that $\phi(x) = \phi(x + \xi)$ for all $\xi \in k$. Let Φ be the function represented by ϕ over the quotient group $\mathbb{A}_k/k$, that is,

$$\Phi(x + k) := \phi(x).$$

The function Φ is then continuous over this quotient group. It is continuous with compact support and is hence contained in $\mathscr{C}_c(\mathbb{A}_k/k)$.

Definition 5.1.1. *We define the following functional on* $\mathscr{C}_c(\mathbb{A}_k/k)$*:*

$$I(\Phi) = \int_D \phi(x)dx,$$

where D is the additive fundamental domain of definition 4.2.8.

As D is relatively compact, this integral is bounded. $I(\Phi)$ is thus a bounded functional on $\mathscr{C}_c(\mathbb{A}_k/k)$ and must correspond to a Radon measure dv on $\mathbb{A}_k/k$ in the sense that

$$I(\Phi) = \int_{\mathbb{A}_k/k} \Phi(v)dv.$$

Here v is actually a coset, say $x + k$, of $\mathbb{A}_k/k$.

Is the measure dv translation independent? Consider the coset $v' = x' + k$ in $\mathbb{A}_k/k$. Let $\Phi'(v) = \Phi(v + v')$. Then

$$I(\Phi') = \int_{\mathbb{A}_k/k} \Phi(v + v')dv = \int_{\mathbb{A}_k/k} \Phi(v)d(v - v').$$

In order to prove translation independence of dv,we must prove that $I(\Phi) = I(\Phi')$. From definition 5.1.1,

$$I(\Phi') = \int_D \phi(x + x')dx. \tag{5.1.1}$$

For a given set S and element b, $S - b$ denotes the set of elements $\{a - b : a \in S\}$. One can check that the steps used in the proof of theorem 4.2.9 will take an element $x - x'$ of $\mathbb{A}_k - x'$ to an element of $D - x'$ in a unique manner, that is, $x - x'$ is congruent to one and only one element of $D - x'$ modulo the field elements. That is,

$$\mathbb{A}_k - x' = \bigcup_{\xi \in k} (D - x' + \xi), \tag{5.1.2}$$

where the union is disjoint. As $\mathbb{A}_k - x'$ is nothing but $\mathbb{A}_k$, we have the disjoint union

$$\mathbb{A}_k = \bigcup_{\xi \in k} (D - x' + \xi).$$

Writing D as $D \cap \mathbb{A}_k$ gives us the disjoint union

$$D = D \cap \mathbb{A}_k = \bigcup_{\xi \in k} \left(D \cap (D - x' + \xi)\right).$$

Hence

$$I(\Phi') = \int_D \phi(x + x')dx = \sum_{\xi \in k} \int_{D \cap (D - x' + \xi)} \phi(x + x')dx.$$

Using the identity

$$x \in D \cap (D - x' + \xi) \Leftrightarrow (x + x') \in D + x' \cap D + \xi,$$

the latter integral can be written as

$$\int_{D\cap(D-x'+\xi)} \phi(x+x')dx = \int_{D+x'\cap D+\xi} \phi(x)d(x-x') = \int_{D+x'\cap D+\xi} \phi(x)dx.$$

Now using $x \in (D + x' \cap D + \xi) \Leftrightarrow (x + \xi) \in ((D + x' - \xi) \cap D)$ along with the periodicity of ϕ and translation invariance of dx, we see that

$$\int_{D+x'\cap D+\xi} \phi(x)dx = \int_{(D+x'-\xi)\cap D} \phi(x+\xi)d(x+\xi) = \int_{(D+x'-\xi)\cap D} \phi(x)dx.$$

This gives

$$I(\Phi') = \sum_{\xi\in k} \int_{(D+x'-\xi)\cap D} \phi(x)dx = \int_{\bigcup_\xi (D+x'-\xi)\cap D} \phi(x)dx = \int_{\bigcup_\xi (D+x'-\xi)\cap D} \phi(x)dx$$

Using the same reasoning as for equation 5.1.1, we have

$$\mathbb{A}_k = \mathbb{A}_k + x' = \bigcup_\xi D + x' - \xi,$$

giving

$$\int_{\bigcup_\xi (D+x'-\xi)\cap D} \phi(x)dx = \int_{\mathbb{A}_k\cap D} \phi(x)dx = \int_D \phi(x)dx = I(\Phi)$$

Thus $I(\Phi') = I(\Phi)$, proving that the Radon measure dv is translation invariant and hence a Haar measure.

As the volume of D is 1 by theorem 4.2.10, we have

$$\int_{\mathbb{A}_k/k} dv = I(1) = \int_D dx = 1.$$

Let us summarize in the following lemma:

Lemma 5.1.2. (theorem 4.2.1, §4.2, [13]) *For a continuous periodic function ϕ on the adèles, $\int_D \phi(x)dx$ is equal to the integral of the corresponding function Φ over $\mathbb{A}_k/k$ with respect to that Haar measure which gives $\mathbb{A}_k/k$ the measure* 1.

Under the identification of $\hat{\mathbb{A}}_k$ with $\mathbb{A}_k$, the subgroup k^* of the group of unitary characters on the adèles gets identified with the field k. The subgroup k^* is simply the set of unitary characters on the adèles which vanish on k. Hence k^* is precisely the unitary character group of the quotient $\mathbb{A}_k/k$. Thus, under the identification between the adèle group and its group of unitary characters, the unitary character group of $\mathbb{A}_k/k$ gets identified with k. One sees that the following notational substitutions are legitimate. We replace:

an element (coset) $v = x + k$ of $\mathbb{A}_k/k$ by the element x of $\mathbb{A}$,

the integral over $\mathbb{A}_k/k$ with respect to dv by the integral over D with respect to dx,

continuous functions Φ on $\mathbb{A}_k/k$ by periodic continuous functions ϕ on the adèles,

unitary characters of $\mathbb{A}_k/k$ by unitary characters of the adèles corresponding to elements ξ of k

This gives us the following expressions for the Fourier transform and the inversion formula for Φ.

Lemma 5.1.3. (lemma 4.2.2, §4.2, [13])

$$\hat{\phi}(\xi) = \int_D \phi(x) e^{-2\pi i \Lambda(\xi x)} dx.$$

If $\sum_{\xi \in k} |\hat{\phi}(\xi)| < \infty$, *then*

$$\phi(x) = \sum_{\xi \in k} \hat{\phi}(\xi) e^{2\pi i \Lambda(\xi x)}.$$

Proof. The first part of the lemma is evident from the discussion preceding the lemma. The latter part is a statement of the inversion formula. As there is an identification between the unitary character group of $\mathbb{A}_k/k$ and the field k, the condition $\sum_{\xi \in k} |\hat{\phi}(\xi)| < \infty$ simply means that the function $\hat{\Phi}$ is contained in $L_1(\widehat{(\mathbb{A}_k/k)})$, ensuring that the inversion formula holds. The expression

$$\phi(x) = \sum_{\xi \in k} \hat{\phi}(\xi) e^{2\pi i \Lambda(\xi x)}$$

is a reformulation of the inversion formula in terms of the notational substitutions discussed prior to the lemma. □

Lemma 5.1.4. (lemma 4.2.3, §4.2, [13]) *Let $f(x)$ be continuous and contained in $L_1(\mathbb{A}_k)$, $\sum_{\eta\in k} f(x+\eta)$ uniformly convergent for $x \in D$ (as k is not ordered, by convergence we mean absolute convergence). Then for $\phi(x) = \sum_{\eta\in k} f(x+\eta)$, we have*

$$\hat{\phi}(\xi) = \hat{f}(\xi).$$

Proof. As $\phi(x)$ is a periodic, continuous function on the adèles, we can use the expression for the Fourier transform, mentioned in lemma 5.1.3. Even though ϕ is technically a function on the adèles, for our purpose it behaves like a function on $\mathbb{A}_k/k$ via Φ.

The function f on the other hand is treated solely as a function on the adèles and satisfies the equation 4.2.1 for its Fourier transform.

$$\begin{aligned}
\hat{\phi}(\xi) &= \int_D \phi(x)e^{-2\pi i\Lambda(\xi x)}dx = \int_D \left(\sum_{\eta\in k} f(x+\eta)e^{-2\pi i\Lambda(\xi x)}\right)dx \\
&= \sum_{\eta\in k}\int_D f(x+\eta)e^{-2\pi i\Lambda(\xi x)}dx = \sum_{\eta\in k}\int_{\eta+D} f(x)e^{-2\pi i\Lambda(\xi x-\xi\eta)}dx \\
&= \sum_{\eta\in k}\int_{\eta+D} f(x)e^{-2\pi i\Lambda(\xi x)}dx = \int_{\bigcup_\eta(\eta+D)} f(x)e^{-2\pi i\Lambda(\xi x)}dx \\
&= \int_{\mathbb{A}_k} f(x)e^{-2\pi i\Lambda(\xi x)}dx = \hat{f}(\xi).
\end{aligned}$$

Note that we have used the fact that the integrand is uniformly convergent on D and that D has finite measure (theorem 4.2.10), to interchange the summation and integral signs. We have used lemma 4.2.12 to obtain $\Lambda(\xi\eta) = 0$ as well as theorem 4.2.9 to obtain $\bigcup_{\eta\in k}(\eta+D) = \mathbb{A}_k$. □

Combining lemmas 5.1.3 and 5.1.4, one obtains the Poisson Formula.

Lemma 5.1.5. (Poisson Formula) (lemma 4.2.4, §4.2, [13]) *If $f(x)$ satisfies the conditions:*

i. *$f(x)$ continuous and contained in $L_1(\mathbb{A}_k)$,*

ii. *$\sum_{\xi\in k} f(x+\xi)$ uniformly convergent for $x \in D$,*

iii. *$\sum_{\xi\in k} |\hat{f}(\xi)|$ convergent,* *then*

$$\sum_{\xi \in k} \hat{f}(\xi) = \sum_{\xi \in k} f(\xi).$$

Proof. Since we want to apply the lemmas 5.1.3 and 5.1.4, let us check whether the hypotheses of these lemmas are satisfied by our conditions on f. The hypotheses of lemma 5.1.4 are satisfied by the first two conditions on f. Putting $\phi(x) = \sum_{\xi \in k} f(x + \xi)$, it is easy to see that ϕ is continuous and periodic. As lemma 5.1.4 states that $\hat{\phi}(\xi) = \hat{f}(\xi)$, the third condition ensures that $\sum_{\xi \in k} |\hat{\phi}(\xi)| < \infty$. Thus all the hypotheses of lemma 5.1.3 are also satisfied and now we can apply both these lemmas.

Putting $x = 0$ in the inversion formula of lemma 5.1.3 gives us

$$\phi(0) = \sum_{\xi \in k} \hat{\phi}(\xi).$$

As $\hat{\phi}(\xi) = \hat{f}(\xi)$ by lemma 5.1.4 and $\phi(0) = \sum_{\xi \in k} f(\xi)$, we obtain

$$\sum_{\xi \in k} f(\xi) = \sum_{\xi \in k} \hat{f}(\xi).$$

□

Replacing x by ax where a is an idèle gives us a result which may be regarded as the number theoretic analogue of the Riemann-Roch theorem.

Theorem 5.1.6. (Riemann-Roch Theorem) (theorem 4.2.1, §4.2, [13]) *If $f(x)$ satisfies the conditions:*

i. *$f(x)$ continuous and contained in $L_1(\mathbb{A}_k)$,*

ii. *$\sum_{\xi \in k} f(a(x + \xi))$ convergent for all idèles a and adèles x, and uniformly convergent for $x \in D$,*

iii. *$\sum_{\xi \in k} |\hat{f}(a\xi)|$ convergent for all idèles a,*

then

$$\frac{1}{|a|} \sum_{\xi \in k} \hat{f}(\xi/a) = \sum_{\xi \in k} f(a\xi).$$

Proof. Consider the function $g(x) = f(ax)$. We want to show that this function satisfies all the conditions of the previous lemma and hence satisfies the Poisson Formula. As multiplication by an idèle is a continuous map, and $f(x)$ is continuous, the function $g(x)$ is also continuous. Moreover,

$$\begin{aligned}\int_{\mathbb{A}_k} g(x)dx &= \int_{\mathbb{A}_k} f(ax)dx = \int_{a(\mathbb{A}_k)} f(x)d(x/a) \\ &= \int_{a(\mathbb{A}_k)} f(x)\frac{dx}{|a|} \quad \text{(using lemma 4.2.4)} \\ &= \frac{1}{|a|}\int_{\mathbb{A}_k} f(x)dx \quad \text{(using lemma 4.2.3).}\end{aligned}$$

This shows that since $f(x) \in L_1(\mathbb{A}_k)$, the same is true for $g(x)$. Thus, $g(x)$ satisfies condition $i.$ of lemma 5.1.5.

As

$$\sum_{\xi\in k} g(x+\xi) = \sum_{\xi\in k} f(a(x+\xi)),$$

which is uniformly convergent for all $x \in D$ as per the hypothesis, $g(x)$ satisfies the condition $ii.$ of lemma 5.1.5.

Let us now compute the Fourier transform of g in terms of that for f using reasoning similar to that used to prove that $g(x) \in L_1(\mathbb{A}_k)$.

$$\begin{aligned}\hat{g}(x) &= \int_{\mathbb{A}_k} f(a\eta)e^{-2\pi i\Lambda(x\eta)}d\eta = \frac{1}{|a|}\int_{a(\mathbb{A}_k)} f(\eta)e^{-2\pi i\Lambda(x\eta/a)}d\eta \\ &= \frac{1}{|a|}\int_{\mathbb{A}_k} f(\eta)e^{-2\pi i\Lambda(x\eta/a)}d\eta = \frac{1}{|a|}\hat{f}(x/a).\end{aligned}$$

Hence,

$$\sum_{\xi\in k} |\hat{g}(\xi)| = \frac{1}{|a|}\sum_{\xi\in k} |\hat{f}(\xi/a)|,$$

which is convergent for all idèles a, as stated in the hypothesis. Thus $g(x)$ satisfies condition $iii.$ of lemma 5.1.5 as well.

Now applying the Poisson Formula to $g(x)$, we get

$$\sum_{\xi\in k} \hat{g}(\xi) = \sum_{\xi\in k} g(\xi),$$

which gives,

$$\frac{1}{|a|}\sum_{\xi\in k}\hat{f}(\xi/a)=\sum_{\xi\in k}f(a\xi).$$

□

5.2 The functional equation of the ζ-function

Let us first examine the characters of I_k. We shall be interested only in those characters which are trivial on $k^\times$. Each such character c can be thought of as a character $\bar{c}$ on $I_k/k^\times$ with $\bar{c}(ak^\times) := c(a)$. As $J_k/k^\times \subseteq I_k/k^\times$ is a compact subgroup, $\bar{c}$ restricted to $J_k/k^\times$ is in fact a unitary character, hence, $|c(b)| = |\bar{c}(bk^\times)| = 1$ for all $b \in J_k$. If a character c of I_k is trivial on the norm one idèles, then $c(a)$ depends only the absolute value of a and is thus a character of the value group $(\mathbb{R}^+, \cdot)$, given by $|a| \mapsto |a|^s$, $s \in \mathbb{C}$. Any character of the idèles thus looks like a unitary character of J_k times $|a|^s$, with $s \in \mathbb{C}$ This discussion is analogous to the one for the characters of $k_\mathfrak{p}^\times$ (section 2.3). Another similar concept is that of the exponent of a character. We have $|c(a)| = |a|^\sigma$ for some real number σ, called the exponent of c. There is also the notion of equivalence: two characters are said to be equivalent if they agree on J_k. Then an equivalence class consists of characters of the form $c(a) = c_0(a)|a|^s$, where $c_0(a)$ is a fixed representative of the class while$s$ varies over $\mathbb{C}$. Then each equivalence class can be viewed as a Riemann surface.

Let us now define the ζ-function for the field k. Let $f(x)$ denote a complex valued function on the adèles, let $f(a)$ be its restriction to the idèles. We denote by $\mathfrak{z}$, the class of all functions satisfying the following conditions:

i. $f(x)$ and $\hat{f}(x)$ (the Fourier transform of f) are continuous and belong to $L^1(\mathbb{A}_k)$

ii. $\sum_{\xi\in k} f(a(x+\xi))$ and $\sum_{\xi\in k} \hat{f}(a(x+\xi))$ are convergent for each idèle a and adèle x, this convergence is uniform in the pair (a, x) for $x \in D$ and a ranging over any fixed compact subset of I_k.

iii. $f(a)|a|^\sigma$ and $\hat{f}(a)|a|^\sigma \in L^1(I_k)$ for $\sigma > 1$.

The first condition ensures that the Fourier inversion formula holds. Because of the first two conditions, the Riemann-Roch theorem is valid for functions of $\mathfrak{z}$.

Definition 5.2.1. (definition 4.4.1,§4.4, [13]) *For each $f \in \mathfrak{z}$, we have a function of characters c, defined for all characters of exponent σ greater than 1, given by*

$$\zeta(f,c) = \int f(a)c(a)d'a.$$

This is called the ζ-function of k.

If we repeat the argument of lemma 3.1.2, we see that the ζ-function is regular in the domain of all characters of exponent greater than 1. We seek analytic continuation of the ζ-function to the domain of all characters. This brings us to the Main Theorem of this thesis, but before stating it, we digress to two lemmas which we will use to prove the Main Theorem.

For c with $\sigma > 1$, define $\zeta_t(f,c)$ as $\int_{J_k} f(tb)c(tb)db$. Then,

$$\zeta(f,c) = \int f(a)c(a)d'a = \int_0^\infty \left[\int_{J_k} f(tb)c(tb)db\right]\frac{dt}{t} = \int_0^\infty \zeta_t(f,c)\frac{dt}{t}.$$

As $\zeta_t(f,c)$ is convergent for some c and $|c(tb)| = |tb|^\sigma = t^\sigma$ is constant for $b \in J_k$, $\zeta_t(f,c)$ is absolutely convergent for c of any exponent, for almost all t. Hence the statement of the following lemma makes sense.

Lemma 5.2.2. (lemma A, §4.4, [13]) *For all characters c, we have*

$$\zeta_t(f,c) + f(0)\int_E c(tb)db = \zeta_{1/t}(\hat{f},\hat{c}) + \hat{f}(0)\int_E \hat{c}\left(\frac{1}{t}b\right)db.$$

Proof.

$$\zeta_t(f,c) + f(0)\int_E c(tb)db = \int_{J_k} f(tb)c(tb)db + \int_E f(0)c(tb)db.$$

As $J_k = \bigcup_{\alpha \in k^\times} E$, and the expression is uniformly convergent, we get that the above expression is equal to

$$\sum_{\xi \in k}\int_{\xi E} f(\xi tb)c(tb)db = \int_E \left(\sum_{\xi \in k} f(\xi tb)\right)c(tb)db$$

The Riemann-Roch theorem (theorem 5.1.6) gives us

$$\int_E \left(\sum_{\xi \in k} \hat{f}(\xi/(tb))\right)c(tb)/|tb|db$$

We can write $c(tb)/|tb| = c^{1/tb}|1/tb| = \hat{c}(1/tb)$. Using the transformation $b \mapsto 1/b$ and $db \mapsto db$ gives

$$\int_E \left(\sum_{\xi \in k} \hat{f}(\xi b/t) \right) \hat{c}(b/t) db.$$

Evaluating the right hand side of the equation in the theorem using the steps prior to the use of the Riemann-Roch theorem brings us to exactly this expression. This completes the proof. □

Lemma 5.2.3. (lemma B, §4.4, [13])

$$\int_E c(tb)db = \begin{cases} \kappa t^s & \textit{if } c(a) = |a|^s, \textit{ that is, } c(tb) = t^s \\ 0 & \textit{if } c(a) = c(tb) \textit{ is non trivial on } J \end{cases}$$

Here $\kappa =$ volume of E computed in theorem 4.3.7*. Similarly,*

$$\int_E \hat{c}(b/t)db = \begin{cases} \kappa t^{s-1} = \kappa(\frac{1}{t})^{1-s} & \textit{if } c(a) = |a|^s, \textit{ that is, } c(tb) = t^s \\ 0 & \textit{if } c(a) = c(tb) \textit{ is non trivial on } J \end{cases}$$

Proof. Denoting a character over I_k as c and an element of I_k as $a = tb$. Let c_0 be the restriction of c to J_k, that is, $c_0(tb) := c(b)$. We see that c/c_0 is a character of the value group and is hence given by $(c/c_0)(tb) = |tb|^s = t^s$. Thus $c(tb) = c_0(b)t^s$. Then the integral in question is simply $t^s \int_E c_0(b)db$. As E is the fundamental domain for J_k mod $k^\times$, we can construct a character $\bar{c}$ on $J_k/k^\times$ from the character c_0 on E as follows: $\bar{c}(bk^\times) := c_0(b)$. Then taking the integral of the character $c_0(b)$ over E is like taking the integral over the cosets $bk^\times$ of $J_k/k^\times$, of the character $\bar{c}(bk^\times)$. As $\bar{c}$ is in fact a unitary character on the compact group $J_k/k^\times$, we know that the integral is κt^s if $\bar{c}$ is trivial on $J_k/k^\times$ and zero otherwise. But $\bar{c}(bk^\times)$ trivial on $J_k/k^\times$ is equivalent to $c_0(b)$ being trivial on J_k, which is equivalent to saying that $c(tb) = c_0(b)t^s$ is trivial on J or that $c(tb) = t^s$. This proves the first result. For the second part, using the transformation $b \mapsto 1/b, db \mapsto db$,

$$\int_E \hat{c}(b/t)db = |b/t| \int_E c(t/b)db = |1/t| \int_E c(tb)db.$$

Using the result for the first part gives us the required result. □

We are now ready to prove the Main Theorem.

Theorem 5.2.4. (Main Theorem) (theorem 4.4.1, §4.4, [13]) *The ζ-function can be analytically continued to the domain of all characters. This extended function is single-valued and regular, except at $c(a) = 1$ and $c(a) = |a|$, where it has simple poles with residues $-\kappa f(0)$ and $\kappa f(0)$ respectively, where κ is the volume of E, computed in theorem 4.3.7. The functional equation is given by*

$$\zeta(f, c) = \zeta(\hat{f}, \hat{c}),$$

where $\hat{c}(a) = |a|c^{-1}(a)$.

Proof. From the discussion prior to lemma 5.2.2,we have for c with $\sigma > 1$,

$$\zeta(f, c) = \int_0^\infty \zeta_t(f, c)dt/t = \int_0^1 \zeta_t(f, c)dt/t + \int_1^\infty \zeta_t(f, c)dt/t.$$

Let us first consider the integral,

$$\int_0^1 \zeta_t(f, c)dt/t = \int_{|a|\geq 1} f(a)c(a)d'a.$$

Is this integrable?

$$\int_{|a|\geq 1} |f(a)c(a)|d'a = \int_{|a|\geq 1} |f(a)||a|^\sigma d'a,$$

where σ is the exponent of c. If $\sigma > 1$ then by condition iii. for the class of functions $\mathfrak{z}$, the integral on the right hand side of the above equation is finite. On the other hand, if $\sigma \leq 1$, then this integral is bounded by the integral with $\sigma > 1$ and is thereby finite. The problem arises when we try to deal with the other integral. We tackle this by using the lemmas 5.2.2 and 5.2.3 to change the limits 0 to 1 over the integral sign to the more manageable limits 1 to ∞:

$$\int_0^1 \zeta_t(f, c)dt/t = \int_0^1 \zeta_{1/t}(\hat{f}, \hat{c})dt/t + \left[\int_0^1 \kappa\hat{f}(0)\left(\frac{1}{t}\right)^{1-s} dt/t - \int_0^1 \kappa f(0)t^s dt/t\right],$$

where the expression in the square brackets is to be included if and only if $c(a) = |a|^s$ (see lemma 5.2.3). If $c(a) = |a|^s$, then exponent of c greater than 1 means that $Re(s) > 1$. This confirms that the integrals in the square brackets make sense. To evaluate $\int_0^1 \zeta_{1/t}(\hat{f}, \hat{c})dt/t$, consider the transformation $t \mapsto 1/t$. Then $d(1/t) =$

$-1/t^2 dt$ means that under this transformation, $dt/t \mapsto td(1/t) = -dt/t$. Hence $\int_0^1 \zeta_{1/t}(\hat{f}, \hat{c})dt/t = \int_\infty^1 \zeta_t(\hat{f}, \hat{c})(-dt/t) = \int_1^\infty \zeta_t(\hat{f}, \hat{c})dt/t$. We also evaluate the integrals in square brackets. This gives

$$\int_0^1 \zeta_t(f, c)dt/t = \int_1^\infty \zeta_t(\hat{f}, \hat{c})dt/t + \left[\frac{\kappa\hat{f}(0)}{s-1} - \frac{\kappa f(0)}{s}\right].$$

Putting the pieces together, we have

$$\zeta(f, c) = \int_1^\infty \zeta_t(f, c)\frac{dt}{t} + \int_1^\infty \zeta_t(\hat{f}, \hat{c})\frac{dt}{t} + \left[\frac{\kappa\hat{f}(0)}{s-1} - \frac{\kappa f(0)}{s}\right].$$

All the terms on the right make sense as the integrals are analytic for all c. This is thus the analytic continuation of $\zeta(f, c)$ to the domain of all characters. $c(a) = 1$ gives $s = 0$ in the square bracket and is thus a simple pole with residue $-\kappa f(0)$. $c(a) = |a|$ gives $s = 1$ and is likewise a simple pole with residue $\kappa\hat{f}(0)$. Noting that the exponent of c is s, that of $\hat{c}$ is $1 - s$ and computing in terms of exponent instead of s, the analytic continuation can be written as

$$\zeta(f, c) = \int_1^\infty \zeta_t(f, c)\frac{dt}{t} + \int_1^\infty \zeta_t(\hat{f}, \hat{c})\frac{dt}{t} + \left[-\frac{\kappa\hat{f}(0)}{\text{exponent}(\hat{c})} - \frac{\kappa f(0)}{\text{exponent}(c)}\right]. \quad (5.2.1)$$

This formulation makes it clear that replacing (f, c) by $(\hat{f}, \hat{c})$ merely interchanges the terms within the square bracket. Since $c(-1) = 1$ as c is trivial on $k^\times$,

$$\zeta_t(\hat{\hat{f}}, \hat{\hat{c}})\frac{dt}{t} = \int_{J_k} \hat{\hat{f}}(tb)\hat{\hat{c}}(tb)db = \int_{J_k} f(-tb)c(-tb)db = \int_{J_k} f(-tb)c(tb)db$$

Using $b \mapsto -b$ and $db \mapsto db$, we find that this is the same as $\zeta_t(f, c)$. Thus if we replace (f, c) by $(\hat{f}, \hat{c})$ in equation 5.2.1, the terms outside the square bracket get interchanged and so do the terms inside the square bracket, leaving the whole expression unchanged! This gives the functional equation $\zeta(f, c) = \zeta(\hat{f}, \hat{c})$. □

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