Applications of light-cone superspace in Quantum Field Theory

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by

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Certificate

This is to certify that this thesis entitled "Applications of light-cone superspace in Quantum Field Theory" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by Sarthak Parikh at the Indian Institute of Science Education and Research Pune, under the supervision of Dr. Sudarshan Ananth during the academic year 2011-2012.

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Abstract

In this thesis we focus on supersymmetric quantum field theories. In particular we study the maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills theory in light-cone superspace. We present two applications of light-cone superspace formalism: 1) we derive a new Lagrangian for $\mathcal{N} = 4$ super Yang-Mills, where the scattering amplitudes are manifest, and 2) we initiate a new formalism for computing correlation functions of gauge invariant operators.

Contents

Chapter 1

Introduction

1.1 Quantum field theory

The physics at small-scales is described using quantum mechanics. We resort to special relativity to explain phenomena at high speeds. Since elementary particles are both small and can travel at high speeds, we need to use relativistic quantum mechanics to describe their dynamics. The combination of quantum mechanics and special relativity inevitably leads to the conclusion that particle number is not conserved, *i.e.* particles can be created or annihilated - a feature regularly observed in collider experiments.

However, quantum mechanics is insufficient for describing systems where the particle number is not fixed. Thus we describe particles using fields, which can intefere constructively or destructively resulting in creation or annihilation of particles. Quantum field theory (QFT) is a quantum, relativistic theory of fields, which is well-suited for explaining the interaction of elementary particles.

In quantum mechanics, we quantize the dynamical variables of classical particle mechanics, *i.e.* we promote the dynamical variables in the theory to operators acting on a Hilbert space. In the same way, QFT is the quantization of classical fields, *i.e.* classical fields are promoted to operators acting on a Hilbert space.

1.2 Symmetries

An important object in QFT is the Lagrangian of the system. The Lagrangian gives the classical equations of motion for the system, and the quantum mechanical probability amplitudes. Various symmetry principles govern the form of the Lagrangian. A Lagrangian may have several symmetries, both continuous and discrete. The symmetries may be global or local; they may be spacetime symmetries or internal symmetries. According to Noether's theorem, to every symmetry is associated a conserved quantity. These conserved quantities, also called conserved charges, are the generators of their respective symmetry transformations. We mention below some of these symmetries.

1.2.1 Poincaré invariance

The Lagrangian of a relativistic theory must be invariant under Lorentz transformations, upto a divergence. A stronger condition is Poincaré invariance, which includes Lorentz invariance as well as spacetime translational invariance.

The four conserved Noether currents associated with translational invariance form the energy-momentum tensor. The conserved charges, associated with the conserved currents form the momentum four-vector, P^μ, where P^0 is the total energy carried by the field and $\vec{P}$ is the total linear momentum of the field. Furthermore, there are six conserved charges associated with Lorentz invariance, three of which give the total angular momentum of the field. The ten generators of the Poincaré group form a closed algebra through their commutators called the Poincaré algebra.

1.2.2 Gauge symmetry

Gauge symmetry, or gauge redundancy is a fundamental principle in modern particle physics. It allows for independent, internal symmetry transformations of a field at every point in spacetime.

The simplest example of a gauge theory is classical electrodynamics. The Lagrangian of the theory is given by

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} , \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\nu .$$

The Lagrangian is invariant under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) .$$

This is also known as gauge redundancy since physical states related to each other by gauge transformations are one and the same. Thus we have freedom to fix the gauge, to get rid of the redundancy in the description. Common examples of gauges are the Lorentz gauge, $\partial_\mu A^\mu = 0$, and the Coloumb gauge, $\vec{\nabla}.\vec{A} = 0$.

The gauge field A_μ describes photons. Note that photons have only two physical degrees of freedom, while the gauge field A_μ has four components. Although it is not manifest, the gauge field has two independent components only (this is because the A_0 component has no kinetic term, *i.e.* it is not dynamical, and hence is completely determined in terms of the other three components; furthurmore, fixing a gauge further reduces the number of independent components by one). As we shall comment later, implementing a particular gauge choice, the *light-cone gauge*, provides a description of the theory solely in terms of the physical degrees of freedom. However, this comes at the expense of manifest Lorentz invariance. We discuss Yang–Mills theory in light-cone gauge in more detail in Chapter 2.

Electromagnetism, which is discussed above, is the simplest example of a Yang–Mills theory (also refered to as gauge theory). Quantum electrodynamics (QED) is a theory of the vector field A_μ (gauge field) and a spin-1/2 field ψ (spinor), interacting with each other. The spinor field gauge transforms as

$$\psi(x) \to \mathrm{e}^{-ie\Lambda(x)} \psi(x) ,$$

and the (gauge invariant) QED Lagrangian is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - m\bar{\psi}\psi\,,$$

where γ^μ are the Dirac matrices, and the covariant derivative $D_\mu \equiv \partial_\mu - ieA_\mu$. The spinor ψ describes electrons. The conserved current associated with gauge invariance in QED is the electric current density, and the conserved charge is the electric charge.

QED is an example of a Yang–Mills theory where the gauge symmetry group is U(1). A non-abelian Yang–Mills theory, with gauge symmetry group SU(N) has the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu)\psi - \frac{1}{4}F^{\mu\nu a}F^a_{\mu\nu} - m\bar{\psi}\psi\,,$$

where ψ is now a multiplet of N spinors, the colour labels $a = 1, \ldots, N^2 - 1$, $D_\mu = \partial_\mu - igA^a_\mu T^a$, and

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu\,,$$

where T^a are the generators of the gauge group SU(N), obeying the commutation relations (or Lie algebra)

$$\left[T^a, T^b\right] = if^{abc}T^c\,,$$

where the real factors f^{abc} are the structure constants of the group. The spinors and guuge fields transform under gauge transformations as

$$\psi(x) \to \mathrm{e}^{-ig\Lambda^a(x)T^a}\psi(x)\,, \quad A^a_\mu(x) \to A^a_\mu(x) + \partial_\mu\Lambda^a(x) + gf^{abc}\Lambda^b A^c_\mu\,.$$

Quantum chromodynamics (QCD) is an example of a non-abelian Yang–Mills theory based on the gauge group SU(3).

1.2.3 Supersymmetry

Supersymmetry is a symmetry that relates bosons (fields which transform as integral spin representations of the Poincaré group) to fermions (fields which transform as half-integer spin representations of the Poincaré group). Supersymmetry transformations mix bosons with fermions, and thus their generators are necessarily fermionic. Supersymmetry is an extension of the spacetime Poincaré symmetry, with the supersymmetry generators expanding the Poincaré algebra to the superPoincaré algebra. We discuss supersymmetry in more detail in Chapter 2.

Supersymmetry plays a role in several candidate theories describing physics beyond the Standard Model. However it has not yet been experimentally verified. Regardless of its fate as a physical theory of Nature, supersymmetry has found several uses as a powerful calculational tool. The work presented in this thesis utilizes this useful feature of supersymmetry.

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This thesis is organised as follows. In Chapter 2 we review light-cone gauge, light-cone superspace, and $\mathcal{N} = 4$ super Yang–Mills (SYM) theory. In Chapters 3 and 4 we present two applications of $\mathcal{N} = 4$ super Yang–Mills (SYM) theory in light-cone gauge.

Chapter 2

Light-cone superspace

2.1 Light-cone gauge

In this section, we shall review light-cone coordinates and light-cone gauge, in the context of electromagnetism and (non-abelian) pure Yang–Mills theory.

2.1.1 Light-cone coordinates

We are free to choose any coordinate system we like to tackle a particular problem. We choose to work in light-cone coordinates, because it makes the formulation of our theory easier.

With the space-time metric $(-,+,+,+)$, the light-cone coordinates and their derivatives are defined as

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3)\,; \qquad \partial^{\pm} = \frac{1}{\sqrt{2}}(-\partial_0 + \partial_3); \tag{2.1}$$

$$x = \frac{1}{\sqrt{2}}(x_1 + ix_2)\,; \qquad \bar{\partial} = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2); \tag{2.2}$$

$$\bar{x} = \frac{1}{\sqrt{2}}(x_1 - ix_2)\,; \qquad \partial = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2), \tag{2.3}$$

such that

$$\partial^+ x^- = \partial^- x^+ = -1\,; \qquad \bar{\partial} x = \partial \bar{x} = +1. \tag{2.4}$$

We can also verify that

$$\Box \equiv \partial^\mu \partial_\mu = 2(\partial\bar{\partial} - \partial_+\partial_-). \tag{2.5}$$

x^+ plays the role of light-cone time, ∂_- is now a spatial derivative, and its inverse $\frac{1}{\partial_-}$, is defined using the prescription in [3],

$$\frac{1}{\partial_-} f(x^-) = \int \mathrm{d}\xi\ \theta(\xi - x^-) f(\xi)\,. \tag{2.6}$$

2.1.2 Electromagnetism in light-cone gauge

Classical electromagnetism (with no sources) follows from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \,, \qquad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{2.7}$$

The vector gauge field A_μ represents the photon.

In the usual space-time metric, the gauge field A_μ has four components $\{A_0, A_1, A_2, A_3\}$. Working in light-cone coordinates, the gauge field has components $\{A, \bar{A}, A^+, A^-\}$ with

$$A^{\pm} = \frac{1}{\sqrt{2}}(A^0 \pm A^3) \, ; \quad A = \frac{1}{\sqrt{2}}(A_1 + iA_2) \, ; \quad \bar{A} = \frac{1}{\sqrt{2}}(A_1 - iA_2) \, . \tag{2.8}$$

The defining path integral of the theory is,

$$\int DA \, D\bar{A} \, DA^+ \, DA^- \; e^{i \int \mathrm{d}x^+ L} \quad , \tag{2.9}$$

where the Lagrangian

$$L = -\frac{1}{4} \int \mathrm{d}x \, \mathrm{d}\bar{x} \, \mathrm{d}x^- \, F^{\mu\nu} F_{\mu\nu} \tag{2.10}$$

is a function of $\{A, \bar{A}, A^+, A^-\}$. Recall from the previous chapter that the gauge field A^μ is arbitrary upto a further fixing, which is called gauge freedom. We use gauge freedom in fixing the gauge, by implementing the light-cone gauge

$$A^+ = 0. \tag{2.11}$$

This is effected in the path integral by introducing a delta function

$$\int DA \, D\bar{A} \, DA^+ \, DA^- \; e^{i \int \mathrm{d}x^+ L} \, \delta(A^+) \quad , \tag{2.12}$$

which leaves us with

$$\int DA \, D\bar{A} \, DA^- \; e^{i \int \mathrm{d}x^+ L} \quad , \tag{2.13}$$

where L is now just a function of $\{A, \bar{A}, A^-\}$. It is now possible to rewrite the path integral as a gaussian in A^-:

$$\int DA \, D\bar{A} \; e^{i \int \mathrm{d}^4x \, (\bar{A} \Box A)} \int DA^- \; e^{-\frac{1}{2}\left(\partial_- A^- - \partial \bar{A} - \bar{\partial} A\right)^2} \quad , \tag{2.14}$$

where the A^- component can now be integrated out (after employing a change of variables).

The Lagrangian is now a function of only two components, A and $\bar{A}$. Thus the advantage of choosing light-cone gauge is that we can eliminate the unphysical degrees of freedom, and only work with the physical degrees (in this case, A and $\bar{A}$ represent the left-circularly polarised and the right-circularly polarised light).

2.1.3 Non-abelian Yang–Mills in light-cone gauge

As mentioned in Chapter 1, the Lagrangian for a (non-abelian) pure Yang–Mills theory is

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}\,(F^{\mu\nu}F_{\mu\nu})\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,[A_\mu, A_\nu]\,, \tag{2.15}$$

where the trace is over colour labels and g is the dimensionless coupling constant in the theory. Note that in contrast to free electromagnetism, this Lagrangian contains cubic and quartic terms in A_μ, which represent the interaction vertices in the theory.

To rewrite this Lagrangian in light-cone gauge, we proceed in exactly the same manner as we did for free electromagnetism. We choose the light-cone gauge, and integrate out the A^- component, so that the effective action in the path integral, which is now just a function of $\{A, \bar{A}\}$, is

$$S = \int dx^+ L \tag{2.16}$$

with $L = L^{-+} + L^{-++} + L^{--+} + L^{--++}$, where

$$\begin{aligned}
L^{-+} &= \mathrm{Tr}\int d^3x\ \bar{A}\Box A, && (2.17)\\
L^{-++} &= -2ig\ \mathrm{Tr}\int d^3x\ \left[\frac{\bar{\partial}}{\partial_-}A, A\right]\partial_-\bar{A}, && (2.18)\\
L^{--+} &= -2ig\ \mathrm{Tr}\int d^3x\ \left[\frac{\partial}{\partial_-}\bar{A}, \bar{A}\right]\partial_- A, && (2.19)\\
L^{--++} &= 2g^2\ \mathrm{Tr}\int d^3x\ \frac{1}{\partial_-}\left[\partial_- A, \bar{A}\right]\frac{1}{\partial_-}\left[\partial_-\bar{A}, A\right]. && (2.20)
\end{aligned}$$

2.2 Supersymmetry: superspace and superfields

The Lagrangian for a supersymmetric theory is invariant under supersymmetry transformations. However the Lagrangian, written in terms of the component bosonic and fermionic fields may not be manifestly supersymmetric, *i.e.* its invariance under supersymmetry transformations may not be obvious. However, it is possible to rewrite the Lagrangian in a manifestly supersymmetric form, by introducing new notation, namely *superspace* and *superfields*.

Several different supersymmetric models have been proposed in the literature. Some of these models have more than one kind of supersymmetry transformation. These are known as extended supersymmetric models. The $\mathcal{N} = 4$ super Yang–Mills (SYM) theory in four dimensions is an extended supersymmetric theory, with maximal supersymmetry allowed (for a theory describing particles upto spin one).

The field content of the $\mathcal{N} = 4$ SYM theory comprises a gauge field, A_μ, four Weyl fermions, ψ^m_α, and their conjugates, $\bar{\psi}_{m\,\dot{\alpha}}$, $m = 1, \ldots, 4$, and six real scalars, φ^i, $i = 1, \ldots, 6$.

We now explain what we mean by superspace and superfields in the context of $\mathcal{N}=4$ SYM theory. A covariant superfield formalism of this theory has not been found, but one in light-cone gauge has been successfully developed [4]. We shall work in light-cone gauge in the following.

2.2.1 Grassmann numbers

Before we introduce the superspace, we shall discuss anticommuting numbers, called Grassmann numbers. If η, η' are Grassmann numbers, then

$$\eta^2 = 0\,, \qquad {\eta'}^2 = 0\,, \qquad \eta\eta' = -\eta'\eta\,.$$

Thus the most general function of a single Grassmann number η is

$$f(\eta) = f_0 + f_1\eta\,.$$

The most general function of two Grassmann numbers, η and η' is

$$g(\eta,\eta') = g_0 + g_1\eta + g_2\eta' + g_3\eta\eta'\,.$$

Grassmann integration is defined as follows

$$\int \mathrm{d}\eta = 0\,, \qquad \int \mathrm{d}\eta\; \eta = 1\,.$$

This implies

$$\int \mathrm{d}\eta\, \mathrm{d}\eta'\; \eta\eta' = -\int \mathrm{d}\eta\; \eta \int \mathrm{d}\eta'\; \eta' = -1\,.$$

Thus

$$\int \mathrm{d}\eta\; f(\eta) = f_1\,, \qquad \int \mathrm{d}\eta\, \mathrm{d}\eta'\; g(\eta,\eta') = -g_3\,.$$

The differentiation operator anticommutes with Grassmann numbers

$$\frac{\partial\eta}{\partial\eta} = 1\,, \qquad \frac{\partial(\eta'\eta)}{\partial\eta} = -\eta'\frac{\partial\eta}{\partial\eta} = -\eta'\,.$$

Thus

$$\frac{\partial f}{\partial\eta} = f_1\,, \qquad \frac{\partial g}{\partial\eta} = g_1 + g_3\eta'\,, \qquad \frac{\partial g}{d\eta'} = g_2 - g_3\eta\,, \qquad \frac{\partial}{\partial\eta'}\frac{\partial g}{\partial\eta} = g_3\,.$$

The anticommuting Dirac delta function has the property

$$\int \mathrm{d}\eta\; \delta(\eta-\eta')f(\eta) = f(\eta')\,.$$

This implies that

$$\delta(\eta-\eta') = \eta - \eta'\,.$$

2.2.2 $\mathcal{N}=4$ light-cone superspace

The light-cone superspace for $\mathcal{N}=4$ SYM theory is obtained by adding eight fermionic (Grassmann) coordinates to the four bosonic spacetime coordinates,

$$z = \{x\,, \bar{x}\,, x^+\,, x^-\,, \theta^1\,, \theta^2\,, \theta^3\,, \theta^4\,, \bar{\theta}_1\,, \bar{\theta}_2\,, \bar{\theta}_3\,, \bar{\theta}_4\,\}. \tag{2.21}$$

This can be compactly written as $z = \{x^\mu\,, \theta^m\,, \bar{\theta}_m\}$, where $m = 1, \ldots, 4$. Note that $\overline{(\bar{\theta}_m)} = \theta^m$. Define Grassmann derivatives

$$\bar{\partial}_m \equiv \frac{\partial}{\partial\theta^m}\,; \qquad \partial^m \equiv \frac{\partial}{\partial\bar{\theta}_m}\,, \tag{2.22}$$

then the various anticommutation relations are

$$\begin{aligned} \{\theta^m, \theta^n\} = \{\theta^m, \bar{\theta}_n\} = \{\theta^m, \bar{\theta}_n\} \;&=\; \{\partial^m, \theta^n\} = \{\bar{\partial}_m, \bar{\theta}_n\} = 0\,; \qquad (2.23)\\ \{\partial^m, \bar{\theta}_n\} = \delta^m_n \quad &; \quad \{\bar{\partial}_m, \theta^n\} = \delta^n_m\,. \qquad (2.24)\end{aligned}$$

We also introduce the superspace chiral derivatives, d^m and $\bar{d}_m$, defined as

$$d^m = -\frac{\partial}{\partial\bar{\theta}_m} + \frac{i}{\sqrt{2}}\theta^m\partial_-\,, \qquad \bar{d}_m = \frac{\partial}{\partial\theta^m} - \frac{i}{\sqrt{2}}\bar{\theta}_m\partial_-\,, \quad m = 1, \ldots, 4\,. \tag{2.25}$$

They obey

$$\{d^m, \bar{d}_n\} = i\sqrt{2}\,\delta^m_n\,\partial_-\,. \tag{2.26}$$

The Grassmann integrals in light-cone superspace are normalised so that

$$\int \mathrm{d}\theta_m\,\theta^n = \delta^n_m\,, \qquad \int \mathrm{d}\bar{\theta}^m\,\bar{\theta}_n = \delta^m_n\,. \tag{2.27}$$

We define

$$\mathrm{d}^4\theta = \frac{1}{(4!)^2}\,\varepsilon^{mnpq}\mathrm{d}\theta_m\mathrm{d}\theta_n\mathrm{d}\theta_p\mathrm{d}\theta_q\,, \qquad \mathrm{d}^4\bar{\theta} = \frac{1}{(4!)^2}\,\varepsilon_{mnpq}\mathrm{d}\bar{\theta}^m\mathrm{d}\bar{\theta}^n\mathrm{d}\bar{\theta}^p\mathrm{d}\bar{\theta}^q\,. \tag{2.28}$$

This, together with (2.27), ensures that

$$\int \mathrm{d}^4\theta\,\delta^{(4)}(\theta) = \int \mathrm{d}^4\bar{\theta}\,\delta^{(4)}(\bar{\theta}) = 1\,, \tag{2.29}$$

where the δ-functions are defined as

$$\delta^{(4)}(\theta) = \langle\theta^4\rangle \equiv \varepsilon_{mnpq}\theta^m\theta^n\theta^p\theta^q\,, \qquad \delta^{(4)}(\bar{\theta}) = \langle\bar{\theta}^4\rangle \equiv \varepsilon^{mnpq}\bar{\theta}_m\bar{\theta}_n\bar{\theta}_p\bar{\theta}_q\,. \tag{2.30}$$

Note that due to the anticommutativity of Grassmann numbers and the antisymmetry of the Levi-Civita symbol,

$$\varepsilon_{mnpq}\theta^m\theta^n\theta^p\theta^q = 4!\;\theta^1\theta^2\theta^3\theta^4\,. \tag{2.31}$$

The following identity can be verified using the normalisation of Grassmann integrals

$$\int \mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}\;\theta^m\theta^n\theta^p\theta^q\bar{\theta}_m\bar{\theta}_n\bar{\theta}_p\bar{\theta}_q = \frac{1}{4!}\,. \tag{2.32}$$

2.2.3 $\mathcal{N}=4$ superfield in light-cone superspace

The following description of the $\mathcal{N}=4$ superfield in light-cone superspace closely follows the descriptions presented in [1, 4].

The irreducible representations of the supersymmetry algebra are referred to as *supermultiplets*. Each supermultiplet contains an equal number of bosons and fermions, which are called the component fields. A superfield is a function of superspace coordinates containing all the component fields.

As previously mentioned, the field content of the $\mathcal{N}=4$ SYM theory comprises a gauge field, A_μ, four Weyl fermions, ψ^m_α, and their conjugates, $\bar{\psi}_{m\,\dot{\alpha}}$, $m=1,\ldots,4$, and six real scalars, φ^i, $i=1,\ldots,6$. The gauge field components are

$$A_\pm = \frac{1}{\sqrt{2}}(A_0 \pm A_3)\,, \quad A = \frac{1}{\sqrt{2}}(A_1 + iA_2)\,, \quad \bar{A} = \frac{1}{\sqrt{2}}(A_1 - iA_2)\,. \tag{2.33}$$

The light-cone gauge description of the theory uses only physical degrees of freedom. As before, we fix the gauge by setting $A^+=0$ and integrating out A^-, leaving the two transverse components, A and $\bar{A}$. Similarly the four Weyl fermions, ψ^m_α, and their conjugates, $\bar{\psi}_{m\,\dot{\alpha}}$, are decomposed according to the projection

$$\psi^m_\alpha \;\rightarrow\; \psi^m_{(\pm)} = \mathcal{P}_\pm \psi^m_\alpha\,, \qquad \bar{\psi}_{m\,\dot{\alpha}} \;\rightarrow\; \bar{\psi}^{(\pm)}_m = \mathcal{P}_\pm \bar{\psi}_{m\,\dot{\alpha}}\,, \tag{2.34}$$

where $\mathcal{P}_\pm = -\frac{1}{\sqrt{2}}\sigma^\pm$, with $\sigma^\pm = \frac{1}{\sqrt{2}}(\sigma^0 \pm \sigma^3)$. We then integrate out the $\psi^m_{(+)}$ and $\bar{\psi}^{(+)}_m$ components, leaving four one-component fermionic fields and their conjugates,

$$\lambda^m \equiv \psi^m_{(-)}\,, \qquad \bar{\lambda}_m \equiv \bar{\psi}^{(-)}_m\,. \tag{2.35}$$

The $\mathcal{N}=4$ multiplet is completed by the six real scalar fields, which we represent as $\mathrm{SU}(4)_R$ bi-spinors, φ^{mn}, $m,n=1,\ldots,4$, satisfying the reality condition

$$\bar{\varphi}_{mn} \equiv (\varphi^{mn})^* = \frac{1}{2}\varepsilon_{mnpq}\varphi^{pq}\,. \tag{2.36}$$

An irreducible representation of the $\mathcal{N}=4$ super-algebra is realised in terms of a single complex superfield, $\Phi(x,\theta,\bar{\theta})$, which contains all the fields $(A\,,\bar{A}\,,\lambda^m\,,\bar{\lambda}_m\,,\varphi^{mn})$ as components. The superfield $\Phi(x,\theta,\bar{\theta})$ is defined by the constraints [4, 5]

$$d^m\Phi(x,\theta,\bar{\theta}) = 0\,, \qquad \bar{d}_m\bar{d}_n\Phi(x,\theta,\bar{\theta}) = \frac{1}{2}\varepsilon_{mnpq}d^pd^q\bar{\Phi}(x,\theta,\bar{\theta})\,, \tag{2.37}$$

where $\bar{\Phi}=\Phi^*$ satisfies $\bar{d}_m\bar{\Phi}(x,\theta,\bar{\theta})=0$, where d^m and $\bar{d}_m$ are the chiral derivatives defined in (2.25). The unique solution to these constraints is a superfield with the following component expansion [4]

$$\begin{aligned}\Phi\,(x,\theta,\bar{\theta}) \;=\; & -\frac{1}{\partial_-}A(y) - \frac{i}{\partial_-}\theta^m\bar{\lambda}_m(y) + \frac{i}{\sqrt{2}}\,\theta^m\theta^n\bar{\varphi}_{mn}(y) \\ & + \frac{\sqrt{2}}{6}\theta^m\theta^n\theta^p\varepsilon_{mnpq}\lambda^q(y) - \frac{1}{12}\,\theta^m\theta^n\theta^p\theta^q\varepsilon_{mnpq}\partial_-\bar{A}(y),\end{aligned} \tag{2.38}$$

where we introduced the chiral variable

$$y = (x^+, y^- = x^- - \frac{i}{\sqrt{2}}\theta^m\bar{\theta}_m, x, \bar{x}) \tag{2.39}$$

and the right hand side is understood to be a power expansion about x^-.

In terms of the superfields Φ and $\bar\Phi$, the $\mathcal{N}=4$ SYM light-cone action is [3, 4]

$$\mathcal{S} = 72\int \mathrm{d}^4x \int \mathrm{d}^4\theta\, \mathrm{d}^4\bar\theta\; \mathcal{L}(\Phi,\bar\Phi,\partial_\mu\Phi,\partial_\mu\bar\Phi)\;, \tag{2.40}$$

where the Lagrangian density, $\mathcal{L}(\Phi,\bar\Phi,\partial_\mu\Phi,\partial_\mu\bar\Phi) \equiv \mathcal{L}_{\Phi,\bar\Phi}$, is

$$\begin{aligned}\mathcal{L}_{\Phi,\bar\Phi} \;=\; \mathrm{Tr}\Bigg\{ & -2\,\bar\Phi\,\frac{\Box}{\partial_-^2}\,\Phi + i\frac{8}{3}g\left(\frac{1}{\partial_-}\bar\Phi\,\left[\Phi,\bar\partial\Phi\right] + \frac{1}{\partial_-}\Phi\,\left[\bar\Phi,\partial\bar\Phi\,\right]\right) \\ & +2g^2\left(\frac{1}{\partial_-}\left[\Phi,\partial_-\Phi\right]\frac{1}{\partial_-}\left[\bar\Phi,\partial_-\bar\Phi\,\right] + \frac{1}{2}\left[\Phi,\bar\Phi\,\right]\left[\Phi,\bar\Phi\,\right]\right)\Bigg\}.\end{aligned} \tag{2.41}$$

The superfields Φ and $\bar\Phi$, just like the component fields in the $\mathcal{N}=4$ multiplet, transform in the adjoint representation of the gauge group SU(N). They can therefore be represented as matrices,

$$\Phi(x,\theta,\bar\theta) = \Phi^a(x,\theta,\bar\theta)T^a\,, \qquad \bar\Phi(x,\theta,\bar\theta) = \bar\Phi^a(x,\theta,\bar\theta)T^a\,, \tag{2.42}$$

where T^a, $a=1,\ldots,N^2-1$, are generators of the fundamental representation of SU(N), satisfying

$$\left[T^a,T^b\right] = if^{abc}T^c\,, \qquad \mathrm{Tr}\left(T^aT^b\right) = \frac{1}{2}\,\delta^{ab}\,, \tag{2.43}$$

and f^{abc} are the structure constants for SU(N).

The superfields Φ and $\bar\Phi$ satisfy additional "hidden" constraints. One verifies that (2.37), together with the supersymmetry algebra (2.26), gives the following relations

$$\bar d_m\,\bar d_n\,\bar d_p\,\bar d_q\,\Phi \;=\; 2\,\varepsilon_{mnpq}\,\partial_-^2\,\bar\Phi\,, \tag{2.44}$$

$$\bar d_m\,\bar d_n\,\bar d_p\,\Phi \;=\; i\sqrt{2}\,\varepsilon_{mnpq}\,d^q\,\partial_-\,\bar\Phi\;, \tag{2.45}$$

$$\bar d_m\,\bar d_n\,\Phi \;=\; \frac{1}{2}\,\varepsilon_{mnpq}\,d^p\,d^q\,\bar\Phi\;, \tag{2.46}$$

$$\bar d_m\,\Phi \;=\; \frac{i}{6\sqrt{2}}\,\varepsilon_{mnpq}\,d^n\,d^p\,d^q\,\frac{1}{\partial_-}\,\bar\Phi \tag{2.47}$$

$$\Phi \;=\; \frac{1}{48}\,\varepsilon_{mnpq}d^m\,d^n\,d^p\,d^q\,\frac{1}{\partial_-^2}\,\bar\Phi\,. \tag{2.48}$$

In fact, the complex conjugate of constraint (2.48) can be used to write the Lagrangian for $\mathcal{N}=4$ SYM (2.41) purely in terms of the superfield Φ,

$$\begin{aligned}\mathcal{L}_{\Phi,\bar\Phi} \;=\; & \frac{1}{2}\Phi^a\left(-3\frac{\langle\bar d^4\rangle\Box}{\partial_-^4}\right)\Phi^a \\ & -2gf^{abc}\left[\left(\frac{\langle\bar d^4\rangle}{\partial_-^3}\Phi^a\right)\Phi^b\bar\partial\Phi^c + \frac{1}{48}\left(\frac{1}{\partial_-}\Phi^a\right)\left(\frac{\langle\bar d^4\rangle}{\partial_-^2}\Phi^b\right)\partial\left(\frac{\langle\bar d^4\rangle}{\partial_-^2}\Phi^c\right)\right] \\ & -\frac{g^2}{32}f^{eab}f^{ecd}\left[\frac{1}{\partial_-}\left(\Phi^a\partial_-\Phi^b\right)\frac{1}{\partial_-}\left(\frac{\langle\bar d^4\rangle}{\partial_-^2}\Phi^c\right)\left(\frac{\langle\bar d^4\rangle}{\partial_-}\Phi^d\right)\right. \\ & \qquad\left. +\,\frac{1}{2}\Phi^a\left(\frac{\langle\bar d^4\rangle}{\partial_-^2}\Phi^b\right)\Phi^c\left(\frac{\langle\bar d^4\rangle}{\partial_-^2}\Phi^d\right)\right]\,,\end{aligned} \tag{2.49}$$

where trace has been performed over the colour indices, using (2.43), and $\langle \bar{d}^4 \rangle \equiv \varepsilon^{mnpq} \bar{d}_m \bar{d}_n \bar{d}_p \bar{d}_q$.

On some occassions it will be convenient to rewrite the $\mathcal{N} = 4$ superfield (2.38) as

$$
\begin{aligned}
\Phi(x, \theta, \bar{\theta}) \;=\; & \mathrm{e}^{-\frac{i}{\sqrt{2}} \theta^m \bar{\theta}_m \partial_-} \left[-\frac{1}{\partial_-} A(x) - \frac{i}{\partial_-} \theta^m \bar{\lambda}_m(x) + \frac{i}{\sqrt{2}} \theta^m \theta^n \bar{\varphi}_{mn}(x) \right. \\
& \left. + \frac{\sqrt{2}}{6} \varepsilon_{mnpq} \theta^m \theta^n \theta^p \lambda^q(x) - \frac{1}{12} \varepsilon_{mnpq} \theta^m \theta^n \theta^p \theta^q \, \partial_- \bar{A}(x) \right] . \qquad (2.50)
\end{aligned}
$$

2.2.4 Position space super Feynman rules

Perturbative evaluation of correlation functions or scattering amplitudes in a quantum field theory becomes mathematically increasingly tedious with increasing orders in the coupling constant. Feynman rules for the theory are a set of rules which specify once and for all how to represent each term in the perturbation series with a pictorial representation, called a Feynman diagram. Feynman diagrams offer a convenient way to keep track of the terms in the double exponential series in the coupling constant and external sources. Feynman rules are then used to translate the pictures back to mathematical expressions.

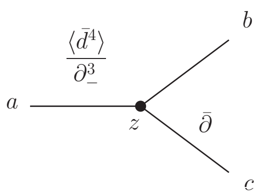
We shall now present the position space Feynman rules for $\mathcal{N} = 4$ SYM theory. We work solely in terms of the chiral superfield, using the action in the form (2.49). The superfield propagator is given by

$$
\begin{aligned}
\Delta^a_b(z - z') \;&=\; \langle \Phi^a(x, \theta, \bar{\theta}) \, \Phi_b(x', \theta', \bar{\theta}') \rangle \\
&=\; -\frac{2}{(4!)^3} \frac{\delta^a_b}{(2\pi)^2} \frac{1}{(x - x')^2} \langle d^4 \rangle \delta^{(4)}(\theta - \theta') \delta^{(4)}(\bar{\theta} - \bar{\theta}'), \qquad (2.51)
\end{aligned}
$$

where $\langle d^4 \rangle \equiv \varepsilon_{mnpq} d^m d^n d^p d^q$, and $\delta^{(4)}(\theta - \theta') \equiv \varepsilon_{mnpq} (\theta^m - \theta'^m)(\theta^n - \theta'^n)(\theta^p - \theta'^p)(\theta^q - \theta'^q)$. The result is derived in appendix A.2.1. In appendix A.2.2 we show that this propagator leads to the correct propagators for the component fields.

The superfield interaction vertices in configuration space can be immediately read off from the superspace action (2.49). They involve a combination of chiral and space-time derivatives and $1/\partial_-$ operators acting on the various legs as well as group theory factors. The two cubic vertices are

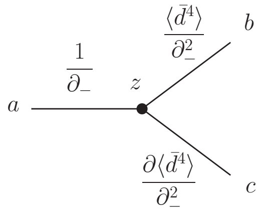
$$
\int \mathrm{d}^{12} z \, (-2g) f^{abc} \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^3} \Phi_a \right) \Phi_b \bar{\partial} \Phi_c \quad \longrightarrow \quad (-2g) f^{abc} \qquad (2.52)
$$



and

$$\int \mathrm{d}^{12}z \, \left(-\frac{g}{24}\right) f^{abc} \left(\frac{1}{\partial_-}\Phi_a\right)\left(\frac{\langle \bar{d}^4\rangle}{\partial_-^2}\Phi_b\right)\partial\left(\frac{\langle \bar{d}^4\rangle}{\partial_-^2}\Phi_c\right)$$

$$\longrightarrow \quad \left(-\frac{g}{24}\right) f^{abc} \tag{2.53}$$

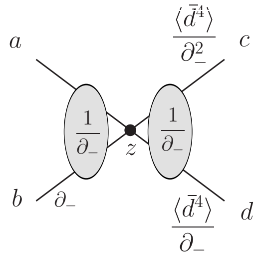


Here we use a black dot to denote interaction vertices, which are integrated over the whole superspace, $z = (x, \theta, \bar{\theta})$, reflecting the fact that all intermediate steps in the calculations are manifestly $\mathcal{N} = 4$ supersymmetric. In the following we will refer to (2.52) and (2.53) as Vertex 3-I and Vertex 3-II respectively.

The two quartic vertices are

$$\int \mathrm{d}^{12}z \, \left(-\frac{g^2}{32}\right) f^{eab} f^{ecd} \left[\frac{1}{\partial_-}(\Phi_a \partial_- \Phi_b)\, \frac{1}{\partial_-}\left(\frac{\langle \bar{d}^4\rangle}{\partial_-^2}\Phi_c \, \frac{\langle \bar{d}^4\rangle}{\partial_-}\Phi_d\right)\right]$$

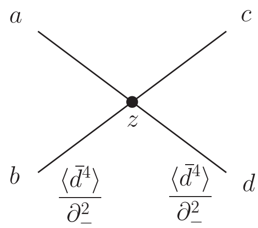
$$\longrightarrow \quad \left(-\frac{g^2}{32}\right) f^{eab} f^{ecd} \tag{2.54}$$



and

$$\int \mathrm{d}^{12}z \, \left(-\frac{g^2}{64}\right) f^{eab} f^{ecd} \left[\Phi_a \left(\frac{\langle \bar{d}^4\rangle}{\partial_-^2}\Phi_b\right)\Phi_c\left(\frac{\langle \bar{d}^4\rangle}{\partial_-^2}\Phi_d\right)\right]$$

$$\longrightarrow \quad \left(-\frac{g^2}{64}\right) f^{eab} f^{ecd} \tag{2.55}$$



In the vertex (2.54) the two $1/\partial_-$ operators in the shaded ovals act on both the adjacent legs. We will refer to (2.54) as Vertex 4-I and to (2.55) as Vertex 4-II.

Chapter 3

MHV Lagrangian

The material in this chapter is largely based on a paper [1] by the author (with Dr. Sudarshan Ananth and Dr. Stefano Kovacs).

3.1 Introduction

Scattering amplitudes carry all the physical information in QFT. The traditional approach to computing scattering amplitudes goes as follows: first we compute the correlation functions in the theory using Feynman diagrams, and then apply the LSZ reduction formula to obtain the amplitudes.

There are several disadvantages of working with the traditional method for a gauge theory. First, the individual Feynman diagrams are not gauge invariant - thus do not reflect the symmetry of the theory. But the sum of all possible Feynman diagrams *is* gauge invariant. However, with growing number of external particles, the number of Feynman diagrams to consider grows very rapidly, *e.g.* for a 9 particle scattering, there are 559,408 diagrams to evaluate - which is impractical. Secondly, explicit computation reveals there are huge cancellations between diagrams, and the final expressions for scattering amplitudes (at tree-level) have a very simple form. The simplicity of the scattering amplitudes hints at the possibility of an alternate way to arrive at the amplitudes.

In particular the simplicity is strikingly evident when one considers planar amplitudes with external states of definite helicity and focusses on the so-called colour-ordered partial amplitudes, as opposed to full cross-sections. They depend only on the momenta and helicities of the n gluons. The simplest non-trivial partial amplitudes are those with two gluons of one helicity and all the others of the opposite helicity. Amplitudes of this type with two gluons of negative helicity and $n-2$ gluons of positive helicity are referred to as maximally helicity violating (MHV).

Cachazo, Svrcek and Witten [7] suggested an approach to perturbative calculations in Yang–Mills theory, refered to as MHV formalism, which makes it possible to construct generic helicity amplitudes by sewing together off-shell MHV amplitudes in the theory. In [9] and [10], it was shown that a “MHV Lagrangian” can be derived from the usual light-cone Yang–Mills Lagrangian, by a suitable field redefinition. The MHV scattering amplitudes are manifest in the new MHV Lagrangian.

We have extended this idea to a supersymmetric Yang–Mills theory, the $\mathcal{N}=4$ supersymmetric Yang–Mills (SYM) theory. In this chapter, we derive a manifestly MHV Lagrangian for the $\mathcal{N}=4$ SYM theory in light-cone superspace. This is achieved by constructing a canonical redefinition which maps the $\mathcal{N}=4$ superfield, Φ, and its conjugate, $\bar{\Phi}$, to a new pair of superfields, χ and $\widetilde{\chi}$. In terms of the new superfields the $\mathcal{N}=4$ Lagrangian takes a manifestly MHV form, *i.e.* involves only MHV vertices. We also identify new constraint relations satisfied by the new superfields. Finally, we test our derivation by showing that an expansion of our superspace Lagrangian in component fields reproduces the correct gluon MHV vertices.

3.2 Helicity assignments

In light-cone gauge we can identify helicity with the U(1) charge associated with rotations in the transverse $(x,\bar{x})$ plane. Complex fields are used to describe particles with helicity. Real fields describe helicity zero particles (Lorentz scalars). In the case of $\mathcal{N}=4$ SYM, the helicity assignments are as given in table 3.1.

Fields	U(1) charge	Helicity	Factors	U(1) charge	Helicity
A	$+1$	$+$	θ^m	$+1/2$	$+$
$\bar{A}$	-1	$-$	$\bar{\theta}_m$	$-1/2$	$-$
λ^m	$-1/2$	$-$	∂	$+1$	$+$
$\bar{\lambda}_m$	$+1/2$	$+$	$\bar{\partial}$	-1	$-$
φ^{mn}	0	0	$\partial^{\pm}$	0	0

Table 3.1: Helicity assignments

As a result the superfield $\Phi(x,\theta,\bar{\theta})$ has definite helicity $+1$, as shown by the component expansion (2.38). Similarly, the expression of the conjugate superfield, $\bar{\Phi}(x,\theta,\bar{\theta})$, in terms of component fields shows that it has helicity -1. This ensures that the $\mathcal{N}=4$ SYM action (2.40) is U(1) neutral as required by Lorentz invariance.

In view of these helicity assignments for the $\mathcal{N}=4$ superfields, Φ and $\bar{\Phi}$, we can write the light-front Lagrangian as

$$L_{\Phi,\bar{\Phi}} = \int_{\Sigma} \mathrm{d}^3x\, \mathrm{d}^4\theta\, \mathrm{d}^4\bar{\theta}\, \left[\mathcal{L}^{(-+)}_{\Phi,\bar{\Phi}} + \mathcal{L}^{(-++)}_{\Phi,\bar{\Phi}} + \mathcal{L}^{(--+)}_{\Phi,\bar{\Phi}} + \mathcal{L}^{(--++)}_{\Phi,\bar{\Phi}}\right]\,, \tag{3.1}$$

where the integration is on a surface of constant x^+, Σ, and the superscripts refer to the number of superfields of helicity $+1$ (Φ) and -1 ($\bar{\Phi}$). Comparing with (2.41) we find

$$\mathcal{L}^{(-+)}_{\Phi,\bar{\Phi}} = -2\,\mathrm{Tr}\left(\bar{\Phi}\frac{\Box}{\partial_-^2}\Phi\right)\,, \tag{3.2}$$

$$\mathcal{L}^{(-++)}_{\Phi,\bar{\Phi}} = i\frac{8}{3}g\,\mathrm{Tr}\left(\frac{1}{\partial_-}\bar{\Phi}[\Phi,\bar{\partial}\Phi]\right)\,,\quad \mathcal{L}^{(--+)}_{\Phi,\bar{\Phi}} = i\frac{8}{3}g\,\mathrm{Tr}\left(\frac{1}{\partial_-}\Phi[\bar{\Phi},\partial\bar{\Phi}]\right) \tag{3.3}$$

and

$$\mathcal{L}^{(--++)}_{\Phi,\bar\Phi} = 2g^2\,\mathrm{Tr}\left(\frac{1}{\partial_-}[\Phi,\partial_-\Phi]\frac{1}{\partial_-}[\bar\Phi,\partial_-\bar\Phi] + \frac{1}{2}[\Phi,\bar\Phi][\Phi,\bar\Phi]\right)\,. \tag{3.4}$$

3.3 Towards a MHV Lagrangian for $\mathcal{N}=4$ SYM

In this section, we identify a superfield redefinition

$$\Phi(x,\theta,\bar\theta) \to \chi(x,\theta,\bar\theta)\,, \qquad \bar\Phi(x,\theta,\bar\theta) \to \widetilde\chi(x,\theta,\bar\theta)\,, \tag{3.5}$$

such that in terms of the new superfields the $\mathcal{N}=4$ action takes a manifestly MHV form. We require the redefinition to be a canonical transformation in superspace, to ensure that the change of variables (3.5) does not give rise to a Jacobian when used in the path integral. The transformation must also preserve the helicity of the superfields, so that $\chi(x,\theta,\bar\theta)$ and $\widetilde\chi(x,\theta,\bar\theta)$ must have the same definite helicities, $+1$ and -1 respectively, as the original superfields.

Our construction of the superfield redefinition (3.5) follows closely that of [10, 11] for the pure Yang–Mills case. As in those papers, we will find that, in order to produce a manifestly MHV Lagrangian, the redefinition (3.5) is necessarily non polynomial. The superfields Φ and $\bar\Phi$ are given by infinite series in the new fields χ and $\widetilde\chi$. We will show that the superfield redefinitions take the form

$$\begin{aligned}\Phi(p) &= \sum_{n=2}^{\infty} g^{n-2}\int \mathrm{d}^3p_1\cdots\mathrm{d}^3p_{n-1}\,\delta^{(3)}(p-p_1-\cdots-p_{n-1})\Gamma(p;p_1,\ldots,p_{n-1})\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\chi(p_1)\cdots\chi(p_{n-1})\\ \bar\Phi(-p) &= -\sum_{n=2}^{\infty}\sum_{s=2}^{n} g^{n-2}\int \mathrm{d}^3p_1\cdots\mathrm{d}^3p_{n-1}\,\delta^{(3)}(p+p_1+\cdots+p_{n-1})\frac{p_-}{(p_s)_-}\\ &\quad\times\Xi^{(s-1)}(p;p_1,\ldots,p_{n-1})\chi(p_1)\cdots\chi(p_{s-1})\widetilde\chi(p_s)\chi(p_{s+1})\cdots\chi(p_{n-1})\end{aligned}$$

where the dependence on the fermionic coordinates, θ and $\bar\theta$, has not been indicated explicitly. Here and in the following d^3p denotes $\mathrm{d}p_-\,\mathrm{d}p\,\mathrm{d}\bar p$ in momentum space integrals. We will outline how to derive the explicit form of the coefficient functions, Γ and Ξ, in these series.

Substituting the expressions of Φ and $\bar\Phi$ in terms of χ and $\widetilde\chi$ gives rise to a Lagrangian of the form

$$L_{\chi,\widetilde\chi} = \int_\Sigma \mathrm{d}^3x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar\theta\left[\mathcal{L}^{(-+)}_{\chi,\widetilde\chi} + \sum_{k=1}^{\infty}\mathcal{L}^{(--\overbrace{+\cdots+}^{k})}_{\chi,\widetilde\chi}\right] \tag{3.6}$$

in which all terms are manifestly MHV in light-cone superspace. Here the k-th term in sum contains two $\widetilde\chi$'s, k χ's and a factor of g^k.

Our superspace analysis presents additional complications, which do not arise in the non-supersymmetric case. The transformed superfields do not satisfy the constraints (2.37) satisfied by the original superfields, and so they are not guaranteed to describe the same degrees of freedom. We shall identify new constraints satisfied by the transformed superfields and prove that χ and $\widetilde\chi$ with these new constraints describe the irreducible $\mathcal{N}=4$ multiplet.

3.3.1 Canonical Transformation

As in the pure Yang–Mills case [9, 10, 11], the aim is to construct the superfield redefinition in such a way as to eliminate the non-MHV cubic vertex, $\mathcal{L}^{(-++)}_{\Phi,\bar\Phi}$, from the Lagrangian. The new superfields, χ and $\widetilde{\chi}$, are thus defined requiring

$$\mathcal{L}^{(-+)}_{\Phi,\bar\Phi} + \mathcal{L}^{(-++)}_{\Phi,\bar\Phi} \to \mathcal{L}^{(-+)}_{\chi,\widetilde{\chi}}\,. \tag{3.7}$$

To ensure the canonicity of the transformation, we define the new superfields, χ and $\widetilde{\chi}$, via a generating functional. In complete analogy with the pure Yang–Mills case [10], we search for a generating functional of the form

$$K(\chi, \pi_\Phi) = \int \mathrm{d}^3x\, \mathrm{d}^4\theta\, \mathrm{d}^4\bar\theta\ \mathrm{Tr}\,[k(\chi)\, \pi_\Phi]\ , \tag{3.8}$$

where π_Φ is the conjugate momentum to Φ. From $K(\chi, \pi_\Phi)$ we construct the canonical change of variables, which is defined by the relations

$$\begin{aligned}
\Phi(x,\theta,\bar\theta) &= \int \mathrm{d}^3x'\mathrm{d}^4\theta'\,\mathrm{d}^4\bar\theta'\ \frac{\delta\big(\mathrm{Tr}\,[k(\chi)\,\pi_\Phi]\,(x',\theta',\bar\theta')\big)}{\delta\pi_\Phi(x,\theta,\bar\theta)} \\
\pi_\chi(x,\theta,\bar\theta) &= \int \mathrm{d}^3x'\mathrm{d}^4\theta'\,\mathrm{d}^4\bar\theta'\ \frac{\delta\big(\mathrm{Tr}\,[k(\chi)\,\pi_\Phi]\,(x',\theta',\bar\theta')\big)}{\delta\chi(x,\theta,\bar\theta)},
\end{aligned} \tag{3.9}$$

where π_χ is the momentum conjugate to χ.

The functional $k(\chi)$ in (3.8) is fixed by the requirement that the Lagrangian written in terms of the new superfields take the desired MHV form. The first equation in (3.9) implies $\Phi = k(\chi)$, *i.e.* it defines the relation between Φ and the new superfield χ. The second equation in (3.9) becomes

$$\pi_\chi(x,\theta,\bar\theta\,) \equiv \frac{1}{\partial_-}\widetilde{\chi}(x,\theta,\bar\theta\,) = \int \mathrm{d}^3x'\mathrm{d}^4\alpha\,\mathrm{d}^4\bar\alpha\, \frac{\delta\,\{\Phi^a(x',\alpha,\bar\alpha)(t^a)^i{}_j\}}{\delta\chi(x,\theta,\bar\theta\,)}\,\frac{1}{\partial_-}\{\bar\Phi^b(x',\alpha,\bar\alpha)(t^b)^j{}_i\}\,. \tag{3.10}$$

and it defines the new superfield $\widetilde{\chi}$ in terms of Φ and $\bar\Phi$. The form of the generating functional (3.8) ensures that the terms involving ∂_+ cancel out in (3.7),

$$\int \mathrm{d}^3x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar\theta\,\mathrm{Tr}\left(2\frac{1}{\partial_-}\bar\Phi\,\partial_+\Phi\right) = \int \mathrm{d}^3x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar\theta\,\mathrm{Tr}\left(2\frac{1}{\partial_-}\widetilde{\chi}\,\partial_+\chi\right)\,. \tag{3.11}$$

We substitute (3.10) into (3.7) to obtain (after Fourier transforming)

$$\omega_1\Phi_1 + i\frac{2}{3}g\int_{23}\delta^{(3)}(p_1-p_2-p_3)\,\frac{\{2{,}3\}}{p_{2-}+p_{3-}}\,\Phi_2\Phi_3 - \int_l \omega_l\frac{\delta\Phi_1}{\delta\chi_l}\,\chi_l = 0, \tag{3.12}$$

where $\Phi_j \equiv \Phi(p_j)$, $\omega_k \equiv \frac{p_k\bar p_k}{p_{k-}}$ and $\{i{,}j\} \equiv (\bar p_j p_{i-} - \bar p_i p_{j-})$ and for the measures we have defined

$$\int_{12\ldots n} \equiv \int\prod_{k=1}^{n}\mathrm{d}p_{k-}\,\mathrm{d}p_k\,\mathrm{d}\bar p_k \qquad \text{and} \qquad \int \mathrm{d}\theta \equiv \int \mathrm{d}^4\theta\,\mathrm{d}^4\bar\theta\,. \tag{3.13}$$

In the following we will also use the notation $(p_i, p_j) = p_j p_{i-} - p_i p_{j-}$.

Condition (3.12) indicates that Φ is a power-series in χ of the following form

$$\Phi_1 = \sum_{n=2}^{\infty} g^{n-2} \int_{2\ldots n} \delta^{(3)}(p_1 - p_2 - \cdots - p_n)\, \Gamma_{1,2,\ldots,n}\, \chi_2 \cdots \chi_n \;. \tag{3.14}$$

$\Gamma_{1,2,\ldots,n} \equiv \Gamma(p_1, p_2, \ldots, p_n)$ are coefficients to be determined order by order. Substituting the ansatz (3.14) back into (3.12), we find the Γ coefficient is

$$\Gamma_{+,2,\ldots,n} = \left(i\frac{2}{3}\right)^{n-2} \frac{p_{2-} p_{3-}^2 p_{4-}^2 \cdots p_{(n-1)-}^2 p_{n-}}{(p_2,p_3)(p_3,p_4)\ldots(p_{n-1},p_n)}\,, \tag{3.15}$$

where $\Gamma_{+,i,j} \equiv \Gamma_{i+j,i,j}$.

Having obtained an all-order expression for the field redefinition for Φ we now turn to $\bar{\Phi}$. We differentiate Φ with respect to χ and substitute the result in (3.10) to obtain the following expression for $\bar{\Phi}$

$$\bar{\Phi}_{-1} = -\sum_{m=2}^{\infty} g^{m-2} \sum_{s=2}^{m} \int_{2\ldots m} \delta^{(3)}(p_1+p_2+\cdots+p_m) \frac{p_{1-}}{p_{s-}}\, \Xi^{s-1}_{1,2,\ldots,m} \chi_2 \cdots \chi_{s-1} \widetilde{\chi}_s \chi_{s+1} \cdots \chi_m, \tag{3.16}$$

where the superscript on Ξ corresponds to the position of $\widetilde{\chi}$ in the string of χ's. To compute the higher order Ξ coefficients, we start with (3.11). From the expansion of Φ in (3.14), since all the fields have the same x^+ dependence and none of the Γ coefficients depend on x^+, we get

$$\partial_+ \Phi_1 = \sum_{n=2}^{\infty} g^{n-2} \sum_{r=2}^{n} \int_{2\ldots n} \delta^{(3)}(p_1 - p_2 - \cdots - p_n)\, \Gamma_{1,2,\ldots,n} \chi_2 \cdots \chi_{r-1} \partial_+ \chi_r \chi_{r+1} \cdots \chi_n. \tag{3.17}$$

We substitute (3.16) and (3.17) in (3.11), and evaluate order by order to find,

$$\Xi^{s-1}_{-,2,\ldots,m} = \frac{p_{1-}}{p_{s-}}\, \Gamma_{+,2,\ldots,m}\,, \qquad (m \geq 3 \;\; ; 2 \leq s \leq m)\,, \tag{3.18}$$

where $\Xi^{s-1}_{-,i,j} = \Xi^{s-1}_{-i-j,i,j}$.

The transformations (3.14) and (3.16) can be inverted to express χ and $\widetilde{\chi}$ in terms of Φ and $\bar{\Phi}$. The details of the calculation in this section are presented in [1].

3.3.2 New constraints

As discussed in section 2.2.3 the original superfields, Φ and $\bar{\Phi}$, are both constrained. They satisfy (anti) chirality conditions,

$$d^m\, \Phi = 0\;, \qquad \bar{d}_m\, \bar{\Phi} = 0\;, \tag{3.19}$$

and constraints (2.44)-(2.48).

The most general superfield in $\mathcal{N} = 4$ superspace does not describe an irreducible multiplet of the $\mathcal{N} = 4$ superalgebra. Imposing the constraints (3.19) and (2.44)-(2.48) reduces the number of independent components in Φ and $\bar{\Phi}$ ensuring that these superfields describe *only* the $\mathcal{N} = 4$ degrees of freedom.

We need to show that the new superfields also describe the $\mathcal{N}=4$ supermultiplet. This is not guaranteed, because in constructing the canonical change of variables, we have treated Φ and $\bar{\Phi}$ as unconstrained. We need to deduce what conditions for χ and $\widetilde{\chi}$ are implied by the constraints on the original superfields and then show that these new conditions give rise to the correct degrees of freedom. This can be achieved starting with the inverse transformations and imposing the conditions (3.19), (2.44)-(2.48) on the right hand side.

From the transformation relating Φ and χ one can verify that the latter is also chiral,

$$d\,\chi = 0\,. \tag{3.20}$$

The remaining constraints on Φ and $\bar{\Phi}$ are, however, not valid for χ and $\widetilde{\chi}$. In particular, the superfield $\widetilde{\chi}$ is not anti-chiral. Moreover, as a consequence of the structure of the field redefinition, we expect the constraints satisfied by χ and $\widetilde{\chi}$ to be modified order by order in the coupling. We will present here the schematic form of the new conditions for χ and $\widetilde{\chi}$ to order g.

We start with the inverse transformations truncated at order g,

$$\chi_p = \Phi_p - g\int_{qr}\delta(p-q-r)\,\Gamma_{p,q,r}\,\Phi_q\,\Phi_r + \mathcal{O}(g^2)\,, \tag{3.21}$$

$$\widetilde{\chi}_p = \bar{\Phi}_p + g\int_{qr}\delta(p-q-r)\left\{\frac{p_-}{q_-}\,\Gamma_{-q,r,-p}\,\bar{\Phi}_q\,\Phi_r\,+\frac{p_-}{r_-}\,\Gamma_{-r,-p,q}\,\Phi_q\,\bar{\Phi}_r\right\} + \mathcal{O}(g^2)\,. \tag{3.22}$$

The expansion (3.21) is consistent with the chirality of χ. Acting with the superspace derivative $\bar{d}_m$ on (3.22) and using (3.19), (2.44)-(2.48) we arrive at the relation

$$\frac{1}{\partial_-^2}\bar{d}\,\widetilde{\chi} \sim g\left(\frac{d^3}{\partial_-}\widetilde{\chi}\right)\widetilde{\chi}\,, \tag{3.23}$$

which replaces the anti-chirality condition for $\bar{\Phi}$.

The additional constraint relations [1], analogous to (2.44)-(2.48), are

$$\frac{d^4}{\partial_-^2}\widetilde{\chi} \;\sim\; \chi + g\chi^2 + g^3\chi^3 + \cdots\,, \tag{3.24}$$

$$\frac{d^3}{\partial_-}\widetilde{\chi} \;\sim\; \bar{d}\chi + g\chi\bar{d}\chi + g^2\chi^2\bar{d}\chi + \cdots\,, \tag{3.25}$$

$$d^2\widetilde{\chi} \;\sim\; \bar{d}^{\,2}\chi + g\bar{d}\chi\bar{d}\chi + g\chi\bar{d}^{\,2}\chi + g^2\chi\bar{d}\chi\bar{d}\chi + g^2\chi^2\bar{d}^{\,2}\chi + \cdots\,, \tag{3.26}$$

$$\partial_- d\widetilde{\chi} \;\sim\; \bar{d}^{\,3}\chi + g\bar{d}\chi\bar{d}^{\,2}\chi + g\chi\bar{d}^{\,3}\chi + \cdots\,, \tag{3.27}$$

$$\partial_-^2\widetilde{\chi} \;\sim\; \bar{d}^{\,4}\chi + g\bar{d}^{\,2}\chi\bar{d}^{\,2}\chi + g\bar{d}\chi\bar{d}^{\,3}\chi + g\chi\bar{d}^{\,4}\chi + \cdots\,. \tag{3.28}$$

Notice that at zero-th order in the coupling χ and $\widetilde{\chi}$ coincide with Φ and $\bar{\Phi}$ respectively. The above conditions are consistent with this observation. The superfield χ is chiral and (3.23) reduces to

$$\bar{d}\,(\widetilde{\chi})_0 = 0\,, \tag{3.29}$$

showing that $\widetilde{\chi}$ is anti-chiral for $g = 0$. Similarly the conditions (3.24)-(3.28) reduce to (2.44)-(2.48) at $g = 0$.

Having obtained the new constraints satisfied by χ and $\widetilde{\chi}$ we proceed to show that they give rise to the correct field content. Since χ is chiral, we can write it as [1]

$$\chi(x,\theta,\bar{\theta}) = \alpha(y) + \beta(y)\,[\theta] + \gamma(y)\,[\theta]^2 + \delta(y)\,[\theta]^3 + \varepsilon(y)\,[\theta]^4\,. \tag{3.30}$$

We find that $\widetilde{\chi}$ satisfying the "inside-out" relations (3.24)-(3.28) is forced to have the structure

$$\begin{aligned} \widetilde{\chi}(x,\theta,\bar{\theta}) \;=\; & A_{00}(y) + A_{10}(y)\,[\theta] + A_{20}(y)\,[\theta]^2 + A_{30}(y)\,[\theta]^3 + A_{40}(y)\,[\theta]^4 \\ & + A_{01}(y)\,[\bar{\theta}] + A_{02}(y)\,[\bar{\theta}]^2 + A_{03}(y)\,[\bar{\theta}]^3 + A_{04}(y)\,[\bar{\theta}]^4\,, \end{aligned} \tag{3.31}$$

where y is the chiral variable (2.39) and all component fields, A_{ij}, $i,j = 0,\ldots,4$, are fully determined in terms of the component fields $\alpha,\beta,\gamma,\delta$ and ε.

The remaining condition on χ and $\widetilde{\chi}$ is (3.23). Imposing this constraint halves the number of independent components in the new superfields. Therefore we conclude that χ and $\widetilde{\chi}$ contain a total of eight bosonic and eight fermionic independent degrees of freedom.

3.4 MHV Lagrangian for $\mathcal{N} = 4$ Yang–Mills

The manifestly MHV Lagrangian in terms of the new superfields χ and $\widetilde{\chi}$ to order g^2 is

$$\begin{aligned} L_{\chi,\widetilde{\chi}} \;=\; & \mathrm{Tr}\int \mathrm{d}\theta\,\mathrm{d}p\; g_0(p)\;\widetilde{\chi}_{-p}\,\chi_p \\ & + g\;\mathrm{Tr}\int \mathrm{d}\theta\,\mathrm{d}p\,\mathrm{d}k\,\mathrm{d}l\;\delta(p+k+l)\;g_1(p,k,l)\;\chi_p\,[\widetilde{\chi}_k\,,\widetilde{\chi}_l] \\ & + g^2\;\mathrm{Tr}\int \mathrm{d}\theta\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r\,\mathrm{d}l\;\delta(p+q+r+l)\;g_{21}(p,q,r,l)\;\;[\chi_p\,,\chi_q]\,[\widetilde{\chi}_r\,,\widetilde{\chi}_l] \\ & + g^2\;\mathrm{Tr}\int \mathrm{d}\theta\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r\,\mathrm{d}l\;\delta(p+q+r+l)\;g_{22}(p,q,r,l)\;\;[\chi_p\,,\widetilde{\chi}_q]\,[\chi_r\,,\widetilde{\chi}_l]\;, \end{aligned} \tag{3.32}$$

where

$$g_0(p) \;=\; -4\,\frac{p\bar{p} - p_+p_-}{p_-^2}\,, \tag{3.33}$$

$$g_1(p,q,r) \;=\; \frac{i4}{3}\,\frac{(q,r)}{p_-(q_- + r_-)}\,, \qquad \boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r} = 0\;, \tag{3.34}$$

$$g_{21}(p,q,r,l) \;=\; \frac{16}{9}\frac{p_-}{q_-^2}\,\frac{(q_- + r_-)^2(q,r)}{l_-(q+r,l)} - \frac{4}{9}\,\frac{q_-p_-(r,l)}{(r_- + l_-)^2(r+l,q)}\,, \tag{3.35}$$

$$g_{22}(p,q,r,l) \;=\; \frac{16}{9}\frac{r_-}{p_-^2}\,\frac{(q_- + r_-)^2(q+r,l)}{q_-(q,r)} - 2\,\frac{(q_-p_- + l_-r_-)}{(q_- + l_-)^2}\,, \tag{3.36}$$

[1] Here we use the notation $[\theta]^n$ to denote n powers of θ without specifying the $\mathrm{SU}(4)_R$ indices.

where $\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}+\boldsymbol{l}=0$.

Using the explicit expressions for the coefficients Γ and Ξ in (3.14) and (3.16) it is possible, though tedious, to derive higher order vertices in the MHV Lagrangian. We will not present these calculations here.

3.5 Component Lagrangian

In this section we discuss the form of the gluon MHV vertices arising from the component expansion of the superspace Lagrangian given in the previous section. These gluon vertices should coincide with those in the pure Yang–Mills MHV Lagrangian [10]. This will thus allow us to test our superspace result. We will carry out the comparison for terms up to order g^2, *i.e.* we will consider cubic and quartic vertices.

We use the inverse transformations writing $(\chi, \widetilde{\chi})$ in terms of $(\Phi, \bar{\Phi})$ to first order, (3.21-3.22). Next we expand the superfields in terms of the component gluon fields., setting all other components in Φ and $\bar{\Phi}$ to zero

$$\begin{aligned} \Phi_p &= -\frac{1}{ip_-}\, A_p\, \mathrm{e}^{-\frac{i}{\sqrt{2}}\theta^m\bar{\theta}_m ip_-} - \frac{1}{12}\varepsilon_{rstu}\theta^r\theta^s\theta^t\theta^u\, ip_-\, \bar{A}_p\,, \\ \bar{\Phi}_p &= -\frac{1}{ip_-}\, \bar{A}_p\, \mathrm{e}^{\frac{i}{\sqrt{2}}\theta^m\bar{\theta}_m ip_-} - \frac{1}{12}\varepsilon^{rstu}\bar{\theta}_r\bar{\theta}_s\bar{\theta}_t\bar{\theta}_u\, ip_-\, A_p\,. \end{aligned} \tag{3.37}$$

In order to make contact with the known form of the MHV gluon couplings we then need to express the component fields, A and $\bar{A}$, in terms of the new fields describing the two helicities of the gluons, B and $\widetilde{B}$. We use the form of the field redefinition derived in [9, 10] for the Yang–Mills case. Using these relations the new fields χ and $\widetilde{\chi}$ can be written in terms of B and $\widetilde{B}$. Substituting these expressions into our superspace Lagrangian (3.32) reproduces exactly the cubic and quartic vertices in the MHV Lagrangian of [9, 10]. The details of the calculation are presented in reference [1].

3.6 Discussion

In this chapter, we constructed a manifestly MHV Lagrangian from the usual Lagrangian for $\mathcal{N}=4$ SYM in light-cone superspace. We found that the canonical field redefinitions which were needed were non-polynomial. We determined the field redefinitions to all orders, and also determined the new constraints satisfied by the redefined fields. We also checked whether these constraints were enough to limit the degree of freedom in the new superfields to describe the $\mathcal{N}=4$ supermultiplet. Finally, we reproduced the Yang–Mills MHV Lagrangian from the MHV Lagrangian for $\mathcal{N}=4$ SYM.

As discussed in the introduction section, a MHV Lagrangian, with MHV vertices manifest, provides a Lagrangian origin to the MHV formalism developed in [7]. This has many consequences, as techniques and lessons from the traditional Lagrangian formulation of field theories may be utilized to understand more deeply properties of scattering amplitudes in QFT.

Chapter 4

Gauge invariant correlation functions

The material in this chapter is largely based on a preprint [2] by the author (with Dr. Sudarshan Ananth and Dr. Stefano Kovacs), which has been submitted for review.

4.1 Introduction

In a conformally invariant gauge theory, the fundamental observables are the correlation functions of gauge-invariant operators. In the case of the $\mathcal{N}=4$ supersymmetric Yang–Mills (SYM) theory such correlation functions play a central role in the context of the AdS/CFT correspondence [12, 13, 14].

In this chapter, we study such correlation functions in $\mathcal{N}=4$ SYM theory using the light-cone superspace formalism. The unique advantage of this formalism is that it provides a description, solely in terms of physical degrees of freedom, in which the full $\mathcal{N}=4$ supersymmetry as well as the SU(4) R-symmetry are manifestly realised. However, this is achieved at the expense of manifest Lorentz invariance.

In this chapter we shall introduce the main features of $\mathcal{N}=4$ light-cone superspace as applied to the study of gauge-invariant correlation functions in position space. We shall specifically present the one-loop calculation of a simple four-point correlator of gauge-invariant scalar operators belonging to the super-multiplet of the energy-momentum tensor. We will reproduce the known tree-level and one-loop results for the four point function of composite operators which are bilinear in the elementary scalars and transform in the $\mathbf{20'}$ representation of the SU(4) R-symmetry of $\mathcal{N}=4$ SYM.

Our primary aim is to develop efficient methods for computing perturbative corrections to correlation functions. We comment on the intriguing possibility that the manifest $\mathcal{N}=4$ supersymmetry in this approach may allow for a compact description of entire multiplets and their correlation functions.

4.2 Perturbative calculations in position space

Gauge-invariant correlation functions in a conformal field theory are most naturally studied in position space rather than momentum space. We now discuss some general aspects of perturbative calculations using the formalism of light-cone superspace. In section 2.2.4, we presented the form of the superfield propagator and summarised the Feynman rules in position space.

Notice that the superfield propagator in position space (2.51) has essentially the same form as in momentum space [5]. Consequently the basic manipulations employed in the calculation of position space super Feynman diagrams are the same as used in momentum space. This represents a distinct feature compared with covariant superspace formalisms, where there are more significant differences between position and momentum space formulations.

Super Feynman diagrams constructed from the interaction vertices and the propagator contain space-time derivatives (∂, $\bar{\partial}$ and ∂_-, but not ∂_+) as well as chiral derivatives d^m and $\bar{d}_m$ defined in (2.25). All these derivatives can be integrated by parts in superspace integrals. They can also be transferred from one end point to the other of the super-propagator they act on, $\Delta(z - z')$, using the fact that the latter is only a function of the difference $(z - z')$. Moreover, the $1/\partial_-$ operators can effectively be "integrated by parts" as explained in (A.45).

The general strategy for the evaluation of position space Feynman diagrams is as follows. The first step consists in computing Grassmann integrals, utilising the fermionic δ-functions in the super-propagator. For this purpose one needs to free up one internal line of all the chiral derivatives, using repeated integrations by parts, and then use the relation (A.46) in appendix A.3.

Once the fermionic integrals at each interaction vertex have been computed, the external super-operators are projected onto specific components, thus drastically reducing the number of non-zero contributions.

At this point the resulting bosonic integrals can be directly compared to the corresponding expressions obtained using Lorentz covariant formulations. In section 4.4 we illustrate these steps in the case of a simple four-point function and we show how the light-cone superspace analysis reproduces the known covariant results prior to the evaluation of the final bosonic integrals.

4.3 Composite operators and correlation functions

In this paper we will only consider examples of correlators of operators constructed from the elementary scalars in the $\mathcal{N} = 4$ multiplet, φ^{mn}. This ensures that the explicit form of the operators remain the same (in light-cone gauge) as in Lorentz covariant formulations. The simplest such operators are scalars of dimension 2 belonging to the super-multiplet of the energy-momentum tensor. They transform in the representation $\mathbf{20'}$ of the SU(4) R-symmetry group and, in terms of the φ^{mn} representation for the elementary scalars, they take the form

$$\begin{aligned} Q^{[mn][pq]} &= \mathrm{Tr}\left(\varphi^{mn}\varphi^{pq}\right) - \frac{1}{12}\varepsilon^{mnpq}\,\mathrm{Tr}\left(\bar{\varphi}_{rs}\varphi^{rs}\right) \\ &= \frac{1}{3}\,\mathrm{Tr}\left(2\varphi^{mn}\varphi^{pq} + \varphi^{mp}\varphi^{nq} - \varphi^{mq}\varphi^{np}\right) . \end{aligned} \tag{4.1}$$

We can express the same operators in terms of the representation φ^i of the scalars as $\mathrm{SU}(4)_R$ vectors as

$$Q^{ij} = \mathrm{Tr}\left(\varphi^i\varphi^j\right) - \frac{1}{6}\delta^{ij}\,\mathrm{Tr}\left(\varphi^k\varphi^k\right)\,. \tag{4.2}$$

The equivalence of the two forms (4.1) and (4.2) can be verified using the identity (A.1).

In order to describe the operators (4.1)-(4.2) in light-cone superspace we introduce composite superfield operators which contain them in their component expansion. For this purpose it is convenient to work with the form (4.2) which, using (A.1) we can rewrite as

$$Q^{ij} = \frac{1}{8}\left(\sigma^{i\,pq}\sigma^{j\,rs} - \frac{1}{3}\delta^{ij}\varepsilon^{pqrs}\right)\mathrm{Tr}\left(\bar\varphi_{pq}\bar\varphi_{rs}\right)\,. \tag{4.3}$$

From the form of the $\mathcal{N}=4$ superfield (2.38) and the definition (2.25) of the chiral derivatives, $\bar d_m$, it is easy to verify that the scalar field $\bar\varphi_{mn}(x)$ in the expansion of $\Phi(z)$ can be isolated as follows

$$\bar\varphi_{mn}(x) = \frac{i}{\sqrt{2}}\left[\bar d_m \bar d_n \Phi(x,\theta,\bar\theta)\right]\big|_{\theta=\bar\theta=0}\,. \tag{4.4}$$

We can then define the super-operator

$$\mathcal{Q}^{ij}(z) = -\frac{1}{16}\left(\sigma^{i\,pq}\sigma^{j\,rs} - \frac{1}{3}\delta^{ij}\varepsilon^{pqrs}\right)\mathrm{Tr}\left[\left(\bar d_p \bar d_q \Phi(z)\right)\left(\bar d_r \bar d_s \Phi(z)\right)\right]\,, \tag{4.5}$$

which contains (4.3) as its $\theta=\bar\theta=0$ component,

$$Q^{ij}(x) = \left[\mathcal{Q}^{ij}(z)\right]\big|_{\theta=\bar\theta=0}\,. \tag{4.6}$$

The only other operator of bare dimension 2 in the $\mathcal{N}=4$ theory is an unprotected one, the superconformal primary operator, $K(x)$, belonging to the long Konishi multiplet [15, 16]. $K(x)$ is a $\mathrm{SU}(4)_R$ singlet and takes the form

$$K = \mathrm{Tr}(\varphi^i\varphi^i) = \frac{1}{4}\varepsilon^{mnpq}\mathrm{Tr}\left(\bar\varphi_{mn}\bar\varphi_{pq}\right)\,. \tag{4.7}$$

Using (4.4) we can construct a super-operator containing $K(x)$ as $\theta=\bar\theta=0$ component. We define

$$\mathcal{K}(z) = -\frac{1}{8}\varepsilon^{mnpq}\mathrm{Tr}\left[\left(\bar d_p \bar d_q \Phi(z)\right)\left(\bar d_r \bar d_s \Phi(z)\right)\right]\,, \tag{4.8}$$

so that

$$K(x) = \left[\mathcal{K}(z)\right]\big|_{\theta=\bar\theta=0}\,. \tag{4.9}$$

In the present paper we consider only correlators of protected operators, focussing on a four-point function of the $\mathcal{Q}^{ij}$ defined in (4.5). In the following section we present the tree-level and one-loop calculations for this four-point function and in deriving our results we will assume the non-renormalisation of two- and three-point functions as it is a gauge-invariant result and thus remains valid when working in light-cone superspace.

4.4 A simple four-point correlation function

The study of four-point correlation functions of protected operators in $\mathcal{N}=4$ SYM provides a good starting point for the application of light-cone superspace techniques to the calculation of off-shell observables.

In the case of four-point functions of $\mathcal{N}=4$ primary operators the dependence on the external points is not fixed by the symmetries of the theory. Quantum corrections to these correlators can be reorganised into functions, $F_4(r,s;g)$, of the coupling constant and two conformally invariant cross ratios, which can be chosen as

$$r=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,,\qquad s=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\,,\tag{4.10}$$

where $x_{ij}^2=(x_i-x_j)^2$.

We consider four-point correlation functions of the operators Q^{ij} given in (4.3),

$$G_4^{(Q)}(x_1,\ldots,x_4)=\langle Q^{i_1j_1}(x_1)\,Q^{i_2j_2}(x_2)\,Q^{i_3j_3}(x_3)\,Q^{i_4j_4}(x_4)\rangle\,,\tag{4.11}$$

which can be obtained from the correlation functions of the corresponding super-operators, $\mathcal{Q}^{ij}$, defined as

$$\mathcal{G}_4^{(Q)}(z_1,\ldots,z_4)=\langle \mathcal{Q}^{i_1j_1}(z_1)\,\mathcal{Q}^{i_2j_2}(z_2)\,\mathcal{Q}^{i_3j_3}(z_3)\,\mathcal{Q}^{i_4j_4}(z_4)\rangle\,,\tag{4.12}$$

by setting to zero the external fermionic coordinates,

$$G^{(Q)}(x_1,\ldots,x_4)=\mathcal{G}_4^{(Q)}(z_1,\ldots,z_4)\Big|_{\theta^{(\alpha)m}=\bar\theta^{(\alpha)}_m=0}\,,\qquad \forall\,\alpha=1,\ldots,4\,,\ m=1,\ldots,4\,,\tag{4.13}$$

where the index α labels the external points.

In this chapter we restrict our attention to a simple four-point function in the class (4.11), which we denote by $G_4^{(H)}(x_1,\ldots,x_4)$. It corresponds to the following choice for the flavour indices

$$G_4^{(H)}(x_1,\ldots,x_4)=\langle Q^{12}(x_1)\,Q^{34}(x_2)\,Q^{34}(x_3)\,Q^{12}(x_4)\rangle\,.\tag{4.14}$$

We re-derive the known tree-level and one-loop contributions to (4.14) working in light-cone superspace. Our starting point is thus

$$\mathcal{G}_4^{(H)}(z_1,\ldots,z_4)=\langle \mathcal{Q}^{12}(z_1)\,\mathcal{Q}^{34}(z_2)\,\mathcal{Q}^{34}(z_3)\,\mathcal{Q}^{12}(z_4)\rangle\,,\tag{4.15}$$

which reduces to (4.14) upon setting to zero the external fermionic coordinates.

The simplifications induced by the choice of $\mathrm{SU}(4)_R$ indices in (4.14) will become apparent in the next subsections where we evaluate this particular four-point function at tree-level and one-loop.

We start by writing (4.14) using the form (4.3) for the Q^{ij} operators,

$$\begin{aligned}G_4^{(H)}(x_1,\ldots,x_4)&=\Big\langle \mathrm{Tr}\left[\left(\varphi^1\varphi^2\right)(x_1)\right]\mathrm{Tr}\left[\left(\varphi^3\varphi^4\right)(x_2)\right]\mathrm{Tr}\left[\left(\varphi^3\varphi^4\right)(x_3)\right]\mathrm{Tr}\left[\left(\varphi^1\varphi^2\right)(x_4)\right]\Big\rangle\\&=\left(\frac{1}{8}\right)^4\sigma^{1m_1n_1}\sigma^{2p_1q_1}\sigma^{3p_2q_2}\sigma^{4m_2n_2}\sigma^{3p_3q_3}\sigma^{4m_3n_3}\sigma^{1m_4n_4}\sigma^{2p_4q_4}\\&\times\Big\langle \mathrm{Tr}\left[(\bar\varphi_{m_1n_1}\bar\varphi_{p_1q_1})(x_1)\right]\mathrm{Tr}\left[(\bar\varphi_{m_2n_2}\bar\varphi_{p_2q_2})(x_2)\right]\mathrm{Tr}\left[(\bar\varphi_{m_3n_3}\bar\varphi_{p_3q_3})(x_3)\right]\mathrm{Tr}\left[(\bar\varphi_{m_4n_4}\bar\varphi_{p_4q_4})\,(x_4)\right]\Big\rangle\,.\end{aligned}\tag{4.16}$$

The explicit form of the super-operator containing (4.16) as its $\theta = \bar\theta = 0$ component is

$$\mathcal{G}_4^{(H)}(z_1,\ldots,z_4) = \frac{1}{16}\left(\frac{1}{8}\right)^4\left(\frac{i}{\sqrt{2}}\right)^8 \sigma^{1m_1n_1}\sigma^{2p_1q_1}\sigma^{3p_2q_2}\sigma^{4m_2n_2}\sigma^{3p_3q_3}\sigma^{4m_3n_3}\sigma^{1m_4n_4}\sigma^{2p_4q_4}$$
$$\times\Big\langle\Big(\bar d^{(1)}_{m_1}\bar d^{(1)}_{n_1}\Phi^a(z_1)\bar d^{(1)}_{p_1}\bar d^{(1)}_{q_1}\Phi^a(z_1)\Big)\Big(\bar d^{(2)}_{m_2}\bar d^{(2)}_{n_2}\Phi^b(z_2)\bar d^{(2)}_{p_2}\bar d^{(2)}_{q_2}\Phi^b(z_2)\Big)$$
$$\times\Big(\bar d^{(3)}_{m_3}\bar d^{(3)}_{n_3}\Phi^c(z_3)\bar d^{(3)}_{p_3}\bar d^{(3)}_{q_3}\Phi^c(z_3)\Big)\Big(\bar d^{(4)}_{m_4}\bar d^{(4)}_{n_4}\Phi^d(z_4)\bar d^{(4)}_{p_4}\bar d^{(4)}_{q_4}\Phi^d(z_4)\Big)\Big\rangle\,. \tag{4.17}$$

Notice that in $G_4^{(H)}(x_1,\ldots,x_4)$ we choose all the Q^{ij} operators with distinct flavour indices, so that when re-writing them in the form (4.3) the second term, which subtracts the $\mathrm{SU}(4)_R$ trace never appears. This leads to simplifications in the calculation since there are fewer contractions to consider.

For compactness of notation, in the following we write the super-propagator as

$$\Delta^i_j(z_1 - z_2) = \frac{k\delta^i_j}{x_{12}^2}\langle d^4\rangle\delta^{(8)}_{12}\,, \tag{4.18}$$

where $k = -2/(2\pi)^2(4!)^3$, $x_{12}^2 = (x_1 - x_2)^2$ and $\delta^{(8)}_{12} = \delta^{(4)}(\theta_1-\theta_2)\delta^{(4)}(\bar\theta_1 - \bar\theta_2)$.

4.4.1 Tree level

At tree level there are multiple contractions possible in (4.17). However, only the one shown in figure 4.1 is non-zero. The reason why all other contractions vanish is evident from the form of $G_4^{(H)}$ in the first line of (4.16): all other contraction are zero because the propagator (A.32) for the elementary scalars is diagonal in flavour space.

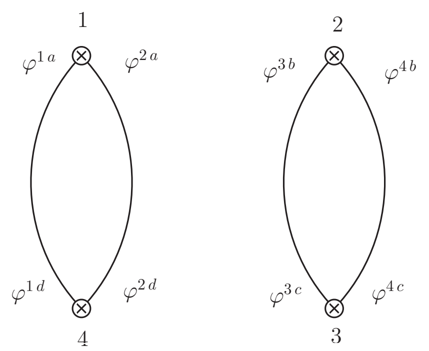


Figure 4.1: Tree-level contribution to $G_4^{(H)}(x_1,\ldots,x_4)$.

It is straightforward to obtain the same result in superspace. A free propagator connecting scalars $\varphi^{a_1\,i_1}(x_1)$ and $\varphi^{a_2\,i_2}(x_2)$ in two Q operators gives rise to the factor

$$\sigma^{i_1m_1n_1}\sigma^{i_2m_2n_2}\left(\bar d_{m_1}\bar d_{n_1}\langle d^4\rangle\,\frac{\delta^{(8)}_{12}}{x_{12}^2}\,\overleftarrow{\bar d_{m_2}\bar d_{n_2}}\,\delta^{a_1a_2}\right)\,, \tag{4.19}$$

which, upon setting to zero the θ and $\bar{\theta}$ coordinates at points z_1 and z_2, reduces to

$$\frac{\delta^{a_1 a_2}}{x_{12}^2}\,\sigma^{i_1 m_1 n_1}\sigma^{i_2 m_2 n_2}\langle d^4\rangle \bar{d}_{m_1}\bar{d}_{n_1}\bar{d}_{m_2}\bar{d}_{n_2}\,\delta_{12}^{(8)}\Big|_{\theta_1=\bar{\theta}_1=0} = (4!)^3\,8\,\frac{\delta^{a_1 a_2}}{x_{12}^2}\,\delta^{i_1 i_2}\,, \tag{4.20}$$

where $\sigma^{i_1 m_1 n_1}\sigma^{i_2 m_2 n_2}\varepsilon_{m_1 n_1 m_2 n_2} = 8\,\delta^{i_1 i_2}$. Thus each external φ^i can only be connected through a free propagator to a φ^j with $i = j$ for a non-vanishing contribution. Therefore at tree level the only allowed contraction in $\mathcal{G}_4^{(H)}(z_1,\ldots,z_4)$ is the one in figure 4.1 which, using (4.18), yields

$$\begin{aligned}
\mathcal{G}_4^{(H)}(z_1,\ldots,z_4) = {}& \frac{1}{2^{20}}\,\sigma^{1 m_1 n_1}\sigma^{2 p_1 q_1}\sigma^{3 p_2 q_2}\sigma^{4 m_2 n_2}\sigma^{3 p_3 q_3}\sigma^{4 m_3 n_3}\sigma^{1 m_4 n_4}\sigma^{2 p_4 q_4} \\
& \times k^4\left(\bar{d}_{m_1}\bar{d}_{n_1}\langle d^4\rangle\frac{\delta_{14}^{(8)}}{x_{14}^2}\overleftarrow{\bar{d}_{n_4}\bar{d}_{m_4}}\delta^{ad}\right)\left(\bar{d}_{p_1}\bar{d}_{q_1}\langle d^4\rangle\frac{\delta_{14}^{(8)}}{x_{14}^2}\overleftarrow{\bar{d}_{q_4}\bar{d}_{p_4}}\delta^{ad}\right) \\
& \times\left(\bar{d}_{m_2}\bar{d}_{n_2}\langle d^4\rangle\frac{\delta_{23}^{(8)}}{x_{23}^2}\overleftarrow{\bar{d}_{n_3}\bar{d}_{m_3}}\delta^{bc}\right)\left(\bar{d}_{p_2}\bar{d}_{q_2}\langle d^4\rangle\frac{\delta_{23}^{(8)}}{x_{23}^2}\overleftarrow{\bar{d}_{q_3}\bar{d}_{p_3}}\delta^{bc}\right)\,.
\end{aligned} \tag{4.21}$$

Setting to zero all the external θ^m's and $\bar{\theta}_m$'s we get

$$\left[G_4^{(H)}(x_1,\ldots,x_4)\right]_{\text{tree}} = \frac{(N^2-1)^2}{16(2\pi)^8}\,\frac{1}{(x_{14}^2)^2(x_{23}^2)^2}\,. \tag{4.22}$$

4.4.2 One-loop

One-loop contributions to $G_4^{(H)}(x_1,\ldots,x_4)$ are of order g^2 and involve either two cubic interaction vertices or a single quartic vertex. Moreover we can distinguish between disconnected diagrams, which factorise into the product of tree-level and one-loop two-point functions, and connected four-point diagrams.

Factorised two-point functions

Figure 4.2 depicts the disconnected one-loop contributions to $G_4^{(H)}$. They factorise as

$$\langle \mathcal{Q}^{12}(z_1)\,\mathcal{Q}^{12}(z_4)\rangle_{1-\text{loop}}\langle \mathcal{Q}^{34}(z_2)\,\mathcal{Q}^{34}(z_3)\rangle_{\text{tree}}\,. \tag{4.23}$$

A second set of diagrams in which the interaction vertices connect to the external points z_2 and z_3 gives rise to a contribution of the form

$$\langle \mathcal{Q}^{12}(z_1)\,\mathcal{Q}^{12}(z_4)\rangle_{\text{tree}}\langle \mathcal{Q}^{34}(z_2)\,\mathcal{Q}^{34}(z_3)\rangle_{1-\text{loop}}\,. \tag{4.24}$$

Both (4.23) and (4.24) vanish thanks to the non-renormalisation of two-point functions of protected operators. Therefore we assume that $G_4^{(H)}(x_1,\ldots,x_4)$ receives no contribution from the sum of all diagrams with the topologies in figure 4.2.

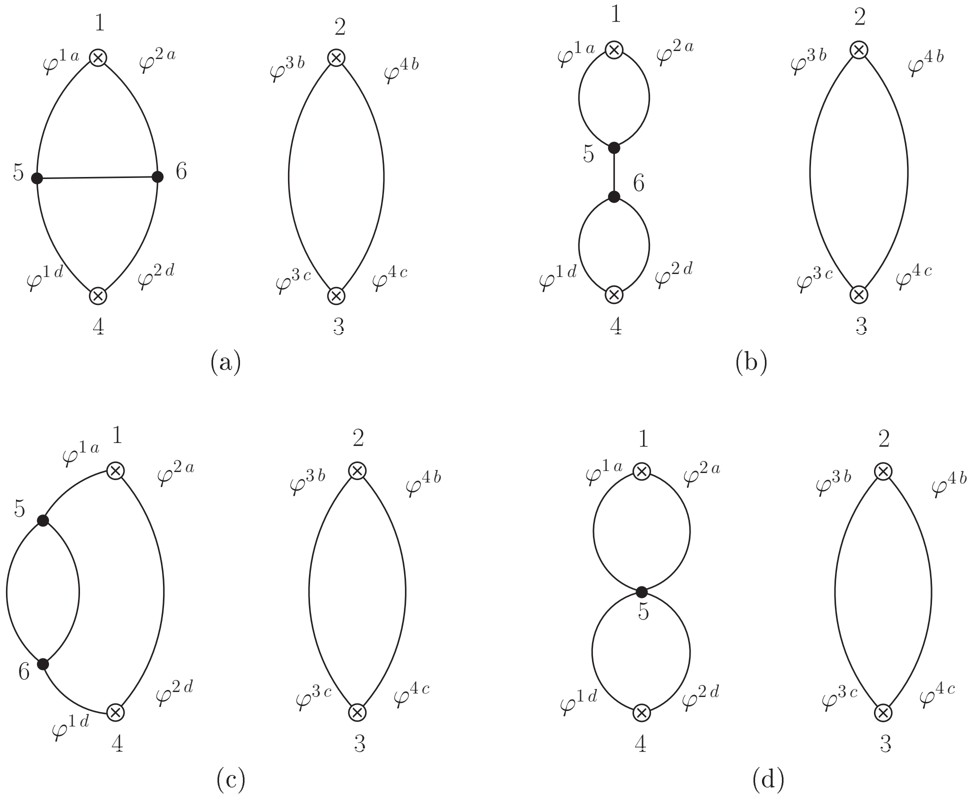


Figure 4.2: Disconnected one-loop contributions to $G_4^{(H)}(x_1, \ldots, x_4)$.

Connected diagrams involving two cubic vertices

The next set of diagrams of order g^2 that we need to consider are connected ones involving two cubic vertices. There are two distinct types of contractions to take into account which are shown in figure 4.3.

The building blocks for these diagrams are the cubic vertices (2.52) and (2.53). Analysing the combinations of chiral derivatives in these vertices one can verify that in order to produce a potentially non-vanishing contribution a diagram must involve one vertex of each type. This is proven in appendix A.4.1.

The contributions from the two diagrams in figure 4.3 vanish individually, but for different reasons.

The vanishing of diagrams of the type in figure 4.3a is straightforward. Since the superfield propagator is diagonal in colour space, the free contractions between points z_1 and z_4 and between points z_2 and z_3, combined with the traces at each external point, force two of the indices of the totally antisymmetric structure constants f^{abc} at the interaction vertices in z_5 and z_6 to be the same. Therefore these diagrams are identically zero. Since this vanishing result follows from the colour structure of the diagram, all other Wick contractions, which differ only in the distribution of flavour indices, give a zero result as well.

Diagrams of the type shown in figure 4.3b also vanish, but the proof is slightly

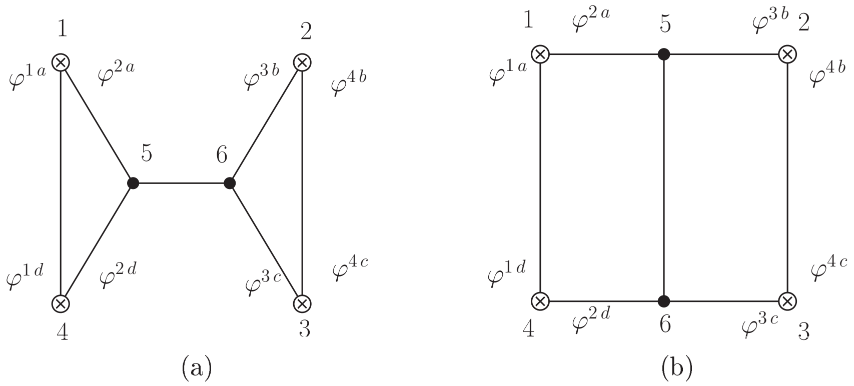


Figure 4.3: Connected one-loop contributions to $G_4^{(H)}(x_1,\ldots,x_4)$ involving cubic vertices.

more involved, requiring manipulations which are described in detail in appendix A.4.1. The vanishing of contributions with this topology follows from the observation that a contraction in which two external fields φ^{i_1} and φ^{i_2} are connected to a cubic interaction vertex gives rise to a factor of $\sigma^{i_1mn}\sigma^{i_2pq}\varepsilon_{mnpq} = 8\delta^{i_1i_2}$. The reason for this is explained under Rule A.4.1 in Appendix A.4.1.

In the case of the diagram in figure 4.3b the internal point z_5 (z_6) connects φ^2 with φ^3, which results in a factor of $\sigma^{2mn}\sigma^{3pq}\varepsilon_{mnpq} = 0$. Other Wick contractions, with a different distribution of flavour indices, vanish for the same reason.

Connected diagrams involving one quartic vertex

The last type of contribution to $G_4^{(H)}(x_1,\ldots,x_4)$ at order g^2 comes from diagrams involving a single quartic vertex. With our choice of external flavours the only allowed topology is depicted in figure 4.4, where the interaction vertex at point z_5 can be either (2.54) or (2.55). The first type of contribution, constructed using the vertex (2.54), vanishes. Therefore the entire one-loop correction to $G_4^{(H)}(x_1,\ldots,x_4)$ comes from diagrams of the type in figure 4.4, with the quartic interaction at point z_5 corresponding to Vertex 4-II (2.55).

We present below the calculation of the contraction shown in the figure, in which the two free propagators connecting points z_1 and z_4 and points z_2 and z_3 carry flavour 1 and 4 respectively. There are additional contributions in which the $z_1 - z_4$ line has flavour 2 and/or the $z_2 - z_3$ line has flavour 3. These produce the same contribution as the diagram we analyse and therefore simply give rise to a multiplicity factor in the final answer.

The vanishing of diagrams involving Vertex 4-I (2.54) follows from Rule A.4.2 in Appendix A.4.2. The requirement that the structure functions be non-zero conflicts with the requirement that the $\sigma\sigma\varepsilon$ contractions be non-zero. Consequently Vertex 4-I does not contribute.

Finally we come to the calculation of the non-zero contribution from diagrams of the topology in figure 4.4 in which the interaction vertex is of type 4-II.

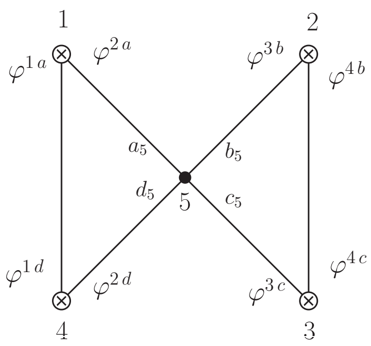


Figure 4.4: Connected one-loop contributions to $G_4^{(H)}(x_1,\ldots,x_4)$ involving a quartic vertex (a_5, b_5, c_5 and d_5 are colour indices).

We factorise the diagram as in figure 4.5. The different Wick contractions correspond to inequivalent ways of gluing together parts (a) and (b) in the figure.

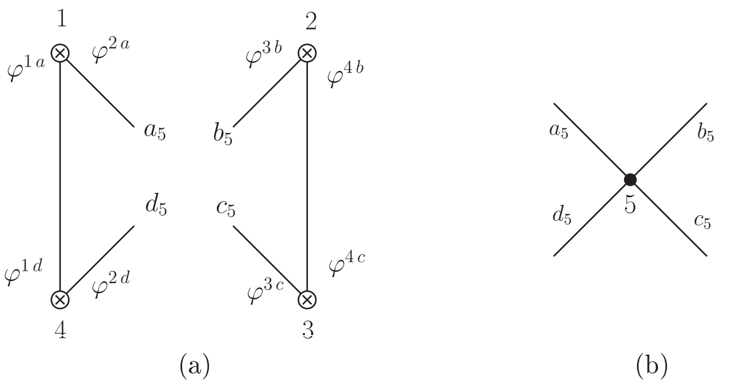


Figure 4.5: Factorisation of diagram involving a quartic vertex.

The following contribution comes from figure 4.5a and is common to all diagrams in this set

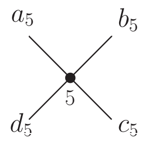
$$E_4[a_5,b_5,c_5,d_5] = \frac{1}{16}\left(\frac{1}{8}\right)^4 \sigma^{1m_1n_1}\sigma^{2p_1q_1}\sigma^{3p_2q_2}\sigma^{4m_2n_2}\sigma^{3p_3q_3}\sigma^{4m_3n_3}\sigma^{1m_4n_4}\sigma^{2p_4q_4}\;k^6\delta^{ad}\delta^{bc}$$
$$\times\left(\frac{i}{\sqrt{2}}\right)^8\left(\bar d_{m_1}\bar d_{n_1}\langle d^4\rangle\frac{\delta^8_{14}}{x^2_{14}}\overleftarrow{\bar d_{n_4}\bar d_{m_4}}\right)\left(\bar d_{m_2}\bar d_{n_2}\langle d^4\rangle\frac{\delta^8_{23}}{x^2_{23}}\overleftarrow{\bar d_{n_3}\bar d_{m_3}}\right)f^{ea_5b_5}f^{ec_5d_5}\left(-\frac{g^2}{64}\right). \quad (4.25)$$

This common portion simplifies to

$$E_4[a_5,b_5,c_5,d_5] = T(\sigma)\left(-\frac{g^2}{2^{26}}\right)k^6\,\delta^{ad}\delta^{bc}f^{ea_5b_5}\;f^{ec_5d_5}\frac{(4!)^6}{8^4}\varepsilon_{m_1n_1n_4m_4}\varepsilon_{m_2n_2n_3m_3}\frac{1}{x^2_{14}}\frac{1}{x^2_{23}}, \quad (4.26)$$

where $T(\sigma)$ denotes the product of the eight σ coefficients in (4.25).

We now need to consider all possible ways of gluing of this factor with the piece resulting from figure 4.5b. We use the following notation,



$$\equiv V_4[a_5, b_5, c_5, d_5]\,,$$

where the order of the arguments in V_4 corresponds to the clockwise labelling in the vertex starting from the top left leg.

The different Wick contractions are analysed in appendix A.4.2. Combining all the non-zero contributions we find that figure 4.4 evaluates to

$$-\,g^2 f^{abc} f^{abc} \frac{1}{8(2\pi)^{12}} \frac{1}{x_{14}^2 x_{23}^2} \int \mathrm{d}^4 x_5 \frac{1}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2}\,. \tag{4.27}$$

Using $f^{abc} f^{abc} = N(N^2-1)$ and including all multiplicity factors the complete one-loop contribution to (4.14) is therefore

$$\Big[G_4^{(H)}(x_1, \ldots, x_4)\Big]_{1\text{-loop}} = -g^2 N(N^2-1) \frac{1}{2(2\pi)^{12}} \frac{1}{x_{14}^2 x_{23}^2} \int \mathrm{d}^4 x_5 \frac{1}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2}\,. \tag{4.28}$$

The box integral in (4.28) is well known [17] and can be expressed in terms of the cross ratios (4.10). Using the form of the box integral in [18], the one-loop contribution to $G_4^{(H)}(x_1, \ldots, x_4)$ takes the form

$$\Big[G_4^{(H)}(x_1, \ldots, x_4)\Big]_{1\text{-loop}} = -g^2 N(N^2-1)\pi^2 \frac{1}{2(2\pi)^{12}} \frac{1}{x_{14}^2 x_{23}^2 x_{13}^2 x_{24}^2}\, F_4^{(H)}(r,s)\,, \tag{4.29}$$

where $F_4^{(H)}(r,s)$ can be expressed as a combination of logarithms and dilogarithms as

$$\begin{aligned} F_4^{(H)}(r,s) \;=\; & \frac{1}{\sqrt{p}} \Bigg\{ \log(r)\log(s) - \left[\log\left(\frac{r+s-1-\sqrt{p}}{2}\right)\right]^2 + \\ & -2\,\mathrm{Li}_2\left(\frac{2}{1+r-s+\sqrt{p}}\right) - 2\,\mathrm{Li}_2\left(\frac{2}{1-r+s+\sqrt{p}}\right)\Bigg\}\,, \end{aligned} \tag{4.30}$$

where $\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ and

$$p = 1 + r^2 + s^2 - 2r - 2s - 2rs\,. \tag{4.31}$$

4.5 Discussion

In this chapter we studied correlation functions of gauge-invariant operators in $\mathcal{N} = 4$ SYM using the light-cone superspace formulation. Our main goals are to develop efficient techniques for the computation of perturbative corrections to correlation functions.

As a computational tool light-cone superspace is particularly promising for a number of reasons. This formulation of the $\mathcal{N}=4$ SYM theory uses only one type of superfield, which carries no space-time or $\mathrm{SU}(4)_R$ indices. Therefore the general structure of super Feynman diagrams and the combinatorial analysis involved in their study are simpler than in other formulations. Moreover we expect that the manifest $\mathcal{N}=4$ supersymmetry will lead to a significant computational advantage, in terms of the number of diagrams to evaluate, at higher orders in the perturbative expansion.

In the case of the simple four-point function $G_4^{(H)}(x_1,\ldots,x_4)$ we reproduced the known result to one-loop order. The light-cone gauge thus yields a manifestly Lorentz covariant result. This is thanks to non-trivial cancellations of derivatives and $1/\partial_-$ factors. It will be important to understand these cancellations in a systematic way for more complicated correlation functions and/or at higher orders in perturbation theory.

One of the benefits of superspace formulations of supersymmetric gauge theories is the possibility of providing a compact description of entire multiplets in terms of superfields. In this respect the light-cone superspace description of $\mathcal{N}=4$ SYM is particularly interesting as it is the only formulation of the theory in which the full $\mathcal{N}=4$ supersymmetry is manifest. Working with super-operators such as (4.5) and (4.8) should make it possible to extract all correlation functions of operators in the same supersymmetry multiplet from a single super-correlator. It will be interesting to study other components in the θ-expansion of the super-correlation function $\mathcal{G}_4^{(Q)}$. These should contain information about correlation functions of the super-partners of the Q^{ij}'s.

Although here we were concerned only with correlation functions of the superconformal primaries (4.1)-(4.2), it is natural to speculate that the light-cone superspace formalism will permit a description of the entire energy-momentum tensor multiplet using a single composite superfield. This will require the addition of terms cubic and quartic in the superfield Φ to the super-operator (4.5). These additional terms should not modify the $\theta=\bar{\theta}=0$ component, while producing the correct cubic and quartic terms in the remaining operators. The exact form of these additional terms in the super-operator should be determined by the entire $\mathcal{N}=4$ superalgebra, including the non-linearly realised dynamical generators. The possibility of constructing such a composite superfield operator is intriguing.

Appendix A

Conventions and useful formulae

A.1 Conventions and notation

The scalar fields in the $\mathcal{N}=4$ multiplet can be represented either as $\mathrm{SU}(4)_R$ bi-spinors, φ^{mn}, satisfying the reality condition (2.36) or as vectors, φ^i, $i=1,\ldots,6$. The two representations are related by

$$\varphi^i = \frac{1}{\sqrt{8}}\Sigma^i_{mn}\varphi^{mn} = \frac{1}{2\sqrt{8}}\,\varepsilon^{mnpq}\Sigma^i_{mn}\,\bar{\varphi}_{pq} = \frac{1}{\sqrt{8}}\,\sigma^{i\,pq}\,\bar{\varphi}_{pq}\ . \tag{A.1}$$

where Σ^i_{mn} ($\bar{\Sigma}^{mn}_i$) are Clebsch-Gordan coefficients relating the product of two $\mathbf{4}$'s ($\bar{\mathbf{4}}$'s) to the $\mathbf{6}$ of SU(4). They are defined as follows

$$\begin{aligned}
\Sigma^i_{mn} &= (\Sigma^I_{mn}, \Sigma^{I+3}_{mn}) = (\eta^I_{mn}, i\bar{\eta}^I_{mn})\,, \\
\bar{\Sigma}^{mn}_i &= (\bar{\Sigma}^{mn}_I, \bar{\Sigma}^{mn}_{I+3}) = (\eta^I_{mn}, -i\bar{\eta}^I_{mn})\,, \qquad I=1,2,3\,,
\end{aligned} \tag{A.2}$$

where $(\eta^I_{mn}, i\bar{\eta}^I_{mn})$ are 't Hooft symbols,

$$\begin{aligned}
&\eta^I_{mn} = \bar{\eta}^I_{mn} = \varepsilon_{Imn}\,, \quad m,n=1,2,3 \\
&\eta^I_{m4} = \bar{\eta}^I_{4m} = \delta^I_m\,, \quad m=1,2,3 \\
&\eta^I_{mn} = -\eta^I_{nm}\,, \quad \bar{\eta}^I_{mn} = -\bar{\eta}^I_{nm}\,.
\end{aligned} \tag{A.3}$$

Splitting up the i index in terms of $I=1,2,3$, the coefficients (A.2) can be written as

$$\begin{aligned}
\Sigma^I_{mn} &= \varepsilon^I{}_{mn4} + (\delta^I_m\delta^4_n - \delta^I_n\delta^4_m)\,, \\
\Sigma^{I+3}_{mn} &= i\varepsilon^I{}_{mn4} - i(\delta^I_m\delta^4_n - \delta^I_n\delta^4_m)\,.
\end{aligned} \tag{A.4}$$

From this we obtain the $\sigma^{i\,mn}$ coefficients

$$\begin{aligned}
\sigma^{Ipq} &= \varepsilon^{Ipq4} + (\delta^{Ip}\delta^q_4 - \delta^p_4\delta^{Iq})\,, \\
\sigma^{(I+3)pq} &= -i\varepsilon^{Ipq4} + i(\delta^{Ip}\delta^q_4 - \delta^p_4\delta^{Iq})\ .
\end{aligned} \tag{A.5}$$

A.2 Superfield and component field propagators

In this appendix we discuss in detail the derivation of the propagator (2.51) for the $\mathcal{N}=4$ superfield. We start with a path integral derivation which will allow us to check the consistency of various conventions for Grassmann integrals and functional derivatives.

A.2.1 Path integral derivation

The superfield propagator can be obtained inverting the kinetic operator in (2.49). We can obtain it constructing the generating functional for Green functions of the $\mathcal{N}=4$ superfield in the free theory limit, $Z_0[J]$.

Functional differentiation of $Z[J]$ with respect to the sources, $J(x,\theta,\bar\theta)$, gives rise to Green functions of the $\mathcal{N}=4$ superfields. Because of the chirality of both Φ and J we need to be careful in defining the rules for functional differentiation in superspace. In defining the functional derivative with respect to a chiral superfield we require the condition that the variation of a chiral superfield be chiral. To satisfy this condition we consider a chiral superfield, $\Psi(x,\theta,\bar\theta)$, written in terms of the chiral variable (2.39) and we impose

$$\frac{\delta\Psi(y',\theta')}{\delta\Psi(y,\theta)} = \delta^{(4)}(y-y')\delta^{(4)}(\theta-\theta')\,. \tag{A.6}$$

To obtain the form of the derivative $\delta\Psi(x',\theta',\bar\theta')/\delta\Psi(x,\theta,\bar\theta)$ in terms of the standard superspace coordinates we consider

$$\frac{\delta}{\delta\Psi(x,\theta,\bar\theta)}\int \mathrm{d}^4x'\mathrm{d}^4\theta'\mathrm{d}^4\bar\theta'\,\Psi(x',\theta',\bar\theta')\,F(x',\theta',\bar\theta')\,, \tag{A.7}$$

where $F(x,\theta,\bar\theta)$ is a generic (non-chiral) superfield. Using (A.6) we can evaluate (A.7) as follows

$$\begin{aligned}
&\frac{\delta}{\delta\Psi(x,\theta,\bar\theta)}\int \mathrm{d}^4x'\mathrm{d}^4\theta'\mathrm{d}^4\bar\theta'\,\Psi(x',\theta',\bar\theta')\,F(x',\theta',\bar\theta')\\
&=\int \mathrm{d}^4y'\mathrm{d}^4\theta'\mathrm{d}^4\bar\theta'\,\frac{\delta\Psi(y',\theta')}{\delta\Psi(y,\theta)}\,F(x'^+,y'^-+\frac{i}{\sqrt2}\theta'\bar\theta',x',\bar x',\theta',\bar\theta')\\
&=\int \mathrm{d}^4\bar\theta'\,F(x^+,y^-+\frac{i}{\sqrt2}\theta\bar\theta',x,\bar x,\theta,\bar\theta')\\
&=\frac{1}{(4!)^2}\langle d^4\rangle F(x,\theta,\bar\theta)\,,
\end{aligned} \tag{A.8}$$

where in the last step we used

$$\int \mathrm{d}\bar\theta^k\,F(x^+,y^-+\frac{i}{\sqrt2}\theta\bar\theta,x,\bar x,\theta,\bar\theta) = d^kF(x^+,x^-,x,\bar x,\theta,\bar\theta)\,,\qquad k=1,\ldots,4\,, \tag{A.9}$$

which can be verified expanding left and right hand sides in components. From (A.8) we deduce the rule for functional differentiation with respect to a chiral superfield,

$$\frac{\delta\Psi^a(x',\theta',\bar\theta')}{\delta\Psi^b(x,\theta,\bar\theta)} = \frac{1}{(4!)^2}\delta^a_b\langle d^4\rangle\delta^{(4)}(x-x')\delta^{(4)}(\theta-\theta')\delta^{(4)}(\bar\theta-\bar\theta')\,, \tag{A.10}$$

which applies in particular to the $\mathcal{N}=4$ superfield, Φ. For its conjugate, $\bar\Phi$, using the complex conjugate of (2.48), we get

$$\frac{\delta\bar\Phi^a(x',\theta',\bar\theta')}{\delta\Phi^b(x,\theta,\bar\theta)} = \frac{1}{2(4!)^3}\delta^a_b\frac{\langle \bar d^4\rangle\langle d^4\rangle}{\partial_-^2}\delta^{(4)}(x-x')\delta^{(4)}(\theta-\theta')\delta^{(4)}(\bar\theta-\bar\theta')\,. \tag{A.11}$$

We can now define the generating functional, $Z[J]$, as follows

$$Z[J] = \frac{\displaystyle\int [\mathrm{d}\Phi]\,\mathrm{e}^{-\mathcal{S}[\Phi]+\int \mathrm{d}^{12}z\,\Phi^a(z)\frac{\langle\bar d^4\rangle}{4\partial_-^4}J_a(z)}}{\displaystyle\int [\mathrm{d}\Phi]\,\mathrm{e}^{-\mathcal{S}[\Phi]}}\,, \tag{A.12}$$

where, as usual, $\mathrm{d}^{12}z = \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar\theta$.

Notice, in particular, the coupling to the sources, $J(z)$, in (A.12). This is chosen so as to produce the correct coupling to external sources in the equations of motion. This can be seen considering the free theory in the presence of external sources,

$$\int \mathrm{d}^{12}z\,\frac{1}{2}\Phi^a(z)\,\mathcal{K}_a^{\ b}\,\Phi_b(z) + \int \mathrm{d}^{12}z\,\Phi^a(z)\frac{\langle\bar d^4\rangle}{4\partial_-^4}J_a(z)\,, \tag{A.13}$$

where the kinetic operator is

$$\mathcal{K}_a^{\ b} = -3\,\delta_a^{\ b}\,\frac{\langle\bar d^4\rangle\Box}{\partial_-^4}\,. \tag{A.14}$$

Varying (A.13) with respect to the superfield Φ gives rise to the correct equations of motion in the presence of an external source,

$$\frac{1}{(4!)^2}\langle d^4\rangle\mathcal{K}_a^{\ b}\,\Phi_b(x,\theta,\bar\theta) = J_a(x,\theta,\bar\theta)\,. \tag{A.15}$$

The right hand side is straightforward to obtain using the definition (A.10),

$$\begin{aligned}
&\frac{\delta}{\delta\Phi^a(z)}\int \mathrm{d}^{12}z'\,\Phi^b(z')\frac{\langle\bar d^4\rangle}{4\partial_-^4}J_b(z') = \frac{1}{(4!)^2}\int \mathrm{d}^{12}z'\,\langle d^4\rangle\delta^{(12)}(z-z')\frac{\langle\bar d^4\rangle}{4\partial_-^4}J_a(z')\\
&= \frac{1}{(4!)^2}\int \mathrm{d}^{12}z'\,\delta^{(12)}(z-z')\frac{\langle d^4\rangle\langle\bar d^4\rangle}{4\partial_-^4}J_a(z') = \int \mathrm{d}^{12}z'\,\delta^{(12)}(z-z')J_a(z') = J_a(z)\,,
\end{aligned}$$

where we used the fact that $\langle d^4\rangle\langle\bar d^4\rangle = 4(4!)^2\,\partial_-^4$ when acting on a chiral superfield such as $J(z)$.

In the free theory limit the exponent in the generating functional (A.12) reduces to

$$-\frac{1}{2}\left(\Phi^a, \mathcal{K}_a^{\ b}\,\Phi_b\right)+\left(\Phi^a, \frac{\langle\bar{d}^4\rangle}{4\partial_-^4}J_a\right) = -\frac{1}{2}\int \mathrm{d}^{12}z\,\Phi^a(z)\,\mathcal{K}_a^{\ b}\,\Phi_b(z)+\int \mathrm{d}^{12}z\,\Phi^a(z)\frac{\langle\bar{d}^4\rangle}{4\partial_-^4}J_a(z)\,. \tag{A.16}$$

The functional integral (A.12) becomes Gaussian and thus straightforward to compute. The result is

$$Z_0[J] = \mathrm{e}^{\frac{1}{2}\left(\widetilde{J}^a,[\mathcal{K}^{-1}]_a^{\ b}\,\widetilde{J}_b\right)} \tag{A.17}$$

where

$$\widetilde{J}^a(z) = \frac{\langle\bar{d}^4\rangle}{4\partial_-^4}J^a(z) \tag{A.18}$$

and $\mathcal{K}^{-1}$ is the inverse of the kinetic operator (A.14). In (A.17) a factor of $\det(\mathcal{K})^{-1/2}$ has been cancelled between numerator and denominator. The free generating functional (A.17) allows to construct the perturbative expansion of the full functional $Z[J]$ in (A.12).

Introducing the kernel, $\Delta(z,z')$, of the operator $\mathcal{K}^{-1}$, we can rewrite (A.17) as

$$Z_0[J] = \mathrm{e}^{\frac{1}{2}\int \mathrm{d}^{12}z\,\mathrm{d}^{12}z'\;\widetilde{J}^a(z)[\Delta(z,z')]_a^{\ b}\widetilde{J}_b(z')}\,. \tag{A.19}$$

$\Delta(z,z')$ is of course the super-propagator we are interested in. Let us denote by $K(z,z')$ the kernel of the kinetic operator (A.14),

$$K(z,z') = -3\,\delta^{(12)}(z-z')\,\frac{\langle\bar{d}^4\rangle\Box}{\partial_-^4}\,, \tag{A.20}$$

where $\delta^{(12)}(z-z') = \delta^{(4)}(x-x')\delta^{(4)}(\theta-\theta')\delta^{(4)}(\bar{\theta}-\bar{\theta}')$. Then $\Delta(z,z')$ is defined by the condition

$$\int \mathrm{d}^{12}z''\,\Delta(z,z'')\,K(z'',z') = \delta^{(12)}(z-z')\,, \tag{A.21}$$

or, introducing a chiral test superfield, $\Psi(z)$,

$$\int \mathrm{d}^{12}z''\int \mathrm{d}^{12}z'\,\Delta(z,z'')\,K(z'',z')\,\Psi(z') = \Psi(z)\,. \tag{A.22}$$

Using the explicit form (A.20) of $K(z,z')$ we have

$$\begin{aligned}\Psi(z) &= \int \mathrm{d}^{12}z'\int \mathrm{d}^{12}z''\,\Delta(z,z'')\delta^{(12)}(z''-z')\left(-3\frac{\langle\bar{d}^4\rangle\Box}{\partial_-^4}\Psi\right)(z')\\ &= \int \mathrm{d}^{12}z'\,\Delta(z,z')\left(-3\frac{\langle\bar{d}^4\rangle\Box}{\partial_-^4}\Psi\right)(z')\,.\end{aligned} \tag{A.23}$$

The solution for $\Delta(z,z')$ is of the form

$$\Delta(z,z') = k\,\frac{\langle d^4\rangle}{(x-x')^2}\delta^{(4)}(\theta-\theta')\delta^{(4)}(\bar{\theta}-\bar{\theta}')\,, \tag{A.24}$$

with k a constant to be fixed. Substituting into the right hand side of (A.23) we get

$$\begin{aligned}
&\int \mathrm{d}^{12}z'\, k\, \frac{\langle d^4\rangle}{(x-x')^2}\delta^{(4)}(\theta-\theta')\delta^{(4)}(\bar\theta-\bar\theta')\left(-3\frac{\langle \bar d^4\rangle\Box}{\partial_-^4}\Psi\right)(z') \\
&= -3k\int \mathrm{d}^{12}z'\,\Box\frac{1}{(x-x')^2}\delta^{(4)}(\theta-\theta')\delta^{(4)}(\bar\theta-\bar\theta')\left(\frac{\langle d^4\rangle\langle \bar d^4\rangle}{\partial_-^4}\Psi\right)(z') \\
&= -3k(2\pi)^2 4(4!)^2\int \mathrm{d}^{12}z'\,\delta^{12}(z-z')\Psi(z') = -\frac{3k(4!)^3(2\pi)^2}{2}\,\Psi(z)\,, \qquad \text{(A.25)}
\end{aligned}$$

where we used integration by parts and the relations

$$\Box\frac{1}{(x-x')^2} = (2\pi)^2\delta^{(4)}(x-x') \qquad \text{(A.26)}$$

and

$$\langle d^4\rangle\langle \bar d^4\rangle\Psi(z) = 4(4!)^2\partial_-^4\,\Psi(z)\,. \qquad \text{(A.27)}$$

The latter is valid for a chiral superfield $\Psi(z)$. From (A.25) we read off the value of the constant k,

$$k = -\frac{2}{(4!)^3(2\pi)^2}\,. \qquad \text{(A.28)}$$

So the superfield propagator is

$$\Delta^a_b(z-z') = -\frac{2}{(4!)^3}\frac{\delta^a_b}{(2\pi)^2}\frac{1}{(x-x')^2}\langle d^4\rangle\delta^{(4)}(\theta-\theta')\delta^{(4)}(\bar\theta-\bar\theta')\,. \qquad \text{(A.29)}$$

A.2.2 Relation to component field propagators

In order to verify that the superfield propagator constructed in the previous subsection contains the correct propagators for the individual fields in the $\mathcal{N}=4$ multiplet we now re-derive the $\Delta(z-z')$ starting from the component expansion of $\Phi(z)$.

In the following it will be convenient to write the $\mathcal{N}=4$ superfield in the form (2.50). The kinetic terms in the $\mathcal{N}=4$ light-cone component action are

$$S_0 = \int \mathrm{d}^4x\left[\bar A(x)\Box A(x) + \frac{1}{2}\varphi_i(x)\Box\varphi^i(x) - \frac{i}{\sqrt2}\bar\lambda_m(x)\frac{\Box}{\partial_-}\lambda^m(x)\right], \qquad \text{(A.30)}$$

where the relation between the six real scalar fields φ^i, $i=1,\ldots,6$ and the φ^{mn}'s, $m,n=1,\ldots,4$ in (2.50) involves Clebsch-Gordan coefficients and it is given explicitly in (A.1).

From (A.30) we get the free propagators for the component fields,

$$(\Delta^{(A)})^a_b\,(x-y) = \langle \bar A^a(x)\,A_b(y)\rangle = \frac{1}{(2\pi)^2}\frac{\delta^a_b}{(x-y)^2} \qquad \text{(A.31)}$$

$$(\Delta^{(\varphi)})^{a\,ij}_b\,(x-y) = \langle \varphi^{a\,i}(x)\,\varphi^j_b(y)\rangle = \frac{1}{(2\pi)^2}\frac{\delta^{ij}\delta^a_b}{(x-y)^2} \qquad \text{(A.32)}$$

$$\Rightarrow\quad (\Delta^{(\varphi)})^{a\,pq}_{b\,mn}\,(x-y) = \langle \bar\varphi_{a\,mn}(x)\,\varphi^{pq}_b(y)\rangle = \frac{1}{(2\pi)^2}\frac{(\delta^q_m\delta^p_n-\delta^p_m\delta^q_n)\delta^a_b}{(x-y)^2}$$

$$(\Delta^{(\lambda)})^{a\,n}_{b\,m}\,(x-y) = \langle \bar\lambda^a_m(x)\,\lambda^n_b(y)\rangle = \frac{i\sqrt2}{(2\pi)^2}\,\partial_-\frac{\delta^n_m\delta^a_b}{(x-y)^2} = \frac{i\sqrt2}{(2\pi)^2}\frac{\delta^n_m\delta^a_b(x^+-y^+)}{(x-y)^4}\,. \qquad \text{(A.33)}$$

We can now consider the superfield two-point function,

$$\Delta^a_b(x,\theta,\bar\theta;x',\theta',\bar\theta') = \langle \Phi^a(x,\theta,\bar\theta)\,\Phi_b(x',\theta',\bar\theta')\rangle\,. \tag{A.34}$$

Using (2.50), we expand this two-point function as

$$\begin{aligned}
\langle \Phi^a(x,\theta,\bar\theta)\,\Phi_b(x',\theta',\bar\theta')\rangle &= \mathrm{e}^{-\frac{i}{\sqrt2}(\theta^m\bar\theta_m\partial_- + \theta'^m\bar\theta'_m\partial'_-)}\langle\Big[-\frac{1}{\partial_-}A^a(x) - \frac{i}{\partial_-}\theta^m\bar\lambda^a_m(x) \\
&+\frac{i}{\sqrt2}\,\theta^m\theta^n\bar\varphi^a_{mn}(x) + \frac{\sqrt2}{6}\,\varepsilon_{mnpq}\theta^m\theta^n\theta^p\lambda^{a\,q}(x) - \frac{1}{12}\,\varepsilon_{mnpq}\theta^m\theta^n\theta^p\theta^q\,\partial_-\bar A^a(x)\Big] \\
&\Big[-\frac{1}{\partial'_-}A_b(x') - \frac{i}{\partial'_-}\theta'^r\bar\lambda_{b\,r}(x') + \frac{i}{2\sqrt2}\,\varepsilon_{rsuv}\theta'^r\theta'^s\varphi^{uv}_b(x') + \frac{\sqrt2}{6}\,\varepsilon_{rsuv}\theta'^r\theta'^s\theta'^u\lambda^v_b(x') \\
&-\frac{1}{12}\,\varepsilon_{rsuv}\theta'^r\theta'^s\theta'^u\theta'^v\partial'_-\bar A_b(x')\Big]\rangle\,,
\end{aligned} \tag{A.35}$$

where $\partial'_- = \partial/\partial x'^-$ and we used the reality condition

$$\bar\varphi_{mn}(x) = \frac12\,\varepsilon_{mnpq}\,\varphi^{pq}(x) \tag{A.36}$$

for the scalar field in the second superfield.

In the superspace two-point function (A.35) the only non-zero contractions correspond to the component two-point functions (A.31)-(A.33). Therefore we get

$$\begin{aligned}
\langle \Phi^a(x,\theta,\bar\theta)\,\Phi_b(x',\theta',\bar\theta')\rangle &= \mathrm{e}^{-\frac{i}{\sqrt2}(\theta^m\bar\theta_m\partial_- + \theta'^m\bar\theta'_m\partial'_-)}\Big[\frac{1}{12}\varepsilon_{mnpq}\theta^m\theta^n\theta^p\theta^q\langle\partial_-\bar A^a(x)\,\frac{1}{\partial'_-}A_b(x')\rangle \\
&+\frac{1}{12}\varepsilon_{mnpq}\theta'^m\theta'^n\theta'^p\theta'^q\langle\frac{1}{\partial_-}A^a(x)\,\partial'_-\bar A_b(x')\rangle - \frac14\varepsilon_{mnpq}\theta^r\theta^s\theta'^m\theta'^n\langle\bar\varphi^a_{rs}(x)\,\varphi^{pq}_b(x')\rangle \\
&-i\frac{\sqrt2}{6}\varepsilon_{mnpq}\theta^r\theta'^m\theta'^n\theta'^p\langle\frac{1}{\partial_-}\bar\lambda^a_r(x)\,\lambda^q_b(x')\rangle - i\frac{\sqrt2}{6}\varepsilon_{mnpq}\theta^m\theta^n\theta^p\theta'^r\langle\lambda^{q\,a}(x)\,\frac{1}{\partial_-}\bar\lambda_{r\,b}(x')\rangle\Big]\,,
\end{aligned} \tag{A.37}$$

Using (A.31)-(A.33) and integration by parts to get rid of the extra ∂_-'s, we find

$$\begin{aligned}
\langle \Phi^a(x,\theta,\bar\theta)\,\Phi_b(x',\theta',\bar\theta')\rangle &= \delta^a_b\,\mathrm{e}^{-\frac{i}{\sqrt2}(\theta^m\bar\theta_m - \theta'^m\bar\theta'_m)\partial_-}\varepsilon_{mnpq}\Big[-\frac{1}{12}\theta^m\theta^n\theta^p\theta^q - \frac{1}{12}\theta'^m\theta'^n\theta'^p\theta'^q \\
&-\frac12\theta^m\theta^n\theta'^p\theta'^q + \frac13\theta^m\theta'^n\theta'^p\theta'^q + \frac13\theta^m\theta^n\theta^p\theta'^q\Big]\,\frac{1}{(2\pi)^2}\,\frac{1}{(x-x')^2} \\
&= -\frac{1}{12(2\pi)^2}\,\delta^a_b\,\mathrm{e}^{-\frac{i}{\sqrt2}(\theta^m\bar\theta_m - \theta'^m\bar\theta'_m)\partial_-}\,\frac{\delta^{(4)}(\theta-\theta')}{(x-x')^2}\,.
\end{aligned} \tag{A.38}$$

where we used the definition (2.30) of the fermionic δ-function. The super-propagator can be put in a more convenient form using the following identity

$$\langle d^4\rangle\delta^{(4)}(\bar\theta-\bar\theta') = (4!)^2\,\mathrm{e}^{-\frac{i}{\sqrt2}(\theta^m\bar\theta_m - \theta^m\bar\theta'_m)\partial_-}\,, \tag{A.39}$$

which can be proven expanding the left hand side as

$$\begin{aligned}\langle d^4\rangle\delta^{(4)}(\bar\theta-\bar\theta') &= \varepsilon_{mnpq}\varepsilon^{rsuv}d^m d^n d^p d^q(\bar\theta_r-\bar\theta'_r)(\bar\theta_s-\bar\theta'_s)(\bar\theta_u-\bar\theta'_u)(\bar\theta_v-\bar\theta'_v)\\ &= (4!)^2\, d^1d^2d^3d^4(\bar\theta_1-\bar\theta'_1)(\bar\theta_2-\bar\theta'_2)(\bar\theta_3-\bar\theta'_3)(\bar\theta_4-\bar\theta'_4)\end{aligned}\tag{A.40}$$

and using (no sum over the repeated index k)

$$d^k(\bar\theta_k-\bar\theta'_k) = -1+\frac{i}{\sqrt2}(\theta^k\bar\theta_k-\theta^k\bar\theta'_k)\partial_- = -\mathrm{e}^{-\frac{i}{\sqrt2}(\theta^k\bar\theta_k-\theta^k\bar\theta'_k)\partial_-}\qquad k=1,\ldots,4\,.\tag{A.41}$$

The identity (A.39) can be rewritten as

$$1=\frac{1}{(4!)^2}\,\mathrm{e}^{+\frac{i}{\sqrt2}(\theta^m\bar\theta_m-\theta^m\bar\theta'_m)\partial_-}\,\langle d^4\rangle\delta^{(4)}(\bar\theta-\bar\theta')\,.\tag{A.42}$$

Inserting (A.42) into the expression for the super-propagator we get

$$\begin{aligned}\langle\Phi^a(x,\theta,\bar\theta)\Phi_b(x',\theta',\bar\theta')\rangle &= -\frac{\delta^a_b}{12(2\pi)^2}\,\mathrm{e}^{-\frac{i}{\sqrt2}(\theta^m\bar\theta_m-\theta^m\bar\theta'_m)\partial_-}\,\frac{\delta^{(4)}(\theta-\theta')}{(x-x')^2}\\ &\quad\times\frac{1}{(4!)^2}\,\mathrm{e}^{+\frac{i}{\sqrt2}(\theta^m\bar\theta_m-\theta^m\bar\theta'_m)\partial_-}\,\langle d^4\rangle\delta^{(4)}(\bar\theta-\bar\theta')\,,\end{aligned}\tag{A.43}$$

where we used the δ-function in (A.38) to change θ'^m into θ^m in the first exponential. The exponential factors in (A.43) cancel and we finally get

$$\langle\Phi^a(x,\theta,\bar\theta)\Phi_b(x',\theta',\bar\theta')\rangle = -\frac{2}{(4!)^3}\,\frac{\delta^a_b}{(2\pi)^2}\,\frac{\langle d^4\rangle\delta^{(4)}(\theta-\theta')\delta^{(4)}(\bar\theta-\bar\theta')}{(x-x')^2}\,.\tag{A.44}$$

in agreement with (A.29).

A.3 Useful superspace relations

We collect in this appendix various relations used in manipulations of super Feynman diagrams in light-cone superspace.

Although $\frac{1}{\partial_-}$ is not a differential operator, it can be "integrated by parts" in superspace expressions. For generic superfields $f(x,\theta,\bar\theta)$ and $g(x,\theta,\bar\theta)$ we have

$$\begin{aligned}\int \mathrm{d}^{12}z\, f(z)\frac{1}{\partial_-}g(z) &= \int \mathrm{d}^{12}z\,\frac{\partial_-}{\partial_-}f(z)\frac{1}{\partial_-}g(z)\\ &= -\int \mathrm{d}^{12}z\,\frac{1}{\partial_-}f(z)\frac{\partial_-}{\partial_-}g(z) = -\int \mathrm{d}^{12}z\,\frac{1}{\partial_-}f(z)g(z)\,.\end{aligned}\tag{A.45}$$

Using the definition (2.25) of the chiral derivatives, d^m and $\bar d_m$, and their commutation relation, it is easy to verify the following identity

$$\int \mathrm{d}^{12}z_2\,\delta^{(8)}(\theta_1-\theta_2)\left[\langle d^4_{(1)}\rangle\langle\bar d^4_{(1)}\rangle\delta^{(8)}(\theta_1-\theta_2)\right] = (4!)^4\,,\tag{A.46}$$

which is used repeatedly to carry out the integrations over the fermionic coordinates at each interaction vertex in superspace Feynman diagrams.

The commutation relation (2.26) for the superspace chiral derivatives implies

$$\overrightarrow{\langle \bar{d}^4\rangle\langle d^4\rangle \bar{d}_p \bar{d}_q} = 4!\,\varepsilon_{abpq}\,\overrightarrow{\partial_-^2\langle \bar{d}^4\rangle d^a d^b}\,, \tag{A.47}$$

$$\overrightarrow{\langle \bar{d}^4\rangle\langle d^4\rangle\langle \bar{d}^4\rangle} = 4(4!)^2\overrightarrow{\partial_-^4\langle \bar{d}^4\rangle}\,. \tag{A.48}$$

A.4 Details of four-point function calculation

A.4.1 Diagrams involving cubic vertices

As pointed out in section 4.4.2 contributions to four-point functions of the Q^{ij} operators cannot be built using two cubic vertices of the same type (Vertex 3-I in (2.52) or Vertex 3-II in (2.53)). This can be seen from a simple counting of chiral derivatives and fermionic coordinates θ and $\bar{\theta}$.

We start by counting the superficial numbers (or powers) of d, $\bar{d}$, θ and $\bar{\theta}$ present in various factors used in constructing a four point function.

Structure	d	$\bar{d}$	θ	$\bar{\theta}$
Propagator	4	0	4	4
Cubic Vertex 3-I	0	4	0	0
Cubic Vertex 3-II	0	8	0	0
External φ field in $\mathcal{Q}$	0	2	0	0

Table A.1: Superficial powers of d, $\bar{d}$, θ, $\bar{\theta}$

The superficial numbers (or powers) of various derivatives and fermionic variables in a four point function as shown in figure A.1, are presented in table A.2 for the three possible cases.

Combination of vertices	d	$\bar{d}$	θ	$\bar{\theta}$	$\mathrm{d}\theta$	$\mathrm{d}\bar{\theta}$
Vertex 3-I and Vertex 3-II	20	20	20	20	8	8
Vertex 3-I twice	20	16	20	20	8	8
Vertex 3-II twice	20	24	20	20	8	8

Table A.2: Superficial powers of d, $\bar{d}$, θ, $\bar{\theta}$, $\mathrm{d}\theta$, $\mathrm{d}\bar{\theta}$ in a four point function

After performing the fermionic integrals in a super Feynman diagram, we are left with an *equal* number of θ's and $\bar{\theta}$'s. Thus when fermionic coordinates are set to zero, a non-vanishing contribution can only arise if there are *equal* numbers of d's and $\bar{d}$'s present to cancel the θ's and $\bar{\theta}$'s. Thus, as can be seen from table A.2, only the combination of one vertex of type 3-I and one of type 3-II can produce a non-zero result, as this is the only way of satisfying the above criterion.

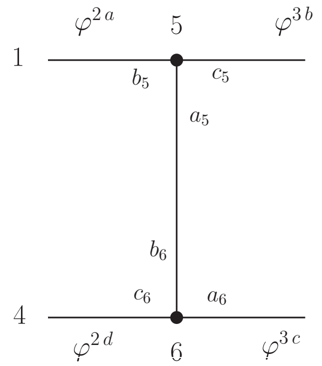


Figure A.1

Rule A.4.1 *In the topology shown in Figure A.1, a cubic vertex cannot have component fields φ^i and φ^j with $i \neq j$, connected to any two of its legs.*

This result can be understood in terms of component fields. The only cubic vertices involving two scalar fields in the $\mathcal{N} = 4$ action – in any gauge, including the light-cone gauge – are the minimal coupling to the gauge field. Since the latter is a flavour singlet, the interaction cannot change the flavour index carried by the scalar field. This result can be obtained [2] in superspace as well.

A.4.2 Diagrams involving quartic vertices

Rule A.4.2 *In the topology shown in Figure A.2, component fields φ^i and φ^j with flavour $i \neq j$, cannot simultaneously attach to those legs of the quartic vertex which are both chiral fields, or both anti-chiral fields* [1].

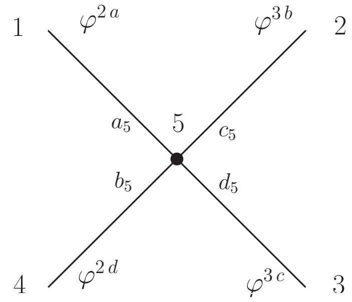


Figure A.2

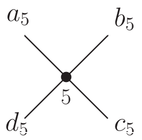
For a four point function constructed using Vertex 4-I (2.54), if the leg with colour index a_5 (chiral field) is connected with the external field φ^i, and the leg with colour index b_5 (chiral field) with the field φ^j, we get a factor of $\sigma^{i\,mn}\sigma^{j\,pq}\varepsilon_{mnpq} \;=\; 8\delta^{ij}$

[1] Here we use the term "anti-chiral" field to refer to superfields associated with legs in a diagram carrying a $\langle\bar{d}^4\rangle/\partial_-^2$ factor. These were originally $\bar{\Phi}$'s before use of the complex conjugate of (2.48).

when evaluating the correlation function. The same thing happens with legs carrying colour indices c_5 and d_5 (anti-chiral fields) connected with external fields φ^i and φ^j. For Vertex 4-II (2.55), if the leg with colour index a_5 (chiral field) is connected with φ^i and the leg with index c_5 (chiral field) with φ^j, we get a factor of $\sigma^{i\,mn}\sigma^{j\,pq}\varepsilon_{mnpq} = 8\delta^{ij}$. The same happens with legs carrying colour indices d_5 and b_5 (anti-chiral fields). Thus for such arrangements with $i \neq j$, the contraction vanishes.

This rule is verified [2] by evaluating each permutation of the interaction vertex in Figure A.2.

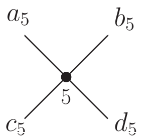
The only non-zero contributions to $G_4^{(H)}(x_1, \ldots, x_4)$ at one loop come from diagrams involving a quartic vertex of type 4-II. As explained in section 4.4.2 there are various inequivalent Wick contractions to consider and we analyse them in detail below. We begin with



$$\equiv V_4[a_5, b_5, c_5, d_5]$$

$$= \int_5 \delta^{aa_5}\delta^{bb_5}\delta^{cc_5}\delta^{dd_5} \left(\bar{d}_{p_1}\bar{d}_{q_1}\langle d^4\rangle \frac{\delta^8_{15}}{x^2_{15}}\right)\left(\frac{\langle\bar{d}^4\rangle}{\partial^2_-}\langle d^4\rangle \frac{\delta^8_{52}}{x^2_{52}}\overleftarrow{\bar{d}_{q_2}\bar{d}_{p_2}}\right)$$
$$\times \left(\bar{d}_{p_3}\bar{d}_{q_3}\langle d^4\rangle \frac{\delta^8_{35}}{x^2_{35}}\right)\left(\frac{\langle\bar{d}^4\rangle}{\partial^2_-}\langle d^4\rangle \frac{\delta^8_{54}}{x^2_{54}}\overleftarrow{\bar{d}_{q_4}\bar{d}_{p_4}}\right)$$
$$\propto \varepsilon_{p_1q_1p_3q_3}\,.$$

Product with the common part $E_4[a_5, b_5, c_5, d_5]$ in (4.25) results in the contraction $\sigma^{2p_1q_1}\sigma^{3p_3q_3}\varepsilon_{p_1q_1p_3q_3} = 0$. The reason why $V_4[a_5, b_5, c_5, d_5]$ leads to this contraction is explained under Rule A.4.2 above.



$$\equiv V_4[a_5, b_5, d_5, c_5]$$

$$= \int_5 \delta^{a_5a}\delta^{b_5b}\delta^{c_5d}\delta^{d_5c} \left(\bar{d}_{p_1}\bar{d}_{q_1}\langle d^4\rangle \frac{\delta^8_{15}}{x^2_{15}}\right)\left(\frac{\langle\bar{d}^4\rangle}{\partial^2_-}\langle d^4\rangle \frac{\delta^8_{52}}{x^2_{52}}\overleftarrow{\bar{d}_{q_2}\bar{d}_{p_2}}\right)\left(\bar{d}_{p_3}\bar{d}_{q_3}\langle d^4\rangle \frac{\delta^8_{35}}{x^2_{35}}\frac{\overleftarrow{\langle\bar{d}^4\rangle}}{\partial^2_-}\right)$$
$$\times \left(\langle d^4\rangle \frac{\delta^8_{54}}{x^2_{54}}\overleftarrow{\bar{d}_{q_4}\bar{d}_{p_4}}\right)$$
$$= \int_5 \delta^{a_5a}\delta^{b_5b}\delta^{c_5d}\delta^{d_5c} \left(\langle d^4\rangle\bar{d}_{q_1}\bar{d}_{p_1}\frac{\delta^8_{51}}{x^2_{51}}\right)\left(4!\varepsilon_{rsq_2p_2}\langle\bar{d}^4\rangle d^r d^s\frac{\delta^8_{52}}{x^2_{52}}\right)\left(4!\varepsilon_{uvq_3p_3}\langle\bar{d}^4\rangle d^u d^v\frac{\delta^8_{53}}{x^2_{53}}\right)$$
$$\times \left(\langle d^4\rangle\bar{d}_{q_4}\bar{d}_{p_4}\frac{\delta^8_{54}}{x^2_{54}}\right)\,, \qquad \text{(A.49)}$$

where we used (A.47).

We now use the following rule for partially integrating $\langle d^4\rangle$ to a product of two terms (disregarding the cases where both the terms are not acted upon by two d's each),

$$\int \mathrm{d}^4\theta(\langle d^4\rangle F)(GH) = 6\,\varepsilon_{m_1n_1m_2n_2}\int \mathrm{d}^4\theta\, F(d^{m_1}d^{n_1}G)(d^{m_2}d^{n_2}H)\,, \qquad \text{(A.50)}$$

and simplify $V_4[a_5,b_5,d_5,c_5]$ to

$$6\int_5 d^{a_5a}\delta^{b_5b}\delta^{c_5d}\delta^{d_5c}\left(\frac{\delta^8_{51}}{x^2_{51}}\right)\left(4!\varepsilon_{rsq_2p_2}d^{m_1}d^{n_1}\langle\bar{d}^4\rangle d^r d^s\frac{\delta^8_{52}}{x^2_{52}}\right)$$
$$\times\left(4!\varepsilon_{uvq_3p_3}d^{m_2}d^{n_2}\langle\bar{d}^4\rangle d^u d^v\frac{\delta^8_{53}}{x^2_{53}}\right)\left(\bar{d}_{q_1}\bar{d}_{p_1}\langle d^4\rangle\bar{d}_{q_4}\bar{d}_{p_4}\frac{\delta^8_{54}}{x^2_{54}}\right)\varepsilon_{m_1n_1m_2n_2}\,. \quad \text{(A.51)}$$

$V_4[a_5,b_5,d_5,c_5]$ as written in (A.51) simplifies to

$$6(4!)^2\int\mathrm{d}^4x_5\, d^{a_5a}\delta^{b_5b}\delta^{c_5d}\delta^{d_5c}\left(\frac{\varepsilon_{m_1n_1m_2n_2}}{x^2_{51}}\right)\left(\frac{(4!)^3\varepsilon_{rsq_2p_2}\varepsilon^{m_1n_1rs}}{x^2_{52}}\right)$$
$$\times\left(\frac{(4!)^3\varepsilon_{uvq_3p_3}\varepsilon^{m_2n_2uv}}{x^2_{53}}\right)\left(\frac{(4!)^3\varepsilon_{q_1p_1q_4p_4}}{x^2_{54}}\right), \quad \text{(A.52)}$$

in the limit $\theta,\bar{\theta}\to 0$. Using the following property of the Levi-Civita symbol

$$\varepsilon_{m_1n_1m_2n_2}\,\varepsilon^{m_1n_1pq} = 2\left(\delta^p_{m_2}\delta^q_{n_2}-\delta^p_{n_2}\delta^q_{m_2}\right)\,, \quad \text{(A.53)}$$

we simplify

$$\left(\varepsilon_{m_1n_1m_2n_2}\,\varepsilon^{m_1n_1rs}\right)\varepsilon_{rsq_2p_2}\left(\varepsilon_{uvq_3p_3}\varepsilon^{m_2n_2uv}\right) = 4\varepsilon_{m_2n_2q_2p_2}\left(\varepsilon_{uvq_3p_3}\varepsilon^{m_2n_2uv}\right) = 16\varepsilon_{p_3q_3p_2q_2}\,. \quad \text{(A.54)}$$

Thus $V_4[a_5,b_5,d_5,c_5]$ (A.52) simplifies to

$$16\times 6\times(4!)^{11}\times d^{a_5a}\delta^{b_5b}\delta^{c_5d}\delta^{d_5c}\times\varepsilon_{p_3q_3p_2q_2}\varepsilon_{p_1q_1p_4q_4}\int\mathrm{d}^4x_5\,\frac{1}{x^2_{51}x^2_{52}x^2_{53}x^2_{54}}\,. \quad \text{(A.55)}$$

Substituting

$$k=(-1).2.\frac{1}{(4!)^3}\frac{1}{(2\pi)^2}\,,\qquad T(\sigma)\varepsilon\varepsilon\varepsilon\varepsilon = 2^{12}\,, \quad \text{(A.56)}$$

we obtain the final expression for $V_4[a_5,b_5,d_5,c_5]$ times the common part (4.26) as

$$-g^2f^{eab}f^{eab}\frac{1}{(2\pi)^{12}64}\frac{1}{x^2_{14}x^2_{23}}\int\mathrm{d}^4x_5\frac{1}{x^2_{51}x^2_{52}x^2_{53}x^2_{54}}\,. \quad \text{(A.57)}$$

All permutations of the arguments in $V_4[a_5,b_5,d_5,c_5]$ of the form $[e_1,g_1,e_2,g_2]$ where $e_i\in\{a_5,c_5\}, g_i\in\{b_5,d_5\}$ or $e_i\in\{b_5,d_5\}, g_i\in\{a_5,c_5\}$, $i=1,2$, will have a non-zero contribution. The reason is explained under Rule A.4.2 above.

From the structure of Vertex 4-II (2.55), it is easy to see that

$$V_4[a_5,b_5,d_5,c_5]=V_4[a_5,d_5,b_5,c_5]=V_4[c_5,b_5,d_5,a_5]=V_4[c_5,d_5,b_5,a_5]$$
$$=V_4[b_5,a_5,c_5,d_5]=V_4[b_5,c_5,a_5,d_5]=V_4[d_5,a_5,c_5,b_5]=V_4[d_5,c_5,a_5,b_5]\,.$$

References

[1] S. Ananth, S. Kovacs and S. Parikh, "A manifestly MHV Lagrangian for $\mathcal{N}=4$ Yang-Mills", *JHEP* **1105** (2011) 051 `[arXiv:hep-th/1101.3540]`.

[2] S. Ananth, S. Kovacs and S. Parikh, "Gauge-invariant correlation functions in light-cone superspace", *preprint submitted for review* `[arXiv:hep-th/1203.5376]`.

[3] S. Mandelstam, "Light-cone superspace and the ultraviolet finiteness of the $N=4$ model", *Nucl. Phys.* **B 213** (1983) 149.

[4] L. Brink, O. Lindgren and B. E. W. Nilsson, "$\mathcal{N}=4$ Yang–Mills Theory on the Light Cone", *Nucl. Phys.* **B 212** (1983) 401.

[5] L. Brink, O. Lindgren and B. E. W. Nilsson, "The Ultraviolet Finiteness of the $\mathcal{N}=4$ Yang–Mills Theory", *Phys. Lett.* **B 123** (1983) 323.

[6] B. S. DeWitt, "Quantum theory of gravity. III. Applications of the covariant theory", *Phys. Rev.* **162** (1967) 1239.

[7] F. Cachazo, P. Svrcek and E. Witten, "MHV Vertices And Tree Amplitudes In Gauge Theory", *JHEP* **0409** (2004) 006, `[arXiv:hep-th/0403047]`.

[8] E. Witten, "Perturbative gauge theory as a string theory in twistor space", *Commun. Math. Phys.* **252** (2004) 189 `[arXiv:hep-th/0312171]`.

[9] A. Gorsky and A. Rosly, "From Yang-Mills Lagrangian to MHV Diagrams", *JHEP* **0601** (2006) 101, `[arXiv:hep-th/0510111]`.

[10] P. Mansfield, "The Lagrangian origin of MHV rules", *JHEP* **0603** (2006) 037, `[arXiv:hep-th/0511264]`.

[11] J. H. Ettle and T. R. Morris, "Structure of the MHV-rules Lagrangian", *JHEP* **0608** (2006) 003 `[arXiv:hep-th/0605121]`.

[12] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity", *Adv. Theor. Math. Phys.* **2** (1998) 231 `[arXiv:hep-th/9711200]`.

[13] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory", *Phys. Lett.* **B 428** (1998) 105 `[arXiv:hep-th/9802109]`.

[14] E. Witten, "Anti-de Sitter space and holography", *Adv. Theor. Math. Phys.* **2** (1998) 253 `[arXiv:hep-th/9802150]`.

[15] K. Konishi, "Anomalous Supersymmetry Transformation of Some Composite Operators in SQCD", *Phys. Lett.* **B 135** (1984) 439.

[16] M. Bianchi, S. Kovacs, G. Rossi and Y. S. Stanev, "Properties of the Konishi multiplet in $\mathcal{N} = 4$ SYM theory", *JHEP* **0105** (2001) 042 `[arXiv:hep-th/0104016]`.

[17] R. Karplus and M. Neuman, "The scattering of light by light", *Phys. Rev.* **83** (1951) 776.
G. 't Hooft and M. J. G. Veltman, "Scalar One Loop Integrals", *Nucl. Phys.* **B 153** (1979) 365.

[18] M. Bianchi, S. Kovacs, G. Rossi and Y. S. Stanev, "On the logarithmic behavior in $\mathcal{N} = 4$ SYM theory", *JHEP* **9908** (1999) 020 `[arXiv:hep-th/9906188]`.