Feedback Control of Chaos and Hyper Chaos

A thesis submitted towards partial fulfilment of
BS MS Dual Degree Programme

by

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under the guidance of

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Indian Institute of Science Education and Research
Pune

Certificate

This is to certify that this thesis entitled **Feedback control of Chaos and Hyperchaos** submitted towards the partial fulfilment of the BS MS dual degree programme at the Indian Institute of Science Education and Research Pune represents the original research carried out by **Joola Madhuri** at **Indian Institute Of Science Education and Research, Pune**, under the supervision of **Dr. G. Ambika** during the academic year 2011-2012.

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Abstract

Most of the natural systems are non-linear in nature and can exhibit irregular and complex behaviour. Such complex dynamics, in general are termed as chaotic or hyper chaotic behaviour, leading to irregular fluctuations and hence are not desirable in most of the situations or applications. Hence there is an increasing interest in methods to control such complex behaviour.

In my project work, we have introduced a new control technique to quench such complex dynamical systems to steady state behaviour. We numerically and analytically study the resultant properties of the system when coupled with an external damped system. We observe that for critical coupling coefficient, the complex behaviour can be controlled to periodic or steady state dynamics. We find the value of the output result depends on the coupling coefficient values and by adjusting the damping coefficient, any desirable behaviour can be obtained. We also study the nature of transition of different systems from a complex behaviour to the steady state behaviour. So also by adding a periodic dynamics in the control system, we can produce periodic behaviour of a desired frequency in such complex systems.

Contents

Chapter 1

Introduction

The dynamics of a nonlinear dynamical system is governed by a set of differential equations that decide the way their variables change in time. In general these systems can have a variety of dynamical behaviours such as steady state, periodicity, quasi periodicity, chaos and hyper chaos[1],[2]. Their dynamics is mathematically represented as

$$\dot{X} = F(X, a) \tag{1.1}$$

where X is an array containing the variable

$$X = [x_1, x_2, ...x_n]^T, x \subset \mathbb{R}^n \tag{1.2}$$

which govern the dynamics of the system, $F(X, a)$ is an array of non linear functions

$$F(X, a) = [f_1(X, a), f_2(X, a), ..., f_n(X, a)] \tag{1.3}$$

that decides the nature of the dynamics of the system and **a** is a set containing all the parameters involved in the construction of system dynamics.

We can obtain the fixed points or steady state of the system by equating each variable

$$\dot{x}_i = 0 \tag{1.4}$$

Then the solution of the system is X^* where X^* is an array

$$X^* = [x_1^*, x_2^*, ...x_n^*]^T \tag{1.5}$$

containing the solutions of each variable governing the system and

$$F(x_i^*, a) = 0 \tag{1.6}$$

Depending on the nonlinearity of the function $F(X)$, we can have m number of solutions, thereby having m number of fixed points for the system but these fixed points can be stable or unstable. The stability of a fixed point is determined by obtaining the eigen values of the Jacobian matrix corresponding to the fixed point. Jacobian matrix of the system Eq[1.1] contains all the partial derivatives of the function $F(X)$ and is

$$J = \begin{pmatrix} \frac{\partial f_1(X,a)}{\partial x_1} & \frac{\partial f_2(X,a)}{\partial x_1} & . & . & . & \frac{\partial f_n(X,a)}{\partial x_1} \\ \frac{\partial f_1(X,a)}{\partial x_2} & \frac{\partial f_2(X,a)}{\partial x_2} & . & . & . & \frac{\partial f_n(X,a)}{\partial x_2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \frac{\partial f_1(X,a)}{\partial x_n} & \frac{\partial f_2(X,a)}{\partial x_n} & . & . & . & \frac{\partial f_n(X,a)}{\partial x_n} \end{pmatrix} \tag{1.7}$$

We obtain the eigen values of the system by solving the characteristic equation

$$Det[J - \lambda I] = 0 \tag{1.8}$$

where λ is the eigen value. This analysis is commonly known as linear stability analysis. From the sign and complexity of the eigen values, the stability of the system can be analysed.[1] A fixed point is said to be stable if all the eigen values have negative real parts.

Chaos is a particular behaviour which the system exhibits for certain values of control parameters. In this state, the system is highly sensitive to the initial conditions. This indicates that even when the system starts from very close initial conditions after a period of time there exists no correlation between the respective states and hence the dynamics of the system over time becomes unpredictable. This behaviour of dynamical systems, being highly sensitive to initial conditions is popularly referred to as *butterfly effect.* Such a behaviour can be detected by calculating an index called Lyapunov exponent, which gives the average rate at which two near by trajectories diverge in time. Chaotic behaviour has been studied in various fields such as mathematical sciences, physical, engineering, biological, physiological, ecological and even economic sciences. There are many mathematical models which exhibit chaos, a few of which are discussed in detail in the following chapters.

Lyapunov exponent is a quantity which characterises the rate of separation of infinitesimally close trajectories, and its positive nature is one of the essential signatures of chaotic behaviour[1]. The number of Lyapunov exponents of a system depends on the number of first order differential equations governing the system.

If δz_0 is the separation of the initial conditions of the two trajectories of the system and $\delta z(t)$ is the separation of the two trajectories of the system after a time interval t, then the lyapunov exponent λ relates the two separations of the system as

$$|\delta z(t)| \approx e^{\lambda} t |\delta z_0| \tag{1.9}$$

This equation clearly indicates that when the λ value for the system is less than zero then the separation between the two trajectories tend to decrease over time. When the λ value is greater than zero, there is divergence of nearby trajectories and so chaos.

Hence for a given system the nature of dynamics can be inferred from the spectrum of lyapunov exponents. If all the lyapunov exponents are less than zero then the system has a steady state or a fixed point behaviour. If one of the lyapunov exponent is equal to zero and rest all are less than zero then the system has a periodic behaviour. If one lyapunov exponent is greater than zero and one is equal to zero and rest all are less than zero then the system exhibits chaos. And if the system has more than one positive lyapunov exponent then the system is hyperchaotic in nature. Hence this index serves as a quantifier to study the nature of the complexity of dynamics.

In general a dynamical system can undergo a sequence of transitions or bifurcations from steady state behaviour to chaotic behaviour as the parameter values are varied. In most of the dynamical systems chaos is reached by a sequence of period doubling bifurcations. There are other routes to attain chaos such as tangent bifurcation, interior crisis, intermittency, etc. We illustrate such transitions by taking specific examples of systems that exhibit chaotic behaviour.

1.1 Chaotic Systems

1.1.1 Rössler System

This is a continuous - time dynamical system with three non linear differential equations governing the dynamics of the system and are represented as follows.[[1]]

$$\begin{aligned}\dot{x} &= -y - x \\ \dot{y} &= x + ay \\ \dot{z} &= \beta + z(x - c)\end{aligned} \tag{1.10}$$

This system exhibits varied kind of dynamical behaviour depending on the parameter values of the system. For example for parameter values a = 0.1, b = 0.1, and c = 18, system exhibits chaotic behaviour Fig[1.1].

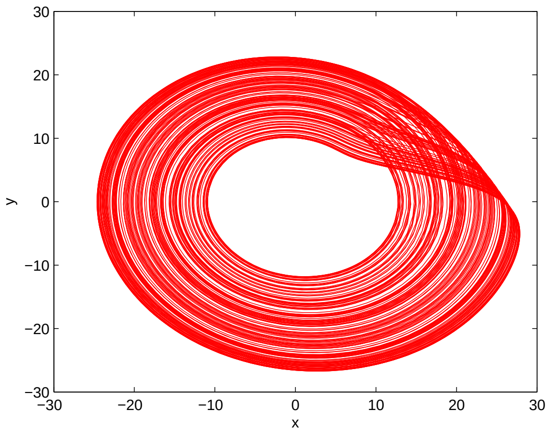

Figure 1.1: Rössler System with parameter values a = 0.1, b= 0.1, c = 18

The fixed points of the system by considering Eq[1.4] are obtained as

$$(x^*, \frac{x^*}{a}, \frac{-x^*}{a}) \tag{1.11}$$

where

$$x^* = \frac{c \pm \sqrt{c^2 - 4ab}}{2} \tag{1.12}$$

The corresponding Jacobian matrix is

$$J = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z^* & 0 & x^* - c \end{pmatrix} \tag{1.13}$$

We can obtain the stability of the fixed points from the characteristic equation Eq[1.8].

From the linear stability analysis we observe that the fixed point of the system exists only when $c^2 > 4ab$, but is not stable for any range of c.

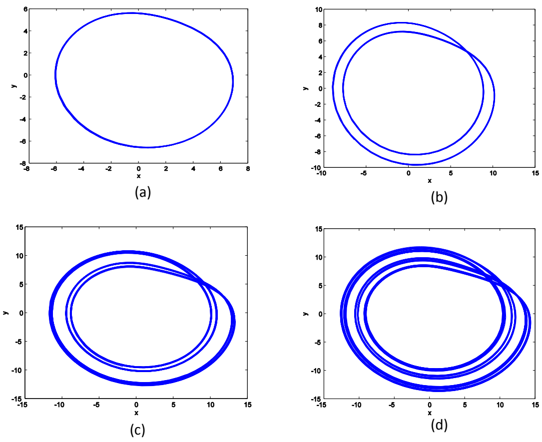

Figure 1.2: Rössler system at a = b = 0.1 and (a) c = 4, (b) c = 6 (c) c= 8 (d) c = 8.7

In order to study the dynamics of the system's behaviour, we gradually increase the parameter value c keeping other parameter values a and b constant. Fig[1.2] At $c = 4$ we observe a stable limit cycle, at $c = 6$, we observe 2 period limit cycle, at $c = 8$, we observe 4 period limit cycle, at $c = 8.7$ we observe 8 period limit cycle and so on and at $c = 18$ we observe that the system has infinite period limit cycles indicating the presence of chaotic behaviour. By gradually increasing the parameter value c the system undergoes sequence of period doubling bifurcation eventually leading to a chaotic state. It is a continuous change from stable fixed point state to chaotic state. The bifurcation diagram of the transition of periodic behaviour to a chaotic state, is shown in Fig[1.3]. Here have plotted all local maxima of x variable at each parameter value c. This pattern is called period doubling scenario.

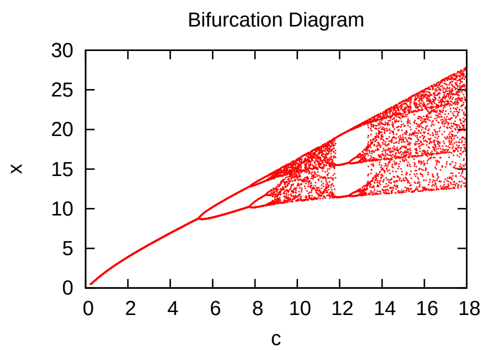

Figure 1.3: Bifurcation Diagram of Rössler System

1.1.2 Lorenz System

Lorenz system is another continuous time non-linear dynamical system exhibiting chaos for particular range of parameter values. This system is derived as a model for study of atmosphere systems. It can also model many other systems like laser, water wheel [1] etc. The equations of Lorenz system are as follows:

$$\begin{aligned} \dot{x} &= \sigma(y + x) \\ \dot{y} &= \gamma x - y - xz \\ \dot{z} &= xy - z\beta \end{aligned} \tag{1.14}$$

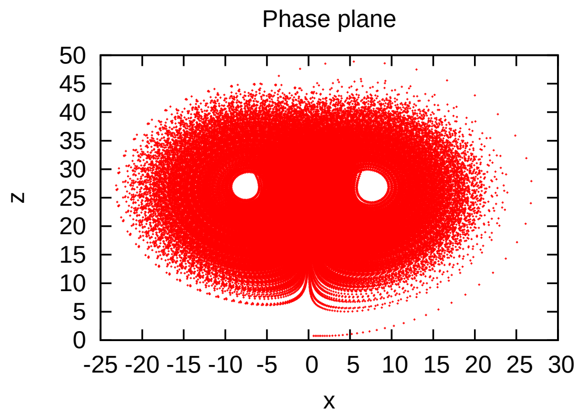

Figure 1.4: Lorenz System at parameter values $\sigma = 10$, $\beta = 8/3$ and $\gamma = 28$

For parameter values $\sigma = 10$, $\beta = 8/3$ and $\gamma = 28$, system exhibits chaotic behaviour. Fig[1.4]

Like in the case of R$\ddot{o}$ssler system we vary parameter value γ, keeping other parameter values constant to study the route to chaos. We observe that unlike R$\ddot{o}$ssler System, Lorenz system has a sudden transition from stable fixed point behaviour to chaotic behaviour.

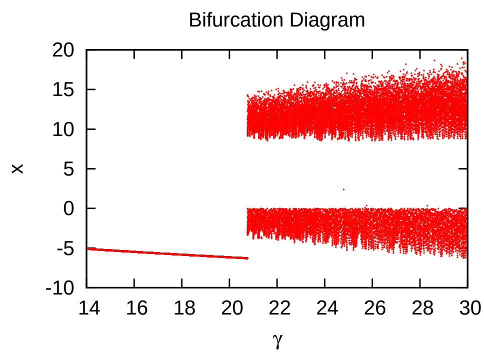

Figure 1.5: Bifurcation Diagram of Lorenz system

We consider parameter values σ and β to be constant and vary parameter value γ, we observe that at $c = 20$ the system has a stable fixed point behaviour and at $c = 21$ there is a sudden transition from fixed point behaviour to chaotic behaviour. The bifurcation diagram shown in Fig[1.5] shows the sudden transition of the Lorenz system evidently.

1.2 Hyper Chaotic Systems

Systems with more than one positive lyapunov exponent are called Hyper-chaotic systems and this behaviour is found in continuous-time n-dimensional systems with $n > 3$. These systems are more complex, highly random and have higher unpredictability than chaotic systems. Hence, such systems are used in secure communication of signals. Chua and Chen are two of the hyperchaotic systems which are discussed in detail below.

1.2.1 Chen System

This hyperchaotic system has been obtained by introducing an additional feedback into a 3D quadratic chaotic system and it can exhibit varied system dynamics depending on the parameter values [20]. This systems is practically used in many communication networks and in cryptography. Following are the equations which govern the system dynamics.

$$\begin{aligned} \dot{x} &= a(y - x) + eyz \\ \dot{y} &= cx - dxz + y + u \\ \dot{z} &= xy - bz \\ \dot{u} &= -ky \end{aligned} \tag{1.15}$$

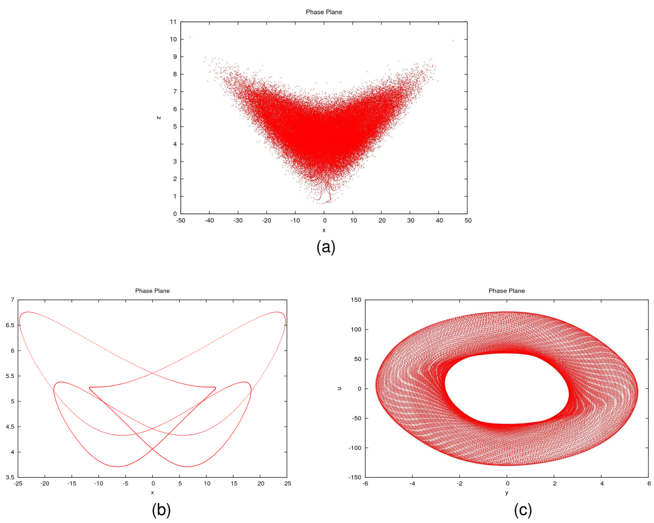

Figure 1.6: Chen System at (a) k = 100, (b) k = 535 (c) k = 700

For the parameter values $a = 35$, $b = 4.9$, $c = 25$, $d = 5$, $e = 35$, $k = 100$ the system exhibits hyperchaos Fig[1.6 (a)]. At higher values of parameter value k keeping the other parameters constant, the system exhibits different dynamical behaviour. For example at $k = 264.5$ we have periodic orbit Fig[1.6 (b)], at $k = 700$ system displays quasi periodicity. Fig[1.6 (c)]

1.2.2 Chua System

Chua system is widely applied in electrical circuits as hyperchaotic chua circuit. [?] It is also used in many laser models, neuron models, etc. Following are the equations governing the nature of the system.

$$\begin{aligned} \dot{x} &= \alpha(y - ax^3 - (1+c)x) \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y - \gamma z + u \\ \dot{u} &= -sx + yz \end{aligned} \tag{1.16}$$

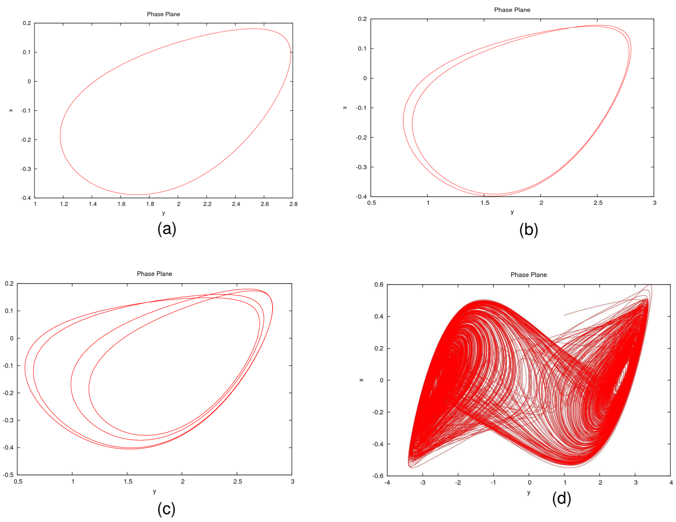

Figure 1.7: Chua System at (a)$\alpha = 20$, (b)$\alpha = 23$, (c)$\alpha = 23.7$, (d) $\alpha = 27$

For parameters values $\alpha = 27$,$\beta = 50$, $c = -1.2$, $a = 0.03$, $s = 0.1$, $\gamma = 0.3$, the system exhibits hyperchaos Fig[1.7(d)]. This system reaches chaos by a period doubling phenomenon. We vary parameter α keeping all the other parameter values constant, at $\alpha = 16$ the system displays a fixed point behaviour at $\alpha = 20$ the system displays a 1-period limit cycle Fig[1.7(a)], at $\alpha = 23$ the system has 2-period limit cycle Fig[1.7(b)] and at $\alpha = 23.7$ the system has 4-period limit cycle Fig[1.7(c)]and so on. Finally at $\alpha = 27$ the system has a hyperchaotic behaviour.

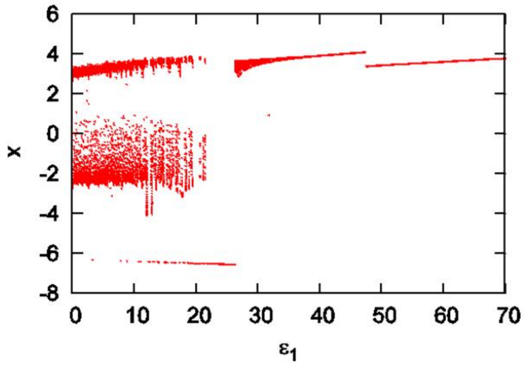

Figure 1.8: Bifurcation Diagram of Chua system

We can observe this transition from the stable fixed point behaviour to the hyperchaotic behaviour from the bifurcation diagram Fig[1.8]

1.3 Time Delay System

Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, robotics, etc where a finite delay in feedback is inevitable. The presence of time lag in the dynamics of the system makes the system ∞ dimension and may cause undesirable system dynamics or even instability. Hence in order to obtain a steady dynamics we need to apply control techniques. We discuss below two typical time delay systems.

1.3.1 Mackey Glass System

This is a non linear time delay dynamical model with τ as the delay parameter of the system. This is applied in modelling many physiological models especially model for bloodflow in Leukemia[30]. Following equation governs the dynamics of this system.

$$\dot{x} = \beta \frac{x_\tau}{1 + x_\tau^n} - \gamma x \tag{1.17}$$

We consider the parameters values as $\beta = 2$, $\tau = 2$, $\gamma = 1$. For different parameter value $'n'$ Mackey Glass displays different dynamical behaviour. This system has a continuous transition from a periodic behaviour to chaotic dynamics and it attains chaos by period doubling bifurcation. Fig[1.9]. In Fig[1.10]; the bifurcation diagram is shown for detailed study of the system

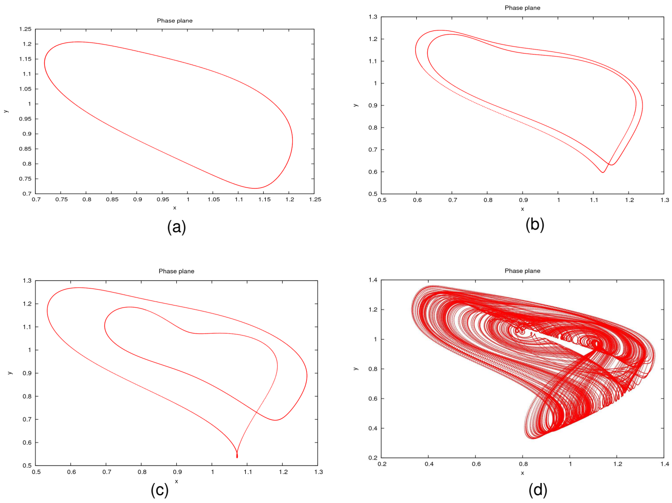

Figure 1.9: Mackey Glass System at (a)n=6, (b)n= 7.4, (c)n=8, (d)n=10

dynamics. In this bifurcation diagram the global maxima and global minima for each parameter value has been plotted Fig[1.10].

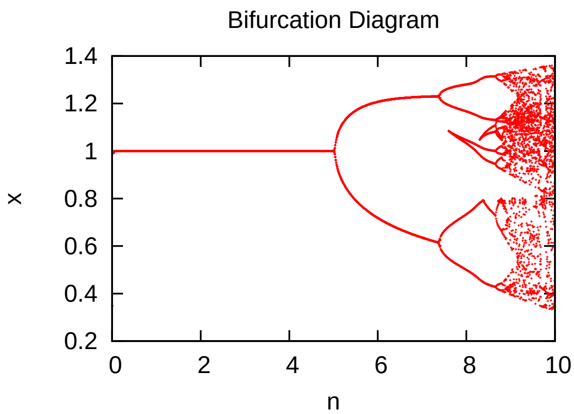

Figure 1.10: Bifurcation Diagram of Mackey Glass system

1.3.2 Ikeda System

This is another model with time delay feedback in the system, with following equations governing the motion of system dynamics. It arises as a model in many laser systems. [23]

$$\dot{x} = asin^2[x(t-\tau) - c] - \beta x(t) \tag{1.18}$$

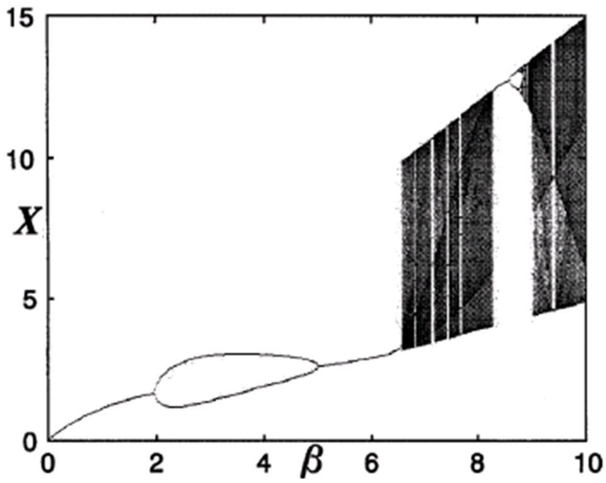

Figure 1.11: Bifurcation diagram of Ikeda system

This system has a continuous transition from a stable fixed point behaviour to chaotic behaviour. Fig[1.11][10] gives different ranges of parameter β value keeping the other parameters constant.

In this chapter we have introduced standard chaotic, hyperchaotic and time delay systems that exhibit complex behaviour. In the next chapter, we will present the feedback control method that can control such complex dynamics to a steady state behaviour.

Chapter 2

Feed back Control of Chaos

As we have described in the previous chapter, most of the nonlinear dynamical systems are capable of displaying different kinds of dynamical behaviours such as fixed point behaviour, periodicity, quasi periodicity, chaos and hyperchaos. For many practical purposes and applications, systems with a stable output is most essential and desirable than irregular fluctuations. Hence controlling the dynamics of complex systems is of considerable interest. For example such steady state behaviour is required in many practical cases such as chemical reactions[3][4][5], for suppression of power fluctuation in coupled laser systems[13][14], relativistic magnetrons, bio oscillators[28], ecological models and many physiological systems like neuronal disorders. Most of the studies are based on theorical and experimental point of view [6][7][8][12]. In all these cases regulation of the system dynamics is of a major concern.

Since the classical work on chaos control was first presented by Ott, Grebogi,and Yorke, there were various techniques introduced to stabilize unstable and complex dynamical systems[18][22][25][11][21]. Few of the effective techniques which were successful in controlling chaos are adaptive control[27][15], time delay feed back control[17], linear feed back control[17][16][26], variation in internal parameters of the system, targeting via linear augmentation [19] etc.

We report a study on how an external system can control a systems dynamics. We have introduced a method of control which is a self adjustable control mechanism where the dynamical system is connected through a linear feedback coupling with an external dynamic system. This system is an over damped system kept alive by the feedback from the system and also the system gets a continuous feedback from the external damped system.

This mechanism is found to work for all kind of complex dynamical systems such as chaotic, hyperchaotic, timedelay hyperchaos. By properly monitoring the coupling coefficients of the control system, all these complex systems can be forced to the desired steady state behaviour or periodic behaviour.

2.1 General Mechanism of Control

The dynamics of a system 'x' in interaction with an external damped control system can in general written by the following equations.

$$\begin{aligned} \dot{x} &= f(x) + \epsilon_1 \gamma y \\ \dot{y} &= -ky - \epsilon_2 \gamma^T x \end{aligned} \tag{2.1}$$

Here x represents n-dimensional dynamical system whose intrinsic dynamics is governed by $f(x)$. The external system is a one dimensional over damped oscillator y with damping parameter k. The external system is kept alive by the feedback from the system as given by the last term is second equation of Eq[2.1] and the system also gets feedback from the external system by the last term in first equation of Eq[2.1]. γ is a column matrix $[m \times 1]$, with elements zeros and ones, it decides the components of x that gets feedback from the external system. γ^T is the tranpose of gamma and it decides the components of x that gives feedback to the external system. We consider ϵ_1 to be the strength of feedback from the external system to the system and ϵ_2 the strength of feedback from system to the external system. By adjusting the values of the coupling strengths ϵ_1 and ϵ_2 and the damping parameter k, we can control the dynamics of the system to a stable fixed point or a steady state behaviour.

We note that the fixed point of the new coupled system is not the same as the fixed point of the intrinsic dynamical system but is a new one created in the presence of the external system.

2.1.1 Linear stability analysis:

We present the linear stability analysis for a general case, coupled with an external system as in Eq[2.1]. For this we write the variation equation formed by linearizing Eq[2.1] as

$$\begin{aligned} \dot{\xi} &= f'(x)\xi + \epsilon_1 \gamma z \\ \dot{z} &= -kz - \epsilon_2 \gamma^T \xi \end{aligned} \tag{2.2}$$

where ξ, z are small deviations of the system and the external system respectively. For the fixed point to be stable, all the eigen values of the Jacobian matrix of the corresponding fixed point should have a negative real part.

It is not easy to proceed further with the analysis of the stability of this coupled system as a general case. However, we can make an approximate analysis by assuming that the time average value of $f'(x)$ as an effective constant value μ. In this case γ and γ^T values will become one. So the above equation would look like

$$\begin{aligned} \dot{\xi} &= \mu\xi + \epsilon_1 z \\ \dot{z} &= -kz - \epsilon_2 \xi \end{aligned} \tag{2.3}$$

and the corresponding Jacobian matrix is

$$J = \begin{pmatrix} \mu & \epsilon_1 \\ -\epsilon_2 & -k \end{pmatrix} \tag{2.4}$$

From the characteristic equation Eq[1.8] we deduce the eigen values as

$$\lambda_{\pm} = \frac{(\mu - k) \pm \sqrt{(\mu - k)^2 - 4(\epsilon_1\epsilon_2 - \mu k)}}{2} \tag{2.5}$$

$$\Longrightarrow \lambda_{\pm} = \frac{(\mu - k) \pm \sqrt{(\mu + k)^2 - 4\epsilon_1\epsilon_2}}{2} \tag{2.6}$$

From 2.6, we can find the stability conditions as follows
(a) If $(\mu + k)^2 < 4\epsilon_1\epsilon_2$, $\lambda_{\pm}$ are complex and the stability condition is $\mu < k$
(b) If $(\mu + k)^2 > 4\epsilon_1\epsilon_2$, $\lambda_{\pm}$ are real and the stability condition is $\mu k < \epsilon_1\epsilon_2$

Hence for a given μ and k value, the transition of dynamics of the system to a stable fixed point behaviour occurs at critical values of ϵ_1 and ϵ_2 and they obey

$$\epsilon_{1c} = \frac{\mu k}{\epsilon_{2c}} \tag{2.7}$$

We numerically analyse this stability criteria in the following sections for specific cases.

2.2 Feed back control of Rössler systems

We have applied this control mechanism on Rössler system which as described in Chapter 1 is chaotic in nature for particular parameter values.

The equations of the system when coupled with an external system are,

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3 + \epsilon_1 y \\ \dot{x}_2 &= x_1 + a x_2 \\ \dot{x}_3 &= b + x_3(x_1 - c) \\ \dot{y} &= -ky - \epsilon_2 x_1\end{aligned} \tag{2.8}$$

As described in the Chapter 1 for the parameter values $a = b = 0.1$ and $c = 18$, R$\ddot{o}$ssler system displays a chaotic behaviour. We observe from Fig[2.1] that by introducing or by coupling the system with the external system the dynamics of the system is quenched to a stable fixed point state. In Fig[2.1] after 500 time steps the external system with coupling coefficient values $\epsilon_1 = 1$, $\epsilon_2 = 1$ and $k = 1$ has been coupled to the R$\ddot{o}$ssler system. Thereby we see a control of the chaotic system to a stable fixed point state of the system.

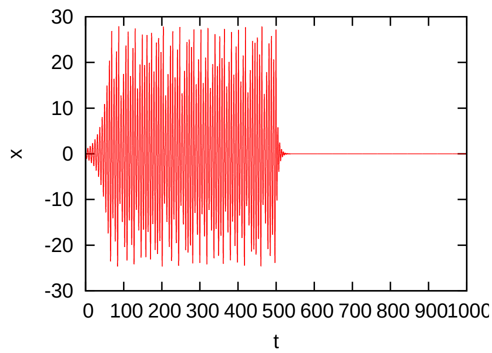

Figure 2.1: Time series of chaotic rossler system for coupling coefficients $\epsilon_1 = 1$, $\epsilon_2 = 1$ and $k = 1$ introduced at t=500 timesteps

2.2.1 Stability analysis of the R$\ddot{o}$ssler System:

We obtain the fixed points of the R$\ddot{o}$ssler system coupled with an external system by equating each $\dot{x} = 0$. By solving this we obtain the fixed points as

$$(x^*, \frac{-x^*}{a}, \frac{-b}{x^* - c}, \frac{\epsilon_2 x^*}{k}) \tag{2.9}$$

where

$$x^* = \frac{c \pm \sqrt{c^2 - 4ab\frac{k}{k + a\epsilon_1\epsilon_2}}}{2} \tag{2.10}$$

This fixed point is different from the intrinsic fixed point of the R$\ddot{o}$ssler system Eq[1.10]. The corresponding Jacobian matrix is

$$J = \begin{pmatrix} 0 & -1 & -1 & \epsilon_1 \\ 1 & a & 0 & 0 \\ \frac{-b}{x^* - c} & 0 & x^* - c & 0 \\ \epsilon_2 & 0 & 0 & -k \end{pmatrix} \tag{2.11}$$

The eigen value of this coupled system can now be obtained by solving the characteristic equation Eq[1.8]

For the above fixed points, using Mathematica we obtain the eigen values for any given set of parameters. For few choosen parameter values k, ϵ_1, ϵ_2 we have all eigen values with negative real part. For instance for $k = 1$, $\epsilon_1 = 1$, $\epsilon_2 = 1$, we have

$$\begin{aligned} \lambda_1 &= -4.28874 + 8.13913i, \\ \lambda_2 &= -4.28874 - 8.13913i, \\ \lambda_3 &= -0.890393, \\ \lambda_4 &= -0.0127554 \end{aligned} \tag{2.12}$$

We thus confirm the stable fixed point state from a complex chaotic behaviour using our control mechanism.

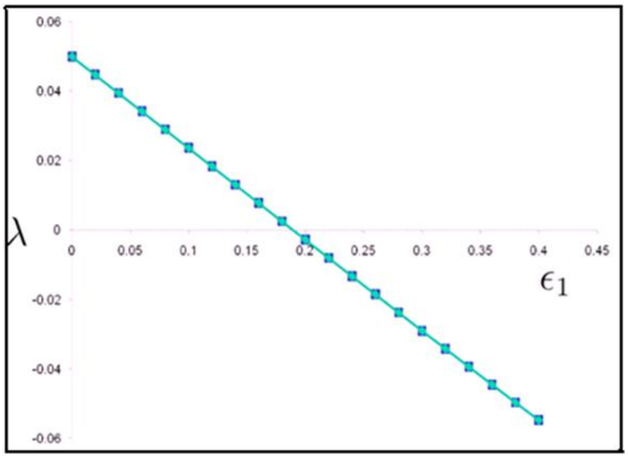

Figure 2.2: Largest eigenvalue of the Jacobian matrix as ϵ_1 is varied

Fig[2.2] is a plot of the largest eigen value varied with respect to a parameter value ϵ_1 keeping all other parameter values constant. We can see that as one of coupling coefficient of the system crosses a critical value, the largest eigen value crossed zero and the system displays a stable fixed point behaviour.

2.2.2 Numerical Analysis of the Rössler System

We do a numerical analysis of the control method by solving Rössler system with 4^{th} order Runge − Kutta method. From the timeseries of the Rössler system Fig[2.1], we clearly observe the quenching of the chaotic dynamics to the stable fixed point behaviour.

It is clear that the stability of the steady state depends on the coupling coefficients. Hence we study the transition of the chaotic behaviour of the system to a stable fixed point behaviour by identifying the regions of stable fixed point in the parameter plane of coupling strengh ϵ and k, where $\epsilon = \epsilon_1 = \epsilon_2$. To characterize the state of stable fixed point region we use an index A, defined as the difference between the global maxima and global minima of the time series of the system over a sufficiently long interval. This index A $= 0$ represents amplitude nearly equal to zero, which means the stable fixed point region, if the index A $\neq 0$ then the system has an oscillatory dynamics.

Using this index, we identify regions of control in the parameter plane ϵ vs k and ϵ_1 and ϵ_2 in Fig[2.3]. This is a 3d plot with x and y axis as the parameters and the colour ranging axis contains values of amplitude of the two systems oscillations over time. In fig[2.3(a)] we observe that for very small values of the damping parameter k, we need higher coupling strengths to obtain stable fixed point behaviour.

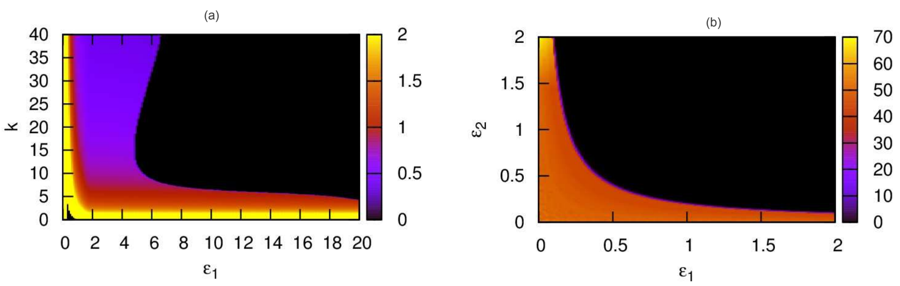

Figure 2.3: Parameter plane (a) ϵ vs k (b)ϵ_1 vs ϵ_2 with black region representing the region of control

In Fig[2.3(b)] ϵ_1 vs ϵ_2 parameter plane is plotted keeping the damping parameter constant $k = 1$. We find that the transition curve agrees with the condition obtained from the stability analysis in Eq[2.7].

In Fig[2.4] we show the transition of the system from the complex behaviour to the stable fixed point behaviour, where the plot gives the amplitude of the oscillations of the system as coupling strenghts ϵ_1 is varied

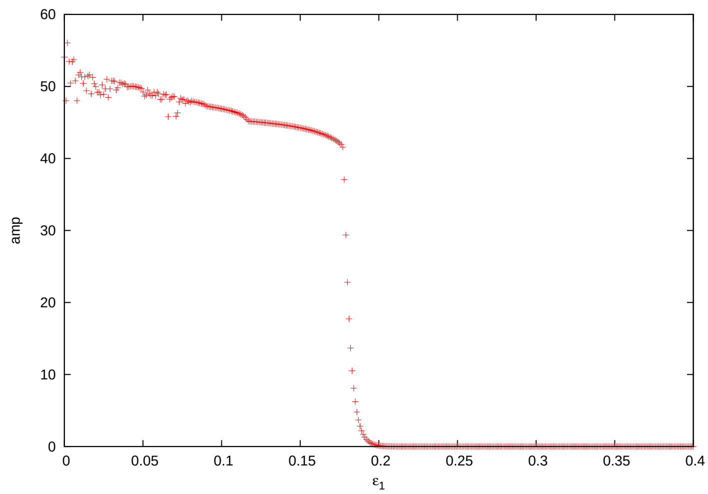

Figure 2.4: Change in amplitude of the systems oscillations as ϵ_1 is varied

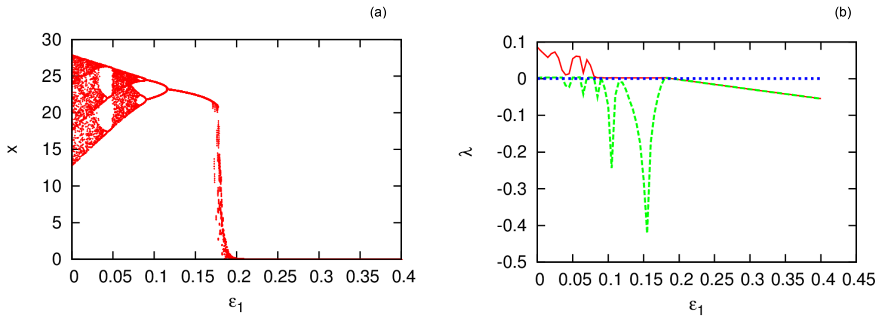

Figure 2.5: (a) Bifurcation diagram of coupled R$\ddot{o}$ssler system as ϵ_1 is varied (b)Largest Lyapunov exponent's of the coupled chaotic R$\ddot{o}$ssler system

(keeping rest of the parameters constant). As the parameter value ϵ_1 increases, amplitude of the systems oscillations gradually decrease to be equal to zero, indicating a stable fixed point behaviour. We observe the same trend by varing any other parameter value ϵ_2 and k.

We notice from the phase space and time series that as the coupling strenghts increases, the system undergoes reverse period doubling behaviour. Fig[2.5(a)]shows the bifurcation plot of the system when varied with the coupling strength ϵ_1 and we observe that the transtion is continuous and is reverse period doubling bifurcation. We also confirm this transition by plotting the largest lyaponov exponent of the system Fig[2.5(b)]. As the coupling coefficient value increases the largest lyapunov exponent changes from positive to negative value justifing the control of complex dynamics to a steady state behaviour.

2.2.3 Feed back control of Lorenz System

We illustrate this control mechanism on another chaotic system, Lorenz system. Following are the equations of this system when coupled with an external control system.

$$\begin{aligned} \dot{x} &= \sigma(y+x) + \epsilon_1 w \\ \dot{y} &= \gamma x - y - xz \\ \dot{z} &= xy - z\beta \\ \dot{w} &= -kw - \epsilon_2 x \end{aligned} \tag{2.13}$$

We obtain the fixed points of this coupled system by equating each $\dot{x} = 0$, as

$$(x^*, x^* + \frac{\epsilon_1\epsilon_2 x^*}{k\sigma}, \frac{x^*}{b}(\frac{\epsilon_1\epsilon_2 x^*}{k\sigma} + x^*), -\frac{\epsilon_2 x^*}{k}) \tag{2.14}$$

where

$$x^* = \pm\sqrt{\frac{\gamma b}{(\frac{\epsilon_1\epsilon_2}{k\sigma} + 1)} - b} \tag{2.15}$$

or

$$x^* = 0 \tag{2.16}$$

Its corresponding Jacobian matrix is

$$J = \begin{pmatrix} -\sigma & \sigma & 0 & \epsilon_1 \\ \gamma - \frac{x^*}{b}(\frac{\epsilon_1\epsilon_2 x^*}{k\sigma} + x^*) & -1 & -x^* & 0 \\ (\frac{\epsilon_1\epsilon_2 x^*}{k\sigma} + x^*) & x^* & -b & 0 \\ \epsilon_2 & 0 & 0 & -k \end{pmatrix} \tag{2.17}$$

We can solve the characteristic equation for obtaining the eigen values Eq[1.8] of the corresponding fixed points of this coupled system. Using Mathematica we have calculated the eigen values of the coupled system for a set of parameter values. For few choosen coupling coefficient values for example $k = 1$, $\epsilon_1 = 20$ and $\epsilon_2 = 20$, fixed point

$$(x^*, x^* + \frac{\epsilon_1\epsilon_2 x^*}{k\sigma}, \frac{x^*}{b}(\frac{\epsilon_1\epsilon_2 x^*}{k\sigma} + x^*), -\frac{\epsilon_2 x^*}{k}) \tag{2.18}$$

where

$$x^* = -\sqrt{\frac{\gamma b}{(\frac{\epsilon_1\epsilon_2}{k\sigma} + 1)} - b} \tag{2.19}$$

has following eigen values

$$\begin{aligned}\lambda_{1,2} &= -4.22031 \pm 44.6456i, \\ \lambda_3 &= -4.53286, \\ \lambda_4 &= -1.69318 \end{aligned} \tag{2.20}$$

This fixed point for few selected values of coupling coefficient and damping parameter has all eigen values with negative real part, indicating the system to have a steady state behaviour.

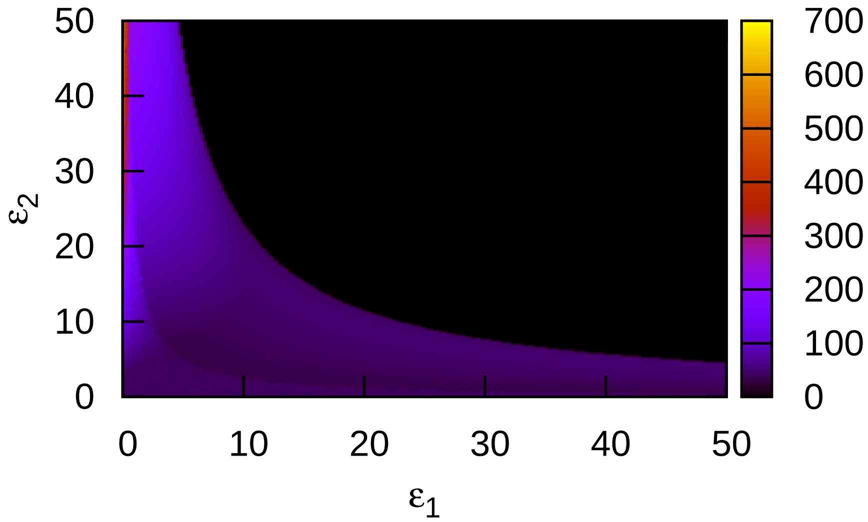

Figure 2.6: ϵ_1 and ϵ_2 Parameter plot of Lorenz system, black region represents the steady state behaviour

Fig[2.6] is the parameter plot with a transition of ϵ_1 and ϵ_2 curve, for coefficient values more than a critical condition, the system is quenched to a stable dynamics. Lorenz system also justifies the stability criteria of the general mechanism Eq[2.7].

In this case, the transition from the chaotic behaviour to stable fixed point behaviour is different from the Rössler system. Hence a fixed point state is reached from complex behaviour by a sudden transition. From fig [2.7], bifurcation diagram and lyapunov exponent we observe the transition is sudden, the largest lyapunov expontent of the system takes a sudden shift from positive to negative.

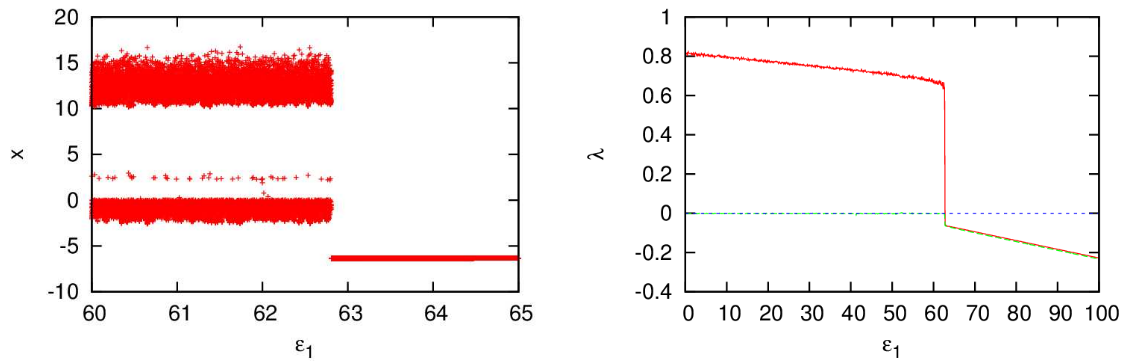

Figure 2.7: Lorenz system (a) Bifurcation Diagram (b) Lyapunov Exponent as ϵ_1 parameter is varied

2.3 Feed back Control of Chaotic behaviour to Periodic oscillations

Periodic behaviour is also a desirable behaviour in many practical applications, periodicity is required in respiration and cardiac activity and is an essential concept in fluid dynamics. Our general mechanism is also capable of controlling any complex dynamical system to a periodic system for few specific coupling coefficient values, in the case of system like Rossler.

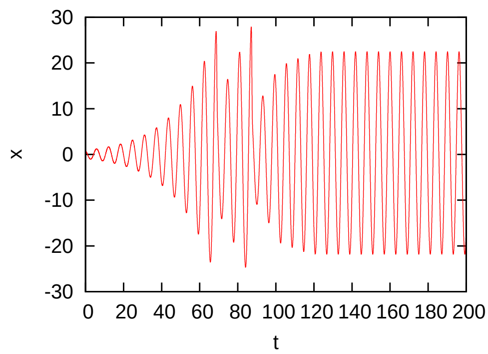

Figure 2.8: Time series of R$\ddot{o}$ssler exhibiting periodic behaviour

In such cases the systems have a continuous transition from the chaotic behaviour to the fixed point behaviour when coupled with an external damped system. In Fig[2.5] as the coupling coefficient ϵ_1 is varied considering all other parameters to be constant the system gradually changes from a chaotic behaviour to a periodic behaviour and then to the stable fixed point behaviour. Hence in order to control the dynamical system to a period state we can choose the ϵ_1 value where the system is controlled to a periodic behaviour. Fig[2.8] gives a periodic behaviour of R$\ddot{o}$ssler system for coupling coefficients $\epsilon_1 = 0.15$, $\epsilon_2 = 1$, $k = 1$.

But for the case of Lorenz system the route taken to control chaotic behaviour to the fixed point behaviour is a sudden transition. So there are no values of ϵ_1 for which the system can settle to a periodic behaviour. For such systems we are not able to control the system dynamics to periodic behaviour through our feedback control mechanism. For this we modify the control by adding a periodic function or drive to the control system. Following are the set of equations of Lorenz system having an additional periodic function in the general mechanism of control.

$$
\begin{aligned}
\dot{x} &= \sigma(y + x) + \epsilon_1 w \\
\dot{y} &= \gamma x - y - xz \\
\dot{z} &= xy - z\beta \\
\dot{w} &= -kw - \epsilon_2 x + asin(u) \\
\dot{u} &= \omega
\end{aligned}
\tag{2.21}
$$

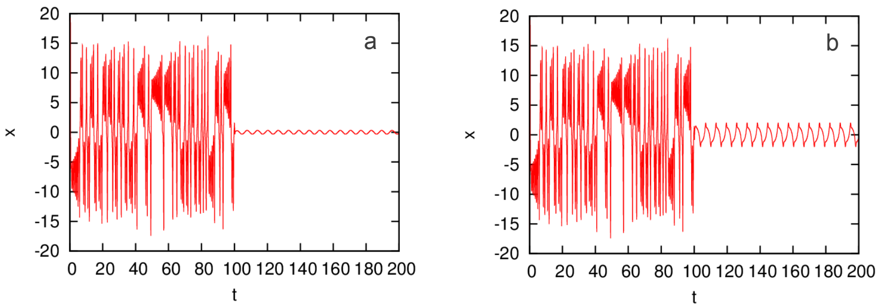

Figure 2.9: Periodic behaviour Lorenz system at (a)a = 1, (b)a = 10

where a is the amplitude of the introduced periodic function and ω is the frequency. For specific values of the coupling coefficient and damping parameter values, the system exhibits a periodic behaviour. The amplitude and the frequency of the desired periodic behaviour can be adjusted by varying the a and w values Fig[2.9]. Hence any desirable periodic orbit can be obtained from a system which in general does not exhibit a periodic property.

In the parameter plane ϵ_1 and ϵ_2 the regions where the periodic behaviour is stable is the same regions where the fixed point behaviour was stable in the case of control mechanism with out the periodic function Fig[2.6].

In conclusion we have shown how the feedback mechanism introduced by us can control chaotic system to periodic or steady state behaviour.

Chapter 3

Feed back Control of Hyper chaotic Systems

The control mechanism introduced in the previous chapter for the control of chaotic systems is applied for the case of hyperchaotic systems in this chapter. We couple the hyperchaotic system with a damped external system in order to quench the complex dynamics.

3.1 Chen System

We consider one of the hyper chaotic systems described in the chapter 1. One set of the parameter values for which the system is hyperchaotic are $a = 35$, $b = 4.9$, $c = 25$, $d = 5$, $e = 35$, $k' = 100$. In order to quench the dynamics of the system we couple this system with an external damped system. Following are the equations governing the dynamics of this coupled hyperchaotic system.

$$\begin{aligned}
\dot{x} &= a(y - x) + eyz \\
\dot{y} &= cx - dxz + y + u + \epsilon_1 w \\
\dot{z} &= xy - bz \\
\dot{u} &= -k'y + \epsilon_1 w \\
\dot{w} &= -kw - \epsilon_2 y - \epsilon_2 u
\end{aligned} \tag{3.1}$$

For particular coupling coefficient ranges the system settles to a stable fixed point behaviour. Fig[3.1] displays the time series of the system without coupling with the external system till 500 time steps and then coupling with the external system. Hence this control mechanism seems to work for the case of hyperchaotic system. Fig3.1 gives a range of parameter values ϵ_1 and k values considering the parameter value ϵ_2 constant, for which the index $A = 0$ (.i.e) the amplitude of the oscillations of the system as described in the chapter 2. The dark region where $A = 0$ in the parameter ϵ_1 and k plot shows the system's complex behaviour settle to a stable fixed point behaviour, after coupling with this damped external system.

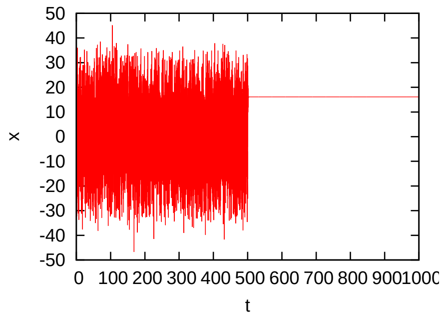

Figure 3.1: Time series of hyperchaotic Chen System at parameters $\epsilon_1 = 6$, $\epsilon_2 = 1$, $k = 2$

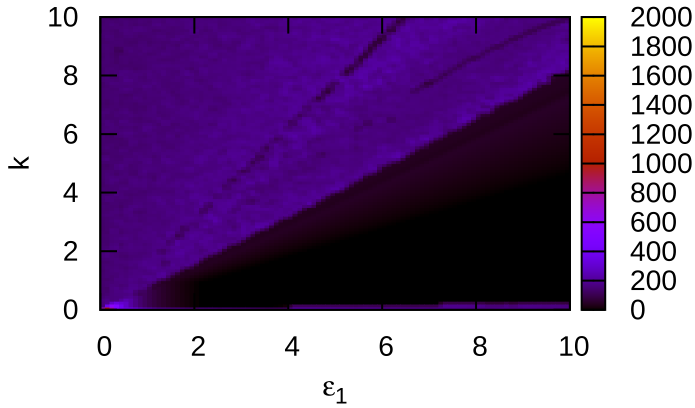

Figure 3.2: Parameter plane of controlled Chen System, the black region shows the fixed point region

This coupled system has a reverse period doubling sequence from the hyperchaotic dynamics to the stable fixed point dynamics thereby having a continuous transition. Hence we can control the complex dynamics even to the periodic behaviour by adjusting the coupling coefficient values accordingly. Fig[3.3(a)] shows the continuous transition of the complex dynamics to a steady state behaviour.

The bifurcation diagram shows the reverse period doubling sequence involved during the control mechanism as in Fig[3.3(b)]

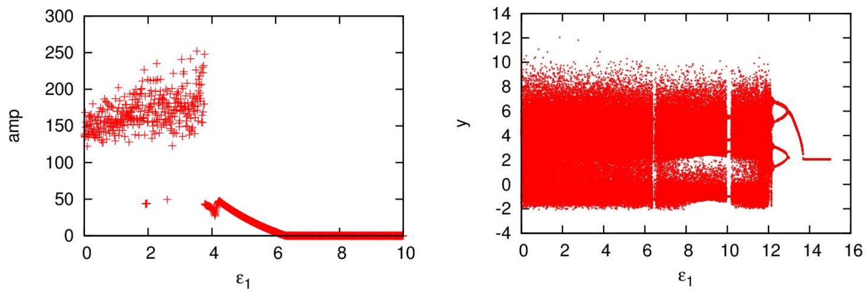

Figure 3.3: (a)Variation of amplitude of the systems oscillations as ϵ_1 is varied, (b)Bifurcation diagram of Chen system as ϵ_1 is varied

3.2 Chua System

Chua system is another complex hyperchaotic system which is generally used in many hyperchaotic circuits for generating complex signals and is described in chapter 1. Inorder to control the dynamics of the complex system we have applied our control mechanism of coupling the system to the external system like in the case of Chen hyper chaotic system. Following are the equations governing the coupled system.

$$\begin{aligned}
\dot{x} &= \alpha(y - ax^3 - (1+c)x) \\
\dot{y} &= x - y + z + \epsilon_1 w \\
\dot{z} &= -\beta y - \gamma z + u \\
\dot{u} &= -sx + yz \\
\dot{w} &= -kw - \epsilon_2 y
\end{aligned} \tag{3.2}$$

We obtain the fixed points of this coupled system by considering each $\dot{x}_i = 0$, as

$$[x^*, y^*, z^*, u^*, w^*] \tag{3.3}$$

where

$$\begin{aligned}
y^* &= ax^3 + x + cx, \\
z^* &= (ax^3 + x + cx)(1 + \frac{\epsilon_1\epsilon_2}{k} - x), \\
u^* &= (ax^3 + x + cx)(\beta + \gamma + \frac{\gamma\epsilon_1\epsilon_2}{k}) - \gamma x, \\
w^* &= -\frac{\epsilon_2(ax^3 + x + cx)}{k}
\end{aligned} \tag{3.4}$$

where x is the solution of 6^{th} order polynomial

$$(ax^3 + x + cx)[(ax^3 + x + cx)(1 + \frac{\epsilon_1\epsilon_2}{k} - x)] - sx = 0 \tag{3.5}$$

Using mathematica we obtain that

$$x = Root[-0.1 + (0.24 + \frac{0.04\epsilon_1\epsilon_2}{k})1 + (-0.042 - \frac{0.012\epsilon_1\epsilon_2}{k})1^3 + (0.009 + \frac{0.0009\epsilon_1\epsilon_2}{k})1^5, 2] \quad (3.6)$$

is the stable solution and for certain coupling coefficient values $k = 1$, $\epsilon_1 = 10$, $\epsilon_2 = 15$, the fixed point mentioned above has all negative real part eigen values, indicating stability of the fixed points.

$$\begin{aligned} \lambda_1 &= -10.5219 \\ \lambda_{2,3} &= -0.0803752 \pm 9.00368i \\ \lambda_4 &= -0.595907 \\ \lambda_5 &= -0.130487 \end{aligned} \quad (3.7)$$

The fixed point is stable only after a critical value of the coupling coefficient values. In fig[3.4] we see a gradual change in the largest two eigen values of the fixed point from a positive value to a negative real part as the one of the coupling coefficient is varied keeping all other parameters constant.

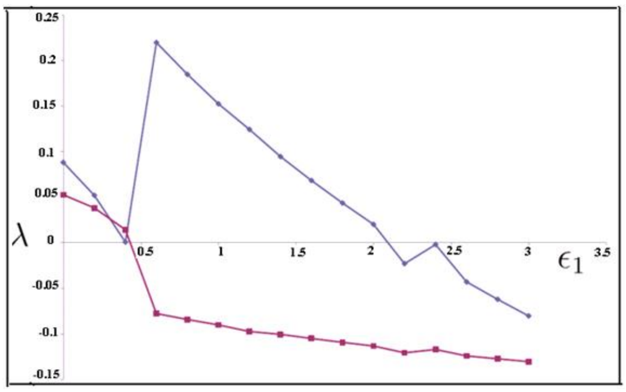

Figure 3.4: Largest Eigen values of Chua System as parameter ϵ_1 is varied

The stability criteria Eq[2.7] as in the case of chaotic systems is justified even for the case of hyperchaotic systems. Only after a threshold value of the coupling coefficient values this mechanism of controlling the complex dynamics to the steady state behaviour is possible. Fig[3.5] gives the critical curve of the parameter values, with these values greater than the critical values the system can be stabilized.

The transition from the hyperchaotic state to the steady state is sudden unlike in the case of chen system.

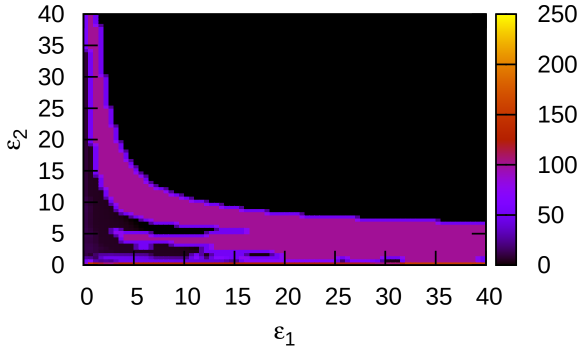

Figure 3.5: ϵ_1 and ϵ_2 Parameter plot of Chua system

The original route of Chua system with out coupling with the external system from a fixed point behaviour to a hyperchaotic behaviour is through a period doubling sequence, which implies that the system had a continuous transition to a complex behaviour. But when this system is coupled to the external environment to control its dynamics it does not take the reverse period doubling intead has a sudden transition from complex behaviour to a steady state behaviour. Fig[3.6] gives the bifurcation plot of this system to observe the sudden transition.

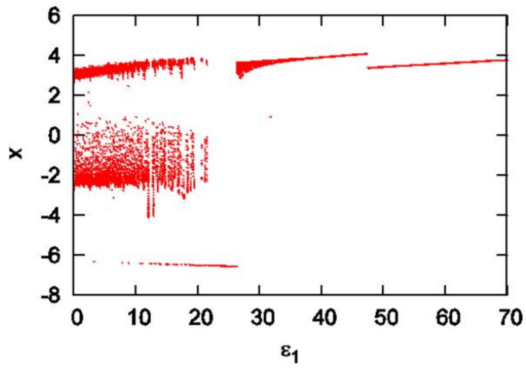

Figure 3.6: Bifurcation diagram of Chua system with varying ϵ_1

In this chapter, we have shown the control of hyperchaotic systems to steady state behaviour using feed back control through an over damped external system.

Chapter 4

Feed back Control of Time delay Systems

4.1 Introduction

Time delay systems are very important systems that occurs often in the real world situations[24]. These system can also be modelled using a set of differential equations called delay differential equations. Two typical time delay systems have been introduced in Chapter 1. In general, dynamics of time delay systems x is represented as

$$\dot{x} = F(x, x_\tau, a) \tag{4.1}$$

where $x_\tau = x(t - \tau)$ and $F(x, x_\tau, a)$ determines the nature of the system including delay variables. The fixed point of the system obeys

$$x^* = x = x_\tau \tag{4.2}$$

where x^* is the fixed point of the system and

$$F(x^*) = 0 \tag{4.3}$$

We note that the general stability analysis derived in chapter 2 is not practical for time delay systems.

Let us assume the solution of this time delay system as

$$x(t) = Ae^{\lambda t} \tag{4.4}$$

where A is an constant and λ is the eigen value of the system. then

$$x(t - \tau) = Ae^{\lambda(t-\tau)} \tag{4.5}$$

$$\Rightarrow x(t-\tau) = Ae^{\lambda t}e^{-\lambda\tau} \tag{4.6}$$

$$\Rightarrow x_\tau = xe^{-\lambda\tau} \tag{4.7}$$

and the characteristic equation for the delay system is

$$|J_0 + e^{-\lambda\tau}J_\tau - \lambda I| = 0 \tag{4.8}$$

with

$$J_0 = \frac{\partial F_i}{\partial x_j} \tag{4.9}$$

and

$$J_1 = \frac{\partial F_i}{\partial x_{\tau j}} \forall i,j = 1,2,..n \tag{4.10}$$

where n is the number of variables involved in the system. We cannot obtain the eigen values directly from the characteristic equation of a time delay system as the characteristic equation is a transcendental equation and has infinitly many eigen values. Hence it is not easy to find the stability of the time delay system analytically. However in specific cases, alternate methods can be used as shown in the following sections.

4.2 Mackey Glass System

The general mechanism of controlling complex systems introduced in this work is applied to the time delay chaotic systems. Mackey glass system exhibits chaos for particular parameter values and inorder to control this chaotic dynamics of this system we apply our general control mechanism. Following are the set of differential equations governing the nature of the chaotic Mackey Glass system coupled with an external damped system.

$$\begin{aligned} \dot{x} &= \beta\frac{x_\tau}{1+x_\tau^n} - \gamma x + \epsilon_1 w \\ \dot{w} &= -kw - \epsilon_2 x \end{aligned} \tag{4.11}$$

Fig[4.1] gives the timeseries of the system with coupling with an external system with control effective from t = 500 time step.

We obtain the fixed points of this system by equating each $\dot{x} = 0$, as

$$(x^*, -\frac{\epsilon_2 x^*}{k}) \tag{4.12}$$

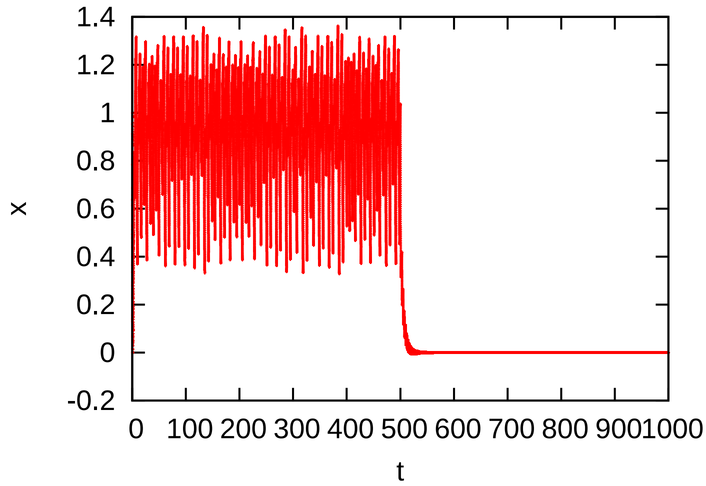

Figure 4.1: Time series of Mackey Glass system at coupling coefficient values $\epsilon_1 = 2$,$\epsilon_2 = 2$ and k =2 from t=500 time steps

where

$$x^* = \sqrt[n]{\frac{\beta k}{\gamma k + \epsilon_1 \epsilon_2} - 1} \tag{4.13}$$

or

$$x^* = 0 \tag{4.14}$$

The fixed point $x^* = 0$ is the only real solution for

$$\frac{\beta k}{\gamma k + \epsilon_1 \epsilon_2} < 1 \tag{4.15}$$

for the paramter values used in our analysis $\beta = 2$, $\gamma = 1$, $k = 2$,$\epsilon_1 = 2$, and $\epsilon_2 = 2$. This explains the timeseries of this system in Fig[4.1] where we observe that the fixed point with $x^* = 0$ is stable when controlled with an external system.

4.3 Stability Analysis

In order to study the stability of this system we write the variation equation formed by linearizing Eq[4.11]

$$\begin{aligned} \delta \dot{x} &= \frac{\beta}{1 + x_\tau^n}[1 - \frac{n x_\tau^n}{1 + x_\tau^n}]\delta x_\tau - \gamma x_\tau + \epsilon_1 \delta w \\ \delta \dot{w} &= -k \delta w - \epsilon_2 \delta x \end{aligned} \tag{4.16}$$

$$\rightarrow \begin{pmatrix} \delta\dot{x} \\ \delta\dot{w} \end{pmatrix} = \begin{pmatrix} \frac{\beta}{1+(x^*)^n}[1-\frac{n(x^*)^n}{1+(x^*)^n}]e^{-\lambda\tau}-\gamma & \epsilon_1 \\ -\epsilon_2 & -k \end{pmatrix} \begin{pmatrix} \delta x \\ \delta w \end{pmatrix} \tag{4.17}$$

as $x_\tau = e^{-\lambda\tau}x$ From Eq[4.7]. Hence the characteristic equation of the above pair of equations is

$$\begin{pmatrix} \frac{\beta}{1+(x^*)^n}[1-\frac{n(x^*)^n}{1+(x^*)^n}]e^{-\lambda\tau}-\gamma & \epsilon_1 \\ -\epsilon_2 & -k \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0 \tag{4.18}$$

$$\rightarrow \frac{\beta}{1+(x^*)^n}[1-\frac{n(x^*)^n}{1+(x^*)^n}]e^{-\lambda\tau}(k+\lambda) = (k+\lambda)(\gamma+\lambda)+\epsilon_1\epsilon_2 \tag{4.19}$$

This is a trancendental equation and so is difficult to obtain the eigen values of the system. Hence to analysis the stability, we follow the geometric method given in [29]. For $x^* = 0$ is given by the eigen value equation.

$$\beta(k+\lambda)e^{-\lambda\tau} = \lambda^2 + \lambda(k+\gamma) + k\gamma + \epsilon_1\epsilon_2 \tag{4.20}$$

First with the time delay parameter $\tau = 0$ we check if the fixed point $x^* = 0$ is stable for choosen coupling coefficient values of the external damped system.

From Eq[4.20] by substituting $\tau = 0$, we have

$$\beta(k+\lambda) = \lambda^2 + \lambda(k+\gamma) + k\gamma + \epsilon_1\epsilon_2 \tag{4.21}$$

This is a quadratic equation and hence the eigen values of the system with out time delay are

$$\lambda_{1,2} = \frac{-(k+\gamma-\beta) \pm \sqrt{(k+\gamma-\beta)^2 - 4(k\gamma+\epsilon_1\epsilon_2-k\beta)}}{2} \tag{4.22}$$

By substituting choosen parameter values $\beta = 2$, $\gamma = 1$, $k = 2$,$\epsilon_1 = 2$, and $\epsilon_2 = 2$ we get

$$\lambda_{1,2} = -\frac{1}{2} \pm i\sqrt{(7)} \tag{4.23}$$

which shows that at delay parameter $\tau = 0$ the fixed point $x^* = 0$ is stable.

Now if τ is non zero, the change in stability of this fixed point can occur only if the eigen value crosses the imaginary axis, (i.e) If the fixed point having negative real part changes to positive real part it has to cross $\lambda = 0 + iw$, where w has a real solution.

To check this let us consider the eigen value of the system to be $\lambda = iw$. By substituting this in Eq[4.20] we obtain

$$\beta(k+iw)e^{-iw\tau} = (iw)^2 + (iw)(k+\gamma) + k\gamma + \epsilon_1\epsilon_2 \quad (4.24)$$

$$\rightarrow e^{-iw\tau} = ((iw)^2 + \frac{(iw)(k+\gamma) + k\gamma + \epsilon_1\epsilon_2)}{(\beta(k+iw)}) \quad (4.25)$$

After factorizing RHS of Eq[4.25] we obtain

$$\begin{aligned} cos(w\tau) - isin(w\tau) &= \frac{k(-w^2 + k\gamma + \epsilon_1\epsilon_2)}{\beta(k^2+w^2)} + \frac{w^2(k+\gamma)}{\beta(k^2+w^2)} \\ &+ i(-\frac{w(-w^2 + k\gamma + \epsilon_1\epsilon_2)}{\beta(k^2+w^2)} + \frac{kw(k+\gamma)}{\beta(k^2+w^2)}) \end{aligned} \quad (4.26)$$

If there exists a solution to Eq[4.26] then the change of stability occurs as there exists a real value of w. For this we look for a solution geometrically by plotting the real and imaginary parts of the LHS and RHS and check if they cross each other.

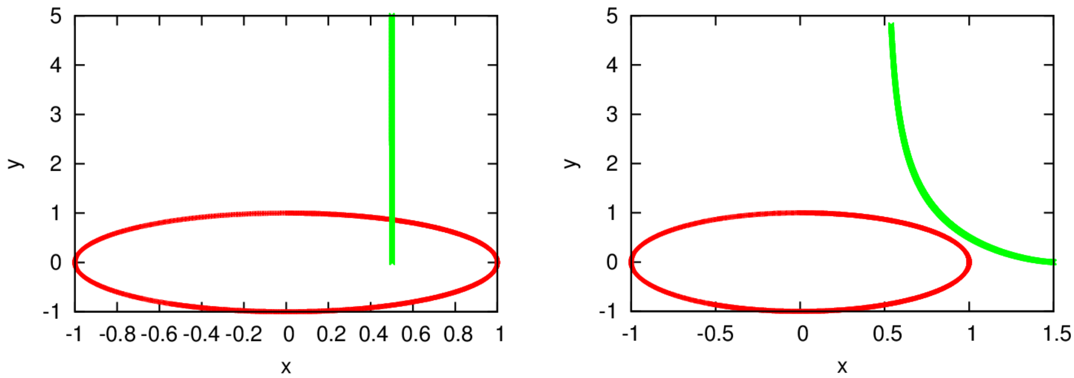

Figure 4.2: Geometric Analysis for (a)$\epsilon_1 = 0.1$ and $\epsilon_2 = 0.1$ (b)$\epsilon_1 = 2$ and $\epsilon_2 = 2$

In Fig[4.2] at $\tau = 2$, for $\epsilon_1 = 2$ and $\epsilon_2 = 2$ there exists no solutions of Eq[4.26], but as we vary the coupling coefficient values to $\epsilon_1 = 0.1$ and $\epsilon_2 = 0.1$ we have a solution for Eq[4.26] indicating a change in stability.

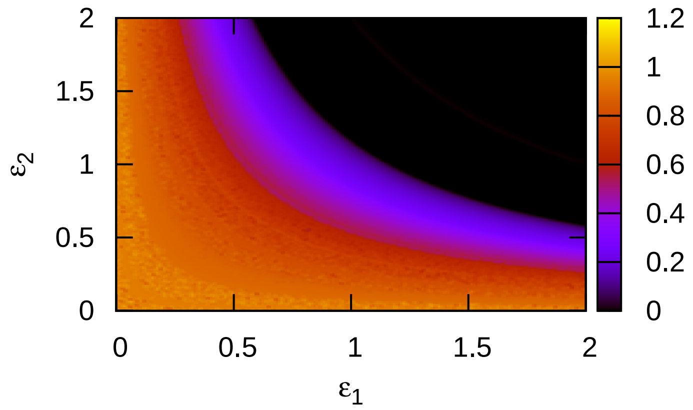

Figure 4.3: Parameter Plane of mackey glass system when coupled with a control

From the parameter plane ϵ_1 and ϵ_2 at delay $\tau = 2$, we observe that as the control parameters attains critical values the chaotic system is quenched to a steady state behaviour. This also justifies the stability analysis for choosen parameter values. Like for $\tau = 2$ and $\epsilon_1 = 2$ and $\epsilon_2 = 2$ we observe no change in stable state of the fixed point and hence is stable from the stability analysis and for $\epsilon_1 = 0.1$ and $\epsilon_2 = 0.1$ we observe a change in stability.

The transition of chaotic Mackey glass system from its chaotic dynamics to a stable fixed point state is a continuous transition of reverse period doupling sequence. Fig[4.4(a)] gives the amplitude of the systems oscillations after a long interval of time with varing ϵ_1 parameter. We observe that as the ϵ_1 parameter values is gradually increasing the amplitude of the systems oscillations change from very high value amplitude value to nearly equal to zero. Amplitude value nearly equal to zero implies the system has settled to a fixed point of the system. The fixed point of the system coupled with an external system is different from the intrinsic fixed points of the chaotic Mackey glass system. The bifurcation of this system as ϵ_1 parameter is varied is shown in Fig[4.4(b)]

We also observe that when we couple Ikeda time delay system with the external system we observe that the chaotic dynamics of this system is quenched to a steady state behaviour.

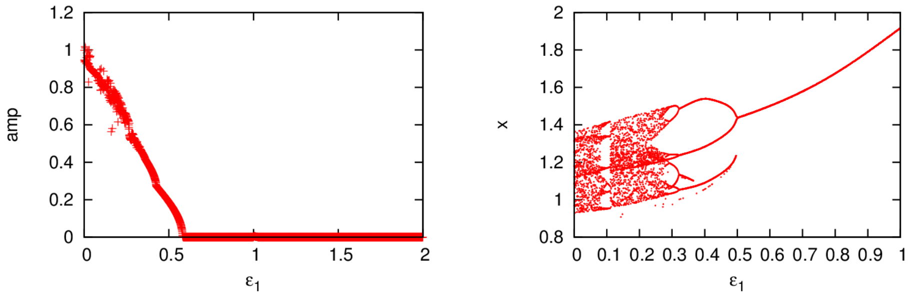

Figure 4.4: (a)Amplitude vs the change in coupling coefficient ϵ_1 (b) Bifurcation Diagram

Thus the control mechanism is found effective in the case of time delay systems also where control from hyperchaos to steady state is possible. The stability in the presence of the delay is analysed using geometric control.

Chapter 5

Feedback Control with Time Delay and Diffusive Coupling

We know that time delay is unavoidable or inherent in any feed back or coupling due to finite transition time. Hence we study the control mechanism used here in the presence of time delay. So also we study another type of coupling called diffusive coupling in our control technique, which is a more realistic type of coupling in many practical situations.

5.1 Control with Time Delay

We introduce time delay in the feedbacks from the external damped system to the chaotic system and vise versa, in our general mechanism of control of complex dynamics Eq[2.1]. We consider chaotic R$\ddot{o}$ssler system for studing this control mechanism with time delay in the construction of the coupled system. Following are the modified equations of the R$\ddot{o}$ssler system coupled with an external damped system, including the delay

$$\begin{aligned} \dot{x} &= -\sigma(y+x) + \epsilon_1 y(t-\tau) \\ \dot{y} &= x + ay \\ \dot{z} &= \beta + z(x-c) \\ \dot{w} &= -kw - \epsilon_2 x(t-\tau) \end{aligned} \tag{5.1}$$

For simplicity we take the time delay in the feedbacks between the external system and R$\ddot{o}$ssler system to be equal.

Fig[5.1(b)] gives the ranges of the parameter values ϵ_1 and ϵ_2 for the steady state behaviour of the R$\ddot{o}$ssler system when controlled by the external damped system in presence of time delay in the feedback terms. From

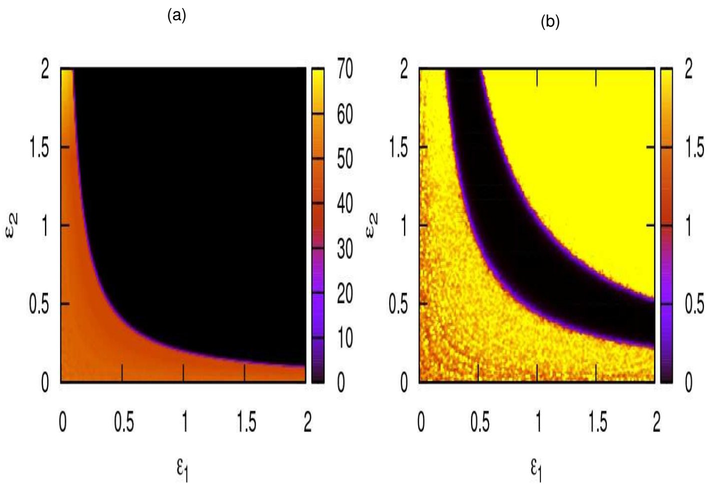

Figure 5.1: Time delay in the feedback for controlling Rö ssler system with $\tau = 0.2$, with dark region representing the stable fixed point region

comparision with no time delay in the system shown in Fig[5.1(a)], the region of control is less in the presence of time delay, observed as the time delay of this coupled system increases the region of steady state behaviour reduces gradually and for lower values of coupling coefficient ϵ_1 and ϵ_2 compared to the values with respect to no time delay, the system reaches instability and escapes eventually compared to the case with out delay the region of steady state behaviour is reduced.

We also observe that the transition curve above which the stable fixed point region is obtained has also shifted to higher values of coupling coefficients in case of delay system. i.e. For those values of coupling coeffients for which we had stable fixed point behaviour without delay in the system has changed to complex behaviour in presence of delay in the system.

5.2 Diffusive coupling

Instead of the the feedback coupling of the chaotic system with the external damped system, we couple the chaotic system by a diffusive coupling to the external damped system. Let us consider chaotic Rö ssler system, following are the equation's governing the nature of the Rö ssler system with a diffusive coupling through a external damped system.

$$\begin{aligned}
\dot{x}_1 &= -x_2 - x_3 + \epsilon_1(y - x) \\
\dot{x}_2 &= x_1 + a x_2 \\
\dot{x}_3 &= b + x_3(x_1 - c) \\
\dot{y} &= -ky + \epsilon_2(x - y)
\end{aligned} \tag{5.2}$$

Fig[5.2] gives the ϵ_1 and ϵ_2 parameter plane for constant $k = 1$, the parameter plot shows the regions of stable fixed point regions of the R$\ddot{o}$ssler system when coupled with an external damped system through a diffusive couling.

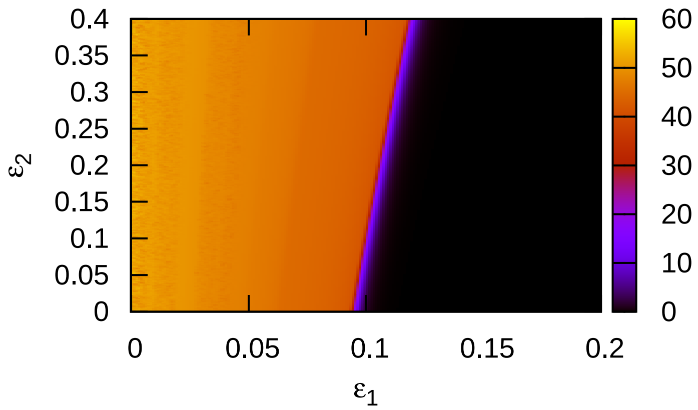

Figure 5.2: R$\ddot{o}$ssler System with diffusive feedback coupling, where the yellow region represents complex behaviour of the system and the black region represents steady state behaviour.

We observe that even in the absence of diffusive feedback coupling from the R$\ddot{o}$ssler system to the external damped system at certain parameter values of ϵ_1 the chaotic dynamics of the system is controlled to a stable fixed point behaviour.

In this chapter, we present the effect of time delay in the control mechanism. We find that the control is still possible for small time delays even though the regions of control in the parameter plane is reduced. The mechanism works with diffusive type of feed back also.

Chapter 6

Feed back with delay and detuning in Coupled Systems

In this section we present the effects of time delay and detuning in controlling the dynamics of two coupled Rössler systems. In a recent work it was shown that an external damped system is coupled to two directly coupled chaotic systems, their dynamics can be suppressed for few coupling coefficient values.[32] This is called amplitude death of the coupled systems.

6.1 Time - delay

We introduce time delay in the feedback terms from systems to the external system and viceversa and also in the feedback terms between the systems. Following are the equations of the coupled Rössler system coupled with an external damped system with time-delay in their coupling.

$$\begin{aligned}
\dot{x}_{11} &= -x_{12} - x_{13} + \epsilon_e \beta_1 y(t-\tau_1) + \epsilon_s (x_{21}(t-\tau_2) - x_{11}) \\
\dot{x}_{12} &= x_{11} + a x_{12} \\
\dot{x}_{13} &= b + x_{13}(x_{11} - c) \\
\dot{x}_{21} &= -x_{22} - x_{23} + \epsilon_e \beta_2 y(t-\tau_1) + \epsilon_s (x_{11}(t-\tau_2) - x_{21}) \\
\dot{x}_{22} &= x_{21} + a x_{22} \\
\dot{x}_{23} &= b + x_{23}(x_{21} - c) \\
\dot{y} &= -ky - \frac{\epsilon_e}{2}[\beta_1 x_{11}(t-\tau_1) + \beta_2 x_{22}(t-\tau_1)]
\end{aligned} \tag{6.1}$$

Here τ_1 is the time delay parameter in the feedback terms between the systems and the external system and τ_2 is the time delay parameter between the two systems.

We first study the case where there is time delay only in the feedback terms between the system and the external system but not in the feedback terms between the two chaotic R$\ddot{o}$ssler systems.

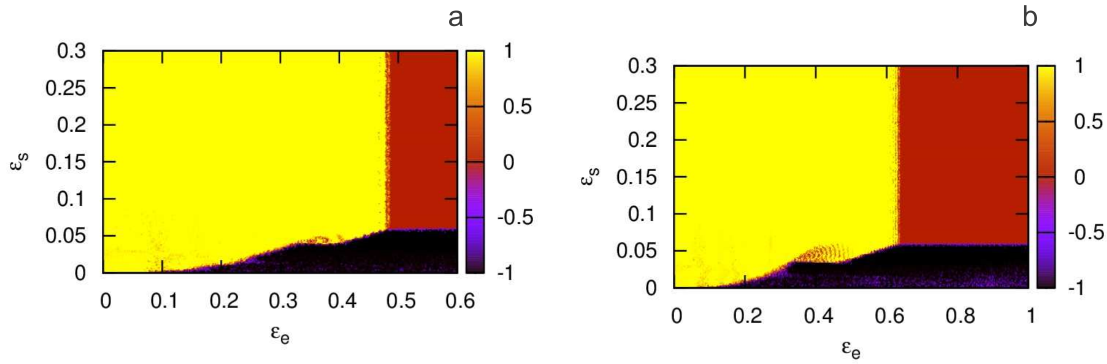

Figure 6.1: Delay between the systems and the external environment with τ_1 = 0.15, where the red region represents the amplitude death region, the yellow region represents the synchronizable region and the black region represents the antisynchronisable region.

Fig[6.1] gives the parameter plane ϵ_e and ϵ_s of the coupled system coupled with an external damped system when introduced with a delay in between the system and the external system alone. We observe that when we introduce time delay even for small increase in time delay, there is significant change in the region of amplitude death. We observe that as the time delay between the systems and the external system increase the region of amplitude death region has shifted to its right. For only for higher values of ϵ_e value the systems reach amplitude death of the coupled systems. But for values lower than ϵ_s for which we had anti syncronization region in the case of no delay in system, displayed amplitude death region. This implies that the antisynchronization regions of the coupled system has moved down and the synchronization region has increased.

Now let us consider time delay in the feedback terms between the two systems alone and not in between the systems and the external system. We observe that for those coupling coefficient values for which we observed synchronization region in the case of no delay system has changed to amplitude death region as we increase the delay parameter. This change is small as compared to the case of change in delay between the systems and the external system. We observe that even the region of anti synchronization increases as the delay parameter increases and the shift from the synchronization region to amplitude death region has a pattern as shown in Fig[6.2] which is the parameter plot ϵ_e and ϵ_s.

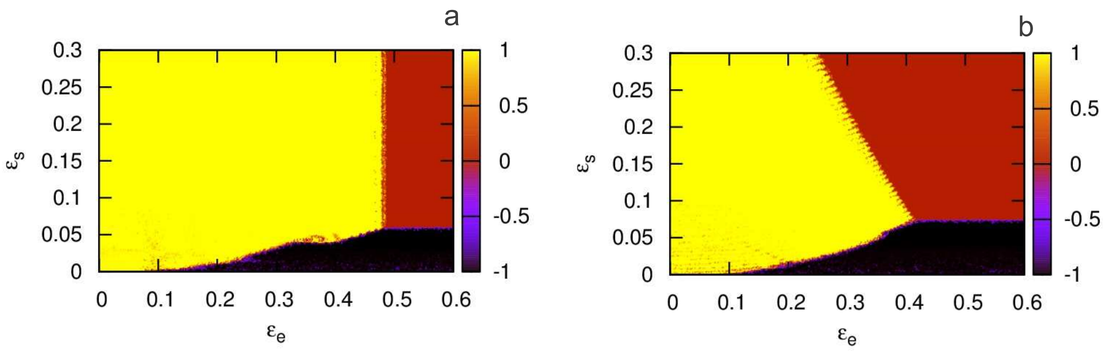

Figure 6.2: Delay between the systems with $\tau_2 = 1$, where the red region represents the amplitude death region, the yellow region represents the synchronizable region and the black region represents the antisynchronisable region.

Hence when there exists delays in both the feedback terms in between the systems and in the feed back term between the systems and the external system, the delay with respect to the feedback term between the system and the external system is more effective. For small changes of this delay vanishes the amplitude death reagions for few coupling coefficient values.

6.2 Detuning

We introduce detuning to the coupled Rössler system coupled with an external damped system. Following are the equations of the system with detuning in the Rössler systems.

$$
\begin{aligned}
\dot{x}_{11} &= -w_1 x_{12} - x_{13} + \epsilon_e \beta_1 y + \epsilon_s (x_{21} - x_{11}) \\
\dot{x}_{12} &= x_{11} + a x_{12} \\
\dot{x}_{13} &= b + x_{13}(x_{11} - c) \\
\dot{x}_{21} &= -w_2 x_{22} - x_{23} + \epsilon_e \beta_2 y + \epsilon_s (x_{11} - x_{21}) \\
\dot{x}_{22} &= x_{21} + a x_{22} \\
\dot{x}_{23} &= b + x_{23}(x_{21} - c) \\
\dot{y} &= -ky - \frac{\epsilon_e}{2}[\beta_1 x_{11} + \beta_2 x_{22}]
\end{aligned}
\tag{6.2}
$$

where w_1 and w_2 are the respective frequencies of the two Rössler systems coupled to the external system. When w_1 and w_2 are equal to one there is no detuning then the system is actually the original coupled Rössler systems coupled with an external system.

As the value of $w = w_1 = w_2$ increases the amplitude death region is at-

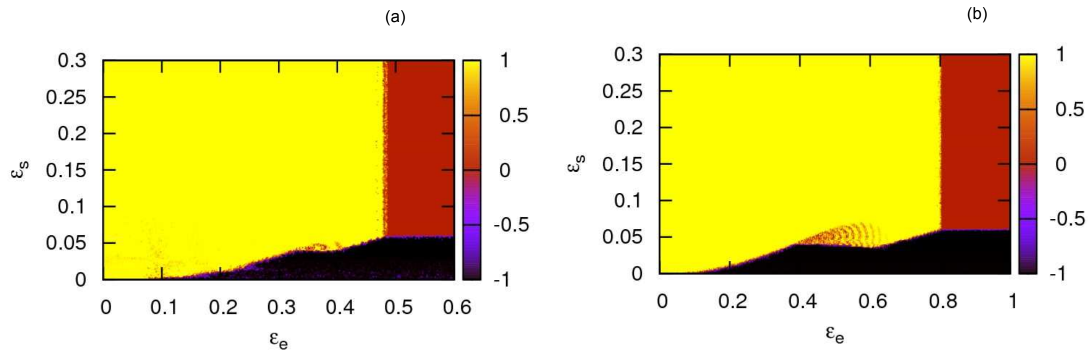

Figure 6.3: Parameter plane ϵ_1 and ϵ_2. Yellow region shows synchronization, red shows amplitude death region and the black region represents the antisynchronisable region.

tained at higher values of the coupling coefficient ϵ_e . The change in the amplitude death region in the parameter plane is not gradual (i.e.) when w = 1 or w = 1.9 the system has the no change in the parameter plane but as w changes to 2 Fig[6.3] there is a shift of amplitude death region in the parameter plane and amplitude death region is attained only at higher coupling coefficient values. Fig[6.3]

When the difference $w_1 \sim w_2 \geq 1$ then for all most all coupling coefficient values the system has amplitude death regions. There exists no regions of synchronization and anti synchronization in this case.

Thus we find that the region of amplitude death observed by coupling two directly coupled Rössler systems through a damped control system, changes significantly when there is delay in the coupling or detuning between the systems.

Chapter 7

Results and Discussions

In this project, we have introduced a control mechanism to control chaos and hyperchaos in dynamical and time delay systems. We work out the stability analysis analytically and numerically using Jacobian matrix, bifurcation diagrams, lyapunov exponents and obtain regions of control in the parameter plane.

The salient features of this control mechanism are the following.

- External control system consists of a simple over damped oscillator.
- Control method is system independent with no direct manipulation of system parameters.
- Control mechanism is very easy to implement in many practical cases such as electrical circuits.
- Effective for large variety of complex dynamical systems such as chaotic, hyperchaotic, time delay systems.
- By choosing appropriate coupling coefficient values, we can quench any complex dynamics to a steady state behaviour.
- Any desired value of the fixed point can be obtained by adjusting the parameters of the control system.
- Control of complex behaviour to a periodic behaviour is also possible for a choosen set of parameters.
- For those systems with sudden transition from complex behaviour to the steady state behaviour such as Lorenz system, we can obtain periodic behaviour by adding a periodic forcing term to the external damped system.

- Any desirable freqency and amplitude of the desired periodic dynamics can be obtained by adjusting the amplitude and frequency of the additional periodic forcing term of the external system.
- Effect of delay in feedback coupling terms between the complex dynamical system and the external damped system is analysed and a diffusive type of feedback is also studied.
- We also studied the effects of time delay and detuning in the case of two coupled chaotic systems when coupled with an external damped system.

Further work in this direction will be to study this control method applied to experimental and higher dimentional systems.

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