Transversal Hypergraphs

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by

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Certificate

This is to certify that this thesis entitled "Transversal Hypergraphs" submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents work carried out by Mr. Ankur Paliwal under the supervision of Dr. Soumen Maity.

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Reader 1

Reader 2

Dedicated to my father; my inspiration.

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Abstract

A hypergraph is a generalization of a graph in which an edge can connect any number of vertices. Formally, a hypergraph H is a pair $H = (X, E)$ where X is a set of elements called vertices, and E is the set of non-empty subsets of X called hyperedges or edges. A transversal (or "hitting set") of a hypergraph $H = (X, E)$ is a set $T \subseteq X$ that has non-empty intersection with every edge. A transversal T is called minimal if no proper subset of T is a transversal. The transversal hypergraph of H is the hypergraph (X, F) whose edge set F consists of all minimal transversals of H. Computing the transversal hypergraph has several applications in combinatorial optimization, in game theory, and in several fields of computer science such as machine learning, data mining and computer program optimization. This thesis is mainly concerned with several properties of transversal hypergraphs and transversal hypergraph generation problem, which asks to generate all minimal transversals of a given hypergraph.

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Chapter 1

Introduction

1.1 Background and Notations

The *basic idea* of the hypergraph concept is to consider a *generalization* of a graph in which *subset of any size* of a given set may be an edge rather than two-element subsets. When *drawing hypergraphs*, edges of size two are curves connecting respective vertices, while edges of size other than two are closed curves separating the respective subsets from the rest of vertices. (See Figure 1.1)

Definition 1: Let $X = \{x_1, x_2, ..., x_n\}$ be a finite set. A *hypergraph* on X is a family $H = (E_1, E_2, ..., E_m)$ of subsets of X such that

$$E_i \neq \emptyset \qquad (i = 1, 2, ..., m) \tag{1.1}$$

and

$$\bigcup_{i=1}^{m} E_i = X \tag{1.2}$$

x_i are called vertices and E_i are called hyperedges of the hypergraph.

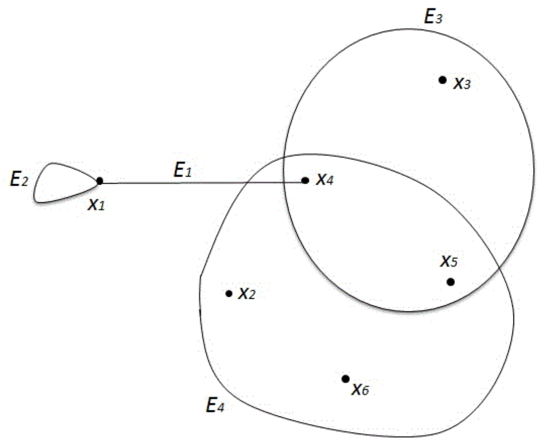

Figure 1.1: Representation of a hypergraph

Definition 2: A *simple hypergraph* (or *sperner family*) is a hypergraph $H = (E_1, E_2, ..., E_m)$ such that

$$E_i \subset E_j \;\Rightarrow\; i = j \tag{1.3}$$

A simple graph is a simple hypergraph each of whose edges has cardinality 2; a multigraph is a hypergraph in which each edge has cardinality less than or equal to 2.

The order of H, denoted by $n(H)$, is the number of vertices; while the number of edges of H is denoted by $m(H)$.

The *incidence matrix* of a hypergraph $H = (E_1, E_2, ..., E_m)$ of order n is a matrix $A = ((a^i_j))$ with n rows that represent the vertices and m columns that represent the edges of H such that

$$a^i_j = \begin{cases} 0 & if \quad x_i \notin E_j \\ 1 & if \quad x_i \in E_j \end{cases}$$

Example 1: Let $X = \{x_1,\, x_2,\, x_3,\, x_4,\, x_5,\, x_6\}$ and $H = (E_1,\, E_2,\, E_3,\, E_4)$ where $E_1 = \{x_1,\, x_4\}$, $E_2 = \{x_1\}$, $E_3 = \{x_3,\, x_4,\, x_5\}$ and $E_4 = \{x_2,\, x_4,\, x_5,\, x_6\}$, then the incidence matrix of H is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(see Figure 1.1)

Definition 3: The *dual* of a hypergraph, $H = (E_1, E_2, ..., E_m)$ on $X = \{x_1, x_2, ..., x_n\}$ is a hypergraph $H^* = (X_1, X_2, ..., X_n)$ whose vertices $e_1, e_2, ..., e_m$ correspond to the edges of H, and with edges

$$X_i = \{e_j \mid x_i \in E_j \text{ in H}\}$$

It can be easily seen that incidence matrix of dual, H^* of a hypergraph, H, is the transpose of incidence matrix of H and so here we have $(H^*)^* = H$.

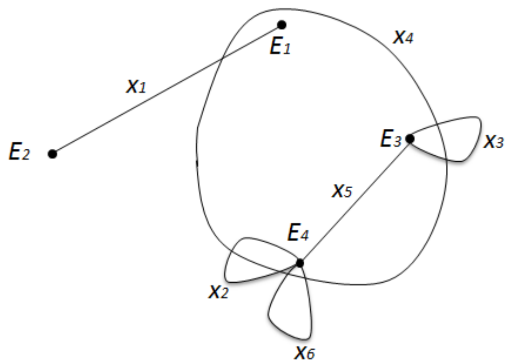

Figure 1.2: Dual of the hypergraph in Figure 1.1

So the dual of the hypergraph H of Example 1 is $H^* = (X_1, X_2, X_3, X_4, X_5, X_6)$ where vertex set is $\{e_1, e_2, e_3, e_4\}$ and $X_1 = \{e_1, e_2\}$, $X_2 = \{e_4\}$, $X_3 = \{e_3\}$, $X_4 = \{e_1, e_3, e_4\}$,

$X_5 = \{e_3, e_4\}$, $X_6 = \{e_4\}$ and its incidence matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Definition 4: For a set $J \subset \{1, 2, ..., m\}$ we call the family $H^{'} = (E_j \mid j \in J)$ the *partial hypergraph generated by the set* J. The set of vertices of $H^{'}$ is a nonempty subset of X.

Definition 5: For a set $A \subset X$, we call the family,

$$H_A = (E_j \cap A \quad | \quad 1 \leq j \leq m, E_j \cap A \neq \emptyset) \tag{1.4}$$

the *sub-hypergraph induced by the set* A

Some definitions from graph theory which may be extended without ambiguity to a hypergraph, $H = (E_1, E_2, ..., E_m)$, are as follows:

Rank: The rank of H is, $r(H) = \max_j |E_j|$

Anti-rank: The anti-rank of H is, $s(H) = \min_j |E_j|$

Uniform hypergraph: Uniform hypergraph is a hypergraph such that $r(H) = s(H)$

r**-uniform hypergraph:** A simple uniform hypergraph of rank r, is called r-uniform hypergraph.

Star: For a vertex x, star $H(x)$ with centre x is the partial hypergraph formed by the edges containing x.

Degree: The degree, $d_H(x)$ of a vertex x is the number of edges in $H(x)$, so, $d_H(x) = m(H(x))$.

Maximum degree: The maximum degree of H is denoted by

$$\Delta(H) = \max_{x \in X} d_H(x).$$

Regular hypergraph: A hypergraph in which all vertices have same degree is called a regular hypergraph.

Linear hypergraph: A hypergraph is linear if $|E_i \cap E_j| \leq 1$ for $i \neq j$.

Intersecting family: We define an intersecting family to be a set of edges having non-empty pairwise intersection. For example, for every vertex x of H, the star, $H(x)$ is an intersecting family of H.

1.2 Sperner Theorem

Theorem 1.2.1 (Sperner [8]; proof by Yamamoto, Meshalkin, Lubell, Bollobas [3]).
Every simple hypergraph H of order n satisfies

$$\sum_{E \in H} \binom{n}{|E|}^{-1} \leq 1. \tag{1.5}$$

Further, the number of edges $m(H)$ satisfies

$$m(H) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \tag{1.6}$$

Proof Let X be a finite set of cardinality n. Consider a directed graph G with vertices the subsets of X, with an arc from $A \subset X$ to $B \subset X$ if $A \subset B$ and $|A| = |B|$ - 1.

For example, the directed graph for $n = 3$ is shown in Figure 1.3.

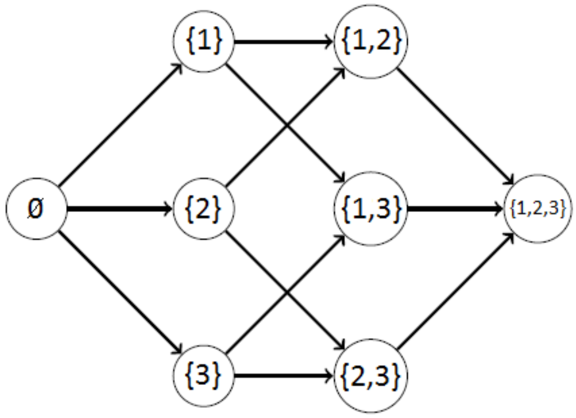

Figure 1.3: Directed Graph for $n = 3$

Let $E \in H$, the number of paths in the graph G from the vertex $\emptyset$ to the vertex E is $|E|!$, thus the total number of paths from $\emptyset$ to X is

$$n! \geq \sum_{E \in H} (|E|)!(n - |E|)!$$

(as H is a simple hypergraph, a path passing through E cannot pass through $E' \in H$, $E' \neq E$). We thus deduce inequality (1.5). For the second part,

$$\binom{n}{|E|} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

whence

$$1 \geq \sum_{E \in H} \binom{n}{|E|}^{-1} \geq m(H) \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1}.$$

We immediately deduce inequality (1.6). ∎

Outline of Thesis

The basic definitions, notations and important results are introduced in Chapter 1. We present the properties about transversal hypergraphs in Chapter 2. Next we present Berge algorithm and two modifications of Berge algorithm based on generalized vertices, on transversal hypergraph generation in Chapter 3.

Chapter 2

Properties of Transversal Hypergraphs

2.1 Introduction

Definition 1: Let $H = (E_1, E_2, ..., E_m)$ be a hypergraph on a set X. A set $T \subset X$ is a *transversal* of H if it meets all the edges, that is to say:

$$T \cap E_i \neq \emptyset \qquad\qquad (i = 1, 2, ..., m)$$

The family of minimal transversals of H constitutes a simple hypergraph on X called the *transversal hypergraph* of H, and denoted by TrH.

Example 1: The complete r-uniform hypergraph K_n^r on X admits as minimal transversals all the subsets of X with $n - r + 1$ elements. Thus

$$Tr(K_n^r) = K_n^{n-r+1}$$

The following lemma gives a necessary and sufficient condition for a hypergraph to be transversal of another. See [1, 9] for details.

Lemma 2.1.1 *Let* $H = (E_1, E_2, ...)$ *and* $H^{'} = (F_1, F_2, ...)$ *be two simple hypergraphs on a set* X. *Then* $H^{'} = TrH$ *if and only if every pair* (A, B) *with* $A, B \subset X, A \cup B = X, A \cap B = \emptyset$, *satisfies:*

(1) *there exists either an* $E \in H$ *contained in* A *or an* $F \in H^{'}$ *contained in* B*;*

(2) *these two cases cannot happen simultaneously.*

Proof ($\Rightarrow$) Let $H^{'} = TrH$ and consider a bipartition (A, B) of X. If an $E \in H$ is contained in A then we have (1). If not, then $X - A = B$ is a transversal of H (since there is no edge contained in A, $X - A$ meets all the edges) and so B contains a minimal transversal $F \in TrH$. F is an edge of $H^{'}$ and F is contained in B. Hence, we again have (1). (2) is rather obvious, if A contains an edge $E \in H$, then B cannot contain an edge of TrH since it won't cut E and if B contains an edge $F \in TrH$, then A cannot contain an edge of H since F is a transversal of H.

($\Leftarrow$) Let $H^{'}$ be a simple hypergraph such that every pair (A, B) satisfies (1) and (2) with H and $H^{'}$. And let $H^{''} = TrH$ for which we have proved that every pair (A, B) satisfies (1) and (2) with H and $H^{''}$. We show that $H^{'} = H^{''}$ and we are done.

If $H^{'} \neq H^{''}$, then there is an edge $F^{'} \in H^{'} - H^{''}$. Consider the pair $(X - F^{'}, F^{'})$. As this pair satisfies (2) with H and $H^{'}$, there is no edge $E \in H$ contained in $X - F^{'}$. Also this pair satisfies (1) with H and $H^{''}$, so there exists an edge $F^{''} \in H^{''}$ contained in $F^{'}$.

Now consider the pair $(X - F^{''}, F^{''})$. As this pair satisfies (2) with H and $H^{''}$, there is no edge $E \in H$ contained in $X - F^{''}$. Also this pair satisfies (1) with H and $H^{'}$, so there exists an edge $F_1^{'} \in H^{'}$ contained in $F^{''}$. Thus we have, $F_1^{'} \subset F^{''} \subset F^{'}$. As $H^{'}$ is a simple hypergraph $F_1^{'} = F^{'}$ and hence $F^{'} \in H^{''}$; a contradiction. By symmetry there cannot exist $F^{''} \in H^{''} - H^{'}$ either.

Therefore $H^{'} = H^{''}$. Since we took $H^{''}$ to be TrH, we get $H^{'} = TrH$, which completes the proof. ∎

Corollary 2.1.2 *Let H and $H^{'}$ be two simple hypergraphs. Then $H^{'} = TrH$ if and only if $H = TrH^{'}$.*

Indeed $H^{'} = TrH$ if and only if every pair (A, B) satisfies (1) and (2) with H, $H^{'}$; that is every pair (B, A) satisfies (1) and (2) with $H^{'}$, H; that is $H = TrH^{'}$.

Corollary 2.1.3 *Let H be a simple hypergraph. Then $Tr(TrH) = H$.*

(From Corollary 2.2.2)

2.2 Application: Problem of the keys of the safe

An administrative council is composed of a set X of individuals. Each of them carries a certain weight in decisions, and it is required that every set $E \subset X$ carrying a total weight greater than some threshold fixed in advance, should have access to documents kept in a safe with multiple locks. The minimal "coalitions" which can open the safe constitute a simple hypergraph H. The problem consists in determining the number of locks necessary so that by giving one or more keys to every individual, the safe can be opened if and only if at least one of the coalitions of H is present.

If $TrH = (F_1, F_2, ..., F_m)$, and if the key to the i-th lock is given to all the members of F_i, it is clear that every coalition $E \in H$ would be able to open the safe; on the other hand, if $A \subset X$ does not contain any edge of H, the individuals making up the set A will not be able to open the safe, since A is not a transversal of TrH. The minimum number of locks that is necessary is therefore $m(TrH)$. In particular if all the n members of the administrative council have the same weight, and if the presence of r individuals is necessary in order to open the safe, the number of locks necessary is

$$m(K_n^{n-r+1}) = \binom{n}{n-r+1}$$

.

2.3 Transversal Hypergraph of an Intersecting Hypergraph

For two simple hypergraphs H and $H^{'}$ on X:

1. We write $H \subset H^{'}$ if every edge of H is also an edge of $H^{'}$. So $H = H^{'}$ if $H \subset H^{'}$ and $H^{'} \subset H$.

2. We write $H < H^{'}$ if every edge of H contains an edge of $H^{'}$. Therefore,
$H \subset H^{'} \Rightarrow H < H^{'}$.

3. We denote by $\chi(H)$ the *chromatic number* of H, that is to say the smallest number of colours necessary to "colour" the vertices of H such that no edge of cardinality > 1 is monochromatic.

Lemma 2.3.1 *If H and $H^{'}$ are simple hypergraphs on X, then*

$$\left.\begin{array}{l} H < H^{'} \\ H^{'} < H \end{array}\right\} \Rightarrow H = H^{'}$$

Proof Indeed, since $H < H^{'}$, every edge E_i of H contains an edge F of $H^{'}$; since $H^{'} < H$, the edge F of $H^{'}$ contains an edge E_j of H. Hence

$$E_i \supset F \supset E_j.$$

Since H is a simple hypergraph, $i = j$, and hence every edge of H is an edge of $H^{'}$. By symmetry, $H = H^{'}$. ∎

Lemma 2.3.2 *A simple hypergraph H without loops satisfies $\chi(H) > 2$ if and only if $TrH < H$.*

Proof Indeed, if $\chi(H) > 2$, we have $TrH < H$. Otherwise there exists a $T \in TrH$ containing no edge of H. But then the bipartition $(T, X - T)$ is such that no edge of H is contained in a single class; it is therefore a bicolouring of H, and that contradicts $\chi(H) > 2$.

Conversely, if $TrH < H$, we have $\chi(H) > 2$. Otherwise there exists a bicolouring (A, B) of the vertices of H. From the vertex colouring lemma, B contains a set $T \in TrH$, and since $TrH < H$, we have also $B \supset E$ for an $E \in H$, which contradicts the fact that (A, B) is a bicolouring of H. ∎

Lemma 2.3.3 *A hypergraph H is intersecting if and only if $H < TrH$.*

Proof For if H is intersecting, every $E \in H$ is a transversal of H, and therefore E contains a minimal transversal $T \in TrH$, so $H < TrH$.

Conversely, if $H < TrH$, every $E \in H$ contains a transversal of H, and therefore meets all the edges of H, that is, H is intersecting. ∎

Theorem 2.3.4 *A simple hypergraph H without loops satisfies $H = TrH$ if and only if:*
(i) $\chi(H) > 2$;
(ii) H is intersecting.

Proof Obvious from Lemmas 2.4.1, 2.4.2 and 2.4.3. ∎

2.4 Hypergraphs with the relation $H = TrH$

Given below are a few examples of hypergraphs H for which $H = TrH$.

Example 2: The complete r-uniform hypergraph K^r_{2r-1} satisfies $Tr(K^r_{2r-1}) = K^r_{2r-1}$.

Example 3: The finite projective plane P_7 on 7 points satisfies $Tr(P_7) = P_7$, for it is an intersecting family and non-bicolourable. (Figure 2.1).

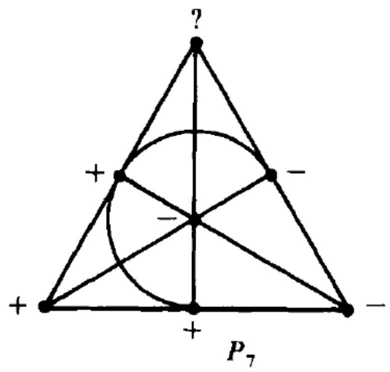

Figure 2.1: Finite projective plane on 7 points

Example 4: The "fan" of rank r is a hypergraph F_r, having r edges of cardinality 2 and one edge of cardinality r. It is an intersecting family and non-bicolourable; therefore $Tr(F_r) = F_r$. (Figure 2.2).

Proposition 2.4.1 *For a simple hypergraph H, the following two conditions are equivalent:*

(1) H has no loops and $\chi(H) > 2$;

(2) TrH is intersecting and is not a star.

Proof For if (1) holds then $TrH < H$ (from Lemma 2.4.2), and the hypergraph $H' = TrH$ is not a star. Thus $H' = TrH < H = TrH'$ and hence H' is intersecting (from Lemma 2.4.3). The converse is proved in the same way. ∎

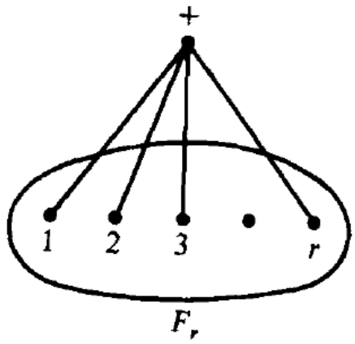

Figure 2.2: Fan of rank r

Proposition 2.4.2 *Every hypergraph H with property (7) satisfies property (8). (Figure 2.3).*

Proof Note that if H satisfies property (7) it has no loops and is simple.

Since $\chi(H-E) = 2$, there exists a bicolouring (A, B) of $H-E$, and E is monochromatic in this bicolouring. Suppose for example that $E \subset A$. If we change the colour of an arbitrary point x of E, a new edge $E' \in H$ will become coloured B, whence $E \cap E' = \{x\}$. From this (8) follows. ∎

Proposition 2.4.3 *Every simple hypergraph H without loops having property(2) satisfies property (8).*

Proof Since every $E \in H$ is a minimal transversal of$H$, the set $E - \{x\}$ is disjoint with some edge $E' \in H$, whence $E \cap E' = \{x\}$. From this (8) follows. ∎

Proposition 2.4.4 (Seymour [7]). *Let H be a hypergraph on X with property (7) and let $A \subset X$; then there is no bipartition (A_1, A_2) of A into two transversal sets of H_A.*

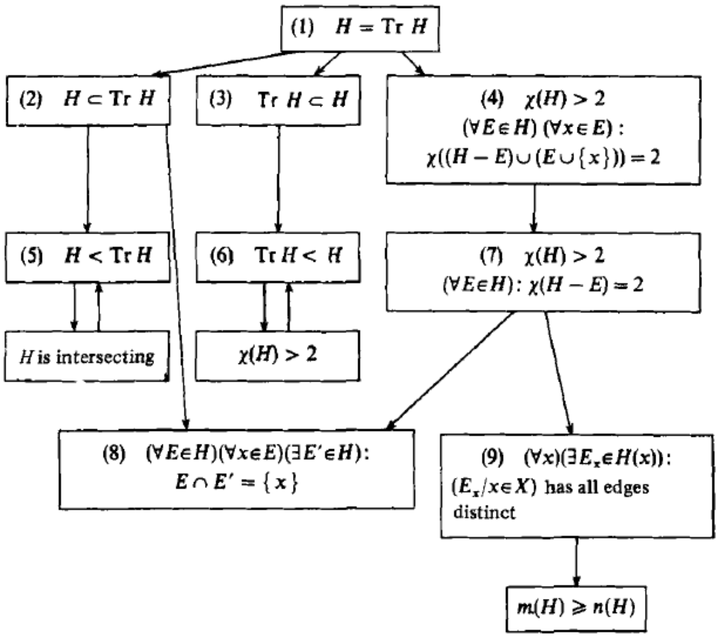

Figure 2.3: Few properties (H simple and without loops)

Proof Note that since H satisfies property (7), it has no loops and is simple. Suppose that such a bipartition (A_1, A_2) exists and consider the partial hypergraph $H' = (E \mid E \in H, E \cap A = \emptyset)$. We have $H' \neq \emptyset$, for if not then (A_1, A_2) would extend to a bicolouring of H. We have $H' \neq H$, since $A \neq \emptyset$. Thus from property (7), the hypergraph H' has a bicouring (B_1, B_2) and $B_1 \cup B_2 \subset X - A$. Since H has no loops, $E \in H'$ implies

$$E \cap B_1 \neq \emptyset, \; E \cap B_2 \neq \emptyset.$$

Furthermore $E \in H - H'$ implies

$$E \cap A_1 \neq \emptyset, \; E \cap A_2 \neq \emptyset.$$

Thus $(A_1 \cup B_1, \; A_2 \cup B_2)$ generates a bicolouring of H, which contradicts (7). ∎

2.5 The coefficients τ and τ'

For a hypergraph H we denote by $\tau(H)$ the *transversal number*, that is to say, the smallest cardinality of a transversal; similarly, we denote by $\tau'(H)$ the largest cardinality of a minimal transversal. Clearly

$$\tau(H) = \min_{T \in TrH} |T| \leq \max_{T \in TrH} |T| = \tau'(H).$$

Example 5: The (n, k, λ)-configuration. This is by definition a k-uniform hypergraph H of order n such that every pair of vertices is contained in exactly λ edges. From this definition we easily deduce that:

1. H is regular and of degree $\Delta(H) = \lambda \frac{n-1}{k-1}$,

2. H has $m(H) = \lambda \frac{n(n-1)}{k(k-1)}$ edges.

For certain known (n, k, λ) configurations, the transversal number τ is given by the following table.

(n, k, λ)	$(13, 3, 1)$	$(10, 4, 2)$	$(9, 4, 3)$	$(11, 3, 3)$	$(12, 4, 3)$
τ	7	4	4	7	6

Theorem 2.5.1 *Let $H = (E_1, E_2, ..., E_m)$ be a hypergraph on X with $\tau'(H) = t$, and let k be an integer ≥ 1. If $k < |E_1| \leq |E_2| \leq ...... \leq |E_m|$, and if every k-tuple of X is contained in at most λ edges of H, then*

$$\sum_{j=1}^{t} \binom{|E_j| - 1}{k} \leq \lambda \binom{n - t}{k}$$

Proof Let T be a minimal transversal of H. For every $x \in T$, there exists an edge E_x such that $E_x \cap T = \{x\}$. Since $E_x \neq E_y$ for $x \neq y$, the family $H' = (E_x \quad | \quad x \in T)$ is a partial hypergraph of H.

By counting in two different ways the pairs (A, E) where $E \in H'$ and where A is a k-tuple of $X - T$ contained in E, we obtain

$$\sum_{x \in T} \binom{|E_x - \{x\}|}{k} = \sum_{A \subset X-T, |A|=k} |\{E_x \quad | \quad E_x \supset A\}| \tag{2.1}$$

from whence, a fortiori,

$$\sum_{j=1}^{t} \binom{|E_j| - 1}{k} \leq \lambda \binom{n-t}{k}. \quad \blacksquare$$

Corollary 2.5.2 *Let H be a hypergraph of order n with no loops, and put $s = \min |E_i|$ and $\Delta = \Delta(H)$. Then $\tau'(H) \leq \left[\frac{n\Delta}{\Delta+s-1}\right]$.*

Proof Indeed, Theorem 2.6.1 with $k = 1$ gives

$$t\binom{s-1}{1} \leq \Delta \binom{n-t}{1}.$$

Whence $\tau'(H) = t \leq \frac{n\Delta}{\Delta+s-1}$. $\blacksquare$

Corollary 2.5.3 *Let H be a linear hypergraph of order n with $\min |E_i| = s > 2$. Then*

$$\tau'(H) \leq n + \frac{1}{2}(s^2 - 3s + 1) - \frac{1}{2}\sqrt{4n(s^2 - 3s + 2) + (s^2 - 3s + 1)^2}.$$

Proof Theorem 2.6.1 with $k = 2$ and $\lambda = 1$ gives

$$t\binom{s-1}{2} \leq \binom{n-t}{2}$$

that is to say

$$t^2 - t(s^2 - 3s + 2n + 1) + (n^2 - n) \geq 0.$$

Equality gives a quadratic equation which has two solutions t' and t'', and we note that $t' < n < t''$. Since $\tau'(H) \leq n$, we have also $\tau'(H) \leq t'$. The result follows. $\blacksquare$

Corollary 2.5.4 (Erdös, Hajnal [4]). *Let H be a linear 3-uniform hypergraph of order n; then*

$$\tau(H) \leq n - \sqrt{2n + \frac{1}{4}} + \frac{1}{2}.$$

Proof This follows from Corollary 2.6.3 with $s = 2$. ∎

Theorem 2.5.5 (Meyer [6]). *Let H be a hypergraph with $\min |E_i| = s > 1$, and suppose that the vertices of X are labelled in such a way that*

$$d_H(x_1) \leq d_H(x_2) \leq ...... \leq d_H(x_n).$$

Then the number $\tau'(H) = t$ satisfies

$$\sum_{i=1}^{t}[d_H(x_i) + s - 1] \leq \sum_{i=1}^{n} d_H(x_i).$$

Proof Using Equation 2.1 of the proof of the Theorem 2.6.1 with $k = 1$, we obtain

$$\sum_{x \in T}(|E_x - \{x\}|) \leq \sum_{x \in X - T} d_H(x).$$

This implies: $t(s-1) \leq \sum_{i=t+1}^{n} d_H(x_i)$. The stated inequality follows. ∎

2.6 τ-critical hypergraphs

We say that a hypergraph $H = (E_1, E_2, ..., E_m)$ is *τ-critical* if the deletion of any edge decreases the transversal number, that is to say, if

$$\tau(H - E_j) < \tau(H) \quad (j = 1, 2, ..., m)$$

Since we cannot have $\tau(H - E_j) < \tau(H) - 1$, this is equivalent to saying that if H is τ-critical with $\tau(H) = t + 1$, then $\tau(H - E) = t$ for every $E \in H$.

Example 6: The hypergraph K^r_{t+r} is τ-critical, since $\tau(K^r_{t+r}) = t+1$ and if E is an edge of K^r_{t+r}, the hypergraph $K^r_{t+r} - E$ has a transversal $X - E$ of cardinality t.

The concept of a τ-critical graph is due to Zykov in 1949. The systematic study started in 1961 with an article by Erdös and Gallai, who showed that a τ-critical graph G without isolated vertices satisfies $2\tau(G) - n(G) \geq 0$. Examples of τ-critical graphs are shown in Figure 2.4 and 2.5.

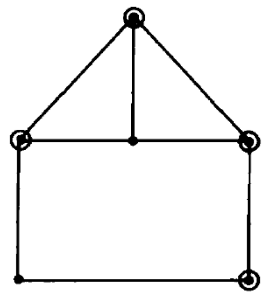

Figure 2.4: $\tau = 4, 2\tau - n = 2$

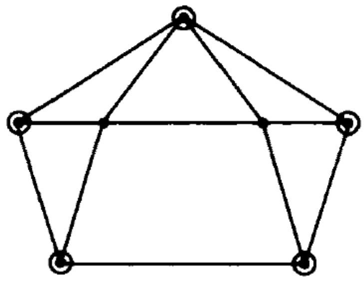

Figure 2.5: $\tau = 5, 2\tau - n = 3$

Proposition 2.6.1 *Every τ-critical hypergraph is simple.*

Proof For if $H = (E_1, ..., E_m)$ is τ-critical and not simple, there exist two indices i and j with $E_i \subset E_j$. An optimal transversal of $H - E_j$ has $\tau(H) - 1$ vertices, and since it meets E_i it also meets E_j. Therefore $\tau(H) \leq \tau(H) - 1$, a contradiction.

Proposition 2.6.2 *Every hypergraph H with $\tau(H) = t + 1$ has a partial hypergraph, a τ-critical hypergraph H' with $\tau(H') = t + 1$.*

Proof Indeed, to obtain H' it is enough to remove from H as many edges as one can without changing the transversal number.

In a hypergraph H a vertex x is said to be critical if
(1) $\tau(H - H(x)) < \tau(H)$.
We note that (1) is equivalent to:
(2) $\tau(H - H(x)) = \tau(H) - l$.
Indeed, if (1) holds then the hypergraph $H_1 = H - H(x)$ has a transversal T_1 of cardinality $\tau(H) - 1$. The set $T_1 \cup \{x\}$ is a transversal of H and, since its cardinality is $\tau(H)$, it is a minimum transversal. From this we obtain (2).

Conversely, if (2) holds, let T be a minimum transversal of H containing x. Then $T - \{x\}$ is a transversal of $H - H(x)$ of cardinality $\tau(H) - l$, from which (1) follows.

Proposition 2.6.3 *Every vertex of a τ-critical hypergraph is critical.*

Proof Let H be a τ-critical hypergraph and let x be one of its vertices. Since x is contained in an edge, E say,

$$\tau(H - H(x)) \leq \tau(H - E) < \tau(H).$$

Thus x is a critical vertex.

2.7 The König property

A *matching* in a hypergraph H is a family of pairwise disjoint edges, and the maximum cardinality of a matching is denoted $\nu(H)$.

A matching can also be defined as a partial hypergraph H_0 with $\Delta(H_0) = 1$.

We note that for every transversal T and for every matching H_0,

$$|T \cap E| \geq 1 \qquad (E \in H_0)$$

Thus $|H_0| \leq |T|$, from whence

$$\nu(H) = \max |H_0| \leq \tau(H).$$

We say that H has the *König property* if $\nu(H) = \tau(H)$.

A *covering* of H will be a family of edges which covers all the vertices of H, that is to say a partial hypergraph H_1 with $\delta(H_1) = \min_{x \in X} d_{H_1}(x) \geq 1$. We write

$$\rho(H) = \min |H_1|.$$

Finally, a *strongly stable set* of H is by definition a set $S \subset X$ such that $|S \cap E_1| \leq 1$ for every $E \in H$, and we write

$$\overline{\alpha}(H) = \max |S|.$$

It is seen immediately that $\rho(H) = \tau(H^*), \overline{\alpha}(H) = \nu(H^*)$; for this reason we say that H has the *dual König property* if $\rho(H) = \overline{\alpha}(H)$. (see [1, 9] for more on König property)

Example 7: The r-partite complete hypergraph. If $n_1 \leq n_2 \leq ...... \leq n_r$, the hypergraph $K^r_{n_1,n_2,......,n_r}$ has the König property since $\tau = n_1$ and $\nu = n_1$. t also has the dual König property since $\rho = n_r$ and $\overline{\alpha} = n_r$.

Example 8: Semi-convex polyominoes. A *polyomino* P is a finite set of unit squares in the plane arranged like a chessboard with some of its squares cut out. With every polyomino P one can associate a hypergraph whose vertices are the unit squares of P and whose edges are the maximal rectangles contained in P.

If P is semi-convex, that is to say if every horizontal line of the plane intersects P in an interval, the hypergraph P has the König property (Berge, Chen, Chvatal, Seow [2]) and the dual König property. The smallest polyomino P with $\nu(P) \neq \tau(P)$ is shown in Figure 2.6.

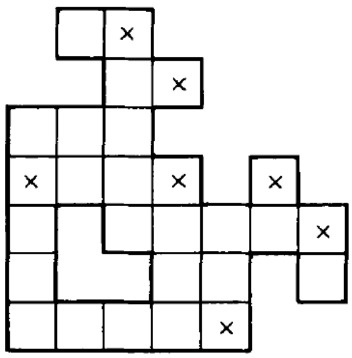

Figure 2.6: Polyomino with $\nu = 6$ and $\tau = 7$.

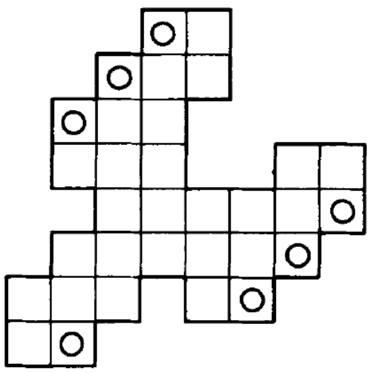

Figure 2.7: Polyomino with $\rho = 8$ and $\overline{\alpha} = 7$.

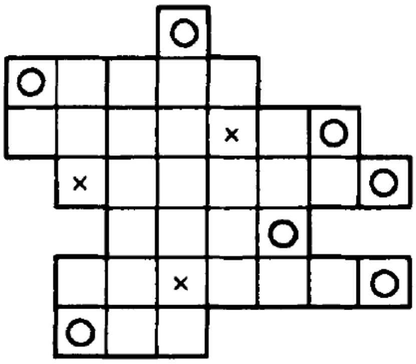

Figure 2.8: Semi-convex polyomino with $\nu = \tau = 3$ and $\rho = \overline{\alpha} = 7$.

In this chapter, we have presented several properties of transversal hypergraphs and matchings.

Chapter 3

Algorithms to Generate Transversal Hypergraph

The Transversal Hypergraph Generation is the problem of generating the transversal hypergraph TrH of a given hypergraph H. Its decisional variant, Transversal Hypergraph, is the problem of deciding whether, given two hypergraphs H and G defined on the same set of vertices, $G = TrH$ holds. Transversal Hypergraph Generation is one of the most important problems on hypergraphs with many practical applications in various areas of Computer Science. The main reason for the large applicability of the Transversal Hypergraph Generation problem is that finding minimal or maximal (with respect to some property) structures or solutions is a common and essential task in many areas. The notion of the transversal is a nice way of modelling these extremal structures. Even more, there are many natural problems that are just disguised form of the Transversal Hypergraph Generation.

3.1 Complexity of Algorithms

It is easy to see that a hypergraph H may have exponentially many (with respect to its size) minimal transversals. Thus, an algorithm that solves a generation problem with large output, like the Transversal Hypergraph Generation, may require exponentially many steps to produce the whole output. There is a surge of interest in defining suitable complexity measures for the efficiency of a generation algorithm. *Total-polynomiality or output-polynomiality* is a measure that takes into account not only the size of the input but the size of the output, too. Stronger requirements for the efficiency of a generation algorithm take into account the size of the input and the size of the output so far (incrementally output-polynomial algorithm) or the delay time between consecutive outputs (polynomial delay algorithm). The exact complexity of the Transversal Hypergraph Generation problem is still open. Its complexity strongly depends on the complexity of its decision version Transversal Hypergraph since there would exist an output-polynomial time algorithm for solving the Transversal Hypergraph Generation problem if and only if the Transversal Hypergraph problem was polynomial time solvable. The Transversal Hypergraph problem is in its generality in co-NP, while several polynomial time cases also exist. Although there are several algorithms that involve, in some manner, the computation of minimal transversals, no output-polynomial time algorithm is known.

3.2 Algorithm of Berge

Definition 1: Let $H = (E_1, E_2, ..., E_m)$ and $H^{'} = (E^{'}_1, E^{'}_2, ..., E^{'}_{m'})$ be two hypergraphs. Then,

$$H \cup H^{'} = (E_1, E_2, ..., E_m, E^{'}_1, E^{'}_2, ..., E^{'}_{m'}), \text{ and}$$

$$H \vee H^{'} = (E_i \cup E_j^{'}, i = 1, ..., m, j = 1, ..., m^{'}).$$

The first operation is the *union* of H and $H^{'}$, i.e, the hypergraph whose hyperedges are the hyperedges of both hypergraphs. The second one is in some sense the *Cartesian product* of them, i.e., the union of all possible pairs of hyperedges, one from the first hypergraph and one from the second one.

Clearly, for two hypergraphs, H and $H^{'}$,

$$Tr(H \cup H^{'}) = Min(TrH \vee TrH^{'}) \tag{3.1}$$

Based on equation (3.1), there is a simple scheme attributed to Berge for generating all minimal transversals of a hypergraph $H = (E_1, ..., E_m)$ on X (*ref.* [1]). Let $H_i = (E_1, ..., E_i)$, $i = 1, ..., m$ be the partial hypergraph of H on X. It holds that $H_i = H_{i-1} \cup (E_i)$, for all $i = 2, ..., m$, while $H_1 = (E_1)$ and $H_m = H$. Thus, $TrH_i = Tr(H_{i-1} \cup (E_i))$ and, according to equation (3.1),

$$\begin{aligned} TrH_i &= Min(TrH_{i-1} \vee Tr(E_i)) \\ &= Min(TrH_{i-1} \vee \{\{v\}, v \in E_i\}) \end{aligned} \tag{3.2}$$

The algorithm of Berge is based on equation (3.2) and computes all minimal transversals of the input hypergraph H recursively, in two steps: First, it computes the minimal transversals of the partial hypergraph H_{i-1} and then it calculates the Cartesian product of the set TrH_{i-1} by the i-th hyperedge E_i of H and removes all elements that are not minimal. Thus, one can compute TrH by starting from the minimal transversals of E_1 (note that the minimal transversals of a hypergraph with a single hyperedge are exactly its vertices) and adding one-by-one the rest of the hyperedges, computing at each step the set of minimal transversals of the new partial hypergraph. The procedure terminates after the addition of the last hyperedge E_m. Algorithm then outputs the transversal hypergraph TrH of the input hypergraph H.

Algorithm 1: The algorithm of Berge

for $i = 2, ..., m$ **do**

Find $Tr(H_{i-1})$

Compute $Tr(H_i) = Min(Tr(H_{i-1}) \vee \{\{v\}, v \in E_i\})$

end for

Return $Tr(H_m)$

The algorithm of Berge is the most simple and direct scheme for computing the minimal transversals of a hypergraph. However, there are several drawbacks that make it inefficient and unsuitable for large problem instances. First of all, notice that all, possibly exponentially many, intermediate transversals of the partial hypergraphs H_i $(i = 1, ..., m-1)$ must be computed (the Cartesian product of the set TrH_{i-1} by the hyperedge E_i) and only the minimal of them must be kept. This means than the total running time of the algorithm may be exponential in both the size of the input and the output. No less important are the memory requirements that also emerge from the above. All these intermediate minimal transversals have to be stored and kept until used for the computation of the new transversal set. Since the number of these intermediate minimal transversals can be exponential, the memory requirements of the algorithm can become devastating. And last but not least, since the computation of the first transversal of the input hypergraph H is accomplished after all minimal transversals of the partial hypergraph H_{m-1} have been computed, the first final minimal transversal is output after exponential delay time.

This is the most severe drawback of the algorithm of Berge in view of the complexity measures for our problem.

3.3 Generalized Vertices

To improve the total running time of the algorithm and reduce its storage requirements, the large number of intermediate partial transversals produced have to be reduced. To do this, the notion of the *generalized vertices* (see [5]) is defined.

Definition 2: Let H be a hypergraph on X. The set $V \subseteq X$ is a *generalized vertex* of H if all the vertices in V belong in exactly the same hyperedges of H.

So the cardinality of a generalized vertex may vary from 1 to $|X|$. If $V_1, V_2, ..., V_k$ are all the generalized vertices of H, then $X = V_1 \cup V_2 \cup ...... \cup V_k$, while $V_i \cap V_j = \emptyset$, for all $i \neq j$, $i, j = 1, ..., k$.

Example 1: Let $H = (\{1,2,3,4,5,6\}, \{3,4,5,7,8,9,10,11,12,13\}, \{5,6,11,12,13,14,15\})$ on $X = \{1, 2, ..., 15\}$. So the generalized vertices of H are

$V_1 = \{1, 2\}$

$V_2 = \{7, 8, 9, 10\}$

$V_3 = \{14, 15\}$

$V_4 = \{3, 4\}$

$V_5 = \{6\}$

$V_6 = \{11, 12, 13\}$

$V_7 = \{5\}$

Note that, all V_i's are maximal, that is if any other element is added to any of V_i, it will no longer be a generalized vertex. Whereas, any subset of V_i is also a generalized vertex.

Definition 3: Let $H = (E_1, E_2, ..., E_m)$ be a hypergraph on X and $V \subseteq X$ be a generalized vertex of H. Then the *generalized hypergraph of H with respect to V* is the hypergraph $H_V^g = (E_1^g, E_2^g, ..., E_m^g)$ on $X_V^g = ((X \setminus V) \cup \{x_V\})$, where x_V is an auxiliary vertex not in X and $E_i^g(1 \leq i \leq m)$ follows from E_i by substituting (if it appears) the set V by the vertex x_V.

The above definition can be generalised for more than one generalized vertices.

Definition 4: If $V_1, V_2, ..., V_k,\ V_i \subseteq X, i = 1, 2, ..., k$, are all the generalized vertices of hypergraph $H = (E_1, E_2, ..., E_m)$ on X, then the *generalized hypergraph of H* is the hypergraph $H^g = (E_1^g, E_2^g, ..., E_m^g)$ on $X^g = (x_{V_1}, x_{V_2}, ..., x_{V_k})$, where $x_{V_1}, x_{V_2}, ..., x_{V_k}$ are the auxiliary vertices not in X and $E_i^g(1 \leq i \leq m)$ follows from E_i by substituting (if they appear) the sets $V_1, V_2, ..., V_k$ by the vertices $x_{V_1}, x_{V_2}, ..., x_{V_k}$, respectively.

Assume that the hypergraph H has a generalized vertex V with cardinality $|V| \geq 2$. Let H_V^g be the generalized hypergraph of H with respect to V and let TrH_V^g be the transversal hypergraph of H_V^g. The importance of the concept of the generalized node follows from the observation that

$$TrH = \left(T^g \in TrH_V^g \mid x_V \notin T^g\right) \cup \left(\{T^g \setminus x_V\} \ \vee \ V,\ T^g \in TrH_V^g \mid x_V \in T^g\right) \qquad (3.3)$$

In other words, the minimal transversals of H follow by taking one by one the minimal transversals of H_V^g that include the vertex x_V and replacing x_V by each (simple) vertex in V, in turn. Obviously, the number of minimal transversals of H produced from a single minimal transversal T^g of H_V^g is exactly $|V|$. The minimal transversals of H_V^g that do not include x_V remain as they are, since they hit H. This procedure can be generalized to any number of generalized vertices.

Lemma 3.3.1 *Let H be a hypergraph on X and $V_1, V_2, ..., V_k$, $V_i \subseteq X$, $i = 1, ..., k$, be its generalized vertices. Let also $T^g = (V_{i_1}, V_{i_2}, ..., V_{i_l})$, $1 \leq i_1, ..., i_l \leq k$, be a minimal transversal of the generalized hypergraph H^g of H. Then,*
(1) every l-tuple of the Cartesian product $(V_{i_1} \vee V_{i_2} \vee ...... \vee V_{i_l})$ is a minimal transversal of H and
(2) no other minimal transversal of H exists.

Proof Let $T = (v_{i_1}, ..., v_{i_l})$ be an l-tuple of the Cartesian product $V_{i_1} \vee V_{i_2} \vee ...... \vee V_{i_l}$ such that $v_{i_j} \in V_{i_j}$, $j = 1, ..., l$. Every simple vertex v_{i_j} is actually a unique representative of V_{i_j} in T. Since T^g is a transversal of H^g and all vertices of every generalized vertex of H belong to exactly the same hyperedges of H, then T is a transversal of H. Moreover, the removal of a simple vertex of T would result in a set that does not hit at least one hyperedge of H since every generalized vertex is represented in T by exactly one simple vertex. Hence, T is a minimal transversal of H.
To prove the second statement, see that if T is a minimal transversal of H, then T has at least one common vertex with every hyperedge of H. Every vertex of T corresponds to exactly one generalized vertex. If T^g is the collection of all these generalized vertices, then T^g is a transversal of H^g since it intersects every hyperedge E_i^g of it. Moreover, T^g is minimal (a proper subset T^g of T^g that intersects every hyperedge of H^g would result, by taking the Cartesian product of its vertices, in a set T that is contained in T and intersects every hyperedge of H, a contradiction). ∎

Example 2: Assume that a hypergraph H has two hyperedges with 50 vertices each: $E_1 = \{1, ..., 50\}$ and $E_2 = \{26, ..., 75\}$. The partial hypergraph $H_2 = (E_1, E_2)$ has 650 minimal transversals (625 with two vertices and 25 with one vertex) which must be kept for the subsequent stage if we use the straightforward scheme. For H_2, three generalized vertices are defined: $V_1 = \{1, ..., 25\}$, $V_2 = \{26, ..., 50\}$, and $V_3 = \{51, ..., 75\}$. Using the generalized vertex approach, we have only 2 minimal transversals to store, namely (V_2) and (V_1, V_3). All minimal transversals of H_2 may occur from these, as Lemma 3.3.1 suggests.

According to Lemma 3.3.1, every minimal transversal T of H is an offspring of some minimal transversal T^g of H^g. Thus, the generation of TrH is now reduced to the generation of $Tr(H^g)$.

3.4 Modified Algorithm of Berge

This section will describe a modification of algorithm of Berge that exploits the concept of the generalized vertex explained above.

Let $V_1, V_2, ..., V_{k_i}$ be the generalized vertices of the partial hypergraph $H_i = (E_1, ..., E_i)$ of H, $k_i \geq 1$. Assume that we have already defined the generalized vertices of H_i and computed $Tr(H_i^g)$. We add now the next hyperedge E_{i+1} to define the partial hypergraph $H_{i+1} = H_i \cup E_{i+1}$. The addition of E_{i+1} imposes the new determination of all previously determined generalized vertices. There are three possible types for every generalized vertex V of H_i:

(α) $V \cap E_{i+1} = \emptyset$. In this case, V is also a generalized vertex of H_{i+1}.

(β) $V \subset E_{i+1}$. In this case, V is also a generalized vertex of H_{i+1}.

(γ) $V \cap E_{i+1} \neq \emptyset$ and $V \notin E_{i+1}$. In this case, V is divided into $V_1 = V \setminus (V \cap E_{i+1})$ and $V_2 = V \cap E_{i+1}$. Both V_1 and V_2 are generalized vertices of H_{i+1}.

Notice that the determination of the new set of generalized vertices depends only on the addition of E_{i+1}. E_{i+1} may also reveal some vertices of H that were unknown until the i-th level. All these vertices will form a new generalized vertex for H_{i+1} (this falls into case (α)).

We next represent $Tr(H_i^g)$ and E_{i+1} according to the new generalized vertices. If (α) or (β) is the case for all generalized vertices of H_{i+1}, then all minimal transversals and E_{i+1} remain as they were. If (γ) is the case, assume that a generalized vertex V is divided into V_1 and V_2. Obviously, E_{i+1} contains only V_2 while every minimal transversal T^g of H_i^g contains both V_1 and V_2. Since one of these vertices suffices for T^g to be a minimal hitting set of H_i^g, two minimal transversals emerge from T^g: one containing V_1 and another one containing V_2 (the generalized vertices of type (α)) and (β) of T also appears in these minimal transversals). If T^g contains κ generalized vertices of type (γ), then T^g corresponds now to 2κ pairwise different minimal transversals of H_i^g, that is, all possible combinations of the two parts in which type (γ) vertices of T^g are divided, along with the generalized vertices of type (α) and (β) of T. Notice that all these *offsprings* of T^g are not necessarily hitting sets of H_{i+1}^g.

$$\begin{aligned} Tr(H_{i+1}^g) &= Tr(H_i^g \cup \{E_{i+1}^g\}) \\ &= Min(Tr(H_i^g) \vee Tr(\{E_{i+1}^g\}) \\ &= Min(Tr(H_i^g) \vee \{\{x_V\} : x_V \in E_{i+1}^g\}) \end{aligned} \tag{3.4}$$

Algorithm 2: The modified algorithm of Berge based on generalized vertices

for $k = 0, ..., m-1$ **do**

Add E_{k+1}

Update the set of generalized vertices

Express TrH_k^g and E_{k+1} as sets of generalized vertices of level $k+1$

Compute $Tr(H_{k+1}^g) = Min(Tr(H_k^g) \vee \{\{x_V\} \ : \ x_V \in E_{k+1}^g\})$

end for

Output $Tr(H_m)$

This algorithm is a modification of the simple scheme of Berge that computes the minimal transversals of the partial generalized hypergraphs according to Equation (3.4). During all intermediate steps, only the generalized transversals are kept which, in turn, are split after the addition of the next hyperedge. Experimental evaluation has shown that this dramatically reduces the number of intermediate transversals (see Example), especially at the early stages (where the generalized nodes are few but large) and greatly improves the time performance and the memory requirements. After the addition of the last hyperedge, this algorithm outputs all minimal transversals of the input hypergraph.

3.5 Depth-First Transversal Computation

Although modified algorithm is more efficient than Berge algorithm, one still may have to wait for a long time for the first final minimal transversal to be output. This happens because it is based on a sort of breadth-first computation: all minimal transversals are computed after a new hyperedge is added and, after the addition of the last one, all final minimal transversals follow almost with zero delay one from the other.

Having in mind the rate of output and the memory requirements, we further improve the modified algorithm by implementing a depth-first computation of the minimal transversals: Suppose that at a certain level k we have computed a minimal transversal T of H_k^g. We add the next hyperedge and determine the generalized vertices, as described above. From T several minimal transversals follow. However, instead of computing them all, we compute one, add the next hyperedge and continue until all hyperedges have been added; in this case we output the final minimal transversal. We then backtrack to the previous level, pick the next minimal transversal, etc.

Algorithm 3: Depth-First Transversal Computation

Add E_1

Update the set of generalized vertices

Express E_1 as set of generalized vertices

Compute $T = TrE_1$

Call `add_next_hyperedge`(T, E_2)

Procedure 4: A procedure for adding the next hyperedge

procedure `add_next_hyperedge`(T, E) {

Update the set of generalized vertices

Express TrH^k and E as sets of generalized nodes of level $k+1$

while `generate_next_transversal`(T, T', l) **do**

  { T' is the l-th offspring of T }

  if E is the last hyperedge **then**

    output T'

  else

    { Let E' be the next hyperedge }

    Call `add_next_hyperedge`(T', E')

    $l = l + 1$

end if

end while

}

Function 5: A function for computing the next minimal transversal

boolean function `generate_next_transversal`(T, T', l) {

if $l \leq |Min(T \vee E)|$ **then**

`generate_next_transversal` = **true**

T' is the l-th element of the set $Min(T \vee E)$

else

`generate_next_transversal` = **false**

end if

}

The whole procedure is described by Algorithm 3. At some level k, procedure `add_next_hyperedge`(T, E) (see Procedure 4) is called for adding the next hyperedge E to the current intermediate minimal transversal T , which, in turn, repeatedly calls the boolean function `generate_next_transversal`$(T, T^{'}, l)$ (see Function 5) that returns the l-th partial minimal transversal $T^{'}$ of the new hypergraph that follows from T. `generate_next_transversal`$(T, T^{'}, l)$ is called until no more minimal transversals follow from T after the addition of E, in which case `generate_next_transversal`() becomes false. After a new minimal transversal $T^{'}$ is returned, `add_next_ hyperedge`() is called recursively for $T^{'}$ and the next hyperedge.

The operation of Algorithm 3 resembles a preorder visit of a tree of transversals with root the single (generalized) minimal transversal of the first hyperedge, and internal vertices at some level, the minimal transversals of the partial generalized hypergraph at that level. The descendants of a minimal transversal are the minimal transversals of the next hypergraph which include this transversal. Finally, the leaves of the tree at level m are the minimal transversals of the original hypergraph.

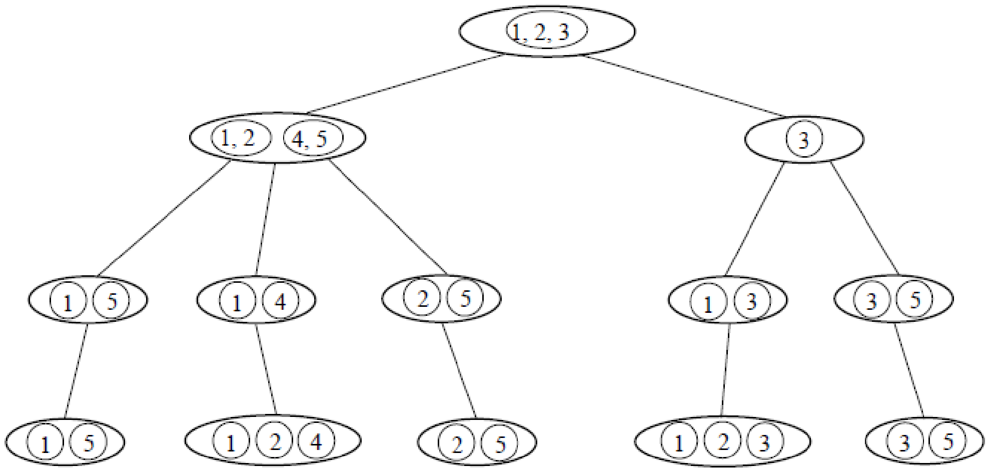

Figure 3.1: Transversal tree of the hypergraph $H = (\{1,2,3\},\{3,4,5\},\{1,5\},\{2,5\})$. The tree is visited in preorder

Example 3: Consider the hypergraph $H = (\{1,2,3\},\{3,4,5\},\{1,5\},\{2,5\})$ of order 5. The tree of transversals which corresponds to the addition of the hyperedges according to the giver order (top to bottom) is shown in Figure 3.1. Generalized vertices are denoted by circles with thin lines. For instance, a partial minimal transversal of the hypergraph consisting of the first two hyperedges is $(\{1,2\},\{4,5\})$.

Remark Notice that there is no need to calculate $Min(T \vee E)$ every time the function `generate_next_transversal`() is called. Instead, in our implementation a more efficient approach was adopted which selects the split parts of the generalized vertices according to the binary expansion of l.

Conclusion

Given a hypergraph as input, the transversal hypergraph problem asks to generate its transversal. In this chapter, we have presented some techniques from literature to solve the transversal hypergraph problem. No polynomial time algorithm for determining TrH is known (it belongs to the class of NP-complete problems). Nonetheless, for hypergraphs with a few vertices we have Berge algorithm and its variants that are sufficiently effective.

Bibliography

[1] Berge, C., *Hypergraphs: Combinatorics of Finite Sets*, North-Holland Mathematical Library. Elsevier Science Publishers B.V., Amsterdam, 1989.

[2] Berge, C., Chen, C. C., Chvátal, V., and Seow, C.S., Combinatorial properties of polyominoes, *Combinatorica*, 1(3), 217-224, 1981.

[3] Bollobás, B., Sperner systems consisting of pairs of complementary subsets, *J. Comb. Theory A*, 15, 363-366, 1973.

[4] Erdös, P., and Hajnal, A., On chromatic number of graphs and set-systems, *Acta Mathematica Hungarica*, 17, 61-99, 1966.

[5] Kavvadias, D. J., and Stavropoulos, E. C., An Efficient Algorithm for the Transversal Hypergraph Generation. *Journal of Graph Algorithms and Applications*, 9(2), 239-264, 2005.

[6] Meyer, J.C., Ensembles stables maximaux dans les hypergraphes, *C.R. Aead. Sci.* Paris, 274, 144-147, 1975.

[7] Seymour, P.D., On the two colouring of hypergraphs, *Quart. J. Math. Oxford (3)*, 25, 303-312, 1974.

[8] Sperner, E., Ein Satz über Untermengen einer endlichen Menge (A theorem on subsets of a finite set). *Math. Z.*, 27, 544-548, 1928.

[9] Voloshin, V. I., *Introduction to Graph and Hypergraph Theory*, Nova Science Publishers Inc., New York, 2009.