MATHZINE

2019

The Annual News letter of the Maths Club, IISER Pune

Acknowledgements

We, the Maths Club coordinators would like to thank all the students and faculty who took time to contribute to the magazine in whatever way. We wish to see it grow for the years to come. We would also like to thank our entire Math Department for being an active part of and for their constants efforts to bring this club together.

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We wish the future Maths Department and the Maths Club the best of luck for all their endeavours.

Number of Odd Binomial Coefficients

Aditya Khanna

Claim: For a given $N \in \mathbb{N}$, $\binom{N}{k}$ $(0 \leq k \leq N)$ is odd for exactly $2^{d(N)}$ values of k, where $d(N)$ is the number of 1's in the binary expansion of N.

Proof. $\binom{N}{k}$ can be interpreted as choosing k dots out of N dots and coloring them. Let D_N denote the set containing N indexed dots.

$$\underbrace{\circ\circ\circ\cdots\circ}_{\text{N dots}}$$

such that the dots are indexed a_1,a_2 ... a_N

Defintion(*Coloring*) A *coloring* defined to be a choice of $\{a_i\}$ for some values of i

Figure 1: A *coloring* $C = \{a_2, a_4, a_5, a_7\}$, of dots for $N = 7$

Defintion(*Partial Coloring*):k is the number of elements in a *coloring* C, then C is called a *partial coloring* if $0 < k < N$.

The example given is a partial coloring

Definition(*Cycle/Step*) For a *coloring*, a *step* results in another (not necessary distinct) coloring such that a_i being colored for original coloring implies a_{i+1} is colored in new coloring, where $a_{N+1} = a_1$

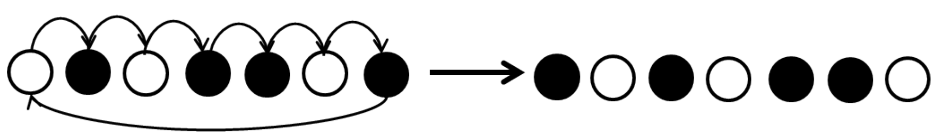


Figure 2: A pictorial representation of a *step* for *coloring* C

In the above example, $C = \{a_2, a_4, a_5, a_7\}$ and $C' = step(C)$, which results in $C' = \{a_3, a_5, a_6, a_1\}$ Say, we have a *coloring*, C, of D_N, then

$$C = step^N(C) \tag{1}$$

Definition(*Equivalent Colorings*): Two colorings are said to be equivalent if they can be converted to each other by repeated applications of the *step* function. That is, if $\exists m \in \mathbb{Z}$ such that

$$C' = step^m(C)$$

Now we are given an N, let us divide it into sets of powers of 2, i.e. D_{2^i}, such that no set of same length occurs twice, which will be unique as binary expansion of N is unique.

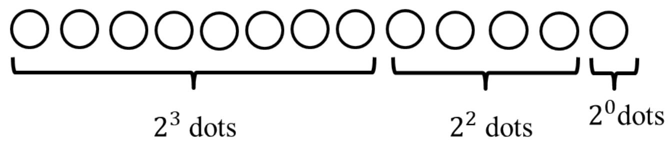


Figure 3: N=13 divided into D_{2^i}'s

Lemma 1: *For every partial coloring of a set with* $2^f (f \in \mathbb{N})$ *elements, the number of equivalent colorings is a power of 2.*

Proof. This can be seen as finding the least r such that $C = step^r(C)$, which after applying k times should give us

$$C = step^{rk}(C)$$

by using equation(1),

$$rk = N$$

. This takes into account the fact that there might exist some colorings before a full cycle that are identical to the first one. N is a power of 2 and r is a factor of N, thus r should also be a power of two, which proves our lemma.

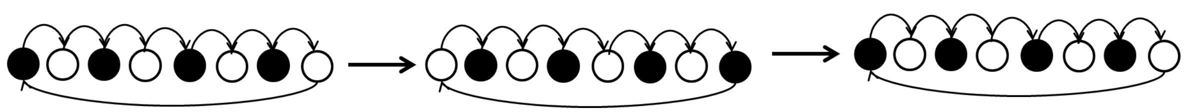


Figure 4: An equivalent class of Coloring of N=8 having only 2 elements

Now, we will talk only about D_i's where $i = 2^j$ for some $j \in \mathbb{N}$.

It can be seen that D_i's can all be colored independently and the number of colorings of k dots is just the product of number of colorings of all D_i's. Number of coloring of k dots out of N is, as seen before, $\binom{N}{k}$. Thus, it can only be odd in the case when the number of elements in colring of all D_i's are odd. As the number of colorings for a D_i is a power of 2, it can only be odd, when the number is 1.

This number corresponds to a fully colored or fully uncolored D_i, as $C = step(C)$ for both of them i.e. $r = 1$. As we have 2 choices for all D_i, to color them fully or to leave them, the number of ways to achieve this is $2^{\text{number of } D_i' s}$. The number of D_i's by construction is the number of 1's in the binary expansion of N. $\square$

Dr. Mousomi Bhakta is an Assistant Professor in the Mathematics Department at IISER Pune. She works on Elliptic PDE, Nonlinear Analysis and Variational Methods. She recently won the **Young Scientist Medal 2018 of the Indian National Academy of Science (INSA)** *for establishing results on Hardy equations with critical and super critical non-linearities, and obtaining significant regularity results on bounded and unbounded domains.*
Lubdhak Mondal and Robin Dev caught up with Dr. Bhakta to have a chat.

- **How do you feel after getting the INSA 2018 Young Scientist Medal?**
  It's certainly a good feeling.

- **What was the research topic for which you received the award?**
  As the name suggests, the award recognises young scientists; scientists under the age of 35. I work on Nonlinear Analysis of PDE. I was a graduate student at TIFR Bangalore, which is a centre for research in Analysis. I have also been working on Hardy Equations since my PhD Days and was awarded the Medal for proving a result involving Hardy Equations. I believe that it was my entire body of work on Hardy Equations that was noted, though the citation refers to a particular theorem.

- **"Hardy" here refers to G.H. Hardy, Ramanujan's mentor, right?**
  Precisely. Hardy was a Number Theorist, but really, when one comes down to it, there is no such thing as "pure" Number Theory. The same can be said about the division between "Pure" and "Applied" Mathematics. These distinctions are rather artificial. Just take a look at a few Fields Medal Lectures; to prove something really nontrivial, one needs to combine many different fields. I would go so far as to say that good Mathematics is where one combine many techniques from various fields. Hardy was a truly remarkable mathematician. He has contributed extensively to Number Theory, but his achievements extend far beyond this one field. For instance, the Hardy Inequality is rather important in the study of PDE. I have spent close to 10 years on Hardy Equations and the Hardy Inequality myself.

- **Hardy even contributed to Biology.**
  Oh, many things! This illustrates once again the idea that divisions are artificial, and not just within Mathematics, but in science in general. A given problem may be motivated by the Physics, Chemistry or Biology involved; mathematicians can still contribute to its solution by bringing in various techniques from their field.

- **Could you tell us something about the practical applications of your field?**
  The field has applications in Celestial Dynamics, the study of motion of celestial objects and in other branches of Astrophysics. You may be

surprised to know that the Heisenberg Uncertainty Principle can be derived from the Hardy Inequality. There are a bunch of other applications in Physics.

- **What do you plan to work on in the future?**
  I plan to continue working in the general area of PDE, particularly on Hardy Equations. There are many open questions waiting to be answered!

- **At IISER, you teach alongside doing your research. Do you enjoy teaching?**
  I love teaching. It doesn't matter what course it is; I have fun teaching all my courses. Whether my students love my teaching or not is quite another matter!*(laughs)* Perhaps the student feedback holds some clues on this one..

- **There are many research institutes in the country, but at IISER, the student gets to interact extensively with the faculty members, see research happening and even make contributions. This is really a fantastic opportunity, isn't it?**
  Definitely. What sets IISER apart is the emphasis on undergraduate education. TIFR, for instance is a research institute where teaching does not play a big role.
  I personally think it is a good thing that I have to teach. Even when I am teaching the same course the third or fourth time, I learn a lot, which is rather fascinating. I try to improve myself every time. Students sometimes ask me questions that have never occurred to me. It's fascinating and it's fun. And one course per semester is hardly a burden. I love teaching!

- **How does the Mathematics department at IISER Pune fare compared to that in other institutes in India and abroad?**
  As far as the IISERs are concerned, I think IISER Pune has been the best so far. IISER Kolkata also has a few very good recruitments. Even here, there are very good candidates waiting to join, so IISER Pune will soon have a great department. IISER Pune is a young institute. We are just a decade old! On the other hand, places such as IISc and TIFR are very old and I think it is a remarkable thing that the IISER department is comparable to departments in these institutes. We have faculty here who produce research that is truly world class. Our colleagues not only get published in very prestigious journals but also are editorial board members for these journals. Having said that, I think our Mathematics Department has room for improvement. But it's great that IISER is already comparable (at least in some areas) to some fantastic research institutes in India and abroad.

- **In India, many students who have the necessary motivation and capability do not opt for the basic sciences due to parental pressure, among other reasons. Even among those who choose a career in research, only a small fraction chooses Mathematics as their subject of interest. What are your views on this trend?**

  I think students are attracted towards Physics and Chemistry since these subjects are far more familiar in terms of applications. Mathematics is different. The fascination lies in the manner of thinking, proving theorems and so on. In some sense, you are in your own world when you think about a Mathematics problem. I understand that this sort of thing is not for everyone. One may like to see applications of their work as soon as possible. Mathematicians, on the other hand, are (mostly) not too worried about the applications of their work. Sometimes, applications appear a century after the mathematical discovery and many times, not at all. It doesn't matter to the mathematician. Mathematicians see beauty in things such as abstraction and generalization. At the end of the day, it depends on the individual's inclination.

- **What is it like to be a Mathematician? When did you decide to become one?**

  I've always liked Mathematics. Some people end up choosing Maths after they realize their manner of thinking does not fit Physics, Engineering, Medicine or whatever else. Others are unwavering in their faith towards Maths right from the start. There are people who see beauty in it and don't care about any other thing. What fascinated me was the logic. The idea that you can prove statements and theorems purely by logic, attracts me. To me, Analysis is just logical proving. I find beauty in logic. That said, I am not fully satisfied with myself and my work. Even today, as a mathematician, I am constantly making an attempt to improve myself. But I love mathematics and I love being a mathematician.

- **Apart from research and teaching, how do you spend your time? What else do you like to do?**

  I love reading fiction, so I try to find time to read. I don't watch much TV. I like cooking too!

MATHS CLUB STUDENT TALKS

1. Stealing necklaces and Slicing Sandwiches - A Topological Perspective

Speaker: Sriram Raghunath

Sriram Raghunath, a 4th year BSMS student, opened the Maths Club Student Talk series with his intriguingly titled talk, which involved aspects of topology, specifically the Borsuk-Ulam Theorem, that illustrated how a theorem about continuous functions can yield a surprisingly elegant solution to a discrete math problem.
Sriram began by introducing concepts such as continuity of functions, spheres and balls. He then went on to state the Borsuk-Ulam theorem. The BorsukUlam theorem states that every continuous function from an n-sphere into Euclidean n-space maps some pair of antipodal points to the same point. Here, two points on a sphere are called antipodal if they are in exactly opposite directions from the sphere's center. In a more intuitive sense, it essentially says that if an n-dimensional sphere is flattened, without tearing, into Euclidean n-space (such as a three-dimensional sphere on a two-dimensional plane), then there is always a pair of antipodal points that land on the same point in the n-space. This theorem leads to the fascinating idea that, at a particular moment, there always exist two places on the opposite sides of the Earth with the same temperature and pressure (since temperature and pressure can be represented on a 2-D plane).
The eponymous problem was then introduced: Two thieves have stolen an open-ended necklace with d types of precious gemstones. What is the minimum number of cuts required to divide it equally amongst themselves? Sriram claimed that d cuts would be enough to solve the problem. This was followed by an even moreastonishing claim - that this solution to a discrete problem could be proven using the Borsuk-Ulam theorem, a theorem based on continuity.
A simplified version of the problem was proven first. The proof goes like this: The discrete problem must first be converted to a continuous one, To do this, the interval [0,1] on the real line is divided into three parts. The lengths of all parts are positive real numbers and add up to 1. This is also true of the coordinates of a point on a three-dimensional sphere with radius 1. So, any two cuts correspond to a point on the sphere. Assign the positive sign to Thief 1 (say, Borsuk) and the negative sign to the other thief (say, Ulam). This means that for two particular cuts, the sign assigned to the square root of the lengths of the segments will go to the corresponding thief. Now, the Borsuk-Ulam theorem guarantees that there exists a set of antipodal points on the sphere such that their mapping (i.e. their distribution to the two thieves) is the same. In other words, on flipping the sign of a triplet of coordinates, the amount given to each thief remains the same, which makes it is a fair distribution. This can be generalized to a problem with d types of gems by using a (d+1)-dimensional sphere. Thus, the Borsuk-Ulam theorem proves that d cuts are indeed sufficient for d types of gemstones.

In a similar fashion, the Borsuk-Ulam theorem was used to solve a continuous problem: the proof for the statement A sandwich made of ham, cheese and bread can be sliced straight such that all three layers are bisected.
The talk concluded with a QA session where the speaker clarified all doubts regarding the concepts talked about.

2. This class is not boring

Speaker: Poornima B.
Finding all the primitive Pythagorean triples (integer solutions of $x^2+y^2=z^2$ having no common factor) is an easy exercise. Consider the equation mod 4, squares either are 0 or 1 mod 4. Sum of two squares can be 0, 1 or 2 mod 4. Since the left hand side of the above equation is also a square(0 or 1 mod 4), and we require that the triple have no common factor, we conclude that z should be odd and exactly one of x or y has to be odd. Not all of them can be even as they would have 2 as a common factor. We now introduce the number field
$Q(i) = 1a + ib : a, b\epsilon Q, i = \mathbb{N}$
We can factorize $x^2 + y$ in Q(i) and obtain
$(x + iy)(x - iy) = z^2$
The problem now reduces to finding factors of z^2 in the ring of Gaussian integers $\mathbb{Z}(i) = 1a + ib : a, b\mathbb{Z}$. We know that $Z(i)$ is a unique factorization domain. We claim that $x + iy$ is of the form $u\alpha^2$ where u is a unit and α is some Gaussian integer. Suppose p is a prime which divides $x + iy$, then p should divide z^2. Since p is a prime, p should divide z, therefore p divides z^2 an even number of times. To show that $p^e|x+iy$ where e is an even integer, assume that p divides both $x+iy$ and $x-iy$, then p divides $2x$(which can be written as $(x+iy)+(x-iy)$). This shows that p is a common factor for both x and z which is a contradiction. Therefore $x + iy$ is of the form $u\alpha^2$ and the solution set is $(x, y) = (1(m^2 - n^2), 2mnl)$ and $z = (m^2 + n^2)$.
We can try to generalize the above method to find solutions to higher degree equations. Fermats last theorem - The equation $x^n + y^n = z^n$ has no solutions in non-zero integers when n $\geq$ 3. The method used above worked because of the phenomenon of unique factorization in Z(i).
We will have to check if unique factorization holds in other rings of consideration and if it fails, we should be able to quantify the extent of failure. Class Number is one such example.

3. Euclidean Geometry in an Orthogonal Frame

Speaker: Neeraj Deshmukh
In this note we describe the structure of Euclidean space using Élie Cartan's method of moving frames. More precisely, we wish to describe trajectory of a point in a "moving frame". For the sake of simplicity we will restrict our attention to the three dimensional Euclidean space, $\mathbb{E}^3$.

Let M be a point in space. Consider an orthogonal frame at M given by three mutually perpendicular vectors I_1, I_2, I_3. This system of a point together with an orthogonal frame gives a trihedron $\mathcal{T} := (M, I_1, I_2, I_3)$ and can be described by six parameters, $u_1, \ldots, u_6$, representing the coordinates of M and the Euler angles defining the orthogonal frame (with respect to the standard axes). We will be interested in the behaviour of $\mathcal{T}$ as we vary the parameters u_i's. The object described by $\mathcal{T}$ with respect to the varying u_i's is what we call a *moving frame*.

Let $\mathcal{T}' = (M', I'_1, I'_2, I'_3)$ be another trihedron which is infinitesimally close to $\mathcal{T}$, i.e,

$$\begin{aligned} M' - M &= \quad dM \\ I'_k - I_k &= \quad dI_k, \quad k = 1, 2, 3 \end{aligned}$$

Projecting the infinitesimal displacements dM and dI_k onto the axes I_k of $\mathcal{T}$, we get a system of differential equations,

$$\begin{aligned} dM &= \omega^i I_i, \\ dI_i &= \omega_i^j I_j \end{aligned} \tag{1}$$

where we use the summation convention on the index repeating above and below. Here, the ω^i, ω_i^j are differential 1-forms in the coordinates (u_i). Also, the vectors I_k's form an orthogonal frame, i.e, $I_i \cdot I_j = \delta_{ij}$. Differentiating we get, $dI_i \cdot I_j + I_i \cdot dI_j = 0$. This gives us another set of equations, $\omega_i^j + \omega_j^i = 0$. So, in fact (1) is described by six differential forms, ω^i, ω_i^j. Thus, given a moving frame in $\mathbb{E}^3$, it satisfies (1), in terms of the six forms ω^i, ω_i^j. Conversely, one may ask,

Question 1. given six forms ω^i, ω_i^j of an infinitesimal displacement of an orthonormal trihedron, does there exist a family of trihedrons admitting these components (as in (1))? Or to put it another way, when is such a system of 1-forms *integrable*?

To answer this we take the above system (1) and apply exterior differentiation,

$$\begin{aligned} dI_i \wedge \omega^i + I_i d\omega^i = 0 \\ dI_j \wedge \omega_i^j + I_j d\omega_i^j = 0. \end{aligned}$$

Substituting the values of dI_i's and rearranging terms, we get,

$$\begin{aligned} d\omega^i &= \omega^j \wedge \omega_j^i \\ d\omega_i^j &= \omega_i^k \wedge \omega_k^j \end{aligned} \tag{2}$$

Hence, if a solution of (1) exists, then it must satify (2). The Theorem of Cartan is that, in fact, this condition is also sufficient.

Theorem 2. [**?**, §26] If the differential forms ω^i, ω_i^j satisfy (2), then for a given initial trihedron $\mathcal{T}_0$ they define a moving frame obtained from the $\mathcal{T}_0$ by an appropriate affine transformation.

The equations (2) are also called the *Maurer-Cartan equations*.

One can also extend this analysis to any manifold. In this setting, however, the system (2) of differential 2-forms may fail to hold, giving rise to non-trivial 2-forms,

$$\begin{aligned} d\omega^i - \omega^j \wedge \omega_j^i &= \Omega^i \\ d\omega_i^j - \omega_i^k \wedge \omega_k^j &= \Omega_i^j \end{aligned} \tag{3}$$

The 2-forms Ω^i, Ω_i^j together describe the curvature of the manifold. In this sense, curvature of a manifold maybe thought of as the failure of a system of differential 1-forms to be integrable (as in Question 1). The components Ω^i describe the translation part of the curvature, whereas the Ω_i^j's describe the associated rotation (cf. [**?**, §36]).

For a Riemannian manifold, the translation part is zero by definition. This is because the Ω^i describe the torsion tensor. Moreover, the Ω_i^j's describe the Riemann curvature tensor (cf. [**?**, Chapter 16]).

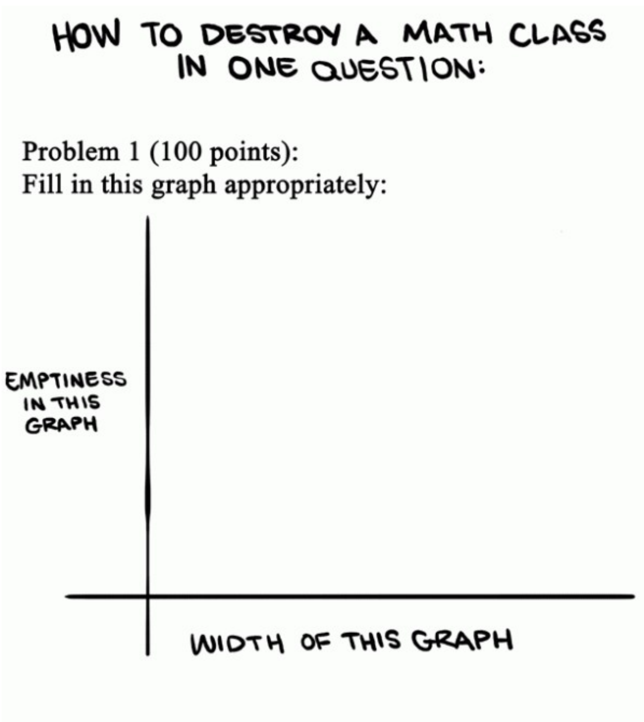
A moving frame, as described above, can be thought of as an integrable submanifold of the isometry group of $\mathbb{E}^3$, $Isom(\mathbb{E}^3)$. Moreover, the differential forms ω^i, ω_i^j, together describe a Lie algebra-valued 1-form ω with values in the Lie algebra of $Isom(\mathbb{E}^3)$ (this ω is known as the Maurer-Cartan form of $Isom(\mathbb{E}^3)$). This point of view admits generalisation to any Lie groups. In this language, Theorem 2 can be stated as,

Theorem 3. [**?**, Theorem 3.6.1] *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Denote by ω_G the Maurer-Cartan form of G. Let ω be a $\mathfrak{g}$-valued 1-form on a smooth manifold M satisfying the Maurer-Cartan equations. Then, for each point $p \in M$, there is a neighbourhood U of p and a smooth map $f : U \to G$ such that $f^*\omega_G = \omega$.*

The Mathematical Sense of Humor

Artist: Zack Wienersmith
SMBC comics

https://www.smbc-comics.com/



Proof that the square root of two is irrational

Anand Rao Tadipatri

On the contrary, suppose $\sqrt{2}$ is a rational number, i.e., $\exists$ p, q $\in \mathbb{N}$ such that $\sqrt{2} = \frac{p}{q}$, and gcd(p, q) = 1.

Squaring both sides and multiplying by q^2,

$$2q^2 = p^2 \Rightarrow q^2 + q^2 = p^2.$$

The ordered triple (q, q, p) is a Pythagorean triple. A well-known method of generating Pythagorean triples involves squaring Gaussian integers (complex numbers of the form $m + ni$, where $m, n \in \mathbb{Z}$). As an example, consider the complex number $2 + i$.

$$(2 + i) \cdot (2 + i) = 3 + 4i.$$

3 and 4 are the base and perpendicular of a right triangle whose sides form a Pythagorean triple. For any general m, n $\in \mathbb{Z}$,

$$(m + ni) \cdot (m + ni) = (m^2 - n^2 + 2mni).$$

$m^2 - n^2$ and $2mn$ are the base and perpendicular of a right triangle whose sides form a Pythagorean triple.

The Pythagorean triple (q, q, p) represents a right triangle with the base and perpendicular measuring q units, and the hypotenuse measuring p units. Since the base and perpendicular are equal, the acute angles of the isosceles right triangle are both $\frac{\pi}{4}$.

In the polar form, the complex number $q+qi$ (which represents the Pythagorean triple (q, q, p)) can be written as $p \cdot e^{i\frac{\pi}{4}}$ (Justification: The magnitude is $\sqrt{(q^2 + q^2)} = p$, and the angle is $\frac{\pi}{4}$, as mentioned above).

Since all Pythagorean triples can be written as squares of Gaussian integers (which, without loss of generality, can be assumed to be in the first quadrant), $\exists$ m, n $\in \mathbb{N}$ such that

$$(m + ni)^2 = q + qi = p \cdot e^{i\frac{\pi}{4}}.$$

Written in the polar form,

$$(m + ni) = \sqrt{p \cdot e^{i\frac{\pi}{4}}} = \sqrt{p} \cdot e^{i\frac{\pi}{8}}.$$

The complex number $e^{i\frac{\pi}{8}}$ can be rewritten, using Euler's formula, as $cos(\frac{\pi}{8}) + isin(\frac{\pi}{8})$.

$$\therefore (m + ni) = \sqrt{p} \cdot cos(\frac{\pi}{8}) + i\sqrt{p} \cdot sin(\frac{\pi}{8}).$$

The real and imaginary parts of the complex numbers can be equated.

$$\therefore \sqrt{p} \cdot cos(\frac{\pi}{8}) = m$$

$$and \quad \sqrt{p} \cdot sin(\frac{\pi}{8}) = n.$$

The values of $cos(\frac{\pi}{8})$ and $sin(\frac{\pi}{8})$ are $\frac{1}{2}\sqrt{2+\sqrt{2}}$ and $\frac{1}{2}\sqrt{2-\sqrt{2}}$ respectively. Comparing the real parts,

$$\sqrt{p} \cdot \frac{1}{2}\sqrt{2+\sqrt{2}} = m$$

$$\Rightarrow \; p \cdot (\frac{1}{4}(2+\sqrt{2}) = m^2$$

$$\Rightarrow \; p \cdot (2 + \frac{p}{q}) = 4m^2$$

$$\Rightarrow \; \frac{p^2}{q} = 4m^2 - 2p.$$

Now, $m \in \mathbb{N}$, and therefore, $4m^2 \in \mathbb{N}$. Similarly, $p \in \mathbb{N}$, and therefore, $2p \in \mathbb{N}$. $\therefore 4m^2 - 2p \in \mathbb{N}\,(or\ \mathbb{Z})$.

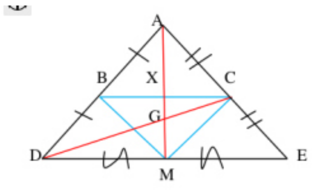
The rational number $\frac{p}{q}$ is definitely not an integer, since 2 is not a perfect square. From the equation, it appears that $\frac{p^2}{q} \in \mathbb{Z}$. This is possible only if $q \mid p$. However, this violates the first assumption in the proof, that gcd(p, q) = 1. A similar contradiction appears when the imaginary parts are equated. Therefore, the assumption must be wrong; $\sqrt{2}$ is irrational. This completes the proof. ■

GEOMETRIC SERIES PROOF FOR A CLASSICAL GEOMETRY PROBLEM

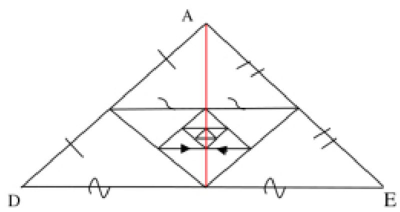
SHUBHAM S. JAISWAL

Various proofs for the centroid dividing the median in figures like the triangle and tetrahedron in a certain fixed ratio have surfaced in the past, including the classical one by construction and using similarity and the one by simply using co-ordinate geometry. The following is my proof which uses elementary geometry as the basis and we arrive at the required result by summing a geometric series obtained in the procedure. Although this proof seems longer, it is quite aesthetic and has advantages over primitive proofs as it takes us through many interesting properties of such figures and some fascinating corollaries can be deduced by the procedure followed in the proof.

Lets begin with our old friend, Tipsy Triangle!

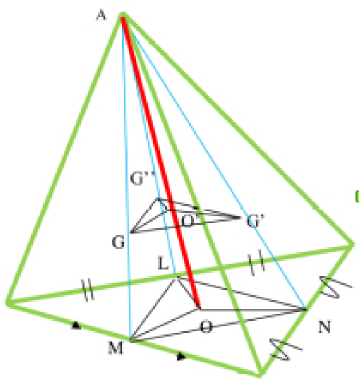


Triangles ABC and ADE are similar hence, BC is parallel to DE. Triangles ABX and ADM are similar and AXC and AME are similar, and hence X is centre of BC and MX is median of triangle MBC. This leads to the conclusion that the triangle which is formed by joining the midpoints of sides of the original triangle has the same centroid as the original one. Also, by similarity, $AX = XM$.

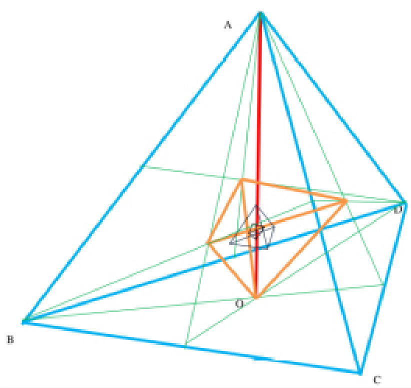


For all the triangles successively formed by joining midpoints of the previously formed triangles, the centroids coincide at G. Hence, distance of centroid from point D, using geometric series, $AM = a, AX = (a/2), a/2 - a/4 + a/8 - a/16 + ... = (a/2)/(1 - (-1/2)) = (a/2)/(3/2) = a/3$ hence, $(2a/3) : (a/3) = 2 : 1$

Its time for our protagonist, the tetrahedron!



Here G, G', G" and O are centroids of the triangles forming original tetrahedron. By similarity, $GG', G'G$" and GG" are parallel to MN, NL and ML respectively, hence we conclude that the base triangles of these tetrahedrons are parallel. Also, by repeated use of similarity, triangles $GG'G$" and MNL are similar and O' is centroid of $GG'G$", and OO' is median for tetrahedron $OGG'G$" and this leads to the conclusion that the centroid of tetrahedron, formed by joining centroids of triangles forming the original tetrahedron, coincides with centroid of the original one. Also, we get $AO' = 2OO'$ by similarity. Join all centroids of triangles forming the tetrahedron to form small tetrahedrons successively, and denote the common centroid of these infinitely many tetrahedrons as G. Hence, distance of the centroid from point O using geometric series is
$AO = a, AO' = (2a/3)$,
$a/3 - a/9 + a/27 - ...... = (a/3)/(1 - (-1/3))$
$= (a/3)/(4/3) = a/4$
Hence, $(3a/4) : (a/4) = 3 : 1$.



A quick corollary

The centroid of a figure is its geometric centre. Physicists generally refer to the centroid of a figure as its centre of mass since the centre of mass of a geometric object of uniform density is the geometric centre of the figure. This proof easily extends to 3-D figures formed by a polygon on a plane (with known position of geometric centre) and a point outside plane forming triangles with sides of polygon. The ratio in which centroid of 3D figure divides line joining the point outside plane and centroid/center of mass of 2D polygon is still 3:1 (e.g. pyramid with base polygon square - we know that the centroid of a square is the intersection of its diagonals). Note the use of lemmas to prove the result. Such lemmas with their proofs form the backbone of proofs of huge and unexpected theorems in Mathematics. They are the stepping stones for the greater good!

THE INVISIBLE SHOWRUNNERS

D.V.S. ABHIJIT

When you are watching a movie all you see on screen is the product of hard work of the people onscreen and people like the director, cinematographer etc. The people off-screen who have dedicated themselves to the cause of the movie are not visible to you. It is only the people/characters on screen that will stay in your memory. You are enjoying the work of the people off-screen, but you do not know them. They are as important if not more important in some cases (cartoons and animation films, cartoons for instance) in running the show.

After working in an Artificial Intelligence (AI) project in the summer of my second year, doing a Mathematical Finance project in the second semester of my 3rd Year, I have realized that mathematicians play the part of people behind the scene in the physical sciences, computer science, and many domains other than mathematics. When I started first working in the Artificial Intelligence, I was apprehensive as to how I would cope with a computer science project, being a student of the basic sciences. The moment I started getting familiar with Deep learning algorithms I realized they were just based on calculus and linear algebra and had little to do with computer science. The implementation of those algorithms was dependent on the current technology which was a part of computer science, but the basis of the field lies in mathematics. To optimize and modify the current algorithms, and to create new ones, mathematicians are needed. The mathematicians who developed the algorithms deserve a lot of credit for the recent boom in AI technology. The scope for progress in the field of AI rests on the shoulders of mathematicians as much as it does on computer scientists.

When we think about financial markets we imagine economists, investors and businesspeople running the show. The above people run their companies based on methods, models and algorithms developed by applied mathematicians. One of the buzzwords in finance in recent times is Hedge Fund. A hedge fund is an investment fund that pools capital from accredited individuals or institutional investors and invests in a variety of assets, often with complex portfolio-construction and risk-management techniques. When you read the definition of a hedge fund it is difficult to realize that it has anything to do with mathematics and mathematicians. It would come as a revelation that the one of the most successful Hedge Fund manager James Simon is a PhD in mathematics and conducted research in mathematics before setting his foot in the financial sector. The mathematicians who run the financial markets in the world are called quantitative analysts. They are playing an increasingly important role in shaping financial decisions of the biggest companies in the world. The people who we see on screen in finance are people from a finance/economics background. Even in financial markets, the mathematicians play the role of the people behind the scene.

Mathematical methods have transformed the area of cancer research in recent years. Mathematics and Statistics has transformed a lot of scientific domains not mentioned in the article (Biology, Physics etc.) over the years in a way that they can never go back to be their old selves. When I was in school, I used to think that Mathematicians were people who speak the language of mathematics, live in their own world and had nothing to do with the real world. After my first two years at IISER, observing the from a scientific perspective, I have come to realize that Mathematicians are the showrunners of the real world, its just that you dont see them. They let their work do the talking.

If students in school are made aware of the importance of mathematicians in solving Global Problems, it would attract a lot of them to pursue a career in mathematics and help in making the world a better place.

Some powerful ways to sum powers

INTRODUCTION

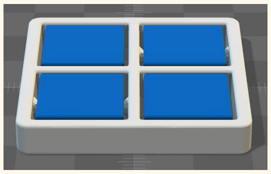
Several methods exist to compute the sum of natural numbers and the summation of squares. Some are purely algebraic, some inductive, a few combinatorial, and others geometric.

In this article, I will be presenting a geometric (and partly algebraic) proof for the sum of squares. With slight modifications, this can be extended to higher powers.

Sum of Squares

$S = 1^2 + 2^2 + 3^2 + ... + n^2$

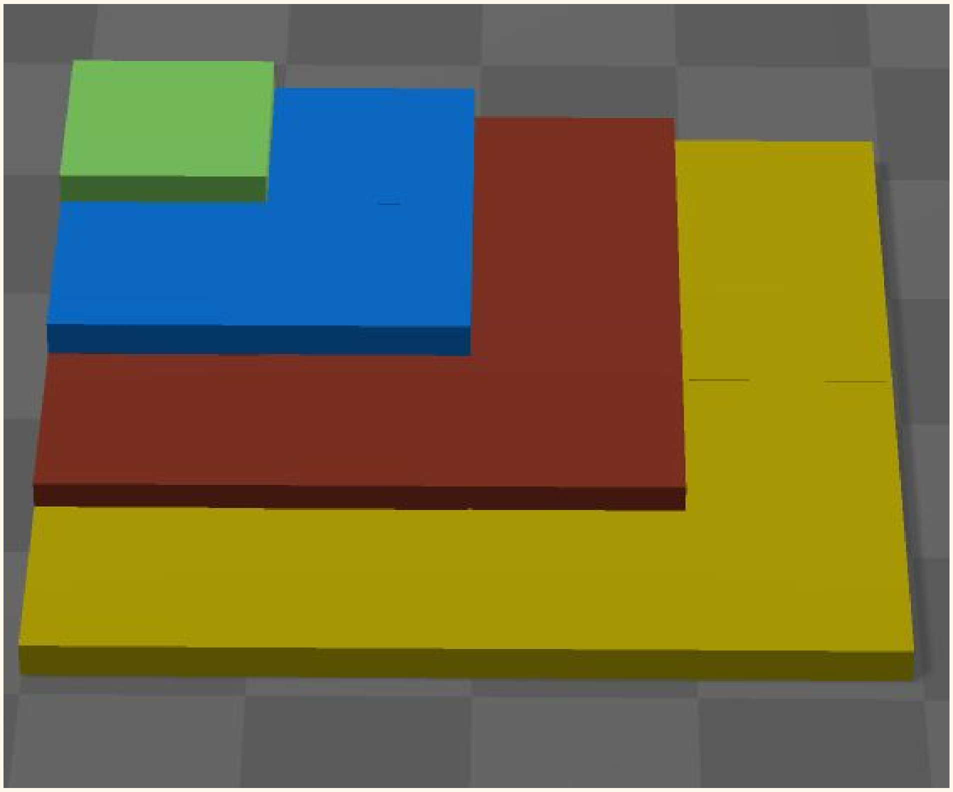
Square numbers are called so because they can be represented as squares.



Representing 2^2 as a square of side 2 units, and in general, n^2 as a square of side n units, is a suitable starting point for a geometric proof.

This representation changes the problem from summing up squares to finding the total area of n squares. The squares can be rearranged freely without altering the total area. Why not line them up in ascending order? Or randomly? Or stack them up like a tower?

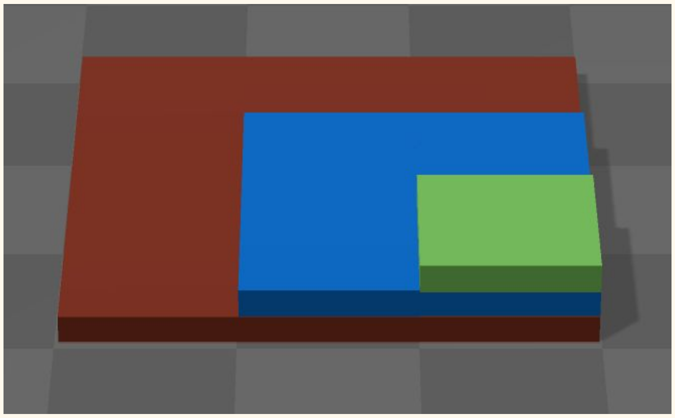
For this proof, consider stacking them up in ascending order, all aligned to one corner. The resulting structure should look something like one-quarter of a step pyramid.



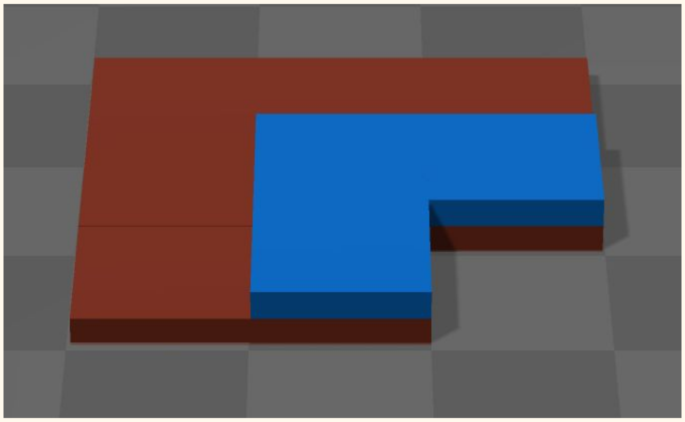
Assuming all squares to have unit thickness, the problem now is to find the volume of the oddly shaped quarter-pyramid structure. One way would be to add up the individual 'levels' of the pyramid - starting with 1^2 on the top, all the way down to n^2. But this is an exercise in futility; it brings us back to the original problem of summing squares. This particular arrangement of the square tiles also opens up another possible way to compute the area - portions of the pyramid at the same height can be grouped together, and their volumes can be added up.

I will be illustrating this approach for $1^2 + 2^2 + 3^2$, before taking the more general case of $1^2 + 2^2 + 3^2 + \ldots + n^2$.

The quarter-step-pyramid structure for $1^2 + 2^2 + 3^2$ is constructed by placing 3^2 at the bottom, 2^2 in the middle and 1^2 at the top, while keeping all squares aligned to some corner.



At the apex of the pyramid is a single square (shown in green) representing 1^2, resting on two bigger squares underneath. The total volume of the vertical column containing this single square is $3 \cdot (1^2)$ (since the total height of the column is three units, and all squares are assumed to have unit thickness). Once the volume of this column has been recorded, it is removed from the structure.



The next highest level of the pyramid (shown in blue) has an area of 2^2 - 1^2 (since the area under the 1^2 column was removed). This area rests on another square and the two have a combined height of 2 units. The volume of this piece is $2 \cdot (2^2 - 1^2)$. Like before, this piece is removed from the structure.

The remaining portion of the structure (shown in red) is a section of the largest square (3^2). An area of 2^2 units has already been removed from it, leaving 3^2 - 2^2. The volume of this piece is $1 \cdot (3^2 - 2^2)$.

This amounts to rewriting $1^2 + 2^2 + 3^2$ as
$3 \cdot (1^2) + 2 \cdot (2^2 - 1^2) + 1 \cdot (3^2 - 2^2)$.

More generally,

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} (n+1-k) \cdot (k^2 - (k-1)^2).$$

The difference of two consecutive squares, k^2 and $(k-1)^2$, is the kth odd number, $2 \cdot k - 1$. Using this, the summation on the right can be simplified further.

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} (n+1-k) \cdot (2k-1) = \sum_{k=1}^{n} (n+1) \cdot (2k-1) - \sum_{k=1}^{n} (2k^2 - k)$$

Notice that a $\sum_{k=1}^{n} k^2$ term appears on the right. The equation can be rearranged to get an expression for $\sum_{k=1}^{n} k^2$.

$$3 \cdot \sum_{k=1}^{n} k^2 = (2 \cdot (n+1) + 1) \cdot \sum_{k=1}^{n} k - (n+1) \cdot \sum_{k=1}^{n} 1$$

Using the fact that $\sum_{k=1}^{n} 1 = n$ and $\sum_{k=1}^{n} k = \frac{n \cdot (n+1)}{2}$ and rearranging the terms, the result simplifies to its final form:

$$\sum_{k=1}^{n} k^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}.$$

Extending to higher powers

The geometric approach to the sum of squares used squares to represent the squares of numbers. How about using cubes to represent the cubes of numbers? And the fourth-dimensional equivalent of a cube (called a tesseract by people who want to sound smart) to represent the fourth powers of numbers? We quickly run into problems with this approach. The human brain is incapable of conceiving four dimensions and beyond. It is impossible to imagine a tesseract in four dimensions, let alone several stacked up in a complicated structure.

With some manipulation, it is possible to make the summation of the p^{th} powers of natural numbers analogous to the summation of squares using square tiles.

Instead of a cube or a tesseract, imagine representing k^p by a **square tile**[1] of area k^p and unit thickness. The side lengths may not always be integers - this is a consequence one must bear with while representing higher dimensional solids in two dimensions. However, this does not affect the approach, and the method used for the sum of squares can be used here too.

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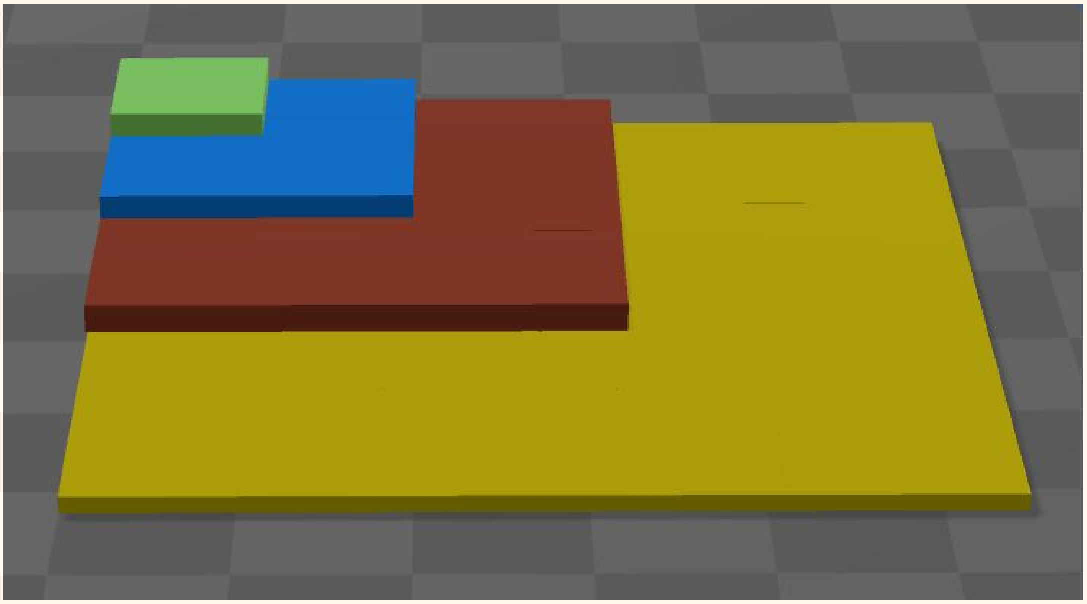
[1] The tiles need not be squares at all. In fact, they can take any shape, as long as the tile representing $(k+1)^p$ entirely covers the tile representing k^p.



(Notice how the difference in the areas of the squares is much larger than those in the p = 2 case. The figure is not an exact representation; it is only illustrative.)

In fact, the approach is exactly the same. To arrive at an expression for $1^p + 2^p + 3^p$, one may re-read the paragraphs on the summation of $1^2 + 2^2 + 3^2$, while mentally replacing 1^2, 2^2 and 3^2 with 1^p, 2^p and 3^p respectively.

In general,

$$\sum_{k=1}^{n} k^p = \sum_{k=1}^{n} (n+1-k) \cdot (k^p - (k-1)^p).$$

At this point, some readers (quite understandably) may be eyeing these sneaky geometric manipulations with some suspicion. The algebraic equivalent of the summation is the following series, in which quite a few terms conveniently cancel out:

$\sum_{k=1}^{n} k^p = 1 \cdot (n^p - (n-1)^p) + 2 \cdot ((n-1)^p - (n-2)^p) + 3 \cdot ((n-2)^p - (n-3)^p) + \ldots + (n-1) \cdot (2^p - 1^p) + n \cdot (1^p)$.

$\sum_{k=1}^{n} k^p = \sum_{k=1}^{n} (n+1-k) \cdot (k^p - (k-1)^p)$.

The geometric approach is more tangible than the algebraic approach, which seems to have been conjured out of thin air with no justification whatsoever.

In the paragraphs that follow, I will be rearranging the terms to get an explicit formula for $\sum_{k=1}^{n} k^p$. At this point, the proof is practically complete, and readers who wish to skip the technical details may directly go to the last section.

To start with, the binomial theorem can be used to expand and evaluate $k^p - (k-1)^p$.

According to the binomial theorem,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

and consequently,

$$(k-1)^p = \sum_{j=0}^{p} \binom{p}{j} (-1)^{p-j} k^j.$$

$$k^p - (k-1)^p = k^p - \left(\sum_{j=0}^{p-1} \binom{p}{j} (-1)^{p-j} k^j + k^p\right) = \sum_{j=0}^{p-1} \binom{p}{j} (-1)^{p-j-1} k^j$$

Returning to the original expression,

$$\sum_{k=1}^{n} k^p = \sum_{k=1}^{n} (n+1-k) \cdot (k^p - (k-1)^p) = (n+1) \cdot \sum_{k=1}^{n} (k^p - (k-1)^p) - \sum_{k=1}^{n} k \cdot (k^p - (k-1)^p)$$

The first summation telescopes to n^p (the remaining terms cancel out), and $k^p - (k-1)^p$ has been computed above.

$$\sum_{k=1}^{n} k^p = n^p \cdot (n+1) - \sum_{k=1}^{n} \sum_{j=0}^{p-1} \binom{p}{j} (-1)^{p-j-1} k^{j+1}.$$

A $\sum_{k=1}^{n} k^p$ term is concealed in the summation on the right. Extracting it from the summation, and then rearranging the terms should give an expression for $\sum_{k=1}^{n} k^p$ in terms of the sum of lower powers.

$$\sum_{k=1}^{n} k^p = n^p \cdot (n+1) - \sum_{k=1}^{n} \sum_{j=0}^{p-2} \binom{p}{j} (-1)^{p-j-1} k^{j+1} - \sum_{k=1}^{n} \binom{p}{p-1} (-1)^0 k^p.$$

The final equation, written in terms of the sums of lower powers, is

$$(p+1) \sum_{k=1}^{n} = n^p \cdot (n+1) - \sum_{k=1}^{n} \sum_{j=0}^{p-2} \binom{p}{j} (-1)^{p-j-1} k^{j+1}.$$

Using this expression, it is possible to write a computer program to obtain a polynomial expression for $\sum_{k=1}^{n} k^p$.

What next?

One disadvantage of the formula derived above is that it is recursive - computing the sum of the p^{th} powers requires knowing the sum of the previous powers. By carefully observing the coefficients of the terms in the summations, certain constants can be introduced, eliminating the need for recursion (at least partly). I have a truly marvelous proof of this, which this article is too small to contain. (The method will probably be published in a follow-up article at a later point in time).

To share your thoughts, criticism and code with the author, send a mail to anand.tadipatri@students.iiserpune.ac.in.

(All the diagrams in this article were made using 3D Builder, an in-built software on Windows 10)

Numbers Chasing Their Own Tails!
Neel Shah

Everyone has divided a natural number by 7 at some point in their life, the most notorious example being 22. But it might be a little surprising that this fraction, that commits the blasphemy of often replacing a famous transcendental number, can also provide material worthy to be in a math magazine.

22/7 is 3.142857142857142857.....

in base 10. Let's play a little with the numerator, by adding 1/7.

23/7 is 3.285714285714.....

Isn't there an uncanny similarity with the previous recurring decimal representation? The digits that recur in 23/7 are exactly the same that recur in 22/7! Shifting these by 3 gives the recurring digits of 1/7 as 1,4,2,8,5,7 and those of 2/7 as 2,8,5,7,1,4. You may verify that the same digits recur in 3/7 upto 6/7 in different cyclic permutations.

The number 142857 is called a cyclic number in base 10. Its consecutive multiples, upto 142857*6, are cyclic permutations of itself. 142857*7 will make you wonder if your calculator is faulty- it is precisely 999999. You would be hard-pressed to find another cyclic number in base 10 (we don't consider the trivial cases of repeating a cyclic number, e.g. 142857142857), and you'll still need to relax a certain condition: you have to allow a leading 0. The next cyclic number we get is: 0588235294117647. That's 16 digits long! The next one is 052631578947368421. I'll leave the counting to you this time.

142857 is the recurring decimal sequence of 1/7, 0588235294117647 is the recurring sequence of 1/17, and the pattern continues such that a cyclic number of length 'k' digits is related to the recurring digits of 1/(k+1). It so happens that all these 'k+1's have to be prime numbers. All cyclic numbers come from recurring digits of ‘1/prime’ factors (this can be used to understand why a (cyclic number*generating prime) is always a sequence of 9s, since (1/prime)*prime = 1 = 0.9999...). However, all primes do not generate cyclic numbers this way. Also, not all bases have cyclic numbers- there are no cyclic numbers in perfect square bases, like base 4 and base 9.

Don't worry, these phenomena aren't unexplained and there are deeper truths pulling the strings. Unfortunately, I can only give glimpses of the intricate mathematical machinery behind cyclic numbers; going in deeper requires knowledge of algebraic number theory which I do not possess.

Let's start by seeing in some detail what actually happens when we divide 1 by a prime. We'll get nothing interesting, for our purposes, by dividing anything by 2 or 5 since they are factors of 10, our base. We get 0.333... on dividing 1 by 3. Let us now divide 1 by 7 by long division. 1 is less than 7, so the 'first' quotient is 0. Similarly, in the next step, our quotient is ‘0.1’. Now the remainder is 3, so we multiply 3 by 10 and repeat the story thus:

30 = 7*4 + 2
20 = 7*2 + 6
60 = 7*8 + 4
40= 7*5 + 5
50 = 7*7 + 1
10 = 7*1 + 3

Now we know what our recurring digits are. But there is something quite interesting about the remainders after every step. We get 6 different remainders in the process of long division, which is actually strange because only 6 different remainders are possible when we divide a non-multiple of 7 by 7. But in general, all those remainders do not appear in the long division of a non-multiple. So what determines the remainders that appear in long division?

If r_0 is our current remainder after performing a step in long division and r_1 is the new remainder, then

$$r_1 = 10r_0 \bmod 7$$

For those who don't know what this means, it means that r_1 and $10r_0$ leave the same remainder when divided by 7. It is just r_1 in this case. Similarly, for later remainders r_2 , r_3 and so on, mod 7 (Don't have too much trouble believing this, it just comes from the way long division works). Since there are finitely many remainders available, after some point we get

$$r_k = r_0$$

and then our sequence of remainders repeats from r_0 (The sequence always repeats eventually, but r_0 may not be part of the repeating cycle. However, r_0 is always part of the repeating cycle for prime divisors.).

Can we find the 'k' for which $r_k = r_0$ i.e.

$$r_k = 10^k r_0 \bmod 7?$$

This means that

$$10^k = 1 \bmod 7.$$

It turns out that we can, to a certain extent. In order to do this, I'll have to introduce you to a famous theorem in elementary number theory: Fermat's Little Theorem. The statement of this theorem is:

If 'p' is a prime number and 'a' is an integer which is not a multiple of 'p', then

$$a^{p-1} = 1 \bmod p.$$

There are truly marvelous proofs of the theorem, but I do not have the space to write one here. I will use the
theorem without proof from now.

In our problem, the prime 'p' is 7 and taking 'a' = 10, we get

$$10^6 = 1 \bmod 7.$$

Thus, whatever our r_0 may be, $r_6 = r_0$. Thus, k=6 (or any multiple of 6, but those aren't important) is definitely a solution to

$$10^k = 1 \bmod 7.$$

It so happens that no ‘k’s that aren’t multiples of 6 satisfy the equation, which is easy for you to verify. If our prime was 3, from the Little Theorem we get k = 2 as one solution of mod 3. But odd numbers, also satisfy the equation. Similarly, for p = 11, 2, not a multiple of 10, is a solution and for p = 13, 6 is a solution. But for p = 17, only k = 16 is a solution. Thus, the period of the cycle of long division remainders for 3 is 1, and it is 3 and 6 for 11 and 13 respectively.

There are many primes p for which k = p – 1 is the only solution to

$$10^k = 1 \text{ mod } p,$$

it is conjectured by Emil Artin that there are infinitely many such primes. 10 is said to be a **primitive root modulo p** (informally, primitive root of p) for such primes. 10 is a primitive root of 7 and 17.
Let us go back to our roots and divide 1 by 7 again. First step: write ‘0.’ in quotient, multiply dividend by 10:

10 = 7*1 + 3, quotient ‘0.1’
30 = 7*4 + 2, quotient ‘0.14’
20 = 7*2 + 6, quotient ‘0.142’
60 = 7*8 + 4, quotient ‘0.1428’
40= 7*5 + 5, quotient ‘0.14285’
50 = 7*7 + 1, quotient ‘0.142857’
10 = 7*1 + 3, quotient 0.‘142857’ recurring.

When we divide any non-multiple of 7 by 7, we write the integer part of the quotient and then come straight to one of the steps in the above cycle. This happens only because every possible nonzero remainder is already in a step of the cycle. Notice that every step of the cycle also gives us a digit of the recurring decimal, which means that the recurring digits of all decimal expansions of non-multiples of 7 come directly from the cycle. Those digits are exactly all possible cyclic permutations of 1,4,2,8,5,7! This proves why the recurring decimals of 2/7 upto 6/7 are cyclic permutations of those of 1/7, and that proves why 142857 must be a cyclic number!

We can similarly prove why the recurring decimal digits of 1/17 give a cyclic number by writing the long division of 1/17. And in the exact same manner, we can go on to prove that all primes like 7 and 17, which have 10 as a primitive root, generate cyclic numbers in the same way as 7 and 17! These are known as the **full reptend primes** in base 10.

Finally, let’s see why other primes like 3, 11, 13 do not produce cyclic numbers. Let us divide 1 by 11.

1 < 11, quotient ‘0.’
10 < 11, quotient ‘0.0’
10*10 = 11*9 + 1, quotient ‘0.01’
1 < 11, quotient 0.‘01’ recurring.

If we divide a general non-multpiple of 11 by 11, we may never encounter this cycle. We only encounter this cycle if our dividend is of the form 11k + 1 or 11k + 10, because only 1 and 10are the remainders that are part of this cycle. This means 2/11 does not have a cyclic permutation of the recurring decimal digits of 1/11, so the recurring digits of 1/11 do not give a cyclic number. We can now understand why any primes that are not full reptend primes do not produce cyclic numbers, that is because their cycles of long division do not contain all possible nonzero remainders.

If our divisor wasn't prime, it so happens that it can't generate a cyclic number. I will leave you to think about what could happen in that case, and also what could happen in bases other than 10.

A SUMMER JOURNEY IN MATHEMATICS AT TIFR

VISHNU N.

An opportunity to do a summer internship at a top institute like TIFR did not everyday, and I was quite excited about it. After getting selected through the VSRP program, until I reached Mumbai on the 3rd of June I had no idea who I was going to study under. On the morning of 4th June after registering at the office I come to know I am going to study under the guidance of Prof Yogish Holla. By the afternoon I got to meet Yogish in his office, and we came to an agreement on what to read in a short period of one month. I was amazed by his dedication and interest towards mathematics. Some days we would discuss for hours together on a particular topic and still, Yogish would be ready for more mathematics. The whole energy in the campus can be very motivating, with seminars to people discussing on the corridors.

During the course of the month, roughly the first two weeks were spent on reading on lifting theorems, fundamental groups (just to get an idea of Algebraic Topology), De-Rahm cohomology, Sheaves and the last two week were spent on reading on sheaf cohomology, Riemann-Roch theorem for compact Riemann surfaces (and its proof using sheaves), Serre Duality and its applications to complex tori. The main highlight of the project was the Riemann-Roch theorem. In simple words, it says that one can determine the genus (the topology) of the surface based on the space of functions on the space.

Some of the other important highlights of the VSRP program include weekly talks by eminent scientist both from TIFR and outside specifically designed for the VSRP students. These talks covered a various multitude of areas in mathematics right from number theory, combinatorics to hyperbolic geometry. The seminars also gave us the opportunity to interact with various faculty at TIFR. One of the ways this internship stood out for me was in terms of the peers. A fixed one month program ensured that all students with similar interests were at TIFR at the same time. I got to interact with various students from diverse colleges but with one interest, Mathematics. In fact, we had a seminar on the last day, where each student was given 15 minutes to present anything interesting related to his project.

One of the spotlights of the TIFR campus is the sea. Even from our rooms we had a view of the sea. An evening walk down the seaside with a cool breeze hitting you is very refreshing. With waves hitting the large rocks during the high tide to the view of sunset to view of the marine drive in the night, the sea always had something to offer. On this opportunity, I would also like to thank Prof Parameswaran and Prof Sandeep Verma, who were the coordinators of the VSRP program and enthusiastically arranged all the talks and the seminar. This experience has been key in making me explore deeper waters in mathematics.

FIELDS MEDAL LECTURE SERIES

The Fields Medal is awarded to young mathematicians (under 40 years of age) for outstanding contributions to their field, at the International Congress of the International Mathematical Union (IMU) - a meeting that takes place every four years. In August 2018, four mathematicians were honoured with the prestigious award - Akshay Venkatesh, Peter Scholze, Alessio Figalli and Caucher Birkar.

Here at IISER Pune, we are anxious to know about any developments of this kind, in great detail. Prof A. Raghuram came up with the idea of having a talk on Akshay Venkatesh, the only Medallist of 2018 of Indian origin, his life and work. Soon, this blossomed into a four-part weekly talk series, one on each mathematician. Each was given by a different professor from IISER Pune, and detailed briefly the work of the mathematicians and led up to their winning contributions.

The following articles summarize these talks, keeping out too much mathematical rigour while retaining a flavour of what was spoken about.

AKSHAY VENKATESH

RITWICK KUMAR GHOSH

This was the first lecture in the four-part Fields Medal series, given by Professor A. Raghuram. The Fields Medallist being discussed was Australian mathematician Akshay Venkatesh, professor at Institute for Advance Study, Princeton. The medal recognises his synthesis of analytic number theory, homogeneous dynamics, topology and representation theory. Amongst many of his research interests, two topics were discussed in the talk.

1. Sub-convexity of L-functions

The first topic was the sub-convexity of L-functions. An L-function is a kind of function on a complex plane which may be obtained by analytic continuation of an L-series (a kind of Dirichlet series usually convergent on a half-plane). The sub-convexity problem is concerned with the size of an L-function, its upper bound being restricted along the critical strip [0,1]. For simplification, the Riemann Zeta function was chosen as an L-function. Various interesting properties of the Riemann Zeta function were touched upon, for example, the functional equation for analytic continuation of the Riemann Zeta Function, the connection of distribution of the primes with the behaviour of the Zeta function on Re(s)=1/2 and something called Lindelof Hypothesis for the Zeta function. In this context Prof. Raghuram also mentioned that Lindelof hypothesis is actually an implication of the famous Riemann hypothesis. An interesting concept called Equidistribution technique was touched upon in the discussion of the resolution of the subconvexity problem of L-functions by Prof. Akshay Venkatesh.

2. Cohomology of arithmetic groups

Cohomology is a sequence of abelian groups associated to a topological space. An arithmetic group is a group obtained as the integer points of an algebraic group, for example $SL_2(\mathrm{Z})$. The study of this topic is linked to automorphic forms and harmonic analysis. They are linked to the area of dynamics by hyperbolic tilings, which involves studying the path of a ball as it travels on a hyperbolic table. This relates geometric and dynamical questions with number theory.

PETER SCHOLZE AND HIS WORLD OF PERFECTOID SPACES

IPSA BEZBARUA

Natural numbers, integers, rational numbers and finally real numbers Dr Debargha Banerjee began his talk on the 17th of January 2019 with a mention of these well-known fields in mathematics. Archimedes was the first to study such algebraic structures in the quest for some compatibility between algebra and topology. Around 1890, Kurt Hensel unveiled the idea of p-adic numbers. The ultimate aim of such studies was in solving polynomial equations of various forms. Galois was the first to study polynomials, with the Galois theory providing a link between group theory and field theory.

Peter Scholzes contribution was that through the introduction of perfectoid spaces, he provided a way to move from the real-like world to the p-adic world via the tilt and untilt operations, effortlessly connecting the two.

Real $< - -_{U} -^{T}_{N} -^{I}_{T} -^{L}_{I} -^{T}_{L} -_{T} - - >$ p-adic

This novel concept had a profound application it yielded a solution to the weight-monodromy conjecture. It also influenced the study of finite dimensional vector spaces and Weil numbers. On the 25 th of December 2018, Scholze showed that elliptic curves

$E : y^2 = x^3 + a_1x^2 + a_2x + a_3, a_i \epsilon Q[i]$

could be reduced to the simple $f(x)dx$ form. Dr Banerjee then presented some slides with short biographies of Scholze, E Galois, Kurt Hensel, Alexander Grothendieck, Pierre Delign, Michael Rapoport and Jean-Marc Fontaine, before concluding his talk with an interview of Scholze at the Centre International de Rencontres Mathmatiques (CIRM).

ALESSIO FIGALLI

MEGHA BHAT

The third lecture in the Fields Medal series was Dr. Anup Biswas talking about Italian mathematician Alessio Figalli. The medal has been awarded to Dr. Figalli for his collective work in the theory of optimal transport, and its application to partial differential equations, metric geometry, and probability.

1. Optimal Transport Problem

According to the Brunn-Minkowski inequality,

$$(vol(A+B))^{\frac{1}{d}} \geq (vol(A))^{\frac{1}{d}} + (vol(B))^{\frac{1}{d}} \tag{1}$$

where A,B $\subset \mathbb{R}^d$ are Borel sets and A + B = $\{x + y \mid x\epsilon$ A, y ϵ B$\}$. Volume of a set refers to its measure, or size. This inequality becomes an equality if A and B are homothetic (one is scaled version of the other).
Suppose we find two Borel sets A and B of equal measure such that upto translation,

$$(vol(A+B))^{\frac{1}{d}} \leq (vol(A))^{\frac{1}{d}} + (vol(B))^{\frac{1}{d}} + \delta \tag{2}$$

for some real δ. One of the problems in transportation theory is to estimate how close A and B are to being convex. Another is to estimate vol(A Δ B). Figalli solved the second of the two problems, and found that vol(A Δ B) $\leq \sqrt{\delta}$.($A\Delta$ B represents the set A $\cup B - A \cap B$.) This means that A and B are nearly the same, upto translation.

2. Free boundary problems and the obstacle problem

A free boundary problem is a partial differential equation to be solved for an unknown function u and an unknown domain Ω. An example is the **Stefan problem**, a free boundary problem that tries to describe a phase boundary moving with time, like the temperature distribution of a melting iceberg floating in water.
The obstacle problem or optimal stopping problem is to find the equilibrium position of a rubber sheet with a fixed boundary, with two fluids of different concentrations on either side. The function u(x) dercribing the sheet is studied for continuity and differentiability. Niremberg and Kinderlehrer proved that if the contact set of u is once differentiable, it is infinitely differentiable. Luis Caffarelli, an Argentine mathematician, proved that the contact set consists exclusively of regular points and singular points. The latter was problem for mathematicians as the function u could not be studied at those points. Figalli and Serra solved this problem.

3. Monge-Ampere equation

In 1781, Gaspard Monge proposed the optimal transport problem, which can be stated as follows: given two distributions with equal masses of a given material $g_0(x), g_1(x)$ (corresponding to an embankment an excavation), find a transport map ψ that carries the first distribution to the second and minimises the transport cost.

Leonid Kantorovich generalised this problem in 1948 for a general metric, and his work in the theory won him the Nobel Prize for Economic Sciences. Brenier and Caffarelli worked on the solution to related Monge-Ampere equation, which Figalli later generalised. This problem is now applied in many different areas including urban planning, imaging and meteorology.

CAUCHER BIRKAR

ANAND TADIPATRI

The last talk in the four-part lecture series, given by Dr. Amit Hogadi, was on the work of the Kurdish mathematician Caucher Birkar. According to the International Mathematical Union (IMU), Caucher Birkar was awarded the Fields Medal in 2018 for the proof of the boundedness of Fano varieties and for contributions to the minimal model program. The Minimal Model Program (MMP), a subject in algebraic geometry, was the focus of the talk.

In algebraic geometry, the basic object of study is an algebraic variety. Algebraic varieties are the set of common solutions to a collection of polynomials, usually defined over an algebraically closed field like the complex numbers. The conic sections are a familiar example of algebraic varieties. There are a plethora of possible geometric structures that these algebraic varieties may possess. However, amidst this diversity, there are similarities that some varieties share. The Minimal Model Program aims to classify algebraic varieties based on certain similarities.

The cumulative work of several mathematicians, including a few Fields medallists, provided the tools necessary for classification of varieties. The number of holes (or handles) in the geometric structure provided a natural criterion for classification. For two- dimensional varieties, Italian mathematicians developed methods that involved blowing up certain regions of the structure to form more complex structures. For instance, a structure in which a point has been replaced by a line is isomorphic to the original structure. Conversely, a structure could be collapsed or blown down into a simpler structure. The two structures belong to the same birational class, meaning that they are essentially the same, apart from a few differences in some subsets. Repeatedly blowing down a structure eventually results in a minimal surface, in which no line can be collapsed. Algebraic varieties are classified based on the minimal surfaces they form when collapsed. The concept of blowing up and blowing down was extended to higher dimensions, but this often resulted in singularities (these can be thought of as points where the surface crosses itself). To avoid singularities, a tool called the flip had to be used - this takes a portion of the surface and replaces it with another curve.

In 2006, Caucher Birkar, in collaboration with three other mathematicians, published a paper now referred to as BCHM, after the authors of the paper: Caucher Birkar, Paolo Cascini, Christopher Hacon, and James McKernan. One of the results proved in this paper was the existence of flips in all dimensions. Consequently, this proved the existence of minimal models for varieties of the general type. The work of Caucher Birkar had an immense impact on the field, and has opened up new areas of research.