Pricing and Hedging in a GBM market with Markov switching: A survey

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Declaration

This is to certify that this thesis entitled Pricing and Hedging in a GBM market with Markov switching: A survey submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune, represents work carried out by Jeeten Patel under the supervision of Dr. Anindya Goswami.

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Reader 1

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This thesis is dedicated to parents and my sir Dr. Anindya Goswami.

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Abstract

This current thesis aims to survey recent development on certain problems in Mathematical Finance. The geometric Brownian motion model for stock price was first proposed by the renowned economist Samuelson in 1965. Later in 1973 Black, Scholes and Merton used that model to find a formula for price of European options. This work commences the application of Stochastic calculus in the research field of quantitative finance. But this model assumes that the basic market parameters, namely, growth rate, volatility and bank interest rate remain constant during the entire period of the option. Numerical data from actual market does not support these assumptions. To overcome these drawbacks, several alternative models are still being proposed in the literature and thereby new mathematical challenges are arising. In recent years a large amount of research is being carried out by considering the market parameters as Markov chains which evolve according to a prescribed transition rate. Markov modulated GBM model is one of that kind. This model can be regarded as straight forward generalization of B-S-M (Black, Scholes and Merton) model. Although such market is proved to have no arbitrage, but the cost paid for this generalization includes features like incompleteness of market, lack of analytic solution, non-uniqueness of option pricing etc. Nevertheless, consideration of the above model opens up a wide range of research topics. The existing literature, those assume above model and related to locally risk minimizing pricing, optimal hedging, portfolio optimization with risk sensitive cost, stability of numerical solutions of associated PDEs, computation of complexity of numerical schemes etc. are thoroughly being surveyed in this current project. Besides, a number of numerical experiments are carried out based on the theoretical results. During thorough study of Springer lecture note on Introduction to stochastic Calculus for Finance" by Dieter Sondermann, as part of prerequisite, a list of errata along with few corrections/suggestion is prepared and enclosed to this thesis.

Contents

Chapter 1

Preliminaries

1.1 Stochastic processes

A *stochastic process* is a family of random variables $\{X(t)|t \in T\}$ defined on a given probability space, indexed by the time variable t, where t varies over index set T. We interpret t as time and call $X(t)$ as the state of process at time t. If index T is a countable set then X a discrete-time stochastic process, and if T is continuum it is a continuous-time process. Any realization of X is called sample path. We are hereby describing those which are relevant for the present thesis.

Consider a discrete time stochastic process $\{X_n, n = 0, 1, 2, ...\}$ with a finite number of states $S = \{1, 2, 3\}$. The dynamics of the process is as follows. You move from state 1 to state 2 with probability 1. From state 3 you move either to 1 or to 2 with equal probability $\frac{1}{2}$, and from 2 you jump to 3 with probability $\frac{1}{3}$, otherwise stay at

1.1.1 Martingale

A stochastic process M is a *martingale* with respect to the filtration $\{\mathcal{F}_t\}$ if it satisfies the following two conditions:

1. M is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$; that is, for every t, $M(t)$ is $\mathcal{F}_t$ measurable.
2. For all $0 < s < t$ we have $E[M(t) \mid \mathcal{F}_s] = M(s)$.

1.1.2 Discrete-time Markov chain

Definition 1.1.1. A *Markov chain* is a sequence of random variables $X_1, X_2, X_3, ...$ with the Markov property,

$$P(X_{n+1} = x | X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P(X_{n+1} = x | X_n = x_n).$$

The possible values of X_i form a countable set S called the state space of the chain.

Example 1.1.1. A famous Markov chain is the so-called simple symmetric random walk, a random walk on one dimension , at each step, the position may change by +1 or 1 with equal probability. From any position there are two possible transitions, to the next or previous integer. The transition probabilities depend only on the current position, not on the manner in which the position was reached. For example, the transition probabilities from 5 to 4 and 5 to 6 are both 1/2, and all other transition probabilities from 5 are 0. These probabilities are independent of whether the system was previously in 4 or 6.

Bernoulli process

A *Bernoulli process* is a finite or infinite sequence of independent random variables $X_1, X_2, X_3, ...,$ such that

1. For each i, the value of X_i is either 0 or 1;
2. For all values of i, the probability that $X_i = 1$ is the same number p.
3. Independence of the trials implies that the process is memoryless.

Interpretations

Several random variables and probability distributions beside the Bernoullis may be derived from the Bernoulli process:

1. The number of successes in the first n trials, which has a binomial distribution B(n, p)
2. The number of trials needed to get r successes, which has a negative binomial distribution NB(r, p)
3. The number of trials needed to get one success, which has a geometric distribution NB(1, p), a special case of the negative binomial distribution.

The negative binomial variables may be interpreted as random waiting times.

1.1.3 Continuous-time Markov chain

The process $\{X(t), t \geq 0\}$ is a *continuous-time Markov chain* with discrete state space if for all $s, t \geq 0$, and non negative integers $i, j, x(u), 0 \leq u \leq s$,

$$P(X(t+s) = j | X(s) = i, X(u) = x(u), 0 \leq u < s) = P(X(t+s) = j | X(s) = i).$$

In other words, a continuous-time Markov chain is a stochastic process having Markovian property that the conditional distribution of the future state at time $t+s$, given the present state at s and all the past states depends only on the present state. If $P(X(t+s) = j|X(s) = i)$ is independent of s, then the continuous-time Markov chain is said to be time homogeneous.

Birth and death process

The birth-death process is a special case of continuous-time Markov process where the state transitions are of only two types: "births" which increase the state variable by one and "deaths" which decrease the state by one. The model's name comes from a common application, the use of such models to represent the current size of a population where the transitions are literal births and deaths. It is continuous-time Markov process for which transitions from state i only happens to either state $i-1$ or state $i+1$. The state of the process can be thought as representing the size of some population so when birth occurs the state increases by 1 and decrease by 1 when death occurs. Let $\{\lambda_i, i \geq 0\}$ be birth rates and $\{\mu_i, i \geq 0\}$ be death rates and as follows:

$$\lambda_i = q_{i,i+1}$$

$$\mu_i = q_{i,i-1}$$

Since $\Sigma_j q_{ij} = v_i$,we see that

$$v_i = \lambda_i + \mu_i,$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} = 1 - P_{i,i-1}$$

Hence we can think when there are i people in the system the time until the next birth is exponential with rate λ_i and is independent of the time until the next death, which is exponential with μ_i.

1.1.4 Brownian Motion With Drift

Brownian Motion

In 1827 the biologist Robert Brown, looking through a microscope at pollen grains in water, noted that the grains moved through the water but was not able to determine the mechanisms that caused this motion. The direction of the force of atomic bombardment is constantly changing, and at different times the pollen grain is hit more on one side than another, leading to the seemingly random nature of the motion. This transport phenomena is named after Robert Brown.

A stochastic process $[X(t), t \geq 0]$ is said to be *Brownian motion* process if:

1. $X(0) = 0$;
2. $\{X(t), t \geq 0\}$ has stationary independent increments;
3. For every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance c^2t.

The Brownian motion process, often called the Wiener process, is one of the most useful stochastic processes in applied probability theory. When c=1, the process is called standard Brownian motion. The mathematical model of Brownian motion has several real-world applications.
Example-

1. Modelling stock prices.
2. Thermodynamics

Brownian Motion With Drift

$\{X(t), t \geq 0\}$ is a *Brownian motion process with drift* coefficient μ :

1. $X(0) = 0$;
2. $\{X(t), t \geq 0\}$ has stationary and independent increments;
3. $X(t)$ is normally distributed with mean μt and variance c^2t.

$X(t) = cB(t) + \mu t$, where $B(t)$ is standard Brownian motion. A Brownian motion with drift is a process that tends to drift off at a rate μ.

1.1.5 Geometric Brownian Motion

A geometric Brownian motion(GBM) also known as exponential Brownian motion is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (also called a Wiener process) with drift. It is an important example of stochastic processes satisfying a stochastic differential equation (SDE), it is used in mathematical finance to model stock prices in the BlackScholes model. If $\{X(t), t \geq 0\}$ is Brownian motion,then the process $\{Y(t), t \geq 0\}$, defined by

$$Y(t) = e^{X(t)}$$

is called *Geometric Brownian motion.*
Now if $X(t)$ is normal with mean 0 and variance t, its moment generating function is given as follows:

$$E[e^{Y(t)}] = e^{\frac{t}{2}}$$

$$Var(Y(t)) = e^{2t} - e^{t}$$

Geometric Brownian motion is useful in modeling when the percentage changes are independent and identically distributed. Suppose that $Y(n)$ is the price of some commodity at time n

$$X_n = Y(n)/Y(n-1)$$

then, taking $Y(0) = 1$,

$$Y(n) = X_1 X_2 ... X_n$$

and so

$$\log Y(n) = \sum_{i=1}^{n} \log X_i,$$

Since the X_i are independent and identically distributed, log $Y(n)$, when normalized, would be approx Brownian motion, and $\{Y(n)\}$ would be approx geometric Brownian motion. A stochastic process S_t is said to follow a GBM if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

where W_t is a Wiener process or Brownian motion and μ (the percentage drift) and σ (the percentage volatility) are constants.

The Lévy characterization of Brownian motion

Let $X(t) = (X_1(t), ..., X_n(t))$ be a continuous stochastic process on a probability space $(\omega, \mathcal{H}, Q)$ with values in $\mathbb{R}^n$. Then the following are equivalent:

1. X(t) is a standard Brownian motion w.r.t Q, i.e. the law of X(t) w.r.t. Q is the same as the law of an n-dimensional Brownian motion.

2. (a) $X(t) = (X_1(t), ..., X_n(t))$ is a martingale w.r.t. Q (and w.r.t. its own filtration) and
   (b) $X_i(t)X_j(t) - \delta_{i,j}t$ is a martingale w.r.t. Q (and w.r.t. its own filtration) $\forall i, j \in \{1, 2, ..., n\}$.

1.1.6 Stochastic volatility model

Stochastic volatility models are one approach to resolve a shortcoming of the Black-Scholes model which assume that the underlying volatility is constant over the life of the derivative, and unaffected by the changes in the price level of the underlying security. By assuming that the volatility of the underlying price is a stochastic process rather than a constant, it becomes possible to model derivatives more accurately. Realised volatility of traded assets displays signicant variability. Any model used for the hedging of derivative contracts on such assets should take into account that volatility is subject to fluctuations.

In constant volatility approach, the derivative's underlying asset price follows standard model for GBM. For stochastic model just replace constant volatility σ with function ν_t that models the variance of S_t . This variance function is also modeled as brownian motion.

$$dS_t = \mu S_t\, dt + \sqrt{\nu_t} S_t\, dW_t$$

$$d\nu_t = \alpha_{S,t}\, dt + \beta_{S,t}\, dB_t$$

where $\alpha_{S,t}$, and $\beta_{S,t}$, are some functions of ν, and S,

1.1.7 Lévy process

Lévy process, named after the French mathematician Paul Lévy, is a stochastic process with independent, stationary increments. A Lévy process can be viewed as the continuous-time analog of a random walk.
Let $X = (X(t), t \geq 0)$ be a stochastic process defined on a probability space $(\omega, \mathcal{F}, P)$. We say that it has independent increments if for each $0 \leq t_1 < t_2 < \cdots < t_{n+1} < \infty$ the random variables $(X(t_{j+1}) - X(t_j), 1 \leq j \leq n)$ are independent and that it has stationary increments if each $X(t_{j+1}) - X(t_j) = X(t_{j+1} - t_j) - X(0)$.
We say that X is a *Lévy process* if it satisfies the following properties:

1. $X(0) = 0$ a.s;
2. X has independent and stationary increments;
3. X is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$;

$$\lim_{t \to s} P(|X(t) - X(s)| > a) = 0.$$

Examples of Lévy processes are Brownian motion, Gaussian processes and the Poisson process. Except Brownian motion, all other Lévy processes have discontinuous paths.

1.1.8 Geometric Lévy process

The geometric Lévy process model is one of improvement in BS model. This model is an incomplete market model, so there are many equivalent martingale measures such as the Esscher martingale measure, the variance optimal martingale measure, the minimal entropy martingale measure etc.

Suppose that a probability space (Ω, F, P) and a filtration $\{F_t, 0 \leq t \leq T\}$ are given. A geometric Lévy precess (GLP) is given by

$$S_t = S_0 e^{Z_t}$$

where Z_t is a Lévy process. We call such a process S_t the geometric Lévy process(GLP).

1.1.9 Markov Modulated GBM

$$dS_t = S_t(\mu(X_t)dt + \sigma(X_t)dW_t)$$

Now Markov modulated GBM is It$\hat{o}$ process.
Take $\mathcal{F}_t$ is a filtration generated by X_t and W_t. Since X_t and W_t are independent. So, W_t remains martingale and S_t is adapted to $\mathcal{F}_t$.

1.1.10 Semi Markov Modulated GBM

X_t is a semi markov process(not Markov) and independent to W_t.
then

$$dS_t = S_t(\mu(X_t)dt + \sigma(X_t)dW_t), \quad t \geq 0.$$

with $S_0 > 0$ is not Lévy process but it is It$\hat{o}$'s process.

Semi-Markov process

A semi-Markov process is one that changes state in accordance with a Markov chain but takes a random amount of time between changes. Consider a stochastic process with states $0, 1, \ldots,$ which whenever enters state $i, i \geq 0$ and follows :

1. The next state it will enter is state j with probability $P_{ij}, i, j \geq 0$.
2. Given that the next state to be entered is state j, the time until the transition from i to j occurs has distributions F_{ij}.

If $Z(t)$ denote the state at time t, then $\{Z(t), t \geq 0\}$ is called a *semi Markov process.* But it doesn't posses the Markovian property. For it prediction the future, it not only requires present state, but also the length of the time that has been spent in that state. A Markov chain is a semi-Markov process in which

$$F_{ij}(t) = \begin{cases} 0, & t < 1 \\ 1, & t \geq 1 \end{cases} \text{ i.e. all transition times of a Markov chain are identically 1.}$$

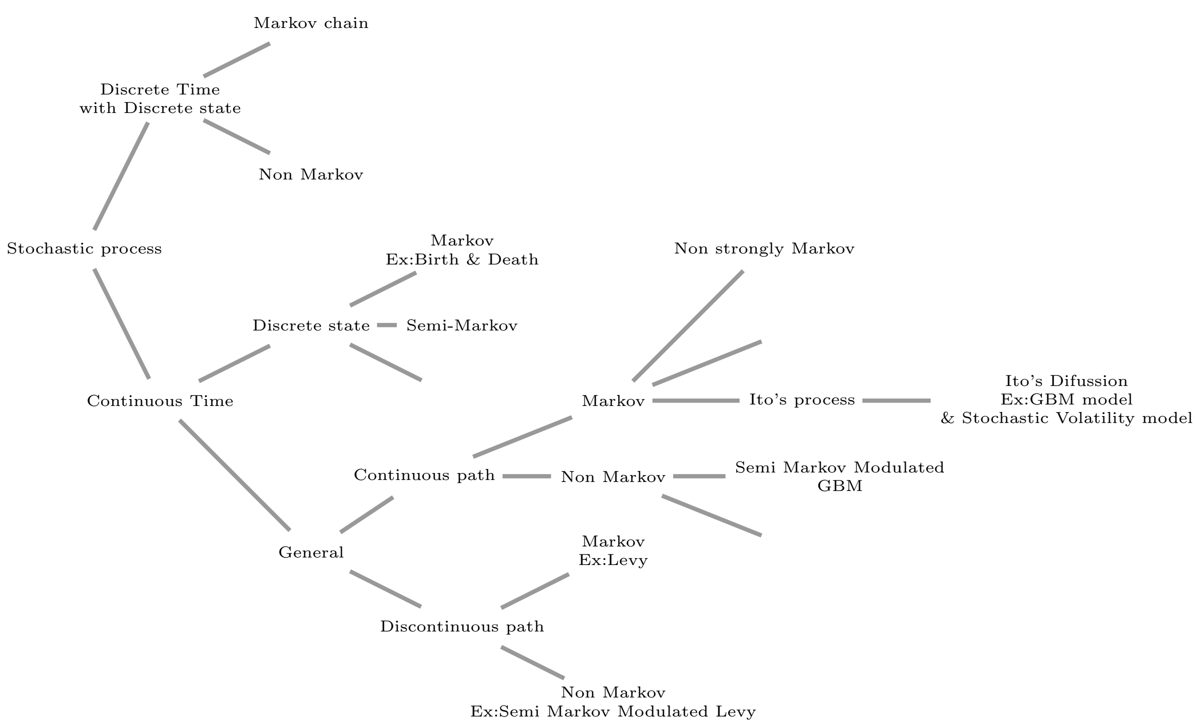

Figure 1.1: Classification Tree of stochastic processes

1.2 Properties of Markov Processes and its Generators

Markov process

Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a probability space, S be a complete separable metric space and $S =$ S-valued stochastic process $X = (X_t,\ t \in T)$ adapted to the filtration is said to possess the Markov property with respect to the $\{\mathcal{F}_t\}$ if, for each $A \in \mathcal{S}$ and each $s, t \in T$ with $s < t$,

$$\mathbb{P}(X_t \in A|\mathcal{F}_s) = \mathbb{P}(X_t \in A|X_s).$$

A Markov process is a stochastic process which satisfies the Markov property with respect to its natural filtration.

Strong Markov property

Suppose that $X = (X_t : t \geq 0)$ is a stochastic process on a probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Then X is said to have the *strong Markov property* if, for each stopping time τ, conditioned on the event $\{\tau < \infty\}$, the process $X_{\tau+}$ (which maybe needs to be defined) is independent from $\mathcal{F}_\tau := \{A \in \mathcal{F} : \tau \cap A \in \mathcal{F}_t,\ \ t \geq 0\}$ and $X_{\tau+t} - X_\tau$ has the same distribution as X_t for each $t \geq 0$.

The strong Markov property is a stronger property than the ordinary Markov property, since by taking the stopping time $\tau = t$, the ordinary Markov property can be deduced. It can be formulated as follows.

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}[f(X_t)|\sigma(X_s)]$$

for all $t \geq s \geq 0$ and $f : S^n \to \mathbb{R}$ bounded and measurable.

X_t be a real valued (Markov) Stochastic process with continuous path. Let $A : V \longrightarrow V_0$ be an operator where V and V_0 are Banach spaces, if A is generator of X_t then

$$f(X_t) - \int_0^t Af(X_s)ds$$

is a martingale for all $f \in$ Dom (A).

Banach spaces

A normed linear space X is said to be a Banach space if for every Cauchy sequence $\{x_n\}_{n=1}^{\infty} \subset X$ there exists an element x in X such that $\lim_{n\to\infty} x_n = x$.

Motivation using a Deterministic process

Let $\{S_t\}_{t\geq 0}$ be a family of operators from $\mathcal{C}^1(\mathbb{R})$ to $\mathcal{C}(\mathbb{R})$ such that

$$S_t f(x_0) = f(x_t), \quad S_0 f(x) = f(x) \quad \forall f \in \mathcal{C}^1(\mathbb{R})$$

$\{S_t\}_{t\geq 0}$ forms a semigroup of operators.

Roughly speaking, one can write $S_t = c^{tA}$ then A is generator of X_t. We illustrate this with an example.

Motivation

Let $x_t = a + ct$ be a deterministic process where a & c are real constants and $t \geq 0$. Using the fundamental theorem of calculus since x_t is of B.V. and chain rule.

$$\begin{aligned} f(x(t)) - f(x(0)) &= \int_0^t f'(x(s))x'(s)ds \\ &= \int_0^t c\frac{d}{dx}f(x(s))ds \quad \forall f \in c'(\mathbb{R}) \\ f(x(t)) - \int_0^t c\frac{d}{dx}f(x(s))ds = f(x_0). \end{aligned}$$

a constant, i.e., a Martingale. $\forall f \in \text{Dom}(A) = C^1(\mathbb{R})$

Hence, $A := c\frac{d}{dx}$ is differential generator of $x_t = a + ct_0$.

Again,

$$\begin{aligned} x_t &= a + ct \\ f(x_t) &= f(a + ct) \\ &= f(a) + f'(a)ct + \frac{1}{2}f''(a)c^2t^2 + \cdots \\ &= (I + ctD + \frac{ct^2}{2!}D^2 + \cdots)f(x)|_a \\ &= e^{ctD}f\ |_{x=x_0} \quad \forall f \in c'(\mathbb{R}) \\ i.e\ S_t &= e^{tcD}, \quad \text{where } D = \frac{d}{dx}. \end{aligned}$$

1.2.1 Markov Semi group and their generators

Consider the matrix Λ which satisfies $\lambda_{ij} := \lim_{h\to 0} \frac{P(X_{t+j}=j|X_t=i)}{h}$ for $j \neq i$

$$P(X_{t+h} = j|X_t = i) = \lambda_{ij}h + o(h)$$
$$\text{and} \quad \sum_{j\neq i} \lambda_{ij} = -\lambda_{ii}.$$

If t=0, then

$$P(X_h = j|X_0 = i) = \begin{cases} \lambda_{ij}h + o(h), & if \;\; i \neq j \\ 1 + \lambda_{ii}h + o(h) \end{cases}$$
$$= (\Lambda h + 1).$$

Let us define the matrix P whose (i, j)th entry gives the probability of one step transition to state j from state i.

$$\begin{aligned} P_{ij} &= P(\{X_t = j\} \cap \{X_s = i, \forall s < t\} \text{ for some } t > 0 \mid X_0 = i) \\ &= \lim_{h\to 0} \sum_{n=0}^{\infty} P(X_{(n+1)h} = j \mid \{X_s = i, \; \forall s \leq nh\} \cap \{X_0 = i\}) P(X_s = i, \; \forall s \leq nh|X_0 = i) \\ &= \lim_{h\to 0} \sum_{n=0}^{\infty} (\lambda_{ij}h + o(h))(1 + \lambda_{ii}h + o(h))^n \\ &= \lim_{h\to 0} (\lambda_{ij}h + o(h)) \frac{1}{1 - (1 + \lambda_{ii}h + o(h))} \qquad [\text{as } 0 < 1 + \lambda_{ii}h + o(h) < 1 \text{ for small h.}] \\ &= \lim_{h\to 0} \frac{\lambda_{ij}h + o(h)}{-\lambda_{ii}h + o(h)} \\ P_{ij} &= \frac{\lambda_{ij}}{-\lambda_{ii}} \end{aligned}$$

then

$$P = I - \text{Diag}\left(\frac{1}{\lambda_i}\right)\Lambda$$
$$\Lambda = \text{Diag}(\lambda_i)(I - P).$$

Markov semigroup

The family $(S(t))_{t\leq 0}$ is called *Markov semigroup* associated to the process $(P_x)_{x\in E}$ which satisfies following conditions:

1. $S(t) : B(E) \to B(E)$ is a bounded linear operator for any $t \geq 0$ and $||S(t)\varphi||_\infty \geq ||\varphi||_\infty$ for any $\varphi \in B(E), t \geq 0$ (that is $||S(t)|| \leq 1$ for any $t, r \geq 0$).
2. $S(0) = \mathbb{I}$.
3. $S(t + r) = S(t)S(r)$, for any $t, r \geq 0$.
4. $S(t)\varphi \geq 0$ a.e. in particular, if $\varphi \leq \psi$ a.e., then $S(t)\varphi \leq S(t)\psi$ a.e..
5. $S(t)1 = 1$ a.e. (here 1 is the function constantly equal to 1).

where, B(E) is the algebra of Borel sets of E.
Now let us define $\mathbb{P}(t)$ which gives the following transition probabilities:

$$\begin{aligned}\mathbb{P}_{ij}(t) &= P(X_t = j|X_0 = i)\\ &= \lim_{n\to\infty_{nh=t}} P(X_{nh} = j|X_s = i)\\ &= \lim_{n\to\infty_{nh=t}} (\Lambda h + I)^n(i,j)\\ &= \lim_{n\to\infty_{nh=t}} \left(I + \frac{t\Lambda}{n}\right)^n (i,j)\\ &= e^{t\Lambda}(i,j)\\ &= \mathbb{P}^t(i,j)(say).\end{aligned}$$

Here $\mathbb{P}$ is Markov semi group which is equal to e^{Λ}.

We see that it is worth mentioning that

$$P \neq \mathbb{P} := e^{\Lambda}$$

which is evident from above.

Actually,
$\mathbb{P}_{ij}$:= Probability of being at j at $t = 1$ given if was at i initially.
P_{ij} = Probability that the next state would be j given it was at i.
$\mathbb{P}_{ii}$ need not be equal to $0 = P_{ii} \quad \forall i$.

1.2.2 Examples

The differential generator A for standard n-dimensional Brownian motion

is given by

$$Af(x) = \frac{1}{2}\sum_{i,j}\delta_{ij}\frac{\partial^2 f}{\partial x_i\,\partial x_j}(x) = \frac{1}{2}\sum_i \frac{\partial^2 f}{\partial x_i^2}(x),$$

i.e., $A = \frac{1}{2}\Delta$, where Δ denotes the Laplace operator.

Itô Diffusion

Stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t,\ X_0 = x_0$$

where $X_t \in \mathbb{R}^n, b(t,x) \in \mathbb{R}^n, \sigma \in \mathbb{R}^{(n\times m)}$ and B_t is m-dimensional Brownian motion, b is the drift coefficient and σ is the diffusion coefficient.
Let X_t be an *Itô* diffusions in $\mathbb{R}^n$. The *generator* A of X_t is obtained by

$$Af(x) = \lim_{t\downarrow 0} \frac{E^x[f(X_t) - f(x)]}{t}; \quad x \in R_n,$$

The set of functions $f : R^n \longrightarrow R$ such that the limit exists at x is denoted by $D_A(x)$. For any compactly-supported C^2 (twice differentiable with continuous second derivative) function f lies in $D_A(x)$ and that

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j}\left(\sigma(x)\sigma(x)^\top\right)_{i,j}\frac{\partial^2 f}{\partial x_i\,\partial x_j}(x).$$

1.2.3 Applications

FeynmanKac formula

The FeynmanKac formula, named after Richard Feynman and Mark Kac, establishes a link between parabolic partial differential equations (PDEs) and stochastic processes. It offers a method of solving certain PDEs by simulating random paths of a stochastic process.

We describe Feynman-Kac formula for Itô processes i.e. of the form

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t, \quad X_0 = x_0$$

with B_t a Brownian motion under the measure P.

Feynman-Kac considers the following problem-

Find a solution $f(x,t) \in C^{2,1}(\mathbb{R} \times [0,T])$ for the PDE

$$Af(x,t) - b(x,t)f(x,t) = 0 \quad (x,t) \in \mathbb{R} \times [0,T)$$

under the boundary condition

$$f(x,T) = g(x) \quad x \in \mathbb{R}$$

where

$$Af(x,t) = f_t(x,t) + f_x(x,t)b(x,t) + \frac{1}{2}\sigma^2(x,t)f_{xx}(x,t).$$

A is the so-called *infinitesimal generator* of X_t, gives the expected rate of change of $f(X_t, t)$, given $X_t = x$, i.e.

$$Af(x,t) = \lim_{\Delta \to 0} \frac{1}{\Delta}(E[f(X(t+\Delta), t+\Delta)|X_t = x] - f(x,t)).$$

From Itôs formula applied to $df(X_t, t)$ and the local martingale property of the Itô-integral one easily derives. then the FeynmanKac formula tells us that the solution can be written as a conditional expectation

$$f(x,t) = E_P\left[\exp\{-\int_t^T b(X_s, s)ds\}g(X_T) \mid X_t = x\right].$$

Chapter 2

Overview of Math Finance

2.1 Derivative Markets and instruments

2.1.1 Basic terminology in option pricing

- Option: An option is the right, but not the obligation to buy (or sell) an asset under specified terms.
- Holder of the Option: Holder has the right without any obligation.
- Writer of the Option: Writer has no right,but is obliged to the holder to fulfill the terms of the option.
- Call Option: It gives the holder the right to buy something from the writer.
- Put Option: It gives the holder the right to sell something from the writer.
- Asset: It can be anything, but we consider only stocks which will be referred to as primary security. An option is a derivative security.
- Strike or Exercise Price: A prescribed amount at which the underlying asset may be bought or sold by the holder.
- Expiration Date: It is a future time(date) after which the option becomes void.
- European Option: It is a contract with the following conditions: At a prescribed time in future (expiration date) the holder of the option may buy a prescribed asset for a prescribed price (strike price).
- American Option: It can be exercised any time before and including the expiration date.
- Short: Selling an asset without actually possessing it.
- Long: buy.

Definition 2.1.1. Derivative contract is a financial instrument with a return that is obtained from the return of another underlying financial instrument. These contracts are created on and traded in two distinct and some related types of markets:

1. **Exchange traded-** These have standard terms and futures and are traded on an organized derivatives trading facility, referred as futures exchange.
2. **Over the counter-** Any transactions created by two parties anywhere else.

Derivative contracts can be classified into two general types:

1. **Forward commitments**
   (a) **Forward contract** is an agreement between two parties in which one party, the buyer agrees to buy from another party,the seller, an underlying asset at a future date at a price established at initial. In some sense it is customized contact. In financial world take place in a large and private market consisting of banking firms, governments and corporations. Its include buy and sell a foreign currency or a commodity at a future date, locking in the exchange rate or commodity at a start.

   (b) **Futures contract** has same basic definition but some additional features that clearly distinguish it from a forward contract. It is public, standardized transaction that takes place on future exchange like stock exchange, which is an organization that provides a facility for engaging in futures transactions and establishes a mechanism through which parties can buy and sell these contracts. The futures exchange implements performance guarantee through an organization called the *clearinghouse* which writes itself in the middle of two parties so each party has a contract with the exchange and not with the other party. The exchange collects payment from one party and disburses payment to other. It also follows process referred as daily settlement or market to market. Where as for forward party only solid, credit worthy parties can generally engage in it.

      i. When the position is established, each party deposits a small amount of money, called the *margin*, with the clearing house.
      ii. When a transaction is initiated, a futures trader puts up a certain amount of money to the initial margin requirement; however this amount of money is not borrowed. It is more like a down payment for the commitments to purchase the underlying at a later date. The money helps the party fulfills his or her obligation. In securities, margin requirements are normally set by federal regulators. In US it is set by the securities exchanges and FINRA.
      iii. To provide a fair market process, the clearinghouse must designate the official price for determining daily gains and losses. This price is called the *settlement price* and represents an average of the final few traders of the day.
      Example: Consider a futures contract in which the current futures price

is \$82. The initial margin requirement is \$5, and the maintained margin requirement is \$2. You go long 20 contracts and meet all margin calls but do not withdraw any excess margin. Assume that on the first day, the contract is established at the settlement price, so there is no market-to-market gain or loss on that day.

Day	Beginning Balance	Funds Deposits	Futures Price	Price Change	Gain/Loss	Ending Balance
0	0	100	82	-	-	100
1	100	0	84	2	40	140
2	140	0	78	-6	-120	20
3	20	80	73	-5	-100	0
4	0	100	79	6	120	220
5	220	0	82	3	60	280
6	280	0	84	2	40	320

Table 2.1: Holder of long position of 20 contracts

On Day 0, you deposits \$100 because the initial margin requirement is \$5 per contract and you go long 20 contracts. At end of Day 2, the balance is down to \$20 below the \$40 maintenance margin requirement(\$2 $\times$ 20). You must deposit enough money to bring the balance up to the initial margin requirement of \$100(\$5 per contract times 20 contracts). So on Day 3, you deposit \$80. The price change on Day 3 causes a loss of \$100, leaving you with a balance of \$0 at end of Day 3. On Day 4, you must deposit \$100 to return the balance to the initial margin level. Moreover, a price decrease to \$79 would trigger a margin call. If future price starts at \$82, it can fall by \$3 to \$79 before it triggers a margin call.

(c) **Forward rate agreement(FRA)** is forward contract in which one party, the long(Buyer), agrees to pay a fixed interest payment at a futures date and receive an interest payment at a rate to be determined at expiration. FRA's are denoted by special notation 3$\times$ 6 FRA in three months; underlying is a Eurodollar deposit that begins in three months and ends three months later, or six months from now. These are forward on interest rates.

Eurodollar time deposits are dollar loans made by one bank to another. Eurodollar deposits accrue interest by adding it on the principal, using 360-day year assumption. The primary Eurodollar rate is called LIBOR(London Interbank Offer Rate). LIBOR, the rate at which London banks are willing to lend to other London banks. Euribor is the rate on a euro time deposit, a loan made by banks to other banks in Frankfurt in which the currency is the euro.

The FRA payoff formula:-
Let Notional principal= N; Underlying rate at expiration= R ; Forward contract rate= F ; Days in underlying rate= D.

$$N\left[\frac{(R-F)(\frac{R}{360})}{1+R(\frac{D}{360})}\right]$$

Example 2.1.1. The treasurer of company A expects to receive a cash inflows of \$15,000,000 in 90 days. The treasurer expects short-term interest rates to fall during the next 90 days. In order to hedge against this risk, the treasurer decides to use an FRA that expires in 90 days and is based on 90-day LIBOR. The FRA is quoted at 5 percent. At expiration, LIBOR is 4.5 percent. Assume that the notional principal on the contract is \$15,000,000.
Here, taking short(seller) position will hedge the interest rate risk for company A. The gain on the contract will offset the reduced interest rate that can be earned when rates fall. Moreover this 3×6 FRA.

$$\$15,000,000\left[\frac{(0.045-0.05)(90/360)}{1+0.045(90/360)}\right]=-\$18,541.41$$

(d) *Swap* is a variation of forward contract that is essentially equivalent to series of a series of forward contracts. It is an agreement between two parties to exchange a series of future cash flows, in other words, one party makes a single fixed payment and the other makes a single floating payment amounts to a forward contract. One party agrees to make known payment to the other hand and receive something unknown in return which depends on some underlying factors like interest rate, exchange rate,stock price, or commodity price.
Types of swap:-

i. In a **Currency swap**, each party makes payments to other in different currencies. A currency swap can have one party a fixed rate in one currency and the other pay a fixed rate in the other currency; have both pay a floating rate in their respective currencies; have the first party pay a fixed in one currency and second party pay a floating rate in other currency; or have the first party pay a floating rate in one currency in one currency and the second pay a fixed rate in other currency. (so in general fixed-fixed; floating-floating; floating-fixed; fixed-floating but all in their respective currencies.)

ii. An **interest rate swap** is that in it is a currency swap in which both currencies are the same. A **plain vanilla swap** is simply an interest rate swap in which one party pays a fixed rate and the other pays a floating rate, with both sets of payments in the same currency. Example Dollar-dominated plain vanilla swap if both currencies are in Dollar and Euro-dominated plain vanilla swap if both currencies in Euro.

iii. The **equity swap** is a swap in which at least one party pays the return on stock or stock index. Equity swaps are commonly used by asset managers.

2. **Contingent claims** are derivatives in which the payoffs occur if a specific event happens. We generally refer to these types of derivatives as options. An **option** is a financial instrument that gives a party the right, but not the obligation, to buy or sell an underlying asset from or to another party at fixed price over a specific period of time. An option that gives the right to buy is referred to as call; an option that gives the right to sell is referred to as put. Calls are **in-the-money** when the underlying value exceeds the exceeds the exercise price and puts are *in-the-money* when the exercise price exceeds the value of the underlying.
S_0, S_T = Price of the underlying asset at time 0 i.e today and time T i.e. expiration.
X = Exercise price
r = Risk-free rate
T = Time of expiration, equal to number of days to expiration divided by 365
c_0, c_T = Price of European call today and at expiration
C_0, C_T = Price of American call today and at expiration
p_0, p_0 = Price of European put today and at expiration
P_0, P_T = Price of American put today and at expiration
European option = It can be exercised only on its expiration day
American option = It can be exercised on any day through the expiration day.

Option	Min.	Max.
European call	$c_0 \geq 0$	$c_0 \leq S_0$
American call	$C_0 \geq 0$	$C_0 \leq S_0$
European put	$p_0 \geq 0$	$p_0 \leq X/(1+r)^T$
American put	$P_0 \geq 0$	$P_0 \leq X$

Table 2.2: Minimum and Maximum Values of Options.

Put-Call Parity is combinations of puts and calls or with a risk-free bond.
Fiduciary Calls and Protective Puts: A fiduciary call is a call plus a risk-free bond maturing on the option expiration day with a face value equal to the exercise price of the option, where as a protective put is the underlying plus a put. Basic put-call parity equation (A fiduciary call is equivalent to a protective put)is following:

$$c_0 + X/(1+r)^T = p_0 + S_0$$

Strategy	Consisting	Worth	=	Strategy	Consisting	Worth
Fiduciary call	Long call + Long bond	c_0 $+ X/(1+r)^T$	=	protective put Long underlying	Long put +	$p_0 + S_0$
Long call	Long call	c_0	=	Synthetic call	Long put+ short bond	$p_0 + S_0 - X/(1+r)^T$ + Long underlying
Long put	Long put	p_0	=	Synthetic put	Long call + Long bond + Short underlying	$c_0 - S_0 + X/(1+r)^T$
Long underlying	Long underlying	S_0	=	Synthetic underlying	Long call + Long bond + Short put	$c_0 + X/(1+r)^T - p_0$
Long bond	Long bond	$X/(1+r)^T$	=	Synthetic bond	Long put + Short call + Long underlying	$p_0 + S_0 - c_0$

Table 2.3: Alternative equivalent combinations of calls, puts, the underlying, and risk free bonds

2.2 Current Practice

Here we enlist some of the major research directions in mathematical finance.

- Advanced methods for pricing and hedging of derivative securities: models with jumps and stochastic volatility, asymptotic methods in option pricing, model calibration, valuation of long-term equity contracts and investment strategies, market with imperfections (proportional transaction costs, delta constraint)
- Stochastic analysis: Functional Ito calculus, path-dependent partial differential equations, Backward Stochastic Differential Equations, Malliavin calculus.
- Interest rate modeling: multi-factor models, multi-curve term structure models, impact of funding on interest rate derivatives.
- Systemic risk: network models of credit contagion, quantitative modeling of feedback effects, metrics for systemic risk, quantitative models of financial stability.
- Counterparty Credit risk, Collateral and Funding: Credit Value Adjustment (CVA), DVA, collateral requirements and their impact on pricing of derivatives, and consistent inclusion of funding costs (FVA); credit derivatives.
- Applications of Stochastic control in finance.
- Liquidity risk: models of price impact and liquidity risk, liquidity-adjusted risk measures, liquidity-based pricing models
- Market microstructure and high frequency modeling: mathematical modeling of limit order markets, statistical modeling of high frequency market data, consequences of high frequency trading for market stability and volatility.
- Rough Differential Equations.
- Numerical Methods for finance: probabilistic methods for non-linear PDE, numerical methods for BSDEs, model calibration.

2.3 Success and Recognition

Mathematical finance is a field of applied mathematics with financial markets as focus. Generally mathematical finance does not require establishing a link to financial theory, observed market prices are taken as input for deriving and extending the mathematical or numerical models. For example, while a financial economist might study the structural reasons for company having a certain share price, a financial mathematician may take the share price as a input, and attempt to use stochastic calculus to obtain the corresponding value of derivatives of the stock. The fundamental theorem of arbitrage-free pricing is one of the key theorems in mathematical finance, while the BlackScholes equation and formula are amongst the key results. Mathematical finance also overlaps heavily with the field of computational finance as well as financial engineering. Often by help of stochastic asset models, modeling and derivation are done (A quantitative analyst). In general, there exist two separate branches of finance that require advanced quantitative techniques: derivatives pricing and risk & portfolio management.

In financial economics, a financial institution is an institution that provides financial services for its clients or members. In modern times the mathematical finance found its unique place in academics. Imperial college London, King's College London Financial Mathematics, LSE Mathematics, Stanford University and Rutgers, The State University of New Jersey deserves to be mentioned among many others for their contribution to research and education in this field. The Mathematical Finance Section of the Department of Mathematics at Imperial College London, is devoted to research on mathematical modeling and computational methods in finance. The Departments of Mathematics and Statistics, in close cooperation with the Departments of Economics and Management, Science and Engineering and the Graduate School of Business, offer an Interdisciplinary Master of Science Degree in Financial Mathematics.

2.4 Dedicated journals

The growth of the subject can be also perceived by looking at the large spectrum of high ranked new journals dedicated to the subject. Needless to mention that SIAM Journal on Financial Mathematics, Quantitative Economics, Quantitative Finance, Annals of Finance, Applied Mathematical Finance, Decisions in Economics and Finance, Finance and Stochastics, International Journal of Theoretical and Applied Finance, Finance Journal of Computational Finance, The Journal of Finance, Mathematical Finance, Finance Mathematical Methods in Economics and Finance, Quantitative Finance, Statistics & Risk Modeling with Applications in Finance and Insurance, Econometric Theory, Econometrica etc are of above category.

2.5 Criticism by practitioners

Over the years, increasingly sophisticated mathematical models and derivative pricing strategies have been developed, but their credibility was damaged by the financial crisis of 2007 – 2010.

Benot Mandelbrot: In the 1960s he discovered that changes in prices do not follow a Gaussian distribution, but are rather modeled better by Lévy alpha-stable distributions. The scale of change, or volatility, depends on the length of the time interval to a power a bit more than 1/2. Large changes up or down are more likely than what one would calculate using a Gaussian distribution with an estimated standard deviation.

Nassim Nicholas Taleb: Contemporary practice of mathematical finance has been subjected to criticism from figures within the field notably by Nassim Nicholas Taleb [48], a professor of financial engineering at Polytechnic Institute of New York University, in his book The Black Swan and Paul Wilmott. Taleb claims that the prices of financial assets cannot be characterized by the simple models currently in use, rendering much of current practice at best irrelevant, and, at worst, dangerously misleading.

Wilmott and Emanuel Derman: In January 2008, they published the Financial Modelers' Manifesto which addresses some of the most serious concerns. Bodies such as the Institute for New Economic Thinking are now attempting to establish more effective theories and methods.

The Great Depression

The Great Depression was a severe worldwide economic depression in the decade preceding World War II. It was the longest, most widespread, and deepest depression of the 20th century.In the 21st century, the Great Depression is commonly used as an example of how far the world's economy can decline. The depression originated in the U.S., after the fall in stock prices that began around September 4, 1929, and became worldwide news with the stock market crash of October 29, 1929 (known as Black Tuesday).

The financial crisis of 2007 - 2008

The financial crisis of 2007 – 2008, which is also known as the global financial crisis and 2008 financial crisis, is considered by many economists to be the worst financial crisis since the Great Depression of the 1930s that resulted in the threat of total collapse of large financial institutions, the bailout of banks by national governments, and downturns in stock markets around the world. Including the housing market which was also suffered, resulting in evictions, foreclosures and prolonged unemployment. The crisis played a significant role in the failure of key businesses, declines in consumer wealth estimated in trillions of US dollars, and a downturn in economic activity leading to the $2008-2012$ global recession and contributing to the European sovereign-debt crisis.

Chapter 3

BSM Market model

3.1 History of Math-Finance

The geometric Browning motion model for stock price was first proposed by the renowned economist Samuelson in 1965 [52]. Before these work Bachelier used Brownian motion with drift to model stock prices in his Ph.D thesis in 1900 [2]. He was perhaps the first person who did mathematical formulation of Brownian motion. In his study Brownian motion was obtained as a limit of random walks. It was noticed after the work of Bachelier that the Brownian motion fails to model the stock price directly due to non-negative feature of stock prices. Then later in 1973 Black, Scholes and Merton [4], found formula for the price of European put and call options. This was the first work where Stochastic calculus is used in the research field of quantitative finance. Black-Scholes model, assumes that the growth rate, volatility and bank interest rate remain constant during the entire period of the option but these assumptions do not support market data. Therefore several alternative models are still being proposed in the literature to overcome these drawbacks which also arises new mathematical challenges. Some of the models include stochastic volatility models, jump-diffusions models, Lévy processes, regime-switching models etc. The market represented by these models are incomplete; that is, a perfect hedge is using self financing strategy is not possible for every.

In 1979 Harrison and Kreps [27], and in 1981 Harrison and Pliska [28], established that the absence of arbitrage of a given market is equivalent to the existence of an equivalent martingale measure under which all discounted price processes are martingales. In 1994 Gerber and Shiu [20], provided a solution to the option pricing problem in an incomplete market by using the Esscher transform. In 1986 Föllmer and Soundermann [18], in 1991 Föllmer and Schweizer [17], in 1991 Schweizer [53], identified a unique equivalent martingale measure by minimizing the quadratic utility of the losses due to imperfect hedging. Later on the minimal martingale measure and risk-minimizing hedging were further developed by several researchers. In 2000 Frey [19], carried out studies on a risk-minimizing strategy when the price process is a pure jump process with a stochastic jump rate and is a martingale under the empirical measure. In 1999 Chan [9], presented a locally risk-minimizing strategy when price proces is driven by general Lévy processes. In 2004 Takuji [58], calculated the density process of the minimal martingale measure and stated the relation to a locally risk-minimizing strategy in jump-diffusion processes. In 2008 Nele and Michèle [47], investigated the locally risk-minimizing hedging strategy for unit linked for life insurance contracts in a financial market. In 2010 Yang and Xiao [63], presented risk-minimizing problems under jump-diffusion models with restricted information and cost.

3.2 Model Description

The BlackScholesMerton model [4] is a mathematical model of a financial market containing certain derivative investment instruments. From the model, one can deduce the BlackScholes formula, which gives the price of European-style options. It is widely used by options market participants. The model was first articulated by Fischer Black and Myron Scholes in their 1973 paper, "The Pricing of Options and Corporate Liabilities", published in the Journal of Political Economy. They derived a partial differential equation, the BlackScholes equation, which governs the price of the option over time. The key idea behind the derivation was to hedge perfectly the option by buying and selling the underlying asset in just the right way and consequently "eliminate risk". This hedge is called delta hedging and is the basis of more complicated hedging strategies such as those engaged in by Wall Street investment banks. The hedge implies there is only one right price for the option and it is given by the Black-Scholes formula. Robert C. Merton was the first to publish a paper expanding the mathematical understanding of the options pricing model and coined the term BlackScholes options pricing model. Merton and Scholes received the 1997 Nobel Prize in Economics for their work.

Assumptions:-

1. There is no arbitrage opportunity.
2. It is possible to borrow and lend cash at a known constant risk-free interest rate.
3. It is possible to buy and sell any amount, even fractional, of stock (this includes short selling).
4. The above transactions do not incur any fees or costs (i.e., frictionless market).
5. The stock price follows a GBM (geometric Brownian motion) with constant drift and volatility.
6. The underlying security does not pay a dividend.

The BlackScholes equation is a partial differential equation, which describes the price of the option over time.

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} + rx\frac{\partial C}{\partial x} - rC = 0 \tag{2.1}$$

3.3 Black - Scholes Theory

We consider a market consisting of one stock and one bond. The stock price $S = \{S_t, t \geq 0\}$ is assumed to follow a geometric Brownian motion given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t, S_0 > 0, \tag{3.1}$$

i.e.

$$\text{i.e.} S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}. \tag{3.2}$$

Since $ES_t = S_0 e^{\mu t}$, μ may be treated as the mean growth rate of return from the stock; $\sigma > 0$ is the volatility. The price of the bond at time t is given by

$$B_t = e^{rt} \tag{3.3}$$

where r is the rate of interest which is assumed to be constant. One can show that *this market is complete.* We assume that the market is *viable*, i.e., there does not exist any arbitrage opportunity with self financing strategy with bounded short selling. We derive the price of an option on the stock S_t.

A trading strategy is a pair $\phi = (\phi^0, \phi^1)$ of predictable processes, where ϕ^0_t is the number of bonds and ϕ^1_t is the number of stocks the investor holds at time t. The value of the portfolio corresponding to the strategy $\phi = (\phi^0_t, \phi^1_t)$ at time t is given by

$$V_t(\phi) := \phi^0_t B_t + \phi^1_t S_t. \tag{3.4}$$

The strategy $\phi = (\phi^0,\ \phi^1)$ is self-financing if no fresh investment is made at any time $t > 0$ and there is no consumption. We work with self-financing strategies only. The gain accrued to the investor via the strategy ϕ up to time t is given by

$$G_t(\phi) := \int_0^t \phi^0_u dB_u + \int_0^t \phi^1_u dS_u. \tag{3.5}$$

For a self-financing strategy ϕ:

$$V_t(\phi) = V_0(\phi) + G_t(\phi). \tag{3.6}$$

Suppose that a European call option with terminal time T and strike price K itself is traded in the market. Let C_t denote the price of this call option at time t. Then

$$C_t = C(t, S_t) \tag{3.7}$$

for some function $C : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$.

3.4 Black-Scholes Partial Differential Equations

We look for a strategy $\phi = (\phi^0, \phi^1)$ such that

$$V_0(\phi) = C_0$$

and

$$V_T(\phi) = (S_T - K)^+$$

In view of the previous $C_t = V_t(\phi)$ for all t. Further we assume that $C(t,x)$is a smooth function of t and x. Now

$$\begin{aligned} V_t = V_t(\phi) &= \phi_t^0 B_t + \phi_t^1 S_t \\ &= V_0(\phi) + \int_0^t \phi_u^0 dB_u + \int_0^t \phi_u^1 dS_u. \end{aligned}$$

Therefore

$$dV_t = \phi_t^0 dB_t + \phi_t^1 dS_t.$$

Using (3.1) and (3.3) it follows that

$$\begin{aligned} dV_t &= r\phi_t^0 B_t dt + \phi_t^1 \left(\mu S_t dt + \sigma S_t dW_t\right) \\ \text{i.e.} \\ dV_t &= \left(r\phi_t^0 B_t + \mu\phi_t^1 S_t\right) dt + \sigma\phi_t^1 S_t dW_t. \end{aligned} \tag{4.1}$$

Since $C(t,x)$ is a smooth function, by Ito's formula

$$\begin{aligned} dC_t &= dC(t, S_t) \\ &= \left\{ \frac{\partial}{\partial t} C(t,S_t) + \mu \frac{\partial}{\partial x} C(t,S_t) S_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2}{\partial x^2} C(t,S_t) \right\} dt \\ &+ \sigma \frac{\partial}{\partial x} C(t,S_t) S_t dW_t. \end{aligned} \tag{4.2}$$

From (4.1) and (4.2), we get

$$\phi_t^1 = \frac{\partial C(t,S_t)}{\partial x} \tag{4.3}$$

and

$$r\phi_t^0 B_t + \mu\phi_t^1 S_t = \frac{\partial C(t,S_t)}{\partial t} + \mu \frac{\partial C(t,S_t)}{\partial x} S_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t,S_t)}{\partial x^2}. \tag{4.4}$$

From (4.3) and (4.4) we get

$$r\phi_t^0 B_t = \frac{\partial C(t,S_t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t,S_t)}{\partial x^2}. \tag{4.5}$$

Again,

$$\phi_t^1 S_t + \phi_t^0 B_t = V_t(\phi) = C(t,S_t).$$

Therefore

$$\phi_t^0 = \frac{1}{B_t}\left[C(t,S_t) - \frac{\partial C(t,S_t)}{\partial x} S_t\right]. \tag{4.6}$$

Therefore

$$\frac{\partial C(t,S_t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t,S_t)}{\partial x^2} + rS_t \frac{\partial C(t,S_t)}{\partial x} - rC(t,S_t) = 0, \qquad 0 \le t < T$$

with $\quad C(T,S_T) = V_T(\phi) = (S_T - K)^+.$

Thus the option price process $C_t = C(t,S_t)$ satisfies the partial differential equation.

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} + rx\frac{\partial C}{\partial x} - rC = 0 \tag{4.7}$$

with the boundary condition :

$$C(T,x) = (x-K)^+, \text{ and } C(t,0) = 0 \quad \forall t \tag{4.8}$$

The equation (4.7) is referred to as the Black-Scholes PDE.

3.5 Analytic Solution of Black-Scholes PDE

Introduce the new variables τ, ς by

$$\tau = \gamma(T-t), \qquad \varsigma = \alpha\left\{\log\frac{x}{K} + \beta(T-t)\right\}, \tag{5.1}$$

Where $\alpha,\ \beta,\ \gamma$ are constants to be chosen later. Define the function $y(\tau,\varsigma)$ by

$$C(t,x) = e^{-r(T-t)} y(\tau,\varsigma). \tag{5.2}$$

Then

$$\begin{aligned} \frac{\partial C}{\partial t} &= re^{-r(T-t)}y + e^{-r(T-t)}\frac{\partial y}{\partial t} \\ \frac{\partial y}{\partial t} &= \frac{\partial y}{\partial \varsigma}\frac{\partial \varsigma}{\partial t} - \frac{\partial y}{\partial \tau}\gamma = -\alpha\beta\frac{\partial y}{\partial \varsigma} - \gamma\frac{\partial y}{\partial \tau}. \end{aligned}$$

Hence

$$\frac{\partial C}{\partial t} = re^{-r(T-t)}y - \alpha\beta e^{-r(T-t)}\frac{\partial y}{\partial \varsigma} - \gamma e^{-r(T-t)}\frac{\partial y}{\partial \varsigma}. \tag{5.3}$$

Also

$$\begin{aligned}\frac{\partial C}{\partial x} &= e^{-r(T-t)}\frac{\partial y}{\partial x} = e^{-r(T-t)}\frac{\partial y}{\partial \varsigma}\frac{\partial \varsigma}{\partial x}\\ &= e^{-r(T-t)}\frac{\partial y}{\partial \varsigma}\ \frac{\alpha}{x}\quad .\end{aligned}$$

Therefore

$$x\frac{\partial C}{\partial x} = \alpha e^{-r(T-t)}\frac{\partial y}{\partial \varsigma}. \tag{5.4}$$

Also

$$\begin{aligned}\frac{\partial^2 C}{\partial x^2} &= e^{-r(T-t)}\frac{\partial}{\partial x}\left[\frac{\partial y}{\partial \varsigma}.\frac{\partial \varsigma}{\partial x}\right]\\ &= e^{-r(T-t)}\left[\frac{\partial^2 y}{\partial \varsigma^2}\left(\frac{\partial \varsigma}{\partial x}\right)^2 + \frac{\partial y}{\partial \varsigma}\frac{\partial^2 \varsigma}{\partial x^2}\right].\end{aligned}$$

Substituting

$$\frac{\partial \varsigma}{\partial x} = \frac{\alpha}{x}, \qquad \frac{\partial^2 \varsigma}{\partial x^2} = -\frac{\alpha}{x^2}$$

in the above relation, we obtain

$$\frac{\partial^2 C}{\partial x^2} = e^{-r(T-t)}\left[\frac{\alpha^2}{x^2}\ \frac{\partial^2 y}{\partial \varsigma^2} - \frac{\alpha}{x^2}\frac{\partial y}{\partial \varsigma}\right].$$

Therefore

$$x^2\frac{\partial^2 C}{\partial x^2} = e^{-r(T-t)}\left[\alpha^2\frac{\partial^2 y}{\partial \varsigma^2} - \alpha\frac{\partial y}{\partial \varsigma}\right]. \tag{5.5}$$

Then we have

$$\begin{aligned}\frac{\partial C}{\partial t} &+ \frac{1}{2}\sigma^2 x^2\frac{\partial^2 C}{\partial x^2} + rx\frac{\partial C}{\partial x} - rC\\ &= e^{-r(T-t)}\left[ry - \alpha\beta\frac{\partial y}{\partial \varsigma} - \gamma\frac{\partial y}{\partial \tau} + \frac{\sigma^2}{2}\left(\alpha^2\frac{\partial^2 y}{\partial \varsigma^2} - \alpha\frac{\partial y}{\partial \varsigma}\right) + r\alpha\frac{\partial y}{\partial \varsigma} - ry\right].\end{aligned}$$

Simplifying and using (4.7), we obtain

$$-\alpha\beta\frac{\partial y}{\partial \varsigma} - \gamma\frac{\partial y}{\partial T} + \frac{\sigma^2}{2}\left[\alpha^2\frac{\partial^2 y}{\partial \varsigma^2} - \alpha\frac{\partial y}{\partial \varsigma}\right] + r\alpha\frac{\partial y}{\partial \varsigma} = 0 \tag{5.6}$$

The coefficient of $\frac{\partial y}{\partial \varsigma}$ in (5.6) is equal to

$$-\alpha\beta - \frac{\alpha\sigma^2}{2} + r\alpha = \alpha\left[r - \beta - \frac{\sigma^2}{2}\right].$$

Choose $\beta = r - \frac{\sigma^2}{2}$. Then (5.6) becomes

$$-\gamma\frac{\partial y}{\partial \tau} + \frac{1}{2}\sigma^2\alpha^2\frac{\partial^2 y}{\partial \varsigma^2} = 0.$$

Now choose $\gamma = \sigma^2\alpha^2$ (where $\alpha \neq 0$ is arbitrary). Then Black - Scholes pde (4.7) becomes

$$\frac{\partial y}{\partial \tau} = \frac{1}{2}\frac{\partial^2 y}{\partial \varsigma^2} \tag{5.7}$$

with the boundary condition:

$$C(T,x) = y\left(0, \log\frac{x}{K}\right).$$

Then

$$y(0,u) = K\left(e^u - 1\right)^+. \tag{5.8}$$

The (unique solution of (5.7) with the boundary condition (5.8) (in the class of function not growing faster than e^{ax^2}) is given by

$$\begin{aligned} y(\tau,\varsigma) &= \frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} e^{-(u-\varsigma)^2/2\tau}K(e^u-1)^+du \\ &= \frac{K}{\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} e^{-(u-\varsigma)^2/2\tau}(e^u-1)^+du. \end{aligned}$$

Thus

$$y(\tau,\varsigma) = Ke^{\varsigma+\frac{1}{2}\tau}\Phi\left(\frac{\varsigma}{\sqrt{\tau}} + \sqrt{\tau}\right) - K\Phi\left(\frac{\varsigma}{\sqrt{\tau}}\right). \tag{5.9}$$

Now in terms of t, x note that

$$\alpha = 1, \qquad \beta = r - \frac{\sigma^2}{2}, \qquad \gamma = \sigma^2.$$

Thus

$$\begin{aligned} e^{\varsigma+\frac{1}{2}\tau} &= \frac{x}{K}e^r(T-t) \\ \frac{\varsigma}{\sqrt{\tau}} + \sqrt{\tau} &= \frac{\log\frac{r}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ &:= g(x, T-t) \end{aligned}$$

and

$$\frac{\varsigma}{\sqrt{\tau}} = g(x, T-t) - \sigma\sqrt{T-t} := h(x, T-t). \tag{5.10}$$

Then

$$C(t, x) = x\Phi(g(x, T-t)) - Ke^{-r(T-t)}\Phi(h(x, T-t)). \tag{5.11}$$

3.6 Completeness and NA

Definition 3.6.1. A *trading strategy* is defined as a (finite) sequence of random variables

$$\phi = \{(\phi_n^0, ..., \phi_n^k), 0 \leq n \leq N\}$$

in $\mathbb{R}^{k+1}$, where ϕ_n^i denotes the number of assets i held at time n, and ϕ is predictable, i.e., for all $i = 0, 1, ..., k$, ϕ_n^i is $\mathcal{F}_0$-measurable, and for $n \geq 1$, ϕ_n^i is $\mathcal{F}_{n-1}$-measurable. This means that the position in the portfolio $(\phi_n^0, ..., \phi_n^k)$ at time n is decided on the basis of the information available at time $(n-1)$, and kept until n when the new quotations are available. The value of the portfolio at time n is given by

$$V_n(\phi) = \langle \phi_n, S_n \rangle = \sum_{i=0}^{k} \phi_n^i S_n^i.$$

Let $\tilde{S}_n^i = \beta S_n^i = (1+r)^{-n} S_n^i$; S_n^i is called the discounted vale of the asset i at time n. The discounted value of the portfolio at time n is given by

$$\tilde{V_n}(\phi) = \beta_n V_n(\phi) = \sum_{i=0}^{k} \phi_n^i \tilde{S_n}^i.$$

Definition 3.6.2. A strategy $\phi = \{(\phi_n^0, ..., \phi_n^k), 0 \leq n \leq N\}$ is called *self-financing* if for all $n = 0, 1, ..., N-1$,

$$\langle \phi_n, S_n \rangle = \langle \phi_n + 1, S_n \rangle$$

$$\sum_{i=0}^{k} \phi_n^i S_n^i = \sum_{i=0}^{k} \phi_{n+1}^i S_{n+1}^i.$$

The interpretation of self-financing strategy is the following: at time n, once the new prices $S_n^0, S_n^1, ..., S_n^k$ are quoted, the investor readjusts his position from ϕ_n to ϕ_{n+1} without bringing in or consuming any wealth.

Definition 3.6.3. A strategy $\phi = \{(\phi_n^0, ..., \phi_n^k), 0 \leq n \leq N\}$ is said to be *admissible* if it is self-financing, and $V_n(\phi) \geq 0$ for any $n = 0, 1, ..., N$.

Definition 3.6.4. A probability measure P^* is said to be *equivalent martingale measure (EMM)* if $P^* \equiv P$ and the discounted asset prices $\{\tilde{S}_n^i\}$ are martingales with respect to P^*. Such a probability measure is also referred to as a *risk neutral measure*.

Definition 3.6.5. An *arbitrage strategy* is an admissible strategy with zero initial value and positive final value with a positive probability or in other words an *arbitrage opportunity* in a market consisting of several securities is a strategy of buying and selling these securities without any investment by the investor, such that it leads to profit with positive without any risk of loss.

Definition 3.6.6. The market is said to be *viable* if there is no arbitrage opportunity. The market is viable (arbitrage free) if and only if there exists an Equivalent martingale measure P^*.

Definition 3.6.7. An $\mathcal{F}_N$-measurable function $H \geq 0$ is called a contingent claim (of maturity N). For example for a European call option on the underlying S^1 with strike price K

$$H = (S_N^1 - K)^+ = \max({S_N}^1 - K, 0).$$

For European put on the same asset with same strike price K,

$$H = (K - S_N^1)^+ = \max(K - {S_N}^1, 0).$$

Definition 3.6.8. A contingent claim defined by H is *attainable* if there exists an admissible strategy ϕ worth H at time N, i.e., $V_N(\phi) = H$.

Definition 3.6.9. The market is said to be *complete* if every contingent claim is attainable, i.e., if H is contingent claim, then there exists an admissible strategy ϕ such that $V_N(\phi) = H$. The strategy ϕ is often referred to a strategy replicating the contingent claim H. Thus any option can be hedged perfectly.

Theorem 3.6.1. A viable market is complete if and only if there exists a unique probability measure $P^* \equiv P$ under which the discounted prices $\{\tilde{S}_n^i\}$ are martingales.

Since a unique EMM exists, the price of an option is uniquely prescribed by the conditional expectation of the discounted price of the stock w.r.t EMM

In an incomplete market there are contingent claims which are not attainable by self-financing strategies. Thus perfect hedging is not possible. At the same time, since there are multiple equivalent measures, the option price is not unique. To overcome this difficulty, option price in an incomplete market is studied by several approaches.

One of the approach in this direction is to assume the existence of a risk neutral measure and carry out the entire analysis under such a measure. A second approach to treat an incomplete market is to make complete market by introducing additional securities known as Arrow-Debreu securities. A third approach in this direction involves superruplicating portfolio and upper-hedging price.

3.7 Criticism

Espen Gaarder Haug and Nassim Nicholas Taleb argue that the BlackScholes model merely recast existing widely used models in terms of practically impossible "dynamic hedging" rather than "risk," to make them more compatible with mainstream neoclassical economic theory. [30] Similar arguments were made in an earlier paper by Emanuel Derman and Nassim Taleb. [11] In response, Paul Wilmott has defended the model, [60], [51].

Chapter 4

Regime Switching Market Model

4.1 Model assumption

Let $(\Omega, \mathcal{F}, P)$ be the underlying complete probability space. Let $\chi = \{1, 2, \ldots, k\}$ be the state space of an irreducible Markov chain $\{X_t, t \geq 0\}$ with transition rule

$$P(X_{t+\delta t} = j | X_t = i) = \lambda_{ij}\delta t + o(\delta t), \quad i \neq j$$

where $\lambda_{ij} \geq 0$ for $i \neq j$; and $\lambda_{ii} = -\sum_{j\neq i}^{k} \lambda_{ij}$. Thus $\Lambda = [\lambda_{ij}]$ denotes the generating Q-matrix of the chain and $p_{ij} := \frac{\lambda_{ij}}{|\lambda_{ii}|}$ are the transition probabilities from state i to state j. We consider a market where the financial parameters, namely interest rate, drift coefficient, volatility coefficient are functions of the observed Markov chain X_t. Let $\{B_t, t \geq 0\}$ be the price of money market account at time t where, spot interest rate is $r(X_t)$ and $B_0 = 1$. We have

$$B_t = e^{\int_0^t r(X_u)du}.$$

We consider a market consisting only one stock as tradeable risky asset. The stock price process S_t solves

$$dS_t = S_t(\mu(X_{t^-})dt + \sigma(X_{t^-})dW_t), \ S_0 > 0 \tag{1.1}$$

where $\{W_t, t \geq 0\}$ is a standard Wiener process independent of $\{X_t, t \geq 0\}$. Let $\mathcal{F}_t$ be a filtration of $\mathcal{F}$ satisfying usual hypothesis and right continuous version of the filtration generated by X_t and S_t. Clearly the solution of above SDE is an $\mathcal{F}_t$ semimartingale with almost sure continuous paths. To price a claim H of European type in the above incomplete market, we would consider the locally risk minimizing pricing approach by Föllmer and Schweizer (see [31], [17]). A hedging strategy is defined as a predictable process $\pi = \{\pi_t =$

$(\xi_t, \varepsilon_t), 0 \leq t \leq T\}$ which satisfies

$$E\left[\int_0^T \xi_t^2 \sigma^2(X_t) S_t^2 dt + \left(\int_0^T |\xi_t||\mu(X_t)|dt\right)^2\right] < \infty \tag{1.2}$$

and

$$E[\varepsilon_t^2] < \infty.$$

The components ξ_t and ε_t denote the amounts invested in S_t and B_t respectively at time t. An optimal strategy is the one for which the quadratic residual risk (see [17] for details) is minimized subject to a certain constraint. It is shown in [17] that the existence of an optimal strategy for hedging an $\mathcal{F}_T$ measurable claim H is equivalent to the existence of Föllmer Schweizer decomposition of discounted claim $H^* := B_T^{-1} H$ in the form

$$H^* = H_0 + \int_0^T \xi_u^{H^*} dS_u^* + L_T^{H^*} \tag{1.3}$$

where $H_0 \in L^2(\Omega, \mathcal{F}_0, P), L^{H^*} = \{L_t^{H^*}\}_{0 \leq t \leq T}$ is a square integrable martingale orthogonal to the martingale part of S_t, $S_t^* := B_t^{-1}$, and $\xi^{H^*} = \{\xi_t^{H^*}\}$ satisfies (1.2). Further, ξ^{H^*} appeared in the decomposition constitutes the optimal strategy. Indeed the optimal strategy $\pi = (\xi_t, \varepsilon_t)$ is given by

$$\begin{aligned} \xi_t &:= \xi_t^{H^*} \\ V_t^* &:= H_0 + \int_0^t \xi_u^{H^*} dS_u^* + L_t^{H^*} \\ \varepsilon_t &:= V_t^* - \xi_t S_t^* \end{aligned}$$

and $B_t V_t^*$ represents the locally risk minimizing price at t of the claim H. Hence the Föllmer Schweizer decomposition decomposition is the key thing to verify.

Now onward we consider a particular claim i.e., a European call option on $\{S_t\}$ with strike price K and maturity time T. In this case the $\mathcal{F}_T$ measurable contingent claim H is given by

$$H = (S_T - K)^+. \tag{1.4}$$

Before stating the main results we recall that in the Black-Schole-Merton model[4] the $\mathcal{F}_T$ measurable claim H is attainable and the price $\eta(t, S_t)$ at time $t \in [0, T]$ is given by

$$\eta(t, S_t) = S_t \Phi\left(\frac{\log(\frac{S_t}{K}) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right) - e^{rt} K^* \Phi\left(\frac{\log(\frac{S_t}{K}) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right) \tag{1.5}$$

where r and σ are constants denoting fixed bank rate and fixed volatility coefficients respectively; $\Phi(x)$ is the CDF of standard normal distribution, $K^* = e^{-rT} K$. The Black-Scholes hedging strategy, called Delta hedging is given by

$$\Delta(t, s) = \frac{\partial \eta(t, s)}{\partial s}$$

where $\Delta(t, s)$ is the number of shares invested in stock.

4.2 Pricing in incomplete Market

The option pricing in a regime switching framework has been studied by several authors using different approaches: Buffington and Elliott [6], DiMasi et al [13], Guo [25], Guo and Zhang [26], Mamon and Rodrigo [44], Roberts and Rogers [50], Tsoi et al. [59], and Yao et al. [64]. Portfolio optimization problem is also studied in a regime switching market [65], [66]. Since the regime switching market is incomplete the option pricing is rather involved. As we know that in *complete market* every contingent claim can be replicated by a self-financing strategy. Thus any option can be hedged perfectly. At the same time since a unique equivalent martingale measure exists, the price of an option is uniquely prescribed by the conditional expectation of the discounted price of the terminal claim with respect to the equivalent martingale measure. But this is not the case in an incomplete market. In an *incomplete market* there are contingent claims which are not attainable by self-financing strategies. Thus perfect hedging is not possible. At the same time since there are multiple equivalent measures the option price is not unique.

To overcome this difficulty option pricing in an incomplete market is studied by several approaches:

1. One of the approaches in this direction is to assume the existence of a risk neutral measure and carry out the entire analysis under such a measure. A *risk neutral measure* is a special kind of an equivalent measure such that the discounted price of every traded item in the market is a martingale with respect to the available information σ-fields. Since options are also traded in the market, the present price of an option is the conditional expectation with respect to the risk neutral measure of the discounted price of the terminal payoff. It may be noted that the risk neutral measure exists in an ideal world whereas the stock price evolves in the real world. As a consequence the parameters of the regime switching market model like $\lambda_{ij}, \sigma(i)$ etc. have to be estimated using specific risk neutral instruments such as federal bonds, treasury bills etc. Also hedging of the option is not emphasized in this approach. For regime switching model this approach has been pursued in several papers including [6], [44], [50], [64] etc.
2. A second approach to treat an incomplete market is to make the market complete by introducing additional securities known as Arrow-Debreu securities [13], [14]. For regime switching market this has been carried out in [25].
3. A third approach in this direction involves superreplicating portfolio and upper-hedging price; see [37] and the references therein. Though this method is very elegant from a mathematical viewpoint, its implementation is rather involved. But this approach for a regime switching model has not been studied thus far as per knowledge.
4. *Mean-variance hedging* is a powerful method in handling non-attainable claims in an incomplete market; see [46] and the references therein. Mean-variance hedging are of two types: variance-minimizing hedging and risk-minimizing hedging. Variance-

minizing hedging involves self-financing strategies only. Since variance-minimizing method is essentially based on self-financing strategies it does not allow additional borrowing or withdrawal of funds before the terminal date. This puts a limitation on the scope of its applicability. For non-attainable claims it is desirable to do away with self-financing strategies and use strategies which allow continuous transfer of funds (borrowing or lending) with the provision of a suitable optimality criterion which focuses on the minimization of the future risk exposure due to additional cash flow at any time. A suitable notion of *risk-minimizing hedging* was introduced by Föllmer and Sondermann [18] to address the option pricing in an incomplete market which was further pursued by Föllmer and Schweizer [17]. By introducing a quadratic residual risk function they obtained an abstract formula for the risk minimizing option price via the minimal martingale measure P^*. Furthermore it is shown in [17] that if the discounted claim admits a certain decomposition known as Föllmer-Schweizer decomposition under the market probability P, one can obtain expressions for the hedging strategy and the residual risk; see [54] for an excellent survey on this method. For regime switching market this has been studied in [13] for European claims which include call and put options. In particular for a two state Markov chain an explicit expression for the European call is obtained in [13]. Further development is done in [3] and [12].

4.3 Equations of Pricing and Hedging

Consider the following system of partial differential equations

$$\frac{\partial\varphi(t,s,i)}{\partial t}+\frac{1}{2}\sigma(i)^2s^2\frac{\partial^2\varphi(t,s,i)}{\partial s^2}+r(i)s\frac{\partial\varphi(t,s,i)}{\partial s}+\sum_{j=1}^{k}\lambda_{ij}\varphi(t,s,i)=r(i)\varphi(t,s,i) \quad (3.1)$$

for $t<T$, $s>0$ and $i=1,2\ldots,k$ with the boundary condition

$$\varphi(T,s,i)=(s-K)^+ \ s\geq 0, \qquad \varphi(t,0,i)=0 \ \forall t\in[0,T], i\in\chi \quad (3.2)$$

where φ is of polynomial growth. Note that if Λ is a null matrix i.e., the case when the Markov chain X_t does not transit almost surely, the equation (3.7) coincides with that of standard B-S-M model. In view of this, the above system can be considered as a generalization of Black-Scholes equation for a Markov modulated market where the extra coupling term represents the correction term arising due to the regime switching. Nevertheless, the fact, the solution of above problem gives locally risk minimizing price, needs a proof. This is given below in 4.3.1 which appears in [3]. In order to state the theorem certain terminologies should be defined.

It would be convenient to represent the Markov chain $\{X_t\}$ as a stochastic integral with respect to a Poisson random measure which would play an important role later.

For a Polish space $\mathcal{S}$, let $\mathcal{B}(\mathcal{S})$ denote its Borel σ-field and $\mathcal{M}(\mathcal{S})$ the set of all nonnegative integer valued σ-finite measures on $\mathcal{B}(\mathcal{S})$. Let $\mathcal{M}_\sigma(\mathcal{S})$ be the smallest σ-field on $\mathcal{M}(\mathcal{S})$ with respect to which the maps $\alpha_B : \mathcal{M}(\mathcal{S}) \to \mathbb{N}\bigcup\{\infty\}$ defined by $\alpha_B(\mu) := \mu(B)$ are measurable for all $B \in \mathcal{B}(\mathcal{S})$; $\mathcal{M}(\mathcal{S})$ is assumed to be endowed with the σ-field $\mathcal{M}_\sigma(\mathcal{S})$.

For $i \neq j \in \mathcal{X}$, let Λ_{ij} be consecutive (with respect to the lexicographic ordering on $\mathcal{X} \times \mathcal{X}$) left closed right open intervals of the real line, each having length λ_{ij}. By embedding $\mathcal{X}$ in $\mathbb{R}^k$ by identifying i with $e_i \in \mathbb{R}^k$ define a function $h : \mathcal{X} \times \mathbb{R} \to \mathbb{R}^k$ by

$$h(i,z) \;:=\; \begin{cases} j-i & \text{if } z \in \Lambda_{ij} \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Then

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} h(X_{u-}, z)\wp(du, dz) \tag{3.4}$$

where the integration is over the interval $(0,t]$ and $\wp(dt, dz)$ is an $\mathcal{M}(\mathbb{R}_+ \times \mathbb{R})$ -valued Poisson random measure with intensity $dtdz$; $\wp(dt, dz), X_0, W$ and S_0, defined on $(\Omega, \mathcal{F}, P)$ are independent.

$\hat{\wp}(dt, dz) := \wp(dt, dz) - dtdz$ is the compensated Poisson random measure.

Theorem 4.3.1. *Let $\{\varphi_c(t, s, i),\ i = 1, 2, \ldots, k\}$ denote the unique solution of the Cauchy problem (3.1), (3.2) in $C([0,T] \times \bar{\mathbb{R}} \times \chi) \cap C^{1,2}((0,T) \times \mathbb{R}_+, \chi)$. Then*
(i) $\varphi_c(t, S_t, X_t)$ is the risk minimizing option price at time t;
(ii) An optimal strategy $\pi^* = \{\xi_t^*, \eta_t^*\}$ is given by

$$\xi_t^* = \frac{\partial \varphi_c(t, S_t, X_{t-})}{\partial s} \tag{3.5}$$

$$\eta_t^* = V_t^* - \xi_t^* S_t^* \tag{3.6}$$

where

$$\begin{aligned} V_t^* &= \varphi_c(0, X_0, S_0) + \int_0^t \frac{\partial \varphi_c(u, S_u, X_{u-})}{\partial s} dS_u^* \\ &\quad + \int_0^t e^{-\int_0^u r(X_v)dv} \int_{\mathbb{R}} [\varphi_c(u, S_u, X_{u-} + h(X_{u-}, z)) - \varphi_c(u, S_u, X_{u-})]\hat{\wp}(du, dz). \end{aligned} \tag{3.7}$$

(iii) The quadratic residual risk at time t is given by

$$R_t(\pi^*) \;=\; E[\int_t^T e^{-2\int_0^u r(X_v)dv} \sum_{j \neq X_u} \lambda_{X_u j} \Big(\varphi_c(u, S_u, j) - \varphi_c(u, S_u, X_u)\Big)^2 du \mid \mathcal{F}_t]. \tag{3.8}$$

Proof. Let $0 \leq t \leq T$. By applying Ito's formula to $e^{-\int_0^t r(X_u)du}\varphi_c(t, S_t, X_t)$ under the measure P and using (3.1), (1.1) and (3.6), we obtain after suitable rearrangement of terms

$$\begin{aligned} e^{-\int_0^t r(X_u)du}\varphi_c(t, S_t, X_t) &= \varphi_c(0, S_0, X_0) + \int_0^t \frac{\partial \varphi_c(u, S_u, X_{u-})}{\partial s} dS_u^* + \int_0^t e^{-\int_0^u r(X_v)dv} \\ & \int_{\mathbb{R}} [\varphi_c(u, S_u, X_{u-} + h(X_{u-}, z)) - \varphi_c(u, S_u, X_{u-})]\hat{\wp}(du, dz). \quad (3.9) \end{aligned}$$

Letting $t \uparrow T$, we obtain

$$\begin{aligned} B_T^{-1}(S_T - K)^+ &= \varphi_c(0, S_0, X_0) + \int_0^T \frac{\partial \varphi_c(u, S_u, X_{u-})}{\partial s} dS_u^* + \int_0^T e^{-\int_0^u r(X_v)dv} \\ & \int_{\mathbb{R}} [\varphi_c(u, S_u, X_{u-} + h(X_{u-}, z)) - \varphi_c(u, S_u, X_{u-})]\hat{\wp}(du, dz). \quad (3.10) \end{aligned}$$

Since we already know that $B_T^{-1}(S_T - K)^+$ admits a *Föllmer-Schweizer decomposition* (1.3), we can argue (as in [17] Theorem 3.14) to conclude (i) and (ii) using the decomposition in (3.10).

(iii) Using Itô's isometry, the residual risk at time t is given by

$$\begin{aligned} R_t(\pi^*) &= E\Big[\Big(\int_t^T \int_{\mathbb{R}} e^{-\int_0^u r(X_v)dv}\Big\{\varphi_c\Big(u, S_u, X_{u-} + h(X_{u-}, z)\Big) \\ &\qquad -\varphi_c(u, S_u, X_{u-})\Big\}\hat{\wp}(du, dz)\Big)^2\Big|\mathcal{F}_t\Big] \\ &= E\Big[\int_t^T \int_{\mathbb{R}} e^{-2\int_0^u r(X_v)dv}\Big\{\varphi_c\Big(u, S_u, X_{u-} + h(X_{u-}, z)\Big) \\ &\qquad -\varphi_c(u, S_u, X_{u-})\Big\}^2 \hat{\wp}(du, dz)\Big|\mathcal{F}_t\Big] \\ &= E[\int_t^T e^{-2\int_0^u r(X_v)dv} \sum_{j \neq X_u} \lambda_{X_u j}\Big(\varphi_c(u, S_u, j) - \varphi_c(u, S_u, X_u)\Big)^2 du \mid \mathcal{F}_t]. \quad (3.11) \end{aligned}$$

It is important to note that the Theorem 4.3.1 assumes existence and uniqueness of solution (3.1) and (3.2) which should be proved. The following theorem settles that along with some more interesting results.

Theorem 4.3.2. *(i) The following integral equation has a unique solution in the class of functions belonging to* $C([0, T] \times \overline{\mathbb{R}_+} \times \chi) \bigcap C^{1,2}((0, T) \times \mathbb{R}_+ \times \chi)$

$$\begin{aligned} \varphi(t, s, i) = &\ e^{-\lambda_i(T-t)}\eta_i(t, s) + \int_0^{T-t} \lambda_i e^{-(\lambda_i + r(i))v} \\ & \times \sum_j p_{ij} \int_0^\infty \varphi(t + v, x, j) \frac{e^{-\frac{1}{2}\left(\left(\ln(\frac{x}{s}) - (r(i) - \frac{\sigma^2(i)}{2})v\right)\frac{1}{\sigma(i)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(i)\sqrt{v}x} dx dv \quad (3.12) \\ \text{with} \quad \varphi(T, s, i) = &\ (s - K)^+, \qquad \varphi(t, 0, i) = 0 \ \forall t \in [0, T], i \in \chi \quad (3.13) \end{aligned}$$

where $\eta_i(t,s)$ is the standard Black-Scholes price of European call option with fixed interest rate $r(i)$ and volatility $\sigma(i)$.

(ii) Moreover, the solution $\varphi(t,s,i)$ of (3.12) *and* (3.13) *is the locally risk minimizing price of H (as in* (1.4)*) at time t with $S_t = s, X_t = i$.*

(iii) The cauchy problem 3.1 and 3.2 has unique classical solution.

Proof: We prove the first part of Theorem 4.3.2 primarily by constructing a smooth solution of (3.12)-(3.13). In order to do that let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be a complete probability space which holds a standard Brownian motion $\tilde{W}$ and a Markov process $\tilde{X}$ independent of $\tilde{W}$ such that the rate matrix of $\tilde{X}$ is the same as that of X. Let $\tilde{S}_t$ be given by

$$d\tilde{S}_t = \tilde{S}_t(r(\tilde{X}_t)dt + \sigma(\tilde{X}_t)d\tilde{W}_t), \;\; \tilde{S}_0 > 0 \tag{3.14}$$

and $\tilde{\mathcal{F}}_t$ be the underlying filtration satisfying usual hypothesis. Thus $\tilde{P}$ is risk-neutral measure for the risky asset $\tilde{S}$ given by (3.14). Let Y_t represent holding time i.e., the amount of time the process $\tilde{X}_t$ is at the current state after the last jump. Let the consecutive jump times be $0 = T_0 < T_1 < T_2 < \cdots$ and $n(t) := \max\{n \geq 0 \mid T_n \leq t\}$. Hence, $T_{n(t)} = t - Y_t$. Clearly, $f(y|i) := \lambda_i e^{-\lambda_i y}$ is the conditional probability density function of holding time and $F(y|i) = 1 - e^{-\lambda_i y}$ is the corresponding CDF where $\lambda_i = -\lambda_{ii}$. Here we recall the following obvious relation

$$\frac{f(y|i)}{1 - F(y|i)} = \lambda_i.$$

Because of Markovity of $(\tilde{S}_t, \tilde{X}_t)$, we know that there is a measurable function $\varphi : [0,T] \times [0,\infty) \times \chi \to \mathbb{R}$ such that $\varphi(t,0,i) = 0$ and

$$\varphi(t, \tilde{S}_t, \tilde{X}_t) = \tilde{E}[e^{-\int_t^T r(\tilde{X}_u)du}(\tilde{S}_T - K)^+ \mid \tilde{\mathcal{F}}_t] \tag{3.15}$$

holds for all $t \in [0,T]$ where $\tilde{E}$ is expectation under $\tilde{P}$. Due to irreducibility of $\tilde{X}_t$, for any fixed $\tilde{X}_0, \tilde{S}_0$, the map φ (as in (3.15)) is defined uniquely almost everywhere on $[0,T] \times [0,\infty) \times \chi$. Now by conditioning at transition times and using the conditional lognormal distribution of stock price process, we have

$$
\begin{aligned}
&\varphi(t,\tilde S_t,\tilde X_t)\\
&= \tilde E[e^{-\int_t^T r(\tilde X_u)du}(\tilde S_T-K)^+|\tilde S_t,\tilde X_t]\\
&= \tilde E[\tilde E[e^{-\int_t^T r(\tilde X_u)du}(\tilde S_T-K)^+|\tilde S_t,\tilde X_t,T_{n(t)+1}]|\tilde S_t,\tilde X_t]\\
&= \tilde P(T_{n(t)+1}>T|\tilde X_t)\tilde E[e^{-\int_t^T r(\tilde X_u)du}(\tilde S_T-K)^+|\tilde S_t,\tilde X_t,\{T_{n(t)+1}>T\}]+\\
&\quad \int_0^{T-t}\tilde E[e^{-\int_t^T r(\tilde X_u)du}(\tilde S_T-K)^+|\tilde S_t,\tilde X_t,T_{n(t)+1}=t+v]\frac{f(t+v-T_{n(t)}\mid \tilde X_t)}{1-F(t-T_{n(t)}|\tilde X_t)}dv\\
&= e^{-\lambda_{\tilde x_t}(T-t)}\eta_{\tilde X_t}(t,\tilde S_t)+\int_0^{T-t}\lambda_{\tilde X_t}e^{-(\lambda_{\tilde X_t}+r(\tilde X_t))v}\sum_j p_{\tilde X_t j}\int_0^\infty \tilde E[e^{-\int_{t+v}^T r(\tilde X_u)du}(\tilde S_T-K)^+|\tilde S_{t+v}=x,\\
&\quad \tilde X_{t+v}=j,\tilde T_{n(t)+1}=t+v]\frac{e^{-\frac12((\ln(\frac{x}{\tilde S_t})-(r(\tilde X_t)-\frac{\sigma^2(\tilde X_t)}{2})v)\frac{1}{\sigma(\tilde X_t)\sqrt v})^2}}{\sqrt{2\pi}\sigma(\tilde X_t)\sqrt v x}dxdv\\
&= e^{-\lambda_{\tilde X_t}(T-t)}\eta_{\tilde X_t}(t,\tilde S_t)+\int_0^{T-t}\lambda_{\tilde X_t}e^{-(\lambda_{\tilde X_t}+r(\tilde X_t))v}\\
&\quad \times\sum_j p_{\tilde X_t j}\int_0^\infty \varphi(t+v,x,j)\frac{e^{-\frac12((\ln(\frac{x}{\tilde S_t})-(r(\tilde X_t)-\frac{\sigma^2(\tilde X_t)}{2})v)\frac{1}{\sigma(\tilde X_t)\sqrt v})^2}}{\sqrt{2\pi}\sigma(\tilde X_t)\sqrt v x}dxdv.
\end{aligned}
$$

where $\eta_i(t,s)$ is the standard Black-Scholes price of European call option with fixed interest rate $r(i)$ and volatility $\sigma(i)$. Again using irreducibility of Markov chain, we can replace $(\tilde S_t,\tilde X_t)$ by generic variable (s,x) in the above relation and thus conclude that φ is a solution of (3.12)-(3.13). The first term on the right hand side is clearly in $C^{1,2}((0,T)\times\mathbb{R}_+\times\chi)$. The continuous differentiability in t of the second term follows from the fact that the term $\varphi(t+v,x,j)$ is multiplied by $C^1((0,\infty))$ function in v and then integrated over $v\in(0,T-t)$. Now twice continuous differentiability in s of the second term follows from direct calculation. Thus $\varphi(t,s,i)$ is in $C^{1,2}((0,T)\times\mathbb{R}_+\times\chi)$. Finally the continuity of φ on $[0,T]\times\overline{\mathbb{R}_+}$ follows trivially. We note that the right side of (3.12) can be considered as the image of φ under a contraction. Hence, uniqueness also follows.

In view of Theorem 4.3.1(i), the proof follows if φ, as above, is the unique classical solution of (3.1) - (3.2). Note that $(\tilde S_t,\tilde X_t)$ is jointly Markov with infinitesimal generator $\tilde{\mathcal{A}}$ given by

$$\tilde{\mathcal{A}}\varphi(t,s,i)=\frac12\sigma(i)^2s^2\frac{\partial^2\varphi(t,s,i)}{\partial s^2}+r(i)s\frac{\partial\varphi(t,s,i)}{\partial s}+\sum_{j=1}^k\lambda_{ij}\varphi(t,s,i).$$

Therefore, (3.1) can be rewritten as $\frac{\partial\varphi}{\partial t}(t,s,i)+\tilde{\mathcal{A}}\varphi(t,s,i)=r(i)\varphi(t,s,i)$. Hence using Feynman-Kac formula, φ as in (3.15) is a mild solution of (3.1) with terminal condition (3.2). It is also shown above that φ is in $C([0,T]\times\mathbb{R}_+)\bigcap C^{1,2}((0,T)\times\mathbb{R}_+)$. Hence φ is a classical solution of (3.1)-(3.2). Uniqueness of the Cauchy problem is asserted from the stochastic representation of its solution. Hence the result follows. □

Theorem 4.3.3. *Consider the following function given in terms of solution of* (3.12) *and* (3.13)

$$\psi(t,s,i) = e^{-\lambda_i(T-t)}\frac{\partial \eta_i(t,s)}{\partial s} + \int_0^{T-t} \lambda_i e^{-(\lambda_i+r(i))v} \sum_j p_{ij} \int_0^{\infty} \varphi(t+v,x,j)$$

$$\times \frac{e^{-\frac{1}{2}\left(\left(\ln(\frac{x}{s})-(r(i)-\frac{\sigma^2(i)}{2})v\right)\frac{1}{\sigma(i)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(i)^3 v^{3/2} x s}\left(\ln\left(\frac{x}{s}\right) - \left(r(i) - \frac{\sigma^2(i)}{2}\right)v\right) dx \quad (3.16)$$

$$\text{with} \quad \psi(T,s,i) = 1_{(K,\infty)}(s), s \geq 0 \qquad \psi(t,0,i) = 0 \; \forall t \in [0,T], i \in \chi. \quad (3.17)$$

The processes $\xi_t := \psi(t, S_t, X_{t-})$ *and* $\varepsilon_t := B_t^{-1}(\varphi(t, S_t, X_{t-}) - \xi_t S_t)$ *comprise the optimal hedging strategy for the claim* H *in* (1.4).

Proof: Let us define

$$\xi := \frac{\partial \varphi(t, S_t, X_{t^-})}{\partial s} \text{ and } \varepsilon_t := e^{-\int_0^t r(X_u)du}(\varphi(t, S_t, X_{t^-}) - \xi_t S_t)$$

where φ solves (3.12)-(3.13). Using the both of Theorem 4.3.1 and Theorem 4.3.2 we get, $\pi := (\xi, \varepsilon)$ is an optimal strategy.

Note that by differentiating both sides of (3.12) with respect to s, (3.16) is obtained. The terminal condition (3.17) is also an almost everywhere derivative of (3.13).

4.4 Numerical Method

To solve (3.12)-(3.13), we use the step by step quadrature method. Let Δt and Δs be the time step and stock price step sizes respectively. For m, m', l positive integers and $i \in \chi$, set

$$\mathcal{G}(m,m',l,i) := \frac{e^{-\frac{1}{2}((\ln(\frac{m'}{m})-(r(i)-\frac{\sigma^2(i)}{2})l\Delta t)\frac{1}{\sigma(i)\sqrt{l\Delta t}})^2}}{\sqrt{2\pi}\sigma(i) m' \Delta s \sqrt{l\Delta t}}$$

$$\varphi_m^n(i) \approx \varphi(T - n\Delta t, m\Delta s, i), \qquad \varphi_0^n(i) = 0, \qquad n = 0, 1, \ldots, N := \lfloor \frac{T}{\Delta t} \rfloor.$$

Now we use the following quadrature rule over successive intervals $[0, n\Delta t]$ for a function ψ on this interval, we use

$$\int_0^{n\Delta t} \psi(v)dv \approx \Delta t \sum_{l=0}^{n} \omega_n(l)\psi(l\Delta t),$$

where $\omega_n(l)$ are weights to be chosen appropriately. Applying the above procedure in (3.12) we obtain the following set of equations

$$\varphi_m^n(i) = e^{-\lambda_i n\Delta t}\eta_i(T-n\Delta t, m\Delta s)+\Delta t\sum_{l=1}^{n}\omega_n(l)e^{-l\Delta t(r(i)+\lambda_i)}\sum_j p_{ij}\Delta s\sum_{m'}\varphi_m^{n-1}(j)\mathcal{G}(m,m',l,i) \tag{4.1}$$

with

$$\varphi_m^0(i) = (m\Delta s - K)^+. \tag{4.2}$$

We choose a repeated trapezium rule by which the weights ω_n are given by

$$\omega_n(l) = \begin{cases} 1, & \text{for } l = 1, 2, \ldots, n-1 \\ \frac{1}{2}, & \text{for } l = 0, n. \end{cases}$$

Convergence of the above scheme is obvious, the issue of stability is addressed below.

Theorem 4.4.1. *Let $a := \max_\chi \lambda_i e^{-(\lambda_i + r(i))}$. For*

$$\Delta t \leq \frac{e^{-aT}}{a} \tag{4.3}$$

the scheme (4.1) *is strictly stable with respect to an isolated perturbation. Moreover, the scheme displays uniformly bounded error propagation.*

Proof. We first note that $\mathcal{G}(m,m',l,i)$ corresponds to a lognormal density and the holding time densities $f(\cdot|\cdot)$ are bounded. Let δ_n be an additive error in $\varphi_m^n(i)$ $\forall$ m and i. Now it is easy to show that the effect of the isolated perturbation δ_n in $\varphi_m^N(i)(N := \lfloor \frac{T}{\Delta t} \rfloor)$ is additive and given by

$$\epsilon_n = a\Delta t(1 + a\Delta t)^{N-n}\delta_n.$$

If Δt is sufficiently small and satisfies (4.3), we get $\epsilon_n < \delta_n$, i.e., the scheme is strictly stable with respect to an isolated perturbation. Let δ_n be bounded by a fixed constant δ. Now the total effect ϵ of the perturbation in the value $\varphi_m^N(i)$ is given by

$$\epsilon := \sum_{n=0}^{N} \epsilon_n < (e^{aT} - 1)\delta.$$

Hence the result follows. □

Now we are ready to prove Theorem 4.4.2.

Theorem 4.4.2. *Given a finite grid of the domain $[0,T] \times \overline{\mathbb{R}_+}$, let N and M be the number of discrete points on $[0,T]$ and $\overline{\mathbb{R}_+}$ respectively. Let $T(N,M)$ denote the computational complexity to solve* (3.12) *and* (3.13) *with above grid using step by step quadrature method. Then we have*

$$T(N,M) = O(N^2M^2). \tag{4.4}$$

Proof: To organize better, before computation of (4.1) we evaluate and store the values of known functions on the entire grid, so that those values can directly be used at later stages. Let C be the number of operations, required to accomplish that. We first estimate C. Let the constant c_η be the number of elementary operations required to evaluate η at a single entry. Similarly, let $c_{\mathcal{G}}$ and $c_{\exp}$ be the constants corresponding to the functions $\mathcal{G}$ and exponential respectively. Hence in view of (4.1), we obtain directly

$$C = kN(c_{\exp}+1) + kN(c_{\exp}+3) + kNMc_\eta + kNM^2c_{\mathcal{G}} = O(NM^2).$$

Let $C_m^{(i)}(n)$ denote the number of additional computational operations which are required to obtain $\varphi_m^n(i)$ from (4.3) for fixed $n(\geq 1)$, m and i assuming that values of $\varphi_m^{n-1}(i)$ are known for all m and i. We allow $C_m^{(i)}(0)$ to represent the computational complexity of initial data at each entry. Hence $C(n,M) := \sum_{i\in\chi, m\leq M} C_m^{(i)}(n)$ represents the total complexity at nth stage for each $n \leq N$.

It is evident from (4.2) that $C_m^{(i)}(0)$ is independent of i and similarly complex(c_0 say) for all m. Hence $C(0,M) = Mc_0$.

From (4.1), it is not difficult to get $C_m^{(i)}(n) = 2n(k(M+1)+1)+2$. Hence,

$$C(n,M) = 2[n(k(M+1)+1)+1]kM$$

for all $n = 1,\ldots,N$. Therefore, total number of operations i.e., $T(N,M)$ is given by

$$\begin{aligned} T(N,M) &= C + \sum_{n=0}^{N} C(n,M) \\ &= C + C(0,M) + \sum_{n=1}^{N} 2[n(k(M+1)+1)+1]kM \\ &= O(N^2M^2). \end{aligned}$$

□

Remark 4.4.1. *In this section we have developed a numerical scheme to compute option price using a quadrature method. It is natural to ask if this has any advantage over the one based on solving the PDE* (3.1)-(3.2) *using Crank-Nicholson implicit scheme. In order to compare the computational complexities, we present a brief description of the corresponding Crank-Nicholson scheme below.*

To solve (3.1)-(3.2), we transform by replacing $t = T - v$ and $s = e^z$ and get a new system of PDEs

$$-\frac{\partial\varphi(v,z,i)}{\partial v}+(r(i)-\frac{1}{2}\sigma(i)^2)\frac{\partial\varphi(v,z,i)}{\partial z}+\frac{1}{2}\sigma(i)^2\frac{\partial^2\varphi(v,z,i)}{\partial z^2}+\sum_{j=1}^{k}\lambda_{ij}\varphi(v,z,j)=r(i)\varphi(v,z,i) \tag{4.5}$$

on the domain $(0, T) \times \mathbb{R}$ with

$$\varphi(0, z, i) = (e^z - K)^+. \tag{4.6}$$

Let Δt be the time mesh length and Δz be the stock mesh length in logarithmic scale. Let $N := [\frac{T}{\Delta t}]$, z_0 a large negative number and M a large positive integer. For $n \leq N, m = 0, 1, \dots M$

$$\varphi_m^n(i) := \varphi(n\Delta t, z_0 + m\Delta z, i).$$

The terminal condition (4.6) gives

$$\varphi_m^0(i) = (e^{z_0+m\Delta z} - K)^+.$$

Let $\varphi^n := [\varphi_0^n(1), \dots, \varphi_0^n(k), \varphi_1^n(1), \dots, \varphi_M^n(1), \dots, \varphi_M^n(k)] \in \mathbb{R}^{k(M+1)}$. If φ_{km+i}^n denotes the $km + i$th component of φ^n, then $\varphi_{km+i}^n = \varphi_m^n(i)$. Now the Crank Nicholson discretization of (4.5) gives

$$A\varphi^{n+1} = (-2I - A)\varphi^n \tag{4.7}$$

where A is an appropriate block diagonal real matrix of size $k(M + 1) \times k(M + 1)$ (see [?] for details). By repeated use of (4.7) the numerical solution to (4.5)-(4.6) is given by

$$\varphi^n = (-2A^{-1} - I)^n\varphi^0.$$

Above scheme essentially involves inversion and multiplication of matrices of order $k(M+1)$. It is known that the computational complexity of such operation is $O(k^3M^3)$. Hence the computational complexity of computing φ^n is $O(nk^3M^3)$. If $T(n, M)$ is the complexity of computing φ^n for $n \leq N$. Then we have

$$T(N, M) = O(NM^3). \tag{4.8}$$

4.5 Numerical Example

In this section we consider an example of a Markov modulated market with three regimes. The state space is $\mathcal{X} = \{1, 2, 3\}$. The drift coefficient, volatility and interest rate at each regime are chosen as follows

$$\Big(\mu(i), \sigma(i), r(i)\Big) := \begin{cases} (1, 0.2, 0.3) & \text{if} \quad i = 1 \\ (1.2, 0.5, 0.8) & \text{if} \quad i = 2 \\ (1.5, 0.7, 1.2) & \text{if} \quad i = 3. \end{cases}$$

The transition rate matrix Λ is assumed to be given by

$$\Lambda = \begin{pmatrix} -1 & 2/3 & 1/3 \\ 1 & -2 & 1 \\ 1/3 & 2/3 & -1 \end{pmatrix}.$$

For this case we compute the price of a European call option where the strike price $K = 5$ and maturity $T = 1$. In order to compute numerically we need to choose space-time discretization. For the above market, the restriction suggested by (4.3)is $\Delta t = T/N$. We consider, in particular

$$\Delta t = T/N = 0.5/16 = 0.3125,$$

$$\Delta x = (10 * K)/M = (10 * 5)/300 = 0.1667$$

M is a large positive integer. We carry out computation for solving standard Black-Scholes pricing call option (3.12)-(3.13) as well as hedging strategy (3.16)-(3.17) for many different large values of M. For each M, the computational elapsed times are recorded for both the cases.

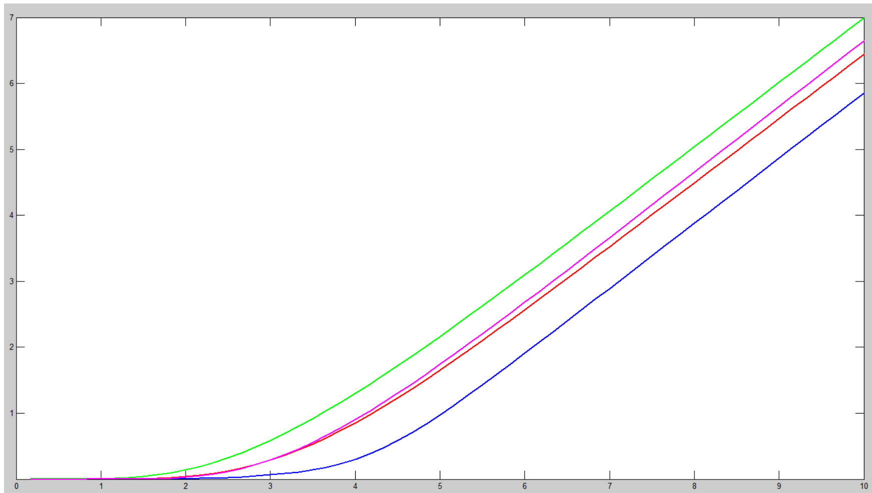

Figure 4.1: We take s along horizontal axis and plot the $\varphi(0, s, i)$ along vertical axis for $i = 1, 2, 3$. In this plot, the magenta line show the Black-Scholes price of call option with interest rate $r = r(2)$ and volatility = sigma(2). The other three curves blue, red and green show the show the European call option prices at time zero for three different initial regimes $1, 2, 3$ respectively with the same maturity and strike price.

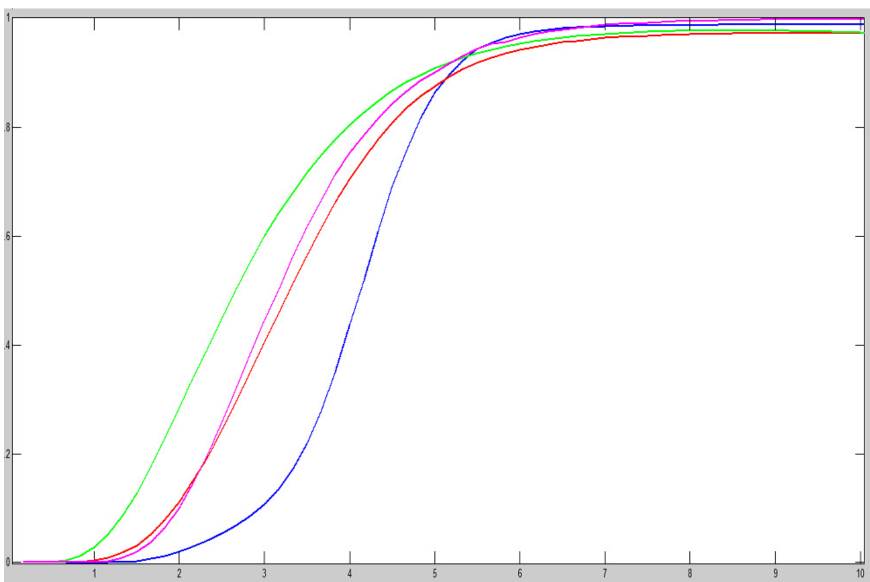

Figure 4.2: We take s along horizontal axis and plot the $\varphi(0, s, i)$ along vertical axis for $i = 1, 2, 3$. In this plot, the magenta line show the Black-Scholes price of call option with interest rate $r = r(2)$ and volatility = sigma(2). The other three curves blue, red and green show the show the hedging components at time zero for three different initial regimes $1, 2, 3$ respectively with the same maturity and strike price.

Appendix A

Errata

Errata on Springer lecture note (LNEMS 579) on Introduction to stochastic Calculus for Finance" by Dieter Sondermann.

1. Page 9 line 8 : Point 1. $\sup_n E[|X|^p]$ it should be replaced with $\sup_n E[|X_n|^p]$.
2. Page 25 line 6 : The equation should be replaced by the following arugment.
   $g(x) = x^2 \; g'(x) = 2x \; g''(x) = 2$,
   $F_1 := g \circ F$ and
   $Y_t := F(X_t)$. Thus
   $$F_1(X_t) = g(Y_t) \;\; \forall t > 0.$$
   Using Ito's formula we get
   $$dY_s = F'(X_s)dX_s + \frac{1}{2}F''(X_s)d\langle X\rangle_s \;\; \forall s > 0$$
   $$F_1(X_t) = F_1(X_0) + \int_0^t F_1'(X_s)dX_s + \frac{1}{2}\int_0^t F_1''(X_s)d\langle X\rangle_s$$
   $$g(Y_t) = g(Y_0) + \int_0^t g'(Y_s)dY_s + \frac{1}{2}\int_0^t g''(Y_s)d\langle Y\rangle_s$$
   Using above equations, it follows
   $d\langle Y\rangle_t = \int_0^t F'^2(X_s)d\langle X\rangle_s$.
3. Page 31 line 15: Y_t should be replaced with X_t.
4. Page 31 line 18: $G_0\mathrm{E}(\alpha X_t)$ should be replaced with $G_0\mathcal{E}(\alpha X_t)$.
5. Page 35 line 10:Whole sentence can be written as:- But, whereas the return $dX_t(\omega)$ depends on ω, the right hand side of (16) does not depend on dX_t, and therefore Π is riskless.

6. Page 42 line 1: This term $L^1 \in (\omega, \mathcal{F}_t, P)$ should be $L^1(\Omega, \mathcal{F}_t, P)$.
7. Page 43 line -2: Proof suggestion as follows :
   $P[T_b < \infty]$
   $=P[B_{T_{-n,b}} = b$ for some $n]$
   $=P[\bigcup_{n=0}^{\infty} \{B_{-n,b} = b\}]$
   $=\lim_{n\to\infty} P[B_{-n,b} = b]$
   $=\lim_{n\to\infty} \frac{n}{b+n} = 1.$
   Hence, $E[T_b] \geq E[T_{a,b}] = |a|.b \underset{a\downarrow-\infty}{\longrightarrow} \infty. \qquad \forall a < 0.$
8. Page 45 line 16: $\mathrm{E}[X_{t_{i+1}} - X_{t_i} \mid X_{t_i}] = 0$. The term $\mid X_{t_i}$ should be there.
9. Page 45 line 19: M_t^n**is a discrete time** martingale.
10. page 47 line 9, line 10 and line 11 : $\forall t$ should be mentioned.
11. Page 49 line 2 : $\mathrm{Var}[M_t] = E[{M_t}^2] - (E[M_t])^2 = E[\langle M\rangle]_t +$ $\mathbf{Var}[M_0]$.

Appendix B

Matlab code

```
clear
d = 1.0/(sqrt(2*pi));
f=1/sqrt(2);
T =0.5;      % MATURITY
N=16;
dt=T/(N-1);
    st= 5.0; % STRIKE PRICE;
     X0=0; % LOWER LIMIT OF S
M=300;
eta= zeros(N,M,3);
u=zeros(N,M,3);
C11=zeros(N,3);
C12=zeros(1,3);
de= zeros(N,M,3);
du=zeros(N,M,3);
C21=zeros(N,3);
C22=zeros(1,3);
C3=zeros(N,3);
LN1=zeros(N,M,M,3);
LN2=zeros(N,M,M,3);
xint=zeros(N,M,3);
dx= (10*st)/M;
     P= [0,2/3,1/3;0.5,0,0.5;1/3,2/3,0];
     lambda=[1,2,1];
     R=[0.3,0.8,1.2];
     SIG=[0.2,0.5,0.7];
% Black-Schole-Merton Solution
```

```
% GENERATE STANDARD NORMAL DISTRIBUTION FUNCTION
SND= 0.5+ 0.5*erf(f*(-4+0.001*(1:4000)));
%CALCULATE eta and de
for k=1:3
    rp=R(k)+(1.0/2)*SIG(k)^2;
    rm=R(k)-(1.0/2)*SIG(k)^2;
            for i=2:N
                tm=(i-1)*dt;    %tm is time to expiry:=T-t
                dn= SIG(k)*sqrt(tm);        %dn is denominator
                for j=1:M
                    s=j*dx;             %s is in (0, 10)
                    x= (log(s/st)+rm*tm)/dn;
                    if x > 4.0
                    ph= 1.0;
                      elseif x<-4.0
                             ph= 0.0;
                      elseif x<0.0
                             xx=floor(1000*(4.0+x))+1;
                             ph= SND(xx);
                      elseif  x>0.0
                             xx=floor(1000.0*(4.0 - x))+1;
                             ph= 1 - SND(xx);
                    end
                    term= st*exp(-R(k)*tm)*ph;
                    y= (log(s/st)+rp*tm)/dn;
                    if y > 4.0
                    ph= 1.0;
                      elseif y<-4.0
                             ph= 0.0;
                      elseif y<0.0
                             xx=floor(1000*(4.0+y))+1;
                             ph= SND(xx);
                      elseif  y>0.0
                             xx=floor(1000.0*(4.0 - y))+1;
                             ph= 1 - SND(xx);
                    end
                    eta(i,j,k)= s*ph - term;
                    de(i,:,k) = gradient(eta(i,:,k),dx);
                end
            end
end
%DEFINE THE INITIAL DATA \Phi^1
for j=1:M
     for i=1:3
         u(1,j,i)=max(0.0,dx*(j-1)-st);
     end
end
%DEFINE THE INITIAL DATA \Psi^1
for j=1:M
     for i=1:3
         du(1,j,i)=max(sign(dx*(j-1)-st),0);
     end
```

```
end
%Construction of Black-Scholes Solution
%Construction of Matrix LN( v, s, x, i)
 for i=1:3
     C12(i)= d/SIG(i);
     C22(i)= d/(SIG(i)^3);
        for kk=2:N
        C11(kk,i) = lambda(i)*(exp(-(R(i)+lambda(i))*((kk-1)*dt)) /
sqrt((kk-1)*dt));
        C21(kk,i) = lambda(i)*(exp(-(R(i)+lambda(i))*((kk-1)*dt)) /
((kk-1)*dt)^(3/2)) ;
        C3(kk,i) = exp(-lambda(i)*(kk-1)*dt);
        for j=1:M
        for jj=1:M
        LN1(kk,j,jj,i)=exp(-0.5*(( log(jj/j)-(R(i)-0.5*SIG(i)^2)*((kk-
1)*dt))/(SIG(i)*sqrt((kk-1)*dt) ))^2);
        LN2(kk,j,jj,i)=(exp(-0.5*(( log(jj/j)-(R(i)-
0.5*SIG(i)^2)*((kk-1)*dt))/(SIG(i)*sqrt((kk-1)*dt) ))^2)) *
(log(jj/j)-(R(i)-0.5*SIG(i)^2)*((kk-1)*dt)) ;
        end
        end
        end
end
%CALCULATING u(t,s,i) FOR ALL t,s,i"
for k=2:N
for j=1:M
for i=1:3
    vint=0;
    for kk=2:k
        jj=1;        % INTEGRATION wrt x starts (trapizoidal rule)
        term =0;
        for ii=1:3
        term=term + u(k-kk+1,jj,ii)* P( i, ii);
        end
        xint(kk,j,i) = 0.5 * term * LN1(kk,j,jj,i)/jj;
        for jj=2:M-1
             term =0;
             for ii=1:3
                term=term + u(k-kk+1,jj,ii)* P( i, ii);
             end
        xint(kk,j,i) = xint(kk,j,i)+term*LN1(kk,j,jj,i)/jj;
        end
            jj=M;
            term =0;
            for ii=1:3
             term=term + u(k-kk+1,jj,ii)* P( i, ii);
            end
         xint(kk,j,i) = xint(kk,j,i)+0.5* term*LN1(kk,j,jj,i)/jj;
         % INTEGRATION wrt x completed
        vint = vint + xint(kk,j,i) * C11(kk,i) * dt;
    end
        u(k,j,i)=C3(k,i)*eta(k,j,i) + vint * C12(i);
```

```
end
end
end

 %CALCULATING du(t,s,i) FOR ALL t,s,i"
for k=2:N
for j=1:M
for i=1:3
    vint=0;
    for kk=2:k
        jj=1;         % INTEGRATION wrt x starts (trapizoidal rule)
        term =0;
        for ii=1:3
        term=term + u(k-kk+1,jj,ii)* P( i, ii);
        end
        xint(kk,j,i) = 0.5 * term * LN2(kk,j,jj,i)/jj;
        for jj=2:M-1
             term =0;
             for ii=1:3
                term=term + u(k-kk+1,jj,ii)* P( i, ii);
             end
        xint(kk,j,i) = xint(kk,j,i)+term*LN2(kk,j,jj,i)/jj;
        end
            jj=M;
            term =0;
            for ii=1:3
             term=term + u(k-kk+1,jj,ii)* P( i, ii);
            end
         xint(kk,j,i) = xint(kk,j,i)+0.5* term*LN2(kk,j,jj,i)/jj;
         % INTEGRATION wrt x completed
        vint = vint + xint(kk,j,i) * C21(kk,i) * dt;
    end
        du(k,j,i)=C3(k,i)*de(k,j,i) + vint * C22(i)/(j*dx);
end
end
end
ss=dx*[1:M];
%ploting
plot(ss,u(N,:,1),'blue', ss,u(N,:,2),'red', ss,u(N,:,3),'green',
ss,eta(N,:,2),'magenta')
plot(ss,du(N,:,1),'blue', ss,du(N,:,2),'red', ss,du(N,:,3),'green',
ss,de(N,:,2),'magenta')
```

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