# Resource Theory of Quantum Coherence

**A Thesis**

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Shubhalakshmi S

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2019

Supervisor: Prof. Ujjwal Sen
Harish-Chandra Research Institute

# Certificate

This is to certify that this dissertation entitled “Resource Theory of Quantum Coherence” towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by “Shubhalakshmi S at Harish-Chandra Research Institute” under the supervision of “Prof. Ujjwal Sen, Professor, Department of Physics” , during the academic year 2018-2019.

( Shubhalakshmi S)
Signature of the student

(Prof. Ujjwal Sen)
Signature of the supervisor

Committee:

Prof. Ujjwal Sen

Prof. T S Mahesh

This thesis is dedicated to all the well-wishers

# Declaration

I hereby declare that the matter embodied in the report entitled "Resource Theory of Quantum Coherence" are the results of the work carried out by me at the Department of Physics, Harish-Chandra Research Institute, under the supervision of Prof. Ujjwal Sen and the same has not been submitted elsewhere for any other degree.

( Shubhalakshmi S)
Signature of the student

(Prof. Ujjwal Sen)
Signature of the supervisor

# Acknowledgments

Writing this report has been entrancing. I take the opportunity to thank all the people who have contributed to the final result in several ways.

I owe my most sincere gratitude to my advisor, Prof. Ujjwal Sen for being friendly, constantly supporting guiding force and for his patience throughout my learning process. The discussions were truly enlightening and highly motivating in gaining immense knowledge.

Besides my supervisor, I am very grateful to Prof. T S Mahesh of IISER Pune for introducing me to the topic and for creating an opportunity to work with the country's one of the foremost group in the field. I would also like to thank the MS thesis committee members for believing in me and hence approving me off to carry out my final year project work at HRI.

I'm highly thankful to the other QIC group members, visitors and others at HRI for accepting me to be a part of the community, for being cheerful and highly encouraging all the time. The discussions and other group activities were intellectually stimulating and were a great fun.

The thesis has been written during my stay at HRI. I thank the Academy for providing excellent study atmosphere and also for its financial support. I would also thank IISER for providing amazing atmosphere for exploring myself personally and professionally during my four years BS-MS coursework.

I am immensely indebted to the leading researchers in this field for sharing their knowledge and making me available the reprints of their publications. Their ideals and concepts have had a remarkable influence on shaping my thought process.

I would like to express my profound gratitude to my family and dearest friends who have been my pillars of strength and support throughout this journey and for their forever love.

# Abstract

Concepts developed in Quantum information have recently been successfully utilized in characterizing various physical phenomena. One of the frameworks developed in studying quantum systems is quantum resource theory. It is a rigorous mathematical framework which identifies a physical characteristic as a resource, and develops a theory in which the building blocks are "free quantum operations" - the operations that do not create the resource, and "free quantum states" - the states that are devoid of the resource. It also often provides ways to optimally convert one resource state into another. This report is a review on the resource theoretic approach to the study of quantum coherence. Along with focusing on the general development of the resource theory of quantum coherence, it discusses about the measures that quantify the resource. In addition, numerics on the quantitative measures for single qubit states is being done and analysed. It also discusses how the theoretical framework gives new perspective to the previous experimental results and demands for new experimental designs for better understanding of the concepts.

# Contents

# List of Figures

# Chapter 1

# Introduction

Quantum information science emerged as a field of study around two decades ago. The study involves in exploiting the quantum properties of a system to perform tasks efficiently. This field is related to many other areas such as mathematics, computer science etc apart from physics. Along with major applications like cryptography, faster and efficient quantum computation, it is also studied as a tool to understand deeper questions involving quantum mechanics and properties of quantum systems. Amongst the various frameworks of study, quantum resource theory(QRT) structure was a very recently developed framework which involves rigorous mathematical equations to bring out the beauty of a physical phenomena or a quantum object.

Each object gets a value assigned according to its characteristic properties. Resource theory is a mathematical framework that characterizes a particular object or a property according to its physical capabilities. From resource theory, we get to know about the allowed actions and the operations that are free or prohibited in the given scenario. Resource theory approach to problems depends on what particular object or an action you are considering at that point of time. Recently this perspective has succeeded in quantum information and communication branch in studying the quantum systems. So, in quantum case, this approach is called quantum resource theory and it deals with the study of systems involving atomic and sub-atomic level particles, objects and phenomena. Depending on the physics of the problem that we are interested in, quantum resource theoretic approach helps optimize the study process. This outlook was a great hit for the entanglement theory. So, this approach has been adopted in characterizing various other physical phenomena in quantum physics and hence building resource theory for all those resources. The examples within quantum information theory include quantum resource theory of entanglement [19, 20], quantum coherence [22] and superposition [1], quantum thermodynamics [21, 23], quantum correlations [24], asymmetry and quantum reference frames [25] , non-locality [26], 'magic states' in stabilizer quantum computation [27] etc. We can hence say that QRTs correspond to physical models.

Resources are classified as follows : classical or quantum, noisy or noiseless and static or dynamic. The term "resource theory" was first coined by Schumacher in article of 2003 that was unpublished. The name was first made official in [6]. That was the first explicit construction of quantum resource theory of something other than that of entanglement. There have been papers earlier discussing about the QRT of information. A precise mathematical definition of resource theory was first given in [29] and [28] as symmetric monoidal cathegory. Later it was in [18] a simpler mathematical generalization of QRTs was given.

We present here a brief discussion of general structure of quantum resource theory. The discussion involves only a finite dimensional Hilbert Space ($\mathscr{H}$), say of the dimension d.
Definition of QRT (minimal mathematical requirements): Let $\mathscr{O}$ be a mapping that assigns two Hilbert spaces into a unique set of completely positive trace preserving (CPTP) operations, that is, $\mathscr{O}(\mathscr{H}_{in},\mathscr{H}_{out}) \subset \mathfrak{B}(\mathscr{H}_{in},\mathscr{H}_{out})$ . Let an induced mapping be defined as $\mathscr{F} := \mathscr{O}(\mathbb{C},\mathscr{H})$ where $\mathbb{C}$ is complex field and $\mathscr{H}$ is an arbitrary Hilbert space. QRT is defined as a tuple $\mathscr{R} = (\mathscr{F},\mathscr{O})$ provided the following conditions hold:

i. Identity map id should be contained in the set $\mathscr{O}(\mathscr{H}) := \mathscr{O}(\mathscr{H},\mathscr{H})$ for any Hilbert space $\mathscr{H}$. Physically, this simply means that identity map is free.

ii. If $\Lambda \in \mathscr{O}(\mathscr{H}^A,\mathscr{H}^B)$ and $\Phi \in \mathscr{O}(\mathscr{H}^B,\mathscr{H}^C)$ then $\Phi \circ \Lambda \in \mathscr{O}(\mathscr{H}^A,\mathscr{H}^C)$. This means that $\Phi \circ \Lambda$ is free only when both $\Phi$ *and* $\Lambda$ are free.

In QRT, we have:

- *Free states*: $\mathscr{F}(\mathscr{H}) \subset S(\mathscr{H})$ defines the set of free states acting on $\mathscr{H}$. Here $S(\mathscr{H})$ is set of states in $\mathscr{H}$. These are states which do not have the resource that is under study.
- *Resource states*:These are also known as static resources. These are the states that belong to set $S(\mathscr{H}) \setminus \mathscr{F}(\mathscr{H})$. These are the states that contain resource of interest and are not available free of cost. These are those states that allow for operational advantage in some task.

- *Free operations*: CPTP maps in $\mathscr{O}(\mathscr{H}_{in}, \mathscr{H}_{out})$. These operations are freely accessible and they map set of free states onto itself.

- *Dynamical resources*: CPTP maps that are not contained in $\mathscr{O}(\mathscr{H}_{in}, \mathscr{H}_{out})$.

There is also *tensor product structure* of QRTs which is a better mathematical description and explains more properties. There are other types of mathematical structures in QRTs which include convex resource theories, non-convex resource theories, affine resource theories, resource theories of quantum processes and QRTs with resource destroying maps. The golden rule of QRTs is that free operations cannot convert any free state to another state which is not contained in the set of free states and also cannot create resource state from any free state. If resource state(s) is available then the parties can certainly overcome the limitations of the allowed(free) operations either partially or completely. Depending on the physical constraint either free operations or free states turn out to be the natural choice. We can say that QRT models the tasks that can be accomplished physically by the parties, given a quantum system with rules/law of physics and the experimental limitations. It basically studies the interconversion of resource states under free or restricted operations, specially it is about interconversion between pure resource states and mixed ones. If conversions between pure states is asymptotically reversible then we can construct a standard unit resource measure. This motivates to study of two basic processes which are resource distillation and resource formation. Resource distillation is transformation from mixed resource state to unit resource and resource formation is the reversion transformation from a unit resource to mixed one. The two well-motivated quantities due to the two basic processes are distillable resource and resource cost. These quantities have a clear operational interpretation and characterising these quantities is the principle objective of the theory.Limits on possible transformations and achievable rates are given by resource monotones and thus reduce the complexity of the characterization problem. In case of reversible QRT, a unique measure exists, everything about resource transformation is clear and simple. But in case of irreversible QRT, the focus is on whether a bound resource states exist or not. Irreversible QRT is the case in which no resource can be extracted (or distilled) but a non-zero amount of resource is needed to create the bound resource states.

Main advantages of QRT framework are:

- This framework allows for comparison of amount of resource present in the different quantum states. This tells us that if a particular resource state can be converted into another through free operations then the both states have same amount of resource. This framework helps us understand how valuable resources are and they cannot be created freely.

- From practical viewpoint, QRT framework help reflect experimental capabilities. Depending on the resource under investigation, practical setup is always in such a way that free operations can be performed within experimental degrees of freedom.

- QRT gives a complete and detailed mathematical description of the physical phenomenon that is under study. It also analyses the physical phenomena and completely characterizes it.

- It is easier to draw similarities and do comparisons amongst the resources studied with this framework. This helps a lot in identifying various applications. This is specially when resource that has already been characterised is having few similar properties with the one being studied.

This report is an overview of a specific physical phenomena as a resource.The resource theory framework of quantum coherence has been discussed in detail and few results have been reproduced. The first section introduces to the concept of quantum coherence and explains the basis-dependent structure of resource theory of quantum coherence that was first developed in [2]. The second section is all about coherence quantifiers. And then there is discussion on some of the numerics done based on the coherence quantifiers. The report closes on briefly summarizing the explained topics.

# Chapter 2

# Quantum Coherence

Study of coherence dates back to the time when people were trying to understand the physics of waves. Coherence concept which is central to interference phenomena has applications beyond ray optics and classical regime. Quantum Mechanics, a theory that unifies waves and particle nature further strengthened the role of coherence in physics. Superposition principle is a signature reason for departure of quantum mechanics from classical physics. Coherence along with energy quantization and tensor product structure is fundamental to interference and multipartite entanglement. Hence has a wide applications in quantum physics and quantum information science.

Quantum theory of coherence is constructed viewing coherence as a quantum physical phenomena that can be exploited as a resource to achieve various tasks that are otherwise not possible in classical physics regime. Framework of the theory resembles classical electromagnetic phenomena and is in terms of phase space distribution and multipoint correlation functions [15, 16, 17]. Quantitative theory was developed capturing the resource character of this physical phenomena in a mathematically rigorous fashion. The constructed theories focus on the ability to create coherence and detect coherence as in, how the presence of coherence makes differences in measurement statistics [14]. The development of such a theory has resulted in number of quantum technologies not just restricted to lasers and its applications but also quantum-enhanced metrology and communication protocols and extends to thermodynamics and certain branches of biology.

Following the early approach of quantifying superposition of orthogonal quantum states [1] and progressing along the lines of the independent related resource theory of asymmetry [5, 6, 7, 8, 9] and trying to draw parallels with resource theory of quantum entanglement, primary work on resource theory on coherence was proposed by in [2] and was further developed in [10, 11, 12],[4] and several others including [13]. The theory is still under development, provides rigorous framework describing quantum coherence in analogy to what has been done with other non classical resources. This theory focuses on answering questions like the resource cost and number of achievable tasks by a classical device that cannot create coherence in preferred basis. There are theoretical frameworks which are constructed on level of operations [14] and they mainly focus on how well coherence be detected by a given operation.

## 2.1 Resource Theory of Quantum Coherence

The resource theory framework of quantum coherence that will be discussed here will be the one which was developed in [2]. This framework is built by considering coherence to be a basis dependent concept. This is related to superposition principle. Intuitively, the amount of coherence present in the system is considered to be a function of the off-diagonal elements of the system's density matrix. The off-diagonal entries of a density matrix are basis dependent. Any density matrix can be diagonalized and hence off-diagonal elements can be made zero which would imply zero coherence. Basically, in this framework, coherence is measured as the minimal distance between the quantum system state and the line passing through the center of the bloch sphere connecting the poles (that is, computational eigenbasis which would be $\{|0\rangle, |1\rangle\}$). There are also works done in the basis-independent direction which is related to purity of the system [30, 31].In basis-independent measure, coherence is the distance measured from the system state to the center of the block sphere. In this report only the basis-dependent framework will be discussed.

### 2.1.1 Incoherent states

These are what are called as the free states , that is , these are the states available free of cost in the resource theory of quantum coherence. These states are defined depending on preferred reference basis. The physics of the problem dictates the choice of the reference basis. These are the states which when written in density matrix formulation are diagonal in the specific basis.

The following is the mathematical definition of incoherent states. Given the Hilbert space dimension to be d denoted as $\mathscr{H}$ (please note that here 'd-the dimension of the considered Hilbert space' is assumed to be finite though some extensions to 'infinite d' can be contemplated). For *single party case*. Let the reference basis of the corresponding Hilbert space be denoted as $\{|\psi_i\rangle_{i=0,1,..,d-1}\}$. Note that these are orthonormal states. All the incoherent density operators in the choice of basis defined above are of the form

$$\rho = \sum_{i=0}^{d-1} p_i |\psi_i\rangle \langle\psi_i|$$

where $p_i$ denotes the probability of $|\psi_i\rangle$.
These form a set of states called Incoherent states denoted by *I*. The set of Incoherent states set are subset of set of bounded trace class operators defined on $\mathscr{H}$ (that is, $I \subset$ B($\mathscr{H}$)).

For *Multi party case*. Coherence is studied with respect to the reference basis that is constructed as the tensor product of the local preferred basis of each corresponding subsystems. The convex combinations of incoherent pure product states are defined to be the general multi party incoherent states. So, for an N-qubit system, all the diagonal density matrices $\rho$ in the composite computational basis, i.e. , $\{|0\rangle, |1\rangle\}^{\otimes N}$ are the only states that form set of incoherent states *I*.

### 2.1.2 Coherent states

In the resource theory of coherence, all those states that do not belong to the set of incoherent states are resource states. These states are called coherent states. These are those states when written in terms of density matrix have non-zero off diagonal elements in the preferred basis.

### 2.1.3 Incoherent operations

Free operations for the resource theory of coherence are called the incoherent operations. An incoherent state (free state) acted upon by an incoherent operation remains incoherent. Mainly coherence (resource) cannot be created, not even probabilistically from incoherent states when acted upon by incoherent operations. These operations are not unique but rather dependent on the practical considerations of the problem under investigation.

The following is the characterization of the set of incoherent physical operations. These are set of completely positive trace preserving (CPTP) maps, $\Lambda$: B($\mathscr{H}$) $\mapsto$ B($\mathscr{H}$) acting on a state as

$\Lambda[\rho] = \sum_n K_n^\dagger \rho K_n$ where $\{K_n\}$ are set of Kraus operators satisfying,

$$\sum_n K_n^\dagger K_n = \mathbb{I}$$
$$K_n^\dagger I K_n \subseteq I \quad \forall n$$

The two mainly distinguished general classes of quantum operations are:

I. Incoherent completely positive and trace preserving quantum operations (ICPTP).

$$ICPTP(\rho) = \sum_n K_n^\dagger \rho K_n$$

where $\{K_n\}$ are all of same dimension $d_{out} \times d_{in}$.
This formulation implies that any information loss about measurement outcome may in principle be available.

II. The quantum operations for which measurement outcomes are preserved and accordingly sub selection is permitted. Corresponding to outcome n, the state is

$$\rho_n = (K_n^\dagger \rho K_n)/p_n$$

occurring with probability $p_n = \mathrm{tr}[K_n \rho K_n^\dagger]$.

Few important classes of incoherent operations are:

- *Maximally incoherent operations (MIOs)*: Also known as incoherence preserving operations was first defined in [1]. These creation-incoherent operations are defined to be completely positive trace preserving and non-selective quantum operations. These constitute the largest set of free operations in the resource theory of quantum coherence.

- *Strictly incoherent operations (SIOs)*: This class of operations were first introduced in [4]. Under the constraints that incoherent operators must follow, each Kraus operator of $m^{th}$ measurement outcome can be written in the following form.$K_m = \sum_i c(i) \left|j(i)\right\rangle \left\langle i\right|$ where $\left|i\right\rangle$ is index set of basis , $\left|j\right\rangle$ is function of the index set and c(i) are coefficients. Now we call the quantum operation SIO when not just $K_m$ is incoherent but $K_m^\dagger$ is also incoherent. And the strictly incoherent Kraus operators, $K_m$ are specified as one-to-one $|j(i)\rangle$ function. SIOs are those operations which have Kraus decomposition $\{K_m\}$ such that measurement outcome in reference basis when applied to output state are independent of the coherence of the input state.

There are many other classes of incoherent operations that aren't discussed here but are constructed from operational point of view which include translationally invariant operations (TIO) [5, 6, 7, 8, 32, 33, 34], physical incoherent operations (PIO) [10, 11], dephasing-covariant incoherent operations (DIO) [10, 11, 12, 33]. Many classes of incoherent operations are constructed for studying the role of energy in the context of coherence and these include genuinely incoherent operations (GIO) [35], energy preserving operations (EPO) [36]. Fully incoherent operations (FIO) [35] are the most general set which turn out to be incoherent for every Kraus decomposition.

### 2.1.4 Maximally Coherent State

A d-dimensional maximally coherent state is defined as the state which allows deterministic generation of all other d-dimensional quantum states via free operations(i.e, incoherent operations). The definition of a maximally coherent state is independent of a particular coherence measure and allows to identify a unit for coherence measure which is termed as coherence bit or cobit [37] to which all measures can be normalized. A canonical example of a maximally coherent state is given by:

$$|\phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle$$

## 2.2 Quantifying Coherence

After qualitatively characterising the resource, the very next topic of interest would be whether the physical trait that has been termed as resource can be quantified or not. To know whether a particular value can be considered as the amount of resource present in the given system. So in the following section, measures and their properties are briefly summarized. Intuitively one can relate a function of the off-diagonal elements of considered quantum state to be quantitative measure of the coherence. This idea simply comes from the way the free states in the basis-dependent resource theory framework of coherence. In case of quantum coherence as a resource, axiomatic approach on quantification was first presented in [1]. Later on in [2], an alternative framework was put forth and there has been lot of work still on going in this area.

Resource theoretic approach to quantification of a resource defines the properties of a functional and states the conditions that it has to satisfy to be a valid measure of the resource under investigation. The following are the postulates of any coherence quantifier must obey [22].

(C1) **Non-negativity**: The very first quick check for measure, C is that it should vanish only for incoherent states and it should vanish for all incoherent states. Otherwise( that is, for all the other states which are termed coherent) the value that C takes should be positive.

$$C(\rho) \geq 0$$

with equality if and only if $\rho \in I$.

(C2) **Monotonicity**: For C to be a proper measure of coherence, it must not increase under any type of incoherent operations.

$$C(\wedge[\rho]) \leq C(\rho)$$

for any IO. Monotonicity under type(I) incoherent operations, that is for ICPTP maps.

$$C(\wedge[\rho]) \leq C(\rho) \;\; \forall \wedge \text{ being ICPTP maps.}$$

(C3) **Strong monotonicity**: Monotonicity under type (**II**) incoherent operations(i.e,selective measurements on average).

$$\sum_n p_n C((\rho_n) \leq C(\rho) \;\; \forall \{K_n\}$$
$$\text{with } \sum_n K_n^\dagger K_n = \mathbb{I} \text{ and}$$
$$K_n^\dagger I K_n \subseteq I \;\; \forall n$$

where $p_n$ =tr[K$_n\rho$K$_n^\dagger$] and the post measurement being $\rho_n = (K_n \rho K_n^\dagger)/p_n$.

(C4) **Convexity**: C should not increase when there's mixing of quantum states. Therefore, C must be a convex function.

$$\sum_n p_n C(\rho_n) \geq C(\sum_n p_n \rho_n)$$

for any $\{\rho_n\}$ and $p_n \in$(0,1) satisfying $\sum_n p_n = 1$

(C5) **Uniqueness for pure states**: The measure C takes special form in case for pure states $|\psi\rangle$.

$$C(|\psi\rangle\langle\psi|) = S(\triangle|\psi\rangle\langle\psi|)$$

where $\triangle$ is dephasing operator that is defined as $\triangle[\rho] = \sum_{j=0}^{d-1} |j\rangle\langle j|\rho|j\rangle\langle j|$ and $S(\rho) = -\mathrm{Tr}[\rho\log(\rho)]$ is the von Neumann entropy.

(C6) **Additivity**: Under tensor products, C is additive.

$$C(\rho\otimes\sigma) = C(\rho) + C(\sigma)$$

Few remarks on the postulates.
We observe that if a quantity C satisfies the postulates C3 and C4, it implies that C also satisfies the postulate C2 [2]. And here's the proof.
Consider

$$\begin{aligned} C(_{ICPTP}(\rho)) &= C(\sum_n p_n\rho_n) \\ \Rightarrow C(_{ICPTP}(\rho)) &\leq \sum_n p_n\, C(\rho_n) \;\; (\because \text{postulate } C4) \\ \Rightarrow C(_{ICPTP}(\rho)) &\leq C(\rho) \;\; (\because \text{postulate } C3) \end{aligned}$$

This is exactly C2 postulate. Hence proved.

Conditions C1 and C2 are the minimal conditions that any measure C to be a meaningful coherence quantifier. Remaining conditions are derived by drawing analogy from the entanglement theory. So, we name coherence quantifiers accordingly. If a quantity C satisfies only postulates C1 and either of C2 or C3 or both, its called a *coherence monotone*. Most of the Coherence monotones are observed to violate the postulates C5 and C6 even if few of them follow C3 and C4 postulates along with C1 and C2.
In 2016, [40] came up we alternate requirements for a coherence monotone. This showed that instead of C3 and C4,along with obeying C1 annd C2 it is enough for a quantity C to be additive on block-diagonal state. Mathematically,

$$C(p\,\rho\oplus(1-p)\,\sigma) = p\,C(\rho) + (1-p)\,C(\sigma)$$

If a quantity satisfies all the conditions together from C1 to C6, it is called a *coherence measure*.

Additional to these postulates, in [41] postulated that a valid quantifier of coherence must be maximal only on the set of pure maximally coherent state $|\phi_d\rangle$. This additional postulate is satisfied by particularly by two coherence measures, i.e, distillable coherence and coherence cost. These two quantifiers follow C1 to C6 and are hence considered to be proper coherence measures.

### 2.2.1 Distance based measures

Distance measures have been natural applicants for resource quantifiers.
Definition [2]: We define coherence measure C as the minimum distance, $\mathfrak{D}$ of the given state $\rho$ to the set of incoherent states $I$. Mathematically,

$$C_{\mathfrak{D}}(\rho) \;=\; \min_{\sigma \in I} \mathfrak{D}(\rho, \sigma)$$

We recheck whether the distance measure defined above follows the postulates that we had defined.

- Consider $\rho$ to be an incoherent state, i.e, $\rho \in I$. In this case, the minimum distance from an incoherent state to incoherent set would be the distance from $\rho$ to itself which is zero.So,the quantifier C vanishes when $\rho$ is incoherent. And for any other state $\rho$(that is not incoherent), the distance is always positive.Hence this defined quantifier follows postulate C1.

- If distance $\mathfrak{D}$ contracts under CPTP maps, that is ,

  $$\mathfrak{D}(\rho, \sigma) \;\geq\; \mathfrak{D}(_{CPTP}(\rho),_{CPTP}(\sigma))$$

  Same follows for ICPTP maps, that is,

  $$\mathfrak{D}(\rho, \sigma) \;\geq\; \mathfrak{D}(_{ICPTP}(\rho),_{ICPTP}(\sigma))$$

Now consider

$$\begin{aligned} C_{\mathfrak{D}}(\rho) \;&=\; \min_{\sigma\in I} \mathfrak{D}(\rho,\sigma) \\ \Rightarrow C_{\mathfrak{D}}(\rho) \;&\geq\; \min_{\sigma\in I} \mathfrak{D}({}_{ICPTP}(\rho),\; {}_{ICPTP}(\sigma)) \\ \text{We have } {}_{ICPTP}(I) \in I &\text{ and } \sigma \in I \\ \Rightarrow_{ICPTP} (\sigma) \;=\; \sigma &\text{ and} \\ \therefore C_{\mathfrak{D}}(\rho) \;&\geq\; \min_{\sigma\in I} \mathfrak{D}({}_{ICPTP}(\rho),\sigma) \\ \Rightarrow C_{\mathfrak{D}}(\rho) \;&\geq\; C_{\mathfrak{D}}({}_{ICPTP}(\rho)) \end{aligned}$$

Hence the defined quantifier in terms of distance between the states follows postulate C2.

- If distance is jointly convex, that is,

$$\mathfrak{D}(\sum_i p_i\rho_i, \sum_j p_j\sigma_j) \;\leq\; \sum_i p_i\, \mathfrak{D}(\rho_i,\sigma_i).$$

Now we have

$$\begin{aligned} C_{\mathfrak{D}}(\sum_n p_n\rho_n) = \min_{\sigma\in I} \mathfrak{D}(\sum_n p_n\rho_n, \sum_n p_n\sigma_n) &\text{ for } \sigma\in I \\ C_{\mathfrak{D}}(\sum_n p_n\rho_n) \leq \min_{\sigma\in I} \sum_n p_n\, \mathfrak{D}(\rho_n,\sigma_n)& \\ \Rightarrow C_{\mathfrak{D}}(\sum_n p_n\rho_n) \leq \sum_n p_n \min_{\sigma\in I} \mathfrak{D}(\rho_n,\sigma_n)& \\ \text{But since } \min_{\sigma\in I} \mathfrak{D}(\rho_n,\sigma_n) = C_{\mathfrak{D}}(\rho_n)& \\ \therefore C_{\mathfrak{D}}(\sum_n p_n\rho_n) \leq \sum_n p_n\, C_{\mathfrak{D}}(\rho_n)& \end{aligned}$$

Hence coherence quantifier $C_{\mathfrak{D}}(\rho)$ follows postulate C2.

- The coherence quantifier $C_{\mathfrak{D}}(\rho)$ is convex and hence follows the postulate C4.

- The proof of distance measure follows postulate C3 includes proof of few conditions that are given in [39] are following :

  (F1) $\mathfrak{D}(\rho\|\,\sigma) \geq 0)$ with equality holding only when $\rho = \sigma$
  Now in the case of coherence quantifier $C_{\mathfrak{D}}(\rho)$, this is true because of the postulate C1.

(F2) Unitary invariance: $\mathfrak{D}(\rho\|\,\sigma)$ should remain invariant under unitary operation.That is,

$$\mathfrak{D}(\rho\|\sigma) \;=\; \mathfrak{D}(U\rho U^{\dagger}\|\,U\sigma U^{\dagger})$$

(F3) Distance doesnot increase under partial tracing.That is,

$$\mathfrak{D}(tr_p\rho\|\,tr_p\sigma) \;\leq\; \mathfrak{D}(\rho\|\,\sigma)$$

where $tr_p$ is partial trace.

(F4) $\sum_i p_i\,\mathfrak{D}((\rho_i/p_i)\|(\sigma_i/q_i)) \;\leq\; \sum_i \mathfrak{D}(\rho_i\|\sigma_i)$
where $p_i = tr(\rho_i)$; $q_i = tr(\sigma_i)$; $\rho_i = V_i\rho V_i^{\dagger}$; $\sigma_i = V_i\sigma V_i^{\dagger}$; where $V_i$'s may not be necessarily local.

(F5a) $\mathfrak{D}(\sum_i P_i\rho P_i \,\|\, \sum_i P_i\sigma P_i) = \sum_i \mathfrak{D}(P_i\rho P_i \,\|\, P_i\sigma P_i)$
where $P_i$ =Any set of orthogonal projectors,i.e, $P_iP_j = \delta_{ij}P_i$

(F5b) $\mathfrak{D}(\rho\otimes P_\alpha \,\|\, \sigma\otimes P_\alpha) \;=\; \mathfrak{D}(\rho\|\sigma)$; $P_\alpha$ is any projector.

Condition F2 ensures that measure is invariant under local unitary. Conditions F2, F3, F4 and F5 ensure that the quantifier does not increase under local general measurement and postselection.

Two distance based measures of coherence will be briefly discussed here and are following :

i. Based on matrix norms.

ii. Relative entropy of coherence.

**Distance measure based on matrix norm**

Matrix norm based distance measures are defined in the following way:

$$\mathfrak{D}(\rho,\sigma) \;=\; \|\rho-\sigma\|$$

with $\|\cdot\|$ being some matrix norm.
Now, for the above defined quantity to be a valid coherence quantifier(i.e, a coherence monotone), it must satisfy the postulates C1 to C4.

We observe that the general definition that is given above turns out to be a jointly convex function provided it satisfies

- Triangle inequality which is in general stated as

  $$\|M1+M2\| \leq \|M1\| + \|M2\|$$

  for any two matrices M1 and M2.

- Absolute homogenity which is stated as

  $$\|\alpha M\| = |\alpha| \, \|M\|$$

  for any $\alpha \in \mathbb{C}$ and any matrix M.

Hence distance measure based on matrix norm is called convex coherence quantifier as it satisfies the postulate C4.

Now we have to check whether in general all the matrix norm satisfy other postulates or only specific ones do. This is done by taking relevant norms and checking for the conditions.And these include the $l_p$ norms and Schatten p-norms.

- $l_p$ norms: These are denoted as $\|\cdot\|_{l_p}$. And the mathematical definition goes as follows:

  $$\|M\|_{l_p} = \Big( \sum_{i,j} |M_{i,j}|^p \Big)^{1/p} \;\; for \;\; p \geq 1.$$

  Corresponding coherence quantifier is denoted as $C_{l_p}$.

  – For p=1 we have $l_1$- *norm of coherence*. This measure of coherence was introduced in [2]. It is mathematically of the form:

    $$C_{l_1} = \min_{\sigma \in I} \|\rho - \sigma\|_{l_1} = \sum_{i \neq j} |\, \rho_{ij} \,|$$

    Now we have to check if the $l_1$ norm follows the postulates and is whether a valid measure or not.

* For a $\rho \in I$ then $\rho$ can be denoted as $\sigma$ and hence we find that for such a state,$l_1$ norm vanishes.Clearly when the system state $\rho$ is a resource state, that is, not a incoherent state,then we actually find that the $l_1$ norm to be always positive. Hence proving that it follows postulate C1.

* Now consider, for a given $\rho$

$$\sum_n p_n\, C_{l_1}(\rho_n) = \sum_n p_n \sum_{i\neq j} |\,[\rho_n]_{i,j}\,|$$

where $\rho_n = \frac{K_n \rho K_n^\dagger}{p_n}$ , p= $\mathrm{tr}\left[K_n \rho K_n^\dagger\right]$ and $\{K_n\}$ are incoherent Kraus operators.

$$\sum_n p_n\, C_{l_1}(\rho_n) \;=\; \sum_n \sum_{i\neq j} |\left[K_n \rho K_n^\dagger\right]_{i,j}|$$

$$\sum_n p_n\, C_{l_1}(\rho_n) \;=\; \sum_n \sum_{i\neq j} |\sum_{k,l} \,[K_n]_{i,k}\, \rho_{k,l} \left[K_n^\dagger\right]_{l,j}|$$

$$\sum_n p_n\, C_{l_1}(\rho_n) \;\leq\; \sum_{k\neq l} |\rho_{k,l}| \sum_n \sum_{i\neq j} |\,[K_n]_{i,k} \left[K_n^\dagger\right]_{l,j}|$$

Now consider

$$\sum_n \sum_{i\neq j} |\,[K_n]_{i,k} \left[K_n^\dagger\right]_{l,j}| \leq \sum_n \sum_i |\,[K_n]_{i,k}| \;\sum_j\; |\left[K_n^\dagger\right]_{l,j}| \;\; (by\ using\ Schwarz\ inequality)$$

$$\sum_n \sum_{i\neq j} |\,[K_n]_{i,k} \left[K_n^\dagger\right]_{l,j}| \leq \sqrt{\sum_n\; (\sum_i |\,[K_n]_{i,k}\,|)^2 \;\sum_m\; (\sum_j |\left[K_n^\dagger\right]_{l,j}|)^2}$$

Finding whether we can simplify RHS further :

$$\sum_n\; (\sum_i |\,[K_n]_{i,k}\,|)^2 = \sum_n\; \sum_{i,j} |\,[K_n]_{i,k} \left[K_n^\dagger\right]_{k,j}|$$

$$\sum_n\; (\sum_i |\,[K_n]_{i,k}\,|)^2 = \sum_n\; \sum_{i,j} |\,\langle i|\, K_n\,|k\rangle\,\langle k|\, K_n^\dagger\,|j\rangle\,| = \sum_n\; \sum_{i,j} \delta_{i,j} |\,\langle i|\, K_n\,|k\rangle\,\langle k|\, K_n^\dagger\,|j\rangle\,|$$

$$\sum_n\; (\sum_i |\,[K_n]_{i,k}\,|)^2 = \sum_n\; \sum_i |\,\langle k|\, K_n\,|i\rangle\,\langle i|\, K_n^\dagger\,|k\rangle\,| = 1$$

Substituting back in the previous inequality gives us

$$\sum_n \sum_{i \neq j} | [K_n]_{i,k} \left[K_n^\dagger\right]_{l,j} | \leq 1$$

Now substituting this back in the $l_1$ norm inequality gives

$$\begin{aligned} \sum_n p_n \, C_{l_1}(\rho_n) \;&\leq\; \sum_{k \neq l} |\rho_{k,l}| \\ \text{but since} \sum_{k \neq l} |\rho_{k,l}| \;&=\; C_{l_1}(\rho) \\ \Rightarrow \sum_n p_n \, C_{l_1}(\rho_n) \;&\leq\; C_{l_1}(\rho) \end{aligned}$$

Hence proved that $C_{l_1}$ norm satisfies postulate C3(strong monotonicity condition).

* $l_1$ norm is jointly convex. Convexity condition C4 is also satisfied.
* We know once a quantity satisfies C3 and C4 postulates, it implies that it satisfies postulate C2.

Hence we conclude that $l_1$ norm is a coherence monotone as it satisfies the postulates C1 to C4.
For it to be a proper coherence measure, additional to the four postulates, it has to satisfy additivity(C6) and uniqueness for pure states(C5). But it is found that it violates these because we can find a counterexample. For maximally coherent state $|\phi_d\rangle$, $C_{l_1}(|\phi_d\rangle) = d-1$. [42] have also shown that for d=2, $C_{l_1}$ norm is a coherence monotone but it violates monotonicity condition and is no longer a coherence monotone for DIO and MIO for $d \geq 2$.

– For p=2, though $C_{l_2}$ seems like a good candidate for coherence quantification but isn't because of monotonicity postulates(C2 and C3) violation. This is explained in [2] with a counter example.

- Schatten p-norm: It is denoted as $\|\cdot\|_p$ and is mathematically of the form

$$\|M\|_p = (Tr\left[(M^\dagger M)^{p/2}\right])^{1/p}$$

Corresponding coherence quantifier is denoted as $C_p$.

– For p=1, we have Schatten p-norm reducing to trace norm. Trace norm is found to follow C1, C2 and C4 postulates for any class of incoherent operations. But it was shown in [40] that it violates C3 for any set of IOs. It is found that for single-qubit states $C_1 = C_{l_1}$. We have also proved that this is a coherence monotone. For all single-qubit states, it is found that $C_p$ is convex coherence monotone for $p \geq 1$.

– For p=2, $C_2$ is equivalent to Hilbert Schmidt norm or $C_{l_2}$ norm and hence is not a coherence monotone. $C_p$ for all $p \geq 1$ is shown to be a coherence monotone for set of GIO in 2017.

But in general for higher dimensional systems, $C_p$ as well as $C_{l_p}$ are observed to violate C2 and C3 for any set of IOs and $p > 1$.

**Relative Entropy of Coherence**

The relative entropy of coherence is a measure induced from the quantum relative entropy. This measure was first introduced in [2]. This measure is denoted by $C_{rel.ent}$.

Consider quantum relative entropy which is

$$S(\rho \| \sigma) = Tr(\rho \log \rho) - Tr(\rho \log \sigma)$$

The induced coherence quantifier is defined as

$$C_{rel.ent} \;=\; \min_{\sigma \in I} S(\rho \| \sigma)$$

Checking if the defined induced quantifier is a valid measure or not.

- We clearly observe that $C_{rel.ent} = 0$ only when $\rho = \sigma$. We observe that for other states $\rho$ which are resource states, $C_{rel.ent} > 0$ provided support of $\sigma$ is greater than that of $\rho$. Hence $C_{rel.ent}$ follows condition C1.
- Since quantum relative entropy is contractive under CPTP maps, it surely follows condition C2.
- For showing that the defined quantity follows C3, we'll follow selective measurement approach. It is found that quantum relative entropy follows F1 to F5 conditions and hence we

get:

$$S(\rho\|\sigma) \;\geq\; \sum_n p_n\, S(\rho_n\|\sigma_n)$$

for all set of $\{K_n\}$ being incoherent Kraus operators. $p_n = Tr\left[K_n\rho K_n^\dagger\right]$ ; $\rho_n = (K_n\rho K_n^\dagger)/p_n$ ; $\sigma_n = (K_n\sigma K_n^\dagger)/Tr\left[K_n\sigma K_n^\dagger\right]$

So, we have

$$C_{rel.ent}(\rho) \;=\; \min_{\sigma\in I} S(\rho\|\sigma)$$

Using previous inequality we can write

$$C_{rel.ent}(\rho) \;\geq\; \min_{\sigma\in I} \sum_n p_n\, S(\rho_n\|\sigma_n)$$

Since all incoherent Kraus operators take incoherent state to another incoherent state,we can denote $\sigma_n$ as $\sigma$. we have,

$$C_{rel.ent}(\rho) \;\geq\; \min_{\sigma\in I} \sum_n p_n\, S(\rho_n\|\sigma)$$

$$\text{But } \min_{\sigma\in I} S(\rho_n\|\sigma) \;=\; C_{rel.ent}(\rho_n$$

$$\therefore C_{rel.ent}(\rho) \;\geq\; \sum_n p_n\, C_{rel.ent}(\rho_n)$$

This proves that relative entropy of coherence quantity follows postulate C3.

- Quantum relative entropy is jointly convex and hence the quantifier defined also follows the postulate C4.

Since the defined relative entropy of coherence follows all the postulates from C1 to C4, it is a coherence monotone.

Closed form solution of $C_{rel.ent}$: Let $\rho_d$ denote the incoherent state with only the diagonal elements of given state $\rho$, That is , $\rho_d = \sum_i \rho_{i,i} |i\rangle \langle i|$

Writing quantum relative entropy in terms of $\rho_d$

$$S(\rho\|\sigma) = S(\rho_d) - S(\rho) + S(\rho_d\|\sigma)$$

Now minimising this over the incoherent states, we get the coherence quantifier. We find that $S(\rho\|\sigma)$ is minimum when $\rho_d = \sigma$.

Final form of relative entropy of coherence is as follows:

$$C_{rel.ent}(\rho) \;=\; S(\rho_d) - S(\rho)$$

For pure states:

$$C_{rel.ent}(\rho) = S(\rho_d) \;\; (\because S(\rho) = 0 \;\text{ for pure state, }\; \rho = |\psi\rangle\langle\psi|)$$

For maximally coherent state($|\phi_d\rangle$):

$$C_{rel.ent}(|\phi_d\rangle\langle\phi_d|) = log(d)$$

We see that this closed form solution follows the postulates C5 and C6 in addition to C1 to C4 and hence is a proper coherence measure. It is mentioned in the review paper [22], the right hand side of the closed form solution was independently proposed as coherence quantifier in [38].

### 2.2.2 Distillable coherence and Coherence cost

_Distillable coherence_ : Under incoherent operation, copies of a given state $\rho$ can be used to produce certain number of maximally coherent single-qubit states $|\phi_2\rangle$. The optimal number of these maximally coherent state obtained per copy of the given state in the asymptotic limit is called distillable coherence. Its mathematical definition was first given in [4] and it goes as follows:

$$C_d(\rho) \;=\; sup\{R : \lim_{n\to\infty} \,(\,\inf_{\Lambda_i} \,\|\, \Lambda_i\left[\rho^{\otimes n}\right] \,-\, |\phi_2\rangle\langle\phi_2|^{\otimes \lfloor nR \rfloor} \,\|_1\,) = 0\}$$

Such complicated looking formula for $\Lambda_i \in I$ simplifies to

$$C_d(\rho) = C_r(\rho) = S(\rho_{diag}) - S(\rho)$$

for any arbitrary mixed state $\rho$. $C_r(\rho)$ is the relative entropy of coherence.

_Coherence cost_ : It quantifies the minimal rate of maximally coherent single-qubit states $|\phi_2\rangle$ required to produce a given state $\rho$ under incoherent operations in the asymptotic limit. [4] was the first one to give mathematical formulation of this quantity in 2016 and is as follows:

$$C_c(\rho) \; = \; inf\{R : \lim_{n\to\infty} (\inf_{\Lambda_i} \parallel \rho^{\otimes n} \; - \; \Lambda_i \left[ \left|\phi_2\right\rangle \left\langle\phi_2\right|^{\otimes \lfloor nR \rfloor} \right] \parallel_1) = 0\}$$

By considering those $\Lambda_i \in I$ we have

$$C_c(\rho) = C_f(\rho) \; = \; \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \, S(\triangle \left[ \left|\psi_i\right\rangle \left\langle\psi_i\right| \right])$$

where $C_f$ is coherence of formation.

In [4], it has been proved rigorously that both distillable coherence and coherence cost follow all six postulates C1 to C6 and hence qualify t be proper coherence measure.
In general,

$$C_d(\rho) \leq C_c(\rho)$$

with equality holding only for pure states.
For pure state $\rho = \left|\psi\right\rangle \left\langle\psi\right|$,

$$C_d(\rho) = S(\rho_{diag}) \;\; (\because S(\rho) = 0 \;\text{ as }\; \rho \;\text{ is a pure state})$$
$$C_c(\rho) = C_f(\rho) = S(\rho_{diag}) \;\text{ for a pure }\; \rho$$

Hence equality is proved for pure states.
Equality implies that interconversion between pure coherent states is asymptotically reversible and standard unit coherence measure exists. This means that a pure state $\left|\phi_1\right\rangle$ of distillable coherence of $c_1$ cobit can be asymptotically converted to another pure state $\left|\phi_2\right\rangle$ at the rate $\frac{c_1}{c_2}$. [4] proves that for mixed states $C_d(\rho)$ is strictly less than $C_c(\rho)$. [4] also prove that there are no states with $C_d(\rho) = 0$ and $C_c(\rho) \neq 0$. This implies that in the resource theory of coherence based on the set of incoherent operations, there does not exist a concept ‘bound coherence’. This is an important result as this is one of the major differences between coherence and entanglement theory.

# Chapter 3

# Numerics and discussion

This section consists of all the first hand numerical results of distance measure of coherence explained in the previous sections. We study the behaviour of plots.
For numerical calculation we have taken a single-qubit state.

$$\rho = \begin{bmatrix} \frac{1+(r\cos(\theta))}{2} & \frac{r\sin(\theta)}{2} \\ \frac{r\sin(\theta)}{2} & \frac{1-(r\cos(\theta))}{2} \end{bmatrix}$$

Single-qubit pure state density matrix would be of the form: $\rho(r=1)$

$$\rho_{pure} = \begin{bmatrix} \frac{1+\cos(\theta)}{2} & \frac{\sin(\theta)}{2} \\ \frac{\sin(\theta)}{2} & \frac{1-\cos(\theta)}{2} \end{bmatrix}$$

which reduces to give the form:

$$\rho_{pure} = \begin{bmatrix} \cos^2(\frac{\theta}{2}) & \frac{\sin(\theta)}{2} \\ \frac{\sin(\theta)}{2} & \sin^2(\frac{\theta}{2}) \end{bmatrix}$$

Section of plots include the behaviour of coherence quantifiers based on distance measure with respect to different single-qubit states.

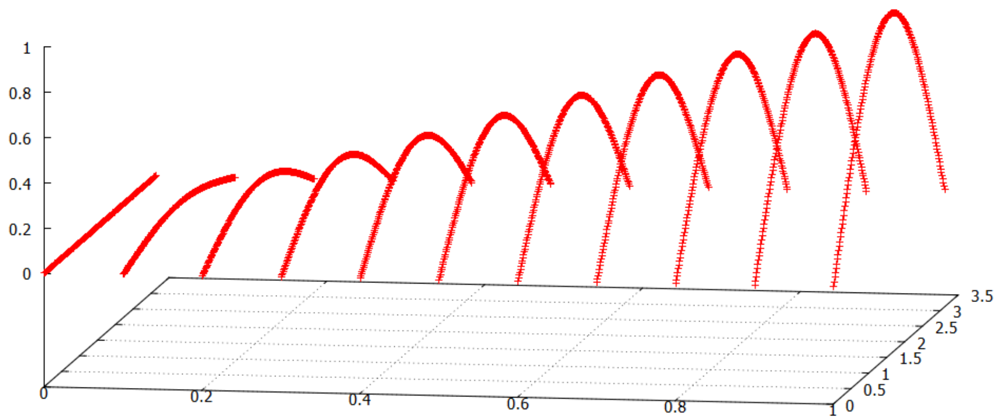


Figure 3.1: $l_1$-norm measure of coherence
x-axis: Radial distance, r
y-axis: Polar angle, $\theta$ (in rad)
z-axis: $C_{l_1}$

**Figure 1**: Numerical calculation of the $C_{l_1}$ measure of coherence as a function of $r \in [0,1]$ and $\theta \in (0,\pi)$ for a given $\rho$
x-axis: r which is the radial distance varies from 0 to 1 for bloch sphere
y-axis: $\theta$ which is the polar angle that varies from 0 to $\pi$
z-axis: $C_{l_1}$ which is the $l_1$ norm coherence quantifier.
Note that the phase factor(polar angle) doesn't play an important role and hence is not considered.

We know, $C_{l_1} = \min_{\sigma \in I} \|\rho - \sigma\|_1$
Clearly, $C_{l_1}$ is minimum when $\sigma = \rho_d$ where $\rho_d = \sum_k \rho_{k,k} |k\rangle \langle k|$, incoherent state with diagonal elements of the system state $\rho$.
$C_{l_1}$=Sum of the off-diagonal elements of the system density matrix. We observe that, as r increases from 0 to 1, the maximum coherence value also increases.
Coherence is minimum when r=0, that is, $C_{l_1} = 0$ for all $\theta$.
Value of Maximum Coherence is highest at r=1, that is, $C_{l_1} = 1$ for $\theta = \frac{\pi}{2}$

In general we observe that the coherence value is maximum for pure state when compared to other states(i.e, mixed states). This is because the pure state is the only state that is superposition of its eigenstates of the system density matrix.

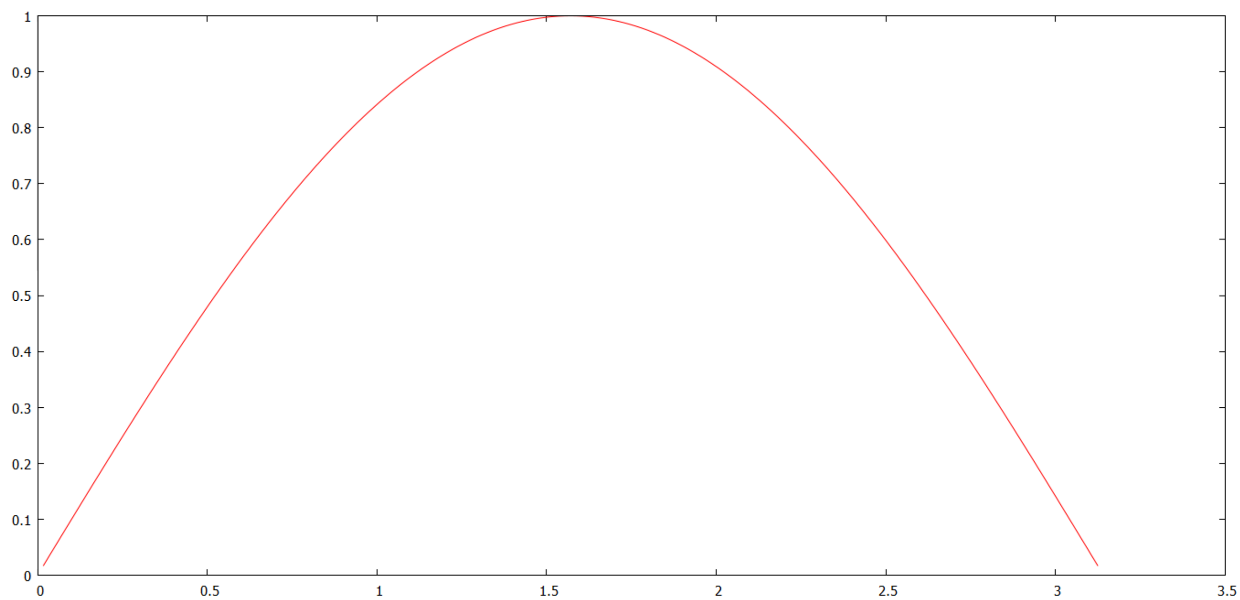


Figure 3.2: $l_1$ norm measure of coherence for pure statex-axis: Polar angle,$\theta$ (in rad)y-axis: $C_{l_1}$

**Figure 2**: Numerical calculation of the $C_{l_1}$ measure of coherence as a function of $\theta \in (0, \pi)$ for a given pure state $\rho$
r is fixed and r=1.
x-axis: $\theta$ which is the Polar angle that varies from 0 to $\pi$
y-axis: $C_{l_1}$ which is the $l_1$ norm coherence quantifier.

Here we focus more on the behaviour of the measure with respect to pure states. Pure states are those that constitute the surface of the bloch sphere. We observe that coherence measure increases from 0 as $\theta$ increases from 0 to $\frac{\pi}{2}$, attains maximum at $\theta = \frac{\pi}{2}$ and again decreases to 0 at $\theta = \pi$. Even in double-slit experiments (3), we observe that interference is maximum(or we observe brightest fringes) when the $\theta = \frac{\pi}{2}$, that is, when the system state is equal superposition of the eigenstates. This result is consistent with experimental result (3) and hence can be considered as a coherence quantifier.

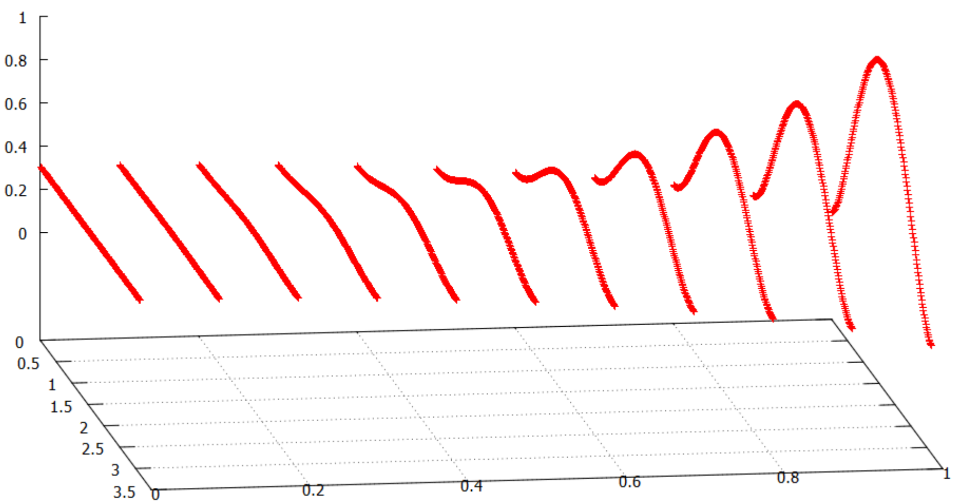


Figure 3.3: Relative entropic norm measure of coherence
x-axis: Radial distance, r
y-axis: Polar angle,$\theta$ (in rad)
z-axis: $C_{rel.ent}$

**Figure 3**: Numerical calculation of the $C_{rel.ent.}$ measure of coherence as a function of $r \in [0,1]$ and $\theta \in (0,\pi)$ for a given $\rho$
x-axis: r which is the radial distance varies from 0 to 1 for bloch sphere
y-axis: $\theta$ which is the Polar angle that varies from 0 to $\pi$
z-axis: $C_{rel.ent}$ which is the relative entropic coherence quantifier.
Note that the phase factor(polar angle) doesn't play an important role and hence is not considered.

$C_{rel.ent.} = S(\rho_d) - S(\rho)$ where $\rho_d = \sum_k \rho_{k,k} |k\rangle \langle k|$, incoherent state with diagonal elements of the system state $\rho$. $S(\rho) = -Tr(\rho \log \rho)$ von Neumann entropy of the state $\rho$
We observe that the measure value $C_{rel.ent.} = 0$ when r=0 irrespective of the value of $\theta$. We observe that the measure value $C_{rel.ent.} = 0$ when $\theta = 0$ *and* $\theta = \frac{\pi}{2}$ irrespective of the value of r. This matches with the intuition, that is, at the center of bloch sphere and at the poles, coherence is minimum and is zero. At each r, as r increases, the maximum value of the measure is attained at $\theta = \frac{\pi}{2}$. Value of Maximum Coherence is highest at r=1, that is, $C_{rel.ent.} = 1$ for $\theta = \frac{\pi}{2}$

In general we observe that the coherence value is maximum for pure state when compared to other states(i.e, mixed states). This is because the pure state is the only state that is superposition of its eigenstates of the system density matrix.

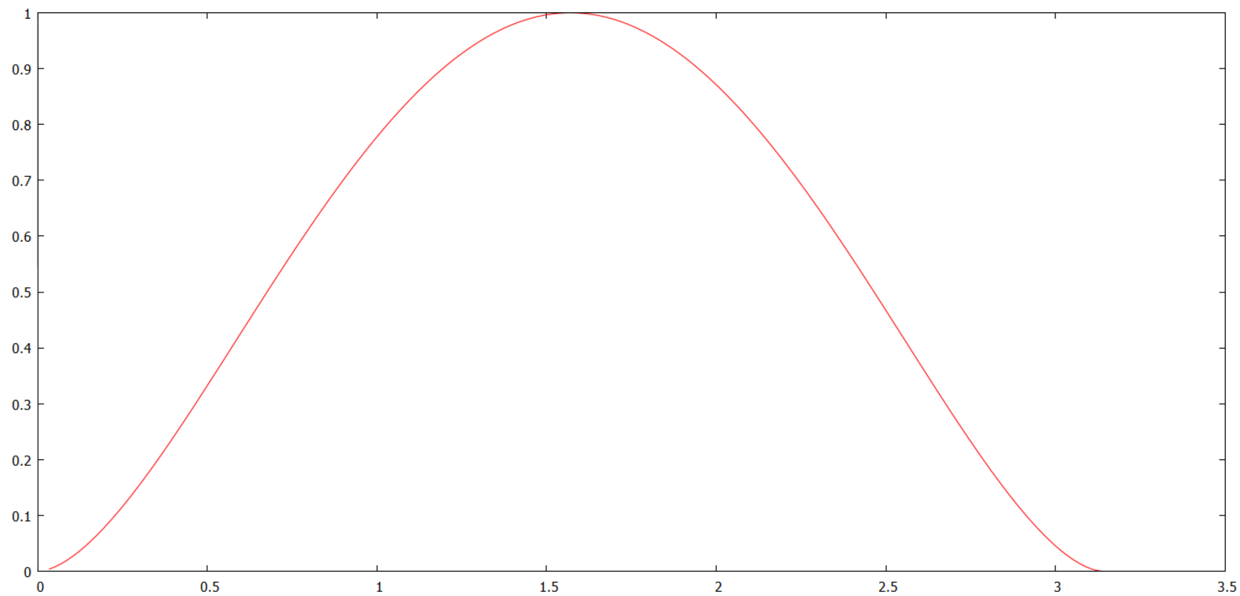


Figure 3.4: Relative entropic norm measure of coherence for pure state
x-axis: Polar angle, $\theta$ (in rad)
y-axis: $C_{rel.ent}$

**Figure 4**: Numerical calculation of the $C_{rel.ent.}$ measure of coherence as a function of $\theta \in (0, \pi)$ for a given pure state $\rho$
r is fixed and r=1.
x-axis: $\theta$ which is the Polar angle that varies from 0 to $\pi$
y-axis: $C_{rel.ent}$ which is the relative entropic coherence quantifier.

We know, $C_{rel.ent.} = S(\rho_d) - S(\rho)$ where $\rho_d = \sum_k \rho_{k,k} |k\rangle \langle k|$, incoherent state with diagonal elements of the system state $\rho$. $S(\rho) = -Tr(\rho \log \rho)$ von Neumann entropy of the state $\rho$
And we also know that for pure states $S(\rho) = 0$.
Hence we have $C_{rel.ent.} = S(\rho_d)$ only for pure states. Here we focus more on the behaviour of the measure with respect to pure states. Pure states are those that reside on the surface of the bloch sphere. We observe that coherence measure increases from 0 as $\theta$ increases from 0 to $\frac{\pi}{2}$, attains maximum at $\theta = \frac{\pi}{2}$ and again decreases to 0 at $\theta = \pi$. Even in double- slit experiments (3),

we observe that interference is maximum(or we observe brightest fringes) when the $\theta = \frac{\pi}{2}$, that is, when the system state is equal superposition of the eigenstates. This result is consistent with experimental result (3) and hence can be considered as a coherence quantifier.

We observe that the numerical results are consistent with not just the experimental observations (3) but also with each other in case of both $l_1$ norm measure and relative entropic measure of coherence.

## Bridge connecting experiments and theory

Theory is considered to be successful only when it can explain the experimental observations and can come up with new ideas on experiments for further strengthening of concepts. I have divided this section into three eras:

1. *Oldest experiment*: Double- slit experiment with electrons, a thought experiment. The experiment that is discussed here is the thought experiment involving electrons explained in [43] which led to the development of quantum mechanics. The aim of the experiment was to understand the behaviour of electrons. The distribution of the electrons ejected from the electron gun aimed in the direction of the two slits is observed. It was found that the probability distribution of electron distribution on the screen was unequal to the sum of the probabilities of electrons passing through each slits. This surprising result paved way for development of quantum mechanics.

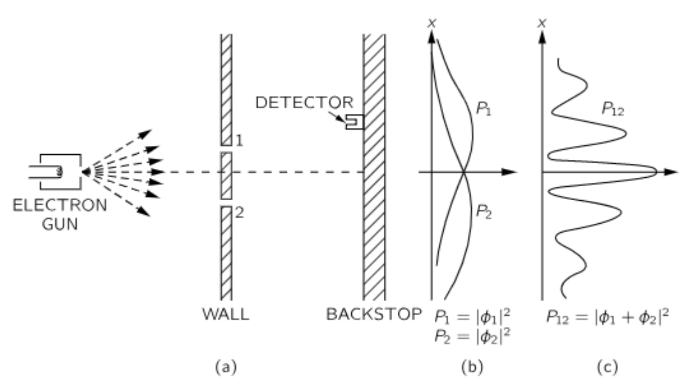


Figure 3.5: Thought experiment for understanding quantum behaviour of electrons [43]

How the theoretical framework explains the observation? The electron wavefunction passes through both the slits. The superposition of the wavefunctions out of the two slits result in the intensity pattern of alternate bright and dark bands on the screen. Superposition being the signature of quantumness, is studied as the quantum coherence. Considering the incident beam to be a pure single qubit state, and split beam after two slits to be superposition of the fixed reference basis, we get the distribution pattern to be consistent with the numerics plotted above for pure single qubit case.

2. *Intermediate era*: Set of those experiments that had observed signatures of quantum coherence but didn't have a well-established resource-theoretic framework. Those experiments that were performed after the oldest thought experiment and before the development of theoretical framework in the year 2014, fall into this category. One such experiment include that described in [44]. In [44] they find a way of saying how long quantumness of a qubit is preserved, by which they mean violation of a Leggett-Garg inequality (LGI). The system is observed to have non-zero off diagonal terms implying non-zero quantum coherence even after that time. Which implies that this quantumness is not detectable by the LGI but can be explained by the resource theoretical framework of quantum coherence. There have been several studies in which it was analysed how destruction of coherence leads to reduction of quantum correlations. An experiment in this direction can be found in [46].

3. *Era after development of theoretical framework*: All those experiments designed based on the theoretical framework for further clarity of concepts, or the experimental observations gotten that can be clearly explained after the theory came into existence belong to this set. One of the experiments that was designed very recently based on the given theoretical framework includes [45]. This experiment aims at answering the question on the efficiency of manipulation and inter-conversion of resource which is one of the key questions of the theory. They find optimal probabilities for mixed state conversions via stochastic incoherent operations for qubit states. They also extend discussion on distributed scenarios. Experimentally they show that in a linear optics set-up, optimal state conversion probabilities can be achieved which would be very helpful for real world applications, specially in quantum technologies. Few other works that fall into this classification include [47, 48, 49, 50, 51].

# Chapter 4

# Conclusion and Outlook

In this report we have seen how the resource theory of quantum coherence was developed and has played a central role in quantum information theory, in general physics. Coherence can be studied as a resource, provided the experimenter is limited to what are called as "free operations" that cannot create any resource.

The report mainly discusses the set of definitions and conditions that any resource theory of coherence must include. Basically the resource theory of coherence should have:

- *Free states (F)* which are called incoherent states in this context, that do not contain any resource (zero coherence) and are freely accessible.
- *Free operations* ($\mathscr{O}$) which are a well-defined class of incoherent operations and are physically justified.
- No free operations defined must be able to create coherence (resource) from set of incoherent states (free states).
- Free operations can take an incoherent state to another incoherent state of same dimension.
- Free operations also allow for conversion of d-dimensional maximally coherent state $|\phi_d\rangle$(termed as a cobit) to any other d-dimensional state.

After discussion of resource theory formulation of coherence, next section of the report discussed on the quantification of coherence. A detailed review of a few important postulates for a valid coherence quantifier is given. In later section, we provide a discussion of two important distance-based quantifiers of coherence, viz, matrix norm measures and relative entropic measures. There's also a brief discussion on distillable coherence and coherence cost. This leads to the concept of 'bound coherence' (or rather, its absence) within coherence resource theory based on the set of IOs that brings out the contrast between entanglement and coherence theory.

Though there is a large body of work on the resource theory of quantum coherence, the theory is still at a nascent stage, and substantial research focusing on various physical aspects of coherence are imminent in near future.

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \tag{4.1}$$

# Bibliography


[1] Åberg, J.(2006),'*Quantifying Superposition*', arXiv:quant-ph/0612146.

[2] Baumgratz, T., M. Cramer, and M. B. Plenio (2014), Phys. Rev. Lett. **113**, 140401.

[3] Bromley, T. R., M. Cianciaruso, and G. Adesso (2015), Phys. Rev.Lett. **114**, 210401.

[4] Winter, A., and D. Yang (2016), Phys. Rev. Lett. **116**, 120404.

[5] Gour, G., I. Marvian, and R. W. Spekkens (2009), Phys. Rev. A **80**, 012307.

[6] Gour, G., and R. W. Spekkens (2008), New J. Phys. **10**, 033023.

[7] Marvian, I., and R. W. Spekkens (2014a), Nat. Commun. **5**, 3821.

[8] Marvian, I., and R. W. Spekkens (2014b), Phys. Rev. A **90**, 062110.

[9] Vaccaro, J. A., F. Anselmi, H. M. Wiseman, and K. Jacobs (2008), Phys. Rev. A 77, 032114.

[10] Chitambar, E., and G. Gour (2016a), Phys. Rev. A **94**, 052336

[11] Chitambar, E., and G. Gou,(2016b), Phys. Rev. Lett. **117**, 030401

[12] Chitambar, E., and G. Gour (2017), Phys. Rev. A **95**, 019902

[13] Yadin, B., J. Ma, D. Girolami, M. Gu, and V. Vedral (2016), Phys. Rev. X **6**, 041028

[14] Thomas Theurer, Dario Egloff, Lijian Zhang, Martin B. Plenio, arXiv:1806.07332

[15] Glauber, R. J. (1963), Phys. Rev. **130**, 2529.

[16] Mandel, L., and E. Wolf (1965), Rev. Mod. Phys. **37**, 231.

[17] Sudarshan, E. C. G. (1963), Phys. Rev. Lett. **10**, 277.

[18] Eric Chitambar, Gilad Gour, arXiv:1806.06107.

[19] Horodecki, R., P. Horodecki, M. Horodecki, and K. Horodecki(2009), Rev. Mod. Phys. **81** (2), 865.

[20] Plenio, M. B., and S. Virmani (2007),Quant. Inf. Comput. **7** (1 and 2), 1, quant-ph/0504163.

[21] Goold, J., M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk(2016), Journal of Physics A: Mathematical and Theoretical **49** (14), 143001.

[22] Streltsov, A., G. Adesso, and M. B. Plenio (2017), Rev. Mod.Phys. **89**, 041003.

[23] Gour, G., M. P. Muller, V. Narasimhachar, R. W. Spekkens, and N. Y. Halpern (2015), Physics Reports **583**, 1

[24] Modi, K., A. Brodutch, H. Cable, T. Paterek, and V. Vedral (2012), Rev. Mod. Phys. **84**, 1655.

[25] Bartlett, S. D., T. Rudolph, and R. W. Spekkens (2007), Rev.Mod. Phys. **79**, 555

[26] Brunner, N., D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner (2014), Rev. Mod. Phys. **86**, 419.

[27] Veitch, V., S. A. H. Mousavian, D. Gottesman, and J. Emerson (2014), New Journal of Physics **16** (1), 013009

[28] Coecke, B., T. Fritz, and R. W. Spekkens (2016), Information and Computation **250**, 59 , quantum Physics and Logic.

[29] Fritz, T. (2017), Mathematical Structures in Computer Science *27*, 850.

[30] Alexander Streltsov, Hermann Kampermann, Sabine Wölk, Manuel Gessner, Dagmar Bruß (2018) , New J. Phys. **20**, 053058.

[31] Chandrashekar Radhakrishnan , Zhe Ding , Fazhan Shi , Jiangfeng Du, Tim Byrnes , arXiv :1805.09263.

[32] Marvian, I., and R. W. Spekkens (2013), New J. Phys. **15**, 033001.

[33] Marvian, I., and R. W. Spekkens (2016), Phys. Rev. A **94**, 052324

[34] Marvian, I., R. W. Spekkens, and P. Zanardi (2016), Phys. Rev. A **93**, 052331.

[35] de Vicente, J. I., and A. Streltsov (2017), J. Phys. A **50**, 045301.

[36] Chiribella, G., and Y. Yang (2015), arXiv:1502.00259.

[37] Chitambar, E., and M.-H. Hsieh (2016), Phys. Rev. Lett. **117**,020402

[38] Herbut, F. (2005), J. Phys. A **38** (13), 2959.

[39] V. Vedral and M. B. Plenio (1998),Phys. Rev.A **57**,1619.

[40] Yu, X.-D., D.-J. Zhang, G. F. Xu, and D. M. Tong (2016b), Phys.Rev. A **94**, 060302.

[41] Peng, Y., Y. Jiang, and H. Fan (2016a), Phys. Rev. A **93**, 032326.

[42] Bu, K., and C. Xiong (2016), arXiv:1604.06524.

[43] Feynman lectures on physics Volume III, Chapter 1, 'Quantum Behavior'.

[44] Vikram Athalye, Soumya Singha Roy and T. S. Mahesh (2011), Phys. Rev. Lett. **107**, 130402.

[45] Kang-Da Wu, Thomas Theurer, Guo-Yong Xiang, Chuan-Feng Li, Guang-Can Guo, Martin B. Plenio and Alexander Streltsov (2019), arXiv: 1903.01479v1.

[46] Jin-Shi Xu, Xiao-Ye Xu, Chuan-Feng Li, Cheng-Jie Zhang, Xu-Bo Zou, and Guang-Can Guo (2010), Nature Commun. **1**, 7.

[47] Davide Girolami (2014), Phys. Rev. Lett. **113**, 170401.

[48] Zong-Quan Zhou, Susana F. Huelga, Chuan-Feng Li, and Guang-Can Guo (2015), Phys. Rev. Lett. **115**, 113002.

[49] Tania Paul and Tabish Qureshi (2017), Phys. Rev. A **95**, 042110.

[50] Wei-Min Lv, Chao Zhang, Xiao-Min Hu, Huan Cao, Jian Wang, Yun-Feng Huang, Bi-Heng Liu, Chuan-Feng Li, and Guang-Can Guo (2018), Phys. Rev. A **98**, 062337.

[51] J. Gao, Z.-Q. Jiao, C.-Q. Hu, L.-F. Qiao, R.-J. Ren, Z.-H. Ma, S.-M. Fei, V. Vedral, and X.-M. Jin (2018), Communications Physics **1**, 89.