Asymptotic Flatness at Timelike Infinity

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by

Aniket Khairnar

IISER PUNE

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

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Supervisor: Dr. Amitabh Virmani

Certificate

This is to certify that this dissertation entitled Asymptotic Flatness at Timelike Infinity towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Aniket Khairnarat Indian Institute of Science Education and Research under the supervision of Dr. Amitabh Virmani, Associate Professor, Department of Physics, Chennai Mathematical Institute , during the academic year 2018-2019.

Dr. Amitabh Virmani

Committee:

Dr. Amitabh Virmani

Dr. Sachin Jain

This thesis is dedicated to CMI's coffee and the blackboards in the discussion area. The project would not have progressed without them.

Declaration

I hereby declare that the matter embodied in the report entitled Asymptotic Flatness at Timelike Infinity are the results of the work carried out by me at the Department of Physics, Chennai Mathematical Institute, under the supervision of Dr. Amitabh Virmani and the same has not been submitted elsewhere for any other degree.

Aniket Khairnar

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Abstract

BMS symmetries have been argued to be the exact symmetries of quantum gravity theory in asymptotically flat spacetimes. If this is the case, then they should also be visible at timelike infinity as well. In this thesis, we introduce a notion of asymptotically flat spacetimes at timelike infinity and discuss boundary conditions that allow for BMS symmetries to act on the space of solutions. The BMS symmetries lead to non-trivial conserved charges and conservation laws.

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Chapter 1

Introduction

The aim of any physical theory is to describe actual systems in the universe. In many such theories, there are class of models which represents "isolated systems". For example in electromagnetism, there are isolated charge distributions whose field falls off at a particular rate in an inertial coordinate system. Usually, one does not expect such isolated systems to be realised physically as it can never be truly isolated from the surrounding, but, they are good approximations to actual physical systems. Also, it is only through such a notion of isolated systems, we can study subsystems in our universe. If we do not have such models, then we would have to describe the system in each and every detail.

General relativity (GR) is a physical theory of spacetime. We want a similar notion of isolated systems in it. An example of an isolated system in GR would be the gravitational field outside a massive body. The gravitational field falls off with distance and hence, the geometry of spacetime become flat far away from the source of curvature. Thus, asymptotically flat spacetimes represents isolated systems in GR. A precise definition of asymptotic flatness in GR is difficult to give due to lack of a global inertial coordinate system, whose radial coordinate could be used to specify falloff behavior of the metric. A natural question is then how do we define asymptotically flat spacetimes?

One way to describe such spacetimes would be to inquire if there exist a coordinate system x^{μ} in which the metric approaches the flat metric at large coordinate values. For $g_{\mu\nu} = \eta_{\mu\nu} + O\left(\frac{1}{r}\right)$ as $r \to \infty$. This is a coordinate dependent notion of describing asymptotically flat spacetimes. Another way would be to check for the possibility of attaching a boundary to the spacetime using a conformal transformation such that the boundary represents points at infinity. This definition is manifestly coordinate independent, and by providing boundary representing infinity, it removes the complication of taking limits as one goes to infinity. Apart from technicalities, both these definitions describe isolated systems in general relativity. In this thesis, we are going to use the former way of describing asymptotically flat spacetimes.

One naively expects that the asymptotic symmetry group of asymptotically flat spacetimes to be the isometries of flat spacetime, i.e., Poincaré group. This expectation is, however, not realised. The asymptotic symmetry group becomes Lorentz group with a distorted translation group, with an infinite number of extra generators. These extra generators are called supertranslations. The full asymptotic symmetry group is an infinite dimensional group which contains Poincaré group. It is known as the Bondi-Metzner-Sachs (BMS) group [1, 2, 3, 4].

A detailed motivation to analyse such spacetimes is to describe conserved charges, especially for supertranslations. These set of charges characterize the asymptotically flat spacetimes. To obtain the asymptotic symmetry group corresponding to these charges, we must know how to reach infinity. There are three different asymptotic region namely null infinity, spacelike infinity and timelike infinity that one can study in flat spacetime.

Most of the early studies of asymptotically flat spacetimes were focussed on null-infinity [1, 2, 3, 4] with the motive to understand the properties of gravitational radiation. Null infinity is the place where massless particles end up eventually; it is the place where dynamics happens, for example, mass loss due to the emitted gravitational radiation in a black hole binary collision. Spatial infinity on the other hand is the place where there is no dynamics. It is the place where one describes notions of conserved quantities, e.g., total mass and angular momentum of the spacetime.

A remarkable outcome of the early studies at null infinity was the discovery of the so-called BMS supertranslations. BMS supertranslations form an abelian symmetry group, known as the BMS group. Intuitively, these are just angle dependent translations. The physical significance of this symmetry was not much appreciated in the 70s and 80s. Several authors tried to find boundary conditions at null infinity so as to remove the "supertranslation ambiguities", but none of these attempts were successful. No boundary conditions were found that allow for gravitational radiation and do not have BMS supertranslations as the allowed asymptotic symmetries [16].

This situation needs to be contrasted with asymptotic symmetries at spacelike infinity [6, 7]. Earlier investigations at spacelike infinity showed no analog of BMS symmetries. Famously, Ashtekar, Bombelli, and Ruela [17] proposed boundary conditions at spacelike infinity where the symmetry group was exactly Poincaré. There were hints to a much larger extensions, the so-called Spi supertranslations, but no analog of the BMS symmetries were discovered in those investigations [8, 9]. The Spi supertranslations are functions of three coordinates rather than two.

In the modern literature it has been realised that the BMS symmetry is a blessing rather than a disadvantage. In fact several extensions of the BMS group has been argued to be of use. These conclusions come from the investigations pioneered by the Strominger's group. They have claimed deep connections between soft theorems, memory effects, asymptotic symmetries [15].

A natural question that has resurfaced from these investigations is how to incorporate BMS symmetries at spacelike infinity? If BMS symmetries are physical, then they should be visible at spatial infinity. This only led to deepening of the puzzle, since boundary conditions allowing for BMS symmetries at spatial infinity were not naturally found in the earlier investigations.

In the years 2008 to 2011 hints start to emerge that Ashtekar, Bombelli, and Ruela boundary conditions are too strong. They could be relaxed in a number of way; perhaps also to incorporate the BMS symmetries of null infinity at spacelike infinity. In an insightful work, Compère and Dehouck proposed one such relaxation [5], though only later with the work of Henneaux and Troessaert [12, 10, 11] the relation to null infinity became clear.

If BMS symmetries are symmetries of the exact theory, they should also be visible at timelike infinity. They would appear as diffeomorphisms leaving the boundary conditions at timelike infinity invariant. The aim of the thesis is to introduce a notion of asymptotically flat spacetimes at timelike infinity and discuss boundary conditions that allow for BMS symmetries to act on the space of solutions and give non-trivial conserved charges and conservation laws.

Chapter 2

Asymptotically flat spacetimes at timelike infinity

In the earlier work carried out by Beig and Schmidt [13, 14] to study asymptotic flatness at spacelike infinity, a coordinate dependent description was used. Their considerations were inspired by the understanding of previous results obtained at null infinity and their will to explore spacelike infinity. Their formalism introduced a coordinate system in which asymptotically flat spacetimes admit an expansion in negative powers of a "radial coordinate" in a neighborhood of spatial infinity. We are going to adopt a similar coordinate dependent way to define our class of asymptotically flat spacetimes.

In section 2.1, we follow steps similar to Beig and Schmidt to arrive at a form of metric which describes asymptotically flat spacetimes at timelike infinity. The Beig Schmidt ansatz for our case admits an expansion in the negative powers of a "timelike coordinate" in the vicinity of timelike infinity. This form describes the kinematical space of metrics. We need to impose Einstein's equations to introduce dynamics in the system. In section 2.2, Einstein's equations are expressed as hierarchy of equations satisfied by the fields present in the metric ansatz. Metrics which can be brought into our asymptotic form and satisfies Einstein's equations forms the space of solutions.

2.1 Definition of asymptotic flatness

Following Beig and Schmidt, we consider metrics that asymptotically approach flat space at timelike infinity

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n=1}^{m} \frac{1}{\tau^n} l^{(n)}_{\mu\nu}\left(\frac{x^\sigma}{\tau}\right) + \dots \tag{2.1.1}$$

where x^σ's are the usual cartesian coordinates on flat space, and τ is

$$\tau^2 = -\eta_{\mu\nu} x^\mu x^\nu. \tag{2.1.2}$$

Given (2.1.1) there is a large freedom to find another set of flat coordinates $\bar{x}^\mu$ and hence a new $\bar{\tau}$ such that equation (2.1.1) holds again. Consider for example,

$$x^\mu = \bar{x}^\mu + \sum_{n=1}^{s} \frac{a^\mu(\bar{x}^\nu/\bar{\tau})}{\bar{\tau}^n}, \tag{2.1.3}$$

with

$$\bar{\tau}^2 = -\eta_{\mu\nu} \bar{x}^\mu \bar{x}^\nu. \tag{2.1.4}$$

Note that

$$\frac{\partial x^\mu}{\partial \bar{x}^\nu} = \delta^\mu_\nu + \sum_{n=1}^{s} \left\{ \frac{n}{\bar{\tau}^{n+1}} a^\mu(\bar{x}^\sigma/\bar{\tau}) \left(\eta_{\sigma\nu} \frac{\bar{x}^\sigma}{\bar{\tau}}\right) + \frac{1}{\bar{\tau}^{n+1}} \frac{\partial a^\mu}{\partial \bar{x}^\lambda}(\bar{x}^\sigma/\bar{\tau}) \left[\delta^\lambda_\nu + \eta_{\sigma\nu} \frac{\bar{x}^\lambda}{\bar{\tau}} \frac{\bar{x}^\sigma}{\bar{\tau}}\right] \right\}. \tag{2.1.5}$$

All terms on the right hand side of this equation are dependent on $(\bar{x}^\sigma/\bar{\tau})$ alone. Therefore,

$$\frac{\partial x^\mu}{\partial \bar{x}^\nu} = \delta^\mu_\nu + \sum_{n=1}^{s} \frac{b^\mu(\bar{x}^\nu/\bar{\tau})}{\bar{\tau}^{n+1}}. \tag{2.1.6}$$

Choosing $s \geq n-1$, where n appears in (2.1.1) we see that these coordinate transformations preserve the form of the asymptotic expansion.

Our main focus of attention in this project will be the so-called supertranslations - direction dependent shifts of the origin which also preserve the form of the metric(2.1.1)

$$x^\mu = \bar{x}^\mu + \xi^\mu\left(\frac{\bar{x}^\nu}{\bar{\tau}}\right), \tag{2.1.7}$$

$$\frac{\partial x^\mu}{\partial \bar{x}^\nu} = \delta^\mu_\nu + \frac{1}{\bar{\tau}}\frac{\partial \xi^\mu}{\partial \bar{x}^\lambda}(\bar{x}^\sigma/\bar{\tau})\left[\delta^\lambda_\nu + \eta_{\sigma\nu}\frac{\bar{x}^\lambda}{\bar{\tau}}\frac{\bar{x}^\sigma}{\bar{\tau}}\right]. \tag{2.1.8}$$

There are more coordinate transformations that preserve the form of the asymptotic expansion, but we will not be concerned with those other ones. Supertranslations are complicated enough for now.

It is convenient to use τ as a coordinate and together with the set of directions x^ν/τ. Let ϕ^a be the coordinates on the manifold of directions, then there exist functions $w^\mu(\phi^a)$ such that

$$w^\mu(\phi^a) = \frac{x^\mu}{\tau}. \tag{2.1.9}$$

We have

$$dx^\mu = w^\mu d\tau + \tau\partial_a w^\mu d\phi^a. \tag{2.1.10}$$

Substituting this into (2.1.1) we have

$$\eta_{\mu\nu}dx^\mu dx^\nu = -d\tau^2 + \tau^2(h^{(0)}_{ab}d\phi^a d\phi^b). \tag{2.1.11}$$

Defining

$$\tilde{\sigma}^{(n)} = -l^{(n)}_{\mu\nu}w^\mu w^\nu, \tag{2.1.12}$$

$$A^{(n)}_a = l^{(n)}_{\mu\nu}w^\mu\partial_a w^\nu, \tag{2.1.13}$$

$$h^{(n)}_{ab} = l^{(n)}_{\mu\nu}\partial_a w^\mu\partial_b w^\nu, \tag{2.1.14}$$

we get

$$\begin{aligned} ds^2 = & -d\tau^2\left[1 + \sum_{n=1}^m \frac{\tilde{\sigma}^{(n)}(\phi)}{\tau^n} + \mathcal{O}(\tau^{-(m+1)})\right] + 2\tau d\tau d\phi^a\left[\sum_{n=1}^m \frac{A^{(n)}_a(\phi)}{\tau^n} + \mathcal{O}(\tau^{-(m+1)})\right] \\ & +\tau^2 d\phi^a d\phi^b\left[h^{(0)}_{ab} + \sum_{n=1}^m \frac{h^{(n)}_{ab}(\phi)}{\tau^n} + \mathcal{O}(\tau^{-(m+1)})\right]. \end{aligned} \tag{2.1.15}$$

It will be more useful to organise asymptotic expansion if we let $\tilde{\sigma}^{(n)}$ replaced with $\sigma^{(n)}$ such that

$$\begin{aligned} ds^2 &= -d\tau^2 \left[\left(1 + \sum_{n=1}^{m} \frac{\sigma^{(n)}(\phi)}{\tau^n} \right)^2 + \mathcal{O}(\tau^{-(m+1)}) \right] + 2\tau d\tau d\phi^a \left[\sum_{n=1}^{m} \frac{A_a^{(n)}(\phi)}{\tau^n} + \mathcal{O}(\tau^{-(m+1)}) \right] \\ &\quad + \tau^2 d\phi^a d\phi^b \left[h_{ab}^{(0)} + \sum_{n=1}^{m} \frac{h_{ab}^{(n)}(\phi)}{\tau^n} + \mathcal{O}(\tau^{-(m+1)}) \right]. \end{aligned} \tag{2.1.16}$$

There exist coordinate transformations that bring the metric (2.1.16) to a form where

$$\sigma^{(n)} = 0, \qquad \text{for} \qquad n \geq 2, \tag{2.1.17}$$

$$A_a^{(n)}(\phi) = 0, \qquad \text{for} \qquad n \geq 1. \tag{2.1.18}$$

A proof proceeds as follows. Take

$$\phi^a = \bar{\phi}^a + \frac{1}{\bar{\tau}} G^{(1)a}(\bar{\phi}^b), \tag{2.1.19}$$

$$\tau = \bar{\tau}. \tag{2.1.20}$$

Then

$$d\phi^a = d\bar{\phi}^a + \frac{1}{\bar{\tau}} \bar{\partial}_b G^{(1)a} d\bar{\phi}^b - \frac{1}{\bar{\tau}^2} d\bar{\tau} G^{(1)a}. \tag{2.1.21}$$

Substituting this into (2.1.16) one gets a mixed term,

$$d\bar{\phi}^a d\bar{\tau} \left(A_a^{(1)} - G^{(1)b} h_{ab}^{(0)} \right), \tag{2.1.22}$$

which can be set to zero by choosing

$$G^{(1)b} = A_a^{(1)} h^{(0)ab}. \tag{2.1.23}$$

Denoting the resulting metric again as (2.1.16), but with $A_a^{(1)} = 0$, we do the following transformation

$$\tau = \bar{\tau} + \frac{F^{(2)}(\phi^a)}{\bar{\tau}}. \tag{2.1.24}$$

The $\frac{1}{\bar\tau^2}$ term in $d\bar\tau^2$ has a piece

$$\frac{2}{\bar\tau^2}\left(-F^{(2)}+\sigma^{(2)}\right), \tag{2.1.25}$$

hence choosing

$$F^{(2)}=\sigma^{(2)}, \tag{2.1.26}$$

removes the $\sigma^{(2)}$ term. In the process no

$$d\bar\phi^a d\bar\tau \tag{2.1.27}$$

terms are generated, so the fact that $A_a^{(1)}$ is already set to zero does not get altered. Now, let us look at the same procedure for $A_a^{(2)}$ and $\sigma^{(3)}$. Having set $A_a^{(1)}$ and $\sigma^{(2)}$ to zero, we do the following change of coordinates

$$\phi^a=\bar\phi^a+\frac{1}{\tau^2}G^{(2)a}(\bar\phi^b). \tag{2.1.28}$$

This generates a cross term

$$\frac{1}{\bar\tau}d\bar\phi^a d\bar\tau\left(A_a^{(2)}-G^{(2)b}h_{ab}^{(0)}\right), \tag{2.1.29}$$

which can be set to zero by choosing

$$G^{(2)b}=A_a^{(2)}h^{(0)ab}. \tag{2.1.30}$$

We do the following transformation

$$\tau=\bar\tau+\frac{F^{(3)}(\phi^a)}{\bar\tau^2}. \tag{2.1.31}$$

The $\frac{1}{\bar\tau^3}$ term in $d\bar\tau^2$ has a piece

$$\frac{2}{\bar\tau^3}\left(-2F^{(3)}+\sigma^{(3)}\right), \tag{2.1.32}$$

hence choosing

$$F^{(3)}=\frac{1}{2}\sigma^{(3)}, \tag{2.1.33}$$

removes the $\sigma^{(3)}$ term. In the process no $\frac{1}{\bar\tau}d\bar\phi^a d\bar\tau$ terms are generated. Continuing this logic one can arrive at a metric which is of the form

$$ds^2=-\left(1+\frac{\sigma}{\tau}\right)^2 d\tau^2+\tau^2\left(h_{ab}^{(0)}+\frac{1}{\tau}h_{ab}^{(1)}+\frac{1}{\tau^2}h_{ab}^{(2)}+\ldots\right)d\phi^a d\phi^b. \tag{2.1.34}$$

This is the final form of the metric we will work with, where σ is a scalar function of ϕ^a and $h^{(1)}_{ab}$ and $h^{(2)}_{ab}$ are successive corrections to the metric in the $\frac{1}{\tau}$ expansion. A spacetime is asymptotically flat at timelike infinity if it can be brought into our Beig-Schmidt form.

2.2 Equation of Motion

We want to impose Einstein's equation on our metric ansatz. We will use the Hamiltonian formalism to split Einstein's equation into a set of three equations. First, we give a review of 3+1 split and then use it for our case.

A 3+1 split is achieved by foliating the spacetime by hypersurfaces. For our case, we use spacelike hypersurfaces. A foliation is specified by a lapse function N and the shift vector N^a which depends on spacetime coordinates x^μ. The choice of foliation is completely arbitrary. We define h_{ab} as the induced metric on the spacelike hypersurface and the full metric of spacetime is given by

$$ds^2 = -N^2 dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt). \tag{2.2.35}$$

The extrinsic curvature is defined as

$$K_{ab} \equiv h^\mu_a h^\nu_a \nabla_\mu n_\nu = \frac{1}{2} h^\mu_a h^\nu_b \pounds_n h_{\mu\nu}, \tag{2.2.36}$$

where h^μ_a is the projection vector and n_μ is the vector normal to the hypersurface. The relation between three dimensional and four dimensional Riemann tensor is given by Gauss-Codazzi equations. We follow the approach describe in [20] to arrive at Gauss-Codazzi equations. The equations are described as follows

$$H = \mathcal{R} + K^2 - K^{ab}K_{ab} = 0, \tag{2.2.37}$$

$$F_a = D_b K^b{}_a - D_a K = 0, \tag{2.2.38}$$

$$F_{ab} = \mathcal{R}_{ab} + \pounds_n K_{ab} - 2K_{ac}K^c{}_b + KK_{ab} - D_{(a}a_{b)} - a_a a_b = 0 \tag{2.2.39}$$

where $\mathcal{R}_{ab}$ is the three dimensional Riemann tensor and a_μ is the acceleration vector to the normal curves.

$$a_\mu = \pounds_n n_\mu \tag{2.2.40}$$

$$a_b = h^\mu_b \pounds_n n_\mu \tag{2.2.41}$$

If we take the trace of (2.2.39) and use equation (2.2.37), we get

$$h^{ab}\pounds_n K_{ab} = K^{ab}K_{ab} + D_a a^a + a_a a^a. \tag{2.2.42}$$

For our case, the metric takes the following form

$$ds^2 = -N^2 d\tau^2 + h_{ab}dx^a dx^b, \tag{2.2.43}$$

where

$$N = 1 + \frac{\sigma}{\tau}, \tag{2.2.44}$$

$$h_{ab} = \tau^2\left(h^{(0)}_{ab} + \frac{1}{\tau}h^{(1)}_{ab} + \frac{1}{\tau^2}h^{(2)}_{ab} + \dots\right). \tag{2.2.45}$$

Comparing (2.2.35) and (2.2.43), we observe that the shift vector N_a is set to zero and the lapse function is $N = 1 + \frac{\sigma}{\tau}$. Then normal to the constant τ hypersurfaces is

$$n^\mu = \frac{1}{N}\delta^\mu_\tau, \tag{2.2.46}$$

as a result

$$K_{ab} = \frac{1}{2N}\partial_\tau h_{ab}, \tag{2.2.47}$$

and

$$\pounds_n K_{ab} = \frac{1}{N}\partial_\tau K_{ab}. \tag{2.2.48}$$

The acceleration vector

$$a_b = \frac{1}{N}D_b N, \tag{2.2.49}$$

where the covariant derivativ D_b is taken with respect to the metric h_{ab} (2.2.45) and therefore

$$D_a a_b + a_a a_b = \frac{1}{N}D_a D_b N. \tag{2.2.50}$$

Combining all the above things we get, the following simplified equations

$$H \equiv h^{ab}\frac{1}{N}\partial_\tau K_{ab} - K_{ab}K^{ab} - h^{ab}\frac{1}{N}D_aD_bN = 0, \tag{2.2.51}$$

$$F_a \equiv D^bK_{ab} - D_aK = 0, \tag{2.2.52}$$

$$F_{ab} \equiv \mathcal{R}_{ab} + \frac{1}{N}\partial_\tau K_{ab} - 2K_{ac}K^c_b + K_{ab}K - \frac{1}{N}D_aD_bN = 0. \tag{2.2.53}$$

We have obtained the expression for equations of motion. Now, we can use the Beig-Schmidt form to obtain equations of motion at zeroth, first and second order respectively. Each quantity appearing the above equation admits an expansion in τ. The metric on constant τ hypersurface is

$$h_{ab} = \tau^2\left(h^{(0)}_{ab} + \frac{1}{\tau}h^{(1)}_{ab} + \frac{1}{\tau^2}h^{(2)}_{ab} + \dots\right). \tag{2.2.54}$$

The inverse metric can be written as

$$h^{ab} = \frac{1}{\tau^2}\left(h^{(0)ab} - \frac{1}{\tau}h^{(1)ab} - \frac{1}{\tau^2}\left(h^{(2)ab} - h^{(1)a}_c h^{(1)cb}\right) + \dots\right). \tag{2.2.55}$$

The tensors $h^{(n)}_{ab}$ with order index are raised and lowered with respect to the background metric $h^{(0)}_{ab}$. The covariant derivative $\mathcal{D}_a$ in the following equations is defined with respect to the $h^{(0)}_{ab}$. Tensors without such order index are raised and lowered using the full metric h_{ab}. Now, we evaluate the asymptotic expansion of quantities required in the equation of motion.

The extrinsic curvature is defined as

$$K_{ab} = \frac{1}{2N}\partial_\tau h_{ab} \tag{2.2.56}$$

$$K_{ab} = \tau h^{(0)}_{ab} + \left(\frac{1}{2}h^{(1)}_{ab} - \sigma h^{(0)}_{ab}\right) + \frac{1}{\tau}\left(\sigma^2 h^{(0)}_{ab} - \frac{1}{2}\sigma h^{(1)}_{ab}\right) + \dots \tag{2.2.57}$$

We also have

$$
\begin{aligned}
K^a_b &= \frac{1}{\tau}\delta^a_b + \frac{1}{\tau^2}\bigg(-\frac{1}{2}h^{(1)a}_b - \sigma\delta^a_b\bigg) \\
&+ \frac{1}{\tau^3}\bigg(-h^{(2)a}_b + \frac{1}{2}h^{(1)a}_c h^{(1)c}_b + \frac{1}{2}\sigma h^{(1)a}_b + \sigma^2\delta^a_b\bigg) + \ldots, \qquad (2.2.58) \\
K^{ab} &= \frac{1}{\tau^3}h^{(0)ab} + \frac{1}{\tau^4}\bigg(-\frac{3}{2}h^{(1)ab} - \sigma h^{(0)ab}\bigg) + \\
&\frac{1}{\tau^5}\bigg(-2h^{(2)ab} + \sigma^2 h^{(0)ab} + 2h^{(1)ac}h^{(1)b}_c + \frac{3}{2}\sigma h^{(1)ab}\bigg) + \ldots. \qquad (2.2.59)
\end{aligned}
$$

The covariant derivative with respect to full metric h_{ab} leads to a similar expansion of the Christoffel connection

$$
\Gamma^a_{\ bc} = \Gamma^{(0)a}_{\ \ \ \ bc} + \frac{1}{\tau}\Gamma^{(1)a}_{\ \ \ \ bc} + \frac{1}{\tau^2}\Gamma^{(2)a}_{\ \ \ \ bc} + \ldots \qquad (2.2.60)
$$

where

$$
\begin{aligned}
\Gamma^{(1)a}_{\ \ \ \ bc} &= \frac{1}{2}\big(\mathcal{D}_c h^{(1)a}_{\ \ \ \ b} + \mathcal{D}_b h^{(1)a}_{\ \ \ \ c} - \mathcal{D}^a h^{(1)}_{\ \ \ ab}\big), \qquad (2.2.61) \\
\Gamma^{(2)a}_{\ \ \ \ bc} &= \frac{1}{2}\big(\mathcal{D}_c h^{(2)a}_{\ \ \ \ b} + \mathcal{D}_b h^{(2)a}_{\ \ \ \ c} - \mathcal{D}^a h^{(2)}_{\ \ \ bc}\big) \\
&- \frac{1}{2}h^{(1)ad}\big(\mathcal{D}_c h^{(1)}_{db} + \mathcal{D}_b h^{(1)}_{dc} - \mathcal{D}_d h^{(1)}_{bc}\big). \qquad (2.2.62)
\end{aligned}
$$

The three-dimensional Ricci tensor also has an expansion in τ.

$$
\mathcal{R}_{ab} = \mathcal{R}^{(0)}_{ab} + \frac{1}{\tau}\mathcal{R}^{(1)}_{ab} + \frac{1}{\tau^2}\mathcal{R}^{(2)}_{ab} + \ldots \qquad (2.2.63)
$$

The zeroth order Ricci tensor is constructed from the metric $h^{(0)}_{ab}$. The first and second order Ricci tensor are

$$
\mathcal{R}^{(1)}_{ab} = \frac{1}{2}\bigg(\mathcal{D}^c\mathcal{D}_b h^{(1)}_{ac} + \mathcal{D}^c\mathcal{D}_a h^{(1)}_{bc} - \mathcal{D}_a\mathcal{D}_b h^{(1)} - \mathcal{D}^c\mathcal{D}_c h^{(1)}_{ab}\bigg), \qquad (2.2.64)
$$

$$\begin{aligned}
\mathcal{R}^{(2)}_{ab} &= \frac{1}{2}\Big(\mathcal{D}^c\mathcal{D}_b h^{(2)}_{ac} + \mathcal{D}^c\mathcal{D}_a h^{(2)}_{bc} - \mathcal{D}_a\mathcal{D}_b h^{(2)} - \mathcal{D}^c\mathcal{D}_c h^{(2)}_{ab}\Big) + \frac{1}{2}\mathcal{D}_b\Big(h^{(1)cd}\mathcal{D}_a h^{(1)}_{cd}\Big) \qquad (2.2.65)\\
&\quad -\frac{1}{2}\mathcal{D}_b\Big(h^{(1)cd}\big(\mathcal{D}_a h^{(1)}_{bc} + \mathcal{D}_b h^{(1)}_{ac} - \mathcal{D}_c h^{(1)}_{ab}\big)\Big) + \frac{1}{4}\mathcal{D}^c h^{(1)}\Big(\mathcal{D}_a h^{(1)}_{bc} + \mathcal{D}_b h^{(1)}_{ac} - \mathcal{D}_c h^{(1)}_{ab}\Big)\\
&\quad -\frac{1}{4}\mathcal{D}_a h^{(1)}_{cd}\mathcal{D}_b h^{(1)cd} + \frac{1}{2}\mathcal{D}_c h^{(1)}_{ad}\mathcal{D}^c h^{(1)d}{}_{b} - \frac{1}{2}\mathcal{D}_c h^{(1)}_{ad}\mathcal{D}^d h^{(1)c}{}_{b}.
\end{aligned}$$

The equations can be expanded as

$$H = \frac{1}{\tau^3}H^{(1)} + \frac{1}{\tau^4}H^{(2)} + \dots, \qquad (2.2.66)$$

$$F_a = \frac{1}{\tau^2}F^{(1)}_a + \frac{1}{\tau^3}F^{(2)}_a + \dots, \qquad (2.2.67)$$

$$F_{ab} = F^{(0)}_{ab} + \frac{1}{\tau}F^{(1)}_{ab} + \frac{1}{\tau^2}F^{(2)}_{ab} + \dots. \qquad (2.2.68)$$

Zeroth order equation of motion

The zeroth order metric is Minkowski metric expressed in hyperbolic coordinates

$$\begin{aligned}
ds^2 &= -d\tau^2 + \tau^2(d\rho^2 + \sinh^2\rho\, d\theta^2 + \sinh^2\rho\,\sin^2\theta\, d\phi^2), \qquad (2.2.69)\\
&= -d\tau^2 + \tau^2 h^{(0)}_{ab}dx^a dx^b, \qquad (2.2.70)
\end{aligned}$$

where $h^{(0)}_{ab}$ is the unit metric on Euclidean AdS space.

The Hamiltonian and momentum equations are trivially satisfied at zeroth order. The equation of motion gives the following

$$\mathcal{R}_{ab} + 2h^{(0)}_{ab} = 0. \qquad (2.2.71)$$

First order equations of motion

The Hamiltonian equation $H^{(1)} = 0$ gives

$$(\mathcal{D}^2 - 3)\sigma = 0. \qquad (2.2.72)$$

The momentum equation $F_a^{(1)} = 0$ gives

$$\mathcal{D}^b k_{ab} = \mathcal{D}_a k \tag{2.2.73}$$

The equation of motion $F_{ab}^{(1)} = 0$ gives

$$(\mathcal{D}^2 + 3)k_{ab} = \mathcal{D}_a \mathcal{D}_b k + k h_{ab}^{(0)} \tag{2.2.74}$$

If we impose that k_{ab} is traceless and divergence free, then the equation of motion becomes.

$$(\mathcal{D}^2 + 3)k_{ab} = 0$$

Second order equations of motion

At second order, the Hamiltonian equation gives $H^{(2)} = 0$,

$$h^{(2)} = 12\sigma^2 + \frac{1}{4}k^{ab}k_{ab} - k^{ab}\mathcal{D}_a\mathcal{D}_b\sigma - \mathcal{D}^c\sigma\mathcal{D}_c\sigma. \tag{2.2.75}$$

The momentum equation $F_a^{(2)}$ gives

$$\mathcal{D}^b h_{ab}^{(2)} = \frac{1}{2}k^{bc}\mathcal{D}_b k_{ac} + \mathcal{D}_a\bigg(8\sigma^2 - \frac{1}{8}k^{bc}k_{bc} - k^{bc}\sigma_{bc} - \sigma^c\sigma_c\bigg). \tag{2.2.76}$$

The equation of motion $F_{ab}^{(2)} = 0$ takes the form,

$$(\mathcal{D}^2 + 2)h_{ab}^{(2)} = NL_{ab}(\sigma,\sigma) + NL_{ab}(\sigma,k) + NL_{ab}(k,k), \tag{2.2.77}$$

where the non linear terms are given by,

$$\begin{aligned}
NL_{ab}(\sigma,\sigma) &= h_{ab}^{(0)}(18\sigma^2 + 4\sigma^c\sigma_c) + \mathcal{D}_a\mathcal{D}_b(5\sigma^2 - \sigma^c\sigma_c) + 4\sigma\sigma_{ab},\\
NL_{ab}(\sigma,k) &= -\mathcal{D}_a\mathcal{D}_b(k^{cd}\sigma_{cd}) - 2h_{ab}^{(0)}(k^{cd}\sigma_{cd}) - 4\sigma k_{ab} + 4\sigma^c(\mathcal{D}_{(a}k_{b)c} - \mathcal{D}_c k_{ab})\\
&\quad +4\sigma_{c(a}k_{b)}^c,\\
NL_{ab}(k,k) &= \mathcal{D}_c k_{d(a}\mathcal{D}_{b)}k^{cd} - \frac{1}{2}\mathcal{D}_b k^{cd}\mathcal{D}_a k_{cd} + \mathcal{D}_c k_{ad}\mathcal{D}^c k_b^d - \mathcal{D}_c k_{ad}\mathcal{D}^d k_b^c\\
&\quad -k_{ac}k_b^c + k^{cd}(\mathcal{D}_c\mathcal{D}_d k_{ab} - \mathcal{D}_c\mathcal{D}_{(a}k_{b)d}).
\end{aligned}$$

Electric part of the Weyl tensor

The Weyl tensor is the trace free part of Riemann tensor and is defined as,

$$C_{\mu\lambda\nu\sigma} = R_{\mu\lambda\nu\sigma} - g_{\mu[\nu}R_{\sigma]\lambda} - g_{\lambda[\nu}R_{\sigma]\mu} + \frac{1}{3}Rg_{\mu[\nu}g_{\sigma]\nu}. \tag{2.2.78}$$

The electric part of the Weyl tensor is defined as

$$\begin{aligned} E_{ab} &= h_a^\mu h_b^\nu C_{\mu\lambda\nu\sigma}n^\lambda n^\sigma, \\ &= h_a^\mu h_b^\nu R_{\mu\lambda\nu\sigma}n^\lambda n^\sigma - \frac{1}{2}h_{ab}R_{\sigma\lambda}n^\lambda n^\sigma - \frac{1}{2}h_a^\mu h_b^\nu R_{\mu\nu} - \frac{1}{6}Rh_{ab}. \end{aligned} \tag{2.2.79}$$

To simplify the above expression, we have used the following relations,

$$\begin{aligned} g_{\mu\nu} &= h_{\mu\nu} - n_\mu n_\nu \\ h_a^\mu n_\mu &= 0 \\ h_a^\mu h_b^\nu g_{\mu\nu} &= h_{ab} \\ n^\mu n_\mu &= -1 \end{aligned}$$

Using the equations of motion for the bulk spacetime $R_{\mu\nu} = 0$, the above expression simplifies to,

$$E_{ab} = -\pounds_n K_{ab} + D_{(a}a_{b)} + a_a a_b + K_{ac}K_b^c. \tag{2.2.80}$$

Inserting the asymptotic expansions for the metric and the extrinsic curvature, it takes the form

$$E_{ab} = \frac{1}{\tau}E_{ab}^{(1)} + \frac{1}{\tau^2}E_{ab}^{(2)} + \dots \tag{2.2.81}$$

After computing each term, we get the following answer

$$\begin{aligned} \pounds_n K_{ab} &= h_{ab}^{(0)} - \frac{1}{\tau}\sigma h_{ab}^{(0)} + \frac{1}{\tau^2}\left(\frac{1}{2}\sigma k_{ab} - \sigma^2 h_{ab}^{(0)}\right)\dots, \\ D_{(a}a_{b)} + a_a a_b &= \frac{1}{\tau}\sigma_{ab} + \frac{1}{\tau^2}\left(2\sigma_a\sigma_b - \sigma\sigma_{ab} - \sigma^c\sigma_c h_{ab}^{(0)} + \frac{1}{2}\sigma^c\mathcal{D}_c k_{ab} - \sigma^c\mathcal{D}_{(a}k_{b)c}\right) + \dots, \\ K_{ac}K_b^c &= h_{ab}^{(0)} - \frac{2}{\tau}\sigma h_{ab}^{(0)} + \frac{1}{\tau^2}\left(-h_{ab}^{(2)} + 4\sigma^2 h_{ab}^{(0)} + \frac{1}{4}k_{ac}k_b^c - \sigma k_{ab}\right) + \dots. \end{aligned} \tag{2.2.82}$$

We get

$$
\begin{aligned}
E^{(1)}_{ab} &= \sigma_{ab} - \sigma h^{(0)}_{ab}, && (2.2.83)\\
E^{(2)}_{ab} &= -h^{(2)}_{ab} + 2\sigma_a\sigma_b - \sigma\sigma_{ab} - \sigma^c\sigma_c h^{(0)}_{ab} + \frac{1}{4}k_{ac}k^c_b + 5\sigma^2 h^{(0)}_{ab} + \frac{1}{2}\sigma^c\mathcal{D}_c k_{ab} - \sigma^c\mathcal{D}_{(a}k_{b)c} - \frac{3}{2}\sigma k_{ab}
\end{aligned}
$$

We note that this tensor is tracefree and divergence free,

$$
D^a E^{(1)}_{ab} = 0, \qquad (2.2.84)
$$

$$
E^{(1)a}{}_a = 0. \qquad (2.2.85)
$$

In the earlier literature this conserved traceless tensor played an important role. It was used to construct conserved quantities for translations Killing vectors. When we discuss construction of the charges later in the thesis, we will discuss the relation between our expression for charges and the expression obtained using this tensor. As of now, we have only computed the electric part of the Weyl tensor. Such a tensor can be useful for simplifying the second order equations of the motion written above, (2.2.75)–(2.2.77).

2.3 Summary

In this section, we summarize our boundary conditions. A spacetime is asymptotically flat at timelike infinity if its metric can be brought into the following form by doing appropriate coordinate transformation,

$$
ds^2 = -\left(1 + \frac{\sigma}{\tau}\right)^2 d\tau^2 + \tau^2\left(h^{(0)}_{ab} + \frac{1}{\tau}h^{(1)}_{ab} + \frac{1}{\tau^2}h^{(2)}_{ab} + \dots\right) d\phi^a d\phi^b. \qquad (2.3.86)
$$

We define k_{ab} as,

$$
k_{ab} \equiv h^{(1)}_{ab} + 2\sigma h^{(0)}_{ab}. \qquad (2.3.87)
$$

k_{ab} is a symmetric, traceless and divergenceless tensor. It satisfies the following equations,

$$
k_{[ab]} = 0, \quad k^a{}_a = 0, \quad \mathcal{D}^b k_{ab} = 0. \qquad (2.3.88)
$$

For our boundary conditions k_{ab} takes the following form,

$$k_{ab} = 2\left(\mathcal{D}_a\mathcal{D}_b\omega - h^{(0)}_{ab}\omega\right). \tag{2.3.89}$$

When we impose the tracefree condition, we get the differential equation satisfied by the supertranslation parameter,

$$\left(\mathcal{D}^2 - 3\right)\omega = 0. \tag{2.3.90}$$

From the first order equation of motion, we see that the mass aspect σ also satisfies the same equation,

$$\left(\mathcal{D}^2 - 3\right)\sigma = 0. \tag{2.3.91}$$

Chapter 3

Supertranslations

The Beig-Schmidt ansatz describes a class of asymptotically flat spacetimes near timelike infinity. The transformations which preserve such an asymptotic form would correspond to the asymptotic symmetries of these spacetimes. There are a set of diffeomorphisms which preserve the form of the metric ansatz.

$$x^{\mu} = L^{\mu}_{\nu} x^{\nu} + T^{\mu} + S^{\mu}(x^{\nu}),$$

where L^{μ}_{ν} are the Lorentz transformations, T^{μ} are translations and S^{μ} are supertranslations. In this project, we are only concerned with supertranslations. Supertranslations are direction dependent shift of origin. They are not gauge redundancies because they act on the physical state of the system. There can be more transformations which preserve the metric form but we do not deal with them now. In the next section, we have obtained the form of supertranslation vector field to first order. This analysis has been extended to second order as well.

3.1 First order Supertranslations

The Beig-Schmidt ansatz has an asymptotic expansion in the timelike coordinate τ. The metric has to be fixed to an order to obtain transformation which can preserve its structure to that order. These transformations will also have asymptotic expansion in τ.

We start with the metric

$$
\begin{aligned}
ds^2 &= -\left[1+\frac{2\sigma}{\bar{\tau}}+\mathcal{O}(1/\bar{\tau}^2)\right]d\bar{\tau}^2+2\bar{\tau}d\bar{\tau}d\bar{\phi}^a\left[\mathcal{O}(1/\bar{\tau}^2)\right] \\
&\quad +\bar{\tau}^2\left(h^{(0)}_{ab}+\frac{1}{\bar{\tau}}h^{(1)}_{ab}+\mathcal{O}(1/\bar{\tau}^2)\right)d\bar{\phi}^a d\bar{\phi}^b, \qquad (3.1.1)
\end{aligned}
$$

and ansatz for the vector field

$$
\xi=\omega(\phi)\partial_\tau+\frac{G^a(\phi)}{\tau}\partial_a .
$$

Here, $\omega(\phi)$ and $G^a(\phi)$ are unknown parameters describing the vector field. They only depend on hyperboloid coordinates. Under this transformation, the coordinate change as

$$
\begin{aligned}
\bar{\tau} &= \tau+\omega(\phi)+\dots, \\
\bar{\phi}^a &= \phi^a+\frac{G^a}{\tau}+\dots
\end{aligned}
$$

We will notice how the field appearing in the metric changes under this transformation.

$$
\begin{aligned}
\sigma(\bar{\phi}^a) &= \sigma\left(\phi^a+\frac{1}{\tau}G^a+\dots\right) \\
&= \sigma(\phi)+\frac{1}{\tau}\sigma_c G^c+\dots \qquad (3.1.2)
\end{aligned}
$$

where we have used the notation $\sigma_c=\partial_c\sigma$ and similarly, we will use it for other scalar functions e.g. $\omega_c=\partial_c\omega$. Also, we get:

$$
\begin{aligned}
\frac{1}{\bar{\tau}} &= \frac{1}{\tau}+\dots \qquad (3.1.3) \\
\frac{1}{\bar{\tau}^2} &= \frac{1}{\tau^2}+\dots \\
d\bar{\tau} &= d\tau+\omega_c d\phi^c+\dots \\
d\bar{\tau}^2 &= d\tau^2+2\tau d\tau d\phi^c\left[\frac{1}{\tau}\omega_c+\mathcal{O}\left(\frac{1}{\tau^2}\right)\right]+\tau^2 d\phi^a d\phi^b\mathcal{O}\left(\frac{1}{\tau^2}\right).
\end{aligned}
$$

Indices are raised and lowered with $h^{(0)}_{ab}$ on all quantities that carry order index. Using (3.1.2) and (3.1.3), we get

$$
\begin{aligned}
-\left[1+\frac{2\sigma}{\bar{\tau}}+\mathcal{O}\left(\frac{1}{\bar{\tau}^2}\right)\right]d\bar{\tau}^2 &= -\left[1+\frac{2\sigma}{\tau}+\mathcal{O}\left(\frac{1}{\tau^2}\right)\right]d\tau^2+2\tau d\tau d\phi^a\left[-\frac{\omega_a}{\tau}+\mathcal{O}\left(\frac{1}{\tau^2}\right)\right] \\
&\quad +\tau^2\mathcal{O}\left(\frac{1}{\tau^2}\right)d\phi^a d\phi^b.
\end{aligned}
$$

The terms of $\mathcal{O}(1/\tau^2)$ and higher has been suppressed. Secondly, we have:

$$
\begin{aligned}
h^{(0)}_{ab}(\bar{\phi}^c) &= h^{(0)}_{ab}\left(\phi^c+\frac{1}{\tau}G^c+\dots\right) \\
&= h^{(0)}_{ab}+\frac{1}{\tau}G^c\partial_c h^{(0)}_{ab}+\dots \qquad (3.1.4)\\
\bar{\tau}^2 &= \tau^2+2\omega\tau+\dots \qquad (3.1.5)
\end{aligned}
$$

Combining (3.1.4) and (3.1.5) we get,

$$
\bar{\tau}^2 h^{(0)}_{ab} = \tau^2 h^{(0)}_{ab}+\tau\big(G^c\partial_c h^{(0)}_{ab}+2\omega h^{(0)}_{ab}+\dots\big)
$$

The coordinate differentials $d\bar{\phi}^a$ transforms as,

$$
d\bar{\phi}^a = d\phi^a + d\phi^c\frac{1}{\tau}\partial_c G^a - d\tau\frac{1}{\tau^2}G^a+\dots
$$

which gives:

$$
d\bar{\phi}^a d\bar{\phi}^b = d\phi^a d\phi^a+\frac{1}{\tau}\left(\partial_c G^a d\phi^c d\phi^b+\partial_c G^b d\phi^c d\phi^a\right)+\frac{d\tau}{\tau^2}\left(-G^a d\phi^b-G^b d\phi^a\right)+\frac{1}{\tau^2}\mathcal{O}\left(\frac{1}{\tau^2}\right)d\tau^2.
$$

All this gives,

$$
\begin{aligned}
\bar{\tau}^2 h^{(0)}_{ab}d\phi^a d\phi^b &= \tau^2\left[h^{(0)}_{ab}+\frac{1}{\tau}(G^c\partial_c h^{(0)}_{ab}+2h^{(0)}_{c(a}\partial_{b)}G^c+2\omega h^{(0)}_{ab})+\mathcal{O}\left(\frac{1}{\tau^2}\right)\right]d\phi^a d\phi^b \\
&\quad +2\tau d\tau d\phi^a\left[-\frac{G_a}{\tau}+\mathcal{O}\left(\frac{1}{\tau^2}\right)\right]+d\tau^2\left[\mathcal{O}\left(\frac{1}{\tau^2}\right)\right]
\end{aligned}
$$

If we now gather all terms we see that:

$$
\left[1+\frac{2\sigma}{\bar{\tau}}+\dots\right]d\bar{\tau}^2 \to \left[1+\frac{2\sigma}{\tau}+\dots\right]d\tau^2,
$$

and we also notice that

$$2\bar{\tau}d\bar{\tau}d\bar{\phi}^a\left[\mathcal{O}\left(\frac{1}{\bar{\tau}^2}\right)\right] \to 2\tau d\tau d\phi^a\left[\frac{1}{\tau}\big(-\omega_a - G_a\big) + \mathcal{O}\left(\frac{1}{\bar{\tau}^2}\right)\right].$$

If we want to keep the form of the metric, then we must fix G_a in terms of ω

$$G_a = -\omega_a.$$

The above condition ensures that the vector field preserves Beig-Schmidt form. Therefore, we have

$$\tau^2 h^{(0)}_{ab}(\bar{\phi})d\bar{\phi}^a d\bar{\phi}^b \quad \to \quad \tau^2\left[h^{(0)}_{ab} + \frac{1}{\tau}(2\mathcal{D}_a\mathcal{D}_b\omega - 2\omega h^{(0)}_{ab}) + \mathcal{O}(1/\tau^2)\right]d\phi^a d\phi^b.$$

After combining all the pieces, we get the change in the first order metric $h^{(1)}_{ab}$ as

$$h^{(1)}_{ab} \to h^{(1)}_{ab} + 2D_aD_b\omega - 2\omega h^{(0)}_{ab}. \tag{3.1.6}$$

So, the vector field which preserves the asymptotic form to first order is,

$$\xi_{ST} = -\omega\partial_\tau + \frac{\omega^a}{\tau}\partial_a + \ldots . \tag{3.1.7}$$

We define $k_{ab} = h^{(1)}_{ab} + 2\sigma h^{(0)}_{ab}$. Under the action of the supertranslations,

$$\sigma \quad \to \quad \sigma \tag{3.1.8}$$

$$k_{ab} \quad \to \quad k_{ab} + 2(\mathcal{D}_a\mathcal{D}_b\omega - h^{(0)}_{ab}\omega) \tag{3.1.9}$$

Our boundary condition states that the tensor k_{ab} has to be traceless and supertranslations must preserve this conditions. Therefore, we obtain a differential equation satisfied by the supertranslation parameter ω as

$$(\mathcal{D}^2 - 3)\omega = 0 \tag{3.1.10}$$

3.2 Second order Supertranslations

We have found the form of the supertranslation vector fields to first order in the asymptotic expansion. We also obtained how the asymptotic fields transform under the action of supertranslations.

This analysis can be extended to second order by following the same steps as above. We again start with the metric ansatz to $\mathcal{O}(1/\tau^3)$

$$ds^2 = -\left(1+\frac{\sigma}{\tau}\right)^2 d\tau^2 + 2\tau d\tau d\phi^a \mathcal{O}(1/\tau^3) + \tau^2\left(h^{(0)}_{ab} + \frac{1}{\tau}h^{(1)}_{ab} + \frac{1}{\tau^2}h^{(2)}_{ab} + \mathcal{O}(1/\tau^3)\right)d\phi^a d\phi^b.$$

We take the following ansatz for the supertranslation vector field,

$$\xi_{ST} = \left(-\omega(\phi) + \frac{1}{\tau}F^{(2)}(\phi) + \dots\right)\partial_\tau + \left(\frac{\omega^a(\phi)}{\tau} + \frac{1}{\tau^2}G^{(2)a}(\phi) + \dots\right)\partial_a, \tag{3.2.11}$$

where $F^{(2)}$ and $G^{(2)a}$ are function (to be determined) of ϕ^a coordinates.

We assess how the coordinate differentials and the fields changes under the above transformation. Then, we impose the condition that the Beig-Schimdt form has to be preserved. This constrains the form of $F^{(2)}$ and $G^{(2)a}$ in terms of ω, σ and $h^{(1)}_{ab}$. These functions are found to be,

$$\begin{aligned} F^{(2)} &= \sigma\omega + \sigma^c\omega_c - \frac{1}{2}\omega^a\omega_a, \\ G^{(2)}_a &= \frac{1}{2}\big(2\omega\omega_a + \sigma\omega_a - \omega\sigma_a - \sigma_{ac}\omega^c - \sigma_c\omega^c_a - h^{(1)}_{ab}\omega^b - \omega^b\omega^c\Gamma_{abc}\big), \end{aligned}$$

where Γ_{abc} is the Christoffel connection with respect to the background metric $h^{(0)}_{ab}$. A tedious calculation then shows that under the action of the supertranslations we find,

$$\begin{aligned} h^{(2)}_{ab} \;\to\; & h^{(2)}_{ab} - \omega h^{(1)}_{ab} + \omega^c\big(\mathcal{D}_c h^{(1)}_{ab} - \mathcal{D}_a h^{(1)}_{bc}\big) + 2h^{(1)}_{c(a}\mathcal{D}_{b)}\omega^c - h^{(1)}_{bc}\omega^c_a \\ & +\sigma\omega_{ab} - \omega\sigma_{ab} + 2\sigma\omega h^{(0)}_{ab} + 2\sigma^c\omega_c h^{(0)}_{ab} - \omega^c\sigma_{bca} - \sigma_{ac}\omega^c_b - \sigma_{bc}\omega^c_a - \sigma^c\omega_{bca} \\ & +\big(\omega^2 h^{(0)}_{ab} - 2\omega\omega_{ab} + \omega_{ac}\omega^c_b\big) \end{aligned} \tag{3.2.12}$$

The above expressions will be useful when we look at the algebra of charges between rotations and supertranslations. Such a computation is not attempted in this thesis.

Chapter 4

Construction of charges

We have constructed the space of solutions which consist of asymptotically flat spacetimes at timelike infinity. Then, we found a set of diffeormorphisms which preserves the boundary condition. These diffeormorphisms are the supertranslations. We now show that they are asymptotic symmetries of spacetime, i.e., they are not gauge symmetries, and one can associate non-trivial conserved charges to them.

4.1 Covariant phase space formalism

In classical mechanics or classical field theory, one starts with a covariant Lagrangian variational principle, transforms to a Hamiltonian (phase space) description of the system. This description necessarily requires a decomposition between space and time which breaks manifest covariance. Phase space can be constructed covariantly by mapping the initial value data to the space of solutions. The space of solutions is called the covariant phase space. In this chapter we very briefly review the formalism to define a symplectic structure on the space of solutions. Using this symplectic structure charges for the asymptotic symmetries are computed.

The formalism for the construction of charges that we are following is the one by Wald [18, 19]. Let M be a d-dimensional manifold. Let $\mathcal{F}$ be the space of "kinematically allowed" Lorentzian metrics on M.

Let $\mathcal{L}$ be a diffeomorphism covariant Lagrangian density d-form on $\mathcal{F}$. The variation of $\mathcal{L}$ is,

$$\delta\mathcal{L} = F(g)\cdot\delta g + d\theta(g,\delta g), \tag{4.1.1}$$

where $F(g) = 0$ is the equation of motion and θ is the the presymplectic potential $(d-1)$ form.

Here, δg is a perturbation in g. For such perturbations, there exists a one-parameter family of metrics g_λ, such that $\delta g = \frac{dg_\lambda}{d\lambda}|_{\lambda=0}$. The perturbation corresponds to a tangent vector in $\mathcal{F}$.

Now, we can define a presymplectic current $(d-1)$ form ω,

$$\omega(g,\delta_1 g,\delta_2 g) = \delta_1\theta(g,\delta_2 g) - \delta_2\theta(g,\delta_1 g), \tag{4.1.2}$$

where $\delta_1 g$ and $\delta_2 g$ are two perturbations off of g.

We can define a presymplectic form on $\mathcal{F}$ by integrating the presymplectic current over a spacelike hypersurface,

$$\Omega_\Sigma(g,\delta_1 g,\delta_2 g) = \int_\Sigma \omega(g,\delta_1 g,\delta_2 g). \tag{4.1.3}$$

We must choose boundary conditions so that the above integral is finite and well defined.

Define $\bar{\mathcal{F}} = \{g\in\mathcal{F} \mid F(g) = 0\}$. The space of solutions, $\bar{\mathcal{F}}$ is called the covariant phase space.

Let ξ^a be an arbitrary vector field on M. This vector field can be used to define metric variation $\delta_\xi g \equiv \mathcal{L}_\xi g$ on $\mathcal{F}$. A function $H_\xi : \mathcal{F}\to\mathbb{R}$ is a Hamiltonian conjugate to ξ on a hypersurface Σ if for all $g\in\bar{\mathcal{F}}$ and all δg tangent to $\mathcal{F}$ we have,

$$\delta H_\xi = \Omega_\Sigma(g,\delta g,\mathcal{L}_\xi g) = \int_\Sigma \omega(g,\delta g,\mathcal{L}_\xi g). \tag{4.1.4}$$

In general, such a function H_ξ does not exist. The necessary and sufficient condition for it to exist is,

$$0 = (\delta_1\delta_2 - \delta_2\delta_1)H_\xi = -\int_{\partial\Sigma} \xi \cdot \omega(g, \delta_1 g, \delta_2 g) \tag{4.1.5}$$

for all perturbations $\delta_1 g$ and $\delta_2 g$ tangent to $\mathcal{F}$ at $g \in \bar{\mathcal{F}}$.

4.2 Symplectic structure

Consider the spacetime manifold M to be 4-dimensional. For vacuum solutions in general relativity, the only dynamical field is the metric g_{ab}. The Einstein-Hilbert lagrangian is a 4-form given by

$$\mathcal{L}_{\mu_1\mu_2\mu_3\mu_4} = \frac{1}{16\pi G}\epsilon_{\mu_1\mu_2\mu_3\mu_4}R. \tag{4.2.6}$$

The variation of this lagrangian gives the field equation

$$F_{\alpha\beta\mu_1\mu_2\mu_3\mu_4} = \frac{1}{16\pi G}\epsilon_{\mu_1\mu_2\mu_3\mu_4}\big(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\big), \tag{4.2.7}$$

and the presymplectic potential comes from $\delta R_{\alpha\beta}g^{\alpha\beta}$ term.

$$\delta R_{\alpha\beta}g^{\alpha\beta} = \nabla_\gamma\big(g_{\alpha\beta}\nabla^\gamma\delta g^{\alpha\beta} - \nabla_\alpha\delta g^{\alpha\gamma}\big) \tag{4.2.8}$$

This term has the form of $\nabla_\gamma v^\gamma$, where

$$v^\gamma = g^{\gamma\alpha}g^{\beta\rho}\big(\nabla_\rho\delta g_{\beta\alpha} - \nabla_\alpha\delta g_{\beta\rho}\big). \tag{4.2.9}$$

Therefore, the expression for presymplectic potential is

$$\theta_{\mu_1\mu_2\mu_3} = \frac{1}{16\pi G}v^\gamma\epsilon_{\gamma\mu_1\mu_2\mu_3}. \tag{4.2.10}$$

The associated presymplectic current is

$$\omega_{\mu_1\mu_2\mu_3} = \frac{1}{16\pi G}\omega^\gamma\epsilon_{\gamma\mu_1\mu_2\mu_3}, \tag{4.2.11}$$

where

$$\omega^\gamma = P^{\gamma\nu\alpha\beta\mu\delta}\big(\delta_2 g_{\nu\alpha}\nabla_\beta\delta_1 g_{\mu\delta} - (1 \leftrightarrow 2)\big), \tag{4.2.12}$$

where

$$P^{\gamma\nu\alpha\beta\mu\delta} = g^{\gamma\mu}g^{\delta\nu}g^{\alpha\beta} - \frac{1}{2}g^{\gamma\beta}g^{\nu\mu}g^{\delta\alpha} - \frac{1}{2}g^{\gamma\nu}g^{\alpha\beta}g^{\mu\delta} - \frac{1}{2}g^{\nu\alpha}g^{\gamma\mu}g^{\delta\beta} + \frac{1}{2}g^{\nu\alpha}g^{\gamma\beta}g^{\mu\delta}. \qquad (4.2.13)$$

This is the expression from Wald and Zoupas [18]. Now, we choose the convention to be $\delta g^{\mu\nu} = -g^{\mu\sigma}g^{\nu\kappa}\delta g_{\sigma\kappa}$. We get

$$\begin{aligned}\omega[\delta_1 g, \delta_2 g]^\gamma = \frac{1}{32\pi G}\Big(&\delta_2 g^{\alpha\beta}\nabla^\gamma\delta_1 g_{\alpha\beta} + \delta_2 g\nabla^\alpha\delta_1 g^\gamma_\alpha + \delta_2 g^\gamma_\alpha\nabla^\alpha\delta_1 g - \delta_2 g\nabla^\gamma\delta_1 g \\ &- 2\delta_2 g_{\alpha\beta}\nabla^\alpha\delta_1 g^{\gamma\beta} - (1\leftrightarrow 2)\Big) \qquad (4.2.14)\end{aligned}$$

Using this expression, we can evaluate the symplectic structure on appropriate hypersurface to get the charge associated to asymptotic Killing vector. In order to facilitate comparison with the published literature, we use the following expression for

$$\begin{aligned}\omega[\delta_1 g, \delta_2 g]^\gamma = \frac{1}{32\pi G}\Big(&\delta_2 g^{\alpha\beta}\nabla^\gamma\delta_1 g_{\alpha\beta} + \delta_2 g\nabla^\alpha\delta_1 g^\gamma_\alpha + \delta_2 g^\gamma_\alpha\nabla^\alpha\delta_1 g - \delta_2 g\nabla^\gamma\delta_1 g \\ &- \delta_2 g_{\alpha\beta}\nabla^\alpha\delta_1 g^{\gamma\beta} - \delta_1 g^{\gamma\alpha}\nabla^\beta\delta_2 g_{\alpha\beta} - (1\leftrightarrow 2)\Big). \qquad (4.2.15)\end{aligned}$$

The two expressions differ by a total derivative term.

In the case of asymptotically flat spacetime without any internal boundaries (i.e., no black hole event horizons), it has been suggested by Hawking, Perry, Strominger that in the absence of massive particles that the full future null infinity can be thought of a surface where data can be provided for the determination of the full spacetime.

Thinking of a limiting process in which we successively deform a spacelike Cauchy surface into null infinity, the above statement suggests that the contribution to the dynamics from the $\tau \to \infty$ limit of the constant τ surface should be identically zero. In the following we show that such a expectation is realised with our notion of the asymptotic flatness at timelike infinity. We show that the presymplectic current computed on $\tau =$ const surface in the limit $\tau \to \infty$ is zero.

The requirement that σ and ω should match to mass aspect and supertranslation respectively at null infinity fixes the form of the general form of the solutions that σ and ω can take. As an added bonus of this matching we also find that integrals of presymplectic current computed on constant ρ hypersurfaces in the asymptotic regions is convergent. This convergence property is the directly related to the the parity condition that Ashtekar, Bombelli, and Reula proposed.

The relation to null infinity is not fully explored in this thesis, this analysis is left for the future. Extrapolating results from the analysis of spacelike infinity we can guess that σ and ω have the properties discussed in the next subsection. The properties proposed give a self-consistent picture.

The presymplectic current on a constant τ hypersurface is as follows,

$$\begin{aligned}\omega[\delta_1 g, \delta_2 g]^\tau =& \frac{1}{32\pi G}\Big(\delta_2 g_{ab}\big(\nabla^\tau \delta_1 g^{ab} - 2\nabla^a \delta_1 g^{\tau b}\big) + \delta_2 g \nabla^a \delta_1 g_a^\tau \\ &+ \big(\delta_2 g - \delta_2 g_\tau^\tau\big)\big(\nabla^\tau \delta_1 g_\tau^\tau - \nabla^\tau \delta_1 g\big) - (1 \leftrightarrow 2)\Big)\end{aligned} \tag{4.2.16}$$

Computing various quantities appearing in the above expression for our notion of the asymptotically flat spacetimes at timelike infinity, we have

$$\begin{aligned}\delta_2 g_{ab}\big(\nabla^\tau \delta_1 g^{ab} - 2\nabla^a \delta_1 g^{\tau b}\big) &= \frac{1}{\tau^3}\big(\delta_1 k^{ab}\delta_2 k_{ab} + 36\delta_1\sigma\delta_2\sigma\big) \\ \delta_2 g \nabla^a \delta_1 g_a^\tau &= 0 \\ \big(\delta_2 g - \delta_2 g_\tau^\tau\big)\big(\nabla^\tau \delta_1 g_\tau^\tau - \nabla^\tau \delta_1 g\big) &= \frac{36\delta_1\sigma\delta_2\sigma}{\tau^3}\end{aligned} \tag{4.2.17}$$

This shows that the presymplectic current is zero in the limit $\tau \to \infty$,

$$\omega[\delta_1 g, \delta_2 g]^\tau = \mathcal{O}\left(\frac{1}{\tau}\right) \tag{4.2.18}$$

We can also calculate the integral of the presymplectic current on a constant ρ hypersurface. This computation proceeds as follows:

$$\begin{aligned}\omega[\delta_1 g, \delta_2 g]^\rho &= \frac{1}{32\pi G}\Big(\delta_2 g^{\alpha\beta}(\nabla^\rho \delta_1 g_{\alpha\beta} - \nabla_\alpha \delta_1 g_\beta^\rho) + \delta_2 g(\nabla_\alpha \delta_1 g^{\rho\alpha} - \nabla^\rho \delta_1 g) \\ &\quad + \delta_2 g^{\rho\alpha}\left(\nabla_\alpha \delta_1 g - \nabla^\beta \delta_1 g_{\alpha\beta}\right) - (1 \leftrightarrow 2)\Big)\end{aligned} \tag{4.2.19}$$

Computing various quantities appearing in this expression for our notion of the asymptotically flat spacetimes, we have

$$\delta_2 g^{\alpha\beta}\big(\nabla^\rho\delta_1 g_{\alpha\beta} - \nabla_\alpha\delta_1 g^\rho_\beta\big) = -\frac{1}{\tau^4}\bigg(12\delta_2\sigma\mathcal{D}^\rho\delta_1\sigma + \delta_2 k^{ab}\mathcal{D}^\rho\delta_1 k_{ab} - \delta_2 k^{ab}\mathcal{D}_a\delta_1 k^\rho_b + 2\delta_2 k^{\rho a}\mathcal{D}_a\delta_1\sigma\bigg)$$
$$\delta_2 g\big(\nabla_\alpha\delta_1 g^{\rho\alpha} - \nabla^\rho\delta_1 g\big) = \frac{8}{\tau^4}\delta_2\sigma\mathcal{D}^\rho\delta_1\sigma$$
$$\delta_2 g^{\rho\alpha}\big(\nabla_\alpha\delta_1 g - \nabla^\beta\delta_1 g_{\alpha\beta}\big) = -\frac{1}{\tau^4}\bigg(-2\delta_2 k^{\rho b}\mathcal{D}_b\delta_1\sigma + 4\delta_2\sigma\mathcal{D}^\rho\delta_1\sigma\bigg). \tag{4.2.20}$$

Adding various contributions, we get

$$\omega[\delta_1 g,\delta_2 g]^\rho = \frac{\tau^{-4}}{32\pi G}\bigg(8\delta_1\sigma\mathcal{D}^\rho\delta_2\sigma + \delta_1 k^{ab}\mathcal{D}^\rho\delta_2 k_{ab} - \delta_1 k^{ab}\mathcal{D}_a\delta_2 k^\rho_b - (1\leftrightarrow 2)\bigg) \tag{4.2.21}$$

When we restrict the form of k_{ab} to be,

$$k_{ab} = 2(\mathcal{D}_a\mathcal{D}_b\omega - h^{(0)}_{ab}\omega), \tag{4.2.22}$$

the symplectic current simplifies to

$$\omega[\delta_1 g,\delta_2 g]^\rho = \frac{\tau^{-4}}{4\pi G}\bigg(\delta_1\sigma\mathcal{D}^\rho\delta_2\sigma - \delta_2\sigma\mathcal{D}^\rho\delta_1\sigma\bigg) \tag{4.2.23}$$

The bulk symplectic structure can be obtained by integrating the symplectic current on an appropriate surface as,

$$\begin{aligned}
\Omega(\delta_1 g,\delta_2 g) &= \int_\Sigma \omega[\delta_1 g,\delta_2 g]^\rho d\Sigma \\
&= \frac{1}{4\pi G}\lim_{\tau\to\infty}\int_\Sigma \frac{(d^3x)_\rho}{\tau^4}\sqrt{-h^{(0)}}\big(\delta_1\sigma\mathcal{D}^\rho\delta_2\sigma - \delta_2\sigma\mathcal{D}^\rho\delta_1\sigma\big), \\
&= \frac{1}{4\pi G}\lim_{\tau\to\infty}\int\frac{d\tau}{\tau}\int_{S^2} d\theta\, d\phi\ \sin\theta\big(\delta_1\sigma\partial_\rho\delta_2\sigma - \delta_2\sigma\partial_\rho\delta_1\sigma\big).
\end{aligned} \tag{4.2.24}$$

The symplectic structure is logarithmically divergence for a generic σ. Upon using the proposed properties of σ and ω, one can readily see that the the above integral is convergent. The properties of σ and ω are described in the following section.

4.3 Properties of σ and ω on Euclidean AdS_3

We observe that σ and ω satisfies the same differential equation. Link between timelike and null infinity fixes the behavior of σ and ω on the hyperboloid.

We express the equation in $x^a = (\rho, x^A)$ coordinates and expand it as

$$\partial_\rho^2\sigma + 2\coth(\rho)\partial_\rho\sigma + \frac{1}{\sinh(\rho)^2}\Delta\sigma - 3\sigma = 0, \tag{4.3.25}$$

where Δ is the sphere Laplacian. We look at solutions that can be schematically separated into product form (separation of variable ansatz),

$$\sigma(\rho, \theta, \phi) = \sigma(\rho)Y(\theta, \phi) \tag{4.3.26}$$

When we substitute this form of solution into the above equation, we get a radial equation and an angular equation

$$\partial_\rho^2\sigma + 2\coth(\rho)\partial_\rho\sigma - \frac{l(l+1)}{\sinh(\rho)^2}\sigma - 3\sigma = 0, \tag{4.3.27}$$

$$\Delta Y(\theta, \phi) = -l(l+1)Y(\theta, \phi). \tag{4.3.28}$$

The solution to equation (4.3.27) can be readily obtained using Mathematica and the solutions to the second equation are the standard spherical harmonics.

The solution to the radial equation (4.3.27) is

$$\sigma_l(\rho) = a_l V_l(\rho) + b_l W_l(\rho).$$

where

$$V_l(\rho) = \cosh\rho\left[(\tanh\rho)^{-l-1}{}_2F_1\left(-\frac{(l+1)}{2}, -1-\frac{l}{2}, \frac{1}{2}-l, \tanh^2\rho\right)\right] \tag{4.3.29}$$

$$W_l(\rho) = \cosh\rho\left[(\tanh\rho)^{l}{}_2F_1\left(\frac{(-l+1)}{2}, \frac{l}{2}, \frac{3}{2}+l, \tanh^2\rho\right)\right] \tag{4.3.30}$$

The most general solution to equation is given by

$$\sigma(\rho,\theta,\phi) = \sum_{l,m} \left(\sigma^V_{lm} V_l(\rho) + \sigma^W_{lm} W_l(\rho)\right) Y_{lm}, \tag{4.3.31}$$

where σ^V_{lm} and σ^W_{lm} are the coefficient of linear combinations.

Let us now investigate the behavior of these two independent solutions as $\rho \to 0$ and $\rho \to \infty$.

$\rho \to 0$ behavior

The $\rho \to 0$ limit takes us to the origin.

$$V_l \rightarrow \frac{1}{\rho^{(l+1)}} \tag{4.3.32}$$

$$W_l \rightarrow \rho^l \tag{4.3.33}$$

It is observed that the function V_l diverges near the origin while, the function W_l approaches zero near the origin. The behaviour of these solutions is simply the behaviour of the massless scalar on the three-dimensional flat space $\mathbb{R}^3$. In the limit $\rho \to 0$, the effect of the Euclidean AdS_3 curvature and the non-zero mass term disappear. We effectively get a Laplace equation on $\mathbb{R}^3$.

$\rho \to \infty$ behavior

The $\rho \to \infty$ limit takes us to null infinity on a constant τ hypersurface. To obtain behavior near infinity, we redefine the radial coordinate,

$$y = e^\rho, \qquad \rho = \log(y).$$

For l = 0,1

$$V_l \rightarrow y + \mathcal{O}\left(\frac{1}{y}\right), \tag{4.3.34}$$

$$W_l \rightarrow \mathcal{O}(y). \tag{4.3.35}$$

For $l > 1$,

$$V_l \rightarrow \mathcal{O}\left(\frac{1}{y^3}\right), \quad (4.3.36)$$

$$W_l \rightarrow \mathcal{O}(y). \quad (4.3.37)$$

We can see that the function V_l approaches zero at null infinity except for $l \leq 1$. Also, the function W_l diverges for all values of l near null infinity. These solutions simply correspond to the two types of solutions on Euclidean AdS_3, the normalizable modes i.e., V_l and the non-normalizable modes i.e., W_l. A more detailed investigation is required to properly understand the relation to null infinity. We plan to undertake this in the near future. Given the already existing literature it seems that the non-normalizable part of the solution – W_l correspond to the supertranslation gauge parameter, while the normalizable part – V_l corresponds to the mass aspect σ.

We have restricted the form of σ to be normalizable and this results in a finite symplectic structure in the following manner. Let us revisit the expression of the symplectic structure evaluated on a constant ρ hypersurface

$$\Omega(\delta_1 g, \delta_2 g) = \frac{1}{4\pi G} \lim_{\tau\to\infty} \int \frac{d\tau}{\tau} \int_{S^2} d\theta\, d\phi\, \sin\theta \big(\delta_1\sigma\partial_\rho\delta_2\sigma - \delta_2\sigma\partial_\rho\delta_1\sigma\big). \quad (4.3.38)$$

In the second line, we have separated the τ integral and the integral on S^2. Now, we substitute σ into the above integral,

$$\Omega(\delta_1 g, \delta_2 g) = \frac{1}{4\pi G} \lim_{\tau\to\infty} \int \frac{d\tau}{\tau} \int d\theta\, d\phi\, \sin\theta \sum_{l,m,l',m'} \Big(a_{lm} b_{l'm'} \big(V_l^1 \partial_\rho V_l^2 Y_{lm}^1 Y_{l'm'}^2 - V_l^2 \partial_\rho V_l^1 Y_{lm}^1 Y_{l'm'}^2\big)\Big). \quad (4.3.39)$$

Now, we integrate over the two sphere and use the integral property of spherical harmonics to get

$$\begin{aligned}
\Omega(\delta_1 g, \delta_2 g) &= \frac{1}{4\pi G} \lim_{\tau\to\infty} \int \frac{d\tau}{\tau} \sum_{l,m,l',m'} \Big(a_{lm} b_{l'm'} \big(V_l^1 \partial_\rho V_l^2 \delta_{ll'}\delta_{mm'} - V_l^2 \partial_\rho V_l^1 \delta_{ll'}\delta_{mm'}\big)\Big) \\
&= \frac{1}{4\pi G} \lim_{\tau\to\infty} \int \frac{d\tau}{\tau} \sum_{l,m} \Big(a_{lm} b_{lm} \big(V_l^1 \partial_\rho V_l^2 - V_l^2 \partial_\rho V_l^1\big)\Big) \\
&= 0 \quad (4.3.40)
\end{aligned}$$

V_l^1 and V_l^2 are equal for a given value of l. Hence we get a finite symplectic structure. This result ensures that the choice of σ function is correct.

4.4 Supertranslation charges

Using the covariant phase space formalism, we can arrive at an expression for supertranslation charge,

$$H = \frac{1}{4\pi}\int_C d^2x\sqrt{h^{(0)}}n_a(\sigma\omega^a - \sigma^a\omega). \tag{4.4.41}$$

The charges are also conserved when we restrict supertranslation parameter to satisfy,

$$(\mathcal{D}^2 - 3)\omega = 0. \tag{4.4.42}$$

We find that,

$$\mathcal{D}_a(\sigma^a\omega - \sigma\omega^a) = 0. \tag{4.4.43}$$

The conservation property means that this quantity can be computed on any 2-sphere cross-section C of the Euclidean hyperboloid. It follows from the fact that ω^τ vanishes and from the fact that as $\rho \to \infty$ the expression remains finite. Although it is not explored in full detail, it is expected that this answer matches with the expression for supertranslation charges at null infinity.

Let us now evaluate the charges using σ and ω. We get,

$$H = \frac{1}{4\pi}\sum_{l,m,l',m'}\sinh^2\rho\int d\theta\, d\phi\, \sin\theta\,(Y^l_m Y^{l'}_{m'} a_{l,m}b_{l',m'})\,[V_l\partial_\rho W_{l'} - W_{l'}\partial_\rho V_l] \tag{4.4.44}$$

Now, we use the orthogonality identity of spherical harmonics,

$$\int d\theta\, d\phi\, \sin\theta\, Y^l_m Y^{l'}_{m'} = \frac{(-1)^{m'}4\pi}{(2l+1)}\delta_{l,l'}\delta_{m,-m'},$$

Substituting this into (4.4.44), we get

$$H = \frac{1}{4\pi}\sum_{l,m,l',m'} \sinh^2\rho \frac{(-1)^{m'}4\pi}{(2l+1)}\delta_{l,l'}\delta_{m,-m'}(a_{l,m}b_{l',m'})\,[V_l\partial_\rho W_{l'} - W_{l'}\partial_\rho V_l]$$
$$= \sum_{l,m}\sinh^2\rho\frac{(-1)^m}{(2l+1)}(a_{l,-m}b_{l,m})\,[V_l\partial_\rho W_l - W_l\partial_\rho V_l] \qquad (4.4.45)$$

Since this expression is independent of ρ we can compute it in the limit $\rho \to 0$. We note that

$$\sinh^2\rho \;\to\; \rho^2 \qquad (4.4.46)$$
$$[V_l\partial_\rho W_l - W_l\partial_\rho V_l] \;\to\; \left(\frac{2l+1}{\rho^2}\right). \qquad (4.4.47)$$

It then follows that

$$H = \sum_{l,m}(-1)^m a_{l,-m}b_{l,m} \qquad (4.4.48)$$

For translations, the expression for charges are simplified because $\omega_{ab} - \omega h^{(0)}_{ab} = 0$. We can write the integrand as

$$2\sigma\omega_b - 2\omega\sigma_b = E^{(1)a}_b\omega_a - 2\mathcal{D}^a\left(\omega_{[a}\sigma_{b]}\right) \qquad (4.4.49)$$

which upon integration over the two sphere simply becomes

$$H = \int_C E^{(1)}_{ab}n^a\omega^b d^2S \qquad (4.4.50)$$

This last expression is same as what was obtained by earlier work of Ashtekar et.al [8, 17].

Chapter 5

Future directions

There are several open problems that we need to work on in relation to the topics covered in this thesis. First and foremost, we would like to to explore the relation of our asymptotic analysis to null infinity. It is desirable to have a better understanding of the conditions that would fix the mass aspect σ and the supertranslation parameter ω to be of the form discussed above.

We have not attempted to express any non-trivial solution, e.g., Schwarzschild solution, in a form that manifests the fact that it satisfies our boundary conditions. It will be useful to do that as well.

Hawking, Perry, and Strominger [21, 22, 23] have argued that supertranslation vector fields from null infinity can be related to supertranslation vector fields at the horizon in a simple way. Our work in principal allows us to achieve this via timelike infinity. We plan to explore this circle of ideas in further details.

Finally, in this work we have only looked at the supertranslation charges. It will be natural to extend these computations to discuss charges for rotation and boost generators.

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