Stochastic analysis on Wiener space and applications to distributional asymptotics

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Certificate

This is to certify that this dissertation entitled Stochastic analysis on Wiener space and applications to distributional asymptotics towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Abhishek Tilva partly at Indian Statistical Institute, Bangalore under the supervision of D. Yogeshwaran and Sreekar Vadlamani and subsequently at Indian Institute of Science Education and Research under the supervision of Anindya Goswami, Associate Professor, Department of Mathematics, during the academic year 2018-2019.

Anindya Goswami

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This thesis is dedicated to my parents.

Declaration

I hereby declare that the matter embodied in the report entitled Stochastic analysis on Wiener space and applications to distributional asymptotics are the results of the work carried out by me partly at the Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore under the supervision of D. Yogeshwaran and Sreekar Vadlamani and subsequently at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Anindya Goswami and the same has not been submitted elsewhere for any other degree.

Abhishek Tilva

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Abstract

In this thesis we study Malliavin calculus on infinite dimensional Wiener space and study properties of Malliavin operators. We then see how these along with what is known as Stein's method for distributional approximation is used to obtain quantitative limit theorems inside a fixed Wiener chaos and also sometimes more generally. In a joint work with David Nualart which is the content of chapter 4, we apply these results to prove an invariance principle for functionals of Gaussian random vector fields on Euclidean space for a large class of covariances. This is an extension of the original famous result by Breuer and Major and recent functional convergence results by Nualart et. al. to the case of vector valued fields. We then briefly also look into further applications in the area of geometry of random fields.

Contents

Introduction

The classical central limit theorem (CLT) states that for a sequence of random variables $\{X_n : n \in \mathbb{N}\}$ such that X_i's are independent and identically distributed with mean zero and variance one, we have that as $n \to \infty$

$$n^{-1/2} \sum_{i=1}^{n} X_i \Rightarrow \mathcal{N}(0,1) \tag{0.0.1}$$

where $\mathcal{N}(0,1)$ denotes the standard normal distribution with mean zero and variance 1 and $\Rightarrow$ denotes convergence in distribution, i.e as $n \to \infty$ for every $u \in \mathbb{R}$, $\mathbb{P}(n^{-1/2} \sum_{i=1}^{n} X_i \leq u) \to \Phi(u)$ where $\Phi(u) = \int_{-\infty}^{u} (2\pi)^{-1/2} e^{-x^2/2} dx$. From this result, one is then interested in quantifying the error that one makes when assuming the distribution of the scaled sum to be normal. The Berry-Esseen bound for the classical CLT gives precisely that. It states that if $S_n = (n)^{-1/2} \sum_{i=1}^{n} X_i$ and $N \sim \mathcal{N}(0,1)$, then

$$d_{\mathrm{Kol}}(S_n, N) \leq \frac{0.4785\mathbb{E}[|X_1|^3]}{\sqrt{n}} \tag{0.0.2}$$

where $d_{\mathrm{Kol}}(F,G) = \sup_{z \in \mathbb{R}} |\mathbb{P}(F \in (-\infty, z]) - \mathbb{P}(G \in (-\infty, z])|$, is called the Kolmogorov distance between laws of random variables F and G. One line of inquiry from this result would be to consider a general case of convergence in distribution, i.e. suppose that for a sequence of random variables $\{F_n : n \in \mathbb{N}\}$, we have $F_n \Rightarrow \mathcal{N}(0,V)$, how do we deduce the rate of convergence and error estimates of the form of equation (0.0.2) when the law of F_n is approximated by a normal law? Charles Stein in his landmark paper [1] essentially gave a method for doing this using a simple approach. This approach since then has been widely researched, applied and extended to other target distributions and the same is now known as Stein's method. For example, L. Chen in [2] extended the method for Poisson approximation. We refer the reader to [3] for complete survey on Stein's method.

In this thesis, we study distributional limit theorems similar to the classical central limit theorem for the case of stochastic processes instead of sequences of random variables. More precisely, we are interested in random fields which are stochastic processes on arbitrary parameter sets. Due to their general nature, for stochastic modelling of phenomenon, they have found applications in a wide variety of sciences from medicine to oceanography. When modelling an activity over a region using random fields, one is usually interested in the extreme regions i.e. regions corresponding to very high or very low activity. These are called excursion or sojourn sets respectively of random fields and recently there has been significant interest in studying distributional asymptotics of various geometric functionals of excursion sets. We describe this area of study in Chapter 5. We will also be interested in rates of convergence similar to the Berry-Esseen rate in these asymptotics.

One other question from the classical central limit theorem that one might ask is what about the case when X_i's are not independent? An answer in this direction was given by Péter Breuer and Péter Major in 1983 in their paper [4] wherein they gave a very general central limit theorem under some conditions on the covariances. Since the appearance of their result, the same has been extended to variety of different settings and it has proved to be one of the most applicable results in stochastic analysis. We discuss this theorem in detail in chapter 5 and we give a proof of it's version for random vector fields. This is joint work with David Nualart, [5]. This version for vector valued random fields finds it's applications in the study of limit theorems for various geometric characteristics of excursion sets of random fields. We discuss this in Chapter 5. For more applications of this theorem, we refer the reader to the introduction in [6].

For our proofs and to obtain our results, we rely on tools from stochastic analysis, mainly Malliavin calculus. Paul Malliavin in [7] had initially developed the theory to give a probabilistic proof for Hormander's theorem ([8]) and to give conditions for regularity of probability laws of random variables. Since then the theory has found many applications in areas including finance and limit theorems. In this thesis, we introduce all the necessary tools that we will need in our study in Chapter 2. For an extensive treatment of the theory, we refer the reader to the monograph by David Nualart, ([9]).

We now briefly give organization of various chapters. Chapter 1 introduces the necessary and preliminary definitions regarding theory of random fields and we state few important results concerning them. In Chapter 2, we introduce the tools from Malliavin calculus that we will require further in our study. Chapter 3 is about an important breakthrough in the approach to prove distributional limit theorems of the form of classical central limit theorem. We will use these techniques to prove our results. Chapter 4 contains our work wherein we prove a version of the aforementioned theorem by Breuer and Major and prove additional results. Finally, in Chapter 5 we discuss applications in area concerning geometry of random fields and we conclude by mentioning further lines of research.

Chapter 1

Random Fields

In this chapter we will describe the necessary framework of random fields and in particular Gaussian random fields which will remain our main focus in this thesis. We will limit our discussion to random fields on metric spaces and real valued random fields. However, the corresponding definitions for complex valued random fields are immediate. We will also often use the term "field" instead of random field and we will never imply it to mean the algebraic structure of a field on a set. We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all random elements are defined.

1.1 First definitions

We recall that a random element is a measurable map, $X : \Omega \to (S, \mathcal{S})$, where $(S, \mathcal{S})$ is measurable space and the distribution or law of X is the probability measure $\mathscr{L}_X$ on $(S, \mathcal{S})$ given by $\mathscr{L}_X(A) = \mathbb{P}(X^{-1}(A))$ for $A \in \mathcal{S}$.

Definition 1.1.1. *A random field ξ on a metric space (T, d) is a map $\xi : \Omega \times T \to \mathbb{R}$ such that $\xi(\cdot, x)$ is a random variable for every $x \in T$.*

We will abbreviate $\xi(\cdot, x)$ as $\xi(x)$ and sometimes we will also use the notation ξ_x to mean the same. We recall that a distribution γ on $\mathbb{R}^m$ is multivariate Gaussian if and only if it is either a dirac mass at some point or there exist $\mu \in \mathbb{R}^m$ and a non-negative definite matrix C such that $\widehat{\gamma}(x) = \int_{\mathbb{R}^m} e^{i\langle t,x\rangle}\gamma(dt) = e^{i\langle x,\mu\rangle - \frac{1}{2}\langle Cx,x\rangle}$.

Definition 1.1.2. *A random field ξ is Gaussian if all the finite dimensional distributions of ξ are multivariate Gaussian, i.e. for every $m \in \mathbb{N}$ and every $(x_1, ..., x_m) \in T^m$, the distribution of the vector $(\xi(x_1), ..., \xi(x_m))$, is multivariate Gaussian.*

For a metric space (T, d), we denote the Borel σ-algebra on T by $\mathcal{B}(T)$.

Definition 1.1.3. *A random field ξ is said to be measurable if the map $\xi : \Omega \times T \to \mathbb{R}$ is measurable, i.e. for every Borel set $B \in \mathcal{B}(\mathbb{R})$, $\xi^{-1}(B) \in \mathcal{F} \otimes \mathcal{B}(T)$, the product σ-algebra.*

Definition 1.1.4. *For a random field $\xi : \Omega \times T \to \mathbb{R}$, the function $m : T \to \mathbb{R}$ given by $m(x) = \mathbb{E}[\xi(x)]$ is called the mean function and $r : T \times T \to \mathbb{R}$ given by $r(x, y) = \mathbb{E}[(\xi(x) - \mathbb{E}[\xi(x)])(\xi(y) - \mathbb{E}[\xi(y)])]$ is called the covariance function of the field.*

Definition 1.1.5. *A field on T is centered if $m(x) = 0$ for all $x \in T$.*

1.2 Regularity

We now define various notions of regularity for random fields.

Definition 1.2.1. *A random field $\xi : \Omega \times T \to \mathbb{R}$ is said to be almost surely continuous if almost all sample paths are continuous, i.e. for almost every $\omega \in \Omega$, $\xi(\omega, \cdot) : T \to \mathbb{R}$ is continuous.*

When $T = \mathbb{R}^n$, the same definition will also be used for stronger almost sure regularity conditions on fields where almost every sample path is of class $C^k(\mathbb{R}^n)$.

Definition 1.2.2. *A random field ξ is said to be stochastically continuous if for every $x \in T$ and any $\epsilon > 0$, $\mathbb{P}(|\xi(x) - \xi(y)| > \epsilon) \to 0$ as $d(x, y) \to 0$.*

Definition 1.2.3. *A random field ξ is said to be mean-square continuous if for every $x \in T$, $\mathbb{E}[(\xi(x) - \xi(y))^2] \to 0$ as $d(x, y) \to 0$.*

Remark 1.2.4. *If ξ is mean-square continuous then it is also stochastically continuous.*

Under the assumption of stochastic continuity of a random field ξ, we have that there exists a measurable random field $\tilde{\xi}$ such that for every $x \in T$, $\xi(x)$ and $\tilde{\xi}(x)$ have the same distribution (Pg. 5, [10]).

1.3 Separability

Definition 1.3.1. *A random field $\xi : \Omega \times T \to \mathbb{R}$ is said to be separable if there exists a countable dense set $I \subset T$ and a fixed event N such that $\mathbb{P}(N) = 0$ and for any closed set $B \subset \mathbb{R}$ and open set $D \subset T$, $\{\omega \in \Omega : \xi(\omega, x) \in B \ \forall\, x \in D\}\Delta\{\omega \in \Omega : \xi(\omega, x) \in B \ \forall\, x \in D \cap I\} \subset N$, where Δ denotes the symmetric difference operator, i.e. $A\Delta B = (A \cap B^c) \cup (A^c \cap B)$.*

For a random field ξ on $\mathbb{R}^n$, there exists a separable random field $\tilde{\xi}$ on $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$, $\xi(x)$ and $\tilde{\xi}(x)$ have the same distribution (Pg. 6, [10]). Apart from the first chapter, in this thesis, we will only consider measurable and separable random fields on $\mathbb{R}^n$.

For a random field ξ on a metric space (T, d), we can define a pseudo metric d' on T given by $d'(x, y) = \sqrt{\mathbb{E}[(\xi(x) - \xi(y))^2]}$. With this, we have the following lemma (an improved version of Lemma 1.3.1 of [11]).

Lemma 1.3.2. *A mean-square continuous, separable random field ξ on (T, d) is almost surely continuous w.r.t. the d metric if and only if it is almost surely continuous w.r.t. d' metric, i.e.*

$$\mathbb{P}(\{\omega \in \Omega : \lim_{d(x,y)\to 0} |\xi(\omega, x) - \xi(\omega, y)| = 0 \ \forall\, x \in T\}) = 1$$

$$\textit{if and only if}$$

$$\mathbb{P}(\{\omega \in \Omega : \lim_{d'(x,y)\to 0} |\xi(\omega, x) - \xi(\omega, y)| = 0 \ \forall\, x \in T\}) = 1.$$

Proof. We make the observation that as (T, d) is a separable metric space, we have that (T, d) is Lindelöf, i.e. every open cover has a countable subcover. Then we conclude the result by arguing in the exact same manner as of the proof of Lemma 1.3.1 of [11]. $\square$

1.4 Stationarity

We now turn to describing the notions of stationarity and isotropy for random fields on $\mathbb{R}^n$. The same notions can be extended to the case when T admits a (abelian) group structure.

Definition 1.4.1. *A random field $\xi : \Omega \times \mathbb{R}^n \to R$ is said to be weakly stationary if for any $t, x, y \in \mathbb{R}^n$, $m(x + t) = m(x)$ and $r(x + t, y + t) = r(x, y)$.*

Definition 1.4.2. *A random field $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is said to be strictly stationary if for any $m \in \mathbb{N}$, $(x_1, ..., x_m) \in R^{nm}$ and $t \in \mathbb{R}^n$, the distribution of $(\xi(x_1 + t), ..., \xi(x_m + t))$ is the same as that of $(\xi(x_1), ..., \xi(x_m))$.*

For a weakly stationary random field on $\mathbb{R}^n$, we will make use of the notation, $r(x, y) = r(x - y)$ and we will also call the function $r : \mathbb{R}^n \to \mathbb{R}$ given by $r(x) = \mathbb{E}[\xi(0)\xi(x)]$ as the covariance function.

Remark 1.4.3. *In general, strict stationarity of a field is a stronger condition but for a Gaussian field, the two notions coincide.*

Definition 1.4.4. *A random field on $\mathbb{R}^n$ is said to be isotropic if for any $x, y \in \mathbb{R}^n$ and any $A \in SO(n)$, $m(Ax) = m(x)$ and $r(Ax, Ay) = r(x, y)$, where $SO(n)$ denotes the group of all rotations on $\mathbb{R}^n$.*

For a weakly stationary and isotropic random field on $\mathbb{R}^n$, we have that the covariance function $r(x, y)$ depends only on the distance $|x - y|$, i.e. we have $r(x, y) = r(|x - y|)$.

1.5 Bochner and Karhunen's Theorem

The following notion of non-negative definite functions is useful in our study of random fields.

Definition 1.5.1. *Given a set T, a function $f : T \times T \to \mathbb{R}$ is said to be non-negative definite if for any $m \in \mathbb{N}$ and any $(x_1, ..., x_m) \in T^m$, the matrix $A = (f(x_i, x_j))_{1 \le i,j \le m}$ is non-negative definite.*

We state the following classical result in harmonic analysis due to Bochner without proof (Theorem 5.4.1 of [11]). In the theorem below, by saying that f is non-negative definite, we imply that $f' : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ given by $f'(x, y) = f(x - y)$ is non-negative definite.

Theorem 1.5.2. *(Bochner) A continuous function $f : \mathbb{R}^n \to \mathbb{C}$ is non-negative definite if and only if there exists a finite measure (called spectral measure) ν on $\mathbb{R}^n$ such that $f(x) = \int_{\mathbb{R}^n} e^{i\langle t,x\rangle} \nu(dt)$.*

Due to Bochner's theorem and the fact that covariance functions are non-negative definite, the following is immediate.

Lemma 1.5.3. *Given a mean-square continuous, stationary random field on $\mathbb{R}^n$, there exists a finite measure ν on $\mathbb{R}^n$ such that $r(x) = \int_{\mathbb{R}^n} e^{i\langle t,x\rangle}\nu(dt)$.*

We now describe complex random measures and representations of random fields as integrals with respect to random measures.

Definition 1.5.4. *Given a σ-finite measure space (T, τ, ν) and set $\tau_\nu = \{A \in \tau : \nu(A) < \infty\}$, a complex random measure W based at ν is a random field $W : \Omega \times \tau_\nu \to \mathbb{C}$ such that for any $A, B \in \tau_\nu$, $\mathbb{E}[W(A)] = 0$ and $\mathbb{E}[W(A)\overline{W(A)}] = \nu(A)$. Moreover, if $A \cap B = \emptyset$, then $\mathbb{E}[W(A)\overline{W(B)}] = 0$ and $W(A \cup B) = W(A) + W(B)$ almost surely. W is called a complex Brownian (or Wiener) measure or white noise if $W(A) \sim \mathcal{N}(0, \nu(A))$ for every $A \in \tau_\nu$.*

We note that the above definition gives us that the covariance of W is given by $\mathbb{E}[W(A)\overline{W(B)}] = \nu(A \cap B)$. As by definition, a random measure W is an $L^2(\Omega)$ valued measure, we can define integrals with respect to W of a function $f \in L^2(T, \tau, \nu)$ using a simple approach. For simple functions f of the form $f(x) = \sum_{i=1}^m a_i \mathbf{1}_{A_i}(x)$ for disjoint sets $A_i \in \tau$, we define

$$I(f) = \int_T f(t)W(dt) = \sum_{i=1}^m a_i W(A_i). \tag{1.5.1}$$

We have that this definition gives us an isometry between the space of simple functions and a subspace of $L^2(\Omega)$. Clearly I is a linear operator. For $f(x) = \sum_{i=1}^m \mathbf{1}_{A_i}(x)$ and $g(x) = \sum_{i=1}^n \mathbf{1}_{B_i}(x)$, we can write f and g in terms of the same partition by taking the new partition to consist of disjoint sets $C_{i,j} = A_i \cap B_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Hence, we can write $f(x) = \sum_{i=1}^l a_i \mathbf{1}_{C_i}(x)$ and $g(x) = \sum_{i=1}^l b_i \mathbf{1}_{C_i}(x)$ for disjoint sets $C_i \in \tau$. We have,

$$\mathbb{E}[I(f)\overline{I(g)}] = \sum_{i=1}^l a_i b_i \mathbb{E}[W(C_i)\overline{W(C_i)}] = \sum_{i=1}^l a_i b_i \nu(C_i) = \langle f, g\rangle_{L^2(T)}, \tag{1.5.2}$$

which gives us that I also preserves inner product. I can now be extended to an isometry from $L^2(T, \tau, \nu)$ onto a closed subspace of $L^2(\Omega)$.

We can now state the following representation for stationary fields due to Karhunen

(Theorem 5.4.2 of [11], Pg. 10 of [10]).

Theorem 1.5.5. *(Karhunen) If $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is mean-square continuous, stationary random field, then by lemma (1.5.3), we have that there exists a finite measure ν on $\mathbb{R}^n$ such that $r(x) = \int_{\mathbb{R}^n} e^{i\langle t,x\rangle}\nu(dt)$. There also exists a complex random measure Z based at ν such that for every $x \in \mathbb{R}^n$, almost surely we have*

$$\xi(x) = \int_{\mathbb{R}^n} e^{i\langle t,x\rangle} Z(dt). \tag{1.5.3}$$

Furthermore, ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$ with density (called spectral density) g if and only if for every $x \in \mathbb{R}^n$, almost surely we have

$$\xi(x) = \int_{\mathbb{R}^n} \widehat{\alpha}(t) W(dt), \tag{1.5.4}$$

where W is a complex random measure based at the Lebesgue measure on $\mathbb{R}^n$ and $\alpha \in L^2(\mathbb{R}^n)$ is such that $|\alpha(x)|^2 = g(x)$. Here $\widehat{\alpha}$ denotes the Fourier-Plancherel transform of α.

Chapter 2

Analysis on Wiener space

In this chapter, we introduce the Malliavin operators associated with an isonormal Gaussian process which is defined as follows. We will further note properties of these operators. We fix a real separable Hilbert space $\mathfrak{H}$ throughout this chapter.

Definition 2.0.1. *An isonormal Gaussian process X on $\mathfrak{H}$ is a centered Gaussian random field $X : \Omega \times \mathfrak{H} \to \mathbb{R}$ such that for every $f, g \in \mathfrak{H}$, we have $\mathbb{E}[X(f)X(g)] = \langle f, g \rangle_{\mathfrak{H}}$.*

The existence of such a process follows from Kolmogorov's theorem. From now on we assume that X is an isonormal Gaussian process on $\mathfrak{H}$ and the σ-field $\mathcal{F}$ is generated by the family $\{X(f) : f \in \mathfrak{H}\}$.

2.1 Hermite polynomials

We introduce the Hermite polynomials which are of central importance in Gaussian analysis. There are many equivalent ways of defining the Hermite polynomials. We define it by means of Rodrigues' formula.

The n-th Hermite polynomial is given by the formula

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}). \tag{2.1.1}$$

Hermite polynomials are also the coefficients which appear in the series expansion of the function $F(x,t) = e^{tx-t^2/2}$ in terms of the powers of t. Indeed we have

$$e^{tx-t^2/2} = e^{x^2/2}e^{-(x-t)^2/2} = e^{x^2/2}\sum_{q=0}^{\infty}\frac{t^q}{q!} \times \frac{d^q}{dt^q}e^{-(x-t)^2/2}\Bigg|_{t=0} \tag{2.1.2}$$

$$= e^{x^2/2}\sum_{q=0}^{\infty}\frac{(-1)^q t^q}{q!} \times \frac{d^q}{dx^q}e^{-x^2/2} \tag{2.1.3}$$

$$= \sum_{q=0}^{\infty} t^q \times \frac{H_q(x)}{q!}. \tag{2.1.4}$$

From this, we note the following properties of the Hermite polynomials.

Lemma 2.1.1. *For every $n \geq 0$ and $x \in \mathbb{R}$, we have*
1) $H_n'(x) = nH_{n-1}(x)$.
2) $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$.
3) $H_n(-x) = (-1)^n H_n(x)$.

The next lemma gives us the relation between Hermite polynomials and Gaussian random variables (Lemma 1.1.1 of [9]).

Lemma 2.1.2. *Let (X,Y) be a bivariate normal vector with $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$ and $\mathbb{E}[XY] = \rho$. Then*

$$\mathbb{E}[H_n(X)H_m(Y)] = \begin{cases} n!\rho^n \text{ if } n = m \\ 0 \text{ if } n \neq m. \end{cases}$$

We next define the Hermite polynomials in the multivariate case. For any multi-index $a = (a_1, \dots, a_m), a_i \in \mathbb{N} \cup \{0\}$, we write $|a| = \sum_{i=1}^m a_i$, $a! = \prod_{i=1}^m a_i!$ and

$$\overline{H}_a(x) = \prod_{i=i}^{m} H_{a_i}(x_i). \tag{2.1.5}$$

The below result is fundamental concerning Hermite polynomials (Lemma 1.3.2 of [12]).

Theorem 2.1.3. *The set* $\{\frac{1}{\sqrt{a!}}\overline{H}_a : a$ *is a multi-index*$\}$ *is an orthonormal basis of* $L^2(\mathbb{R}^m, \gamma_m)$, *where* γ_m *denotes the standard Gaussian measure on* $\mathbb{R}^m$.

Let $G : \mathbb{R}^m \to \mathbb{R}$ be such that G is not a constant and $G \in L^2(\mathbb{R}^m, \gamma_m)$. Denoting, $\mathcal{I}_q = \{a \in \mathbb{Z}^m : a_i \geq 0, |a| = q\}$, we have the following expansion of G where the convergence of the series is in L^2 sense,

$$\sum_{q=0}^{\infty} \sum_{a \in \mathcal{I}_q} c(G, a)\overline{H}_a(x) = G(x). \tag{2.1.6}$$

In above expansion $c(G, a) = \frac{1}{\sqrt{a!}} \int_{\mathbb{R}^m} G(x)\overline{H}_a(x)\gamma_m(dx)$. We now give the definition of Hermite rank.

Definition 2.1.4. *For* $G \in L^2(\mathbb{R}^m, \gamma_m)$ *with expansion in equation* (2.1.6)*, the smallest integer* $d \geq 1$ *such that there exist a multi-index* a *such that* $|a| = d$ *and* $c(G, a) \neq 0$ *is called the rank of* G.

2.2 Malliavin derivative

We now proceed to move ahead with our main goal of developing Malliavin calculus on Wiener space, $(\Omega, \mathcal{F}, \mathbb{P}, \mathfrak{H}, X)$. We omit proofs in our discussion and the same can be found in monographs [9] and [13]. We first define the notion of Wiener chaos which is due originally to Norbert Wiener, [14].

Definition 2.2.1. *For a fixed* $n \geq 1$*, the closed linear span in* $L^2(\Omega)$ *of the set of variables* $\{H_n(X(f)) : \|f\|_{\mathfrak{H}} = 1, f \in \mathfrak{H}\}$ *is called the* n*-th Wiener chaos. We denote it by* $\mathcal{H}_n$.

From lemma (2.1.2), we have that for $q_1 \neq q_2$, $\mathcal{H}_{q_1}$ and $\mathcal{H}_{q_2}$ are orthogonal. With this we have the following decomposition of $L^2(\Omega)$ which is known as the Wiener chaos decomposition.

Theorem 2.2.2. *Let* $F \in L^2(\Omega)$*. Then there exist* $F_q \in \mathcal{H}_q$ *such that the following expansion holds in* $L^2(\Omega)$*,* $F - \mathbb{E}[F] = \sum_{q=1}^{\infty} F_q$*. Therefore, we have* $L^2(\Omega) = \bigoplus_{q=1}^{\infty} \mathcal{H}_q$.

For $F \in L^2(\Omega)$, we will use the notation $J_q F = F_q$, the projection of F onto the q-th Wiener chaos.

Definition 2.2.3. *For any Hilbert space $\mathscr{H}$, $L^q(\Omega; \mathscr{H})$ is the set of all $\mathscr{H}$-valued random elements F such that $\mathbb{E}[\|F\|^q_{\mathscr{H}}] < \infty$.*

We denote by $\mathcal{S}_n$, the set of all smooth functions f on $\mathbb{R}^n$ such that f and all it's partial derivatives have at most polynomial growth. Further, we denote by $\mathscr{L}$ as the set of all smooth and cylindrical random variables F of the form $F = f(X(h_1), ..., X(h_n))$ where $f \in \mathcal{S}_n$, $h_1, ..., h_n \in \mathfrak{H}$ and $n \geq 1$. We have that $\mathscr{L}$ is dense in $L^p(\Omega)$ for all $p \geq 1$ (Lemma 2.3.1 of [13]). We will also denote by $\mathfrak{H}^{\odot k}$ as the set of all symmetric tensors in $\mathfrak{H}^{\otimes k}$.

Definition 2.2.4. *For $F \in \mathscr{L}$ given by $F = f(X(h_1), ..., X(h_n))$, the k-th Malliavin derivative of F is $\mathfrak{H}^{\odot k}$-valued variable given by*

$$D^k F = \sum_{i_1, ..., i_k = 1}^{n} \frac{\partial^k f}{\partial x_{i_1} ... \partial x_{i_k}} (X(h_1), ..., X(h_n)) h_{i_1} \otimes ... \otimes h_{i_k}. \tag{2.2.1}$$

We note that from the definition of the derivative, we have $D(FG) = FDG + GDF$ for $F, G \in \mathscr{L}$. We now give the definition of closability of a linear operator from functional analysis.

Definition 2.2.5. *A linear operator $A : \mathcal{D}(A) \to \mathscr{H}$ from some domain $\mathcal{D}(A)$ into a Hilbert space $\mathscr{H}$ is said to be* closable *if it admits a closed extension $B : \mathcal{D}(B) \to \mathscr{H}$ where $\mathcal{D}(A) \subset \mathcal{D}(B)$. By closed we mean that if $x_n \in \mathcal{D}(B) \to x$ and $Bx_n \to x'$ as $n \to \infty$, then $x \in \mathcal{D}(B)$ and $Bx = x'$.*

Definition 2.2.6. *The set $\mathbb{D}^{k,p}$ which is the closure of the operator D^k in $L^p(\Omega)$ with respect to the norm $\| \cdot \|_{k,p}$ defined by*

$$\|F\|^p_{k,p} = \left(\mathbb{E}(|F|^p) + \sum_{i=1}^{k} \mathbb{E}(\|D^i F\|^p_{\mathfrak{H}^{\otimes i}}) \right)^{1/p},$$

for any natural number k and any real number $p \geq 1$ is called the domain of the operator D^k in $L^p(\Omega)$.

The above closure operation is valid due to the result that for any $p \geq 1$, $D^k : \mathscr{L} \subset L^p(\Omega) \to L^p(\Omega; \mathfrak{H}^{\odot k})$ is closable (Proposition 2.3.4 of [13]). We note that the above definition implies that $F \in \mathbb{D}^{k,p}$ if and only if there exists a sequence $\{F_n\}_{n=1}^{\infty} \in \mathscr{L}$ such that $F_n \to F$ in $L^p(\Omega)$ and for every $j = 1, ..., k$, the sequence $D^j F_n$ is a cauchy sequence in $L^p(\Omega)$. We will use the notation, $\cap_{k \geq 1} \mathbb{D}^{k,p} = \mathbb{D}^{\infty,p}$.

The space $\mathbb{D}^{k,2}$ which gives us the domain of D^k in $L^2(\Omega)$ will be of most interest to us and the following result is characterization of $\mathbb{D}^{k,2}$ using the Wiener chaos expansion (Proposition 1.2.2 of [9]).

Lemma 2.2.7. *Let $F \in L^2(\Omega)$ with expansion $F = \sum_{q=0}^{\infty} J_q F$. Then $F \in \mathbb{D}^{k,2}$ if and only if $\sum_{q=1}^{\infty} q^k \left\| J_q F \right\|_2^2 < \infty$.*

Moreover, we have that the Malliavin derivative verifies the following chain rule (Proposition 2.3.7 of [13]).

Lemma 2.2.8. *Let $F = (F_1, ..., F_m)$ is such that for every i, $F_i \in \mathbb{D}^{1,p}$ for some $p \geq 1$ and let $\psi : \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable function with bounded partial derivatives. Then $\psi(F) \in \mathbb{D}^{1,p}$ and*

$$D\psi(F) = \sum_{i=1}^{m} \frac{\partial \psi}{\partial x_i}(F) DF_i.$$

The next lemma is an important integration-by-parts formula (Lemma 1.2.1 of [9]).

Lemma 2.2.9. *For any $F \in \mathscr{L}$ and $h \in \mathfrak{H}$, we have $\mathbb{E}[\langle DF, h \rangle_{\mathfrak{H}}] = \mathbb{E}[FX(h)]$. In particular, we have for $F, G \in \mathscr{L}$, $\mathbb{E}[G\langle DF, h \rangle_{\mathfrak{H}}] = \mathbb{E}[-F\langle DG, h \rangle_{\mathfrak{H}} + FGX(h)]$.*

We next extend the Malliavin derivative to Hilbert space valued variables.

Let V be a real separable Hilbert space and denote by $\mathscr{L}_V$ the space of all smooth and cylindrical V-valued random variables F of the form $F = \sum_{i=1}^{n} F_i v_i$ where $F_i \in \mathscr{L}$ and $v_i \in V$.

Definition 2.2.10. *The k-th Malliavin derivative of $F \in \mathscr{L}_V$ of the form $F = \sum_{i=1}^{n} F_i v_i$ is $\mathfrak{H}^{\odot k} \otimes V$-valued variable given by*

$$D^k F = \sum_{i=1}^{n} D^k F_i \otimes v_i.$$

We will also denote by $\mathbb{D}^{k,p}(V)$, the closure of D^k with respect to the norm

$$\|F\|_{k,p,V}^{p} = \left(\mathbb{E}(||F||_V^p) + \sum_{i=1}^{k} \mathbb{E}(\|D^i F\|_{\mathfrak{H}^{\otimes i} \otimes V}^{p}) \right)^{1/p}.$$

2.3 Malliavin divergence

Our next goal is to define δ^k, the Malliavin divergence operator of order k. It is the adjoint of the derivative operator D^k. The divergence operator δ is also known as the Skorokhod integral.

Definition 2.3.1. *The set* Dom $\delta^k \subset L^2(\Omega; \mathfrak{H}^{\otimes k})$ *consisting of elements u such that*

$$|\mathbb{E}[\langle D^k F, u\rangle_{\mathfrak{H}^{\otimes k}}]| \leq c_u \sqrt{\mathbb{E}[F^2]} \tag{2.3.1}$$

for every $F \in \mathbb{D}^{k,2}$ is called the domain of the operator δ^k.

The relation (2.3.1) implies that the operator $F \mapsto \mathbb{E}[\langle D^k F, u\rangle_{\mathfrak{H}^{\otimes k}}]$ on $\mathbb{D}^{k,2}$ equipped with $L^2(\Omega)$ norm is continuous and by Riesz representation theorem, there exists a unique element $\delta^k(u) \in L^2(\Omega)$ such that the following relation holds,

$$|\mathbb{E}[\langle D^k F, u\rangle_{\mathfrak{H}^{\otimes k}}]| = \mathbb{E}[F \delta^k(u)]. \tag{2.3.2}$$

The above duality relationship is also known as the integration-by-parts formula.

Definition 2.3.2. *For $u \in$ Dom δ^k, the k-th divergence of u, $\delta^k(u)$ is given by the unique element obtained in the relationship* (2.3.2).

Putting $F = 1$ in equation (2.3.2), we have that $\mathbb{E}[\delta^k(u)] = 0$ for all $u \in$ Dom δ^k and from lemma (2.2.9), we have that $\mathfrak{H} \subset$ Dom δ and $\delta(h) = X(h)$ for every $h \in \mathfrak{H}$. One also has that $\mathfrak{H}^{\otimes k} \subset$ Dom δ^k.

Similar to the derivative operator, for a real separable Hilbert space V, we define δ^k for $\mathfrak{H}^{\otimes k} \otimes V$ valued random elements.

Definition 2.3.3. *For $u \in \mathfrak{H}^{\otimes k} \otimes V$ such that $u = \sum_{i=1}^{n} h_i \otimes v_i$ with $h_i \in \mathfrak{H}^{\otimes k}$ and $v_i \in V$, $\delta^k(u)$ is V-valued variable given by*

$$\delta^k(u) = \sum_{i=1}^{k} \delta^k(h_i) v_i. \tag{2.3.3}$$

It is known that δ^k can be extended to bounded operator on $\mathfrak{H}^{\otimes k} \otimes V$ (section 2.6 of [13]). Due to this definition, we have for $f \in \mathfrak{H}^{\otimes k}$, $\delta^k(f) = \delta^{k-q}(\delta^q(f))$.

2.4 Ornstein-Uhlenbeck semigroup and Meyer inequalities

We next introduce the Ornstein-Uhlenbeck semigroup of operators. Our interest will mainly be in it's infinitesimal generator L and what we define as it's pseudo inverse L^{-1}.

Definition 2.4.1. *For $F \in L^2(\Omega)$ with expansion $F = \sum_{q=0}^{\infty} J_q F$, the semigroup of operators $\{P_t : t \geq 0\}$ acting on F by*

$$P_t F = \sum_{q=0}^{\infty} e^{-qt} J_q F \tag{2.4.1}$$

is called the Ornstein-Uhlenbeck semigroup of operators.

We recall the generator of the semigroup of operators is given by $\lim_{t \to 0} \frac{P_t F - F}{t}$ where the limit is in $L^2(\Omega)$.

Definition 2.4.2. *For $F \in \mathbb{D}^{2,2}$, we define $LF = -\sum_{q=0}^{\infty} q J_q F$.*

The operator L with domain $\mathbb{D}^{2,2}$ coincides with the infinitesimal generator of the semigroup $\{P_t : t \geq 0\}$.

We next have a crucial relationship between the operators D, δ and L on which we will rely heavily in our subsequent discussion including our proof of a functional Breuer-Major theorem presented in Chapter 4 (Proposition 2.8.8 of [13]).

Lemma 2.4.3. *For $F \in L^2(\Omega)$, we have $F \in \text{Dom } L$ if and only if $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom } \delta$ and in this case we have $LF = -\delta DF$.*

We now define the pseudo inverse of L.

Definition 2.4.4. *For $F \in L^2(\Omega)$ with expansion $F = \sum_{q=0}^{\infty} J_q F$, we set $L^{-1}F = -\sum_{q=0}^{\infty} \frac{1}{q} J_q F$.*

It is immediate that $L^{-1}F \in \text{Dom } L$ and $LL^{-1}F = F - \mathbb{E}[F]$. Therefore, for any $F \in \mathbb{D}^{1,2}$ such that $\mathbb{E}[F] = 0$, we have that F can be expressed as

$$F = LL^{-1}F = -\delta D L^{-1} F. \tag{2.4.2}$$

We next state an important result due to P. A. Meyer which gives us equivalence in $L^p(\Omega)$ for any $p > 1$ of operators D^k and the operator C^k acting on $L^2(\Omega)$ as following (see Theorem 1.5.1 of [9]). The operator C^k is also denoted as $-(-L)^{k/2}$.

Definition 2.4.5. *For $F \in \mathbb{D}^{k,2}$ with expansion $F = \sum_{q=0}^{\infty} J_q F$, we define*

$$C^k F = -\sum_{q=1}^{\infty} q^{k/2} J_q F.$$

Theorem 2.4.6. *(Meyer) For any $p > 1$ and $F \in \mathbb{D}^{k,2}$, there exists constants $c_{p,k}$ and $C_{p,k}$ such that*

$$c_{p,k}\mathbb{E}[\left\|D^k F\right\|_{\mathfrak{H}^{\otimes k}}^p] \leq \mathbb{E}[|C^k F|^p] \leq C_{k,p}\left(\mathbb{E}[\left\|D^k F\right\|_{\mathfrak{H}^{\otimes k}}^p] + \mathbb{E}[|F|^p]\right). \tag{2.4.3}$$

One other consequence of the Meyer inequalities is the continuity of the operator δ^k from $\mathbb{D}^{k',q}(\mathfrak{H}^{\otimes k})$ to $\mathbb{D}^{k'-k,q}$ for $k' \geq k \geq 1$ and $q \in [1, \infty)$ (Proposition 1.5.7 of [9]).

Theorem 2.4.7. *For $k' \geq k \geq 1$ and $q \in [1, \infty)$, there exists a constant $c_{k',k,q}$ such that for any $u \in \mathbb{D}^{k,q}(\mathfrak{H}^{\otimes k})$, we have*

$$\left\|\delta^k(u)\right\|_{\mathbb{D}^{k'-k,q}} \leq c_{k',k'q} \left\|u\right\|_{\mathbb{D}^{k,q}(\mathfrak{H}^{\otimes k})}. \tag{2.4.4}$$

2.5 Multiple Wiener-Itô integrals

We now turn to discussing another fundamental objects in this thesis called multiple Wiener-Itô integrals. For now we will assume that $\mathfrak{H} = L^2(T, \tau, \nu)$ for some σ-finite measure space (T, τ, ν) with no atoms and we let W denote a complex Wiener measure based at ν (see definition (1.5.4)). The isonormal process X on $\mathfrak{H}$ is given by $X(f) = \int_T f(t)W(dt)$ as defined by equation (1.5.1). For $f \in L^2(T^k, \tau^k, \nu^k)$ of the form

$$f(x) = \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1 \cdots i_k} \mathbf{1}_{A_{i_1} \times \cdots \times A_{i_k}}(x), \tag{2.5.1}$$

we define

$$I_k(f) = \int_{T^k} f(x_1, ..., x_k) W(dx_1)...W(dx_k) = \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1 \cdots i_k} W(A_{i_1}) \cdots W(A_{i_k}). \tag{2.5.2}$$

From this definition, the following properties of the operator I_k are true (see [9]). We recall that for a function $f(x_1, ..., x_k)$ in k variables, the symmetrization of f is given by

$$\widetilde{f}(x_1, ..., x_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{G}_k} f(x_{\sigma(1)}, ..., x_{\sigma(k)}) \tag{2.5.3}$$

where $\mathfrak{G}_k$ is the group of permutations of k elements.

Lemma 2.5.1. *1) $\mathbb{E}[I_k(f)] = 0$.*
2) I_k is linear.
3) $I_k(f) = I_k(\widetilde{f})$.
4) $\mathbb{E}[I_k(f)I_{k'}(g)] = 0$ if $k \neq k'$ and equals $k\langle \widetilde{f}, \widetilde{g} \rangle_{L^2(T^k)}$ otherwise.

Since functions of the form of equation (2.5.1) are dense in $L^2(T^k)$, the above lemma gives us that I_k can be consistently extended to $L^2(T^k)$ and moreover I_k equipped with the norm $\sqrt{k!}\|\cdot\|$ is an isometry from $L_S^2(T^k)$ onto a closed subspace of $L^2(\Omega)$ where $L_S^2(T^k)$ denotes the subset of $L^2(T^k)$ consisting of all symmetric functions.

An equivalent way of defining the operator I_k on $\mathfrak{H}^{\otimes k}$ is by the divergence operator. For a general Hilbert space $\mathfrak{H}$ and an isonormal process X on $\mathfrak{H}$, we define the k-th multiple Wiener-Itô integral as follows.

Definition 2.5.2. *For an integer k and $f \in \mathfrak{H}^{\odot k}$, the k-th multiple Wiener Itô integral of f, $I_k(f)$ is defined as $I_k(f) = \delta^k(f)$.*

We can then define $I_k(f) = I_k(\widetilde{f})$ for $f \in \mathfrak{H}^{\otimes k}$. We have that this definition of the integral coincides with the multiple stochastic integral with respect to a complex Wiener measure in the case when $\mathfrak{H} = L^2(T, \tau, \nu)$ and satisfies the same properties as of lemma (2.5.1) (see exercise 2.7.6 of [13]). By the property of the divergence operator, we have that $I_k(f)$ is infinitely differentiable (Proposition 2.7.4 of [13]).

Lemma 2.5.3. *For any $q \geq 1$ and $f \in \mathfrak{H}^{\odot q}$, we have for all $p \geq 1$, $I_q(f) \in \mathbb{D}^{\infty,p}$ and*

$$D^r I_q(f) = \begin{cases} \frac{q!}{(q-k)!} I_{q-r}(f) \text{ if } r \leq q \\ 0 \text{ if } r > q. \end{cases} \tag{2.5.4}$$

We next state an important result which gives us that I_k from $\mathfrak{H}^{\odot k}$ is an isometry onto the k-th Wiener chaos $\mathcal{H}_k$ (Theorem 2.7.7 of [13]).

Lemma 2.5.4. *For $f \in \mathfrak{H}$ such that $\|f\|_{\mathfrak{H}} = 1$, $I_k(f^{\otimes k}) = H_k(X(f))$. As a result, $I_k : \mathfrak{H}^{\odot k} \to \mathcal{H}_k$ is an isometry and for every $F \in L^2(\Omega)$, we have the expansion $F - \mathbb{E}[F] = \sum_{q=1}^{\infty} I_q(f_q)$ for some $f_q \in \mathfrak{H}^{\odot q}$.*

We next intend to state what is known as the product formula for multiple Wiener Itô integrals. It will remain a fundamental tool in our analysis. We first define the notion of contraction of tensors.

Definition 2.5.5. *For two tensors $f = \sum_{j_1,\dots,j_p=1}^{\infty} a_{j_1,\dots,j_p} e_{j_1} \otimes \cdots \otimes e_{j_p} \in \mathfrak{H}^{\otimes p}$ and $g = \sum_{k_1,\dots,k_q=1}^{\infty} b_{k_1,\dots,k_q} = e_{k_1} \otimes \cdots \otimes e_{k_q} \in \mathfrak{H}^{\otimes q}$, the l-th contraction of f and g ($l \leq \min(p,q)$) is* *the element of* $\mathfrak{H}^{\otimes p+q-2l}$ *given by*

$$f \otimes_l g = \sum_{j_1,\ldots,j_p=1}^{\infty} \sum_{k_1,\ldots,k_q=1}^{\infty} a_{j_1,\ldots,j_p} b_{k_1,\ldots,k_q} \prod_{i=1}^{l} \langle e_{j_i}, e_{k_i} \rangle_{\mathfrak{H}} e_{j_{l+1}} \otimes \cdots \otimes e_{j_p} \otimes e_{k_{l+1}} \otimes \cdots \otimes e_{k_q}. \quad (2.5.5)$$

When $\mathfrak{H} = L^2(T, \tau, \nu)$ *for measure* ν *without atoms, the above definition is equivalent to the following,*

$$f \otimes_l g(a_1, ..., a_{p+q-2l}) = \quad (2.5.6)$$
$$\int_{T^l} f(x_1, ..., x_r, a_1, ..., a_{p-l}) \times g(x_1, ..., x_r, a_{p-l+1}, ..., a_{p+q-2l}) \nu(dx_1)...\nu(dx_l). \quad (2.5.7)$$

Notice that even if f and g are symmetric, the contraction $f \otimes_l g$ is not necessarily a symmetric tensor and we denote it's symmetrization by $f \widetilde{\otimes}_l g$. We recall that for a real separable Hilbert space $\mathscr{H}$ with orthonormal basis $\{e_j\}_{j\geq 1}$, we have that for an element $f \in \mathscr{H}^{\otimes q}$ given by $f = \sum_{j_1,\ldots,j_q} a_{j_1,\ldots,j_q} e_{j_1} \otimes \cdots \otimes e_{j_q}$, the symmetrization of f is given by

$$\widetilde{f} = (q!)^{-1} \sum_{\sigma} \sum_{j_1,\ldots,j_q} a_{j_1,\ldots,j_q} e_{j_{\sigma(1)}} \otimes \cdots \otimes e_{j_{\sigma(q)}}. \quad (2.5.8)$$

We can now state the product formula.

Theorem 2.5.6. *For* $f \in \mathfrak{H}^{\odot p}$ *and* $g \in \mathfrak{H}^{\odot q}$ *where* $p, q \geq 1$, *we have*

$$I_p(f) I_q(g) = \sum_{l=0}^{p \wedge q} l! \binom{p}{l} \binom{q}{l} I_{p+q-2l}(f \widetilde{\otimes}_l g). \quad (2.5.9)$$

We next state a final result regarding multiple Wiener Itô integrals (Theorem 2.10.1 of [13]).

Theorem 2.5.7. *For any* $k \geq 1$ *and* $f \in \mathfrak{H}^{\otimes k}$ *such that* $\|f\|_{\mathfrak{H}^{\odot k}} > 0$, *the law of* $I_k(f)$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}$.

Chapter 3

Fourth Moment Theorem

3.1 Introduction

In this chapter, we discuss what is now known as the Fourth moment phenomena. The Fourth moment theorem was first proved by David Nualart and Giovanni Peccati in 2005 (see [15]). It gives equivalence of convergence in distribution of a sequence of multiple Wiener Itô integrals of fixed order to a Gaussian distribution and convergence of the second and fourth moments of the sequence to the corresponding moments of a Gaussian variable. This provided a huge simplification from what is known as the Markov method of moments which requires proving convergence of every moment to that of the Gaussian variable for the convergence in distribution to hold (see Proposition 5.2.2 of [13]).

Namely in the following theorem, Nualart and Peccati in [15] established the equivalence of 1) and 2) using tools from classical stochastic analysis. After that in [16], Nualart and Ortiz-Latorre used properties of Malliavin operators to give another equivalent condition, namely condition 4) below which paved the way for normal approximation using Malliavin calculus. Nourdin and Peccati in [17] then gave a simple proof by combining Stein's method and Malliavin calculus. Here $\Rightarrow$ denotes convergence in distribution.

Theorem 3.1.1. *For fixed $q \geq 1$ and a sequence $\{F_k\}_{k=1}^{\infty}$ such that $F_k = I_q(f_k)$ for some $f_k \in \mathfrak{H}^{\odot q}$, if $\lim_{k\to\infty} \mathbb{E}[F_k^2] = 1$, then the following are equivalent as $k \to \infty$.*
1) $F_k \Rightarrow \mathcal{N}(0,1)$.

2) $\mathbb{E}[F_k^4] \to 3$.
3) $\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes 2q-2r}} \to 0$ *for every* $1 \le r \le q-1$.
4) $\mathrm{Var}\Big(q^{-1}\,\|DF_k\|_{\mathfrak{H}}^2\Big) \to 0$.

We will here sketch a proof using Nourdin and Peccati's approach.

3.2 Stein's method

We first define notion of distances between laws of random variables.

Definition 3.2.1. *A class $\mathscr{C}$ of Borel measurable complex-valued functions on $\mathbb{R}^m$ is called separating if for any two $\mathbb{R}^m$-valued variables $F, G \in L^1(\Omega)$, $\mathbb{E}[f(F)] = \mathbb{E}[f(G)]$ for all $f \in \mathscr{C}$ implies that F and G have the same distribution.*

Given a separating class $\mathscr{C}$, we can define distance induced by class $\mathscr{C}$, between laws of two random variable F and G such that $f(F), f(G) \in L^1(\Omega)$ for every $f \in \mathscr{C}$ given by

$$d_{\mathscr{C}}(F,G) = \sup_{f\in\mathscr{C}} |\mathbb{E}[f(F)] - \mathbb{E}[f(G)]|. \tag{3.2.1}$$

One can check that the above defined distance is indeed a metric on a subset of probability laws on $\mathbb{R}^m$.

The four commonly used distances are given below.

- Kolmogorov: $\mathscr{C}_{\mathrm{Kol}} = \{f : f(x_1, ..., x_m) = \mathbf{1}_{(-\infty,c_1]}(x_1)...\mathbf{1}_{(-\infty,c_m]}(x_m)$ where $c_i \in \mathbb{R}\}$
- Total Variation: $\mathscr{C}_{TV} = \{f : f(x) = \mathbf{1}_B(x)$ for a Borel set $B\}$
- Wasserstein: $\mathscr{C}_W = \{f :\ f$ is Lipschitz with Lipschitz constant $\|f\|_{\mathrm{Lip}} \le 1\}$
- Fortet-Mourier: $\mathscr{C}_{FM} = \{f :\ f$ is bounded and Lipschitz with $\|f\|_{\mathrm{Lip}} + \|f\|_\infty \le 1\}$

We will use the notation $d_{\mathrm{Kol}}, d_{TV}, d_W$ and d_{FM} to denote distances induced by these various classes. We gather the properties of these distances in the lemma below (Appendix C.3 of [13]). In below lemma we consider a sequence of variables $\{F_k\}_{k=1}^{\infty}$.

Lemma 3.2.2. *1) If $d_{\mathscr{C}}(F_k, F) \to 0$ as $k \to \infty$ for any of the four distances, then $F_k \Rightarrow F$.*
2) As $k \to \infty$, $F_k \Rightarrow F$ if and only if $d_{FM}(F_k, F) \to 0$.
3) If $z \mapsto \mathbb{P}(F \le z)$ is continuous for every $z \in \mathbb{R}$, then as $k \to \infty$, $F_k \Rightarrow F$ if and only if $d_{\mathrm{Kol}}(F_k, F) \to 0$.
4) $d_{TV}(F, G) = \frac{1}{2} \sup_{\|f\|_\infty \le 1} |\mathbb{E}[f(F)] - \mathbb{E}[f(G)]|$.
5) We have $d_W(F, G) \ge d_{FM}(F, G)$ and $d_{TV}(F, G) \ge d_{\mathrm{Kol}}(F, G)$ for any variables F and G.

We now briefly introduce what is known as Stein's method for distributional approximation. Specifically, we will introduce the method for normal approximation.

We start with what is known as Stein's lemma (Lemma 3.1.2 of [13]).

Lemma 3.2.3. *A random variable F has standard normal distribution if and only if for every differentiable function f such that $\mathbb{E}[f(F)] < \infty$, we have $\mathbb{E}[Ff(F)] = \mathbb{E}[f'(F)] < \infty$.*

Based on this lemma, and the heuristic that if the quantity $|\mathbb{E}[Ff(F)] - \mathbb{E}[f'(F)]|$ is small for a suitable class of functions f, it should imply that the law of F is close to standard normal law, Stein constructed the following ordinary differential equation for normal approximation. For a given Borel function $h : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}[h(N)] < \infty$ where $N \sim \mathcal{N}(0, 1)$, one considers the equation

$$f'(x) - xf(x) = h(x) - \mathbb{E}[h(N)]. \tag{3.2.2}$$

Denoting f_h to be a solution of equation (3.2.2), we can replace the dummy variable x by variable F whose law we are interested in approximating and take expectation on both sides to write

$$\sup_{h \in \mathscr{C}} |\mathbb{E}[f_h'(F)] - \mathbb{E}[Ff_h(F)]| = \sup_{h \in \mathscr{C}} |\mathbb{E}[h(F)] - \mathbb{E}[h(N)]|. \tag{3.2.3}$$

We can thus then try to estimate the right hand side of above which is what is required for approximating law of F to law of N in view of distance in equation (3.2.1) for a suitable class $\mathscr{C}$, by means of estimating the left hand side. Notice that in the above equation, the left hand side does not involve the target variable.

We have that the equation (3.2.2) is straighforward to solve.

Lemma 3.2.4. *Every solution of the equation* (3.2.2) *is of the form*

$$f(x) = ce^{x^2/2} + e^{x^2/2}\int_{-\infty}^{x} \Big(h(y) - \mathbb{E}[h(N)]\Big)e^{-y^2/2}dy, \tag{3.2.4}$$

for $c \in \mathbb{R}$*. Further, the solution* f *with* $c = 0$ *is the unique solution satisfying* $\lim_{x\to\pm\infty} e^{-x^2/2}f(x) = 0$.

Proof. We can write equation (3.2.2) as

$$e^{x^2}\frac{d}{dx}\Big(e^{-x^2/2}f(x)\Big) = h(x) - \mathbb{E}[h(N)].$$

It then follows that every solution is indeed of the form equation (3.2.4). Further by dominated convergence theorem we have that

$$\lim_{x\to\pm\infty}\int_{-\infty}^{x}\Big(h(y) - \mathbb{E}[h(N)]\Big)e^{-y^2/2}dy = 0$$

which yields the remaining claim in the lemma. $\square$

We will next consider the class $\mathscr{C}_{\mathrm{Kol}}$ in equation (3.2.3) to illustrate the method and to give a proof of the Fourth Moment Theorem (3.1.1). In this case, the function $h(x) = \mathbf{1}_{(-\infty,c]}(x)$ for some $c \in \mathbb{R}$. We will the denote the solution of equation (3.2.2) with this h as f_c. It is easy to show that in this case the solution in equation (3.2.4) becomes

$$f_c(x) = \begin{cases} \sqrt{2\pi}e^{x^2/2}\Phi(x)(1-\Phi(c)) \text{ if } x \leq c \\ \sqrt{2\pi}e^{x^2/2}\Phi(c)(1-\Phi(x)) \text{ if } x \geq c \end{cases} \tag{3.2.5}$$

where $\Phi(c) = \mathbb{P}(N \leq c) = \mathbb{E}[h(N)]$ denotes the cumulative distribution function of a standard normal variable.

With this, one can deduce the following estimates for the solution f_c (Theorem 3.4.2 of [13]).

Theorem 3.2.5. *For* $c \in \mathbb{R}$*, the function* f_c *in equation* (3.2.5) *is such that* $\|f_c\|_\infty \leq \frac{\sqrt{2\pi}}{4}$ *and* $\|f_c'\|_\infty \leq 1$.

3.3 Nourdin and Peccati's result

Using the previous estimates on solution of Stein's equation, we have the following result which is due to Nourdin and Peccati.

Theorem 3.3.1. *For $F \in \mathbb{D}^{1,2}$ such that $\mathbb{E}[F] = 0$ and the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, we have*

$$d_{\text{Kol}}(F, N) \leq \mathbb{E}[|(1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}|] \leq \sqrt{\mathbb{E}[(1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})^2]} \tag{3.3.1}$$

where $N \sim \mathcal{N}(0, 1)$.

Proof. One has from equation (3.2.3) that

$$d_{\text{Kol}}(F, N) \leq \sup_{c\in\mathbb{R}} |\mathbb{P}(F \leq c) - \Phi(c)| = \sup_{c\in\mathbb{R}} |\mathbb{E}(f_c'(F)) - \mathbb{E}[F f_c(F)]|. \tag{3.3.2}$$

Using equation (2.4.2), lemma (2.2.8) and equation (2.3.2) we have

$$\mathbb{E}[F f_c(F)] = -\mathbb{E}[\delta(DL^{-1}F) f_c(F)] = \mathbb{E}[f_c'(F)\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}]. \tag{3.3.3}$$

The claim now follows by applying Cauchy-Schwarz inequality. □

Sketch of proof of theorem (3.1.1)
In Nourdin and Peccati's result, for the case when $F = I_q(f)$, for some $f \in \mathfrak{H}^{\odot q}$, we have using $L^{-1}I_q(f) = -q^{-1}I_q(f)$ that $\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}} = q^{-1}\|DF\|_{\mathfrak{H}}^2$. Using the product formula for multiple Wiener-Itô integrals, we can compute the following (Lemma 5.2.4 of [13] along with Lemma 1 of [16]).

$$q^{-1}\|DF\|_{\mathfrak{H}}^2 = \mathbb{E}[F^2] + q\sum_{r=1}^{q-1}(r-1)!\binom{q-1}{r-1}^2 I_{2q-2r}(f\widetilde{\otimes}_r f). \tag{3.3.4}$$

We note that this implies that $q^{-1}\mathbb{E}[\|DF\|_{\mathfrak{H}}^2] = \mathbb{E}[F^2]$. One can further compute the following optimal bound and expression in terms of contractions (Lemma 5.2.4 of [13]).

$$\mathrm{Var}\left(q^{-1}\left\|DF\right\|_{\mathfrak{H}}^{2}\right)=q^{-2}\sum_{r=1}^{q-1}r^{2}r!^{2}\binom{q}{r}^{4}(2q-2r)!\left\|f\widetilde{\otimes}_{r}f\right\|_{\mathfrak{H}^{\otimes 2q-2r}}^{2} \tag{3.3.5}$$

$$\mathrm{Var}\left(q^{-1}\left\|DF\right\|_{\mathfrak{H}}^{2}\right)\leq\frac{q-1}{3q}\left(\mathbb{E}[F^{4}]-3\mathbb{E}[F^{2}]^{2}\right)\leq(q-1)\mathrm{Var}\left(q^{-1}\left\|DF\right\|_{\mathfrak{H}}^{2}\right). \tag{3.3.6}$$

From equation (3.3.1), we have that

$$d_{\mathrm{Kol}}(F_k,N)\leq|1-\mathbb{E}[F_k^2]|+\sqrt{\mathrm{Var}\left(q^{-1}\left\|DF_k\right\|_{\mathfrak{H}}^{2}\right)}. \tag{3.3.7}$$

The result can now be concluded easily.

3.4 Chaotic Central Limit Theorem

After the appearance of Nualart and Peccati's Fourth moment theorem, Peccati and Tudor in [18] gave the following important multivariate version of the theorem.

Theorem 3.4.1. *Consider a sequence of vectors $F_k=(F_k^1,...,F_k^m)$ where $F_k^i=I_{q_i}(f_k^i)$ for some $f_k^i\in\mathfrak{H}^{\odot q_i}$ and a non-negative definite symmetric matrix C. If $\mathbb{E}[F_k^iF_k^j]\to C(i,j)$ as $k\to\infty$ then the following are equivalent.*
1) $F_k\Rightarrow\mathcal{N}(0,C)$
2) For every $1\leq i\leq m$, $F_k^i\Rightarrow\mathcal{N}(0,C(i,i))$.

Combining the Fourth Moment theorem and the above multivariate limit theorem, we can prove the following powerful general limit theorem. The below is also known as chaotic central limit theorem. It was proved in the univariate setting by Nualart and Hu in [19]. Here we consider a multivariate version.

Theorem 3.4.2. *Let $F_k=(F_k^i)_{1\leq i\leq m}$ be a sequence of vectors such that $F_k^i\in L^2(\Omega)$ with $\mathbb{E}[F_k^i]=0$ for every $k\in\mathbb{N}$ and $1\leq i\leq m$. Then we have the expansions, $F_k^i=\sum_{q=1}^{\infty}I_q(f_{k,q}^i)$ for some $f_{k,q}^i\in\mathfrak{H}^{\odot q}$. Further assume the following conditions.*
1) For every $1\leq i,j\leq m$ and every $q\geq1$, $\mathbb{E}[I_q(f_{k,q}^i)I_q(f_{k,q}^j)]\to V_q^{i,j}$.

2) For every $1 \leq i,j \leq m$, $b_{i,j} = \sum_{q=1}^{\infty} V_q^{i,j} < \infty$.
3) For every $1 \leq i \leq m$, every $q \geq 1$ and every $1 \leq r \leq q-1$, $\left\|f_{k,q}^i \otimes_r f_{k,q}^i\right\|_{\mathfrak{H}^{\odot 2q-2r}} \to 0$ as $k \to \infty$.
4) For every $1 \leq i \leq m$, $\sup_{k\geq 1} \sum_{q=l+1}^{\infty} \mathbb{E}[I_q(f_{k,q}^i)^2] \to 0$ as $l \to \infty$.
Then as $k \to \infty$ we have $F_k \Rightarrow \mathcal{N}(0,B)$ where $B = (b_{i,j})_{1\leq i,j\leq m}$.

Proof. Let $N \sim \mathcal{N}(0,B)$. We first note that for the desired joint convergence to hold, it suffices to show the pointwise convergence of the characteristic function, i.e. for $t \in \mathbb{R}^m$, we need to show $|\mathbb{E}[e^{it^T F_k}] - \mathbb{E}[e^{it^T N}]| \to 0$ as $k \to \infty$.

Let $N_l \sim \mathcal{N}(0, B_l)$, where $B_l(i,j) = \sum_{q=1}^{l} V_q^{i,j}$ and let $F_{k,l} = (F_{k,l}^i)_{1\leq i\leq m}$ where $F_{k,l}^i = \sum_{q=1}^{l} I_q(f_{k,q}^i)$. Now for $t \in \mathbb{R}^m$ we have,

$$|\mathbb{E}[e^{it^T F_k}] - \mathbb{E}[e^{it^T N}]| \leq |\mathbb{E}[e^{it^T F_k}] - \mathbb{E}[e^{it^T F_{k,l}}]| + |\mathbb{E}[e^{it^T F_{k,l}}] - \mathbb{E}[e^{it^T N_l}]| + |\mathbb{E}[e^{it^T N_l}] - \mathbb{E}[e^{it^T N}]| \tag{3.4.1}$$

Considering the first term in the above expression we have,

$$\begin{aligned}
|\mathbb{E}[e^{it^T F_k}] - \mathbb{E}[e^{it^T F_{k,l}}]| &\leq \mathbb{E}[||e^{it^T F_k} - e^{it^T F_{k,l}}||] \\
&\leq \mathbb{E}[|t^T F_k - t^T F_{k,l}|] \\
&\leq \mathbb{E}[|\sum_{j=1}^{n} t_j (F_k^j - F_{k,l}^j)|] \\
&\leq \sum_{j=1}^{n} |t_j|\ \mathbb{E}[|F_k^j - F_{k,l}^j|] \\
&\leq \sum_{j=1}^{n} |t_j|\ \sqrt{\mathbb{E}[|F_k^j - F_{k,l}^j|^2]} \to 0
\end{aligned} \tag{3.4.2}$$

as by condition 4) we have for every $1 \leq j \leq m$, $\sup_{k\geq 1} \mathbb{E}[|F_k^j - F_{k,l}^j|^2] = \sup_{k\geq 1} \sum_{q=l+1}^{\infty} \mathbb{E}[(I_q(f_{k,q}^j)^2] \to 0$ as $l \to \infty$. Considering the third term we have,

$$|\mathbb{E}[e^{it^T N_l}] - \mathbb{E}[e^{it^T N}]| = |e^{-\frac{1}{2}\langle B_l t, t\rangle} - e^{-\frac{1}{2}\langle Bt, t\rangle}| \to 0 \tag{3.4.3}$$

as for every $1 \leq i, j \leq m$, $B_l(i,j) \to B(i,j)$ as $l \to \infty$. Now, considering the second term, we have that conditions 1), 2), 3) along with theorems (3.1.1) and (3.4.1) and orthogonality of multiple Wiener Itô integrals of different order imply that as $k \to \infty$,

$$(I_1(f^1_{k,1}), ..., I_l(f^1_{k,l}), ..., I_1(f^m_{k,l}), ..., I_l(f^m_{k,l})) \Rightarrow \mathcal{N}(0, P) \tag{3.4.4}$$

where P is the appropriate covariance matrix governed by condition 1). Therefore, by continuous mapping theorem, we also have that $|\mathbb{E}[e^{it^T F_{k,l}}] - \mathbb{E}[e^{it^T N_l}]| \to 0$ as $k \to \infty$ completing the proof.. $\square$

Chapter 4

Breuer-Major Theorem

This chapter is based on a joint work with David Nualart ([5]).

4.1 Introduction

The classical Breuer-Major theorem in its primitive form, as proved first by Péter Breuer and Péter Major in their seminal paper [4] in 1983, states that, under an appropriate condition involving the covariances, the sum of a functional of a stationary sequence of Gaussian variables, scaled by the square root of the number of terms, converges in distribution to a Gaussian variable. A formal statement is as follows. For a centered stationary sequence of Gaussian variables $\{\xi_k : k \in \mathbb{Z}\}$ with unit variance and a function $G \in L^2(\mathbb{R}, \gamma_1)$ of Hermite rank d (see definition (2.1.4)), where γ_1 denotes the standard Gaussian measure on $\mathbb{R}$, if $\sum_{k\in\mathbb{Z}} |\mathbb{E}[\xi_1\xi_{1+k}]|^d < \infty$, then the following convergence in law holds

$$\frac{1}{\sqrt{n}}\Big[\sum_{k=1}^{n} G(\xi_k) - n\mathbb{E}[G(\xi_1)]\Big] \Rightarrow \mathcal{N}(0, V)$$

as $n \to \infty$, for some $V \in [0, \infty)$.

The theorem has now become one of the most celebrated and widely applicable results in stochastic analysis. An extension of the original version to sequences of vectors was done by Arcones in [20] and continuous versions of the theorem for real valued fields are found in

[21, 22, 10].

A continuous version of this theorem (see Theorem 2.3.1 of [10]) asserts that for a zero mean, stationary, isotropic Gaussian random field $\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ with covariance function $r(x) = \mathbb{E}[\xi(0)\xi(x)]$, if $r \in L^d(\mathbb{R}^n)$ and $r(x) \to 0$ as $|x| \to \infty$, then as $s \to \infty$, the finite dimensional distributions of the processes

$$Z_s(t) = \frac{1}{s^{n/2}} \int_{B_n(st^{1/n})} \Big[G(\xi(x)) - \mathbb{E}[G(\xi(x))]\Big] dx, \quad t \in [0, \infty)$$

converge to those of a scaled Brownian motion. Here $B_n(a)$ denotes the ball of radius a centered at the origin in $\mathbb{R}^n$.

Estrade and León have partially addressed the case of random vector fields on the Euclidean space in [23] where they mention adapting the Breuer-Major theorem to prove a Central Limit Theorem for the Euler characteristic of an excursion set (see Proposition 2.4 of [23]).

In this chapter we obtain a multidimensional extension of the continuous Breuer-Major theorem for random fields, including the corresponding invariance principle. We will use the n-cubes $[-s, s]^n$ instead of balls as expanding sets and we prove it without the assumption of isotropy. We will also give a proof for the convergence of Z_s to hold in a functional sense, i.e. convergence in law in $C([0, \infty))$ under the condition that $G \in L^p(\mathbb{R}^m, \gamma_m)$ for some $p > 2$, where γ_m denotes the standard normal distribution on $\mathbb{R}^m$. This remains an unaddressed question in the literature in the case of vectors. The approach here is similar to the method that has been employed in [24] and [25], namely using the representation by means of the Malliavin divergence operator, which is obtained through a shift operator, and applying Meyer inequalities to show tightness. However, in the case of vectors fields, this approach is more involved and requires the introduction of weighted shift operators.

The modern proof of the Breuer-Major theorem is based on the Stein-Malliavin approach and is presented in [13]. We will rely on this methodology for the proofs. We refer the reader to the monographs [13] or [9] for unexplained usage of terms.

We organize the chapter as follows. Section 2 describes the necessary framework and notations. The third section contains the statements of our results. In Section 4 we write the Wiener chaos expansions of variables of interest. Finally, Section 5 contains the proofs.

4.2 Setup

Let $\xi_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$ be zero mean, mean-square continuous, stationary Gaussian random fields which are *jointly stationary*, i.e., for $1 \leq i, j \leq m$, the cross covariance functions, $r_{i,j}(x, y) = \mathbb{E}[(\xi_i(x)\xi_j(y)] = r_{i,j}(x - y)$ (in an abuse of notation), depend only on $x - y$. Then the function $r : \mathbb{R}^n \to M_m(\mathbb{R})$, $r(x) = (r_{i,j}(x))_{1 \leq i,j \leq m}$ is the covariance function for the vector valued field,

$$\xi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m, \;\; \xi(x) = (\xi_i(x))_{1 \leq i \leq m}.$$

Let $G : \mathbb{R}^m \to \mathbb{R}$ be such that G is not a constant and $G \in L^2(\mathbb{R}^m, \gamma_m)$. Denoting, $\mathcal{I}_q = \{a \in \mathbb{Z}^m : a_i \geq 0, |a| = q\}$, we have the following expansion of G where the convergence of the series is in L^2 sense,

$$\sum_{q=0}^{\infty} \sum_{a \in \mathcal{I}_q} c(G, a)\overline{H}_a(x) = G(x). \tag{4.2.1}$$

In above expansion $c(G, a) = \frac{1}{\sqrt{a!}} \int_{\mathbb{R}^m} G(x)\overline{H}_a(x)\gamma_m(dx)$. Let $G_0 = \int_{\mathbb{R}^m} G(x)\gamma_m(dx) = 0$ and let Hermite rank of G be d (see definition (2.1.4)). Therefore,

$$\sum_{q=d}^{\infty} \sum_{a \in \mathcal{I}_q} c(G, a)\overline{H}_a(x) = G(x). \tag{4.2.2}$$

For any integer $q \geq 1$, we will make use of the notation

$$G_q(x) = \sum_{a \in \mathcal{I}_q} c(G, a)\overline{H}_a(x). \tag{4.2.3}$$

We are interested in the asymptotic behavior as $s \to \infty$ of the random variables defined by

$$L_s = \frac{1}{(2s)^{n/2}} \int_{[-s,s]^n} G(\xi(x))dx. \tag{4.2.4}$$

For any integer $q \geq 1$, we put

$$L_s^{(q)} = \frac{1}{(2s)^{n/2}} \int_{[-s,s]^n} G_q(\xi(x))dx. \tag{4.2.5}$$

Also we denote the variances of L_s and $L_s^{(q)}$ by $\mathrm{Var}(L_s) = V_s$ and $\mathrm{Var}(L_s^{(q)}) = V_s^{(q)}$, respectively. Set

$$C_G(x, y) = \mathbb{E}[G(\xi(x))G(\xi(y))]$$

as the covariance function of $G(\xi(x))$. We ignore the degenerate case when $V_s = 0$ for all $s > 0$.

Remark 4.2.1. *We will use Fubini-Tonelli's theorem to exchange integrals and expectation and everytime its use will be justified by Theorem 1.1.1 of [10]. We will also use it to interchange the multiple Wiener-Itô integral and Lebesgue integral.*

We will impose the following condition on the covariances. As noted in the proof of Theorem 1 of [20], given that $r(0)$ is invertible, by a linear transformation we can assume that $r(0) = \mathrm{Id}_{m\times m}$ ($m \times m$ identity matrix). Moreover, recall that $d \geq 1$ is the Hermite rank of our functional G.

Condition (C1). $r(0) = \mathrm{Id}_{m\times m}$ and for every $1 \leq j, k \leq m$, $r_{j,k} \in L^d(\mathbb{R}^n)$.

Remark 4.2.2. *Since by Cauchy-Schwarz inequality and stationarity, $\mathbb{E}[\xi_j(x)\xi_k(0)] \leq 1$,* ***(C1)*** *implies that $r_{j,k} \in L^p(\mathbb{R}^n)$ for all $p \geq d$.*

4.3 Statements

We are now in a position to state the main results. The lemma below provides a simple characterization for the asymptotic variance of L_s defined in equation (4.2.4). Note here that we have assumed $\mathbb{E}[G(\xi(0))] = 0$, that means the Hermite rank of G is $d \geq 1$.

Lemma 4.3.1. *Under* ***(C1)****, the random field $G \circ \xi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is weakly stationary, i.e. $C_G(x, y) = \mathbb{E}[G(\xi(x))G(\xi(y))] = C_G(x - y)$ is a function of $x - y$ and $C_G \in L^1(\mathbb{R}^n)$. The following also holds,*

$$V := \lim_{s\to\infty} V_s = \int_{\mathbb{R}^n} C_G(x)dx < \infty, \tag{4.3.1}$$

where we recall that V_s denoted the variance of the random variable L_s defined in equation (4.2.4).

Theorem 4.3.2. *Under **(C1)**,*

$$L_s = \frac{1}{(2s)^{n/2}} \int_{[-s,s]^n} G(\xi(x))dx \quad \Rightarrow \quad \mathcal{N}(0, V) \ \text{ as } \ s \to \infty.$$

Here V is as in Lemma (4.3.1) *and $\Rightarrow$ denotes convergence in law.*

The above statement is a continuous version of Theorem 4 of [20].

Theorem 4.3.3. *Under **(C1)** as $s \to \infty$, the finite dimensional distributions of the process*

$$Z_{s,y} = \frac{1}{(2s)^{n/2}} \int_{[-sy^{1/n}, sy^{1/n}]^n} G(\xi(x))dx, \quad y \in [0, \infty),$$

converge to those of $\sqrt{V} B_y$ on $[0, \infty)$, where $B = \{B_y, y \geq 0\}$ is a standard Brownian motion.

The above statement is a multi-dimensional extension of Theorem 2.3.1 of [10]. The above two theorems are presented separately for better elucidation and to save on unnecessary notation. Clearly Theorem (4.3.3) contains Theorem (4.3.2).

Theorem 4.3.4. *Assume **(C1)** and $G \in L^p(\mathbb{R}^m, \gamma_m)$ for some $p > 2$. As $s \to \infty$, the probability measures $\{P_s : s > 0\}$ on $C([0, \infty))$ induced by $\{Z_s : s > 0\}$ (as defined in Theorem* (4.3.3)*) converge weakly to the probability measure induced by $\sqrt{V} B_y$ on $C([0, \infty))$, where again B denotes a standard Brownian motion.*

The above result is multi-dimensional counterpart of Theorem 1.1 of [24].

Consider the $m \times m$ symmetric matrix $C = (c_{j,k})_{1 \leq j,k \leq m}$ given by

$$\begin{cases} c_{j,k} &= \int_{\mathbb{R}^n} G(x) x_j x_k \phi_m(x) dx, \quad \text{for } j \neq k \\ c_{j,j} &= \int_{\mathbb{R}^n} G(x)(x_j^2 - 1)\phi_m(x) dx, \quad \text{for } j = k. \end{cases} \tag{4.3.2}$$

We have the following lemma which gives an expression for the asymptotic variance of the second chaos component.

Lemma 4.3.5. *Let G be of Hermite rank* 2 *and assume* **(C1)**. *Let C be the matrix defined* *in equation* (4.3.2). *Then,*

$$\lim_{s\to\infty} V_s^{(2)} = V^{(2)} = \frac{1}{2} \left\| \mathrm{Tr}[rCrC] \right\|_{L^1(\mathbb{R}^n)}. \tag{4.3.3}$$

Suppose in addition that for every $1 \leq j, k \leq m$, $r_{j,k} \in L^1(\mathbb{R}^n)$. Note that due to the stationarity and mean-square continuity of the fields ξ_js, we have, by Bochner's theorem (Theorem 5.4.1 of [11] or equation 1.2.1 of [10]), that there exist finite measures ν_js (called the spectral measures) such that

$$r_{j,j}(x) = \int_{\mathbb{R}^n} e^{i\langle t,x\rangle} \nu_j(dt). \tag{4.3.4}$$

Moreover, due to the integrability of the covariances, we have that the ν_js are absolutely continuous with respect to the Lebesgue measure and admit densities (called spectral densities). Denote the spectral density of ξ_j as f_j and $\alpha_j = \sqrt{f_j}$. Set $\alpha(x) = (\alpha_i(x))_{1\leq i\leq m}$ and let $H(x) = \alpha^T(-x)C\alpha(x)$. Under these conditions, equation (4.3.3) can be written as

$$V^{(2)} = \frac{(2\pi)^{-n}}{2} \|H\|^2_{L^2(\mathbb{R}^n)}. \tag{4.3.5}$$

This formula has been motivated by the result obtained in [26] in the context of the Central Limit Theorem for the number of critical points, where $V^{(2)}$ is obtained as the L^2-norm of a function.

4.4 Chaos expansions

We claim that there exist a Hilbert space $\mathfrak{H}$ and elements $\beta_{j,x} \in \mathfrak{H}$, $1 \leq i, j \leq m$, $x \in \mathbb{R}^n$, such that

$$r_{i,j}(x-y) = \langle \beta_{i,x}, \beta_{j,y} \rangle_{\mathfrak{H}}$$

for all $x, y \in \mathbb{R}^n$ and $1 \leq i, j \leq m$. Indeed, it suffices to choose as $\mathfrak{H}$ the Gaussian subspace of $L^2(\Omega)$ generated by the random field ξ and take $\beta_{i,x} = \xi_i(x)$. Consider an isonormal Gaussian process X on $\mathfrak{H}$. That is, $X = \{X(h) : h \in \mathfrak{H}\}$ is a Gaussian centered family of random variables, defined in a probability space $(\Omega, \mathcal{F}, P)$, such that $\mathbb{E}[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}$ for any $g, h \in \mathfrak{H}$. In this situation, $\{\xi_i(x) : x \in \mathbb{R}^n, 1 \leq i \leq m\}$ has the same law as

$\{X(\beta_{i,x}) : x \in \mathbb{R}^n, 1 \leq i \leq m\}$. Therefore, without loss of generality we can assume the existence of an isonormal process X on $\mathfrak{H}$ such that

$$\xi_j(x) = X(\beta_{j,x}). \tag{4.4.1}$$

We will also assume that the σ field $\mathcal{F}$ is generated by ξ.

We now turn to giving the chaos expansions for L_s given by equation (4.2.4). For $\beta_{j,x}$ as introduced in equation (4.4.1), we have that for any $x \in \mathbb{R}^n$ and $j \neq k$, under **(C1)**,

$$\langle \beta_{j,x}, \beta_{k,x} \rangle_{\mathfrak{H}} = \mathbb{E}[\xi_j(x)\xi_k(x)] = 0. \tag{4.4.2}$$

Now consider any multi-index a such that $|a| = q$. By the previous facts, equation (2.1.5), equation (4.4.1) and taking into account the product formula in theorem (2.5.6) and equation (4.4.2), we can write

$$\overline{H}_a(\xi(x)) = \prod_{j=1}^{m} I_{a_j}(\beta_{j,x}^{\otimes a_j}) = I_q(\beta_{1,x}^{\otimes a_1} \otimes \cdots \otimes \beta_{m,x}^{\otimes a_m})$$

We introduce the elements ρ_x^q and χ_s^q which characterize the expansions. Let

$$\rho_x^q = \sum_{a \in \mathcal{I}_q} c(G,a) \beta_{1,x}^{\otimes a_1} \otimes \cdots \otimes \beta_{m,x}^{\otimes a_m}. \tag{4.4.3}$$

Notice that, although for each $a \in \mathcal{I}_q$, the tensor $\beta_{1,x}^{\otimes a_1} \otimes \cdots \otimes \beta_{m,x}^{\otimes a_m}$ is not necessarily symmetric, the element ρ_x^q is symmetric because $c(G,a)$ is a symmetric function of the multiindex a. Set

$$\chi_s^q = \frac{1}{(2s)^{n/2}} \int_{[-s,s]^n} \rho_x^q dx. \tag{4.4.4}$$

By linearity of the multiple Wiener-Itô integral and Fubini's theorem for multiple Wiener-Itô integral, we have that

$$G_q(\xi(x)) = I_q(\rho_x^q); \quad L_s^{(q)} = I_q(\chi_s^q),$$

where G_q and $L_s^{(q)}$ are defined in equations (4.2.3) and (4.2.5), respectively. Therefore, we have the chaos expansion

$$L_s = \sum_{q=d}^{\infty} I_q(\chi_s^q) = \sum_{q=d}^{\infty} L_s^{(q)}. \tag{4.4.5}$$

This is true because,

$$\mathbb{E}\Big[\Big(L_s - \sum_{q=d}^{l} L_s^{(q)}\Big)^2\Big] = \frac{1}{(2s)^n}\mathbb{E}\Bigg[\Big(\int_{[-s,s]^n} G(\xi(x)) - \sum_{q=d}^{l} G_q(\xi(x))dx\Big)^2\Bigg]$$

$$= \frac{1}{(2s)^n}\int_{[-s,s]^n}\int_{[-s,s]^n}\mathbb{E}\Big[\Big(G(\xi(x)) - \sum_{q=d}^{l} G_q(\xi(x))\Big)\Big(G(\xi(y)) - \sum_{q=d}^{l} G_q(\xi(y))\Big)\Big]dxdy$$

$$\leq \frac{\mathbb{E}\big[(G(\xi(0)) - \sum_{q=d}^{l} G_q(\xi(0)))^2\big]}{(2s)^n}\int_{[-s,s]^n}\int_{[-s,s]^n} dxdy \to 0$$

as $l \to \infty$. The last step follows from stationarity and Cauchy-Schwarz inequality.

Remark 4.4.1. *Due to properties of the multiple Wiener-Itô integrals, we have* $\mathbb{E}[L_s^q] = \mathbb{E}[I_q(\chi_s^q)] = 0$ *and* $\mathbb{E}[G_q(\xi(x))] = \mathbb{E}[I_q(\rho_x^q)] = 0$*. Also* $\mathbb{E}[G_{q_1}(\xi(x))G_{q_2}(\xi(y))] = 0$ *for all* $q_1 \neq q_2$*.*

4.5 Proofs

4.5.1 Proof of Lemma 3.1

Let us first prove the weak stationarity of the random field $G \circ \xi$. Taking into account that $G_q(\xi(x))$ is the projection on the qth Wiener chaos of $G(\xi(x))$, we can write, for any $x, y \in \mathbb{R}^n$,

$$\mathbb{E}[G(\xi(x))G(\xi(y))] = \sum_{q=d}^{\infty}\mathbb{E}[G_q(\xi(x))G_q(\xi(y))].$$

Furthermore, in view of the Diagram formula (see [20]) we have that $C_{G_q}(x, y)$ depends on the covariances $r_{i,j}(x-y)$ and hence $C_{G_q}(x, y)$ is a function of $x - y$. As a consequence, we get that $C_G(x, y) = C_G(x - y)$ is a function of $x - y$.

To show equation (4.3.1) we will make use of Lemma 1 of [20] and condition **(C1)**. We have

$$(2s)^n V_s = \mathbb{E}\Big[\Big(\int_{[-s,s]^n} G(\xi(x))dx\Big)^2\Big] = \int_{[-s,s]^n}\int_{[-s,s]^n} C_G(x-y)dxdy.$$

Since by Cauchy-Schwarz inequality and stationarity, $\mathbb{E}[G(\xi(x))G(\xi(y))] \leq \mathbb{E}[(G(\xi(0)))^2]$, we have $V_s < \infty$ for all $s > 0$ and

$$\begin{aligned} V_s &= \frac{1}{(2s)^n} \int_{[-s,s]^n} \int_{[-s,s]^n} C_G(x-y)dxdy \\ &= \int_{[-2s,2s]^n} C_G(x) \prod_{i=1}^{n} \Big(1 - \frac{|x_i|}{2s}\Big)dx \\ &= \int_{\mathbb{R}^n} C_G(x) \prod_{i=1}^{n} \Big(1 - \frac{|x_i|}{2s}\Big)\mathbf{1}_{[-2s,2s]^n}(x)dx \\ &=: \int_{\mathbb{R}^n} C_G(x) I_{2s}(x)dx. \end{aligned} \tag{4.5.1}$$

We set

$$\psi(x) = \left(\sup_{1\le i\le m} \sum_{j=1}^{m} |r_{i,j}(x)| \right) \vee \left(\sup_{1\le j\le m} \sum_{i=1}^{m} |r_{i,j}(x)| \right). \tag{4.5.2}$$

By Lemma 1 of [20], on the set $\{x : \psi(x) \le 1\}$, we have

$$|C_G(x)| = |\mathbb{E}[G(\xi(0))G(\xi(x))]| \le \psi^d(x) \|G\|^2_{L^2(\mathbb{R}^m,\gamma_m)}.$$

Also $\int_{\mathbb{R}^n} \psi^d(x)dx < \infty$ as $\int_{\mathbb{R}^n} |r_{i,j}(x)|^d dx < \infty$ for all $1 \le i, j \le m$. On the other hand, on the set $\{x : \psi(x) > 1\}$ we can write, taking into account that $|C_G(x)| \le \|G\|^2_{L^2(\mathbb{R}^m,\gamma_m)}$,

$$\begin{aligned} \int_{\{\psi(x)>1\}} |C_G(x)|dx &\le \sum_{i,j=1}^{m} \int_{\{|r_{i,j}(x)|>\frac{1}{m}\}} |C_G(x)|dx \\ &\le \|G\|^2_{L^2(\mathbb{R}^m,\gamma_m)} m^d \sum_{i,j=1}^{m} \int_{\mathbb{R}^n} |r_{i,j}(x)|^d dx < \infty. \end{aligned}$$

Observe that $|I_{2s}(x)| = |\Pi_{i=1}^n (1 - \frac{|x_i|}{2s})\ \mathbf{1}_{[-2s,2s]^n}| \le 1$ for all $s > 0$ and as $s \to \infty, I_{2s}(x) \to 1$. Therefore by dominated convergence theorem,

$$V = \lim_{s\to\infty} V_s = \int_{\mathbb{R}^n} C_G(x)dx < \infty.$$

$\square$

4.5.2 Proof of Theorem 3.2

We will apply Nualart and Hu's criteria for convergence in distribution to a normal variable (see theorem (3.4.2)). As a consequence, the theorem follows if the following conditions hold,

1) For every $q \geq d$, $V_s^{(q)} \to V_q < \infty$ as $s \to \infty$.
2) $V = \sum_{q=d}^{\infty} V^{(q)} < \infty$.
3) For every $q \geq d$ and every $1 \leq b \leq q-1$, $||\chi_s^q \otimes_b \chi_s^q||_{\mathfrak{H}^{\otimes(2q-2b)}} \to 0$ as $s \to \infty$.
4) $\sup_{s>0} \sum_{q=l+1}^{\infty} V_s^{(q)} \to 0$ as $l \to \infty$.

Here χ_s^q is given by equation (4.4.4). Conditions 1), 2) hold by Lemma (4.3.1). For condition 3), by equation (4.4.4) we have that

$$\chi_s^q \otimes_b \chi_s^q = \frac{1}{(2s)^n} \int_{[-s,s]^n} \int_{[-s,s]^n} \rho_x^q \otimes_b \rho_y^q dxdy. \tag{4.5.3}$$

Denoting for a multi-index $i = (i_1, ..., i_q)$, $\zeta_{i,x} = \beta_{i_1,x} \otimes \cdots \otimes \beta_{i_q,x}$, for the desired convergence to hold, we have, by equation (4.4.3), that it suffices to show that for any multi-indices i and j,

$$J_s := \left\| \frac{1}{(2s)^n} \int_{[-s,s]^n} \int_{[-s,s]^n} \zeta_{i,x} \otimes_b \zeta_{j,y} dxdy \right\|^2_{\mathfrak{H}^{\otimes(2q-2b)}} \to 0 \tag{4.5.4}$$

as $s \to \infty$. We have, using equation (2.5.5),

$$J_s = \frac{1}{(2s)^{2n}} \int_{[-s,s]^{4n}} \left(\prod_{k=1}^{b} r_{i_k,j_k}(x-y) r_{i_k,j_k}(z-w) \right.$$
$$\left. \times \left\langle \left(\otimes_{\ell=b+1}^{q} \beta_{i_\ell,x}\right) \otimes \left(\otimes_{\ell=b+1}^{q} \beta_{j_\ell,y}\right), \left(\otimes_{\ell=b+1}^{q} \beta_{i_\ell,z}\right) \otimes \ \left(\otimes_{\ell=b+1}^{q} \beta_{i_\ell,w}\right) \right\rangle_{\mathfrak{H}^{\otimes(2q-2b)}} \right) dxdydzdw.$$

In the above expression, pairing together $\beta_{i_{b+k},x}$ and $\beta_{i_{b+k},z}$ and similarly with the index j, we get that,

$$J_s = \frac{1}{(2s)^{2n}} \int_{[-s,s]^{4n}} \left(\prod_{k=1}^{b} r_{i_k,j_k}(x-y) r_{i_k,j_k}(z-w) \prod_{k=b+1}^{q} r_{i_k,i_k}(x-z) r_{j_k,j_k}(y-w) \right) dxdydzdw$$
$$\leq \frac{1}{(2s)^{2n}} \int_{[-s,s]^{4n}} \psi^b(x-y) \psi^b(z-w) \psi^{q-b}(x-z) \psi^{q-b}(y-w) dxdydzdw.$$

where ψ is as defined by equation (4.5.2). In what follows, the value of constant C is immaterial and changes with each step. By Hölder's inequality and the fact that $\psi \in L^q(\mathbb{R}^n)$ for all $q \geq d$ we have that

$$J_s \leq Cs^{-2n} \int_{[-s,s]^{3n}} \psi^b(x-y)\psi^{q-b}(y-w)dxdydw. \tag{4.5.5}$$

By the change of variables $(x, y, w) \mapsto (x - y, y - w, w)$ we have

$$J_s \leq Cs^{-n} \int_{[-2s,2s]^{2n}} \psi^b(u)\psi^{q-b}(v)dudv.$$

We proceed in a manner similar to [21]. For $k > 0$ denote $T_k = [-k, k]^{2n}$ and T_k^c to be its complement in $\mathbb{R}^{2n}$. Consider the decomposition

$$J_s \leq Cs^{-n} \int_{[-2s,2s]^{2n}\cap T_k} \psi^b(u)\psi^{q-b}(v)dudv + Cs^{-n} \int_{[-2s,2s]^{2n}\cap T_k^c} \psi^b(u)\psi^{q-b}(v)dudv.$$

For any fixed k, since ψ is bounded, we have that the first term tends to zero as $s \to \infty$. For the second term, by Hölder's inequality we can write

$$\begin{aligned} s^{-n} &\int_{[-2s,2s]^{2n}\cap T_k^c} \psi^b(u)\psi^{q-b}(v)dudv \\ &\leq Cs^{-n}\left(s^n \int_{\mathbb{R}^n\setminus[-k,k]^n} \psi^q(u)du\right)^{b/q}\left(s^n \int_{\mathbb{R}^n\setminus[-k,k]^n} \psi^q(v)dv\right)^{(q-b)/q} \\ &\leq C \int_{\mathbb{R}^n\setminus[-k,k]^n} \psi^q(x)dx \to 0 \end{aligned}$$

as $k \to \infty$ yielding the desired conclusion.

Condition 4) also holds as we have, by equation (4.5.1) in Lemma 3.1,

$$\sum_{q=l+1}^{\infty} V_s^{(q)} = \sum_{q=l+1}^{\infty} \int_{\mathbb{R}^n} C_{G_q}(x)\; I_{2s}(x)dx \leq \sum_{q=l+1}^{\infty} \int_{\mathbb{R}^n} C_{G_q}(x)dx = \sum_{q=l+1}^{\infty} V^{(q)} \to 0$$

as $l \to \infty$ uniformly in s.

□

4.5.3 Proof of Theorem 3.3

As defined in the statement,

$$Z_{s,y} = \frac{1}{(2s)^{n/2}} \int_{[-sy^{1/n}, sy^{1/n}]^n} G(\xi(x))dx, \quad y \in [0, \infty).$$

We gather the necessary notation for the Wiener chaos expansions for the new variables. By the Wiener chaos expansions in equation (4.4.5), we have for any $y > 0$, $Z_{s,y} = \sum_{q=d}^{\infty} Z_{s,y}^{(q)} = \sum_{q=d}^{\infty} I_q(\chi_{s,y}^q)$. Here

$$Z_{s,y}^{(q)} = \frac{1}{(2s)^{n/2}} \int_{[-sy^{1/n}, sy^{1/n}]^n} G_q(\xi(x))dx$$

and

$$\chi_{s,y}^q = \frac{1}{(2s)^{n/2}} \int_{[-sy^{1/n}, sy^{1/n}]} \rho_x^q dx.$$

Due to theorem (3.4.2), the convergence of the finite dimensional distributions of Z_s to those of the Brownian motion $\sqrt{V} B_y$ follows if we show that the covariances of the corresponding projections on each Wiener chaos converge. Namely for any $q \geq d$ and $y_1, y_2 > 0$,

$$\mathbb{E}[Z_{s,y_1}^{(q)} Z_{s,y_2}^{(q)}] \to V^{(q)} y_1 \wedge y_2$$

as $s \to \infty$, where $V^{(q)} = \lim_{s \to \infty} V_s^{(q)}$.

Let $y_1 \leq y_2$ and set $s_1 = sy_1^{1/n}$ and $s_2 = sy_2^{1/n}$. Denote $A_s = [-s_1 - s_2, s_1 + s_2]^n$ and $C_s = [s_1 - s_2, s_2 - s_1]^n$. By the change of variables $(x, y) \mapsto (x - y, y)$ we have,

$$\begin{aligned}
\mathbb{E}[Z_{s,y_1}^{(q)} Z_{s,y_2}^{(q)}] &= \frac{1}{(2s)^n} \int_{[-s_1,s_1]^n} \int_{[-s_2,s_2]^n} C_{G_q}(x - y)dxdy \\
&= \frac{1}{(2s)^n} \int_{C_s} C_{G_q}(u)(2s_1)^n du \\
&+ \frac{1}{(2s)^n} \int_{A_s \setminus C_s} C_{G_q}(u) \prod_{i=1}^{n} \Big(s_1 + s_2 - |u_i| \Big) du.
\end{aligned}$$

Due to Lemma 3.1 applied to the random field $G_q(\xi(x))$, we have that as $s \to \infty$

$$\frac{1}{(2s)^n} \int_{[s_1 - s_2, s_2 - s_1]^n} C_{G_q}(u)(2s_1)^n du \to V^q y_1$$

and by dominated convergence theorem the second term converges to zero, that is,

$$\int_{\mathbb{R}^n} C_{G_q}(u) \prod_{i=1}^{n} \Big(\frac{y_1^{1/n} + y_2^{1/n}}{2} - \frac{|u_i|}{2s} \Big) \mathbf{1}_{[A_s \setminus C_s]} du \to 0.$$

Therefore, the theorem follows. □

4.5.4 Proof of Theorem 3.4

Since we have established the convergence of the finite dimensional distributions, it now suffices to show that the family of probability measures $\{P_s : s > 0\}$ is tight. By problem 4.11 of [27], it suffices to show that for some $p > 2$ and for every $T > 0$, the following holds for $0 \leq y_1 \leq y_2 \leq T$,

$$\sup_{s>0} \|Z_{s,y_2} - Z_{s,y_1}\|_{L^p(\Omega)} \leq C_T |y_2 - y_1|^{1/2}. \tag{4.5.6}$$

The desired estimate will be obtained by employing a weighted shift operator and obtaining a representation using the divergence operator. We proceed to define the shift operator.

If $G \in L^2(\mathbb{R}^m, \gamma_m)$ has rank $d \geq 1$ with the expansion given in equation (4.2.2), for any index $i = 1, \dots, m$, we define the operator T_i by

$$T_i(G)(x) = \sum_{q=d}^{\infty} \sum_{a \in \mathcal{I}_q} c(G,a) \frac{a_i}{q} H_{a_i - 1}(x_i) \prod_{j=1, j \neq i}^{m} H_{a_j}(x_j)\,. \tag{4.5.7}$$

We know that $G(\xi(x))$ has the Wiener chaos expansion

$$G(\xi(x)) = \sum_{q=d}^{\infty} \sum_{a \in \mathcal{I}_q} c(G,a) I_q(\beta_{1,x}^{\otimes a_1} \otimes \cdots \otimes \beta_{m,x}^{\otimes a_m}). \tag{4.5.8}$$

The shift operator allows us to represent $G(\xi(x))$ as a divergence. Notice that this operator is more complicated than the shift operator considered in the one-dimensional case (see [25]) because we need the weights a_i/q in order to have the representation as a divergence. Actually, we are interested in representing $G(\xi(x))$ as an iterated divergence.

For any $2 \le k \le d$ and indexes $i_1, \dots, i_k \in \{1, \dots, m\}$, we can define the iterated operator

$$T_{i_1,\dots,i_k} = T_{i_1} \circ \overset{k)}{\cdots} \circ T_{i_k}.$$

The following result is our representation theorem.

Lemma 4.5.1. *For any $2 \le k \le d$, we have*

$$G(\xi(x)) = \delta^k \left(\sum_{i_1,\dots,i_k=1}^{m} T_{i_1,\dots,i_k} G(\xi(x)) \beta_{i_1,x} \otimes \cdots \otimes \beta_{i_k,x} \right).$$

Proof. Using the Wiener chaos expansion in equation (4.5.8) and the operator L^{-1} introduced in definition (2.4.4), we can write

$$\begin{aligned} L^{-1}G(\xi(x)) &= -\sum_{q=d}^{\infty} \sum_{a \in \mathcal{I}_q} c(G,a) \frac{1}{q} I_q(\beta_{1,x}^{\otimes a_1} \otimes \cdots \otimes \beta_{m,x}^{\otimes a_m}) \\ &= -\sum_{q=d}^{\infty} \frac{1}{q} \sum_{a \in \mathcal{I}_q} c(G,a) \overline{H}_a(\xi(x)). \end{aligned}$$

This implies, taking into account that $H'_m = m H_{m-1}$, that

$$\begin{aligned} -DL^{-1}G(\xi(x)) &= \sum_{q=d}^{\infty} \sum_{a \in \mathcal{I}_q} c(G,a) \sum_{i=1}^{m} \frac{a_i}{q} H_{a_i - 1}(\xi_i(x)) \prod_{j=1, j \ne i}^{m} H_{a_j}(\xi_j(x)) \beta_{i,x} \\ &= \sum_{i=1}^{m} T_i G(\xi(x)) \beta_{i,x}. \end{aligned} \tag{4.5.9}$$

Iterating k times this procedure, we can write

$$(-DL^{-1})^k G(\xi(x)) = \sum_{i_1,\dots,i_k=1}^{m} T_{i_1,\dots,i_k} G(\xi(x)) \beta_{i_1,x} \otimes \cdots \otimes \beta_{i_k,x}. \tag{4.5.10}$$

Taking into account that $-\delta D L^{-1}$ is the identity operator on centered random variables, we obtain

$$\begin{aligned} \delta^k (-DL^{-1})^k G(\xi(x)) &= \delta^{k-1} \delta(-DL^{-1})[(-DL^{-1})^{k-1} G(\xi(x))] \\ &= \delta^{k-1} (-DL^{-1})^{k-1} G(\xi(x)). \end{aligned}$$

Iterating this relation and using equation (4.5.10), yields

$$G(\xi(x)) = \delta^k(-DL^{-1})^k G(\xi(x)) = \delta^k\left(\sum_{i_1,\dots,i_k=1}^{m} T_{i_1\dots,i_k} G(\xi(x))\beta_{i_1,x}\otimes\cdots\otimes\beta_{i_k,x}\right).$$

Then, the statement in the lemma is a consequence of equation (2.4.2). This completes the proof. □

The next result is the regularization property of the shift operator $T_{i_1,\dots,i_k}$.

Lemma 4.5.2. *Let $p \geq 2$. Suppose that $G \in L^p(\mathbb{R}^m, \phi_m)$. Then $T_{i_1,\dots,i_k}G(\xi(x))$ belongs to $\mathbb{D}^{k,p}$ for any $k \leq d$ and, moreover,*

$$\sup_{x\in\mathbb{R}^n}\sup_{1\leq i_1,\dots,i_k\leq m}\|T_{i_1,\dots,i_k}G(\xi(x))\|_{k,p} < \infty. \tag{4.5.11}$$

Proof. Because $\langle\beta_{i,x},\beta_{j,x}\rangle_{\mathfrak{H}} = \delta_{ij}$, using equation (4.5.10), we can write for any $x \in \mathbb{R}^n$,

$$T_{i_1,\dots,i_k}G(\xi(x)) = \langle (DL^{-1})^k G(\xi(x)), \beta_{i_1,x}\otimes\cdots\otimes\beta_{i_k,x}\rangle_{\mathfrak{H}^{\otimes k}}.$$

Then, by Meyer inequalities, which imply the equivalence in L^p of the operators D and $(-L)^{1/2}$, we can estimate the $\mathbb{D}^{k,p}$-norm of $T_{i_1,\dots,i_k}G(\xi(x))$ by a constant times the $L^p(\Omega)$-norm of $G(\xi(x))$. □

Let $s_i = sy_i^{1/n}$ and $S_i = [-s_i, s_i]^n$ for i=1,2. We now have

$$\begin{aligned}
&\|Z_{s,y_2} - Z_{s,y_1}\|_{L^p(\Omega)}\\
&\quad= \frac{1}{(2s)^{n/2}}\left\|\int_{S_2\setminus S_1} G(\xi(x))dx\right\|_{L^p(\Omega)}\\
&\quad= \frac{1}{(2s)^{n/2}}\left\|\int_{S_2\setminus S_1}\delta^d\left(\sum_{i_1,\dots,i_d=1}^{m} T_{i_1,\dots,i_d}G(\xi(x))\beta_{i_1,x}\otimes\cdots\otimes\beta_{i_d,x}\right)dx\right\|_{L^p(\Omega)}\\
&\quad= \frac{1}{(2s)^{n/2}}\left\|\delta^d\left(\sum_{i_1,\dots,i_d=1}^{m}\int_{S_2\setminus S_1} T_{i_1,\dots,i_d}G(\xi(x))\beta_{i_1,x}\otimes\cdots\otimes\beta_{i_d,x}dx\right)\right\|_{L^p(\Omega)}.
\end{aligned}$$

Applying Meyer inequalities (see theorem (2.4.7)), we obtain

$$
\begin{aligned}
&\|Z_{s,y_2} - Z_{s,y_1}\|_{L^p(\Omega)} \\
&\leq C_{p,d} \sum_{j=0}^{d} \frac{1}{(2s)^{n/2}} \left\| D^j \left(\sum_{i_1,\dots,i_d=1}^{m} \int_{S_2 \setminus S_1} T_{i_1,\dots,i_d} G(\xi(x)) \beta_{i_1,x} \otimes \cdots \otimes \beta_{i_d,x} dx \right) \right\|_{L^p(\Omega;\mathfrak{H}^{\otimes(j+d)})} \\
&= C_{p,d} \sum_{j=0}^{d} \frac{1}{(2s)^{n/2}} \left(\mathbb{E} \left| \sum_{i_1,\dots,i_d=1}^{m} \sum_{j_1,\dots,j_d=1}^{m} \int_{S_2 \setminus S_1} \int_{S_2 \setminus S_1} \langle D^j T_{i_1,\dots,i_d} G(\xi(x)), D^j T_{i_1,\dots,i_d} G(\xi(x)) \rangle_{\mathfrak{H}^{\otimes j}} \right. \right. \\
&\qquad \left. \left. \times r_{i_1,j_1}(x-y) \cdots r_{i_d,j_d}(x-y) dx dy \right|^{p/2} \right)^{1/p}.
\end{aligned}
$$

Now, using Minkowski's inequality and the estimate obtained in equation (4.5.11), we can write

$$
\begin{aligned}
&\|Z_{s,y_2} - Z_{s,y_1}\|_{L^p(\Omega)} \\
&\leq C_{p,d} \sup_{j=0,\dots,d} \sup_{x \in \mathbb{R}^n} \sup_{i_1,\dots,i_d=1,\dots,m} \|D^j T_{i_1,\dots,i_d} G(\xi(x))\|_{L^{p/2}(\Omega;\mathfrak{H}^{\otimes j})} \\
&\quad \times \frac{1}{(2s)^{n/2}} \left(\sum_{i_1,\dots,i_d=1}^{m} \sum_{j_1,\dots,j_d=1}^{m} \int_{S_2 \setminus S_1} \int_{S_2 \setminus S_1} |r_{i_1,j_1}(x-y) \cdots r_{i_d,j_d}(x-y)| dx dy \right)^{1/2} \\
&\quad \leq C s^{-n/2} \sum_{i,j=1}^{m} \left(\int_{S_2 \setminus S_1} \int_{S_2 \setminus S_1} |r_{i,j}(x-y)|^d dx dy \right)^{1/2}.
\end{aligned}
$$

Therefore, we finally obtain

$$
\|Z_{s,y_2} - Z_{s,y_1}\|_{L^p(\Omega)} \leq C_T |y_1 - y_2|^{1/2} \sum_{i,j=1}^{n} \left(\int_{\mathbb{R}^n} |r_{i,j}(x)|^d dx \right)^{1/2}.
$$

□

4.5.5 Proof of Lemma 3.5

Recall that $C = (c_{j,k})_{1 \leq j,k \leq m}$ is the matrix given by equation (4.3.2). For any $j \neq k$, we denote by $a^{(j,k)}$ the multiindex in $\mathcal{I}_2$ such that $a_j^{(j,k)} = 1$, $a_k^{(j,k)} = 1$ and $a_\ell^{(j,k)} = 0$ for any $\ell \neq j,k$. Also $a^{(j,j)}$ will denote the multiindex in $\mathcal{I}_2$ such that $a_j^{(j,j)} = 2$ and $a_\ell^{(j,j)} = 0$ for any

$\ell \neq j$. Then,

$$\mathcal{I}_2 = \{a^{(j,k)}, 1 \leq j, k \leq m\}.$$

Moreover, from the definition of the matrix C, it follows that for any $j, k = 1, \ldots, m$, $j \neq k$,

$$c(G, a^{(j,k)}) = c_{j,k}$$

and for all $j = 1, \ldots, m$, $c(G, a^{(j,j)}) = \frac{1}{2} c_{j,j}$. With this notation we can write

$$\begin{aligned} V^{(2)} &= \int_{\mathbb{R}^n} \mathbb{E}[G_2(\xi(0)) G_2(\xi(x))] dx \\ &= \frac{1}{4} \int_{\mathbb{R}^n} \sum_{i,j,k,\ell=1}^{m} c_{i,j} c_{\ell,k} \mathbb{E}\left[\overline{H}_{a^{(i,j)}}(\xi(0)) \overline{H}_{a^{(\ell,k)}}(\xi(x))\right] dx. \end{aligned}$$

The computation of the expectations $\mathbb{E}\left[\overline{H}_{a^{(i,j)}}(\xi(0)) \overline{H}_{a^{(\ell,k)}}(\xi(x))\right]$ depends on the indexes i, j, ℓ, k. Consider the following cases:

(i) Case $i \neq j$ and $\ell \neq k$: In this case, we have

$$\begin{aligned} \mathbb{E}\left[\overline{H}_{a^{(i,j)}}(\xi(0)) \overline{H}_{a^{(\ell,k)}}(\xi(x))\right] &= \mathbb{E}\left[\xi_i(0)\xi_j(0)\xi_\ell(x)\xi_k(x)\right] \\ &= r_{i,\ell}(x) r_{j,k}(x) + r_{i,k}(x) r_{j,\ell}(x). \end{aligned}$$

(ii) Case $i \neq j$ and $\ell = k$: In this case, we have

$$\begin{aligned} \mathbb{E}\left[\overline{H}_{a^{(i,j)}}(\xi(0)) \overline{H}_{a^{(\ell,\ell)}}(\xi(x))\right] &= \mathbb{E}\left[\xi_i(0)\xi_j(0)(\xi_\ell^2(x) - 1)\right] \\ &= 2 r_{i,\ell}(x) r_{j,\ell}(x). \end{aligned}$$

(iii) Case $i = j$ and $\ell = k$: In this case, we have

$$\begin{aligned} \mathbb{E}\left[\overline{H}_{a^{(i,i)}}(\xi(0)) \overline{H}_{a^{(\ell,\ell)}}(\xi(x))\right] &= \mathbb{E}\left[(\xi_i^2(0) - 1)^2 (\xi_\ell^2(x) - 1)\right] \\ &= 2 r_{i,\ell}(x)^2. \end{aligned}$$

As a consequence, taking into account the symmetry of the matrix C, we obtain

$$V^{(2)} = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j,k,\ell=1}^{m} c_{i,j}c_{\ell,k}r_{i,\ell}(x)r_{j,k}(x)dx = \frac{1}{2}\int_{\mathbb{R}^n} \mathrm{Tr}[r(x)Cr(x)C]dx.$$

This completes the proof of Lemma (4.3.5). $\square$

Finally, we will show formula in equation (4.3.5), assuming that the covariances are integrable. To do this, it is convenient to choose a different underlying isonormal Gaussian process. Let W denote a complex Brownian measure on $\mathbb{R}^n$ and define the isonormal process X on $L^2(\mathbb{R}^n)$ by

$$X(f) = \int_{\mathbb{R}^n} \mathcal{F}[f](t)W(dt), \tag{4.5.12}$$

where $\mathcal{F}[f]$ denotes the Fourier transform of $f \in L^2(\mathbb{R}^n)$. Recall the following properties of the Fourier transform:

$$\int_{\mathbb{R}^n} f(t)\mathcal{F}[g](t)dt = \int_{\mathbb{R}^n} \mathcal{F}[f](t)g(t)dt, \tag{4.5.13}$$

for $f, g \in L^2(\mathbb{R}^n)$ and $\mathcal{F}[\mathcal{F}[f]](x) = (2\pi)^{-n}f(-x)$.

Due to the assumption that for $1 \le i,j \le m$, $r_{i,j} \in L^1(\mathbb{R}^n)$, we have that the spectral measures ν_js of ξ_js are absolutely continuous with respect to the Lebesgue measure and hence ξ_js admit spectral densities. Denoting the spectral density of ξ_j as f_j, we have that the following representation holds (see equation 1.2.16 of [10]).

$$\xi_j(x) = \int_{\mathbb{R}^n} \mathcal{F}[\alpha_j](t-x)\, dW(t), \tag{4.5.14}$$

where $\alpha_j \in L^2(\mathbb{R}^n)$ are such that $|\alpha_j(t)|^2 = f_j(t)$. Denoting $\beta'_{j,x}(t) = e^{i\langle t,x\rangle}\alpha_j(t)$, we get that $\xi_j(x) = X(\beta'_{j,x})$ and so we have an "embedding" of the field into the isonormal process given by equation (4.5.12). Moreover, we have

$$r_{j,k}(x) = \mathbb{E}[\xi_j(x)\xi_k(0)] = \langle \beta'_{j,x}, \beta'_{k,0}\rangle_{L^2(\mathbb{R}^n)} = \mathcal{F}[\alpha_j\overline{\alpha_k}](x). \tag{4.5.15}$$

As a consequence, we can write

$$V^{(2)} = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j,k,\ell=1}^{m} c_{i,j}c_{\ell,k} r_{i,\ell}(x) r_{j,k}(x)dx$$
$$= \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j,k,\ell=1}^{m} c_{i,j}c_{\ell,k} \mathcal{F}[\alpha_i\overline{\alpha_\ell}](x)\mathcal{F}[\alpha_j\overline{\alpha_k}](x)dx.$$

By Plancherel's theorem,

$$\int_{\mathbb{R}^n} \mathcal{F}[\alpha_i\overline{\alpha_\ell}](x)\mathcal{F}[\alpha_j\overline{\alpha_k}](x)dx = \int_{\mathbb{R}^n} \mathcal{F}[\alpha_i\overline{\alpha_\ell}](x)\overline{\mathcal{F}[(\alpha_j\overline{\alpha_k})\circ \mathrm{sign}]}(x)dx$$
$$= (2\pi)^{-n}\int_{\mathbb{R}^n} \alpha_i(x)\alpha_\ell(x)\alpha_j(-x)\alpha_k(-x)dx,$$

where $\mathrm{sign}(x) = -x$. This implies

$$V^{(2)} = \frac{(2\pi)^{-n}}{2}||H||^2_{L^2(\mathbb{R}^n)},$$

where

$$H(x) = \sum_{j,k=1}^{m} c_{j,k}\alpha_j(-x)\alpha_k(x) = \alpha^T(-x)C\alpha(x). \tag{4.5.16}$$

This completes the proof of equation (4.3.5). □

Chapter 5

Further applications and conclusion

In this chapter, we will briefly describe some applications to distributional asymptotics of various geometric characteristics of excursion sets of random fields.

5.1 Geometry of random fields

We first give an brief overview of the area concerning geometry of random fields.

Random fields with parameter spaces even other than the Euclidean space as such have found their applications in a variety of settings like Neuroimaging, Oceanography and in Cosmological applications. For motivations coming from cosmological applications random fields are considered over sphere ($\mathbb{S}^2$), for example, as in the monograph by Marinucci and Peccati [28] and also a theory concerncing random fields over vector bundles, [29]. Results concerning geometry of random fields in this setting include asymptotics and limit theorems for Euler characteristic and lipschitz killing curvatures as they appear in [30] and [31] and other studies concerning critical points, as in [32]. One interesting result in this setting (invariant random fields over $\mathbb{S}^2$), as described and extended in [33] gives impossibility of simulating a non-Gaussian random field using i.i.d. coefficients in it's Fourier expansion.

For random fields on Euclidean spaces, other than asymptotics of stationary points and volume functionals as obtained in [34], [35] and [26], other topological aspects like the persistant homology of excursion sets which keeps track of appearance and disappearance of the various homology elements as the level 'u' is increased and questions of the sort concerning probability that two points lie in same connected component of an excursion set have been considered by Robert Adler et. al in [36], [37], [38] and [39]. And the behaviour of the field when the entire field lies above a certain level has been considered recent enough by Chakraborty et. al in [40].

5.1.1 Euler characteristic and number of critical points

We consider random fields on Euclidean space and the central limit theorem like behaviour of the Euler characteristic and number of critical points. The CLT for Euler characteristic was established by Estrade and Leone ([34]) and then that for number of critical points was done by Nicolaescu ([26]) using the same approach as had been done in the previous one. Here the asymptotics has been considered in the following sense.

For a centered (i.e. $\mathbb{E}[\xi(x)] = 0 \ \forall\ x$) stationary Gaussian random field ξ on $\mathbb{R}^n$, if Z_s^u denotes the Euler characteristic or number of critical points of the field in $A_u \cap [-s,s]^n$ where $A_u = \{x \in \mathbb{R}^n : \xi(x) \geq u\}$ denotes the excursion set.

$$\frac{Z_s^u - \mathbb{E}[Z_s^u]}{(2s)^{n/2}} \Rightarrow \mathcal{N}(0, \sigma_u^2)$$

as $s \to \infty$, where $\Rightarrow$ denotes convergence in law and $\mathcal{N}$ denotes the normal distribution. The above two are proved by using the following integral representation of the number of critical points which holds true in the deterministic case and hence in almost sure sense (assuming ξ is at least almost surely $\mathcal{C}^2$). We state the general result (Theorem 11.2.3 of [11]). The below result is also known as Kac-Rice formula. We denote by $N_T(f, u; g, B)$ the number of times $f(x) = u$ and $g(x) \in B$ for $x \in T$.

Theorem 5.1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and $g : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Let $T \subset \mathbb{R}^n$ be closed and $B \subset \mathbb{R}^m$ be open. Assume that for every $x \in \partial T$, $f(x) \neq u$ and there exists no $x \in T$ satisfying both $f(x) = u$ and $g(x) \in \partial B$ or $\det \nabla f(x) = 0$. Then,*

$$N_T(f,u;g,B) = \lim_{\epsilon\to 0}\int_T \delta_\epsilon(f(x)-u)\boldsymbol{1}_B(g(x))|\det\nabla f(x)|dx. \tag{5.1.1}$$

where $\delta_\epsilon(x) = (2\epsilon)^{-n}\boldsymbol{1}_{[-\epsilon,\epsilon]^n}(x)$.

With this we would have that the number of critical points N_s^u in the set $A_u \cap [-s,s]^n$ of an almost surely $\mathcal{C}^2$ Gaussian (or non-Gaussian) random field ξ on $\mathbb{R}^n$ is given by the random variable (almost surely)

$$N_s^u = \lim_{\epsilon\to 0}\int_{[-s,s]^n} \delta_\epsilon(\nabla\xi(x))\mathbf{1}_{[u,\infty)}(\xi(x))|\det\nabla^2(\xi(x))|dx, \tag{5.1.2}$$

where $\nabla^2(\xi(t))$ denotes the Hessian of ξ and the convergence is in almost sure sense.

Using this formula, one can define a modified Euler characteristic μ_s^u of the set $A_u\cap[-s,s]^n$ as $\mu_s^u = \sum_{i=1}^n(-1)^i N_{s,i}^u$ where $N_{s,i}^u$ denotes the number of critical points of the field in the set $A_u\cap[-s,s]^n$ of index 'i'. We recall that the index of a critical points is given by the number of negative eigenvalues of the Hessian of the function at that point. With this we get that almost surely (notice the disappearance of the mod)

$$\mu_s^u = (-1)^n\lim_{\epsilon\to 0}\int_{[-s.s]^n} \delta_\epsilon(\nabla\xi(x))\mathbf{1}_{[u,\infty)}(\xi(x))\det\nabla^2(\xi(x))dx, \tag{5.1.3}$$

where convergence is in almost sure sense.

After considering the above representations, one fundamental question is to know whether N_s^u and μ_s^u have finite variance. This question of determining the admissibility of moments by these random variables itself has rich history in literature which we refrain from discussing here. We point the reader to the original paper [23] for some discussion.

Estrade and Leon in [23] show that the convergence in equations (5.1.2) and (5.1.3) also holds in $L^2(\Omega)$ under the condition that almost every sample path of the field is of class $\mathcal{C}^3$ (Proposition 1.1 of [23]). They then consider the $m = 1+n+n(n+1)/2$ dimensional vector field $\widetilde{\xi}(x) = (\xi(x),\nabla\xi(x),\nabla^2(\xi(x))$ and the function $G_\epsilon(x,y,z) = \delta_\epsilon(y)\mathbf{1}_{[u,\infty)}(x)\det\nabla^2(z)$ where $x\in\mathbb{R}, y\in\mathbb{R}^n$ and $z\in\mathbb{R}^{n(n+1)/2}$. We have that the function G_ϵ is square integrable with respect to the standard Gaussian measure on $\mathbb{R}^m$ and hence admits a Hermite expansion. With this, one can prove that N_s^u or μ_s^u admits an expansion in $L^2(\Omega)$ of the form (Proposition

1.3 of [23])

$$\mu_s^u = (-1)^n \sum_{q=1}^{\infty} \sum_{|a|=q} c(a) \int_{[-s,s]^n} \overline{H}_a(\widetilde{\xi}(x))dx. \tag{5.1.4}$$

Here due to the dirac-delta function in formula given in equation (5.1.3), the coefficiants $c(a)$ are such that $\sum_{q=1}^{\infty} \sum_{|a|=q} c(a)^2 = \infty$ and hence a central limit theorem for μ_s^u cannot directly be concluded by means of the Breuer-Major theorem (4.3.2). To show the central limit theorem for μ_s^u, after an intermediatery step which consists in showing that the asymptotic variance of μ_s^u is finite, it suffices to show a central limit theorem for finitely many terms in the expansion in equation (5.1.4) which indeed follows from theorem (4.3.2). We refer the reader to the paper [23] for complete details.

5.1.2 Remarks

We note that from theorem (4.3.2), we obtain a central limit theorem for volume of excursion set by taking the function G to be $G(x) = \mathbf{1}_{[u,\infty)}(x)$. Also, in this case the functional covergence also holds. In [41], the author obtains rates of convergence for the central limit theorem for volume of excursion sets by means of Malliavin calculus.

5.2 Conclusion and future directions

In this thesis, we studied distributional limit theorems by means of Malliavin calculus and Stein's method. Using these tools, we had given a proof of the Breuer-Major theorem for the case of vector valued fields on a Euclidean space of arbitrary dimension. We also proved that the functional convergence holds under mild extra assumptions on the function. We then saw how this is used to obtain some limit theorems for geometric characteristics excursion sets of random fields.

One thing that we had not pursued in this thesis was to obtain rates in the Breuer-Major theorem in the case of random fields. In the discrete case for sequences of vectors, this was done by Nourdin, Peccati and Podolskij in [6]. We wish to pursue this in future.

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