Intersection Theory in Algebraic Geometry

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

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November, 2018

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Certificate

This is to certify that this dissertation entitled Intersection Theory in Algebraic Geometry towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Abhishek Gupta at Indian Institute of Science Education and Research under the supervision of Dr. Vivek Mohan Mallick, Assistant Professor, Department of Mathematics, during the academic year 2017-2018.

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This thesis is dedicated to my family, teachers and my friends

Declaration

I hereby declare that the matter embodied in the report entitled Intersection Theory in Algebraic Geometry are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Vivek Mohan Mallick and the same has not been submitted elsewhere for any other degree.

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Acknowledgments

At first, I want to thank my supervisor Dr. Vivek Mohan Mallick to let me work under his supervision. He has been a nice guide for pointing out my mistakes and being supportive in the time of tense situation. It was a nice experience to explore the ideas of geometry into terms of algebra.

Second, I would like to thank my TAC member Prof. Amit Hogadi being helpful to me throughout my project and guiding me in the right direction.

Third, I would like to thank phd students to help me understanding the concepts and problems throughout the project.

At last, I would like to thank to my family to keep me motivated.

Abstract

In the algebraic geometry, we solve the geometrical problems using the algebraic technique, mainly commutative algebra. Here I have included the basic discussion of Noetherian ring and Hilbert basis theorem. After that I start by understanding some basic notations in algebraic geometry such as varieties, morphism between varieties and its some properties. Along this way, we study about the plane algebraic curve and look for the singularities and its intersection multiplicities at intersection point that is given by Bézout's theorem which shows how it depends on degree of curves. Conveniently, we correlate it with the local ring to acquire good understanding. Along this way, the last section brings us to presheaves and sheaves with its some properties such as morphism defined between them, stalks and universal property of sheafification. Here I will always be working over an algebraically closed field.

Contents

Introduction

My curiosity in this project was how to study geometry in the perspective of algebra. In the algebraic geometry, there are many intuitive ideas to understand the properties of curves in different spaces and their behaviour in perspective of locally and globally. The aim to do project in algebraic geometry to get understanding of its applications in the area of computational Mathematics. I was fascinated by affine space and its local behaviour describing the geometry in terms of commutative algebra. First We have started reading the basic portions such as Noetherian ring, Nullstellansatz theorem, Hilbert basis theorem in algebraic geometry to understand the higher concept. After we define varieties in affine and projective space and its applications which has been required further to prove other theorems. Then we proceed to morphisms of varieties there we see how it relates to ring homomorphism between k-algebras. We define rational maps and nonsingular varieties with its properites. After that we study the plane algebraic curves and its singularities and methods to find multiplicities of curve at point and study its local behaviour such as local ring etc. Then we go through concept of intersection multiplicities between curves. We introuduce singularities, multiplicities and the main part of whole picture, the intersection number in global situation which we find out by B$\acute{e}$zout's theorem. The last chapter is about sheaf and its morphism is used as application to define other properties. In the below, k is assumed algebraically closed field.

Chapter 1

Preliminaries

Proposition 1.0.1. *Let R be a ring, then the following conditions are equivalent:*

1. *R satisfies the ascending chain conditions on ideals.*

2. *Every nonempty family of ideals in R has a maximal element.*

3. *Every ideal of R has finite generating set.*

   A ring R satisfying any of these conditions is called Noetherian ring.

Exercise 1. Let R be a ring with ideals I, J and K then prove that

1. $I(J + K) = IJ + IK$
2. $I \cap (J + K) \supseteq (I \cap J) + (I \cap K)$
3. If $J \subseteq I$ then equality holds in (2).

Exercise 2. Assume R is a ring with an ideal I. Show that in the canonical projection map given by $\pi : R \longrightarrow R/I$ sending r to $r + I$, there is one to one correspondence between set of ideals of R containing I and set of ideals of R/I.

Corollary 1.0.2. *If R is a Noetherian ring with an ideal I then prove that R/I is also Noetherian ring.*

Theorem 1.0.3. ***(Hilbert basis theorem)*** *For a Noetherian ring R, $R[X_1, \ldots, X_n]$ is also Noetherian ring.*

Proof. First, we prove it for one variable, say $R[X_1]$ by showing that every ideal $I \in R[X_1]$ is finitely generated. Then by induction on number of variables, we prove $R[X_1, \ldots, X_n]$ is a Noetherian ring. $\square$

Theorem 1.0.4. *Let A be an integral domain which is a finitely generated k-algebra where k is a field then*

1. *The dimension of A is equal to the transcendence degree of the fraction field $K(A)$ of A over k.*

2. *If p is a prime ideal of A, then $ht(p) + \dim A/p = \dim A$*

Chapter 2

Affine and Projective Space

2.1 Affine Varieties

Here k is assumed algebraically closed field.

An affine space is a set of points with no distinguished point. For given a vector space V, its associated affine space A is obtained by adding each $v \in V$ with a fixed point p in A. An affine n-space over k is denoted by $\mathbb{A}_k^n$.

Definition 2.1.1. An subset of $\mathbb{A}_k^n$ is called an affine algebraic set if it defined as $V(S) = \{p \in \mathbb{A}_k^n \mid f(p) = 0 \ \forall f \in k[X_1, \ldots, X_n]\}$ for a set S of polynomials in $k[X_1, ..., X_n]$.

Definition 2.1.2. An affine n-space over k is a topological space with Zariski topology formed by algebraic sets satisfying axioms for closed set.

Remark 2.1.1. Let $k[X_1, \ldots, X_n]$ be denoted by R. Assume S be set of polynomials in R then $V(S) = V(I)$ for some ideal $I \subseteq R$ where $S \subseteq I$. Since R is a Noetherian ring so I is generated by finitely many polynomials.

Remark 2.1.2. Every affine algebraic set is in one to one correspondence with radical ideal. There is inclusion-reversing between them.

Definition 2.1.3. An affine algebraic set is called an affine variety if it can't be written as union of two nonempty proper affine algebraic sets. An open subset of affine variety is called a quasi-affine variety.

Theorem 2.1.1. *An algebraic set V is an affine variety in $\mathbb{A}_k^n$ iff $I(V)$ is a prime ideal in $k[X_1, \ldots, X_n]$.*

Remark 2.1.3. There is one to one correspondence between left side in $\mathbb{A}_k^n$ and right side in $R = k[X_1, \ldots, X_n]$:

$$\{Affine\ algebraic\ set\} \longleftrightarrow \{Radical\ ideals\ in\ R\}$$
$$\{Affine\ Varieties\} \longleftrightarrow \{Prime\ ideals\ in\ R\}$$
$$\{Points\} \longleftrightarrow \{Maximal\ ideals\ in\ R\}$$

There is inclusion reversing between them.

Proposition 2.1.2. *Let F and G be two affine curves in $k[X, Y]$ s.t. they don't have common factor then $V(F, G) = V(F) \cap V(G)$ is a finite set.*

Proof. Given F and G don't contain common components in $k[X, Y] \cong k[X][Y]$ then they also are coprime to eachother in $k(X)[Y]$. Since $k(X)[Y]$ is a Euclidean domain. It follows that $\gcd(F, G) = 1$, so there exists some $C, D \in k(X)$ s.t. $CF + DG = 1$. To clear the denominator part, choose $H \in k[X]$ s.t. $HC = C'$ and $HD = D'$ where C' and $D' \in k[X][Y]$ then we get $C'F + D'G = H$. If $p \in V(F, G)$ then $H(p) = C'(p)F(p) + D'(p)G(p) = 0$. Since H has only finite number of zeros which gives finitely many possible values for X coordinate in $V(F, G)$. Similarly it follows there can be only finitely many possible values for Y coordinate in $V(F, G)$. Thus $V(F, G)$ is a finite set. □

Theorem 2.1.3. *Prove that all the maximal ideal in $k[X_1, \ldots, X_n]$ are of the form $\langle X_1 - p_1, \ldots, X_n - p_n\rangle$ for some $p_i's \in k$.*

Theorem 2.1.4. *Every algebraic set V in $\mathbb{A}_k^n$ can be decomposed into finite union of affine varieties uniquely up to order.*

Proof. (Existence)

We prove it by contradiction. Suppose there is a set S of nonempty algebraic set of $\mathbb{A}_k^n$ that can't be written as a finite union of irreducible algebraic sets and $S \neq \emptyset$. Since $\mathbb{A}_k^n$ is noetherian topological space, so $\exists$ a minimal element, say X_0. Note that X_0 is not irreducible then it can be decomposed into $X_1 \cup X_2$ where X_1 and X_2 are nonempty algebraic set which contradicts the minimality of S thus $S = \emptyset$.

(Uniqueness)

Let X can be written into two decompositions, say $X = X_1 \cup \ldots \cup X_n$ and $X = Y_1 \cup \ldots \cup Y_m$. Consider $X_i = X \cap X_i = \cup_{j=1}^{m}(Y_j \cap X_i)$ since X_i is irreducible it implies $X_i \subseteq Y_j$ for some j. Similarly, on the other hand $Y_j \subseteq X_k$ for some k it follows that $X_i = Y_j = X_k$ for $i = k$. Removing X_i, Y_j and continue the process on the rest and it will end finite times because of $n \neq m$ then on one side, no set will be remained and on the other side there will be left nonempty set, which is not possible. □

2.2 Dimension of Algebraic Set

Definition 2.2.1. The Krull dimension of a ring R is the length of maximal chain of prime ideals in R. For any prime ideal P in R, its codimension or height is length of maximal chain of prime ideals contained in P.

For example, $\{0\} \subset \langle X_1 - a_1 \rangle \subset \langle X_1 - a_1, X_2 - a_2 \rangle \subset \langle X_1 - a_1, X_2 - a_2, X_3 - a_3 \rangle$ is a maximal chain of prime ideals in $k[X_1, X_2, X_3]$ for any point $(a_1, a_2, a_3) \in k^3$ so its dimension is 3.

Proposition 2.2.1. *Let V be an algebraic set in $\mathbb{A}_k^n$, then $\dim V$ is defined as dimension of affine coordinate ring of V.*

Proposition 2.2.2. *Let V be a quasi-affine variety then $\dim V = \dim \overline{V}$.*

2.3 Projective Varieties

Definition 2.3.1. Assume V is a vector space over k then projectivization of V is the set of all 1-dimensional linear subspaces of V. It is denoted by $\mathbb{P}(V)$. If $V = k^n$, then $\mathbb{P}(k^n) = \mathbb{P}_k^{n-1}$ is called projective $(n-1)$-space over k.

$\mathbb{P}_k^n$ is also considered as set of equivalence classes of points $(X_0, \ldots, X_n)$ in k^{n+1} denoted as $[X_0 : \cdots : X_n]$ with the equivalence relation $\{\sim\}$ such that $(X_0, \ldots, X_n) \sim (Y_0, \ldots, Y_n) \Leftrightarrow X_i = \lambda Y_i$ for some $\lambda \in k \setminus \{0\}$ $\forall i$. We call $X_0, \ldots, X_n$, the projective or homogeneous co-ordinates of point $[X_0 : \ldots : X_n]$ where X_i's are not all zero.

$\mathbb{P}_k^n$ makes a topological space by defining Zariski topology, taking closed sets as projective algebraic set.

Let $U_i = \{[X_0 : \ldots : X_i : \ldots : X_n] \in \mathbb{P}_k^n \mid X_i \neq 0\}$. Since it is an open set in $\mathbb{P}_k^n$ because of being complement of zeros of X_i $\forall\, i = 0, \ldots, n$ where each point in U_i can be represented as $[\frac{X_0}{X_i} : \ldots : \frac{X_{i-1}}{X_i} : 1 : \frac{X_{i+1}}{X_i} : \ldots : \frac{X_n}{X_i}]$

Define the map $\phi_i : U_i \longrightarrow \mathbb{A}_k^n$ sending $[\frac{X_0}{X_i} : \ldots : 1 : \ldots : \frac{X_n}{X_i}]$ to $(\frac{X_0}{X_i}, \ldots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \ldots, \frac{X_n}{X_i})$, ϕ_i is well defined for each i and we can see it is a bijective. $\mathbb{P}_k^n$ is covered by open sets $U_i's$ because if $p \in \mathbb{P}^n$, the there is at least one $X_i \neq 0$ in p that implies $p \in U_i$. Hence projective space has an open covering by affine n-space and topology on $\mathbb{P}_k^n$ is formed by glueing topology on each set U_i which comes from by glueing topology on each set U_i which comes from Zariski topology on affine n-space.

Let H_i be $\{[X_0 : \ldots : X_n] \mid X_i = 0\} = \mathbb{P}_k^n \backslash U_i$. It is a hyperplane which shows isomorphism with $\mathbb{P}_k^{n-1}$ via this map $\psi_i : H_i \longrightarrow \mathbb{P}_k^{n-1}$ sending $[X_0 : \ldots : X_{i-1} : 0 : X_{i+1} : \ldots : X_n]$ to $[X_0 : \ldots : X_{i-1} : X_{i+1} : \ldots : X_n]$ thus $\mathbb{P}_k^n = U_i \cup H_i \cong \mathbb{A}_k^n \cup \mathbb{P}_k^{n-1}$. In particular, we often denote H_{n+1} as H_∞ and it s called the hyperplane at infinity.

For example, $\mathbb{P}_k^1 = \mathbb{A}_k^1 \cup \mathbb{P}_k^0 = \mathbb{A}_k^1 \cup \{\infty\}$ where $\mathbb{A}_k^1$ is affine part of $\mathbb{P}_k^1$ consists of $[1 : x_1]$ for all $x_1 \in k$ and $\mathbb{P}_k^0$ is infinite part of $\mathbb{P}_k^1$ or called the points at infinity. Similarly, $\mathbb{P}_k^2 = \mathbb{A}_k^2 \cup \mathbb{P}_k^1$ where $\mathbb{P}_k^1$ is referred as line at infinity.

Definition 2.3.2. A subset X of $\mathbb{P}_k^n$ is called projective algebraic set if it is zeros of some finite set of homogeneous polynomials.

Proposition 2.3.1. *The map defined above $\phi_i : U_i \longrightarrow \mathbb{A}_k^n$ is a homeomorphism with induced topology defined on U_i and Zariski topology on $\mathbb{A}_k^n$*

Proof. Since ϕ_i is bijective, we just need to show that image of closed set in U_i under ϕ_i is a closed set. Let $R = k[Y_0, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n]$ and S^h be set of all homogeneous polynomials in $k[X_0, \ldots, X_n]$.

We define a map $\alpha_i : S^h \longrightarrow R$ takes $f(X_0, \ldots, X_i, \ldots, X_n)$ to $f(Y_0, \ldots, Y_{i-1}, 1, Y_{i+1}, \ldots, Y_n)$ and a map $\beta_i : R \longrightarrow S^h$ takes a polynomial $f(Y_0, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$ of degree d to ${X_i}^d f(\frac{X_0}{X_i}, \ldots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \ldots, \frac{X_n}{X_i})$.For any closed subset $V \in U_i$ its projective closure $\overline{V}$ in $\mathbb{P}^n$ is zeros of some finite set S in S^h. Let $\alpha(S) = S'$ then it is easy to see that $\phi_i(V) = Z(S')$.

Conversely, consider closed set $W \in \mathbb{A}^n(k)$, then $W = Z(T')$ where $T' \in R$. Let $\beta(T') = T$, then ${\phi_i}^{-1}(W) = Z(\beta(T')) \cap U_i$ is a closed set. Hence, ϕ_i, ${\phi_i}^{-1}$ are closed maps, therefore ϕ_i is homeomorphism. □

Definition 2.3.3. Let R be $k[X_0, \ldots, X_n]$ and for any subset $Y \in \mathbb{P}^n_k$, $I(Y)$ in R is generated by $\{f \in R \mid f\, is\, homogeneous\, and\, f(p) \neq 0\, \forall\, p \in Y\}$

Definition 2.3.4. A projective variety is an irreducible algebraic set in $\mathbb{P}^n_k$ with induced topology.

Lemma 2.3.2. *For an ideal I in $k[X_0, \ldots, X_n]$, the following are equivalent:*

1. *I has a finite generating set of homogeneous polynomial.*
2. *For every $f \in I$, its all homogeneous component of f is in I.*

   An ideal satisfying any of these is called homogeneous ideal.

Proof. $(i) \Rightarrow (ii)$ Let I be $\{f_1, \ldots, f_s\}$ where $f_i's$ are homogeneous. For any $f \in \{f_1, \ldots, f_s\}$, express $f = g_1 f_1 + \ldots + g_s f_s$ with $g_i's$ in $k[X_0, \ldots, X_n]$ then j^{th} form of f can be written as ${g_1}^{j-deg(f_1)} f_1 + \ldots + {g_s}^{j-deg(f_s)} f_s$ which implies j^{th} form of f belongs to I for all $j \geq 0$.

$(ii) \Leftarrow (i)$ Since $k[X_0, \ldots, X_n]$ is Noetherian, I has a finite generating set, say $\{f_1, \ldots, f_s\}$. Suppose each f_i has deg d_i and f_{ij} be j^{th} form for all $j = 0, \ldots, d_i$. Since $f_i \in \{f_{i0}, \ldots, f_{id_i}\}$ but from condition (2), $\{f_{i0}, \ldots, f_{id_i}\} \subseteq I$, similarly it follows for other $f_i's$. It implies that $I = \{f_1, \ldots, f_s\} \subseteq \{f_{10}, \ldots, f_{1d_1}, \ldots, f_{s0}, \ldots, f_{sd_s}\} \subseteq I$. Thus, I can be generated by forms. □

Remark 2.3.1. Every projective algebraic set is in one to one correspondence with homogeneous radical ideal of S (polynomial ring) not equal to $S_+ = \bigoplus_{d>0} S_d$ where S_d is set of all homogeneous polynomials of degree d. There is inclusion-reversing between them.

Remark 2.3.2. Let F be a projective curve in $k[X_0, \ldots, X_n]$, then its affine set of points is given by $V_{\mathbb{A}}(F(X_0 = 1))$, and we define affine part of F as F_h which is dehomogenization of F under $X_0 = 1$ and the points at infinity are given by $V_{\mathbb{P}}(F(X_0 = 0))$

Remark 2.3.3. Let F be a projective curve in $K[X_0, \ldots, X_n]$, then its projective closure is defined as homogenization of F under X_0, denoted by F^h.

Definition 2.3.5. A subset U of $\mathbb{P}_k^n$ is called projective subspace if there exist a linear subspace U' in k^{n+1} such that $U = \mathbb{P}(U')$. Moreover, U is zeros of set S of homogeneous linear polynomials in $k[X_0, \ldots, X_n]$ denoted as $U = V_{\mathbb{P}}(S)$(projective algebraic set) where $U' = V_{\mathbb{A}}(S)$(affine algebraic set) and since U' is of finite dimension, implies that $\dim U = \dim U' - 1$.

For example, let $U' = V_{\mathbb{A}}(X_0 - X_1 - X_2)$ in $\mathbb{A}_k^3$, then its projectivization U is $\mathbb{P}(U') = V_{\mathbb{P}}(X_0 - X_1 - X_2) \in \mathbb{P}_k^2$.

2.4 Morphism between Varieties

Definition 2.4.1. For any quasi-affine variety V in $\mathbb{A}_k^n$, a function $f : V \longrightarrow k$ is called a regular at a point p on V if there is an open neighborhood W around p on V and the polynomials f_1,f_2 in $k[X_1, \ldots, X_n]$ s.t. $f_2(W) \neq 0$ and f is in the quotient form $\frac{f_1}{f_2}$ on W.

Definition 2.4.2. For any quasi-projective variety V in $\mathbb{P}_k^n$, a function $f : V \longrightarrow k$ is called a regular at a point p on V if there is an open neighborhood W around p on V and homogeneous polynomials f_1, f_2 in $k[X_0, \ldots, X_n]$ of the same degree s.t. $f_2(W) \neq 0$ and f is in the quotient form $\frac{f_1}{f_2}$ on W.

Definition 2.4.3. If V and W are varieties (affine, quasi-affine, projective or quasi-projective) then a morphism of varieties $\phi : V \longrightarrow W$ is a continuous map s.t. for every open set $U \subseteq W$ and for every regular function f on U, the function $f \circ \phi : \phi^{-1}(U) \longrightarrow k$ is regular.

Remark 2.4.1. Every morphism between two affine varieties $\phi : V \longrightarrow W$ induces a ring homomorphism $\phi^* : k[W] \longrightarrow k[V]$ called as pullback of ϕ.

Definition 2.4.4. A morphism between two varieties $\phi : V \longrightarrow W$ is called a isomorphism if there is an inverse morphism $\psi : W \longrightarrow V$ with $\psi \circ \phi = id_V$ and $\phi \circ \psi = id_W$. An isomorphism is a bijective and bicontinuous but a bijective and bicontinuous morphism need not be an isomorphism.

Proposition 2.4.1. *If V and W are affine varieties of $\mathbb{A}^n(k)$ and $\mathbb{A}^m(k)$ respectively and a k-algebra homomorphism $\psi : k[W] \longrightarrow k[V]$ is given, then there exists a unique polynomial map $\phi : V \longrightarrow W$ such that ϕ^*(pullback of ϕ) $= \psi$.*

Proof. Assume $k[V] = k[X_1, \ldots, X_n]/I(V)$ and $k[W] = k[Y_1, \ldots, Y_m]/I(W)$. Define $\varphi : k[Y_1, \ldots, Y_m] \longrightarrow k[V]$ and k-algebra homomorphism $\psi : k[W] \longrightarrow k[V]$ satisfying $\varphi(f) = \psi(f+I(W))$. Since φ is a k-algebra homomorphism because $\varphi = \psi \circ \theta$ where $\theta : k[Y_1, \ldots, Y_m] \longrightarrow k[W]$. Let $\varphi(Y_i) = \bar{g}_i$ for all $i = 1, \ldots, m$, then $\varphi(Y_i) = \psi(Y_i + I(W)) = g_i + I(V)$. For any polynomial f,

$$\varphi(f) = \psi \circ \theta(f) = \psi(f+I(W)) = f(g_1, \ldots, g_m)+I(V) = f(\varphi(Y_1), \ldots, \varphi(Y_m)) = \varphi(f(Y_1, \ldots, Y_m))$$

Define a polynomial map $\phi : V \longrightarrow \mathbb{A}^m$ if φ is restricted to range W, we show that $\phi(V) \subseteq W$. Let $h \in I(W)$

$$\begin{aligned}
&\varphi(h) = h(g_1, \ldots, g_m) + I(V) = \psi \circ \theta(h) = \psi(0) = 0 + I(V) \\
&\Rightarrow \varphi(h) = h(g_1, \ldots, g_m) \in I(V) \\
&\Rightarrow h(g_1(p), \ldots, g_m(p)) = 0 \ \forall p \subseteq V \\
&\Rightarrow (g_1(p), \ldots, g_m(p)) \in W \\
&\Rightarrow \phi(V) \subseteq W
\end{aligned}$$

Thus ϕ is a polynomial map from V to W and

$$\begin{aligned}
\phi^*(f + I(W)) &= f \circ \phi + I(V) \\
&= f(g_1, \ldots, g_m) + I(V) \\
&= \psi(f + I(W)) \\
\Rightarrow \phi^* &= \psi
\end{aligned}$$

Uniqueness follows easily, let ϕ_1 and ϕ_2 be two polynomial maps defined by $\phi_1 = (g_1, \ldots, g_m)$ and $\phi_2 = (h_1, \ldots, h_m)$ where g_i, $h_j \in k[V]$. So for any polynomial $f \in k[W]$

$$\begin{aligned}
\psi f &= \phi_1^* f = f \circ \phi_1 \\
\psi f &= \phi_2^* f = f \circ \phi_2 \\
\Rightarrow f \circ \phi_1 &= f \circ \phi_2 \\
f(g_1, \ldots, g_m) &= f(h_1, ..., h_m)
\end{aligned}$$

Let $f = Y_i$ then $Y_i(g_1, \ldots, g_m) = Y_i(h_1, \ldots, h_m) \Rightarrow g_i = h_i \ \forall i \Rightarrow \phi_1 = \phi_2$ $\square$

Theorem 2.4.2. *If V is an affine variety in $\mathbb{A}_k^n$ and $A(V)$ is its affine coordinate ring then*

1. $\mathcal{O}(V) \cong A(V)$

2. *There is one to one correspondence between points of* V *and maximal ideals of* $A(V)$.

3. *For each point* p, $\mathcal{O}_{p,V} \cong A(V)_{m_p}$ *and* $\dim \mathcal{O}_{p,V} = \dim V$.

4. *The Function field* $K(V)$ *of* V *is isomorphic to quotient field of* $A(V)$ *and hence* $K(V)/k$ *is a finitely generated field extension of transcendence degree equal to* $\dim V$.

Proof. We define a map $\alpha : k[X_1, \ldots, X_n] \longrightarrow \mathcal{O}(V)$. Since every polynomial in $k[X_1, \ldots, X_n]$ is regular on $\mathbb{A}_k^n$ and hence on V, then $\ker \alpha = I(V)$ therefore $\alpha' : A(V) \longrightarrow \mathcal{O}(V)$ is injective homomorphism. In the map α, there is a one to one correspondence between ideal containing $I(V)$ in $k[X_1, \ldots, X_n]$ and ideal in $A(V)$, so there is a one to one correspondence between maximal ideals containing $I(V)$ and maximal ideals in $A(V)$ which is again in one to one correspondence with points of V, which proves (2).

For each point p in V, there is a natural map $A(V)_{m_p} \longrightarrow \mathcal{O}_{p,V}$ and it is surjective by definition. Since α' is injective this implies it is injective. Thus $A(V)_{m_p} \cong \mathcal{O}_{p,V}$. By this isomorphism, $\dim \mathcal{O}_{p,V}$ =ht m_p. By theorem 1.0.4(2), ht $m_p + \dim A(V)/m_p = \dim V$ and hence $\dim \mathcal{O}_{p,V} = \dim V$ which proves (3).

Since $K(A(V)) = K(A(V)_{m_p}) = K(\mathcal{O}_{p,V})$ for each point $p \in V$ and $K(\mathcal{O}_{p,V}) = K(V)$. $A(V)$ is finitely generated k-algebra therefore $K(V)/k$ is a finitely generated field extension. By theorem 1.0.4(1), transcendence degree of $K(V)/k = \dim V$. This proves (4).

To prove (1), since $\mathcal{O}(V) \subseteq \mathcal{O}_{p,V}$ for each point $p \in V \Rightarrow \mathcal{O}(V) \subseteq \cap_{p\in V}\mathcal{O}_{p,V}$.

$$from\,(3) \quad \mathcal{O}(V) \subseteq \cap_{p\in V}\mathbb{A}(V)_{m_p}$$

$$from\,(2) \quad \mathbb{A}(V) \subseteq \mathcal{O}(V) \subseteq \cap_{p\in V}\mathbb{A}(V)_{m_p}$$

Since $A(V)$ is an integral domain then $\mathbb{A}(V) = \cap_{p\in V}\mathbb{A}(V)_{m_p}$. Thus $A(V) \cong \mathcal{O}(V)$. □

Theorem 2.4.3. *If* V *is an projective variety in* $\mathbb{P}_k^n$ *and* $S(V)$ *is its homogeneous coordinate ring then*

1. $\mathcal{O}(V) \cong k$

2. *For each point $p \in V, \mathcal{O}_{p,V} = S(V)_{(m_V(p))}$ where $m_V(p)$ is a maximal ideals of $S(V)$ corresponding to p.*

3. *Function field $K(V)$ of V is isomorphic to localization $S(V)_{((0))}$ of $S(V)$ at zero ideal.*

Proof. In previous $U_i \subseteq \mathbb{P}_k^n$ is open set defined as $X_i \neq 0$ for each $i = 0, 1 \ldots, n$. U_i is isomorphic to $\mathbb{A}_k^n$ via ϕ_i. Let $\phi_i(V \cap U_i)$ be V_i where V_i is affine variety by the isomorphism ϕ_i. We define a map $\theta : k[Y_1, \ldots, Y_n] \longrightarrow k[X_0, \ldots, X_n]_{(X_i)}$ by sending $f(Y_1, \ldots, Y_n)$ to $f(\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i})$ leaving out X_i/X_i. we claim that θ is an isomorphism. This can be simply checked. Consider an ideal $I(V_i) \subset k[Y_1, \ldots, Y_n]$ which will go to $I(\overline{V_i})S_{(X_i)}$ the same as $I(V)S_{(X_i)}$ via θ. So passing to the quotient we get $\theta' : A(V_i) \cong S(V)_{(X_i)}$.

Let $p \in V$ then we choose such i s.t. $p \in U_i$ and let $\phi_i(p) = p'$ then we know $A(V_i)_{(M_{V_i}(p'))} \cong \mathcal{O}_{p',V_i}$ where $M_{V_i}(p')$ is a maximal ideal of $A(V_i)$ corresponding to p'. Let $M_{p'}$ and m_p be the maximal ideals of $k[Y_1, \ldots, Y_n]$ and $k[X_0, \ldots, X_n]$ corresponding to $p' \in V_i$ and $p \in V$ respectively. Since $\theta(M_{p'}) = m_p S_{(X_i)}$

$$\theta'(M_{p'}/I(V_i)) = m_p/I(V)S(V)_{(X_i)}$$

$$\theta'(M_{V_i}(p')) = m_V(p)S(V)_{(X_i)}$$

Now we know X_i does not belong to $m_V(p)$ and localization is transitive. so we get $A(V_i)_{(M_{V_i}(p'))} \cong (S(V)_{(X_i)})_{(\theta'(M_{V_i}(p')))} \cong S(V)_{(m_V(p))}$. This proves (2).

Since $K(V) \cong K(V \cap U_i) \cong K(V_i) \cong K(A(V_i))$ and $\theta' : A(V_i) \cong S(V)_{(X_i)}$. So we get $K(V) = K(S(V)_{(X_i)}) \cong S(V)_{((o))}$, which proves (3).

Let f be a regular function on V_i. Since $\phi_i : V \cap U_i \cong V_i$ for each i and $V = \cup_{i=0}^n V \cap U_i$ then f is regular on V_i for each i. Since $\mathcal{O}(V_i) \cong A(V_i) \cong S(V)_{(X_i)}$ so we deduce that there exists q_i s.t. $X_i^{q_i} f \in S(V)$ for each i. This means $X_i^{q_i} f \in S_{q_i}(V)$ where $S_{q_i}(V)$ consists of all homogeneous polynomials of degree q_i in $S(V)$. Put $q \geq \sum_i q_i$ then $S_q(V)$ spans as vector space by monomials of degree q in $X_0, \ldots, X_n$ and we can find at least one monomials in X_i of degree$\geq q_i$. So we get $S_q(V)f \subseteq S_q(V)$

$$since, \;\; S_q(V)f \subseteq S_q(V)$$

$$S_q(V)f^2 \subseteq S_q(V)$$

by iterating, we have $S_q(V)f^r \subseteq S_q(V)$ for all $r > 0$. For any i, $X_i^q f^r \subseteq S_q(V)$ for all $r > 0 \Rightarrow S(V)[f] \subset X_i^{-q}S(V) \subset K(S(V))$. $X_i^{-q}S(V)$ is the $S(Y)$-module of $K(S(Y))$ generated by X_i^q which implies that it is finitely generated. Since $S(V)$ is Noetherian. $S(V)[f]$ is finitely generated as a module over $S(V)$. By the property of integral domain, f is integral over $S(V) \Rightarrow$ there exists a monic polynomial with $S(V)$ coefficients which f satisfies.

$$f^t + a_1 f^{t-1} + \ldots + a_t = 0 \quad where \;\; a_i \in S(V)$$

Since $f \in S(V)_{(X_i)} \Rightarrow deg(f) = 0$. We can replace a_i by their homogeneous components of degree 0.

$$f^t + a_1^{(0)} f^{t-1} + \ldots + a_t^{(0)} = 0 \quad where \;\; a_i^{(0)} \in S_0(V)$$

$\Rightarrow f$ is algebraic over k. But k is algebraically closed field so $f \in k$. This proves (1). □

Proposition 2.4.4. *For any variety V and an affine variety W, there is a natural bijection between*

$$Mor_{var}(V, W) \longrightarrow Mor_{k-alg}(A(W), \mathcal{O}(V))$$

where $Mor_{var}(V, W)$ is set of all morphisms of varieties and $Mor_{k-alg}(A(W), \mathcal{O}(V))$ is set of all homomorphism of k-algebras.

Proof. (Injectivity):

$$\begin{aligned} \alpha : Mor_{var}(V, W) &\longrightarrow Mor_{k-alg}(A(W), \mathcal{O}(V)) \\ \phi : V \to W &\longmapsto \phi^* : A(W) \to \mathcal{O}(V) \end{aligned}$$

$$and \qquad \begin{aligned} \phi^* : A(W) &\to \mathcal{O}(V) \\ f &\mapsto \phi * f = f \circ \phi \end{aligned}$$

Suppose $\alpha(\phi_1) = \alpha(\phi_2)$(i.e. $\phi_1^* = \phi_2^*$) we need to show that $\phi_1 = \phi_2$. Define a map

$$V \rightrightarrows W \hookrightarrow \mathbb{A}_k^n$$

given $\phi_1(y) = (\lambda_1, \ldots, \lambda_n)$ and $\phi_2(y) = (\mu_1, \ldots, \mu_n)$ for arbitrary $y \in V$.

$$\begin{aligned}
\alpha(\phi_1) = \alpha(\phi_2) \Rightarrow & \phi_1^* = \phi_2^* \\
\Rightarrow & \phi_1^* \circ f = \phi_2^* \circ f \\
\Rightarrow & f \circ \phi_1 = f \circ \phi_2 \\
\Rightarrow & f \circ (\phi_1(y)) = f \circ (\phi_2(y)) \ \forall y \in V
\end{aligned}$$

Let $A(\mathbb{A}_k^n)$ be $k[T_1, \ldots, T_n]$ and $\overline{T_i} \in A(W) = \mathcal{O}(W)$ then from above,

$$\begin{aligned}
\overline{T_i}(\phi_1(y)) =& \overline{T_i}(\phi_2(y)) \ \forall\, y \in V, \ \forall\, i \\
\lambda_i =& \mu_i \ \forall\, i \\
\Rightarrow \ \phi_1(y) =& \phi_2(y)
\end{aligned}$$

which proves injectivity of α.

(Surjectivity):

For any $\zeta \in Mor_{k-alg}(A(W), \mathcal{O}(V))$ we need to find $\phi \in Mor_{var}(V, W)$ s.t. $\alpha(\phi) = \zeta$

$$\begin{aligned}
\zeta : A(W) &\longrightarrow \mathcal{O}(V) \\
\overline{T_i} &\longmapsto f_i \ \ \forall\, i
\end{aligned}$$

then we can define a map

$$\begin{aligned}
\psi : V \longrightarrow & \mathbb{A}_k^n \\
y \longmapsto & (f_1(y), \ldots, f_n(y))
\end{aligned}$$

we claim, ψ maps into W. Take $h \in I(W) \Rightarrow \overline{h} = 0 \in A(W)$

$$\begin{aligned}
\zeta(\overline{h}) =& \zeta(h(\overline{T_1}, \ldots, \ldots, \overline{T_n})) = 0 \\
\Rightarrow & h(f_1, \ldots, f_n) = 0 \ \ in \ \mathcal{O}(V) \\
\Rightarrow & h(f_1, \ldots, f_n)(y) = 0 \ \ \forall\, y \in V \\
\Rightarrow & h(f_1(y), \ldots, f_n(y)) = 0 \ \ \forall\, y \in V \\
\Rightarrow & \forall\, y \in V, \ \ (f_1(y), \ldots, f_n(y)) \in V(h)
\end{aligned}$$

$\Rightarrow$ for every $h \in I(W)$, $\psi(y)$ is zero of h for all $y \in V$ $\Rightarrow \psi(y) \in W$ Now what is left to prove that ψ is a continuous map and induces ζ. It is easy to check that continuity of ψ and $\psi^* = \zeta$ and ψ^* goes into $\mathcal{O}(V)$. $\square$

Lemma 2.4.5. *For any variety V and an affine variety $W \subseteq \mathbb{A}^n_k$, a map of sets $\phi : V \longrightarrow W$ is a morphism iff $T_i \circ \phi$ is a regular function on V for each i where $T_1, \dots, T_n$ are the coordinate functions on $\mathbb{A}^n_k$.*

Proposition 2.4.6. *Let $\phi : V \longrightarrow W$ be morphism between affine varieties then ϕ is an isomorphism iff $\phi^* : A(W) \longrightarrow A(V)$ is an isomorphism. Moreover, $(\phi^*)^{-1} = (\phi^{-1})^*$.*

Proof. ($\Rightarrow$) Let ϕ is an isomorphism. Then by definition 2.4.4, $\exists$ a inverse polynomial map $\phi^{-1} : W \longrightarrow V$ such that $\phi \circ \phi^{-1} =\mathrm{id}_W$ and $\phi^{-1} \circ \phi =\mathrm{id}_V$. Since for any polynomial $f \in k[W]$, there induced pullback $(\phi \circ \phi^{-1})^*(f) = f \circ \phi.\phi^{-1} = (\phi^{-1})^*(f \circ \phi) = ((\phi^{-1})^* \circ \phi^*)(f)$ that is equal to $\mathrm{id}_{k[W]}$. Similarly, $(\phi^{-1} \circ \phi)^* = \phi^* \circ (\phi^{-1})^* =\mathrm{id}_{k[V]}$. Thus, ϕ^* is an isomorphism because its inverse map is $(\phi^{-1})^*$.

($\Leftarrow$) if ϕ^* is an isomorphism. Then again by definition, $\exists$ a inverse k-algebra homomorphism $(\phi^*)^{-1} : k[V] \longrightarrow k[V]$ such that $\phi^* \circ (\phi^*)^{-1} =\mathrm{id}_{k[V]}$ and $(\phi^*)^{-1} \circ \phi^* =\mathrm{id}_{k[W]}$. From above proposition 2.4.1, $\exists$ a unique polynomial map $\varphi : W \longrightarrow V$ for ϕ^* such that $\varphi^* = (\phi^*)^{-1}$. It follows $(\phi \circ \varphi)^* = \varphi^* \circ \phi^* =\mathrm{id}_{k[W]}$, Similarly, $(\varphi \circ \phi)^* =\mathrm{id}_{k[V]}$. Using again proposition 2.4.1 we get $(\phi \circ \varphi) =\mathrm{id}_W$ and $(\varphi \circ \phi) =\mathrm{id}_V$. Hence ϕ is an isomorphism and $\phi^{-1} = \varphi$. $\square$

Proposition 2.4.7. *Let $\phi : V \longrightarrow W$ be a polynomial map between algebraic sets. If V is irreducible, then closure $\overline{\phi(V)}$ is irreducible.*

Definition 2.4.5. Let V and W be affine varieties, then a morphism $\phi : V \longrightarrow W$ is called a dominant if $\phi(V)$ is dense in W.

Theorem 2.4.8. *If ϕ is a polynomial map between two affine varieties $V \subset \mathbb{A}^n(k)$ and $W \subset \mathbb{A}^m(k)$. Then*

1. *The pullback $\phi^* : k[W] \longrightarrow k[V]$ is injective iff polynomial map ϕ is dominant.*

2. *The pullback ϕ^* is surjective iff ϕ has a left inverse polynomial map.*

Proof. (1) ($\Rightarrow$) Assume ϕ is not a dominant map then it follows $\phi(V) \subsetneq W$, by inclusion reversing $I(W) \subsetneq I(\phi(V))$ so there $\exists$ a polynomial function H in $k[W]$ s.t. $H \in I(\phi(V))$ but does not belong to $I(W)$.

Define a pullback $\phi^* : k[W] \longrightarrow k[V]$ induced by ϕ, Choose H_1 and H_2 from $I(\phi(V))$ s.t. it does not belong to $I(W)$ and $H_1|_{k[W]} \neq H_2|_{k[W]}$, that implies H_1, $H_2 \neq 0$ in $k[W]$ therefore $\phi^* H_1 = H_1 \circ \phi = 0 \Leftrightarrow H_1 \circ \phi(x) = 0$ where $x \in V$. Similarly, $\phi^* H_2 = H_2 \circ \phi = 0 \Leftrightarrow H_2 \circ \phi(V) = 0$ so $\phi^* H_1 = \phi^* H_2 = 0$ but $H_1|_{k[W]} \neq H_2|_{k[W]}$ so ϕ^* is not injective that contradicts the assumption of injectivity of ϕ^*.

($\Leftarrow$) Suppose ϕ^* is injective that means $I(\phi(V)) = I(W)$ so $W = Z(I(\phi(V)) = \overline{\phi(V)}$ thus $\phi(V)$ is dense in W. $\square$

Remark 2.4.2. Relation between polynomial map between affine varieties and its pull back is given below

1. Injectivity on polynomial map $\not\Rightarrow$ Surjectivity on k-algebra homomorphism
2. Injectivity on polynomial map $\Leftarrow$ Surjectivity on k-algebra homomorphism
3. Surjectivity on polynomial map $\Rightarrow$ Injectivity on k-algebra homomorphism
4. Surjectivity on polynomial map $\not\Leftarrow$ Injectivity on k-algebra homomorphism

2.5 Rational Maps between Varieties

Definition 2.5.1. A rational map $\phi : V \longrightarrow W$ between varieties is an equivalence class of pairs $\langle V', \phi|_{V'}\rangle$ where V' is a nonempty open subset of V and $\phi|_{V'}$ is a morphism from V' to W where two pairs $\langle V_1', \phi|_{V_1'}\rangle$ and $\langle V_2', \phi|_{V_2'}\rangle$ are equivalent if $\phi|_{V_1'} = \phi|_{V_2'}$ on $V_1' \cap V_2'$.

Definition 2.5.2. A rational map $\phi : V \longrightarrow W$ between varieties is called dominant if for some(and every hence) pair $\langle V', \phi|_{V'}\rangle$, the image of $\phi|_{V'}$ is dense in W where V' is a nonempty open subset of V.

Notes 2.5.1. *A rational map is not a map. We can't always compose rational maps. However we can compose dominant rational maps, so we can define category of varieties with dominant rational maps as morphisms.*

Definition 2.5.3. A rational map $\phi : V \longrightarrow W$ between varieties is called a birational map if there exists an inverse rational map $\psi : W \longrightarrow V$ s.t. their compositions $\phi \circ \psi = id_W$ and $\psi \circ \phi = id_V$ as rational maps. And these varieties are called birationally equivalent or birational.

Proposition 2.5.2. *For any variety V there is a base for the topology consists of open affine subsets.*

Theorem 2.5.3. *If V and W are any two varieties then there is a bijection between set of dominants rational maps from V to W and set of k-algebra homomorphisms from $K(W)$ to $K(V)$.*

Proof. Let $\phi : V \longrightarrow W$ be a dominant rational map and $\langle V', \phi|_{V'} \rangle$ represents ϕ. Assume $\langle g, W' \rangle$ is a rational function in $K(W)$ where W' is open set in W and g is regular function on W. Since $\phi_{V'}(V')$ is dense in W that gives $\Rightarrow \phi_{V'}^{-1}(W')$ as a nonempty open subset of V and therefore by morphism of this map $\phi_{V'} : V' \longrightarrow W$, $(g \circ \phi)|_{V'}$ is a regular function on $\phi_{V'}^{-1}(W')$. We get a rational function on V which gives a k-algebra homomorphism from $K(W)$ to $K(V)$.

Conversely if we have a k-algebra homomorphism $\theta : K(W) \longrightarrow K(V)$. By proposition 2.5.1, W is a finite union of affine varieties. W.L.O.G. we may assume W as an affine variety and we define generators for $A(W)$, say $y_1, \ldots, y_n$. Then $\theta(y_1), \ldots, \theta(y_n)$ define a rational functions on V. By definition, there exists an open set V'' in V s.t. for all i, $\theta(y_i)$ is a regular function on V'' which gives an injective k-algebra homomorphism. By proposition 2.4.4, this corresponds to a morphism $\phi : V'' \longrightarrow W$ and it is dense. Hence we get a dominant rational map from V to W. □

Corollary 2.5.4. *If V and W are any two varieties then the following are equivalent.*

1. *V and W are birational to each other.*
2. *There exist open subsets $V' \subseteq V$ and $W' \subseteq W$ s.t. $V' \cong W'$.*
3. *$K(V) \cong K(W)$ as k-algebras.*

Proof. (1) $\Rightarrow$ (2) Given V and W are birationally equivalent it implies that $\phi : V \longrightarrow W$ and $\psi : W \longrightarrow V$ are rational maps with $\phi \circ \psi = id_W$ and $\psi \circ \phi = id_V$ as rational maps.

Assume $\langle V', \phi\rangle$ represents ϕ and $\langle W', \psi\rangle$ represents ψ then $\psi\circ\phi$ is the identity on $\phi^{-1}(W')$ as a rational map. Similarly we get identity map $\phi\circ\psi$ on $\psi^{-1}(V')$ as a rational map. Now we consider these open sets $\phi^{-1}(\psi^{-1}(V'))$ in V and $\psi^{-1}(\phi^{-1}(W'))$ in W which gives isomorphism between these two open sets via ϕ and ψ.

(2) $\Rightarrow$ (3) Given $V' \cong W'$ it implies that $K(V') \cong K(W')$ as k-algebra. Since $K(V') \cong K(V)$ and $K(W') \cong K(W)$ which gives $K(V) \cong K(W)$ as k-algebra.

(3) $\Rightarrow$ (1) By the theorem 2.5.2, we get two dominant rational maps from V to W and from W to V which are inverse to eachother. This gives the required condition (1). □

Proposition 2.5.5. *Any variety V of $\dim r$ is birational to a hypersurface H in $\mathbb{P}_k^{r+1}$.*

Proof. Since we know this field extension $K(V)/k$ is a finitely generated and k is an algebraically closed field that implies $K(V)/k$ is also separably generated. Hence by definition of separably generated, we can find transcendence basis for $K(V)/k$. Transcendence degree of $K(V)/k = \dim V = r$, say transcendence base is $\{x_1, \dots x_r\} \subset K(V)$. By theorem of primitive element, we can find one element $y \in k$ s.t. $K(V) = k(x_1, \dots, x_r, y)$ where y generates $k(V)$ over $k(x_1, \dots, x_r)$. We clear the denominators of quotient of polynomials in coefficients from $k(x_1, \dots, x_r)$. So after clearing the denominators what we get, y satisfies irreducible polynomials $f(T_1, \dots, T_r, S)$ in $r+1$ variables where $T_1, \dots, T_r, S$ are just variables. This polynomial defines a hypersurface in $\mathbb{A}_k^{r+1}$. Thus function field of it is $K(V)$. If two varieties have the same function field they are the birational equivalent which implies that V is birational to hypersurface in $\mathbb{A}_k^{r+1}$ which again tells us V is birational to projective closure of hypersurface in $\mathbb{A}_k^{r+1}$ and that projective closure is required hypersurface H in $\mathbb{P}_k^{r+1}$. □

2.5.1 Blowing up of affine varieties at a point

Assume $X_1, \dots, X_n$ are the affine coordinates of $\mathbb{A}_k^n$ and $Y_1, \dots, Y_n$ are the homogeneous coordinates of $\mathbb{P}_k^{n-1}$. First we define blowing up of $\mathbb{A}_k^n$ at point 0. It is a closed subset V of $\mathbb{A}_k^n \times \mathbb{P}_k^{n-1}$ defined as

$$V = \{(p \times l) \mid p \in l\} \subseteq \mathbb{A}_k^n \times \mathbb{P}_k^{n-1}$$

It is zeros of the equations $\{X_iY_j = X_jY_i \mid i,j = 1,\ldots,n\}$ and $\pi : V \longrightarrow \mathbb{A}_k^n$ obtained by restricting the projection map from $\mathbb{A}_k^n \times \mathbb{P}_k^{n-1}$ onto $\mathbb{A}_k^n$.

Propeties :

1. If $Q \in \mathbb{A}_k^n$, $Q \neq 0$ then $\pi^{-1}(Q) = (Q \times Q') \subset V \times \mathbb{P}_k^{n-1}$ where Q' is a equivalence class of Q in $\mathbb{P}_k^{n-1}$.

2. If $Q = 0 \in \mathbb{A}_k^n$ then $\pi^{-1}(0) = \{0\} \times \mathbb{P}_k^{n-1} \subset V \times \mathbb{P}_k^{n-1}$.

3. The map $\pi : V \longrightarrow \mathbb{A}_k^n$ sends $((X_1,\ldots,X_n) \times [y_1,\ldots,Y_n])$ to $(X_1,\ldots,X_n)$ and it is surjective. Since $\pi^{-1}(0) = \{0\} \times \mathbb{P}_k^{n-1}$ therefore $\pi|_{V\setminus\pi^{-1}(0)} : V \setminus \pi^{-1}(0) \simeq \mathbb{A}_k^n \setminus \{0\}$.

4. V is irreducible.

Definition 2.5.4. Let W be affine variety of $\mathbb{A}_k^n$ containing the zero point, then blowing-up of W at 0 is defined as projective closure of $(\pi^{-1}(W \setminus \{0\}))$ in $\pi^{-1}(W)$ where $\pi : V \longrightarrow \mathbb{A}_k^n$ described above.

2.6 Nonsingular Varieties

Definition 2.6.1. Let V be an affine variety and $I(V) = \langle f_1,\ldots,f_t\rangle$ then V is called nonsingular at a point $p \in V$ if rank of this matrix $\left(\frac{\partial f_i}{\partial X_j}(p)\right)$ is $n - \dim V$. V is nonsingular if it is nonsingular at every point otherwise singular.

Remark 2.6.1. This definition of nonsingularity is independent of chosen different set of generators of $I(V)$ and it is dependent on the embedding of V in affine space. Despite that we can describe it intrinsically in terms of local ring.

Definition 2.6.2. A Noetherian local ring R is called a regular local ring if $\dim_k m/m^2 = \dim R$ where m is maximal ideal of R and residue field k is R/m.

Theorem 2.6.1. *For any affine variety $V \subseteq \mathbb{A}_k^n$ and a point $p \in V$, V is nonsingular at p iff the local ring $\mathcal{O}_{p,V}$ is a regular local ring.*

Proof. Let p be a point $(\lambda_1, \ldots, \lambda_n)$ in $V \subseteq \mathbb{A}^n_k$ and $M_p = \langle X_1 - \lambda_1, \ldots, X_n - \lambda_n \rangle$ be maximal ideal in $k[X_1, \ldots, X_n]$. We define a k-linear map

$$\theta : k[X_1, \ldots, X_n] \longrightarrow k^n$$

$$f(X_1, \ldots X_n) \longmapsto \left(\frac{\partial f}{\partial X_1}(p) \ldots \frac{\partial f}{\partial X_n}(p) \right)$$

θ is surjective because $\theta(X_i - \lambda_i) = (\ldots, 1, \ldots)$(i.e. only i^{th} position has 1 and in other places, it is zero) and it is clear that $\forall i\ \theta(X_i - \lambda_i)$ forms a basis of k^n. It is simple to check that $\theta(M_p^2) = 0$. We claim that $ker(\theta|_{M_p}) = M_p^2$

Since $M_p^2 \subseteq \ker \theta|_{M_p} \subseteq \ker \theta$. Consider $f \in \ker \theta|_{M_p} = \ker \theta \cap M_p$ then $f = \sum_{i=1}^n g_i(X_i - \lambda_i)$. W.l.O.G. we choose j^{th} coordinate $\left(\theta|_{M_p}(f)\right)_j$ of $\theta|_{M_p}(f)$

$$\left(\theta|_{M_p}(f)\right)_j = \sum_{i=1}^n \left(\frac{\partial g_i}{\partial X_j}(p) \right) (X_i - \lambda_i)|_p + \sum_{i=1}^n g_i(p) \frac{\partial (X_i - \lambda_i)}{\partial X_j}|_p$$

Since $\theta(f) = (0, \ldots, 0)$

$$\left(\theta|_{M_p}(f)\right)_j = 0\ \forall j = 1, \ldots, n$$

$$0 + g_j(p) = 0 \quad \forall j = 1, \ldots, n$$

$$\Rightarrow g_j \in M_p \quad \forall j = 1, \ldots, n$$

which implies that $f \in M_p^2$ and therefore $\ker \theta|_{M_p} \subseteq M_p^2$. Thus θ induces an isomorphism $\theta' : M_p/M_p^2 \cong k^n$

Since $p \in V \Rightarrow I(V) \subseteq M_p \Rightarrow I(V) + M_p^2 \subseteq M_p \Rightarrow \frac{I(V)+M_p^2}{M_p^2} \subseteq \frac{M_p}{M_p^2}$. We claim that rank of this matrix $\left(\frac{\partial f_i}{\partial X_j}(p)\right)$ is the same as $\dim \theta(I(V))$ which is the same as $\dim \theta\left(\frac{I(V)+M_p^2}{M_p^2}\right)$. Let $M_V(p)$ be maximal ideal of $A(V)$ and $m_p = M_V(p)\mathcal{O}_{p,V}$ be maximal ideal of $\mathcal{O}_{p,V}$. We get

$$\frac{M_p/M_p^2}{(I(V) + M_p^2)/M_p^2} \cong m_p/m_p^2$$

$$\begin{aligned} \dim_k m_p/m_p^2 &= \dim_k \left(M_p/M_p^2\right) - \dim \left((I(V) + M_p^2)/M_p^2\right) \\ &= \dim_k(k^n) - rank \left(\frac{\partial f_i}{\partial X_j}(p) \right) \end{aligned}$$

If V is nonsingular at p then $rank\left(\frac{\partial f_i}{\partial X_j}(p)\right) = n - \dim V$

$$\begin{aligned}\dim_k m_p/m_p^2 &= n - (n - \dim V)\\ &= \dim V\\ &= \dim \mathcal{O}_{p,V}\end{aligned}$$

$\Rightarrow \mathcal{O}_{p,V}$ is a regular local ring.
Conversely if $\mathcal{O}_{p,V}$ is a regular local ring then

$$\dim \mathcal{O}_{p,V} = \dim_k m_p/m_p^2$$

$$\dim V = \dim_k m_p/m_p^2$$

$$\dim V = n - rank\left(\frac{\partial f_i}{\partial X_j}(p)\right)$$

which implies that $rank\left(\frac{\partial f_i}{\partial X_j}(p)\right) = n - \dim V$, hence V is nonsingular at p. $\square$

Definition 2.6.3. For any variety V and a point $p \in V$, V is nonsingular at p if the local ring $\mathcal{O}_{p,V}$ is a regular local ring.

Theorem 2.6.2. *For any variety V, set $Sing\, V$ of singular points of V is a proper closed subset of V.*

Proof. First we will prove that $Sing\, V$ is a closed subset. Since V can be covered by finite no. of open sets (say $V = \cup_i V_i$). $Sing\, V$ is closed in V iff $(Sing\, V) \cap V_i$ is closed in V_i for each i. Thus we just need to show that $(Sing\, V) \cap V_i$ is closed in V_i. Since we know V_i is affine and rank of jacobian matrix $\left(\frac{\partial f_j}{\partial X_l}(p)\right) < n - r$ for each point $p \in Sing\, V_i$ where $I(V_i) = \langle f_1, \ldots, f_s\rangle$ and $\dim V_i = r$ then we take all the $n - r \times n - r$ minor of $\left(\frac{\partial f_j}{\partial X_l}(p)\right)$. Since their determinants are zero it implies that all the $n - r \times n - r$ minor determinants of $\left(\frac{\partial f_j}{\partial X_l}(p)\right)$ vanish at $p \in Sing\, V_i$. $Sing\, V_i$ is the algebraic set defined by the ideal generated by $I(V_i)$ together with all determinants of $n - r \times n - r$ submatrices or minors of the matrix $\left(\frac{\partial f_j}{\partial X_l}(p)\right)$. Hence, $Sing\, V_i$ is closed.

To prove properness, we use this prop V is birational to hypersurface H in $\mathbb{A}_k^{r+1}$ then $\exists U(open) \subset V$, $\exists W(open) \subset H$ s.t. $U \cong W$. Since H is irreducible and if set of smooth points of H is nonempty then (we have just proved) it is open, irreducible and dense. It means W contains nonempty set of smooth points which implies under the isomorphism of varieties, U contains nonempty collection of smooth points and therefore $Sing\, V$ is a proper subset. So we just need to show that $H \subset \mathbb{A}_k^{r+1}$ has a proper set of singular points. Let H be defined by irreducible polynomial $f(Y_1, \ldots, Y_{r+1}) = 0$. For any point $p \in Sing\, H$, $\frac{\partial f}{\partial Y_i}(p) = 0 \; \forall \; i = 1, \ldots, r+1$. If $Sing\, H = H$, then the function $\frac{\partial f}{\partial Y_i}$ vanish on H and hence $\frac{\partial f}{\partial Y_i} \in I(H)$ for each i. But $I(H)$ is principal ideal generated by f and $\frac{\partial f}{\partial Y_i}$ has degree lesser than $deg(f)$ for each i. So we must have $\frac{\partial f}{\partial Y_i} = 0$ for each i. But in the characteristic 0 it is already impossible to have $\frac{\partial f}{\partial Y_i} = 0$ for each i. In the characteristic $p > 0$, $\frac{\partial f}{\partial Y_i} = 0$ implies that f is actually a polynomial in Y_i^p's. Since k is algebraically closed field we can take p^{th} roots of all the coefficients and we get a polynomial $g(Y_1, \ldots, Y_{r+1})$ s.t. $f = g^p$. But that contradicts the irreducibility of f. $\square$

Chapter 3

Local Properties of Affine plane Curves

3.1 Singular points and Tangent Lines

Here k is assumed algebraically closed field.

Definition 3.1.1. Let F be affine curve in $k[X,Y]$ and point $p \in V(F)$ then p is called nonsingular point of curve F if either partial derivative $F_X(p) \neq 0$ or $F_Y(p) \neq 0$. It is also called simple point or regular point.

If curve has only simple points then the curve is called nonsingular curve. A point that is not simple is called multiple point or singular point.

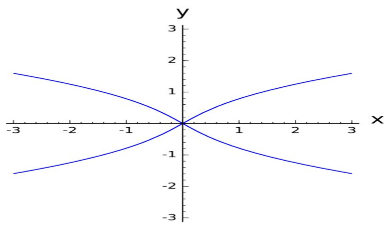
Definition 3.1.2. Let F be an affine curve in $k[X,Y]$ and $p = (a,b) \in V(F)$ be a simple point. Then the tangent line to F at p is defined by $F_X(p).(X-a) + F_Y(p).(Y-b) = 0$

3.2 Multiplicity of Affine Curves at Point

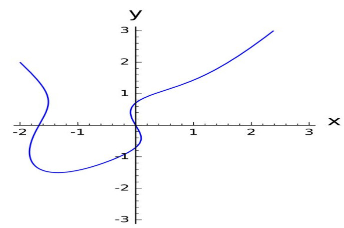
Definition 3.2.1. Let $F \in k[X,Y]$ be an affine curve. Write $F = F_m + F_{m+1} + ... + F_n$ where each F_i is a form of degree i with $F_m \neq 0$. Then multiplicity of F at $p = (0,0)$ is defined to be m and it is denoted by $m_p(F)$.

Theorem 3.2.1. *For given an affine curve $F \in k[X,Y]$ and a point p in F, $m_p(F) = 1$ iff p is a simple point of F.*

For example, the point $(0,0)$ is singular with multiplicity 2 and simple in the below graph (a) and (b) respectively.



(a) $f(X,Y) = Y^4 - X^2 + Y^2$



(b) $f(X,Y) = X^4 - 2Y^3 + X^2Y - X^2 + 3X + Y$

Remark 3.2.1. If $p = (0,0)$ is a simple point of F then the tangent line to F at P is the form F_1 of F.

Remark 3.2.2. If p is a nonzero point of F then $m_p(F) = m_p(F^T)$ where T is translation taking $(0,0)$ to p.

Theorem 3.2.2. *For given an affine curve F in $\mathbb{A}_k^2$, the set of singular points in F is finite.*

Theorem 3.2.3. *Let F be an curve in $\mathbb{A}_k^2$ and a point $p \in F$ then*

$$m_p(F) = dim_k(M_p^n / M_p^{n+1})$$

for large value $n \in \mathbb{N}$, where M_p is a maximal ideal of $\mathcal{O}_p(F)$

Proof. Consider the sequence

$$0 \longrightarrow {M_p}^n / {M_p}^{n+1} \xrightarrow{\phi} \mathcal{O}_p(F)/{M_p}^{n+1} \xrightarrow{\psi} \mathcal{O}_p(F)/{M_p}^n \longrightarrow 0$$

As you can see ,it is exact sequence because ${M_p}^n / {M_p}^{n+1} \subset \mathcal{O}_p(F)/{M_p}^{n+1}$ therefore ϕ is injective and ${M_p}^{n+1} \subset {M_p}^n$ that implies that $\mathcal{O}_p(F)/{M_p}^{n+1} \supset \mathcal{O}_p(F)/{M_p}^n$ therefore ψ is surjective.

In this exact sequence, each of rings is finite dimensional vector space. Since the sequence is exact, then

$$\dim \mathcal{O}_p(F)/{M_p}^{n+1} = \dim\ {M_p}^n/{M_p}^{n+1} + \dim\ \mathcal{O}_p(F)/{M_p}^n$$

$$\dim M_p^n/M_p^{n+1} = \dim \mathcal{O}_p(F)/{M_p}^{n+1} - \dim \mathcal{O}_p(F)/{M_p}^n$$

W.L.O.G. Let p be $(0,0)$ then the maximal ideal correspond to p is $I = \langle X, Y\rangle$ by using affine change of co-ordinate to p and F. Since ${M_p}^n = I^n\mathcal{O}(F)$ and $V(\langle I^n, F\rangle) = \{p\}$ then

$$\dim[\mathcal{O}_p(F)/{M_p}^n] = \dim[\mathcal{O}_p(\mathbb{A}_k^2)/\langle I^n, F\rangle \mathbb{O}_p(\mathbb{A}_k^2)] = \dim[k[X,Y]/\langle I^n, F\rangle].$$

So, we just need to compute $\dim k[X,Y]/\langle I^n, F\rangle$. Let $m = m_p(F)$ then $F \in I^m$.

If $n \geq m$, then $FH \in I^n$ whenever $H \in I^{n-m}$.

Define $\varphi : k[X,Y]/I^{n-m} \longrightarrow k[X,Y]/I^n$ by sending $\overline{A}$ to $\overline{FA}$.

Define $\acute{\varphi} : k[X,Y]/I^n \longrightarrow k[X,Y]/\langle I^n, F\rangle$ by sending $\overline{B}$ to $\overline{B}$ mod F. It is easy to check that the sequence

$$0 \longrightarrow k[X,Y]/I^{n-m} \xrightarrow{\varphi} k[X,Y]/I^n \xrightarrow{\acute{\varphi}} k[X,Y]/\langle I^n, F\rangle \longrightarrow 0$$

is exact because φ is injective and $\acute{\varphi}$ is surjective and $\ker\acute{\varphi} = \mathrm{Im}\varphi$. Thus

$$\begin{aligned}\dim_k([X,Y]/\langle I^n, F\rangle) &= \dim_k(k[X,Y]/I^n) - \dim_k(k[X,Y]/I^{n-m}) \\ &= \frac{n(n+1)}{2} - \frac{(n-m)(n-m+1)}{2} = nm - \frac{m(m+1)}{2}\end{aligned}$$

if we see clearly, the second term $\frac{-m(m+1)}{2}$ does not have any relation with n, so just say it some c

$$\begin{aligned}&= nm + c \\ &= n.m_p(F) + c\end{aligned}$$

therefore

$$\dim(\mathcal{O}_p(F)/{M_p}^n) = n.m_p(F) + c$$
$$\dim(\mathcal{O}_p(F)/{M_p}^{n+1}) = (n+1).m_p(F) + c$$

Hence, $\dim({M_p}^n/{M_p}^{n+1}) = m_p(F)$ with condition $n \geq m_p(F)$. $\square$

Lemma 3.2.4. *Let F be affine curve in $k[X,Y]$ and $p \in V(F)$ with $m_p(F) = m$. Prove that for $0 \leq n < m$, $\dim_k(M_p^n/M_p^{n+1}) = n+1$ where M is a maximal ideal of $\mathcal{O}_p(F)$. Moreover, if p is a simple point iff $\dim_k(M_p/M_p^2) = 1$.*

3.3 Intersection Multiplicities

Definition 3.3.1. Let F and G be curves in $\mathbb{A}^2(k)$ and a point $p \in \mathbb{A}^2(k)$ then intersection multiplicity between the both curves at p is denoted by $\mu_p(F,G)$ that satisfies the following properties :

1. If the curves F and G intersect properly at p then $\mu_p(F,G)$ is nonnegeative integer. Otherwise the value is ∞ if they don't.

2. If the curves F and G intersect transversally at p, then$\mu_p(F,G) = 1$.

3. The intersection multiplicity $\mu_p(F,G) = 0$ iff either p doesn't belong to $F \cap G$ or (F or G) is a nonzero constant.

4. $\mu_p(F,G) = \mu_q(F^T, G^T)$ after affine change of coordinate T on $\mathbb{A}^2(k)$, F and G where $T(p) = q$.

5. $\mu_p(F,G) = \mu_p(G,F)$

6. If $F = \prod\limits_{i=1} {F_i}^{r_i}$ and $G = \prod\limits_{j=1} {G_j}^{s_j}$, then $\mu_p(F,G) = \sum\limits_{i,j} \mu_p(F_i, G_j).r_i.s_j$.

7. $\mu_p(F,G) = \mu_p(F, G + AF)$ for any $A \in k[X,Y]$.

8. $\mu_p(F,G) \geq m_p(F).m_p(G)$, equality occurs if only if F and G have no common tangent lines at p.

Theorem 3.3.1. *Let F and G be any given curves in $k[X,Y]$ and any point $p \in \mathbb{A}_k^2$, then their is unique intersection multiplicity $\mu_p(F,G)$ satisfying above properties $(1)-(7)$ that is defined by this formula*

$$\mu_p(F,G) = \dim_k(\mathcal{O}_p(\mathbb{A}_k^2)/\langle F,G\rangle \mathcal{O}_p(\mathbb{A}_k^2)).$$

Proof. (proof of existence):

(1)If F and G intersect properly at p then $\mu_p(F,G) = \dim(\mathcal{O}_p(\mathbb{A}_k^2)/\langle F,G\rangle)$ is finite because $\dim_k(k[X_1,X_2]/\langle F,G\rangle) \geq \dim_k(\mathcal{O}_p(\mathbb{A}_k^2)/\langle F,G\rangle)$. By prop $V(F,G)$ is finite because there is no common component between F and G.

Suppose F and G have a common component A that implies that $\langle F,G\rangle \subseteq \langle A\rangle$. Thus $\dim_k \mathcal{O}_p(\mathbb{A}_k^2)/\langle F,G\rangle \geq \dim \mathcal{O}_p/\langle A\rangle$. If we show that $\dim \mathcal{O}_p/\langle A\rangle$ is infinite then we are done. Since

$$\mathcal{O}_p(\mathbb{A}_k^n)/\langle A\rangle \cong \mathcal{O}_p(A) \supseteq k[V(A)].$$

So, $\dim_k \mathcal{O}_p(\mathbb{A}_k^n)/\langle A\rangle \geq \dim_k[V(A)]$. Since $V(A)$ is infinite therefore $k[V(A)]$ is infinite dimensional. Thus $\mu_p(F,G) = \infty$.

(2) If F and G intersect transversally at p it means that p is a simple point and the tangent line to F at p is different from the tangent line to G at p then

$$\dim_k(\mathcal{O}_p(\mathbb{A}_k^2)/\langle F,G\rangle) = \dim_k \mathcal{O}_p(V(\langle F,G\rangle)) = 1$$

(3) Assume $\dim_k(\mathcal{O}_p(\mathbb{A}_k^2)/\langle F,G\rangle) = 0 \;\Rightarrow \langle F,G\rangle = \mathcal{O}_p(\mathbb{A}_k^2) \;\Rightarrow CF + DG = 1$ for some $C, D \in \mathcal{O}_p(\mathbb{A}_k^2)$ and it follows that either $F(p) \neq 0$ or $G(p) \neq 0$ so, p doesn't belong to $F \cap G$ and vice versa. If F is nonzero constant then $\langle F,G\rangle = \mathcal{O}_p(\mathbb{A}_k^2)$. Thus $\dim_k(\mathcal{O}_p(\mathbb{A}_k^2)\langle F,G\rangle) = 0$.

(4) Consider $p = (a,b)$, $q = (0,0)$ and the affine change of co-ordinates $T(X,Y) = (aX+b, cY+d)$. Since it is invertible, implies that T is an isomorphism and induces k-algebra isomorphism $\tilde{T} : \mathcal{O}_p(\mathbb{A}_k^2) \longrightarrow \mathcal{O}_q(\mathbb{A}_k^2)$. Thus $\dim_k(\mathcal{O}_p(\mathbb{A}_k^2)\langle F,G\rangle) = \dim_k(\mathcal{O}_q(\mathbb{A}_k^2)\langle F^T, G^T\rangle)$.

(5) It is obvious.

(6) It's enough to show that $\mu_p(F,GB) = \mu_p(F,G) + \mu_p(F,B)$ where F, G and B are any

curves such that F and G have no common component through p then there exists a exact sequence

$$0 \longrightarrow \mathcal{O}_p(\mathbb{A}_k^2)/\langle F, b\rangle \longrightarrow \mathcal{O}_p(\mathbb{A}_k^2)/\langle F, GB\rangle \longrightarrow \mathcal{O}_p(\mathbb{A}_k^2)/\langle F, G\rangle \longrightarrow 0$$

thus, we have

$$\mu_p(F, GB) = \mu_p(F, B) + \mu_p(F, G)$$

(7) Since $\langle F, G\rangle = \langle F, G + AF\rangle$ for any $A \in k[X, Y]$ that follows $\mu_p(F, G) = \mu_p(F, G + AF)$.

(8) Let $m_p(F) = m$, $m_p(G) = n$ we have to show that $\mu_p(F, G) \geq mn$ Since we have proved property (3), by applying this we may assume that $p = (0, 0)$ and ideal $I = \langle X, Y\rangle$ in $k[X, Y]$. Consider a sequence of vector spaces

$$k[X, Y]/I^n \times k[X, Y]/I^m \xrightarrow{\phi} k[X, Y]/I^{m+n} \xrightarrow{\psi} k[X, Y]/\langle I^{m+n}, F, G\rangle \longrightarrow 0$$

where ψ is a natural map and it is surjective. We can easily check this sequence is exact by sending $(\bar{C}, \bar{D})$ to $\overline{CF + DG}$ in ϕ. Then by prop

$$\begin{aligned}
\dim_k k[X, Y]/I^n + \dim_k k[X, Y]/I^m &= \dim_k(k[X, Y]/I^n \times k[X, Y]/I^m) \\
&= \dim \operatorname{Im} \phi + \dim \ker \phi \\
&\geq \dim \operatorname{Im} \phi = \dim \ker \psi
\end{aligned}$$

Since

$$\begin{aligned}
\dim_k k[X, Y]/I^{m+n} &= \dim \ker \psi + \dim \operatorname{Im} \psi \\
&= \dim \ker \psi + \dim_k k[X, Y]/\langle I^{m+n}, F, G\rangle
\end{aligned}$$

therefore

$$\begin{aligned}
\mu_p(F,G) &= \dim_k \mathcal{O}(\mathbb{A}_k^2)/\langle F,G\rangle \\
&\geq \dim_k \mathcal{O}_p(\mathbb{A}_k^2)/\langle I^{m+n},F,G\rangle = \dim_k k[X,Y]/\langle I^{m+n},F,G\rangle \\
&\geq \dim_k k[X,Y]/I^{m+n} - \dim\ker\psi \\
&\geq \dim_k k[X,Y]/I^{m+n} - \dim_k k[X,Y]/I^m - \dim_k k[X,Y]/I^n \\
&\geq \frac{(m+n)(m+n-1)}{2} - \frac{m(m+1)}{2} - \frac{n(n+1)}{2} \\
&\geq mn
\end{aligned}$$

(proof of uniqueness)

See in Theorem 4.4 in [2]. □

Remark 3.3.1. When this equality $\mu_p(F,G) = mn$ follows iff two inequalities above hold equality.

1. To hold first equality, $I^{m+n} \subseteq \langle F,G\rangle$ (in $\mathcal{O}_p(\mathbb{A}_k^2)$).
2. To hold second equality, $\ker\phi$ must be 0.

Lemma 3.3.2. *Let F and G be affine curves in $k[X,Y]$ having no common tangents at point p, then*

$$I^r \subseteq \langle F,G\rangle\mathcal{O}_p(\mathbb{A}_k^2) \quad for\ r \geq m+n-1$$

where I is a maximal ideal in $k[X,Y]$ corresponding to p and $m_p(F)=m$, $m_p(G)=n$

Proof. Let p be $(0,0)$ and $I = \langle X,Y\rangle$. Assume $U_1,...,U_m$ and $V_1,...,V_n$ are tangents of F and G at p. Let $U_i = U_m$ if $i > m$ and $V_j = V_n$ if $j > n$. Define $A_{ij} = U_1...U_i.V_1...V_j$ $\forall i,j \geq 0$ with $A_{00} = 1$. Since the set $\{A_{ij} | i+j = r\}$ forms a basis for the the forms of degree r in $k[X,Y]$ because there are $r+1$ forms in this set and they are linearly independent. It implies that I^r is generated by this set.

If we show that $A_{ij} \in \langle F,G\rangle\mathcal{O}_p$ for $i+j = r \geq m+n-1$ then $I^r \subseteq \langle F,G\rangle\mathcal{O}_p$. Assume $i+j \geq m+n-1$ then either $i \geq m$ or $j \geq n$. W.l.O.G. say $i \geq m$ then $A_{ij} = A_{m0}N$ where N is a form of degree $i+j-m$. Since $A_{m0} = F_m$ so $F = A_{m0} + F'$ where F' has degree at least $m+1$. Then $A_{ij} = FN + F'N$ where $F'N$ has degree at least $i+j+1$ Now, if we show that

$F'N \in \langle F, G\rangle \mathcal{O}_p$ then we are done. First we decompose $F'N$ into homogeneous components $F_i'N$ then we proceed through the same way since $i > m$. we can see $F'N \in \langle F, G\rangle$ if for large value of r, $I^r \in \langle F, G\rangle \mathcal{O}_p$

Is there any such r? To see that, let $V(F, G) = \{p, q_1, ...q_s\}$. Choose a polynomial h that vanishes at p so hX and $hY \in I(V(F, G))$. Since h is unit in $\mathcal{O}$ for some M so $\frac{1}{h^M}(hX)^M = X^M \in \langle F, G\rangle \mathcal{O}$. Similarly, $Y^m \in \langle F, G\rangle \mathcal{O}$, it implies that $I^{2M} \subseteq \langle X^M, Y^M\rangle \subseteq \langle F, G\rangle \mathcal{O}$. $\square$

Lemma 3.3.3. *The map above mentioned in Theorem*

$$k[X, Y]/I^n \times k[X, Y]/I^m \xrightarrow{\phi} k[X, Y]/I^{m+n}$$

defined by $\phi(\bar{C}, \bar{D}) = \overline{CF + DG}$ is one to one iff F and G have no common tangents at p.

Proof. ($\Rightarrow$)Assume that $m_p(F) = m$ and $m_p(G) = n$. Let ϕ be one to one. Suppose there is a common tangent line to F and G at p, say it L then the form $F_m = LF'$ and $G_n = LG'$. It follows that $\phi(G', -F') = \overline{G'F - FG'}$ has degree at least $m+n$ which means $\phi(G', -F') = 0$, so ϕ is not one to one which contradicts our assumption.

($\Leftarrow$)If F and G have no common tangents at p. Define $\phi(\bar{C}, \bar{D}) = \overline{CF + DG}$. Write $C = C_r + C_{r+1} + ...$ and $D = D_s + D_{s+1} + ...$ where C_i and D_j are forms of degree i and j, then $CF + DG = F_mC_r + G_nD_s + ...$ Suppose $r < n$ and $s < m$. If $\phi(\bar{C}, \bar{D}) = 0$, it means $CF+DG$ has at least degree $m+n$ then $F_mC_r = -G_nD_s$. Since F_m and G_n have no common factors, it implies $F_m \mid D_s$ and $G_n \mid C_r$ that means $r \geq n$ and $s \geq m$, so $\ker \phi = 0$ $\square$

Problem 3.3.4. *If $f = Y^2 - X^2 - X^3$ and $G = Y - X + XY^2$ are affine curve in $k[X, Y]$ find the $\mu_p(F, G)$ where $p = (0, 0)$*

Solution 3.3.5.

$$
\begin{aligned}
\mu_p(F,G) &= \mu_p(F^{(1)},G) && \textit{where } F^{(1)} = F - X^2G \\
&= \mu_p(F^{(2)},G) && , F^{(2)} = F^{(1)} - XG \\
&= \mu_p(F^{(3)},G) + \mu_p(Y,G) && , F^{(2)} = Y.F^{(3)} \\
&= \mu_p(F^{(4)},G) + \mu_p(Y,G) && , F^{(4)} = F^{(3)} - G \\
&= \mu_p(F^{(5)},G) + 2\mu_p(Y,G) + \mu_p(-X,G) && , F^{(4)} = -X.F^{(5)} \\
&= \mu_p(F^{(6)},G) + 2\mu_p(Y,G(X,0)) + \mu_p(-X,G(0,Y)) && , F^{(6)} = F^{(5)} + G \\
&= 0 + 2\mu_p(Y,-X) + \mu_p(-X,Y) \\
&= 0 + 2 + 1 = 3
\end{aligned}
$$

Chapter 4

Local Properties of Projective Plane Curves

4.1 Intersection Multiplicities between Curves

Definition 4.1.1. Let F and G be projective curves in $k[X_0, X_1, X_2]$ then at a point $p = [1 : X_1 : X_2]$ in affine part of $\mathbb{P}^2_k$, intersection multiplicity $\mu_P(F, G)$ is same as $\mu_{p'}(F_h, G_h)$ where F_h and G_h are dehomogenization of F and G respectively and $p^{'} = (X_1, X_2)$. If the point p belong to infinite part of $\mathbb{P}^2_k$, then we choose another nonzero co-ordinate to set the line at infinity and we continue with the same process above.

For example, let $F = X_2X_0 + X_1^2$ and $G = X_1 + X_2$ be projective curve then at $p = [0 : 1 : -1]$, choose $X_2 = 0$ be line at infinity and $F_h(X_0, X_1, 1) = X_0 + X_1^2$ and $G_h(X_0, X_1, 1) = X_1$ then $\mu_p(F, G) = \mu_{(0,1)}(F_h, G_h) = 1$

Lemma 4.1.1. *Let F be homogeneous polynomial in $k[X, Y, Z]$ of degree d, then $X.\frac{\partial F(X,Y,Z)}{\partial X} + Y.\frac{\partial F(X,Y,Z)}{\partial Y} + Z.\frac{\partial F(X,Y,Z)}{\partial Z} = d.F(X, Y, Z)$*

Proof. The outline of proof is that we decompose F and write it as $\sum_{i+j+k=d} a_{(i,j,k)}X^iY^jZ^k$ as given $\deg F = d$. Now, evaluate $X\frac{\partial F}{\partial X}$, $Y\frac{\partial F}{\partial Y}$ and $Z\frac{\partial F}{\partial Z}$ then sum it up which gives $dF(X, Y, Z)$. □

Proposition 4.1.2. *Let F be a projective curve in $k[X,Y,Z]$ and a point $p \in V_p(F)$. Prove that p is singular iff $\frac{\partial F(p)}{\partial X} = \frac{\partial F(p)}{\partial Y} = \frac{\partial F(p)}{\partial Z} = 0$. Moreover, if p is nonsingular, then tangent to F at p is defined by $X.\frac{\partial F(p)}{\partial X} + Y.\frac{\partial F(p)}{\partial Y} + Z.\frac{\partial F(p)}{\partial Z} = 0$*

Proof. To prove it, we use Lemma 4.2.1. $\square$

Notes 4.1.3. *In the case of affine space, for a point $p \in \mathbb{A}_k^2$ if the conditions $\frac{\partial F(p)}{\partial X} = \frac{\partial F(p)}{\partial Y} = 0$ hold then to get singularity at that point, $F(p)$ must be zero. But in the case of projective space, if the above conditions hold then we need not to check explicitly that $F(p)$ is zero or not.*

4.2 B*é*zout's Theorem

Lemma 4.2.1. *If F and G are two affine curve in $k[X,Y]$ of degree r and s respectively such that there is no common component between leading terms F_r and G_s of F and G respectively then any polynomial f in $\{F,G\}$ of $\deg d$ can be expressed as $CF + DG$ s.t. $\deg C \leq d - r$ and $\deg D \leq d - s$ where $C, D \in k[X,Y]$*

Proof. Consider any polynomial f in $\{F,G\}$ of $\deg d$ and write it as $CF + DG$ with minimal $\deg C$ where $C, D \in k[X,Y]$. Assume either $\deg C > d - r$ or $\deg D > d - s$, this means the leading terms of CF and DG must cancel. SO $C'F + D'G = 0$ where C' and D' are leading terms of C and D respectively. Since F_r and G_s have no common component, it implies that $F_r \mid D'$ and $G_s \mid C'$. Thus $D' = HF_r$ and $C' = -HG_s$ where $H \in k[X,Y]$. Rewriting f as $(C + HG)F + (D - HF)G$. This gives contradiction of minimality of $\deg C$ because $\deg(C + HG) < \deg C$. $\square$

Lemma 4.2.2. *If F and G are two affine curves in $k[X,Y]$ of degree r and s respectively having no common components. Prove that*

1. *The dimension of $k[X,Y]/\langle F,G\rangle$ is at most $r.s$.*
2. *$\dim k[X,Y]/\langle F,G\rangle = r.s$ if there is no common component between leading terms F_r and G_s of F and G respectively.*

Proof. (i) Choose such $d \geq r+s$, look at a sequence of vector subspace homomorphism given as

$$k[X,Y]_{\leq d-r} \times k[X,Y]_{\leq d-s} \xrightarrow{\phi} k[X,Y]_{\leq d} \xrightarrow{\psi} k[X,Y]/\langle F,G\rangle$$

defined by $\phi(C,D) = CF + DG$ and ψ is a quotient map where $k[X,Y]_{\leq t}$ means a vector subspace of all polynomials in $k[X,Y]$ having degree at most t. Since we know, its dimension is $\binom{t+2}{2}$. If $\phi(C,D) = CF + DG = 0 \Rightarrow CF = -DG$ but there is no common component between F and G, so $F \mid D$ and $G \mid C$ thus $D = HF$ and $C = -HG$ where $H \in k[X,Y]_{\leq d-r-s}$ this means $\ker\phi = H.(-G,F)$ where $H \in k[X,Y]_{\leq d-r-s}$. Since $\operatorname{Im}\phi$ is in form of $CF+DG$ which maps to 0 under ψ. So, $\operatorname{Im}\phi \subset \ker\psi \;\; \longrightarrow (i)$

By theorem,

$$\operatorname{Im}\psi + \dim\ker\psi = \dim k[X,Y]_{\leq d}$$

$$\begin{aligned}\operatorname{Im}\psi &= \binom{d+2}{2} - \dim\ker\psi \\ \dim\operatorname{Im}\psi &\leq \binom{d+2}{2} - \dim\operatorname{Im}\phi \quad (from(i))\end{aligned}$$

By theorem,

$$\dim\operatorname{Im}\phi + \dim\ker\phi = \binom{d-r+2}{2} + \binom{d-s+2}{2}$$

$$\begin{aligned}\dim\operatorname{Im}\psi &\leq \binom{d+2}{2} - \binom{d-r+2}{2} - \binom{d-s+2}{2} + \dim\ker\psi \\ \dim\operatorname{Im}\psi &\leq \binom{d+2}{2} - \binom{d-r+2}{2} - \binom{d-s+2}{2} + \binom{d-r-s+2}{2} \\ \dim\operatorname{Im}\psi &\leq \binom{d+2}{2} - \binom{d-r+2}{2} - \binom{d-s+2}{2} + \binom{d-r-s+2}{2} \\ &\leq rs + 3(r+s-d) \\ &\leq rs\end{aligned}$$

as we have $d \geq r+s$. Since this map $k[X,Y] \longrightarrow k[X,Y]/\langle F,G\rangle$ is surjective. So, for large value of $d \geq r+s$, $\dim k[X,Y]/\langle F,G\rangle = \operatorname{Im}\psi \leq r.s.$

(ii) In above, if we show that $\operatorname{Im}\phi \supset \ker\psi$ then we are done and it follows from previous Lemma 4.2.1. $\square$

Corollary 4.2.3. ***(Bézout's Theorem)*** *If F and G are projective curve in $k[X,Y,Z]$ with no common components then* $\sum_{p\in V(F,G)} \mu_p(F,G) = \deg F.\deg G$

Proof. Suppose F and G are projective curve in $k[X,Y,Z]$. Define F^{dh} and G^{dh} are dehomogenization of F and G respectively. Since K is A.C.F. so we can find a point q in affine part of $\mathbb{P}^2{}_k$ which does not belong to $V(F^{dh}) \cup V(G^{dh})$ ($\Rightarrow q \neq V(F) \cup V(G)$). Since F and G have no common component so $V(F,G)$ is a finite set. We choose a line L passing through q s.t. it does not intersect $V(F,G)$. By projective co-ordinate transformation, we make L as line at infinity, say $Z = 0$. It follows that $\deg(F^{dh}) = \deg(F)$ and $\deg(G^{dh}) = \deg(G) \;\Rightarrow\; V(F,G)$ lies only in affine part of $V(F)$ and $V(G)$. Since all points of $V(F,G)$ are in affine part $\Rightarrow\; V(F,G) \cong V(F^{dh},G^{dh}) \;\Rightarrow\; \sum_{p\in V(F,G)} \mu_p(F,G) = \sum_{p\in V(F^{dh},G^{dh})} \mu_p(F^{dh},G^{dh}).\; \longrightarrow (1)$

As F^{dh} and G^{dh} are homogeneous polynomials in $k[X,Y]$. Hence they can be factorized into linear components but there is no common points at infinity of F and G because of taking such a line L. Thus by lemma 4.2.2, $\dim k[X,Y]/\langle F^{dh},G^{dh}\rangle = \deg F^{dh}.\deg G^{dh} = \deg F.\deg G.\; \longrightarrow (2)$.

from (1) and (2),

$$\sum_{p\in V(F^{dh},G^{dh})} \mu_p(F^{dh},G^{dh}) = \dim k[X,Y]/\langle F^{dh},G^{dh}\rangle$$
$$(\Rightarrow) \sum_{p\in V(F,G)} \mu_p(F,G) = \deg F.\deg G.$$

Hence it is proved. ☐

Chapter 5

Sheaves of Abelian Groups

5.1 Presheaf and Sheaf

Definition 5.1.1. (**Presheaf**) Let X be a topological space. A presheaf $\mathcal{F}$ on X is a function such that for each open set $U \in X$, we have an abelian group $\mathcal{F}(U)$ that is called section of $\mathcal{F}$ over U and for each inclusion map $V \hookrightarrow U$ of open sets V, $U \subset X$ there is a restriction map $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ s.t. it satisfying the following conditions:

1. $\mathcal{F}(\emptyset) = 0$

2. The restriction map ρ_{UU} is an identity map for each open set $U \subset X$.

3. If there is a chain of inclusion of open sets, say $W \hookrightarrow V \hookrightarrow U$ then $\rho_{UW} = \rho_{VW}.\rho_{UV}$.

Definition 5.1.2. (**Sheaf**) A presheaf is a sheaf $\mathcal{F}$ if it satisfies the following conditions:

1. If for each open set U, there is a open cover $\{U_i\}$ and s_1, $s_2 \in \mathcal{F}(U)$ such that $\rho_{UU_i}(s_1) = \rho_{UU_i}(s_2)$ for all U_i then $s_1 = s_2$

2. If for each open set U, there is a open cover $\{U_i\}$ and for all i, a section s_i is given such that $\rho_{U_i,U_i\cap U_j}(s_i) = \rho_{U_j,U_i\cap U_j}(s_j)$ $\forall i, j$ then there is a section $s \in \mathcal{F}(U)$ such that $\rho_{UU_i}(s) = s_i$ for all i.

For example, let X be a topological space, say $\{a, b\}$ with the discrete topology. A presheaf $\mathcal{F}$ is defined as $\mathcal{F}(\emptyset) = 0$, $\mathcal{F}(X) = \mathbb{R}^3$, $\mathcal{F}(\{a\}) = \mathcal{F}(\{b\}) = \mathbb{R}$ with the restriction map as a projection map but it is not a sheaf.

Definition 5.1.3. (**Using equalizer**) A presheaf $\mathcal{F}$ is a sheaf iff for each open set U and all covering $\{U_i\}$ of U, the sequence of abelian groups, the sequence of abelian groups

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_i \mathcal{F}(U_i) \xrightarrow{\beta} \prod_{i,j} \mathcal{F}(U_i \cap U_j) \longrightarrow 0$$

$$s \xmapsto{\alpha} (\rho_{UU_i(s)})_i$$
$$(s_i)_i \xmapsto{\beta} (\rho_{U_i, U_i \cap U_j}(s_i)) - \rho_{U_j, U_i \cap U_j}(s_j))_{i,j}$$

is exact.

Definition 5.1.4. (**Constant presheaf**) A constant presheaf with a abelian group G is an presheaf that assign G to each nonempty open subset of X and whose restriction maps are id_G.

For example, let X be $\{a, b\}$ with the discrete topology and $A = \mathbb{Z}$ then a presheaf $\mathcal{G}$ defined as $\mathcal{G}(\emptyset) = 0,\ \mathcal{G}(X) = \mathcal{G}(\{a\}) = \mathcal{G}(\{b\}) = \mathbb{Z}$ with the restriction map as identity is a constant presheaf.

Definition 5.1.5. (**Constant sheaf**) Suppose X is topological space and G is given an abelian group with the discrete topology, then a constant sheaf is set of all continuous maps from U into G. Moreover if U is connected then $\mathcal{F}(U) \cong G$.

For example, let X be $\{a, b\}$ with the discrete topology and $A = \mathbb{Z}$ then a sheaf $\mathcal{F}$ defined as $\mathcal{F}(\emptyset) = 0,\ \mathcal{F}(X) = \mathbb{Z} \oplus \mathbb{Z},\ \mathcal{F}(\{a\}) = \mathcal{F}(\{b\}) = \mathbb{Z}$ is a constant sheaf.

Definition 5.1.6. (**Subpresheaf of presheaf**) A subpresheaf of presheaf $\mathcal{G}$ is a presheaf $\mathcal{G}'$ such that $\mathcal{G}'(U)$ is a subgroup of $\mathcal{G}(U)$ and its restriction map is the restriction map of $\mathcal{G}$ restricted to $\mathcal{G}'(U)(\rho'_{UV} = \rho_{UV}|_{\mathcal{G}'(U)})$. In the above if $\mathcal{G}$ is a sheaf then $\mathcal{G}'$ is a subsheaf.

for example, every sheaf has zero sheaf as a subsheaf.

Remark 5.1.1. Let $\mathcal{G}$ and $\mathcal{G}'$ be subpresheaves of a presheaf $\mathcal{F}$ on X, then the both are equal iff $\mathcal{G} = \mathcal{G}'\ \forall U \subset X$.

Remark 5.1.2. If $\mathcal{G}$ and $\mathcal{G}'$ are subsheaves of a sheaf $\mathcal{F}$, then they are equal iff both have the same stalks $\forall\, p \in X$.

5.2 Morphism of presheaves

Definition 5.2.1. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X then a morphism of presheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ consists of a family of homomorphism of presheaves $\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for each open set U and if $V \hookrightarrow U$ then the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ {\scriptstyle \rho_{UV}} \downarrow & & \downarrow {\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow[\varphi_V]{} & \mathcal{G}(V) \end{array}$$

Definition 5.2.2. (**Stalk**) let $\mathcal{F}$ be a presheaf on X, then stalk of (F) at a point $p \in X$ is a set of equivalence classes of pairs of the form (U, s) where U is a neighborhood of P and $s \in \mathcal{F}(U)$ with the equivalence relation that is $(U, s) \sim (V, t)$ in $\mathcal{F}_p$ iff there is an open neighborhood W containing p with $W \subseteq U \cap V$ such that $\rho_{UW}(s) = \rho_{VW}(t)$

Definition 5.2.3. (**Using direct limit**) If $\mathcal{F}$ is a presheaf on topological space X then stalk $\mathcal{F}_p$ of $\mathcal{F}$ at p is defined as direct limit of $\mathcal{F}(U)$ where range is all open set U containing p or $\mathcal{F}_p = \lim\limits_{U \ni p} \mathcal{F}(U)$.

By definition of direct limit, for each neighborhood U of p , there exists a canonical morphism $\theta_U : \mathcal{F}(U) \longrightarrow \mathcal{F}_p$ s.t. $\theta_V.\rho_{UV} = \theta_U$ where $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ whenever $p \in V \subseteq U$.

Notes 5.2.1. *If φ is a morphism of presheaves on X then it induces a morphism of presheaves on the level of stalk. For any point $p \in X$,*

$$\varphi : \mathcal{F} \longrightarrow \mathcal{G}$$

$$\varphi_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$$

where $\varphi_p(U, s) = (U, \varphi_U(s))$ and it is well defined (using Def 5.2.1).

Remark 5.2.1. If $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of presheaves on X then

1. $\ker \varphi$ is a presheaf defined by $\ker(\varphi(U))$ for each open set $U \in X$.

2. $\operatorname{Im} \varphi$ is a presheaf defined by $\operatorname{Im}(\varphi(U))$ $\forall$ open set $U \in X$.

3. Coker φ is a presheaf defined by $\text{Coker}(\varphi(U)) = \mathcal{G}(U)/\operatorname{Im}(\varphi(U))$ for each open set $U \in X$.

   If in map φ, sheaves are given instead of presheaves then $\ker\varphi$ is a sheaf.

Lemma 5.2.2. *Consider a morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ on a topological space X, then the following are the equivalent :*

1. *φ is a monomorphism of sheaves of sets.*
2. *$\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is injective for all open set $U \subset X$.*
3. *$\varphi_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$ is injective for each point $p \in X$.*

Lemma 5.2.3. *Consider a morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ on a topological space X, then the following are the equivalent :*

1. *φ is epimorphism of sheaves of sets.*
2. *$\varphi_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$ is surjective for each point $p \in X$.*

Remark 5.2.2. If φ is epimorphism then for each open set $U \in X$ the map φ_U need not be surjective.

Remark 5.2.3. If φ_p is surjective on the level of stalks, then φ_U need not be surjective on the each open set U in X.

Proposition 5.2.4. *A morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ on a topological space X is isomorphism iff morphism of stalks $\varphi_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$ is isomorphism for each point $p \in X$.(It is not true for general presheaves.)*

Proof. ($\Rightarrow$) It is clear. ($\Leftarrow$) Let φ_p be an isomorphism $\forall\, p \in X$. To show φ is an isomorphism, we just need to show that $\varphi(U)$ is isomorphism $\forall$ open set $U \in X$. To see injectivity of $\varphi(U)$, we assume a section $s \in \mathcal{F}(U)$ such that $\varphi_U(s) = 0$. If $s = 0$ we are done. for every point $p \in U$, image of (s, U) under φ_p is $(\varphi_U(s), U) = (0, U)$. But φ_p is injective for each point $p \in U$ tha implies $(s, U) \sim (0, U)$ in $\mathcal{F}_p$, so there exists a open set $V_p \subseteq U$ s.t. $\rho_{UV_p}(s) = \rho_{UV_p}(0) = 0$ for each point $p \in U$. It follows that $\{V_p\}$ is covering of U, by the sheaf condition (1) it implies that $s = 0$ on $\mathcal{F}(U)$.

To show surjectivity of $\varphi(U)$, we assume $s \in \mathcal{G}(U)$ and define s_p is its germ in $\mathcal{G}_p$ for each point $p \in U$. Due to surjectivity of φ_p, we can find section r_p from $\mathcal{F}_p$ for each point $p \in U$ s.t.$\varphi_p(r_p) = s_p$. Write $r_p = (r(p), U_p)$ where $r(p)$ is a section of $\mathcal{F}(U_p)$ for each $p \in U_p \subset U$. Since $\varphi_{U_p}(r(p))$, $\rho_{UU_p}(s) \in \mathcal{G}(U_p)$ and they have the same germs at p. So we can proceed by assuming $\varphi_{U_p}(r(p)) = {\rho'}_{UU_p}(s)$ in $\mathcal{G}(U_p)$. It follows that $\{U_p\}$ is covering of U. For any two points p and q in U, consider two sections of $\mathcal{F}(U_p \cap U_q)$ which are $\rho_{U_p,U_p\cap U_q}(r(p))$ and $\rho_{U_q,U_p\cap U_q}(r(q))$, sent by $\varphi_{U_p\cap U_q}$ to $\rho_{U,U_p\cap U_q}(s)$, Using the injectivity on $\varphi_{U_p\cap U_q}$, we get $\rho_{U_p,U_p\cap U_q}(r(p)) = \rho_{U_q,U_p\cap U_q}(r(q))$. By the sheaf condition (2), for each point $p \in U$ there exists a section $r \in \mathcal{F}(U)$ s.t. $\rho_{UV_p}(r) = r(p)$. Since $\varphi_U(r)$, $s \in \mathcal{G}(U)$ from above for each p, $\rho'_{UU_p}\varphi_U(r) = \varphi_{U_p}(r(p)) = \rho'_{UU_p}(s)$.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\rho_{UU_p}"'] & \mathcal{G}(U) \arrow[d, "\rho'_{UU_p}"] \\
\mathcal{F}(U_p) \arrow[r, "\varphi_{U_p}"'] & \mathcal{G}(U_p)
\end{tikzcd}
\qquad
\begin{tikzcd}
r \arrow[r, "\varphi_U"] \arrow[d, "\rho_{UU_p}"'] & \varphi_U(r) \arrow[d, "\rho'_{UU_p}"] \\
r(p) \arrow[r, "\varphi_{U_p}"'] & \varphi_{U_p}(r(p))
\end{tikzcd}$$

Again by the sheaf conditions (2), $\rho_U(r) = s$ in $\mathcal{G}(U)$. $\Rightarrow$ $\exists\, r \in \mathcal{F}(U)$ s.t. $\varphi_U(r) = s$ $\forall$ (open)$U \subset X$. $\Rightarrow$ φ_U is surjective $\forall$ (open)$U \subset X$.

$\square$

Lemma 5.2.5. *Suppose $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of sheaves, then it is epimorphism iff there exists an open cover $\{U_i\}$ of every open set U and for every section $s \in \mathcal{G}(\mathcal{U})$, $\rho'_{UU_i}(s)$ lies in the image of φ_{U_i} for each i where $\rho'_{UU_i} : \mathcal{G}(U) \longrightarrow \mathcal{G}(U_i)$.*

Lemma 5.2.6. *Suppose $\mathcal{G}$ is a sheaf on a topological space X then for every open set U in X, the map $\psi : \mathcal{G}(U) \longrightarrow \prod_{p\in U} \mathcal{G}_p$ is injective.(It is not true for general presheaf.)*

Proof. fix an open set U ,now choose such s, t from $\mathcal{G}(U)$ such that they maps to the same element in $\mathcal{G}(U) \longrightarrow \mathcal{G}_p$ for each $p \in U$. So $(U, s) \sim (U, t)$ in every stalk $\mathcal{G}_p$ for all point $p \in U$ that implies $\exists$ an open set $V_p \subseteq U$ containing p such that $\rho_{UV_p}(s) = \rho_{UV_p}(t)$ for each $p \in U$ then $\{V_p\}$ makes an open cover of U. Thus by the sheaf condition (1), $s = t$, therefore the map ψ is injective. $\square$

Definition 5.2.4. (**Sheafification of presheaf**) The sheafification of a presheaf $\mathcal{F}$ is the sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ and for each open set U, $\mathcal{F}^+$ is defined as set of functions $f : U \longrightarrow \coprod_{p\in U} \mathcal{F}_p$ satisfying two conditions :

1. For each $p \in U$, $f(p) \in \mathcal{F}_p$
2. For each $p \in U$, there is open set V around p contained in U and a section $g \in \mathcal{F}(V)$ such that $\forall\, q \in V$, $(V, g) = f(q)$.

Theorem 5.2.7. *(**Universal property of sheafification**) Given a presheaf $\mathcal{F}$, there exists a sheaf $\mathcal{F}^+$ and a morphism $\iota : \mathcal{F} \longrightarrow \mathcal{F}^+$ with the property that for any sheaf $\mathcal{G}$ and any morphism $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ there is a unique morphism $\varphi : \mathcal{F}^+ \longrightarrow \mathcal{G}$ such that $\psi = \varphi \circ \iota$.*

Proof. See in proposition 13 in [8] □

Bibliography

[1] Robin Hartshorne, *"Algebraic Geometry"*

[2] Margaret E. Nichols,*"Intersection number of plane curve"*, 2013. available at `https //etd.ohiolink.edu/rws_etd/document/get/oberlin1385137385/inline`

[3] Andreas Gathmann, *"Plane Algebraic Curves"*, 2017. available at `http //www.mathematik.uni-kl.de/∼gathmann/class/curves-2017/curves-2017.pdf`

[4] Ruxandra Moraru. available at `http://www.math.uwaterloo.ca/∼moraru/764AffineVarieties_2013.pdf`

[5] W. Fulton, *"Algebraic Curves"*, Mathematics Lecture Note Series, W.A. Benjamin, 1974.

[6] The stacks Project, *"Chapter 6: Sheaves on Spaces"*. available at `https //stacks.math.columbia.edu/chapter/6`

[7] Ian Coley, *"Exercises from Vakil"*, 2015. available at `https //www.math.ucla.edu/∼iacoley/hw/vakil.pdf`

[8] Patrick J. Morandi, *"Sheaf cohomology", Mathematical Notes.* available at `http //sierra.nmsu.edu/morandi/notes/SheafCohomology.pdf`