A study of the Bergman kernel and metric

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

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April, 2019

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Certificate

This is to certify that this dissertation entitled A study of the Bergman kernel and metric towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Mukul Rai Choudhuri at Indian Institute of Science Education and Research under the supervision of Dr. Diganta Borah, Assistant Professor, Department of Mathematics, during the academic year 2018-2019.

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This thesis is dedicated to Dida

Declaration

I hereby declare that the matter embodied in the report entitled A study of the Bergman kernel and metric are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Diganta Borah and the same has not been submitted elsewhere for any other degree.

Mukul Rai Choudhuri

Acknowledgments

I would firstly like to thank Prof. Diganta Borah for his invaluable guidance and support throughout this project. During a challenging year, my interactions with him gave me a much needed feeling of calmness and balance. I thank my parents and my brother for their constant support and encouragement. I thank Prof. Krishna Kaipa for always willing to help me and giving me sound advice for my PhD application. I would like to thank Prof. Debraj Chakrabarti for his guidance as a TAC member. He was kind, patient and I always felt I could approach him. He also gave me some problems to work on. I was able to solve one of them (a simple one) and I have presented it in Proposition 4.3.1. Finally I would like to thank Prof. Amit Hogadi for writing me a letter of recommendation and teaching so well that I took up mathematics in my first year at IISER.

I also am grateful for having made some very dear friends and for being able to play so much football over here.

Abstract

The main focus of this project was on the Bergman kernel and metric. After studying some basic theory of several complex variables, various properties of Bergman kernel and metric were studied. I calculated the Bergman kernel for many domains and attempted to calculate it for some domains for which it was not known. Bell's proof of Fefferman's theorem using the condition R was studied.

Contents

Introduction

This project had several objectives. The first one was to learn some several complex variables (SCV). This is a rich subject full of interesting mathematical phenomena. I learnt the basic theory of SCV mainly from the introductory textbooks by R. M. Range [17] and S. G. Krantz [12]. Some important topics of SCV were covered that were not directly related to the Bergman kernel such as the concept of finite type.

After achieving a degree of comfort with the basics of SCV, I started working on the Bergman kernel and metric. The Bergman space associated with a domain in $\mathbb{C}^n$ is the set of square integrable holomorphic functions on the domain. It forms a Hilbert space with the standard L^2 inner product and the evaluation maps are bounded linear functionals. Using the Reisz representation theorem one obtains a reproducing kernel called the Bergman kernel. It is a difficult task to calculate this kernel for even simple domains such as the unit ball. During this project, I studied the computation of the Bergman kernels for many domains. Some of these computations were from recent research papers. I also attempted to calculate for some new kernels and was successful in computing it for one simple case (Hartog's triangle in n-dimensions). Using the Bergman kernel, we can define a Kähler metric called the Bergman metric. This metric is invariant under biholomorphisms. Other important properties related to the Bergman kernel were studied including Bergman representative coordinates and Ramadanov's theorem. The Lu Qi-Keng problem is to find out/characterize which domains have non-vanishing Bergman kernels. There are many interesting results related to this problem. Another interesting paper was studied on domains with finite dimensional Bergman spaces.

The scaling method is a powerful technique that reduces boundary problems pertaining to strongly pseduconvex domains to interior problems pertaining to the unit ball. This technique was used to study boundary behavior of Bergman kernel for strongly pseudoconvex domains.

Apart from being studied in its own right, the Bergman kernel and metric have found many deep applications in SCV. The most famous example is the central role it plays in the proof of Fefferman's theorem which states that biholomorphisms between two strongly pseudoconvex domains extend smoothly up to the boundary. I studied a simplified version of this proof due to Bell. He in fact, generalized the theorem to a wider class of domains. A useful reference for Bergman theory is the book written by Krantz on the Bergman kernel and metric [13].

Chapter 1

D'Angelo's finite type

We start by defining the notion of Levi pseudoconvexity and strong pseudoconvexity. These notion are the complex analogues of convexity and strong convexity respectively.

Definition 1.0.1. *Let D be a smooth bounded domain in $\mathbb{C}^n$ with defining function ρ. D is said to be Levi pseudoconvex if for every $p \in \partial D$, we have*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z_k}}(p)\, t_j \bar{t_k} \geq 0$$

for all t in the complex tangent space to the boundary of D at p.

Now for strong pseudoconvexity we demand a strict inequality for the Levi form.

Definition 1.0.2. *Let D be a smooth bounded domain in $\mathbb{C}^n$ with defining function ρ. D is said to be strongly pseudoconvex if for every $p \in \partial D$, we have*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z_k}}(p)\, t_j \bar{t_k} > 0$$

for all $t \neq 0$ in the complex tangent space to the boundary of D at p.

1.1 Concept of finite type

The concept of finite type is important in many sub areas of several complex variables, not just Bergman theory. It was introduced by D'Angelo [9]. In this section we give two equivalent definitions of finite type. We start by defining holomorphic curves and multiplicity of a function at a point.

Definition 1.1.1. *A holomorphic curve is a holomorphic function $\phi : \mathbb{D} \to \mathbb{C}^n$. A holomorphic curve is said to be non-singular if $\phi'(0) \neq 0$.*

Definition 1.1.2. *We denote the multiplicity of a function f at the point p by $v_p(f)$. We will now define $v_p(f)$ in various scenarios.*

1. *Suppose $U \subset \mathbb{C}$ is a domain and that $f : U \to \mathbb{C}$ is holomorphic. Let $p \in U$. Then we define $v_p(f)$ to be the least positive integer k such that $f^{(k)} \neq 0$.*

2. *Suppose $U \subset \mathbb{C}$ is a domain and that $f : U \to \mathbb{C}^m$ is holomorphic. Let $p \in U$ and f_i be the component functions of f. Then we define $v_p(f)$ to be $\min(v_p(f_1), ..., v_p(f_m))$.*

3. *Suppose $U \subset \mathbb{R}^n$ is a domain and that $f \in C^\infty(U)$. Let $p \in U$. Then we define $v_p(f)$ to be the least positive integer k such that $D^\alpha f(p) \neq 0$ for some α satisfying $|\alpha| = k$.*

Before we define finite type, we need to define real hypersurfaces in $\mathbb{C}^n$.

Definition 1.1.3. *Let $M \subset \mathbb{C}^n$. M is said to be a real hypersurface if there exists a real valued smooth function ρ defined on a neighborhood of M such that $\rho^{-1}(0) = M$ and $\nabla\rho \neq 0$ on M.*

The function ρ is called a defining function for M. Examples of real hypersurfaces include boundaries of smoothly bounded domains in $\mathbb{C}^n$. Having defined these objects, we are ready to define the finite type of a point on a hypersurface.

Definition 1.1.4. *Let M be a real hypersurface in $\mathbb{C}^n$ and $p \in M$. Let ρ be a defining function for M. We say that p is a point of finite type if*

$$\Delta(M,p) = \sup_{\phi \in E_p} \left\{ \frac{v_0(\rho \circ \phi)}{v_0(\phi)} \right\} < \infty$$

where E_p is the set of all non-constant, holomorphic curves ϕ such that $\phi(0) = p$. Further if p is a point of finite type, then the quantity $\Delta(M,p)$ is called the finite type of p.

One can easily show that the quantity $\Delta(M,p)$ is independent of choice of defining function. Also the finite type of a point is a bihomorphically invariant property. We can define the finite type of a point in a purely geometric way. We give this alternative definition in the following proposition.

Proposition 1.1.1. *Suppose $M \subset \mathbb{C}^n$ is a hypersurface and $p \in M$. Then*

$$\Delta(M,p) = \sup_{V \in R_p} \sup \left\{ a \in \mathbb{R}^+ : \lim_{V \ni z \to p} \frac{dist(z,M)}{|z-p|^a} \quad exists \right\}$$

where R_p is the collection of one dimensional varieties passing through p.

Proof. Let δ_M denote the signed distance to M in a neighborhood of M. We can show that this is a smooth defining function for M. We also have

$$\frac{v_0(\delta_M \circ \phi)}{v_0(\phi)} = \sup \left\{ a \in \mathbb{R}^+ : \lim_{\zeta \to 0} \frac{dist(\phi(\zeta),M)}{|\phi(\zeta)-p|^a} \quad exists \right\} \tag{1.1}$$

where ϕ is a non-constant holomorphic curve such that $\phi(0)=p$. Taking supremum with ϕ varying over E_p on both sides completes the proof since we can identify E_p and R_p. □

So we see the quantity $\Delta(M,p)$ measures the maximum degree of contact of holomorphic curves with M at p. In the next section we will evaluate finite type for some examples.

1.2 Finite type of some domains

If a domain is smoothly bounded, then its boundary is a real hypersurface. If the finite type of all boundary points is equal to k, then we say that the domain has finite type k.

Proposition 1.2.1. *The unit ball $\mathbb{B}^n = \left\{ z \in \mathbb{C}^n : \sum_i |z_i|^2 < 1 \right\}$ has finite type 2.*

Proof. Let $M = \partial \mathbb{B}^n$ and $p \in M$. Suppose ϕ is a non-constant holomorphic curve such that $\phi(0)=p$. Let $v_0(\phi)=m$. Taking the standard defining function for M, we have

$$(\rho \circ \phi)(z) = \sum_i \phi_i(z)\overline{\phi_i(z)} - 1 \,.$$

We differentiate as follows:

$$\left(\frac{\partial^2}{\partial z \partial \bar{z}}\right)^m (\rho \circ \phi) = \sum_i |\phi_i^{(m)}(z)|^2 \,. \tag{1.2}$$

Since $v_0(\phi) = m$, the above expression will be non-zero at the origin. This gives us $v_0(\rho \circ \phi) \leq 2m$. Hence $\big(v_0(\rho \circ \phi)/v_0(\phi)\big) \leq 2$ for all ϕ. This tells us that $\Delta(M, p) \leq 2$. Next we will show $\Delta(M, p) \geq 2$. Now take $\tilde{\phi}(z) = p + z \cdot v$, where v is a non-zero vector in the complex tangent space to the boundary of the ball at p. Note that

$$\frac{\partial}{\partial z}\bigg|_0 (\rho \circ \phi) = \sum_i \frac{\partial \rho}{\partial z_i}(p)\phi_i'(0) \,. \tag{1.3}$$

Obviously $v_0(\tilde{\phi}) = 1$ and since $\tilde{\phi}'(0)$ is in the complex tangent space, the above equation tells us that $v_0(\rho \circ \phi) \geq 2$. Hence this implies $\Delta(M, p) \geq 2$. This proves the proposition. □

Next example is strongly pseudoconvex domains.

Proposition 1.2.2. *A strongly pseudoconvex domain has finite type 2.*

Proof. Let D be a strongly pseudoconvex domain with global defining function ρ and suppose $p \in \partial D$. It is sufficient to consider only non-singular holomorphic curves. So $v_0(\phi) = 1$. If $\phi'(0)$ does not belong to the complex tangent space at p, then $v_p(\rho \circ \phi) = 1$ by (1.3). If it does, then $v_p(\rho \circ \phi) > 1$. We will now show that in this case $v_p(\rho \circ \phi) = 2$, proving the proposition. Observe that

$$\frac{\partial^2}{\partial z \partial \bar{z}}(\rho \circ \phi) = \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \phi_i'(0)\overline{\phi_j'(0)} \,,$$

hence by strong pseudoconvexity this will be positive. Hence proved. □

Chapter 2

The Bergman kernel and metric

2.1 The Bergman kernel function

Let D be a domain in $\mathbb{C}^n$. Let

$$A^2(D) = \left\{ f \in \mathscr{O}(D) : \int_D |f|^2 dV < \infty \right\}.$$

So $A^2(D)$ is the set of square integrable holomorphic functions on D and is also called the Bergman space of D. Assume for the moment that $A^2(D)$ is non trivial. We will shortly prove that $A^2(D)$ is a closed subspace of the Hilbert space $L^2(D)$ with norm and inner product given by

$$||f|| = \left(\int_D |f|^2 dV \right)^{1/2} \quad \text{and} \quad (f, g) = \int_D f(\zeta)\overline{g(\zeta)} dV(\zeta)$$

and hence it is a Hilbert space as well. Further $A^2(D)$ is a seperable Hilbert space so it will have a countable orthonormal basis. If D is bounded, then the space $A^2(D)$ will contain all polynomials and hence it will be non trivial. It is also clear that the space of polynomials will be infinite dimensional. This gives us the following remark.

Remark 2.1.1. *For bounded domains, the Bergman space is an infinite dimensional Hilbert space. So it will have a countably infinite orthonormal basis.*

Lemma 2.1.1. *Suppose $r > 0$ and $f \in A^2(P(a,r))$, then*

$$|f(a)| \leq \frac{1}{\pi^{n/2} r^n} ||f||_{P(a,r)} .$$

Proof. From the Taylor expansion of f on $P(a,r)$

$$f(z) = \sum_{\nu \in \mathbb{N}^n} c_\nu (z-a)^\nu ,$$

we have

$$||f||_{P(a,r)}{}^2 = \int_{P(a,r)} \sum_{\nu,\mu} c_\nu \overline{c_\mu} (z-a)^\nu \overline{(z-a)}^\mu dV(z) . \tag{2.1}$$

Changing to polar coordinates and taking the integral inside we get

$$\begin{aligned} ||f||_{P(a,r)}{}^2 &= \sum_{\nu,\mu} c_\nu \overline{c_\mu} \int_{[0,2\pi]^n} \int_{[0,r]^n} \rho^{\nu+\mu+1} e^{i\theta(\nu-\mu)} d\rho_1 ... d\rho_n d\theta_1 ... d\theta_n \\ &= \sum_{\nu,\mu} c_\nu \overline{c_\mu} \prod_{k=1}^{n} \int_0^r \rho_k^{\nu_k+\mu_k+1} d\rho_k \int_0^{2\pi} e^{i\theta_k(\nu_k-\mu_k)} d\theta_k \\ &= \sum_\nu |c_\nu|^2 (2\pi)^n \prod_{k=1}^{n} \frac{r^{2\nu_k+2}}{2\nu_k+2} . \end{aligned}$$

Therefore, by taking the $\nu = 0$ term of the summation we get

$$||f||_{P(a,r)}{}^2 \geq |f(a)| \pi^n r^{2n} \tag{2.2}$$

and this finishes the proof. □

Remark 2.1.2. *Another way to obtain this inequality would be to use the Cauchy Integral formula on polydiscs.*

Remark 2.1.3. *The above inequality shows $A^2(\mathbb{C}^n) = \{0\}$.*

Corollary 2.1.2. *Let D be a domain in $\mathbb{C}^n$ and $a \in D$. Then there exists a positive constant C which depends only on n such that*

$$|f(a)| \leq C \delta_D^{-n}(a) ||f||_D \qquad for\, all\, f \in A^2(D).$$

Corollary 2.1.3. *Let D be a domain in $\mathbb{C}^n$ and $K \subset D$ is compact. Then there exists a* *positive constant C_K which depends only on K such that*

$$|f|_K = \sup_{z\in K} |f(z)| \le C_K ||f||_D \qquad for\, all\, f \in A^2(D).$$

Proof. Fix $0 < \delta < dist(K, \partial D)$. By the previous corollary, we have $|f(z)| \le C\delta^{-n}||f||_D$ for all $z \in K$. So if we put $C_K = C\delta^{-n}$, we get the required result. □

This corollary allows us to show that the Bergman space is a Hilbert space.

Lemma 2.1.4. *Let D be a domain in $\mathbb{C}^n$. Then $A^2(D)$ is a closed subspace of $L^2(D)$ and therefore is itself a Hilbert space.*

Proof. Suppose we have a sequence f_m in $A^2(D)$ which converges to f in $L^2(D)$. Let $K \subset D$ be compact, by Cauchy criterion and the previous corollary, we get $|f_l - f_m|_K \le C_K||f_l - f_m|| \le \epsilon$ for sufficiently large l, m. So f_m converges uniformly on compact subsets to f. This implies f is holomorphic and hence belongs to $A^2(D)$. □

Now suppose we have a domain D in $\mathbb{C}^n$ such that $A^2(D)$ is non trivial. Consider the evaluation map $\tau_a : A^2(D) \to \mathbb{C}$ defined by $\tau_a(f) = f(a)$. By Corollary 2.1.2, τ_a is a bounded linear functional on $A^2(D)$. Therefore, by the Riesz representation theorem for Hilbert spaces, we have a unique element $K(., a) \in A^2(D)$ such that

$$\tau_a(f) = f(a) = (f, K_D(., a)) \qquad for\, all\, f \in A^2(D).$$

Definition 2.1.1. *The function $K_D : D \times D \to \mathbb{C}$ is called the Bergman kernel for D.*

Lemma 2.1.5. $K_D(\zeta, z) = \overline{K_D(z, \zeta)}$ *for all $z, \zeta \in D$.*

Proof. Since $K_D(., z)$ is in $A^2(D)$, we have

$$K_D(\zeta, z) = (K_D(., z), K_D(., \zeta)) = \overline{(K_D(., \zeta), K_D(., z))} = \overline{K_D(z, \zeta)}$$

as required. □

Therefore the Bergman kerenel is holomorphic in the first argument and conjugate holomorphic in the second. By Hartog's separate analyticity theorem, we have the following corollary.

Corollary 2.1.6. *Let $K'_D(\zeta, z) = K_D(\zeta, \overline{z})$. Then $K'_D(\zeta, z)$ is a holomorphic function on $D \times \overline{D}$. Also $K_D \in C^\infty(D \times D)$.*

Corollary 2.1.7. *Let $D \subset \mathbb{C}^n$ be a domain with $A^2(D) \neq 0$, then*

$$f(z) = \int_D f(\zeta)\overline{K_D(\zeta, z)}\, dV(\zeta) = \int_D K_D(z, \zeta) f(\zeta)\, dV(\zeta) \quad \textit{for all } f \in A^2(D), z \in D.$$

This property is called the reproducing property of the Bergman kernel.

The Riesz representation theorem gives an isometry between the Hilbert space and its dual space. Therefore we get $||K_D(., a)|| = ||\tau_a|| = \sup\big\{|f(a)| : f \in A^2(D), ||f|| \leq 1\big\}$, where $||\tau_a||$ refers to the operator norm of τ_a. Also

$$K_D(z, z) = \big(K_D(., z), K_D(., z)\big) = ||K_D(., z)||^2.$$

We usually denote $K_D(z, z)$ by $K_D(z)$. This proves the following lemma.

Lemma 2.1.8. *We have $K_D(z) = \sup\big\{|f(z)|^2 : f \in A^2(D), ||f|| \leq 1\big\}$. Consequently if $D_1 \subset D_2$ are domains in $\mathbb{C}^n$. Then*

$$K_{D_2}(z) \leq K_{D_1}(z) \quad \textit{for all } z \in D_1.$$

We can represent the Bergman kernel using an orthonormal basis (complete orthonormal system) of $A^2(D)$. Let $\{\phi_j\}_{j \in \mathbb{N}}$ be an orthonormal basis and $f \in A^2(D)$. Then we have

$$f = \sum_{j=1}^{\infty} (f, \phi_j)\phi_j\,.$$

The convergence is in $A^2(D)$, but using Corollary 2.1.3 we can easily show that the above series converges uniformly on compact sets to f. Before we prove the representaion formula for Bergman kernel we need the following lemma, which is a slight variant of Montel's theorem.

Lemma 2.1.9. *Let D be a domain in $\mathbb{C}^n$. Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of holomorphic functions on D that converges pointwise to a function f on D. Suppose $\{f_j\}_{j\in\mathbb{N}}$ is uniformly bounded on every compact subset of D. Then $f_j \to f$ uniformly on compact subsets of D.*

Proof. Assume to the contrary that f_j does not converge uniformly to f on compact set K. We can find an $\epsilon > 0$ and a subsequence f_{j_i} such that $|f_{j_i}(z_i) - f(z_i)| > \epsilon$ for some $z_i \in K$. Hence f_{j_i} does not have a subsequence which converges uniformly to f on K. But because $f_j \to f$ pointwise, by Montel's theorem we will have a subsequence of f_{j_i} which converges uniformly to f on all compact sets. This is a contradiction. Hence proved. $\square$

Theorem 2.1.10. *Let $\{\phi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $A^2(D)$. We then have the following representation of the Bergman kernel:*

$$K_D(\zeta, z) = \sum_{j=1}^{\infty} \phi_j(\zeta)\overline{\phi_j(z)}$$

where the series converges absolutely and uniformly on compact subsets of $D \times D$.

Proof. Since $K_D(., z) \in A^2(D)$ we have

$$K_D(., z) = \sum_{j=1}^{\infty} (K_D(., z), \phi_j)\phi_j \,. \tag{2.3}$$

By the reproducing property, we get

$$K_D(\zeta, z) = \sum_{j=1}^{\infty} \phi_j(\zeta)\overline{\phi_j(z)}\,. \tag{2.4}$$

By reordering $\{\phi_j\}$ we would still get convergence in (2.3) to $K_D(., z)$. Hence all rearrangements of the series above converge to $K_D(\zeta, z)$. Therefore the series converges absolutely. Note that

$$K_D(z, z) = \sum_{j=1}^{\infty} |\phi_j(z)|^2. \tag{2.5}$$

For the uniform convergence, let $K \subset D$ be a compact set. Then by Hölder's inequality on

$\mathbb{N}$ with the counting measure, we get

$$\sum_{j=1}^{\infty} |\phi_j(\zeta)||\phi_j(z)| \leq \left(\sum_{j=1}^{\infty} |\phi_j(\zeta)|^2\right)^{1/2} \left(\sum_{j=1}^{\infty} |\phi_j(z)|^2\right)^{1/2} = K_D(\zeta,\zeta)K_D(z,z) \leq C_K \quad (2.6)$$

for all z, ζ in K, since $K_D(z,z)$ is a continuous function of z.

So let $K'_D(\zeta, z) = K_D(\zeta, \overline{z})$ and $f_n(\zeta, z) = \sum_{j=1}^{n} \phi_j(\zeta)\overline{\phi_j(\overline{z})}$. These functions are holomorphic. We have seen that f_n converges to K'_D pointwise, also f_n is uniformly bounded on compact subsets of $D \times \overline{D}$ by (2.6). Therefore by Lemma 2.1.9, $f_n \to K'_D$ uniformly on compact subsets. The result follows easily from this. □

Remark 2.1.4. *For bounded domains, constant functions are square integrable and so at a particular point not all the orthonormal basis elements can vanish. Hence $K(z,z) > 0$ throughout the domain.*

2.2 The Bergman kernel for a few standard domains

2.2.1 Bergman kernel for the unit disc

The orthonormal basis representation formula is a useful tool to calculate the Bergman kernel for some domains. We will first calculate the Bergman kernel for the unit disc in $\mathbb{C}$, denoted by $\mathbb{D}$.

First we need to construct an orthogonal basis. Let n, m be nonzero integers. Consider

$$\begin{aligned}
(z^n, z^m) &= \int_{\mathbb{D}} \zeta^n \overline{\zeta^m} dV(\zeta) \\
&= \int_0^1 \int_0^{2\pi} r^{n+m+1} e^{i\theta(n-m)} d\theta dr \\
&= \left(\int_0^1 r^{n+m+1} dr\right)\left(\int_0^{2\pi} e^{i\theta(n-m)} d\theta\right) \\
&= \frac{1}{n+m+2}\delta_{mn} 2\pi\,.
\end{aligned} \tag{2.7}$$

Note that the above calculation shows that $||z^n|| = \sqrt{\pi/(n+1)}$. Let

$$\phi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n.$$

Then $\{\phi_n\}_{n=0}^{\infty}$ forms an orthonormal system in $A^2(\mathbb{D})$. Suppose $f \in A^2(\mathbb{D})$, then

$$f(z) = \sum_{k=0}^{\infty} c_k z^k.$$

Here the partial sums converge to f uniformly on compact subsets of $\mathbb{D}$. From this we can deduce the above series converges in norm i.e in $A^2(\mathbb{D})$. This implies $\{\phi_n\}_{n=0}^{\infty}$ is a complete orthonormal system in $A^2(\mathbb{D})$. We may now apply the formula from Theorem 2.1.10 to get

$$K_{\mathbb{D}}(z,\zeta) = \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(\zeta)} = \sum_{n=0}^{\infty} \frac{n+1}{\pi} z^n \overline{\zeta^n} = \frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)(z\bar{\zeta})^n \,. \tag{2.8}$$

Differentiating $1/(1-z) = 1 + z + z^2 + ...$, we get

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + ... = \sum_{n=0}^{\infty}(n+1)z^n$$

for $|z| < 1$. Therefore

$$K_{\mathbb{D}}(z,\zeta) = \frac{1}{\pi}\frac{1}{(1-z\bar{\zeta})^2} \,. \tag{2.9}$$

2.2.2 Bergman kernel for ellipsoid and unit ball in $\mathbb{C}^n$

The next domain one would naturally consider would be the unit ball in $\mathbb{C}^n$ or the polydisc $\mathbb{D}^n$. We shall shortly prove that the Bergman kernel of a product domain will be the product of the Bergman kernels. This will give us the Bergman kernel for $\mathbb{D}^n$. Let us now calculate the Bergman kernel for the unit ball. We will actually calculate the Bergman kernel for a broader class of domains which includes the unit ball. The set

$$E = \left\{ z \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^{2p_i} < 1 \right\} \tag{2.10}$$

where $p_i > 0$ for all i is called a complex ellipsoid or egg-shaped domain. D'Angelo calculated the Bergman kernel for ellipsoids in [7] and [8] and we now present his calculation.

Lemma 2.2.1. *The set* $\left\{z^\alpha/||z^\alpha||\right\}_{\alpha\in\mathbb{N}^n}$ *forms a complete orthonormal system for* $A^2(E)$*, where* E *is a complex ellipsoid.*

Proof. Let $\mu,\nu \in \mathbb{N}^n$. Consider

$$(z^\mu, z^\nu) = \int_E z^\mu \overline{z^\nu} dV(z)\,.$$

Now after making the change of variables $z_i = r_i e^{i\theta}$, where $0 \le \theta_i < 2\pi$ and $r \in E' = \{r \in \mathbb{R}^n : r_i > 0,\ \sum_i r_i^{2p_i} < 1\}$, we get

$$(z^\mu, z^\nu) = \int_{E'} \int_0^{2\pi} \dots \int_0^{2\pi} r^{\mu+\nu+1} e^{i\theta(\mu-\nu)} d\theta_1 \dots d\theta_n dV(r) = (2\pi)^n \delta_{\mu\nu} \int_{E'} r^{\mu+\nu+1} dV(r)\,. \quad (2.11)$$

Therefore $\left\{z^\alpha/||z^\alpha||\right\}_{\alpha\in\mathbb{N}^n}$ is a orthonormal system in $A^2(E)$.

Let $f \in A^2(E)$. Because these domains are complete Reinhardt domains, the Taylor series of f around the origin converges everywhere on E. So we can write $f(z) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha z^\alpha$ for all $z \in E$. Any ordering of the series converges uniformly on compact subsets, thus we have convergence in norm. This proves the completeness. ☐

The next step would be to evaluate $||z^\alpha||$. For this purpose we will need some theory about gamma and beta functions. The gamma function is defined as

$$\Gamma : (0,\infty) \to \mathbb{R}, \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt\,.$$

Note that by applying the change of variable $t = s^2$, we get

$$\Gamma(x) = 2\int_0^\infty e^{-s^2} s^{2x-1} ds\,. \quad (2.12)$$

Below are few of the basic properties of the gamma function

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1) = 1, \quad \Gamma(n+1) = n!\,.$$

Now we define the beta function. Let $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i > 0\, \forall i\}$. The beta function is defined on $\mathbb{R}_+^m$ as

$$\beta(\alpha) = \frac{\prod_i \Gamma(\alpha_i)}{\Gamma(|\alpha|)} \quad \forall \alpha \in \mathbb{R}_+^m \,. \tag{2.13}$$

Before we state the next proposition we need to define a few objects

$$B_+^m = \mathbb{R}_+^m \cap B^m, \quad S_+^{m-1} = \mathbb{R}_+^m \cap S^{m-1}$$

where B^m and S^{m-1} are the unit ball and unit sphere in $\mathbb{R}^m$ respectively.

Proposition 2.2.2. *Suppose* $\alpha \in \mathbb{R}_+^m$. *Then*

$$\int_{B_+^m} r^{2\alpha-1} dV(r) = \frac{\beta(\alpha)}{2^m|\alpha|}$$

and

$$\int_{S_+^{m-1}} \omega^{2\alpha-1} d\sigma(\omega) = \frac{\beta(\alpha)}{2^{m-1}} \,.$$

Proof. By (2.12), we get

$$\prod_{i=1}^n \Gamma(\alpha_i) = 2^m \int_{\mathbb{R}_+^m} e^{-||x||^2} x^{2\alpha-1} dV(x) \,.$$

Switching to spherical coordinates we get

$$\begin{aligned}
\prod_{i=1}^n \Gamma(\alpha_i) &= 2^m \int_0^\infty \int_{S_+^{m-1}} e^{-r^2} r^{2|\alpha|-m} \omega^{2\alpha-1} r^{m-1} d\sigma(\omega) dr \\
&= 2^m \left(\int_0^\infty e^{-r^2} r^{2|\alpha|-1} \right) \left(\int_{S_+^{m-1}} \omega^{2\alpha-1} d\sigma(\omega) \right) \\
&= 2^{m-1} \Gamma(|\alpha|) \left(\int_{S_+^{m-1}} \omega^{2\alpha-1} d\sigma(\omega) \right) .
\end{aligned}$$

Hence

$$\int_{S_+^{m-1}} \omega^{2\alpha-1} d\sigma(\omega) = \frac{\prod_{i=1}^n \Gamma(\alpha_i)}{2^{m-1}\Gamma(|\alpha|)} = \frac{\beta(\alpha)}{2^{m-1}} \,. \tag{2.14}$$

Again using spherical coordinates, we have

$$\begin{aligned}\int_{B_+^m} r^{2\alpha-1}dV(r) &= \int_0^1\int_{S_{m-1}^+} r^{2|\alpha|-m}\omega^{2\alpha-1}r^{m-1}d\sigma(\omega)dr \\ &= \left(\int_0^1 r^{2|\alpha|-1}dr\right)\left(\int_{S_+^{m-1}}\omega^{2\alpha-1}d\sigma(\omega)\right) = \frac{\beta(\alpha)}{2^m|\alpha|}\end{aligned} \tag{2.15}$$

which proves the proposition. □

With these formulas we can calculate the norms of the monomials z^α ($\alpha \in \mathbb{N}^n$) in $A^2(E)$.

Lemma 2.2.3. *Let* $E = \{z \in \mathbb{C}^n : \sum_{i=1}^n |z_i|^{2p_i} < 1\}$ *and* $\alpha \in \mathbb{N}^n$*, then*

$$||z^\alpha||^2_{A^2(E)} = \frac{\pi^n}{\prod_i p_i}\frac{\beta\left(\frac{\alpha+1}{p}\right)}{\left|\frac{\alpha+1}{p}\right|}\,.$$

Proof. By (2.11), we have

$$(z^\alpha, z^\alpha) = (2\pi)^n\int_{E'} r^{2\alpha+1}dV(r)\,, \tag{2.16}$$

where $E' = \{r \in \mathbb{R}_+^n : \sum_i r_i^{2p_i} < 1\}$. Hence

$$\begin{aligned}||z^\alpha||^2 &= (2\pi)^n\int_{E'} r^{2\alpha+1}dV(r) \\ &= (2\pi)^n\int_{E'} r_1^{2\alpha_1+1}...r_n^{2\alpha_n+1}dr_1..dr_n \\ &= \frac{(2\pi)^n}{\prod_i p_i}\int_{E'} r_1^{2\alpha_1+1-(p_1-1)}...r_n^{2\alpha_n+1-(p_n-1)}(p_1r_1^{p_1-1}dr_1)...(p_nr_n^{p_n-1}dr_n)\,.\end{aligned}$$

Now we make the substitution $t_i = r_i^{p_i}$ i.e $r_i = t_i^{\frac{1}{p_i}}$. Also $dt_i = p_ir_i^{p_i-1}dr_i$. The integrals transforms into

$$||z^\alpha||^2 = \frac{(2\pi)^n}{\prod_i p_i}\int_{B_+^n} t^{\frac{2\alpha+2-p}{p}}dV(t)\,. \tag{2.17}$$

We write $(2\alpha+2-p)/p = 2\gamma-1$. Solving for γ, we get $\gamma_i = (\alpha_i+1)/p_i > 0$. So

$$||z^\alpha||^2 = \frac{(2\pi)^n}{\prod_i p_i}\int_{B_+^n} t^{2\gamma-1}dV(t) = \frac{(2\pi)^n}{\prod_i p_i}\frac{\beta(\gamma)}{2^n|\gamma|} = \frac{\pi^n}{\prod_i p_i}\frac{\beta\left(\frac{\alpha+1}{p}\right)}{\left|\frac{\alpha+1}{p}\right|} \tag{2.18}$$

as required. $\square$

Using the orthonormal basis representation for the Bergman kernel, we get the following proposition.

Proposition 2.2.4. *For $E = \{z \in \mathbb{C}^n : \sum_{i=1}^n |z_i|^{2p_i} < 1\}$, where $p_i > 0$, we have*

$$K_E(z,\zeta) = \sum_{\alpha \in \mathbb{N}^n} \frac{\prod_i p_i}{\pi^n} \frac{\left|\frac{\alpha+1}{p}\right|}{\beta\left(\frac{\alpha+1}{p}\right)} (z\overline{\zeta})^\alpha .$$

Theorem 2.2.5. *Denote the unit ball in $\mathbb{C}^n$ i.e. $\{z \in \mathbb{C}^n : \sum_{i=1}^n |z_i|^2 < 1\}$ by $\mathbb{B}^n$, then*

$$K_{\mathbb{B}^n}(z,\zeta) = \frac{n!}{\pi^n} \frac{1}{[1-(z,\zeta)]^{n+1}} ,$$

where (z,ζ) is the Hermitian inner product of z and ζ in $\mathbb{C}^n$.

Proof. The unit ball is a complex ellipsoid where $p_i = 1$ for all i. Hence, we get

$$\begin{aligned}
K_{\mathbb{B}^n}(z,\zeta) &= \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\pi^n} \frac{|\alpha+1|}{\beta(\alpha+1)} (z\overline{\zeta})^\alpha \\
&= \frac{1}{\pi^n} \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{m+n}{\frac{\prod_i \Gamma(\alpha_i+1)}{\Gamma(m+n)}} (z\overline{\zeta})^\alpha \\
&= \frac{1}{\pi^n} \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{(m+n)!}{\alpha!} (z\overline{\zeta})^\alpha \qquad (2.19) \\
&= \frac{1}{\pi^n} \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (z\overline{\zeta})^\alpha \\
&= \frac{1}{\pi^n} \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} \left(\sum_{i=0}^{n} z_i \overline{\zeta_i} \right)^m .
\end{aligned}$$

The last step uses the multinomial theorem. Note that the power series

$$\frac{1}{1-u} = 1 + u + u^2 + ...$$

converges in $\mathbb{D}$, and differentiating n times we get

$$\frac{n!}{(1-u)^{n+1}} = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} u^m \,.$$

Again this equation holds in $\mathbb{D}$. We have $\left|\sum_{i=0}^{n} z_i \overline{\zeta_i}\right| \leq ||z||\,||\zeta|| < 1$. Therefore

$$K_{\mathbb{B}^n}(z,\zeta) = \frac{n!}{\pi^n \left[1 - \left(\sum_{i=0}^{n} z_i \overline{\zeta_i}\right)\right]^{n+1}} \tag{2.20}$$

which proves the theorem. □

2.2.3 Bergman kernel for product domains

Proposition 2.2.6. *Let D_1 and D_2 be domains in $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$ respectively. Then the Bergman kernel for $D = D_1 \times D_2$ is given by*

$$K_D((z_1,z_2),(\zeta_1,\zeta_2)) = K_{D_1}(z_1,\zeta_1) K_{D_2}(z_2,\zeta_2) \,.$$

Proof. Let $G((z_1,z_2),(\zeta_1,\zeta_2)) = K_{D_1}(z_1,\zeta_1)K_{D_2}(z_2,\zeta_2)$. Then $G(.,(a_1,a_2))$ is an element of $A^2(D)$ by Hartog's separate analyticity theorem and Tonelli's theorem. For $f \in A^2(D)$, we have

$$\begin{aligned}
\int_D f(\zeta)\overline{G(\zeta,a)}\, dV_\zeta &= \int_{D_1 \times D_2} f(\zeta_1,\zeta_2)\overline{G((\zeta_1,\zeta_2),(a_1,a_2))} dV_\zeta \\
&= \int_{D_2}\int_{D_1} f(\zeta_1,\zeta_2)\overline{G((\zeta_1,\zeta_2),(a_1,a_2))} dV_{\zeta_1} dV_{\zeta_2} \\
&= \int_{D_2} \overline{K_{D_2}(\zeta_2,a_2)} \int_{D_1} f(\zeta_1,\zeta_2)\overline{K_{D_1}(\zeta_1,a_1)} dV_{\zeta_1} dV_{\zeta_2} \\
&= \int_{D_2} f(a_1,\zeta_2)\overline{K_{D_2}(\zeta_2,a_2)} dV_{\zeta_2} \\
&= f(a_1,a_2)\,.
\end{aligned} \tag{2.21}$$

The proposition clearly follows. □

Corollary 2.2.7. *The Bergman kernel for the polydisc $\mathbb{D}^n$ is given by*

$$K_{\mathbb{D}^n}(z,\zeta) = \frac{1}{\pi^n}\prod_{j=1}^{n}\frac{1}{(1-z_j\overline{\zeta_j})^2}\,.$$

2.3 The Bergman kernel and minimum integrals

It is possible to define the Bergman kernel in an alternative way, using minimum integrals. We however state it here as a property of the Bergman kernel. Consider the extremal problem:

$$\text{minimize } ||\phi||^2 = \int_D |\phi|^2 dV \text{ subject to } \phi \in A^2(D) \text{ and } \phi(\zeta) = 1.$$

Theorem 2.3.1. *$K_D(.,\zeta)/K_D(\zeta,\zeta)$ is the unique solution to the above extermal problem, provided that the set $S = \{\phi \in A^2(D) : \phi(\zeta) = 1\}$ is non empty or equivalently $K(\zeta,\zeta) > 0$.*

Proof. Let $f \in S$ and consider its orthnormal basis expansion

$$f = \sum_{j=1}^{\infty} a_j\phi_j$$

where $a_j = (f,\phi_j)$. Since $f(\zeta) = 1$, we have $\sum_{j=1}^{\infty} a_j\phi_j(\zeta) = 1$. Also $||f||^2 = \sum_{j=1}^{\infty}|a_j|^2$. Let $K_D(\zeta,\zeta) = \sum_{j=1}^{\infty}|\phi_j(\zeta)|^2 = \sigma$. Note that $\sigma > 0$. We now introduce new constants γ_j, defined by $a_j = \big(\gamma_j + \overline{\phi_j(\zeta)}\big)/\sigma$. Then

$$1 = \sum_{j=1}^{\infty}\left(\frac{\gamma_j + \overline{\phi_j(\zeta)}}{\sigma}\right)\phi_j(\zeta) = \left(\sum_{j=1}^{\infty}\gamma_j\phi_j(\zeta)\right) + 1\,.$$

This gives us

$$\sum_{j=1}^{\infty}\gamma_j\phi_j(\zeta) = 0\,. \tag{2.22}$$

Using the above equation, we get

$$||f||^2 = \sum_{j=1}^{\infty}|a_j|^2 = \sum_j\left(\frac{\gamma_j + \overline{\phi_j(\zeta)}}{\sigma}\right)\left(\frac{\overline{\gamma_j} + \phi_j(\zeta)}{\sigma}\right)$$

$$= \frac{1}{\sigma^2}\left(\sum_j \left(|\gamma_j|^2 + |\phi_j(\zeta)|^2\right)\right)$$

$$= \frac{1}{\sigma^2}\left(\sigma + \sum_j |\gamma_j|^2\right).$$

So the unique solution to the extremal problem, call it ψ, is when $\gamma_j = 0$ for all j and consequently $a_j = \overline{\phi_j(\zeta)}/\sigma$ for all j. So the solution is given by

$$\psi(z) = \sum_j \frac{\overline{\phi_j(\zeta)}}{\sigma}\phi_j(z) = \frac{1}{\sigma}\sum_j \phi_j(z)\overline{\phi_j(\zeta)} = \frac{K_D(z,\zeta)}{K_D(\zeta,\zeta)} \tag{2.23}$$

as claimed. $\square$

Note that the minimal norm is given by $\left|\left|\dfrac{K_D(.\zeta)}{K_D(\zeta,\zeta)}\right|\right|^2 = \dfrac{1}{K_D(\zeta,\zeta)}$.

2.4 The Bergman projection operator

Since $A^2(D)$ is a closed subspace of $L^2(D)$, we have $L^2(D) = A^2(D)\bigoplus A^2(D)^\perp$, i.e every $f \in L^2(D)$ can be written in a unique way as $f = f_1 + f_2$ where $f_1 \in A^2(D)$ and $f_2 \in A^2(D)^\perp$. The map $P_D : L^2(D) \to A^2(D)$ which sends f to f_1 is called the Bergman projection operator.

Theorem 2.4.1. *The Bergman projection operator $P_D : L^2(D) \to A^2(D)$ is given by*

$$P_D f(z) = \int_D K(z,\zeta)f(\zeta)dV_\zeta$$

for all $f \in L^2(D)$.

Proof. Let $f \in L^2(D)$. As discussed above we can write $f = f_1 + f_2$, where $f_1 \in A^2(D)$ and $f_2 \in A^2(D)^\perp$. Using the fact that $f_2 \perp K(.,z)$, we get

$$P_D f(z) = f_1(z) = (f_1, K(.,z)) = (f_1, K(.,z)) + (f_2, K(.,z)) = (f, K(.,z)) = \int_D K(z,\zeta)f(\zeta)dV_\zeta\,.$$

$\square$

2.5 Transformation formula for the Bergman kernel

The following theorem tells how the Bergman kernel transforms under biholomorphisms. Later on this formula will be used to show that biholomorphisms are isometries of the Bergman metric. We shall derive a similar transformation formula for proper holomorphic maps in Chapter 4.

Theorem 2.5.1. *Let D_1, D_2 be two domains in $\mathbb{C}^n$ and $f : D_1 \to D_2$ be a biholomorphism. Then*

$$K_{D_1}(z,\zeta) = \det f'(z)\ K_{D_2}(f(z), f(\zeta))\ \overline{\det f'(\zeta)}$$

for all $z, \zeta \in D_1$. This formula is called the transformation formula for Bergman kernel.

Proof. It is a basic fact of several complex variables that

$$\det J_{\mathbb{R}} f(z) = |\det f'(z)|^2 \,. \tag{2.24}$$

Therefore by the change of variables formula for integration we get

$$\int_{D_2} |h(w)|^2 dV(w) = \int_{D_1} |h(f(z))|^2 |\det J_{\mathbb{R}} f(z)| dV(z) = \int_{D_2} |h(f(z))|^2 |\det f'(z)|^2 dV_z \,. \tag{2.25}$$

So the function $h \mapsto (h \circ f) \det f'$ is an isometry from $L^2(D_2)$ to $L^2(D_1)$. Consequently we have

$$\det f'(.) K_{D_2}(f(.), f(\zeta)) \overline{\det f'(\zeta)} \in A^2(D_1)$$

for all $\zeta \in D_1$. Let $g \in A^2(D_1)$. Then we have

$$\begin{aligned}
\Big(g, \det f'(.) K_{D_2}(f(.), f(\zeta)) \overline{\det f'(\zeta)}\Big) &= \int_{D_1} g(z) \overline{\det f'(z) K_{D_2}(f(z), f(\zeta))} \det f'(\zeta) dV_z \\
&= \det f'(\zeta) \int_{D_2} g(f^{-1}(y)) \overline{\det f'(f^{-1}(y)) K_{D_2}(y, f(\zeta))} |\det (f^{-1})'(y)|^2 dV_y \\
&= \det f'(\zeta) \int_{D_2} g(f^{-1}(y)) \det (f^{-1})'(y) \overline{K_{D_2}(y, f(\zeta))} dV_y \\
&= \det f'(\zeta) g(\zeta) \det (f^{-1})'(f(\zeta)) \\
&= g(\zeta) \,.
\end{aligned}$$

This proves the result. $\square$

Proposition 2.5.2. *Let D_1, D_2 be two domains in $\mathbb{C}^n$ and $f : D_1 \to D_2$ be a biholomorphism. Let $u(z) = \det f'(z)$. Then*

$$P_{D_1}(u \cdot (h \circ f)) = u \cdot (P_{D_2}(h) \circ f)$$

for all $h \in L^2(D_2)$.

Proof. Since $h \in L^2(D_2)$, we have $u \cdot (h \circ f) \in L^2(D_1)$ by (2.25). By Theorem 2.4.1, we have

$$\begin{aligned}
[P_{D_1}(u \cdot (h \circ f))](z) &= \int_{D_1} K_{D_1}(z, \zeta) u(\zeta) h(f(\zeta)) dV_\zeta \\
&= \int_{D_1} u(z) K_{D_2}(f(z), f(\zeta)) |u(\zeta)|^2 h(f(\zeta)) dV_\zeta \\
&= u(z) \int_{D_2} K_{D_2}(f(z), y) h(y) dV_y \\
&= u(z)(P_{D_2}(h) \circ f)(z) .
\end{aligned}$$

□

2.6 The Bergman metric

Using the Bergman kernel we will give a metric on D i.e at every point $p \in D$ we will give an inner product on $T_p(D) = \mathbb{C}^n$. First we prove the following theorem.

Theorem 2.6.1. *Let D be a bounded domain in $\mathbb{C}^n$. Then $\log K(z)$ is a strictly plurisubharmonic function where $K(z) = K_D(z, z)$.*

Proof. From Remark 2.1.4, we see that $K(z) > 0$ and we may take real logarithm of such a number. We need to show

$$\sum_{j,k} \frac{\partial^2 \log K}{\partial z_j \partial \bar{z}_k}(p) t_j \overline{t_k} > 0 \quad for\, all\, p \in D,\; for\, all\, t \in \mathbb{C}^n \setminus \{0\} . \tag{2.26}$$

Using the chain rule for complex differential operators, we have

$$\frac{\partial \log K}{\partial \bar{z}_k} = \frac{1}{K} \frac{\partial K}{\partial \bar{z}_k} .$$

Further

$$\begin{aligned}\frac{\partial^2 \log K}{\partial z_j \partial \bar{z}_k} &= \frac{1}{K}\frac{\partial^2 K}{\partial z_j \partial \bar{z}_k} + \left(-\frac{1}{K^2}\frac{\partial K}{\partial z_j}\right)\frac{\partial K}{\partial \bar{z}_k} \\ &= \frac{1}{K^2}\left(K\frac{\partial^2 K}{\partial z_j \partial \bar{z}_k} - \frac{\partial K}{\partial z_j}\frac{\partial K}{\partial \bar{z}_k}\right).\end{aligned} \tag{2.27}$$

Also

$$K = \sum_{\mu} \phi_\mu \overline{\phi_\mu}\,.$$

Due to Theorem 2.1.10 it is permissible to take the derivative operators inside. Therefore we get

$$\frac{\partial K}{\partial \bar{z}_k} = \sum_{\mu} \phi_\mu \overline{\frac{\partial \phi_\mu}{\partial z_k}} \quad \mathit{and} \quad \frac{\partial K}{\partial z_j} = \sum_{\mu} \frac{\partial \phi_\mu}{\partial z_j}\overline{\phi_\mu} \tag{2.28}$$

and

$$\frac{\partial^2 K}{\partial z_j \partial \bar{z}_k} = \sum_{\mu} \frac{\partial \phi_\mu}{\partial z_j}\overline{\frac{\partial \phi_\mu}{\partial z_k}}\,. \tag{2.29}$$

So combining (2.27), (2.28) and (2.29), we get

$$\frac{\partial^2 \log K}{\partial z_j \partial \bar{z}_k} = \frac{1}{K^2}\left[\left(\sum_{\mu} |\phi_\mu|^2\right)\left(\sum_{\mu} \frac{\partial \phi_\mu}{\partial z_j}\overline{\frac{\partial \phi_\mu}{\partial z_k}}\right) - \left(\sum_{\mu} \frac{\partial \phi_\mu}{\partial z_j}\overline{\phi_\mu}\right)\left(\sum_{\mu} \phi_\mu \overline{\frac{\partial \phi_\mu}{\partial z_k}}\right)\right] \tag{2.30}$$

and so

$$\sum_{j,k} \frac{\partial^2 \log K}{\partial z_j \partial \bar{z}_k} t_j \overline{t_k} = \frac{1}{K^2}\left[\left(\sum_{\mu} |\phi_\mu|^2\right)\left(\sum_{\mu,j,k} \frac{\partial \phi_\mu}{\partial z_j} t_j \overline{\frac{\partial \phi_\mu}{\partial z_k} t_k}\right) - \left(\sum_{\mu,j} \frac{\partial \phi_\mu}{\partial z_j} t_j \overline{\phi_\mu}\right)\left(\sum_{\mu,k} \phi_\mu \overline{\frac{\partial \phi_\mu}{\partial z_k} t_k}\right)\right].$$

Let $\phi'_\mu = \sum_{i=1}^{n} \dfrac{\partial \phi_\mu}{\partial z_i} t_i$. Then the above equation becomes

$$\sum_{j,k} \frac{\partial^2 \log K}{\partial z_j \partial \bar{z}_k} t_j \overline{t_k} = \frac{1}{K^2}\left[\left(\sum_{\mu} |\phi_\mu|^2\right)\left(\sum_{\mu} |\phi'_\mu|^2\right) - \left|\sum_{\mu} \phi_\mu \overline{\phi'_\mu}\right|^2\right]. \tag{2.31}$$

By Cauchy-Schwarz inequality the above expression is always non-negative. Suppose that

$$\sum_{j,k} \frac{\partial^2 \log K}{\partial z_j \partial \bar{z}_k}(p) t_j \overline{t_k} = 0$$

for some $p \in D$ and $t \neq 0$. Then the sequences $\phi_\mu(p)$ and $\phi'_\mu(p,t)$ are linearly dependent. We have already seen that the sequence $\phi_\mu(p)$ is non-zero, now we will see that $\phi'_\mu(p,t)$ is a non-zero sequence. Let $f = \sum a_\mu \phi_\mu$ be an element in $A^2(D)$ and $l(\zeta) = p + \zeta t$ is a function defined in a neighborhood of 0 in the complex plane. Consider

$$\left.\frac{\partial}{\partial \zeta}\right|_{\zeta=0} (f \circ l) = \sum_\mu a_\mu (\phi_\mu \circ l)'(0) = \sum_\mu a_\mu \phi'_\mu(p,t)\,. \tag{2.32}$$

Let $f(z) = (z-p,t)$. Note that $f \in A^2(D)$, also $(f \circ l)(\zeta) = \zeta |t|^2$. So if we choose such an f, then

$$\left.\frac{\partial}{\partial \zeta}\right|_{\zeta=0} (f \circ l) = |t|^2 \neq 0\,.$$

Therefore from (2.32), we get that $\phi'_\mu(p,t)$ is a non zero sequence. Thus there exists non zero complex number λ such that $\phi_\mu(p) = \lambda \phi'_\mu(p,t)$ for all μ. (2.32) implies $f(p) = \lambda (f \circ l)'(0)$ for all $f \in A^2(D)$. Again put $f(z) = (z-p,t)$ to get $0 = \lambda |t|^2$. But both λ and t are non zero. So we get a contradiction. So $\log K$ is strictly plurisubharmonic. □

Consider the matrix $H_{jk}(p) = (\partial^2 \log K)/(\partial z_j \partial \bar{z}_k)(p)$. This matrix is Hermitian (i.e $H^* = H$) and positive definite (i.e $t^* H t > 0$ for all $t \neq 0$) for all $p \in D$. So we get an inner product on $T_p(D) = \mathbb{C}^n$ for each point $p \in D$ given by

$$B_p^D(u,v) = \sum_{j,k} \frac{\partial^2 \log K}{\partial z_j \partial \bar{z}_k}(p) u_j \overline{v_k} = v^* [H(p)]^T u\,. \tag{2.33}$$

This is called the Bergman metric. The length of $v \in T_p(D)$ is given by

$$L_p^D(v) = \sqrt{B_p^D(v,v)}\,. \tag{2.34}$$

If $\gamma : [0,1] \to D$ is a piece-wise smooth C^1 curve in D, then its length w.r.t the Bergman metric is given by

$$l_B^D(\gamma) = \int_0^1 L^D_{\gamma(t)}(\gamma'(t))\, dt\,. \tag{2.35}$$

We define $d_B^D(p_1,p_2)$ to be the infimum of lengths (w.r.t the Bergman metric) of all piece-wise smooth C^1 curves joining p_1 and p_2. A geodesic is a path between two points of minimal length. So $d_B^D(p_1,p_2)$ can be defined as length of geodesic. It can be shown that $d_B^D(p_1,p_2)$

defines a topological metric on D.

Next we will use the transformation formula of the Bergman kernel to show that biholomorphisms are isometries of the Bergman metric.

Theorem 2.6.2. *Let $f : D_1 \to D_2$ be a biholomorphisms between two bounded domains in $\mathbb{C}^n$. Then*

$$B_p^{D_1}(u,v) = B_{f(p)}^{D_2}(f'(p)u, f'(p)v)$$

So f is an isometry of the Bergman metric.

Proof. Let $K_1(z) = K_{D_1}(z,z)$ and $K_2(w) = K_{D_2}(w,w)$. By the transformation formula, we get $K_1(z) = K_2(f(z)) \det f'(z)\overline{\det f'(z)}$ for all $z \in D_1$. Given any point $p \in D_1$, we can find a neighborhood of p such that the following equation holds there:

$$\log K_1 = \log(K_2 \circ f) + \log\det f' + \log\overline{\det f'} + C \tag{2.36}$$

where log is an appropriately chosen holomorphic branch of the logarithm. Hence

$$\frac{\partial^2 \log K_1}{\partial z_j \partial \bar{z}_k} = \frac{\partial^2 \log(K_2 \circ f)}{\partial z_j \partial \bar{z}_k} + \frac{\partial^2 \log\det f'}{\partial z_j \partial \bar{z}_k} + \frac{\partial^2 \log\overline{\det f'}}{\partial z_j \partial \bar{z}_k}.$$

The last two terms will vanish leaving us with

$$\begin{aligned}\frac{\partial^2 \log K_1}{\partial z_j \partial \bar{z}_k} &= \frac{\partial^2 \log(K_2 \circ f)}{\partial z_j \partial \bar{z}_k} \\ &= \frac{1}{(K_2 \circ f)^2}\left((K_2 \circ f)\cdot \frac{\partial^2 (K_2 \circ f)}{\partial z_j \partial \bar{z}_k} - \frac{\partial (K_2 \circ f)}{\partial z_j} \cdot \frac{\partial (K_2 \circ f)}{\partial \bar{z}_k}\right).\end{aligned} \tag{2.37}$$

The last equation comes the same way as (2.27) comes. Using chain rule, we get

$$\frac{\partial (K_2 \circ f)}{\partial \bar{z}_k} = \sum_{i=1}^n \Big(\frac{\partial K_2}{\partial \bar{z}_i} \circ f\Big)\frac{\partial \bar{f}_i}{\partial \bar{z}_k} = \sum_{i=1}^n \Big(\frac{\partial K_2}{\partial \bar{z}_i} \circ f\Big)\overline{\frac{\partial f_i}{\partial z_k}}. \tag{2.38}$$

Differentiating with respect to z_j, we get

$$\frac{\partial^2 (K_2 \circ f)}{\partial z_j \partial \bar{z}_k} = \sum_{i=1}^n \frac{\partial}{\partial z_j}\Big(\frac{\partial K_2}{\partial \bar{z}_i} \circ f\Big)\overline{\frac{\partial f_i}{\partial z_k}} = \sum_{i,l}\Big(\frac{\partial^2 K_2}{\partial z_l \partial \bar{z}_i} \circ f\Big)\frac{\partial f_l}{\partial z_j}\,\overline{\frac{\partial f_i}{\partial z_k}}. \tag{2.39}$$

Lastly

$$\frac{\partial(K_2\circ f)}{\partial z_j}=\sum_{l=1}^{n}\Big(\frac{\partial K_2}{\partial z_l}\circ f\Big)\frac{\partial f_l}{\partial z_j}\,. \tag{2.40}$$

Combining (2.37) to (2.40), observe that $B_p^{D_1}(u,v)$ is

$$\frac{1}{[K_2(f(p))]^2}\left[K_2(f(p))\sum_{j,k,i,l}\frac{\partial^2 K_2}{\partial z_l\partial\bar z_i}\Big|_{f(p)}\frac{\partial f_l}{\partial z_j}\Big|_p u_j\,\overline{\frac{\partial f_i}{\partial z_k}}\Big|_p\overline{v_k}-\sum_{j,k,i,l}\frac{\partial K_2}{\partial z_l}\Big|_{f(p)}\frac{\partial f_l}{\partial z_j}\Big|_p u_j\frac{\partial K_2}{\partial \bar z_i}\Big|_{f(p)}\overline{\frac{\partial f_i}{\partial z_k}}\Big|_p\overline{v_k}\right]$$

$$=\sum_{i,l}\left[\frac{1}{[K_2(f(p))]^2}\left(K_2(f(p))\frac{\partial^2 K_2}{\partial z_l\partial\bar z_i}\Big|_{f(p)}-\frac{\partial K_2}{\partial z_l}\Big|_{f(p)}\frac{\partial K_2}{\partial \bar z_i}\Big|_{f(p)}\right)\right]\left(\sum_j\frac{\partial f_l}{\partial z_j}\Big|_p u_j\right)\left(\sum_k\overline{\frac{\partial f_i}{\partial z_k}}\Big|_p\overline{v_k}\right).$$

By (2.27), the above is equal to

$$\sum_{i,l}\frac{\partial^2\log K_2}{\partial z_l\partial\bar z_i}(f(p))[f'(p)u]_l\overline{[f'(p)v]_i}=B_{f(p)}^{D_2}(f'(p)u,\ f'(p)v)$$

which completes the proof. □

Corollary 2.6.3. *Let $f:D_1\to D_2$ be a biholomorphisms between two bounded domains in $\mathbb{C}^n$. Then*

$$d_B^{D_1}(p,q)=d_B^{D_2}(f(p),f(q))$$

for all $p,q\in D_1$.

Proof. Let $\gamma:[0,1]\to D_1$ be a path. We have $(f\circ\gamma)'(t)=f'(\gamma(t))\gamma'(t)$. By above theorem, $L_{\gamma(t)}^{D_1}(\gamma'(t))=L_{f(\gamma(t))}^{D_2}((f\circ\gamma)'(t))$. This implies $l_B^{D_1}(\gamma)=l_B^{D_2}(f\circ\gamma)$. So we get a length (w.r.t Bergman metric) preserving bijection between paths in D_1 and D_2. This proves the result. □

2.7 The Bergman metric for a few standard domains

In this section, we will calculate the Bergman metric for the unit ball and for the upper half plane $\mathbb{H}=\{z\in\mathbb{C}:\operatorname{Im}z>0\}$.

2.7.1 Bergman metric for the unit ball in $\mathbb{C}^n$

Proposition 2.7.1. *The Bergman metric for $\mathbb{B}^n$ is given by*

$$B_p^{\mathbb{B}^n}(u,v) = \frac{n+1}{[1-(p,p)]^2}\Big[(u,v)\big[1-(p,p)\big] + (u,p)(p,v)\Big]$$

where $p \in \mathbb{B}^n$ and $u, v \in \mathbb{C}^n$.

Proof. By Theorem 2.2.5, we get

$$K(z) = \frac{n!}{\pi^n}\frac{1}{\big(1-\sum_i z_i\bar{z}_i\big)^{n+1}}\,.$$

Differentiating, we get

$$\frac{\partial K}{\partial \bar{z}_k} = \frac{(n+1)!}{\pi^n}\frac{z_k}{\big(1-\sum_i z_i\bar{z}_i\big)^{n+2}} \quad \textit{and} \quad \frac{\partial K}{\partial z_j} = \frac{(n+1)!}{\pi^n}\frac{\bar{z}_j}{\big(1-\sum_i z_i\bar{z}_i\big)^{n+2}}\,.$$

Also

$$\frac{\partial^2 K}{\partial z_j \partial \bar{z}_k} = \frac{(n+1)!}{\pi^n}\frac{1}{\big(1-\sum_i z_i\bar{z}_i\big)^{n+2}}\left[\delta_{jk} + \frac{z_k\bar{z}_j(n+2)}{\big(1-\sum_i z_i\bar{z}_i\big)}\right].$$

The above equations together with (2.27) gives

$$\begin{aligned}\frac{\partial^2 \log K}{\partial z_j \partial \bar{z}_k} &= \frac{n+1}{1-(z,z)}\left[\delta_{jk} + \frac{z_k\bar{z}_j(n+2)}{1-(z,z)}\right] - \frac{(n+1)^2 z_k\bar{z}_j}{[1-(z,z)]^2} \\ &= \frac{(n+1)\delta_{jk}}{1-(z,z)} + \frac{(n+1)z_k\bar{z}_j}{[1-(z,z)]^2} \\ &= \frac{n+1}{[1-(z,z)]^2}\Big[\delta_{jk}\big[1-(z,z)\big] + \bar{z}_j z_k\Big].\end{aligned} \tag{2.41}$$

Now we are ready to calculate the metric. Let $u, v \in \mathbb{C}^n$, then

$$\begin{aligned}B_p^{\mathbb{B}^n}(u,v) &= \frac{n+1}{[1-(p,p)]^2}\left[\sum_{j,k}\delta_{jk}\big[1-(p,p)\big]u_j\bar{v}_k + \sum_{j,k}u_j\bar{p}_j\bar{v}_k p_k\right] \\ &= \frac{n+1}{[1-(p,p)]^2}\Big[(u,v)\big[1-(p,p)\big] + (u,p)(p,v)\Big].\end{aligned} \tag{2.42}$$

$\square$

Corollary 2.7.2. *The Bergman metric for unit disc in $\mathbb{C}$ is given by*

$$B_p^{\mathbb{D}}(u,v) = \frac{2u\bar{v}}{(1-|p|^2)^2}\,.$$

2.7.2 The Bergman metric for $\mathbb{H}$

Proposition 2.7.3. *The Bergman metric for the upper half plane is given by*

$$B_p^{\mathbb{H}}(u,v) = \frac{u\bar{v}}{2(\operatorname{Im} p)^2}$$

where $p \in \mathbb{H}$ and $u, v \in \mathbb{C}$.

Proof. The map $f : \mathbb{H} \to \mathbb{D}$ defined by

$$f(z) = \frac{i-z}{i+z}$$

is a biholomorphism and therefore by Theorem 2.6.2, we get

$$\begin{aligned} B_p^{\mathbb{H}}(u,v) = B_{f(p)}^{\mathbb{D}}(f'(p)u, f'(p)v) &= \frac{2}{(1-|f(p)|^2)^2} f'(p)u\,\overline{f'(p)v} \\ &= \frac{2}{\left(1-\left|\frac{i-p}{i+p}\right|^2\right)^2}\left|\frac{-2i}{(i+p)^2}\right|^2 u\bar{v} = \frac{u\bar{v}}{2(\operatorname{Im} p)^2}\,. \end{aligned} \tag{2.43}$$

This proves the proposition. $\square$

Chapter 3

Bergman representative coordinates and Lu Qi-Keng problem

3.1 Bergman representative coordinates

The Bergman representative coordinates is a set of local holomorphic coordinates that we define on bounded domains in $\mathbb{C}^n$ using the Bergman kernel. Let D be a bounded domain in $\mathbb{C}^n$ with Bergman kernel given by K and let p be a point in D. Since $K(p,p) > 0$, there exists a neighborhood U_p of p such that $K(z,w) \neq 0$ for all $z, w \in U_p$. We define $b_j : U_p \to \mathbb{C}$ as follows:

$$b_j(z) = \frac{\partial}{\partial \bar{w}_j}\Big|_p \log \frac{K(z,w)}{K(w,w)}. \tag{3.1}$$

The above expression does not depend on which branch of the logarithm we choose. Using the orthonormal basis expansion of the Bergman kernel, we get

$$\begin{aligned} b_j(z) &= \frac{\partial}{\partial \bar{w}_j}\Big|_p \log K(z,w) - \frac{\partial}{\partial \bar{w}_j}\Big|_p \log K(w,w) \\ &= \frac{1}{K(z,p)}\left(\frac{\partial}{\partial \bar{w}_j}\Big|_p K(z,w)\right) - \frac{1}{K(p,p)}\left(\frac{\partial}{\partial \bar{w}_j}\Big|_p K(w,w)\right) \\ &= \frac{1}{K(z,p)}\left(\sum_\mu \phi_\mu(z)\overline{\frac{\partial \phi_\mu}{\partial w_j}(p)}\right) - \frac{1}{K(p,p)}\left(\sum_\mu \phi_\mu(p)\overline{\frac{\partial \phi_\mu}{\partial w_j}(p)}\right). \end{aligned} \tag{3.2}$$

This tells us that b_j is holomorphic function on U_p and further $b_j(p) = 0$. Now we define the function $B_p(z) : U_p \to \mathbb{C}^n$ as $B_p(z) = (b_1(z), b_2(z), ..., b_n(z))$. Note that $B_p(p) = 0$. Differentiating b_j with respect to z_k, we get

$$\begin{aligned}
\frac{\partial b_j}{\partial z_k}(p) &= \frac{\partial}{\partial z_k}\Big|_p \left[\frac{1}{K(z,p)} \left(\sum_\mu \phi_\mu(z) \overline{\frac{\partial \phi_\mu}{\partial z_j}(p)} \right) \right] \\
&= \frac{-1}{K(p,p)^2} \left(\frac{\partial}{\partial z_k}\Big|_p \sum_\mu \phi_\mu(z) \overline{\phi_\mu(p)} \right) \left(\sum_\mu \phi_\mu(p) \overline{\frac{\partial \phi_\mu}{\partial z_j}(p)} \right) + \frac{1}{K(p,p)} \left(\sum_\mu \frac{\partial \phi_\mu}{\partial z_k}(p) \overline{\frac{\partial \phi_\mu}{\partial z_j}(p)} \right) \\
&= \frac{1}{K(p,p)} \left(\sum_\mu \frac{\partial \phi_\mu}{\partial z_k}(p) \overline{\frac{\partial \phi_\mu}{\partial z_j}(p)} \right) - \frac{1}{K(p,p)^2} \left(\sum_\mu \frac{\partial \phi_\mu}{\partial z_k}(p)\, \overline{\phi_\mu(p)} \right) \left(\sum_\mu \phi_\mu(p) \overline{\frac{\partial \phi_\mu}{\partial z_j}(p)} \right) \\
&= \frac{\partial^2}{\partial z_k \partial \bar{z}_j}\Big|_p \log K(z,z)\,.
\end{aligned} \tag{3.3}$$

The last equality is a consequence of (2.30). This shows that the matrix $B_p'(p)$ is a positive definite matrix and hence it will have positive determinant. By the inverse function theorem, there is a neighborhood of p which we denote by V_p , such that the restriction of B_p to V_p is a biholomorphism i.e. $B_p : V_p \to W_p$ gives a set of coordinates around p where W_p is a neighborhood of 0. This set of coordinates is called the Bergman coordinates centered at p. The following theorem gives a very interesting property of these coordinates.

Theorem 3.1.1. *Suppose D_1, D_2 are two bounded domains in $\mathbb{C}^n$ with $p_1 \in D_1$ and $p_2 \in D_2$. Further suppose we have a biholomorphism $f : D_1 \to D_2$ such that $f(p_1) = p_2$. Then the biholomorphic mapping f is linear when expressed in the Bergman representative coordinates centered at p_1 and p_2 i.e. $B_{p_2} \circ f \circ B_{p_1}^{-1}$ is a linear map.*

Proof. The map $\tilde{f} = B_{p_2} \circ f \circ B_{p_1}^{-1}$ is a map that takes points from W_{p_1} to W_{p_2} via the domains V_{p_1} and V_{p_2}. Observe that the range and codomain W_{p_1} and W_{p_2} are neighborhoods of 0 and clearly we have $\tilde{f}(0) = 0$. Now let $z, w \in V_{p_1}$, we have

$$\frac{K_{D_1}(z,w)}{K_{D_1}(w,w)} = \frac{\det f'(z) K_{D_2}(f(z), f(w)) \overline{\det f'(w)}}{\det f'(w) K_{D_2}(f(w), f(w)) \overline{\det f'(w)}}\,.$$

Taking log on both sides, we get

$$\log \frac{K_{D_1}(z,w)}{K_{D_1}(w,w)} = \log K_{D_2}(f(z), f(w)) - \log K_{D_2}(f(w), f(w)) + \log f'(z) - \log f'(w) + C\,. \tag{3.4}$$

Now we will differentiate both sides with respect to $\bar{w}_j$ at p_1. So

$$\frac{\partial}{\partial \bar{w}_j}\Big|_{p_1} \log \frac{K_{D_1}(z,w)}{K_{D_1}(w,w)} = \frac{\partial}{\partial \bar{w}_j}\Big|_{p_1} \log K_{D_2}(f(z),f(w)) - \frac{\partial}{\partial \bar{w}_j}\Big|_{p_1} \log K_{D_2}(f(w),f(w))$$
$$= \left[\sum_{k=1}^{n}\left(\frac{\partial}{\partial \bar{y}_k}\Big|_{f(p_1)} \log K_{D_2}(f(z),y)\right)\frac{\partial \bar{f}_k}{\partial \bar{w}_j}(p_1)\right] - \left[\sum_{k=1}^{n}\left(\frac{\partial}{\partial \bar{y}_k}\Big|_{f(p_1)} \log K_{D_2}(y,y)\right)\frac{\partial \bar{f}_k}{\partial \bar{w}_j}(p_1)\right]$$
$$= \sum_{k=1}^{n}\overline{\frac{\partial f_k}{\partial w_j}}(p_1)\left(\frac{\partial}{\partial \bar{y}_k}\Big|_{p_2} \log \frac{K_{D_2}(f(z),y)}{K_{D_2}(y,y)}\right). \tag{3.5}$$

The above equation tells us that if $\tilde{z} \in W_{p_1}$, then

$$\tilde{z} = [f'(p)^*](B_{p_2} \circ f \circ B_{p_1}^{-1})(\tilde{z})$$

where $f'(p)^*$ is the Hermitian of the matrix $f'(p)$. Therefore we get

$$\tilde{f}(\tilde{z}) = [f'(p)^*]^{-1}\tilde{z} \tag{3.6}$$

for all $\tilde{z}$ in W_{p_1}. This proves that $\tilde{f}$ is linear. □

3.2 Lu Qi-Keng problem

If we wish to define a global set of Bergman coordinates, then it is necessary that the Bergman kernel does not vanish. This motivates the following definition.

Definition 3.2.1. *A domain $D \subset \mathbb{C}^n$ is said to be a Lu Qi-Keng domain if $K_D(z,\zeta) \neq 0$ for all $z, \zeta \in D$.*

As a consequence of the transformation formula for Bergman kernel, we see that the property of being a Lu Qi-Keng domain is invariant under biholomorphism. The problem of finding/characterizing which domains are Lu Qi-Keng is called the Lu Qi-Keng problem. From Corollary 2.2.7 and Theorem 2.2.5 we see that balls and polydiscs are Lu Qi-Keng domains. The first examples of non Lu Qi-Keng domains were annuli in $\mathbb{C}$. Suita and Yamada proved in [18] that every bounded, non simply connected domain in $\mathbb{C}$ with smooth boundary is non Lu Qi-Keng. This together with the Riemann mapping theorem implies a smooth bounded domain in the complex plane is Lu Qi-Keng iff it is simply connected. Boas proved in [4] that this result cannot be generalized to higher dimensions by showing that for each $n > 2$, there exists a smooth bounded domain in $\mathbb{C}^n$ which is strongly pseudoconvex, contractible and whose Bergman kernel has zeroes. This was quite surprising as a domain with so many nice properties was expected to be Lu Qi-Keng. Another result of Boas [5] is that the set of Lu Qi-Keng, bounded domains of holomorphy is a nowhere dense subset of bounded domains of holomorphy. This was again a most surprising result because it was expected that the non Lu Qi-Keng domains would be sparser. It is not completely known which complex ellipsoids are Lu Qi-Keng and it is a topic of current research. We shall give some examples of non Lu Qi-Keng domains in $\mathbb{C}^2$ in the next chapter.

Chapter 4

Some more Bergman kernels

4.1 Punctured disc

Let $\mathbb{D}^*$ denote the punctured unit disc in the complex plane i.e. $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. We will calculate the Bergman kernel for this domain.

Proposition 4.1.1. *If $f \in A^2(\mathbb{D}^*)$, then f has a removable singularity at 0.*

Proof. Since f is holomorphic on $\mathbb{D}^*$, f has a Laurent series expansion $f = \sum_{n\in\mathbb{Z}} a_n z^n$. Now

$$\int_{\mathbb{D}^*} |f|^2 dV = \int_{\mathbb{D}^*} \Big(\sum_{n\in\mathbb{Z}} a_n z^n\Big)\overline{\Big(\sum_{n\in\mathbb{Z}} a_n z^n\Big)} dV \;\; = \int_{\mathbb{D}^*} \sum_{m,n} a_m \overline{a_n} z^m \overline{z}^n dV = \sum_{m,n} a_m \overline{a_n} \int_{\mathbb{D}^*} z^m \overline{z}^n dV \, .$$

Using polar coordinates we can see that $\int_{\mathbb{D}^*} z^m \overline{z}^n dV = \big(\int_0^1 r^{n+m+1}\, dr\big)\delta_{mn} 2\pi$. Therefore

$$\int_{\mathbb{D}^*} |f|^2 dV = \sum_{n\in\mathbb{Z}} |a_n|^2 2\pi \int_0^1 r^{2n+1} dr \, . \tag{4.1}$$

However if $n < 0$, then $\int_0^1 r^{n+m+1}$ are all $+\infty$. Therefore $a_n = 0$ for all $n < 0$. So f has a removable singularity at 0. $\square$

Hence the Bergman spaces $A^2(\mathbb{D})$ and $A^2(\mathbb{D}^*)$ can be canonically identified. Therefore

$$f(z) = \int_{\mathbb{D}^*} K_{\mathbb{D}}(z,\zeta) f(\zeta) dV_\zeta \text{ for every } f \in A^2(\mathbb{D}^*) \text{ and } z \in \mathbb{D}^*.$$

From this we can deduce the following corollary.

Corollary 4.1.2. *$K_{\mathbb{D}^*}(z,\zeta) = K_{\mathbb{D}}(z,\zeta)$ for all $z,\zeta \in \mathbb{D}^*$.*

4.2 Generalized Hartog's triangle in $\mathbb{C}^2$

The domain $\{(z_1,z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$ is called the Hartog's triangle. It is an important and frequently studied domain. We also study some variations of this domain which we now define.

Definition 4.2.1. *Let $\gamma > 0$. The generalized Hartog's triangle in $\mathbb{C}^2$ of exponent γ is defined as the domain $\mathbb{H}_\gamma = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1\}$.*

First we will calculate the Bergman kernel for the Hartog's triangle i.e. $\mathbb{H}_1$ followed by $\mathbb{H}_{1/m}$ and $\mathbb{H}_m$ where $m \in \mathbb{N}$. These calculations are due to Edholm who recently calculated the Bergman kernel for these cases [10]. One way to try to do this would be to imitate the method we have used before to calculate the kernel for the disc, ball and ellipsoid i.e. to construct an orthonormal basis of the Bergman space using monomials and then using Theorem 2.1.10. Since $\mathbb{H}_\gamma$ is a connected Reinhardt domain, holomorphic functions would have Laurent series expansions, allowing this method to work. But it is more straightforward to use the transformation formula for the Bergman kernel for biholomorphic maps and for proper holomorphic maps (to be proved soon) to calculate Bergman kernel for the cases we are considering. We denote $K_{\mathbb{H}_\gamma}$ by K_γ for the sake of the brevity.

Proposition 4.2.1. *The Bergman kernel for the Hartog's triangle is given by*

$$K_1(z,w) = \frac{z_2\bar{w}_2}{\pi^2(z_2\bar{w}_2 - z_1\bar{w}_1)^2(1-z_2\bar{w}_2)^2}.$$

Proof. Let $\psi : \mathbb{H}_1 \to \mathbb{D} \times \mathbb{D}^*$ be the map given by $\psi(z_1,z_2) = (z_1/z_2, z_2)$. This map is an biholomorphism since it has an inverse which is given by $\phi(u_1,u_2) = (u_1u_2, u_2)$. So by the transformation formula we get

$$
\begin{aligned}
K_1(z,w) &= \det\psi'(z) K_{\mathbb{D}\times\mathbb{D}^*}(\psi(z),\psi(w))\overline{\det\psi'(w)} \\
&= \frac{1}{z_2}\cdot K_{\mathbb{D}}\left(\frac{z_1}{z_2},\frac{w_1}{w_2}\right)\cdot K_{\mathbb{D}^*}(z_2,w_2)\cdot\frac{1}{\bar{w}_2} \\
&= \frac{1}{\pi^2}\frac{1}{z_2\bar{w}_2}\cdot\frac{1}{\left(1-\dfrac{z_1\bar{w}_1}{z_2\bar{w}_2}\right)^2}\cdot\frac{1}{(1-z_2\bar{w}_2)^2} \\
&= \frac{z_2\bar{w}_2}{\pi^2(z_2\bar{w}_2-z_1\bar{w}_1)^2(1-z_2\bar{w}_2)^2}\,.
\end{aligned}
\tag{4.2}
$$

□

Proposition 4.2.2. *The Bergman kernel for the generalized Hartog's triangle $\mathbb{H}_{1/m}$ ($m\in\mathbb{N}$) is given by*

$$K_{1/m}(z,w)=\frac{z_2^m\bar{w}_2^m}{\pi^2(z_2^m\bar{w}_2^m-z_1\bar{w}_1)^2(1-z_2\bar{w}_2)^2}\,.$$

Proof. This family of domains is also biholomorphic to $\mathbb{D}\times\mathbb{D}^*$. The biholomorphism is given by $\psi_m:\mathbb{H}_{1/m}\to\mathbb{D}\times\mathbb{D}^*$ where $\psi_m(z_1,z_2)=\big((z_1/z_2^m),z_2\big)$. To see that this map is indeed a biholomorphism, we can easily check that $\phi_m(u_1,u_2)=(u_1u_2^m,u_2)$ is an inverse for ψ_m. So using transformation formula we get

$$
\begin{aligned}
K_{1/m}(z,w) &= \det\psi_m'(z) K_{\mathbb{D}\times\mathbb{D}^*}(\psi_m(z),\psi_m(w))\overline{\det\psi_m'(w)} \\
&= \frac{1}{z_2^m}\cdot K_{\mathbb{D}}\left(\frac{z_1}{z_2^m},\frac{w_1}{w_2^m}\right)\cdot K_{\mathbb{D}^*}(z_2,w_2)\cdot\frac{1}{\bar{w}_2^m} \\
&= \frac{z_2^m\bar{w}_2^m}{\pi^2(z_2^m\bar{w}_2^m-z_1\bar{w}_1)^2(1-z_2\bar{w}_2)^2}\,.
\end{aligned}
\tag{4.3}
$$

□

We now prove Bell's transformation rule [2] for the Bergman kernel for proper holomorphic mappings. We shall use the fact that any surjective proper holomorphic mapping is a branched covering of finite order.

Theorem 4.2.3. *Suppose D_1, D_2 are two bounded domains in $\mathbb{C}^n$ and that $F:D_1\to D_2$ is a proper holomorphic mapping of D_1 onto D_2 of order m. Let $\phi_1,\phi_2,...,\phi_m$ denote the m local inverses to F, defined locally on $D_2\setminus V$ where $V=\{F(z):z\in D_1,\det F'(z)=0\}$.*

Then

$$\sum_{k=1}^{m} K_{D_1}(z, \phi_k(w))\, \overline{\det \phi_k'(w)} = \det F'(z)\, K_{D_2}(F(z), w)$$

for all $z \in D_1$ *and for all* $w \in D_2 \setminus V$.

Proof. Let $u = \det F'$ and $U_k = \det \phi_k'$. We start by proving the following claim.

<u>Claim</u>: $P_1(u \cdot (h \circ F)) = u \cdot ((P_2 h) \circ F)$ for all $h \in L^2(D_2)$ where P_i denotes the Bergman projection operator corresponding to D_i.

Note that we had the same result for biholomorphic mappings (Proposition 2.5.2). Since F is a branched covering of order m, we have $||u \cdot (h \circ F)||_{L^2(D_1)} = \sqrt{m}\, ||h||_{L^2(D_2)}$ for all $h \in L^2(D_2)$. So $u \cdot (h \circ F)$ will be an element of $L^2(D_1)$. The claim obviously holds if $h \in A^2(D_2)$. Therefore it is sufficient to prove the claim for $h \in A^2(D_2)^{\perp}$ i.e. we wish to show $P_1(u \cdot (h \circ F)) = 0$ for every $h \in A^2(D)^{\perp}$. If $g \in C_0^{\infty}(D_2 \setminus V)$, then integrating by parts we have $\dfrac{\partial g}{\partial w_j} \in A^2(D)^{\perp}$. We first prove for this class of functions. So let $f \in A^2(D_1)$ and consider

$$\begin{aligned}
&\int_{D_1} f(z) \overline{u(z) \Big(\frac{\partial g}{\partial w_j} \circ F\Big)(z)}\, dV_z \\
&= \int_{D_2} \sum_{k=1}^{m} f(\phi_k(w)) \overline{u(\phi_k(w))} \overline{\frac{\partial g}{\partial w_j}(w)}\, |U_k(w)|^2 dV_w \\
&= \int_{D_2} \sum_{k=1}^{m} f(\phi_k(w))\, U_k(w)\, \overline{\frac{\partial g}{\partial w_j}(w)} dV_w\, .
\end{aligned} \tag{4.4}$$

By integration by parts this will be zero. So $u \cdot \Big(\dfrac{\partial g}{\partial w_j} \circ F\Big) \in A^2(D_1)^{\perp}$ and therefore we get $P_1\Big(u \cdot \Big(\dfrac{\partial g}{\partial w_j} \circ F\Big)\Big) = 0$. Now let H be the span of $\Big\{\dfrac{\partial g}{\partial w_j} : g \in C_0^{\infty}(D_2 \setminus V), 1 \le j \le n\Big\}$. To finish the proof of the claim we will show that H is a dense subset of $A^2(D_2)^{\perp}$. Suppose $h \in A^2(D)^{\perp} \cap H^{\perp}$, then

$$\int_{D_2 \setminus V} h(w) \overline{\frac{\partial g}{\partial w_j}(w)}\, dV_w = 0 \qquad \textit{for all } g \in C_0^{\infty}(D_2 \setminus V),\ j = 1, 2, ..., n.$$

So h satisfies the Cauchy Riemann equations on $D_2 \setminus V$ in the sense of weak derivatives, which implies that h is holomorphic on $D_2 \setminus V$. As V is a complex variety, we have that h

extends holomorphically to all of D, so $h \in A^2(D)$. Therefore $h = 0$, and so the orthogonal complement of H in $A^2(D)^\perp$ is $\{0\}$. This proves H is dense in $A^2(D_2)^\perp$.

Let $w \in D_2 \setminus V$. We can find a C^∞ positive function θ_w supported in $D_2 \setminus V$ which is radially symmetric around w with $\int \theta_w = 1$. For any function $g \in A^2(D_2)$, we have $g(w) = \int_{D_2} g\overline{\theta_w}$ by the mean value property of holomorphic functions. This implies $P_2\theta_w = K_{D_2}(., w)$. Let $z \in D_1$. Now applying the claim to θ_w we get

$$\begin{aligned}
u(z)K_{D_2}(F(z), w) &= [u \cdot (P_2\theta_w \circ f)](z) = [P_1(u \cdot (\theta_w \circ F)](z) \\
&= \int_{D_1} K_{D_1}(z, \zeta)\ u(\zeta)\ \theta_w(F(\zeta))\ dV_\zeta \\
&= \int_{D_2} \sum_{k=1}^{m} K_{D_1}(z, \phi_k(y))\, u(\phi_k(y))\, \theta_w(y)\, |U_k(y)|^2\, dV_y \\
&= \int_{D_2} \sum_{k=1}^{m} K_{D_1}(z, \phi_k(y))\, \overline{U_k(y)}\, \theta_w(y)\, dV_y \\
&= \sum_{k=1}^{m} K_{D_1}(z, \phi_k(w))\, \overline{U_k(w)}\,.
\end{aligned} \tag{4.5}$$

The last equation again uses the mean value property of holomorphic functions. This proves the theorem. ∎

The above transformation formula is a useful tool in computing Bergman kernels. We will use it to calculate the Bergman kernel for the Hartog's triangle $\mathbb{H}_m$.

Theorem 4.2.4. *Let $s = z_1\bar{w}_1$ and $t = z_2\bar{w}_2$. The Bergman kernel for the generalized Hartog's triangle $\mathbb{H}_m$ $(m \in \mathbb{N})$ is given by*

$$K_m(z, w) = \frac{p_m(s)t^2 + q_m(s)t + s^m p_m(s)}{m\pi^2(1-t)^2(t-s^m)^2}\,,$$

where p_m and q_m are the polynomials

$$p_m(s) = \sum_{k=1}^{m} k(m-k)s^{k-1},$$

$$q_m(s) = \sum_{k=1}^{m} (k^2 + (m-k)^2 s^m)s^{k-1}\,.$$

Proof. Let $F : \mathbb{H}_1 \to \mathbb{H}_m$ be the map defined by $F(z_1, z_2) = (z_1, z_2^m)$. This is a proper holomorphic mapping of order m. The function $u(z) = \det F'(z) = m z_2^{m-1}$ never vanishes and so local inverses of F can be defined at every point of $\mathbb{H}_m$. Let $(p_1, p_2) \in \mathbb{H}_m$. Let us now take a branch of log defined in a neighborhood of p_2. The maps $\phi_k(w_1, w_2) = (w_1, \zeta^k w_2^{1/m})$ where $w_2^{1/m} = e^{\log w/m}$ and $\zeta = e^{2\pi i/m}$, define local inverses to F in a neighborhood of (p_1, p_2). Note that $U_k(w) = \det \phi_k'(w) = \zeta^k w_2^{1/m}/m w_2$, and so applying the transformation formula from the previous theorem, we get

$$m z_2^{m-1} K_m\big((z_1, z_2^m), (w_1, w_2)\big) = \sum_{k=1}^{m} K_1\Big(\big(z_1, z_2\big), \big(w_1, \zeta^k w_2^{1/m}\big)\Big) \overline{\left(\frac{\zeta^k w_2^{1/m}}{m\, w_2}\right)}, \tag{4.6}$$

which implies

$$K_m\big((z_1, z_2^m), (w_1, w_2)\big) = \frac{1}{m z_2^{m-1}} \sum_{k=1}^{m} \frac{z_2\big(\overline{\zeta^k w_2^{1/m}}\big)}{\left[\pi^2 \Big(z_2\big(\overline{\zeta^k w_2^{1/m}}\big) - z_1 \bar{w}_1\Big)^2 \Big(1 - z_2\big(\overline{\zeta^k w_2^{1/m}}\big)\Big)^2\right]} \overline{\left(\frac{\zeta^k w_2^{1/m}}{m\, w_2}\right)}. \tag{4.7}$$

This expression requires multiple pages of algebra to simplify and obtain the final result. This simplification is omitted here and can be found in [10]. □

Corollary 4.2.5. *If $m \geq 2$ is an integer, then $\mathbb{H}_m$ is not a Lu Qi-Keng domain.*

Proof. We first consider the case $m \geq 3$. Let $z = \big(0, i/\sqrt{m-1}\big)$ and $w = \big(0, -i/\sqrt{m-1}\big)$. We can easily check that $K_m(z, w) = 0$. For the case $m = 2$, let $z = \big(i/\sqrt{2}, (\sqrt{7}+i)/4\big)$ and $w = \big(-i/\sqrt{2}, (\sqrt{7}-i)/4\big)$. We can again check by direct computation that $K_2(z, w) = 0$. □

4.3 Generalized Hartog's triangle in $\mathbb{C}^n$

The domain $H = \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_1| < |z_2| < ... < |z_n| < 1\}$ is the generalization of the Hartog's triangle to higher dimensions. We now compute its Bergman kernel.

Proposition 4.3.1. *The Bergman kernel for H is given by*

$$K_H(z, w) = \frac{(z_2\bar{w}_2)(z_3\bar{w}_3)...(z_n\bar{w}_n)}{\pi^n (z_2\bar{w}_2 - z_1\bar{w}_1)^2 ... (z_n\bar{w}_n - z_{n-1}\bar{w}_{n-1})^2 (1 - z_n\bar{w}_n)^2}\,.$$

Proof. Let $\psi : H \to \mathbb{D} \times \mathbb{D}^* \times ... \times \mathbb{D}^*$ be the map given by

$$\psi(z_1, ..., z_n) = \left(\frac{z_1}{z_2}, \frac{z_2}{z_3}, ..., \frac{z_{n-1}}{z_n}, z_n \right).$$

This is a biholomorphism since the map $\phi : \mathbb{D} \times \mathbb{D}^* \times ... \times \mathbb{D}^* \to H$ given by $\phi(u_1, ..., u_n) = (u_1 u_2 ... u_n, u_2 u_3 ... u_n, ..., u_n)$ is an inverse to the map ψ. Note that $\det \psi'(z) = 1/(z_2 z_3 ... z_n)$. Applying the transformation formula for biholomorphisms, we get

$$\begin{aligned}
K_H(z,w) &= \frac{1}{z_2 z_3 ... z_n} K_{\mathbb{D}\times\mathbb{D}^*\times...\times\mathbb{D}^*}\left(\left(\frac{z_1}{z_2}, \frac{z_2}{z_3}, ..., \frac{z_{n-1}}{z_n}, z_n \right) \left(\frac{w_1}{w_2}, \frac{w_2}{w_3}, ..., \frac{w_{n-1}}{w_n}, w_n \right) \right) \frac{1}{\bar{w}_2 \bar{w}_3 ... \bar{w}_n} \\
&= \frac{1}{\pi^n (z_2\bar{w}_2)(z_3\bar{w}_3)...(z_n\bar{w}_n)} \frac{1}{\left(1 - \dfrac{z_1\bar{w}_1}{z_2\bar{w}_2}\right)^2} \cdots \frac{1}{\left(1 - \dfrac{z_{n-1}\bar{w}_{n-1}}{z_n\bar{w}_n}\right)^2} \frac{1}{(1 - z_n\bar{w}_n)^2} \\
&= \frac{(z_2\bar{w}_2)(z_3\bar{w}_3)...(z_n\bar{w}_n)}{\pi^n (z_2\bar{w}_2 - z_1\bar{w}_1)^2 ... (z_n\bar{w}_n - z_{n-1}\bar{w}_{n-1})^2 (1 - z_n\bar{w}_n)^2}.
\end{aligned}$$

This proves the result. $\square$

Chapter 5

Ramadanov's theorem

5.1 Some versions of Ramadanov's theorem

Ramadanov's theorem, roughly speaking, tells us that if we have a sequence of domains D^j in $\mathbb{C}^n$ which converges in a certain sense to the domain D, then the Bergman kernels K_{D^j} will also converge to K_D. There are different versions of this theorem which deal with different types of convergence of the domains. In this section we will look at two versions of Ramadanov's theorem. The first is the original form [16] given by Ramadanov himself.

Theorem 5.1.1. *Let $D^1 \subset D^2 \subset \cdots$ be an increasing sequence of bounded domains in $\mathbb{C}^n$ and let $D = \cup_{j=1}^{\infty} D^j$. Suppose that D is also bounded. Then $K_{D^j} \to K_D$ uniformly on compact subsets of $D \times D$.*

We shall now prove a different version of Ramadanov's theorem and modify the proof to prove the original version. The following version appeared in [1].

Theorem 5.1.2. *Let D be a domain in $\mathbb{C}^n$ which is star-convex with respect to the origin and has non vanishing Bergman kernel along the diagonal. Suppose that D^j is a sequence of domains in $\mathbb{C}^n$ all of which contain the origin and have non vanishing Bergman kernel along the diagonal. Further suppose that the sequence D_j converges to D in the following way:*

1. *if $K \subset D$ is compact, then $K \subset D^j$ for j sufficiently large and*

2. for every $\epsilon > 0$ there exists $j_\epsilon \in \mathbb{N}$ such that $D^j \subset (1+\epsilon) \cdot D$ for all $j \geq j_\epsilon$.

Then $K_{D^j} \to K_D$ uniformly on compact subsets of $D \times D$.

Proof. Let $D_0 \subset\subset D$ be a relatively compact subdomain of D. For sufficiently large j, we have $D_0 \subset D^j$. Therefore we have for $z, w \in D_0$ and j large

$$|K_{D^j}(z,w)| \leq \sqrt{K_{D^j}(z)}\sqrt{K_{D^j}(w)} \leq \sqrt{K_{D_0}(z)}\sqrt{K_{D_0}(w)}\,.$$

This tells us that K_{D^j} (j large) is uniformly bounded on compact subsets of $D_0 \times D_0$. By Montel's theorem we get a subsequence $K_{D^{j_r}}$ which converges uniformly on compact subsets of $D_0 \times D_0$. By taking a compact exhaustion of D, we can get a subsequence which we again denote by $K_{D^{j_r}}$ which converges uniformly on compact subsets of $D \times D$. Call this limit function as K_∞. Observe that $K_\infty(z,w)$ is holomorphic in z and anti holomorphic in w. We will now show $K_\infty = K_D$ using the minimum integral property of Bergman kernel from Theorem 2.3.1. Let $w \in D$, note that for j_r large, we have $w \in D^{j_r} \subset 2D$. It follows that $K_{D^{j_r}}(w) \geq K_{2D} > 0$. This implies that $K_\infty(w) > 0$. By Fatou's lemma

$$\begin{aligned} \int_{D_0} \left| \frac{K_\infty(z,w)}{K_\infty(w)} \right|^2 dV_z &\leq \liminf_{r\to\infty} \int_{D_0} \left| \frac{K_{D^{j_r}}(z,w)}{K_{D^{j_r}}(w)} \right|^2 dV_z \\ &\leq \liminf_{r\to\infty} \int_{D^{j_r}} \left| \frac{K_{D^{j_r}}(z,w)}{K_{D^{j_r}}(w)} \right|^2 dV_z = \liminf_{r\to\infty} \frac{1}{K_{D^{j_r}}(w)}\,. \end{aligned} \tag{5.1}$$

We can find a further subsequence $D^{j_{r_k}}$ such that $D^{j_{r_k}} \subset \left(1+\frac{1}{k}\right)D$. Therefore

$$K_{D^{j_{r_k}}} \geq K_{\left(1+\frac{1}{k}\right)D}\,.$$

By (5.1), we get

$$\int_{D_0} \left| \frac{K_\infty(z,w)}{K_\infty(w)} \right|^2 dV_z \leq \liminf_{r\to\infty} \frac{1}{K_{D^{j_r}}(w)} \leq \liminf_{k\to\infty} \frac{1}{K_{D^{j_{r_k}}}(w)} \leq \liminf_{k\to\infty} \frac{1}{K_{\left(1+\frac{1}{k}\right)D}(w)}\,. \tag{5.2}$$

Let $f \in A^2(D)$ with $f(w) = 1$. For k large, define

$$g_k : \Big(1 + \frac{1}{k}\Big) D \to \mathbb{C} \quad , \quad g_k(z) = \frac{f\left(\dfrac{z}{1+1/k}\right)}{f\left(\dfrac{w}{1+1/k}\right)} .$$

Note that $g_k(w) = 1$ and

$$\int_{(1+\frac{1}{k})D} |g_k(z)|^2 dV_z = \frac{(1+1/k)^{2n}}{\left|f\Big(\dfrac{w}{1+1/k}\Big)\right|^2} \int_D |f(\zeta)|^2 dV_\zeta .$$

Now by Theorem 2.3.1, we get

$$\frac{1}{K_{\left(1+\frac{1}{k}\right)D}(w)} \leq \frac{(1+1/k)^{2n}}{\left|f\Big(\dfrac{w}{1+1/k}\Big)\right|^2} \int_D |f(\zeta)|^2 dV_\zeta . \tag{5.3}$$

Taking $\liminf$ on both sides and plugging it back in (5.2), we get

$$\int_{D_0} \left|\frac{K_\infty(z,w)}{K_\infty(w)}\right|^2 dV_z \leq \liminf_{k\to\infty} \frac{(1+1/k)^{2n}}{\left|f\Big(\dfrac{w}{1+1/k}\Big)\right|^2} \int_D |f(\zeta)|^2 dV_\zeta = \int_D |f(\zeta)|^2 dV_\zeta .$$

As D_0 is an arbitrary relatively compact subdomain of D, we get

$$\int_D \left|\frac{K_\infty(z,w)}{K_\infty(w)}\right|^2 dV_z \leq \int_D |f(\zeta)|^2 dV_\zeta . \tag{5.4}$$

Theorem 2.3.1 now implies $K_\infty = K_D$. Now suppose $z, w \in D$ and $K_{D^j}(z,w)$ does not converge to $K_D(z,w)$, then there exists a subsequence $K_{D^{j_s}}(z,w)$ which converges but not to $K_D(z,w)$. Using the above procedure we can find a subsequence of $K_{D^{j_s}}$ which converges uniformly on compact subsets to K_D, which is a contradiction. This implies K_{D^j} converges pointwise to K_D. Using 2.1.9, we get $K_{D^j} \to K_D$ uniformly on compact subsets of $D \times D$. □

Proof of Theorem 5.1.1. We shall adapt the proof of the above theorem to prove this theorem. We obtain (5.1) by reasoning just like in the above theorem. The next step is much simpler in this case. Note that $D^{j_r} \subset D$ for all r. So we get

$$\int_{D_0} \left| \frac{K_\infty(z,w)}{K_\infty(w)} \right|^2 dV_z \leq \liminf_{r\to\infty} \frac{1}{K_{D^{j_r}}(w)} \leq \frac{1}{K_D(w)} \, . \tag{5.5}$$

But $1/K_D(w)$ is the minimal squared norm of the elements in $\{f \in A^2(D) : f(w) = 1\}$. Therefore again by Theorem 2.3.1, we get $K_\infty = K_D$. The rest of the proof is same as in the above theorem. $\square$

Chapter 6

Pinchuk's Scaling method and Boundary behaviour of the Bergman kernel

Definition 6.0.1. *The Siegel upper half-space (also called unbounded realization of unit ball) in $\mathbb{C}^n$ is the domain*

$$H_\infty = \{(z', z_n) \in \mathbb{C}^n \mid 2\operatorname{Re} z_n < -|z'|^2\}$$

where $z' = (z_1, ..., z_{n-1}) \in \mathbb{C}^{n-1}$.

Consider the map

$$\Phi(z', z_n) = \left(\frac{\sqrt{2}z'}{1 - z_n}, \frac{1 + z_n}{1 - z_n} \right) \tag{6.1}$$

defined on the Siegel upper half-space. This map is a biholomorphism onto the unit ball in $\mathbb{C}^n$. Note that it sends the point $(0', -1)$ to the origin.

6.1 Scaling Method

Let $D \subset\subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain. Let $p \in \partial D$. After doing a change of coordinates around p we can shift p to the origin and get a global defining function for D of the form

$$r(z) = 2\operatorname{Re} z_n + |z'|^2 + \epsilon(z)$$

where $\epsilon(z) = o(|z'|^2, |z_1|)$. Suppose we have a sequence $p_j = (0', -\delta_j)$ where each δ_j is positive and $\delta_j \to 0$, i.e p_j converges to p along the real normal to ∂D. For each j, consider the change of coordinates

$$z' = \sqrt{\delta_j}\widetilde{z}' \quad and \quad z_n = \delta_j \widetilde{z_n}\,.$$

Note that this sends the point a_j to $(0', -1)$. Dropping the tildes, we get that a defining function in the new coordinates is

$$r'_j(z) = 2\delta_j \operatorname{Re} z_n + \delta_j |z'|^2 + \epsilon(\sqrt{\delta_j} z', \delta_j z_n)\,.$$

Dividing by δ_j, we get the equivalent defining function

$$r_j(z) = 2\operatorname{Re} z_n + |z'|^2 + \frac{\epsilon(\sqrt{\delta_j} z', \delta_j z_n)}{\delta_j}\,. \tag{6.2}$$

The above defining function gives us a new domain D_j biholomorphic to D. Note that r_j converges uniformly on compact subsets of $\mathbb{C}^n$ to the defining function of H_∞. Since the defining function of D_j converges uniformly on compact subsets of $\mathbb{C}^n$ to the defining function of the H_∞, we can deduce that the domains D_j converge to the Siegel upper half space in the local Hausdorff sense. Let us denote the maps corresponding to the change of coordinates as $\{\phi_j\}$ i.e $\phi_j(z', z_n) = \left(z'/\sqrt{\delta_j}, z_n/\delta_j\right)$. Then by the transformation formula we get $K_D(p_j) = K_{D_j}(0, -1)\delta_j^{-(n+1)}$ and so $K_D(p_j)\delta_j^{n+1} = K_{D_j}(0, -1)$. Thus the problem of studying $K_D(p_j)$ is reduced to the problem of studying $K_{D_j}(0, -1)$. Obseverve that the former is a boundary problem whereas the latter is an interior problem. So to summarize what we have done, we constructed us a sequence of biholomorphisms $\{\phi_j\}$ on D such that the sequence $\phi_j(D) = D_j$ converges to the Siegel upper half-space. This allowed us to reduce a boundary problem to an interior problem. This procedure which is used to convert boundary problems to interior problems is called the scaling method. The scaling method is due to Pinchuk ([14],[15]).

In the above case, we only worked with sequences converging along the normal direction. To carry out the scaling method for arbitrary sequences we first prove the following lemma from [14].

Lemma 6.1.1. *Let $D \subset\subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain with* *smooth global defining function ρ, and p_0 is an arbitrary point of ∂D. Then there are a neighborhood U of p_0 and a family of biholomorphic mappings $\{h^\zeta\}$ from $\mathbb{C}^n$ to $\mathbb{C}^n$, depending continuously on $\zeta \in \partial D \cap U$ satisfying the following conditions:*

1. *$h^\zeta(\zeta) = 0$.*
2. *The defining function $\rho^\zeta = \rho \circ (h^\zeta)^{-1}$ of the domain $D^\zeta = h^\zeta(D)$ has the form*

   $$\rho^\zeta(z) = 2\,\mathrm{Re}(z_n + K^\zeta(z)) + H^\zeta(z) + \alpha^\zeta(z)$$

   where

   $$\alpha^\zeta(z) = o(|z|^2) \quad , \quad K^\zeta(z) = \sum_{j,k=1}^{n} a_{jk}(\zeta) z_j z_k \quad , \quad H^\zeta(z) = \sum_{j,k=1}^{n} b_{jk}(\zeta) z_j \overline{z_k}$$

   with $K^\zeta(z', 0) \equiv 0$ and $H^\zeta(z', 0) \equiv |z'|^2$.
3. *The mapping h^ζ takes the real normal n_ζ to ∂D at the point ζ into the real normal $\{z' = \mathrm{Im}\, z_n = 0\}$ to ∂D^ζ at the origin.*

Proof. The gradient of ρ at $\zeta \in \partial D$ given by

$$\nabla\rho(\zeta) = 2\left(\frac{\partial\rho}{\partial\bar{z_1}}(\zeta), ..., \frac{\partial\rho}{\partial\bar{z_n}}(\zeta)\right)$$

is perpendicular to $T_\zeta(\partial D)$. Hence $n_\zeta = \{z \in \mathbb{C}^n : z = \zeta + t\nabla\rho(\zeta), t \in \mathbb{R}\}$. Without loss of generality we may assume that

$$\frac{\partial\rho}{\partial z_j}(p_0) = 0 \quad \text{for} \quad j = 1, 2, ..., n-1 \quad \text{and} \quad \frac{\partial\rho}{\partial z_n}(p_0) \neq 0 \tag{6.3}$$

For each $\zeta \in \partial D \cap U$ we will represent the mapping h^ζ as a superposition of simpler mappings. We interpret ρ^ζ and D^ζ respectively as the function ρ and the domain D in the new coordinates $h^\zeta(z)$. Motivated by the equations

$$\rho(z) = 2\,\mathrm{Re}\left(\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(\zeta)(z_j - \zeta_j)\right) + o(|z - \zeta|) \tag{6.4}$$

and

$$h^{\zeta}(n_{\zeta}) = \{z \in \mathbb{C}^n : z = t(0', 1), t \in \mathbb{R}\} \tag{6.5}$$

we make the following change of coordinates

$$\begin{aligned} z_j^* &= \frac{\partial \rho}{\partial \bar{z}_n}(\zeta)(z_j - \zeta_j) - \frac{\partial \rho}{\partial \bar{z}_j}(\zeta)(z_n - \zeta_n) \quad for\, j = 1, ..., n-1 \\ z_n^* &= \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(\zeta)(z_j - \zeta_j) \ . \end{aligned} \tag{6.6}$$

By (6.3) this transformation is an automorphism of $\mathbb{C}^n$ for all ζ in $\partial D \cap U$, if the neighborhood U is sufficently small . Also it can be directly verified that ζ gets mapped to the origin, the real normal to ∂D^{ζ} at the origin is the $\operatorname{Re} z_n$ axis and (6.5) is satisfied i.e conditions 1 and 3 are satisfied. To satisfy the condition 2, we make some further transformations.

The domain D has the defining function

$$\rho(z) = 2 \operatorname{Re} \Big(z_n + \sum_{j,k=1}^{n} a_{jk}(\zeta) z_j z_k \Big) + H^{\zeta}(z) + \alpha^{\zeta}(z) \tag{6.7}$$

in the new coordinates (*s have been dropped). Let us again make a change of coordinates

$$z_j^* = z_j \text{ for } j = 1, ..., n-1 \quad \text{and} \quad z_n^* = z_n + \sum_{j,k=1}^{n-1} a_{jk}(\zeta) z_j z_k$$

This is a biholomorphic change of coordinates because it has the inverse

$$z_j = z_j^* \text{ for } j = 1, ..., n-1 \quad \text{and} \quad z_n = z_n^* - \sum_{j,k=1}^{n-1} a_{jk}(\zeta) z_j^* z_k^*$$

Substituting we get the expression of the defining function is

$$\rho(z^*) = 2 \operatorname{Re} \Big(z_n^* + \sum_{j=n\, or\, k=n} a_{jk}^*(\zeta) z_j^* z_k^* + higher\, order\, terms \Big) + H^{\zeta}(z) + \alpha^{\zeta}(z)$$

Dropping the *s we get that in these coordinates the domain D has the form as in (6.7) with $a_{jk}(\zeta) = 0$ whenever $j, k = 1, ..., n-1$. Also note that this transformation keeps the $\operatorname{Re} z_n$ axis fixed. It only remains to arrange $H^{\zeta}(z', 0) \equiv |z'|^2$. As D is strictly pseudoconvex and because the complex tangent space at ζ is given by $z_n = 0$ in the present coordinates, the form $H^\zeta(z', 0)$ is positive definite on $\mathbb{C}^{n-1}$. So it gives us an inner product on $\mathbb{C}^{n-1}$ and we can use Gram-Schmidt process to get an orthonormal basis with respect to the form $H^\zeta(z', 0)$. Therefore we can find an complex linear transformation P_ζ of $\mathbb{C}^n$ which fixes the z_n axis and is such that $P_\zeta^* \circ H^\zeta \circ P_\zeta$ acting on $\mathbb{C}^{n-1}$ gives us the Euclidean inner product. So if we make the change of coordinates $P_\zeta z^* = z$, in the new coordinates $H^\zeta(z', 0) \equiv |z'|^2$. Also note that this transformation preserves the previously constructed properties. So with the combination of all of these transformations we achieve 1, 2 and 3. $\square$

Now we have the tools to carry out scaling method for arbitrary convergence. As before we let $D \subset\subset \mathbb{C}^n$ be a smoothly bounded strongly pseudoconvex domain and $p \in \partial D$, further we take a sequence $\{p_j\}$ in D which converges to p. Let ζ_j be the point on ∂D which is of minimum distance from p_j. Because D is smoothly bounded the existence of such a point is guaranteed, further ζ_j converges to p. The above lemma supplies us with maps h^{ζ_j} which we denote by h^j. By condition 3 of the lemma $h^j(p_j) = (0', -\delta_j)$ where δ_j is a sequence of positive numbers that goes to 0. Just like in the case of normal convergence, we now apply the transformations $\{\phi_j\}$, where $\phi_j(z', z_n) = \left(z'/\sqrt{\delta_j}, z_n/\delta_j\right)$. We denote $(\phi_j \circ h^j)(D)$ as D^j. Note that p_j is sent to $(0', -1)$ by this transformation. D_j will have defining function

$$\rho^j(z) = 2\,\mathrm{Re}\left(\delta_j z_n + 2\delta_j z_n \sum_{\mu=1}^{n-1} a_{\mu,j}\sqrt{\delta_j} z_\mu + a_{n,j}(\delta_j z_n)^2\right) + H^j(\sqrt{\delta_j} z', \delta_j z_n) + \alpha^j(\sqrt{\delta_j} z', \delta_j z_n) \tag{6.8}$$

where

$$\alpha^j(z) = o(|z|^2) \quad , \quad H^j(z) = \sum_{\mu,\nu=1}^{n} b_{\mu\nu,j} z_\mu \overline{z_\nu} \quad \text{with} \quad H^j(z', 0) \equiv |z'|^2 \,.$$

More explicitly

$$H^j(\sqrt{\delta_j} z', \delta_j z_n) = \delta_j |z'|^2 + 2\,\mathrm{Re}\left(\delta_j \overline{z_n} \sum_{\mu=1}^{n-1} b_{\mu,j}\sqrt{\delta_j} z_\mu\right) + b_{n,j}\delta_j^2 |z_n|^2 \,.$$

If we divide ρ^j by δ_j, we still get a defining function for D^j(we continue to denote it by ρ^j). So dividing, we get

$$\rho^j(z) = 2\operatorname{Re}\left(z_n + 2\sqrt{\delta_j} z_n \sum_{\mu=1}^{n-1} a_{\mu,j} z_\mu + a_{n,j}\delta_j z_n^2\right) + |z'|^2$$
$$+ 2\operatorname{Re}\left(\sqrt{\delta_j}\bar{z_n} \sum_{\mu=1}^{n-1} b_{\mu,j} z_\mu\right) + b_{n,j}\delta_j|z_n|^2 + \frac{\alpha^j(\sqrt{\delta_j}z', \delta_j z_n)}{\delta_j}. \quad (6.9)$$

The coefficients and the function α^j in above expression vary continuously on ∂D, hence the above function converges to the defining function of H_∞ uniformly on compact subsets of $\mathbb{C}^n$. So we get a sequence of biholomorphisms $\psi_j = \phi_j \circ h^j$ from D to D_j where $\psi_j(p_j) = (0, -1)$ and D_j converges to the Siegel upper half space in the Hausdorff sense. This completes the scaling method for arbitrary sequences.

Two applications of the scaling method will be discussed in this chapter, one will be a proof of Wong-Rosay theorem and the other will the be boundary behaviour of the Bergman kernel. The latter will be discussed in the next section. The Wong-Rosay theorem gives a sufficient condition for a domain to biholomorphically equivalent to the unit ball. Pinchuk gave a proof of the Wong-Rosay theorem using the scaling method in [15]. We close this section by giving the statement of Wong-Rosay theorem.

Theorem 6.1.2. *Let $D \subset\subset \mathbb{C}^n$ be a domain which has smooth boundary and is strictly pseudoconvex near the point $p \in \partial D$. Suppose there exists a point $p_0 \in D$ and a sequence ϕ_j of automorphisms of D such that $\phi_j(p_0)$ converges to p. Then D is biholomorphic to the the unit ball $\mathbb{B}^n$.*

6.2 Boundary behaviour of the Bergman kernel

Suppose $D \subset\subset \mathbb{C}^n$ is a smoothly bounded strongly pseudoconvex domain and that $p \in \partial D$. Without loss of generality we may assume that $p = 0$ and that D has a global defining function of the form

$$r(z) = 2\operatorname{Re} z_n + |z'|^2 + \epsilon(z)$$

where $\epsilon(z) = o(|z'|^2, |z_1|)$. Now let p_j be a sequence in D that converges to p. Consider the following question: what is the behavior of the sequence $K_D(p_j)$? We will answer this question using the scaling method. By transformation formula we have

$$K_D(p_j) = K_{D_j}(0, -1)\left|\det(\phi_j \circ h_j)'(p_j)\right|^2 = K_{D_j}(0, -1)\,\delta_j^{-(n+1)}|\det h_j'(p_j)|^2. \quad (6.10)$$

It can be checked that $h_j'(p_j) \to I$. Moreover $\delta_D(p_j)/\delta_j \to 1$ where $\delta_D(z)$ is the distance of z to the boundary of D. Now from Ramadanov's theorem it follows that

$$K_D(p_j)\delta_D(p_j)^{n+1} \to K_{H_\infty}(0,-1) = \frac{n!}{\pi^n 2^{n+1}}\,.$$

This gives us the following theorem.

Theorem 6.2.1. *If $D \subset\subset \mathbb{C}^n$ is a smoothly bounded strongly pseudoconvex domain, then*

$$K_D(z) \backsim \delta_D(z)^{-(n+1)}$$

near the boundary of D, where $\delta_D(z)$ is the distance of z to the boundary of D.

Chapter 7

Condition R and Bell's proof of Fefferman's theorem

Using the Bergman kernel and metric, Fefferman proved that biholomorphisms between two strongly pseudoconvex domains extend smoothly up to the boundary [11]. This was a breakthrough work which had a big impact in the field. Bell and Ligocka defined a property called condition R, using which they generalized Fefferman's theorem and simplified its proof [3]. In this chapter, we give a sketch of this proof and we refer the reader to [6] for more details.

7.1 Sobolev spaces

In this section certain class of Sobolev spaces are defined and some of their properties are stated.

Definition 7.1.1. *Let s be a positive integer and D be a bounded domain in $\mathbb{R}^N$, we define the Sobolev space $W^s(D)$ to be the space of all functions u in $L^2(D)$ such that the weak derivatives upto order s exist and $D^\alpha u \in L^2(D)$ for all $|\alpha| \leq s$. $W^s(D)$ forms a Hilbert space with inner product given by*

$$(u, v)_s = \sum_{|\alpha| \leq s} \int_D D^\alpha u \, \overline{D^\alpha v} \, .$$

Consequently the norm is given by

$$||u||_s^2 = \sum_{|\alpha|\le s} ||D^\alpha u||^2_{L^2(D)}\,.$$

We denote the closure of $C_0^\infty(D)$ in $W^s(D)$ by $W_0^s(D)$. If D is a domain in $\mathbb{C}^n$, then we denote $W^s(D)\cap\mathscr{O}(D)$ by $H^s(D)$. Also $H^\infty(\overline{D})$ is defined as $C^\infty(\overline{D})\cap\mathscr{O}(D)$. The following lemma is called Sobolev's Embedding lemma.

Lemma 7.1.1. *Suppose D is a smooth bounded domain in $\mathbb{C}^n$. Then for $s\in\mathbb{N}$, we have*

$$W^{s+n}(D)\subset C^s(\overline{D})$$

and

$$\sup_{|\alpha|\le s,\xi\in\overline{D}} |D^\alpha f(\xi)| \le c\,||f||_{W^{s+n}(D)} \quad \mathit{for\,all}\; f\in W^{s+n}(D)\,.$$

As a corollary of above lemma we get $C^\infty(\overline{D}) = \cap_{s\in\mathbb{N}} W^s(D)$ and $H^\infty(\overline{D}) = \cap_{s\in\mathbb{N}} H^s(D)$.

Lemma 7.1.2. *For smoothly bounded domains $D\subset\subset\mathbb{R}^N$, if f is in $C^\infty(\overline{D})$ and vanishes up to order $s-1$ on the boundary (i.e $D^\alpha f(\xi)=0$ for $|\alpha|\le s-1$ and $\xi\in\partial D$), then $f\in W_0^s(D)$.*

7.2 Proof of Fefferman's theorem

Proposition 7.2.1. *Let D be a smooth bounded domain in $\mathbb{C}^n$. Let us denote K_D by K and P_D by P for brevity. Then the following conditions are equivalent.*

1. *P maps $C^\infty(\overline{D})$ into $H^\infty(\overline{D})$.*
2. *For each $s\in\mathbb{N}$, there exists a non-negative integer $m=m(s)$ such that P is bounded from $W_0^{s+m}(D)$ to $H^s(D)$.*
3. *For each multi-index α, there are constants $c=c_\alpha$ and $m=m_\alpha$, such that*

$$\sup_{z\in D}\left|\frac{\partial^\alpha}{\partial z^\alpha}K(z,w)\right| \le c d(w)^{-m},$$

where $d(w)$ is the distance from w to ∂D.

Definition 7.2.1. *A smooth bounded domain is said to satisfy condition R if it satisfies any of these equivalent criteria.*

We skip the proof of this proposition. Condition R holds for bounded strongly pseudonconvex domains. Before we get to the main theorem, we state one more lemma.

Lemma 7.2.2. *Let $f : D_1 \to D_2$ be a biholomorphic map between two smooth bounded pseudoconvex domains. Then for any $s \in \mathbb{N}$, there is an integer $j = j(s)$ such that the mapping $\phi \mapsto \det f' \cdot (\phi \circ f)$ is bounded from $W_0^{s+j(s)}(D_2)$ to $W_0^s(D_1)$.*

Theorem 7.2.3. *Let D_1 and D_2 be two smooth bounded pseudoconvex domains in $\mathbb{C}^n$ $(n \geq 2)$ which satisfy condition R (in particular they are both strongly pseudoconvex). Suppose $f : D_1 \to D_2$ is a biholomorphism, then f extends smoothly to the boundary.*

Proof. Let $s \in \mathbb{N}$ and $u = \det f'$. Since condition R holds on D_1, there exists $m(s)$ such that P_1 maps $W_0^{s+m(s)}(D_1)$ boundedly into $H^s(D_1)$. We can show that there exists $g \in W_0^{s+m(s)+j}(D_2)$ such that $P_2 g \equiv 1$, where $j = j(s + m(s))$ is chosen as per previous lemma. Applying Proposition 2.5.2, we get $u = P_1\big(u \cdot (g \circ f)\big)$. But the previous lemma implies $u \cdot (g \circ f) \in W_0^{s+m(s)}(D_1)$. By condition R, we get $u \in H^s(D_1)$. As s was arbitrary, by Lemmma 7.1.1 we get $u \in C^\infty(\overline{D_1})$. Similarly we can show $\det(f^{-1})' \in C^\infty(\overline{D_2})$. Therefore the chain rule will imply that u does not vanish over $\overline{D_1}$.

We may choose $g_k \in W_0^{s+m(s)+j}(D_2)$ such that $P_2 g_k$ is the k-th coordinate function on D_2. Repeating above arguments, we get $u \cdot f_k = P_1\big(u \cdot (g_k \circ f)\big)$ is in $C^\infty(\overline{D_1})$. We have already seen that $u \in C^\infty(\overline{D_1})$ and that u is non vanishing on $\overline{D_1}$. Hence we get $f_k \in C^\infty(\overline{D_1})$ for $k = 1, 2, ..., n$. Thus $f \in C^\infty(\overline{D_1})$. This proves the theorem. $\square$

Chapter 8

Domains with Finite Dimensional Bergman Space

Recall that bounded domains have infinite dimensional Bergman space as all polynomials are square integrable. Also we have seen in Remark 2.1.3 that $A^2(\mathbb{C}^n) = \{0\}$. So it is natural to ask do there exist domains in $\mathbb{C}^n$ which have finite dimensional non-trivial Bergman spaces. The answer to this question was given by Wiegerinck in [19]. He showed that such domains do not exist in the complex plane but do exist in $\mathbb{C}^n$ for $n \geq 2$.

8.1 Reinhardt domains in $\mathbb{C}^2$ with k-dimensional Bergman space

We will prove the following theorem from [19] in this section.

Theorem 8.1.1. *For every $k > 0$ there exists a Reinhardt domain in $\mathbb{C}^2$ with k-dimensional Bergman space.*

To prove this theorem we will explicitly construct required domains in several steps. These constructions can be generalized to higher dimensions. We start by defining the following domains

$$D_1 = \left\{ (z,w) \in \mathbb{C}^2 : |w| < \frac{1}{|z| \log |z|},\ |z| > e \right\},$$

$$D_2 = \left\{ (z,w) \in \mathbb{C}^2 : |z| < \frac{1}{|w| \log |w|},\ |w| > e \right\},$$

and let

$$D = D_1 \cup D_2 \cup \big\{ (z,w) \in \mathbb{C}^2 : |z| < 2e,\ |w| < 2e \big\}.$$

Lemma 8.1.2. *The only monomials contained in $A^2(D)$ are those of the form $az^k w^k$, where $a \in \mathbb{C}$ and k is a non-zero integer.*

Proof. Observe that by use of polar coordinates we get

$$\begin{aligned} \int_{D_1} |z|^{2p} |w|^{2q} dV &= (2\pi)^2 \int_{r_1=e}^{\infty} \int_{r_2=0}^{\frac{1}{r_1 \log r_1}} r_1^{2p+1} r_2^{2q+1} \, dr_2 \, dr_1 \\ &= 4\pi^2 \int_e^{\infty} \frac{r_1^{2p-2q-1}}{(\log r_1)^{2q+2}} \, dr_1 \,. \end{aligned} \tag{8.1}$$

If $p > q$, then by applying l'Hôpital's rule we can see the integrand goes to infinity and so the integral cannot be finite. Note that

$$\int_e^{\infty} \frac{1}{x(\log x)^2} = \left[\frac{-1}{\log x} \right]_e^{\infty} = 1\,.$$

If $p \leq q$, then the integrand in (8.1) will be dominated by $1/\big(r_1 (\log r_1)^2\big)$. The above calculation tells us that in this case the integral in (8.1) will be finite. So we get $az^p w^q \in A^2(D_1)$ iff $p \leq q$. Similarly if we integrate over D_2, we will get $az^p w^q \in A^2(D_1)$ iff $q \leq p$. Thus $az^p w^q \in A^2(D)$ iff $q = p$. This proves the lemma. $\square$

Next we enlarge D in the direction $|z| = |w|$ to eliminate monomials of higher degree. Let

$$H_m = \left\{ (z,w) : |z| + |w| > 2,\ \Big| |z| - |w| \Big| < \frac{1}{(|z| + |w|)^m} \right\} \quad \textit{and} \quad D_k = D \cup H_{4k}$$

for $k \in \mathbb{N}$. We will eventually show that D_k has k-dimensional Bergman space and this will prove the theorem. But first we need to prove the following lemma.

Lemma 8.1.3. *The monomial $z^p w^p$ will be contained in $A^2(D_k)$ iff $p < k$.*

Proof. Let $B = \left\{ (r_1, r_2) : r_i > 0,\, r_1 + r_2 > 2, |r_1 - r_2| < \dfrac{1}{(r_1 + r_2)^{4k}} \right\}$. Now consider

$$||z^p w^p||^2_{H_{4k}} = \int_{H_{4k}} |z|^{2p} |w|^{2p} dV = (2\pi)^2 \int_B r_1^{2p+1} r_2^{2p+1} dV_r \,. \tag{8.2}$$

We now make the change of variables $r_1 + r_2 = t$ and $r_1 - r_2 = s$. More precisely, let $A = \{(t, s) : t > 2, -1/t^{4k} < s < 1/t^{4k}\}$. The map $g : B \to A$ defined by $g(r_1, r_2) = (r_1 + r_2, r_1 - r_2)$ and the map $f : A \to B$ defined by $f(t, s) = \big((t + s)/2, (t - s)/2\big)$ are diffeomorphisms which are inverses of each other. Note that $|\det f'| = 1/2$. Now applying the change of variable formula we get that $||z^p w^p||^2$ is equal to

$$4\pi^2 \int_A \frac{1}{2} \left(\frac{t^2 - s^2}{4} \right)^{2p+1} dV = \frac{2\pi^2}{4^{2p+1}} \int_{t=2}^{\infty} \int_{s=-1/t^{4k}}^{1/t^{4k}} (t^2 - s^2)^{2p+1} \, ds \, dt \,. \tag{8.3}$$

After expanding and simplifying using binomial theorem, we can show that this integral converges iff $p \leq k - 1$ which proves the lemma. □

We are now ready for the proof the theorem.

Proof of Theorem 8.1.1. As we have already mentioned we will prove that D_k has a k dimensional Bergman space. Let $f \in A^2(D_k)$. As D_k is a connected Reinhardt domain, f has a Laurent series expansion in D_k. Also since D_k contains the origin, it will have a power series expansion allowing us to extend f to the smallest complete Reinhardt domain containing D_k which is clearly $\mathbb{C}^2$. So we have a power series $\sum_{p,q} a_{pq} z^p w^q$ which converges throughout $\mathbb{C}^2$ and agrees with f on D_k. Observe that this power series converges uniformly on compact subsets of $\mathbb{C}^2$. Now

$$||f||^2_{D_k} \geq \int_{D_k \cap B(0,R)} |f|^2 dV = \sum_{p,q} |a_{pq}|^2 \int_{D_k \cap B(0,R)} |z^{2p} w^{2q}| \, dV \tag{8.4}$$

where $B(0, R)$ denotes the ball of radius R around the origin. Letting $R \to \infty$, the previous lemmata gives us that $a_{pq} = 0$ unless $p = q < k$. Hence $f = \sum_{j=0}^{k-1} a_j z^j w^j$. So the monomials $\{1, zw, ..., (zw)^{k-1}\}$ form a basis for $A^2(D_k)$. This proves the theorem. □

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