Scale Invariant Power Spectrum of Inflation using Expanding BEC: Co-ordinate Dependence of Expanding State

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Prajwal Udupa V

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2019

Supervisor: Dr. Arijit Bhattacharyay

Certificate

This is to certify that this dissertation entitled Scale Invariant Power Spectrum of Inflation using Expanding BEC: Co-ordinate Dependence of Expanding State towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Prajwal Udupa V at Indian Institute of Science Education and Research under the supervision of Dr. Arijit Bhattacharyay, Associate Professor, Department of Physics, during the academic year 2018-2019.

Arijit Bhattacharyay

Dr. Arijit Bhattacharyay

Committee:

Dr. Arijit Bhattacharyay

Dr.Prasad Subramanian

This thesis is dedicated to my parents.

Declaration

I hereby declare that the matter embodied in the report entitled Scale Invariant Power Spectrum of Inflation using Expanding BEC: Co-ordinate Dependence of Expanding State are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Arijit Bhattacharyay and the same has not been submitted elsewhere for any other degree.

Prajwal Udupa V

Acknowledgments

I would like to thank my supervisor Dr. Arijit Bhattacharyay for being a patient and understanding guide. My discussions and interactions with him have been instrumental in shaping the course of the project and my overall view of physics. I would also like to thank Dr.Prasad Subramanian for kindly agreeing to be on my Thesis Advisory Committee.

I would like to take this oppurtunity to also thank Supratik Sarkar for his crucial insights during the initial phase of the project. A special thanks to my friend Pavan for the many useful discussions during the course of the project.

I am ever grateful to my family and friends for their constant love and support, which has ensured that I continue in my endeavours despite the hardships and difficulties.

Finally, I would like to mention that the work done in this thesis is supported by the INSPIRE grant from the Department of Science and Technology, Government of India. Hence, I would like to thank the people of India for their support to Basic Sciences.

Abstract

In [1], it has been shown that for an expanding BEC model with constant background density, the scale invariance of the inflationary power spectrum breaks down when strong dipole-dipole interactions are involved. So, with the aim of studying the scale invariance of the power spectrum, in this work, a Gaussian background density is introduced into the expanding BEC model and the dipole-dipole interaction term is ignored. Within such a setting, it is found that the scale invariance of the inflationary power spectrum is retained in the long wavelength limit. Continuing further, the terms ignored in the long wavelength limit are treated perturbatively using the approach of multiple scale analysis. It is shown that the resulting inflationary power spectrum due to first order amplitude corrections still remains scale invariant.

Contents

Chapter 1

Introduction

The inflationary model of the universe, which suggests that the cosmos expanded rapidly at the early phase of the universe, theoretically answers many questions about the flatness and homogeneity of the universe [2], and even explains the origin of large-scale structure of the cosmos. However this model presents a new problem - if the period of inflation lasts long so as to be concordant with the observations, the scales of physical wavelength which get involved at early times are much smaller than Planck length [1, 3], the length scale below which effects of quantum gravity potentially become apparent [4]. In such a case, trans-Planckian energies become involved, for which the physics is speculative at the best currently. So, in order to tackle this problem, we will take the approach of analogue gravity and model the cosmos as considered by [1] on the basis of an expanding Bose Einstein Condensate (BEC), a known condensed matter system, whose properties can be experimentally investigated.

Since trans-Planckian energies become involved, Planck length becomes an important length scale. Healing length of BEC, ξ_{hl} which characterizes the transition from phononic to free particle mode, happens to provide this crucial analogy to Planck length [5, 6]. In BECs, for length scales much greater than healing length i.e small momenta ($k << 1/\xi_{hl}$), the energy dispersion relation is linear, whereas for length scales much smaller than healing length i.e large momenta ($k >> 1/\xi_{hl}$), the energy dispersion is quadratic. Also, the Lorentz invariance of the field equations is violated below the healing length of BEC. This motivates further to model the cosmic inflation using the expanding BEC model as proposed by [1].

In particular, we are interested in studying the scale invariance of the inflationary power spectrum and where it breaks down. So, first we will consider a BEC with a constant background density and see under what domain the scale invariance of its power spectrum remains. In the later chapters, we will introduce a co-ordinate dependent background density and see what effect it has on scale invariance of the power spectrum.

1.1 The BEC Model - Lagrangian to GP Equation

We want to model the cosmos by an expanding dipolar Bose Einstein Condensate (BEC). This has done successfully in [1] by *Seok-Young* and *Fischer*. In their work, they build the expanding BEC model for cosmos starting from the lagrangian density of the condensate, considering a constant background density. From here, they derive the Gross-Pitaevskii equation (in (2+1)-D), which they later use to generate the phase fluctuation equation. Then, they proceed to solve this equation and use the resultant solution to generate the inflationary power spectrum. On doing this, they have shown that the scale invariance of power spectrum breaks down when strong dipole-dipole interactions are considered. In the rest of this chapter, the details of their work will be described to get a thorough understanding of the model.

Consider the Langrangrian density, $\mathcal{L}$, of a Bose condensate whose state is described by the wavefunction, Ψ (such that $|\Psi|^2$ gives the number density of the condensate), and its bosons have a mass, m.

$$\mathcal{L} = \frac{i\hbar}{2}(\Psi^*\partial_t\Psi - \partial_t\Psi^*\Psi) - \frac{\hbar^2}{2m}|\nabla\Psi|^2 - V_{ext}|\Psi|^2 - \frac{1}{2}\int d^3\mathbf{R}' V_{int}(\mathbf{R}-\mathbf{R}')|\Psi(\mathbf{R}')|^2 \tag{1.1}$$

($\mathbf{R} \equiv (\mathbf{r}, z)$ is the 3-D spatial spherical co-ordinate)

where,
$V_{ext} = \frac{m\omega^2 r^2}{2} + \frac{m\omega_z^2 z^2}{2}$ is the confining potential and

$V_{int}(\mathbf{R}-\mathbf{R}') = g_c\delta^{(3)}(\mathbf{R}-\mathbf{R}') + V_{dd}(\mathbf{R}-\mathbf{R}')$ is the interaction potential between bosons.

Separating the axial and the radial part, one can take an ansatz for Ψ as:

$$\Psi(R,t) = \Psi_r(r,t)\Phi_z(z)e^{-i\omega_z t/2} \tag{1.2}$$

where,

$$\Phi_z(z) = \frac{1}{(\pi d_z^2)^{\frac{1}{4}}}\exp\Big[-\frac{z^2}{2d_z^2}\Big]$$

Putting the expression (1.2) back into the expression (1.1) (and substituting for V_{ext} and V_{int}), and integrating the z-dependent part of $\mathcal{L}$, one gets the modified lagrangian, $\mathcal{L}_r$, in (2+1)-D.

$$\begin{aligned}\mathcal{L}_r = {} & \frac{i\hbar}{2}(\Psi_r^*\partial_t\Psi_r - \partial_t\Psi_r^*\Psi_r) - \frac{\hbar^2}{2m}|\nabla_r\Psi_r|^2 - \frac{m\omega^2 r^2}{2}|\Psi_r|^2 \\ & - \frac{g_c^{2D}}{2}|\Psi_r|^4 - \frac{1}{2}|\Psi_r|^2\int d^2\mathbf{r}' V_{dd}^{2D}(\mathbf{r}-\mathbf{r}')|\Psi_{r'}(\mathbf{r}')|^2\end{aligned} \tag{1.3}$$

where,

$$g_c^{2D} = \frac{g_c}{\sqrt{2\pi}d_z}$$

$$V_{dd}^{2D}(\mathbf{r}-\mathbf{r}') = \int dz dz' V_{dd}(\mathbf{R}-\mathbf{R}')\Phi_z^2\Phi_{z'}^2$$

At this stage, one makes a scale transformation defined by:

$$\mathbf{x} = \frac{\mathbf{r}}{b(t)} \tag{1.4}$$

$$\tau = \int_0^t \frac{1}{b^2(t')}dt' \tag{1.5}$$

Under such a scale transformation, ψ, can be defined as:

$$\Psi_r(\mathbf{r},t) = \frac{e^{i\Phi}\psi(\mathbf{x},\tau)}{b} \tag{1.6}$$

where,

$$\Phi = \frac{1}{2}\frac{mr^2}{\hbar}\frac{\partial_t b}{b}$$

As a result of incorporating all these transformations, one has to see how each term of the $\mathcal{L}_r$ changes.

$$\begin{aligned}
\frac{i\hbar}{2}(\Psi_r^*\partial_t\Psi_r - \partial_t\Psi_r^*\Psi_r) &= \frac{i\hbar}{2}\Big(\frac{\psi^*\partial_\tau\psi}{b^4} + \frac{1}{b^2}\frac{\partial x}{\partial t}\psi^*\frac{\partial\psi}{\partial x} + i\frac{\partial\Phi}{\partial t}\frac{|\psi|^2}{b^2} \\
&\quad - \frac{\partial_\tau\psi^*\psi}{b^4} - \frac{1}{b^2}\frac{\partial x}{\partial t}\frac{\partial\psi^*}{\partial x}\psi + i\frac{\partial\Phi}{\partial t}\frac{|\psi|^2}{b^2}\Big) \\
&= \frac{i\hbar}{2}\Big(\frac{\psi^*\partial_\tau\psi}{b^4} + \frac{1}{b^2}\frac{\partial x}{\partial t}\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial_\tau\psi^*\psi}{b^4} - \frac{1}{b^2}\frac{\partial x}{\partial t}\frac{\partial\psi^*}{\partial x}\psi\Big) \\
&\quad - \frac{mx^2}{2b}\frac{\partial^2 b}{\partial t^2}|\psi|^2 + \frac{mx^2}{2}\Big(\frac{\partial b}{\partial t}\Big)^2\frac{|\psi|^2}{b^2}
\end{aligned} \tag{1.7}$$

$$\frac{\hbar^2}{2m}|\nabla_r\Psi_r|^2 = \frac{mx^2}{2}\Big(\frac{\partial b}{\partial t}\Big)^2\frac{|\psi|^2}{b^2} + \frac{i\hbar}{2}\Big(\frac{1}{b^2}\frac{\partial x}{\partial t}\psi^*\frac{\partial\psi}{\partial x} - \frac{1}{b^2}\frac{\partial x}{\partial t}\frac{\partial\psi^*}{\partial x}\psi\Big) + \frac{\hbar^2}{2mb^4}|\nabla_x\psi|^2 \tag{1.8}$$

$$\begin{aligned}
&\frac{m\omega^2r^2}{2}|\Psi_r|^2 + \frac{g_c^{2D}}{2}|\Psi_r|^4 + \frac{1}{2}|\Psi_r|^2\int d^2\mathbf{r}' V_{dd}^{2D}(\mathbf{r}-\mathbf{r}')|\Psi_{r'}(\mathbf{r}')|^2 \\
&\qquad = \frac{m\omega^2x^2}{2}|\psi|^2 + \frac{g_c^{2D}}{2}\frac{|\psi|^4}{b^4} + \frac{1}{2}|\frac{\psi|^2}{b^4}\int d^2\mathbf{x}' V_{dd}^{2D}(\mathbf{x}-\mathbf{x}')|\psi(\mathbf{x}')|^2
\end{aligned} \tag{1.9}$$

Using the equations (1.7), (1.8) and (1.9), $\mathcal{L}_r$ is obtained in the transformed co-ordinates. Now $\mathcal{L}_r$ should be multiplied by a factor of b^4 (as $dtd^2r = b^4 d\tau d^2x$) to get the effective lagrangian density, $\mathcal{L}_s$, in the transformed co-ordinates.

$$\begin{aligned}\mathcal{L}_s =& \frac{i\hbar}{2}(\psi^*\partial_\tau\psi - \partial_\tau\psi^*\psi) - \frac{\hbar^2}{2m}|\nabla_x\psi|^2 - \frac{mx^2}{2}b^3\frac{\partial^2 b}{\partial t^2}|\psi|^2 - \frac{mx^2}{2}\omega^2 b^4|\psi|^2 \\ &- \frac{g_c^{2D}}{2}|\psi|^4 - \frac{1}{2}|\psi|^2\int d^2x' V_{dd}^{2D}(\mathbf{x}-\mathbf{x}')|\psi(\mathbf{x}')|^2\end{aligned} \tag{1.10}$$

Now one can define:

$$f^2 = \frac{b^3\partial_t^2 b + b^4\omega^2}{\omega_0^2} = \frac{g_c^{2D}}{g_{c,0}^{2D}} = \frac{g_d^{2D}}{g_{d,0}^{2D}} \tag{1.11}$$

Also it is known that,

$$\frac{V_{dd}^{2D}}{V_{dd,0}^{2D}} = \frac{g_d^{2D}}{g_{d,0}^{2D}} \tag{1.12}$$

Therefore one gets,

$$\begin{aligned}\mathcal{L}_s =& \frac{i\hbar}{2}(\psi^*\partial_\tau\psi - \partial_\tau\psi^*\psi) - \frac{\hbar^2}{2m}|\nabla_x\psi|^2 \\ &- f^2\Big(\frac{mx^2\omega_0^2}{2}|\psi|^2 - \frac{g_{c,0}^{2D}}{2}|\psi|^4 - \frac{1}{2}|\psi|^2\int d^2x' V_{dd,0}^{2D}(\mathbf{x}-\mathbf{x}')|\psi(\mathbf{x}')|^2\Big)\end{aligned} \tag{1.13}$$

Using the variational principle on the action obtained from this lagrangian density, one arrives at the non-local Gross-Pitaevskii (GP) equation (1.14).

$$i\hbar\partial_\tau\psi = -\frac{\hbar^2}{2m}\nabla_x^2\psi + f^2\Big(\frac{mx^2\omega^2}{2} + \frac{g_{c,0}^{2D}}{2}|\psi|^2 + \int d^2x' V_{dd,0}^{2D}(\mathbf{x}-\mathbf{x}')|\psi(\mathbf{x}')|^2\Big)\psi \tag{1.14}$$

1.2 GP Equation to the Phase Fluctuation Equation

One can always write ψ as

$$\psi(\mathbf{x}) = \sqrt{\rho(\mathbf{x})}e^{i\phi(\mathbf{x})} \tag{1.15}$$

Putting the equation (1.15) back into (1.14), and then separating the real and imaginary part, one arrives at:

Imaginary Part:

$$\partial_\tau \rho(\mathbf{x}) = -\frac{\hbar}{m}(\nabla_x \phi(\mathbf{x}) \cdot \nabla_x \rho(\mathbf{x}) + \rho \nabla_x^2 \phi(\mathbf{x})) \tag{1.16}$$

Real Part:

$$-\hbar\partial_\tau \phi(\mathbf{x}) = -\frac{\hbar^2}{2m\sqrt{\rho(\mathbf{x})}}\nabla_x^2\sqrt{\rho(\mathbf{x})} + \frac{\hbar^2}{2m}(\nabla_x\phi(\mathbf{x}))^2 + f^2\frac{mx^2\omega_0^2}{2} + f^2\int d^2x' V_{int,0}^{2D}(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}') \tag{1.17}$$

where,

$$V_{int,0}^{2D}(\mathbf{x}-\mathbf{x}') = g_{c,0}^{2D}\delta^{(2)}(\mathbf{x}-\mathbf{x}') + V_{dd,0}^{2D}(\mathbf{x}-\mathbf{x}') \tag{1.18}$$

Now, one can linearize the fields around the background stationary solution i.e $\rho(\mathbf{x}) = \rho_0(\mathbf{x}) + \delta\rho(\mathbf{x})$ and $\phi(\mathbf{x}) = \phi_0(\mathbf{x}) + \delta\phi(\mathbf{x})$, where $\rho_0(\mathbf{x})$ and $\phi_0(\mathbf{x})$ are the density and the phase of the stationary background solution respectively, and $\delta\rho(\mathbf{x})$ and $\delta\phi(\mathbf{x})$ are the small variation of the density and the phase with respect to the background solution. If one puts the linearized field into the equations (1.16) and (1.17) and take the terms of the first order, the following two equations are obtained.

$$\partial_\tau \delta\rho(\mathbf{x}) = -\frac{\hbar}{m}\Big(\nabla_x\phi_0(\mathbf{x}) \cdot \nabla_x\delta\rho(\mathbf{x}) + \nabla_x\delta\phi(\mathbf{x}) \cdot \nabla_x\rho_0(\mathbf{x}) + \rho_0(\mathbf{x})\nabla_x^2\delta\phi(\mathbf{x}) + \delta\rho(\mathbf{x})\nabla_x^2\phi_0(\mathbf{x})\Big) \tag{1.19}$$

$$\begin{aligned}\partial_\tau \delta\phi(\mathbf{x}) =& \frac{h}{4m}\left(\frac{1}{\rho_0(\mathbf{x})}\nabla_x^2\delta\rho(\mathbf{x}) - \frac{\delta\rho(\mathbf{x})}{\rho_0(\mathbf{x})^2}\nabla_x^2\rho_0(\mathbf{x}) - \frac{1}{\rho_0(\mathbf{x})^2}\nabla_x\rho_0(\mathbf{x}) \cdot \nabla_x\delta\rho(\mathbf{x}) + \frac{\delta\rho(\mathbf{x})}{\rho_0(\mathbf{x})^3}(\nabla_x\rho_0(\mathbf{x}))^2\right) \\ &- \frac{\hbar}{m}\nabla_x\phi_0(\mathbf{x}) \cdot \nabla_x\delta\phi(\mathbf{x}) - f^2\int d^2x' V_{int,0}^{2D}(\mathbf{x}-\mathbf{x}')\delta\rho(\mathbf{x}')\end{aligned} \tag{1.20}$$

Now, if $\rho_0(\mathbf{x})$ is assumed to be a constant in space - ρ_0, then all the space derivatives of $\rho_0(\mathbf{x})$ can be ignored. Once this is done, $\delta\rho(\mathbf{x})$ and $\delta\phi(\mathbf{x})$ are written in terms of their fourier components (this is done so that a specific set of momentum values can be chosen later), i.e:

$$\delta\rho \equiv \delta\rho(\mathbf{x}) = \int \delta\rho_k e^{i\mathbf{k}.\mathbf{x}} d\mathbf{k} \tag{1.21}$$

$$\delta\phi \equiv \delta\phi(\mathbf{x}) = \int \delta\phi_k e^{i\mathbf{k}.\mathbf{x}} d\mathbf{k} \tag{1.22}$$

After putting the equations (1.21) and (1.22) back into (1.19) and (1.20), and setting the space derivatives of $\rho_0(\mathbf{x})$ to be zero, one arrives at the coupled fluctuation equations.

$$(\partial_\tau + iv_{com} \cdot \mathbf{k})\delta\rho_k = \frac{\hbar\rho_0}{m} k^2 \delta\phi_k \tag{1.23}$$

$$(\partial_\tau + iv_{com} \cdot \mathbf{k})\delta\phi_k = -\frac{f^2 g_0^{eff}}{\hbar} \mathcal{W}_k \delta\rho_k \tag{1.24}$$

where,

$$v_{com} = \frac{\hbar}{m} \nabla_x \phi_0(\mathbf{x})$$

$$\mathcal{W}_k = \frac{\zeta^2}{4Af^2} + \frac{V_{int,0}^{2D}(\zeta)}{g_0^{eff}}$$

$\Big($where $A = \frac{mc_0^2}{\hbar\omega_{z,0}}$, $c_0 = \sqrt{g_0^{eff}\rho_0/m}$, $\zeta = \mathbf{k}d_{z,0}$ and $V_{int,0}^{2D}(\zeta)$ is the fourier transformation of $V_{int,0}^{2D}\Big)$

Now, if the co-moving frame velocity, v_{com}, is further assumed to be vanishingly small, it can be set to zero. Then, the equations (1.23) and (1.24) reduce to:

$$\partial_\tau \delta\rho_k = \frac{\hbar\rho_0}{m} k^2 \delta\phi_k \tag{1.25}$$

$$\partial_\tau \delta\phi_k = -\frac{f^2 g_0^{eff}}{\hbar} \mathcal{W}_k \delta\rho_k \tag{1.26}$$

Then, equation (1.25) can be substituted into equation (1.26) along with rescaling in the following way - $\delta\tilde{\phi} = \Omega^{-1/2}\delta\phi$ (where $\Omega = c_0^2 m^2/\hbar^2\rho_0$), so as to get the de-coupled fluctuation equation.

$$\delta\ddot{\tilde{\phi}}_k + \Big(2\frac{\dot{a}}{a} - \frac{\dot{\mathcal{W}}_k}{\mathcal{W}_k}\Big)\delta\dot{\tilde{\phi}}_k + \Big(\frac{c_0 k}{a}\Big)^2 \mathcal{W}_k \delta\tilde{\phi}_k = 0 \tag{1.27}$$

(where the overdot indicates the τ derivaties and $a = 1/f$)

In the long wavelength limit (k $\rightarrow$ 0), $\mathcal{W}_k$ tends to 1, and thus equation (1.27) reduces to:

$$\delta\ddot{\tilde{\phi}}_k + 2\frac{\dot{a}}{a}\delta\dot{\tilde{\phi}}_k + \Big(\frac{c_0 k}{a}\Big)^2 \delta\tilde{\phi}_k = 0 \tag{1.28}$$

This is the phase fluctuation equation.

1.3 Scale Invariant Power Spectrum

Before the power spectrum can be calculated, one has to first find a solution to the equation (1.28), which will give the phase fluctuations, from which the power spectrum can be determined. In order to that, one has to start by first defining the conformal time, η (which will run from $-\infty$ to 0) in the following way:

$$\eta = \int_{\infty}^{\tau} \frac{c_0}{a(\tau')} d\tau' \tag{1.29}$$

As one is working in an effective de-Sitter space, a has the form, $a = e^{H\tau}$ (where H is the Hubble constant). Putting this back in the definition of conformal time, we get:

$$\eta = -\frac{c_0}{aH} \tag{1.30}$$

Note

To get $a = e^{H\tau}$, one needs to tune the scale parameter, $b(t)$, and the angular frequency, $\omega(t)$, of the BEC in the experimental setup. In order to do this, the scale parameter has to be set in the following way : $b(t) = a^2(t)$. Then, using equations (1.5) and (1.11), and the fact that $a(t) = 1/f(t)$, one finds that [7, 8]:

$$b(t) = \sqrt{4Ht + 1} \tag{1.31}$$

$$\omega^2(t) = \frac{\omega_0^2(t)}{b^5} + \frac{4H^2}{b^4} \tag{1.32}$$

Thus, if one tunes $b(t)$ and $\omega(t)$ as prescribed by the above equations in the experimental set-up, $a(t)$ can be made to behave like the cosmological factor in the de-Sitter space.

At this stage, one has to define another parameter, s, which is the ratio of Hubble radius to the physical wavelength of the chosen mode.

$$s = \frac{c_0/H}{a/k} = \frac{c_0 k}{aH} \tag{1.33}$$

Therefore, it can be seen be that s is related to η by the simple relation $s = -k\eta$ (hence s runs from ∞ to 0). Now using this relation and equation (1.29), equation (1.28) can be rewritten using the parameter s instead of τ, and thus one obtains:

$$\delta\tilde{\phi}_k'' - \frac{1}{s}\delta\tilde{\phi}_k' + \delta\tilde{\phi}_k = 0 \tag{1.34}$$

(where prime denotes the derivative with respect to s)

We want to find $\delta\phi_k$ which satisfies the equation (1.34). In order to do this, let us first consider the equation in large s limit, where it reduces to $\delta\tilde{\phi}_k'' + \delta\tilde{\phi}_k = 0$. In such a scenario, the solution to the equation is given by:

$$\delta\tilde{\phi}_k = \sqrt{\frac{\hbar V}{2ma^2Hs}} \exp{(is)} \tag{1.35}$$

where the co-efficient was determined by imposing the normalization condition:

$$\left(\delta\tilde{\phi}_k e^{\mathbf{ik.x}}/V, \delta\tilde{\phi}_{k'} e^{\mathbf{ik'.x}}/V\right)_{KG} = \delta^{(2)}_{\mathbf{k},\mathbf{k'}} \tag{1.36}$$

$\Big($The conserved Klein-Gordon(KG) product is defined in the following way:

$$\begin{aligned}(f,g)_{K,G} &= i\frac{mc_0}{\hbar}\int d^2\mathbf{x}\sqrt{|\gamma|}f^*(\mathbf{x},\tau)\overleftrightarrow{\partial_n}g(\mathbf{x},\tau)\\ &= i\frac{mc_0}{\hbar}\int d^2\mathbf{x}\frac{a^2}{c_0^2}f^*(\mathbf{x},\tau)\overleftrightarrow{\partial_\tau}g(\mathbf{x},\tau)\end{aligned}$$

where γ is the determinant of the metric in the spatial slice $\tau =$ constant, n_μ its normal and $f^*(\mathbf{x},\tau)\overleftrightarrow{\partial_n}g(\mathbf{x},\tau) = f^*(\mathbf{x},\tau)n^\mu\partial_\mu(g(\mathbf{x},\tau)) - n^\mu\partial_\mu(f^*(\mathbf{x},\tau))g(\mathbf{x},\tau)\Big)$

In order to determine the analytic solution of $\delta\tilde{\phi}_k$ over the entire range of s, a function, $F(s)$, has to defined in the following way:

$$F(s) = \frac{1}{s}\delta\tilde{\phi}_k(s) \tag{1.37}$$

Putting this back in equation (1.34), one gets:

$$s^2F'' + sF' + (s^2-1)F = 0 \tag{1.38}$$

This is a Bessel equation of order 1, whose general solution can be written as a linear combination of Bessel functions, J_1 and Y_1. Thus, one obtains:

$$\delta\tilde{\phi}_k(s) = s[A(k)J_1(s) + B(k)Y_1(s)] \tag{1.39}$$

In order to determine the co-efficients, $A(k)$ and $B((k)$, the above solution needs to be compared to the solution at $s \to \infty$ limit [9, 10].

$$\lim_{s\to\ \infty} J_1(s) = \sqrt{\frac{2}{\pi s}} \cos{(s)}$$
$$\lim_{s\to\ \infty} Y_1(s) = \sqrt{\frac{2}{\pi s}} \sin{(s)}$$

Putting this back in expression (1.39) and comparing it with solution (1.35), one finds that

$$B(k) = iA(k)$$

$$A(k) = \sqrt{\frac{\pi\hbar V H}{4mc_0^2 k^2}}$$

Therefore,

$$\delta\tilde{\phi}_k(s) = s\sqrt{\frac{\pi\hbar V H}{4mc_0^2 k^2}}[J_1(s) + iY_1(s)] = h_k(s) \tag{1.40}$$

Using the expression (1.40), one can write the mode expansion for the phase fluctuation as:

$$\delta\hat{\tilde{\phi}}(\mathbf{x},\tau) = \sum_{\mathbf{k}} \hat{a_k} f_k^{(0)}(\mathbf{x},\tau) + \sum_{\mathbf{k}} \hat{a_k}^\dagger f_k^{(0)*}(\mathbf{x},\tau) \tag{1.41}$$

where $\hat{a_k}$ and $\hat{a_k}^\dagger$ are time independent creation and annihilation operators obeying the commutation relation, $[\hat{a_k}, \hat{a_{k'}}^\dagger] = \delta^{(2)}_{\mathbf{k},\mathbf{k'}}$. Also the mode function, $f_k^{(0)}(\mathbf{x},\tau)$ is written as:

$$f_k^{(0)}(\mathbf{x},\tau) = \frac{1}{V} h_k(s) e^{i\mathbf{k}.\mathbf{x}} \tag{1.42}$$

Now using the definition of the mode expansion of phase fluctuation, one can calculate the correlation function, ξ.

$$\begin{aligned}\xi(\mathbf{x}-\mathbf{y}) &= \langle 0|\delta\hat{\tilde{\phi}}(\mathbf{x},\tau)\delta\hat{\tilde{\phi}}(\mathbf{y},\tau)|0\rangle \\ &= \frac{1}{V}\sum_{\mathbf{k}}\frac{|h_k|^2}{V}e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}\end{aligned} \tag{1.43}$$

Power spectrum is defined as the fourier transform of the correlation function. Hence from the above definition of correlation function, $|h_k|^2/V$ can be identified as the power spectrum, $P(k)$ i.e:

$$P(k) = \frac{|h_k|^2}{V} \tag{1.44}$$

The power spectrum has be to calculated at late times i.e at $s \to 0$ (or $\tau \to \infty$), hence h_k needs to known as $s \to 0$ [9, 10].

$$\lim_{s\to\ 0} h_k = -i\sqrt{\frac{\hbar V H}{\pi m c_0^2}\frac{1}{k}} \tag{1.45}$$

Thus one gets the power spectrum, $P(k)$, to be:

$$P(k) = \frac{\hbar H}{\pi m c_0^2}\frac{1}{k^2} \tag{1.46}$$

At this point, one define a quantity,

$$\Delta^2(k) = k^2 P(k) \tag{1.47}$$

It can clearly see that $\Delta^2(k)$ is independent of k for the $P(k)$ calculated above. This quantity is what will be referred to as the Scale Invariant Power Spectrum (SIPS). In [1], it has been shown through numerical simulations that SIPS breaks down when strong dipole-dipole interactions are considered.

Chapter 2

Co-ordinate Dependent Background Density and its Implications

In all of the calculations in the previous chapter, the background density of the BEC, $\rho_0(\mathbf{x})$, was assumed to be a constant in space, which need not be the case in a generalized setting. So in our work, we will relax this assumption and see how the co-ordinate dependence of $\rho_0(\mathbf{x})$ will affect the calculations even in the absence of the strong dipole-dipole interactions, which was primarily responsible for the breakdown of SIPS in the expanding BEC model of [1].

2.1 Choosing a Suitable Background Density

The background density, ρ_0 we want to choose should be such that it satisfies the zeroth order GP equation, which reads

$$\mu\sqrt{\rho}_0(\mathbf{x}) = -\frac{\hbar^2}{2m}\nabla_x^2\sqrt{\rho_0(\mathbf{x})} + f^2\frac{mx^2\omega_0^2}{2}\sqrt{\rho_0(\mathbf{x})} + f^2\left[\int d^2\mathbf{x}' V_{int,0}^{2D}(\mathbf{x}-\mathbf{x}')\rho_0(\mathbf{x}')\right]\sqrt{\rho_0(\mathbf{x})} \tag{2.1}$$

where,

$$\mu = -\hbar\partial_\tau\phi_0$$

If we ignore the confining potential term $(f^2\frac{mx^2\omega_0^2}{2}\sqrt{\rho_0(\mathbf{x})})$ - as we are interested in the region of small x - we see that $\sqrt{\rho_0(\mathbf{x})} = \sqrt{\rho_0}\cos{(\mathbf{p}\cdot\mathbf{x})}$ (where ρ_0 is a constant) would solve the equation (2.1). So at a first glance this seems like a viable solution. But in order to see the problem with this choice of background density, we have to write out the resulting fluctuation equation, which reads

$$\partial_\tau \delta\phi = \frac{1}{\rho_0\cos^2{(\mathbf{p}\cdot\mathbf{x})}}\frac{\hbar}{4m}\Big(\nabla_x^2\delta\rho + p^2\delta\rho + \frac{2\sin{(\mathbf{p}\cdot\mathbf{x})}}{\cos{(\mathbf{p}\cdot\mathbf{x})}}\ \mathbf{p}\cdot\nabla_x\delta\rho\Big)$$
$$- \frac{f^2}{\hbar}\int d^2x' V_{int,0}^{2D}(\mathbf{x}-\mathbf{x}')\delta\rho(\mathbf{x}') \qquad (2.2)$$

Now, if we observe the equation (2.2) closely, we see that there are terms with $\cos{(\mathbf{p}.\mathbf{x})}$ in the denominator. Whenever $(\mathbf{p}\cdot\mathbf{x}) = n\frac{\pi}{2}$ (where n is an odd integer), these terms would blow up and as a result the fluctuation equation becomes problematic. Moreover, neglecting the confining potential term goes against the concept of controlled expansion of the gas, making this choice uninteresting. All of this forces us to abandon the choice of $\sqrt{\rho_0(\mathbf{x})} = \sqrt{\rho_0}\cos{(\mathbf{p}\cdot\mathbf{x})}$ for background density.

Having seen that our initial choice of $\rho_0(\mathbf{x})$ does not work, let us adopt a slightly different approach in choosing the background density. If we ignore the interaction term $(V_{int,0}^{2D})$, we see that equation (2.1) has the structure of the linear harmonic oscillator. This motivates us to choose $\sqrt{\rho_0(\mathbf{x})}$ to be the solution of the harmonic oscillator i.e,

$$\sqrt{\rho_0(\mathbf{x})} = \sqrt{\rho_0}e^{-\frac{\mathbf{x}^2}{\alpha}} \qquad (2.3)$$

(where ρ_0 is a constant)

In order to determine the constants μ and α, we substitute equation (2.3) back into equation (2.1). From this, we get

$$\mu = \frac{\hbar\omega_0}{f} \qquad \alpha = \frac{2\hbar f}{m\omega_0}$$

We are interested in the behaviour of $\rho_0(\mathbf{x})$ near the centre of the cloud, where we are dealing with small values of x. In this region, our choice of a Gaussian background density can be approximated to be of the form $R - Sx^2$ (where R and S are constants), which is similar in structure to the density determined from the Thomas-Fermi approximation, where the kinetic energy term is ignored (appendix of [1]). As these two varied approaches to determine the background density predict a similar structure of $\rho_0(\mathbf{x})$, we feel more confident in our choice.

2.2 New Fluctuation Equations

Now, we need to incorporate the co-ordinate dependence of $\rho_0(\mathbf{x})$ into the fluctuation equations. In order to do that, let us first recall the fluctuation equation from the previous before $\rho_0(\mathbf{x})$ was set to be a constant i.e:

$$\partial_\tau \delta\rho = -\frac{\hbar}{m}(\nabla_x \delta\phi \cdot \nabla_x \rho_0(\mathbf{x}) + \rho_0 \nabla_x^2 \delta\phi) \tag{2.4}$$

$$\begin{aligned}\partial_\tau \delta\phi = \frac{h}{4m}\Big(\frac{1}{\rho_0(\mathbf{x})}\nabla_x^2\delta\rho - \frac{\delta\rho}{\rho_0(\mathbf{x})^2}\nabla_x^2\rho_0(\mathbf{x}) - \frac{1}{\rho_0(\mathbf{x})^2}\nabla_x\rho_0(\mathbf{x})\cdot\nabla_x\delta\rho + \frac{\delta\rho}{\rho_0(\mathbf{x})^3}(\nabla_x\rho_0)^2\Big)\\ - f^2\int d^2x' V_{int,0}^{2D}(\mathbf{x}-\mathbf{x}')\delta\rho(\mathbf{x}')\end{aligned} \tag{2.5}$$

Now, we can substitute the expression (2.3) into the equations (2.4) and (2.5). Then, we get

$$\partial_\tau \delta\rho = \frac{\hbar\rho_0}{m} e^{\frac{-2\mathbf{x}^2}{\alpha}}\Big(\ \frac{4}{\alpha}\mathbf{x}\cdot\nabla_x\delta\phi - \nabla_x^2\delta\phi\ \Big) \tag{2.6}$$

$$e^{\frac{-2\mathbf{x}^2}{\alpha}}\partial_\tau\delta\phi = \frac{\hbar}{4m\rho_0}\Big(\ \nabla_x^2\delta\rho + \frac{8}{\alpha}\delta\rho + \frac{4}{\alpha}\mathbf{x}\cdot\nabla_x\delta\rho\ \Big) - e^{\frac{-2\mathbf{x}^2}{\alpha}}\frac{f^2}{\hbar}\int d^2x' V_{int,0}^{2D}(\mathbf{x}-\mathbf{x}')\delta\rho(\mathbf{x}') \tag{2.7}$$

Taking the Fourier transformation of these equations, we arrive at

$$\partial_\tau \delta\rho_k = \frac{\hbar\rho_0}{m}\frac{\alpha}{8\pi}\int d^2\mathbf{p}\ e^{-\frac{\alpha}{8}(\mathbf{k}-\mathbf{p})^2}\ \mathbf{k}\cdot\mathbf{p}\ \delta\phi_p \tag{2.8}$$

$$\begin{aligned}\frac{\alpha}{8\pi}\int d^2\mathbf{p}\ e^{-\frac{\alpha}{8}(\mathbf{k}-\mathbf{p})^2}\partial_\tau\delta\phi_p =& \frac{\hbar}{4m\rho_0}\Big(-k^2\delta\rho_k + \frac{8}{\alpha}\delta\rho_k - \frac{4}{\alpha}\int d^2\mathbf{p}\ \nabla_{\mathbf{k}-\mathbf{p}}\cdot(\delta\rho_p\ \mathbf{p})\ \Big)\\ &- \frac{f^2}{\hbar}\frac{\alpha}{8\pi}\int d^2\mathbf{p}\ e^{-\frac{\alpha}{8}(\mathbf{k}-\mathbf{p})^2} V^{2D}_{int,0}(\mathbf{p})\ \delta\rho_p\end{aligned} \tag{2.9}$$

Using the divergence theorem in 2-D, notice that we can write,

$$\int d^2\mathbf{p}\ \nabla_{\mathbf{k}-\mathbf{p}}\cdot(\delta\rho_p\ \mathbf{p}) = -\oint d\mathbf{p}_l\cdot\mathbf{p}\ \delta\rho_p$$

(where $d\mathbf{p}_l$ is the line element along the loop)

But for large enough $\mathbf{p}$, we can approximate

$$\begin{aligned}-\oint d\mathbf{p}_l\cdot\mathbf{p}\ \delta\rho_p &\simeq -p\sum\delta\rho_{\mathbf{p}}\\ &\simeq 0\end{aligned}$$

where the last equality can be written because the sum of the fluctuations on a closed loop can be approximated to add up to 0.

Also notice that the Gaussian function goes towards zero as we move away from the centre and hence the majority contribution to the integral terms involving the Gaussian comes from near the centre. Thus, in this region, we can can approximate $\delta\rho_p$, $\delta\phi_p$ and $V^{2D}_{int,0}(\mathbf{p})$ to $\delta\rho_k$, $\delta\phi_k$ and $V^{2D}_{int,0}(\mathbf{k})$ respectively. Using this, we can write

$$\begin{aligned}\frac{\hbar\rho_0}{m}\frac{\alpha}{8\pi}\int d^2\mathbf{p}\ e^{-\frac{\alpha}{8}(\mathbf{k}-\mathbf{p})^2}\ \mathbf{k}\cdot\mathbf{p}\ \delta\phi_p &\simeq \frac{\hbar\rho_0}{m}\frac{\alpha}{8\pi}\ \delta\phi_k\ \int d^2\mathbf{p}\ e^{-\frac{\alpha}{8}(\mathbf{k}-\mathbf{p})^2}\ \mathbf{k}\cdot\mathbf{p}\\ &= \frac{\hbar\rho_0}{m}k^2\delta\phi_k\end{aligned}$$

$$\begin{aligned}\frac{\alpha}{8\pi}\int d^2\mathbf{p}\ e^{-\frac{\alpha}{8}(\mathbf{k}-\mathbf{p})^2}\partial_\tau\delta\phi_p &\simeq \frac{\alpha}{8\pi}\partial_\tau\delta\phi_k\int d^2\mathbf{p}\ e^{-\frac{\alpha}{8}(\mathbf{k}-\mathbf{p})^2}\\ &= \partial_\tau\delta\phi_k\end{aligned}$$

$$\begin{aligned}\frac{f^2}{\hbar}\frac{\alpha}{8\pi}\int d^2\mathbf{p}\ e^{-\frac{\alpha}{8}(\mathbf{k}-\mathbf{p})^2}V^{2D}_{int,0}(\mathbf{p})\ \delta\rho_p &\simeq \frac{f^2}{\hbar}\frac{\alpha}{8\pi}\ V^{2D}_{int,0}(\mathbf{k})\ \delta\rho_k\int d^2\mathbf{p}\ e^{-\frac{\alpha}{8}(\mathbf{k}-\mathbf{p})^2}\\ &= \frac{f^2}{\hbar}V^{2D}_{int,0}(\mathbf{k})\ \delta\rho_k\end{aligned}$$

Once these approximations are put in place, equations (2.8) and (2.9) reduce to

$$\partial_\tau\delta\rho_k = \frac{\hbar\rho_0}{m}k^2\delta\phi_k \tag{2.10}$$

$$\partial_\tau\delta\phi_k = \Big(-\frac{\hbar k^2}{4m\rho_0}+\frac{2\hbar}{m\rho_0\alpha}-\frac{f^2}{\hbar}V^{2D}_{int,0}(\mathbf{k})\ \Big)\delta\rho_k \tag{2.11}$$

Comparing equations (2.10) and (2.11) with equations (1.25) and (1.26), we see that $\frac{2\hbar}{m\rho_0\alpha}\delta\rho_k$ is the additional term which arises because of the co-ordinate dependence of the background density. In the next chapter, we will see how this additional term plays into the de-coupled phase fluctuation and then the power spectrum of the BEC.

Chapter 3

Power Spectrum of the Modified BEC System

In the previous chapter, we had obtained the new coupled fluctuation equations as a result of accounting for the co-ordinate dependence of background density ($\rho_0(\mathbf{x})$). Now using those equations, we will first arrive at the de-coupled phase fluctuation equation. Then, using the solution of these phase fluctuation equation, we arrive at the power spectrum of the modified BEC system.

3.1 New Phase Fluctuation Equation

Once we substitute for α in the equation (2.11), the fluctuation equation for BEC reads

$$\partial_\tau \delta\rho_k = \frac{\hbar\rho_0}{m} k^2 \delta\phi_k \tag{3.1}$$

$$\partial_\tau \delta\phi_k = \Big(- \frac{\hbar k^2}{4ma^2\rho_0} + \frac{a\omega_0}{\rho_0} - \frac{1}{a^2\hbar} V^{2D}_{int,0}(\mathbf{k}) \Big) \delta\rho_k \tag{3.2}$$

In our calculations, we will ignore the dipole-dipole interaction (i.e $V^{2D}_{dd,0}$), and hence the interaction term, $V^{2D}_{int,o} = g^{2D}_{eff,0}$. With this in mind, let us define a quantitiy,

$$M = \frac{g^{2D}_{eff,0}}{a^2} - \frac{\hbar\omega_0}{\rho_0}a \tag{3.3}$$

Re-writing (3.2) in terms of M, we have

$$\partial_\tau \delta\phi_k = \frac{1}{\hbar}\Big(M - \frac{\hbar^2 k^2}{4ma^2\rho_0}\Big)\delta\rho_k \tag{3.4}$$

Let us estimate the order of the term $\frac{\hbar^2k^2}{4ma^2M\rho_0}$, considering a^2 to be of order unity. Using the data from [11, 12] $\frac{\hbar^2k^2}{4ma^2M\rho_0} \sim \frac{(10^{-68}\text{kg}^2\text{m}^4\text{s}^{-2})(1\text{m}^{-2})}{(4)(10^{-27}\text{kg})(1)(10^{-30}\text{kgm}^2\text{s}^{-2})} \sim 10^{-11}$*. With this rough estimate in mind, it is reasonable to make the approximation, $1 >> \frac{\hbar^2k^2}{4ma^2M\rho_0}$. Then, we can substitute the expression for $\delta\rho_k$ from the equation (3.4) into (3.1) and rescale $\delta\tilde{\phi} = \Omega^{-1/2}\delta\phi$ (where $\Omega = g^{2D}_{eff,0}m/\hbar^2$), we have

$$\Big(1 - \frac{\hbar^2k^2}{4ma^2M\rho_0}\Big)\delta\ddot{\tilde{\phi}}_k + \Big(\frac{2\hbar^2k^2}{4ma^2M\rho_0}\frac{\dot{a}}{a} + \frac{2\hbar^2k^2}{4ma^2M\rho_0}\frac{\dot{M}}{M} - \frac{\dot{M}}{M}\Big)\delta\dot{\tilde{\phi}}_k + \frac{\rho_0 M}{m}k^2\delta\tilde{\phi}_k = 0 \tag{3.5}$$

$$\implies \delta\ddot{\tilde{\phi}}_k + \Big(\frac{2\hbar^2k^2}{4ma^2M\rho_0}\frac{\dot{a}}{a} + \frac{\hbar^2k^2}{4ma^2M\rho_0}\frac{\dot{M}}{M} - \frac{\dot{M}}{M}\Big)\delta\dot{\tilde{\phi}}_k + \frac{\rho_0 M}{m}k^2\Big(1 + \frac{\hbar^2k^2}{4ma^2M\rho_0}\Big)\delta\tilde{\phi}_k = 0 \tag{3.6}$$

*In order to determine the order of magnitude of the factor $M\rho_0$, we take the data from [11, 12] to get $M\rho_0 \sim g^{2D}_{eff,0}\rho_0 \sim 10^{-50} \times 10^{20}$ kgm^{2}s^{-2} $\sim 10^{-30}$ kgm^{2}s^{-2}. We have used the data of $g_{eff,0}$ and ρ_0^{3D} and this could be done, since we can relate these quantities to $g^{2D}_{eff,0}$ and ρ_0 as follows: $g_{eff,0}\ \rho_0^{3D} \sim \frac{g^{2D}_{eff,0}}{d_z}\rho_0 d_z$, where d_z is the length scale associated with the problem

Equation (3.6) can be written as a wave equation for a massless and minimally coupled free scalar field in a curved space time i.e

$$\Box\delta\phi = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\delta\phi) = 0 \tag{3.7}$$

where the metric $g_{\mu\nu}$ is given by:

$$g_{\mu\nu} = \begin{bmatrix} c_0^2 & 0 & 0 \\ 0 & -\frac{g^{2D}_{eff,0}}{M}\left(1-\frac{\hbar^2k^2}{4ma^2M\rho_0}\right) & 0 \\ 0 & 0 & -\frac{g^{2D}_{eff,0}}{M}\left(1-\frac{\hbar^2k^2}{4ma^2M\rho_0}\right) \end{bmatrix} \tag{3.8}$$

(The metric is described by a 3×3 matrix as the z-part has been integrated out and we are in the (2+1)-D space-time)

In the long wave length limit, the terms having $\frac{\hbar^2k^2}{4ma^2M\rho_0}$ can be dropped and thus (3.6) reduces to

$$\ddot{\delta\tilde{\phi}_k} - \frac{\dot{M}}{M}\dot{\delta\tilde{\phi}_k} + \frac{\rho_0 M}{m}k^2\delta\tilde{\phi}_k = 0 \tag{3.9}$$

Now, if substitute the expression for M from (3.3), we get

$$\ddot{\delta\tilde{\phi}_k} + \frac{\frac{2g^{2D}_{eff,0}}{a^2} + \frac{h\omega_0 a}{\rho_0}}{\frac{g^{2D}_{eff,0}}{a^2} - \frac{h\omega_0 a}{\rho_0}}\dot{\delta\tilde{\phi}_k} + \frac{\rho_0}{m}\left(\frac{g^{2D}_{eff,0}}{a^2} - \frac{h\omega_0 a}{\rho_0}\right)k^2\delta\tilde{\phi}_k = 0 \tag{3.10}$$

The above equation can be re-written as a free scalar field equation using the metric given below.

$$g_{\mu\nu} = \Omega^{-2}\begin{bmatrix} c_0^2 & 0 & 0 \\ 0 & -\frac{g^{2D}_{eff,0}}{M} & 0 \\ 0 & 0 & -\frac{g^{2D}_{eff,0}}{M} \end{bmatrix} \tag{3.11}$$

In the metric (3.11) , only the term $\frac{g^{2D}_{eff,0}}{M}$ has time(τ) dependence. Thus let us define:

$$a'^2(\tau) = \frac{g^{2D}_{eff,0}}{M} \tag{3.12}$$

Now, writing the metric in terms of $a'(\tau)$, we have

$$g_{\mu\nu} = \Omega^{-2} \begin{bmatrix} c_0^2 & 0 & 0 \\ 0 & -a'^2(\tau) & 0 \\ 0 & 0 & -a'^2(\tau) \end{bmatrix} \tag{3.13}$$

Also, re-writing (3.9) in terms of $a'(\tau)$, we get the new phase fluctuation equation.

$$\delta\ddot{\tilde{\phi}}_k + 2\frac{\dot{a}'}{a'}\delta\dot{\tilde{\phi}}_k + \left(\frac{c_0 k}{a'}\right)^2 \delta\tilde{\phi}_k = 0 \tag{3.14}$$

3.2 Obtaining the Power Spectrum

The new phase fluctuation equation has a structure similar to equation (1.28) with the difference that $a(\tau)$ is replaced by $a'(\tau)$. Hence, in order to obtain the power spectrum from here we will follow an approach very similar to the one used in section (1.3). So, let us start by defining a conformal time, η' (which will run from $-\infty$ to 0) in the following way:

$$\eta' = \int_{\infty}^{\tau} \frac{c_0}{a'(\tau')} d\tau' \tag{3.15}$$

The metric in (3.13) looks like the FRW metric with $a'(\tau)$ as the scaling factor. Thus, in an effective de-Sitter space we can write $a'(\tau) = e^{H\tau}$ (where H is the Hubble constant). Putting this back in the definition of conformal time, we get:

$$\eta' = -\frac{c_0}{a'H} \tag{3.16}$$

Note

For our calculations to be experimentally viable, we need to check if we can get $a' = e^{H\tau}$ by tuning the scale parameter, $b(t)$, and the angular frequency, $\omega(t)$, of the BEC in the experimental setup. To do that, let us first write $a'(\tau)$ in terms of $a(\tau)$ using the equations (3.3) and (3.12) and then substitute $a' = e^{H\tau}$. On doing that, we arrive at

$$\frac{\hbar\omega_0}{\rho_0}a^3 + \frac{g^{2D}_{eff,0}}{e^{2H\tau}}a^2 - g^{2D}_{eff,0} = 0 \tag{3.17}$$

Looking at the equation (3.17), we can see that there is atleast one positive real value of $a(\tau)$, which solves the equation. The existence of this positive real value of $a(\tau)$ combined with the fact that $b = a^2$ and $a = 1/f$ will allow us to get an expression for $b(t)$ and $\omega(t)$ using the equations (1.5) and (1.11). This will give us the experimental conditions necessary to get $a'(\tau)$ to behave like the cosmological scale factor in de-Sitter space.

Let us also define another parameter, s_{new}, which is the ratio of Hubble radius to the physical wavelength of the chosen mode.

$$s_{new} = \frac{c_0/H}{a'/k} = \frac{c_0 k}{a'H} \tag{3.18}$$

Therefore, we can see that s is related to η' by the simple relation $s_{new} = -k\eta'$ (hence s_{new} runs from ∞ to 0). Now using this relation and (3.15), we can rewrite (3.14) using the parameter s_{new} instead of τ, and thus we obtain:

$$\delta\tilde{\phi}''_k - \frac{1}{s_{new}}\delta\tilde{\phi}'_k + \delta\tilde{\phi}_k = 0 \tag{3.19}$$

(where prime denotes the derivative with respect to s_{new})

We want to find $\delta\tilde{\phi_k}$ which satisfies the equation (3.19). In order to this, let us first consider the equation in large s_{new} limit, where it reduces to $\delta\tilde{\phi}_k'' + \delta\tilde{\phi_k} = 0$. In such a scenario, the solution to the equation is given by:

$$\delta\tilde{\phi_k} = \sqrt{\frac{\hbar V}{2ma'^2 H s_{new}}} \exp\left(i s_{new}\right) \tag{3.20}$$

The co-efficient of $e^{is_{new}}$ was determined by imposing the normalization condition:

$$\left(\delta\tilde{\phi_k} e^{\mathbf{ik.x}}/V, \delta\tilde{\phi_{k'}} e^{\mathbf{ik'.x}}/V\right)_{KG} = \delta^{(2)}_{\mathbf{k},\mathbf{k'}} \tag{3.21}$$

$\Big($The conserved Klein-Gordon(KG) product is defined in the following way:

$$\begin{aligned}(f,g)_{K,G} &= i\frac{mc_0}{\hbar}\int d^2\mathbf{x}\sqrt{|\gamma|} f^*(\mathbf{x},\tau)\overleftrightarrow{\partial_n} g(\mathbf{x},\tau) \\ &= i\frac{mc_0}{\hbar}\int d^2\mathbf{x}\frac{a'^2}{c_0} f^*(\mathbf{x},\tau)\overleftrightarrow{\partial_\tau} g(\mathbf{x},\tau)\end{aligned}$$

where γ is the determinant of the metric in the spatial slice $\tau =$ constant, n_μ its normal and $f^*(\mathbf{x},\tau)\overleftrightarrow{\partial_n} g(\mathbf{x},\tau) = f^*(\mathbf{x},\tau) n^\mu \partial_\mu(g(\mathbf{x},\tau)) - n^\mu\partial_\mu(f^*(\mathbf{x},\tau)) g(\mathbf{x},\tau)\Big)$

In order to determine the analytic solution of $\delta\tilde{\phi_k}$ over the entire range of s_{new}, let us define a function, $F(s_{new})$:

$$F(s_{new}) = \frac{1}{s_{new}}\delta\tilde{\phi_k}(s_{new}) \tag{3.22}$$

Putting this back in equation (3.19), we get:

$$s_{new}^2 F'' + s_{new}F' + (s_{new}^2 - 1)F = 0 \tag{3.23}$$

This is a Bessel equation of order 1, whose general solution can be written as a linear combination of Bessel functions, J_1 and Y_1. Thus, we obtain:

$$\delta\tilde{\phi_k}(s_{new}) = s_{new}[A(k)J_1(s_{new}) + B(k)Y_1(s)] \tag{3.24}$$

In order to determine the co-efficients, $A(k)$ and $B((k)$, we need to compare the above solution to the solution at $s_{new} \to \infty$ limit [9, 10].

$$\lim_{s_{new}\to\ \infty} J_1(s_{new}) = \sqrt{\frac{2}{\pi s}} \cos{(s_{new})}$$
$$\lim_{s_{new}\to\ \infty} Y_1(s_{new}) = \sqrt{\frac{2}{\pi s}} \sin{(s_{new})}$$

Putting this back in the expression (3.24) and comparing it with solution (3.20), we find that

$$B(k) = iA(k)$$
$$A(k) = \sqrt{\frac{\pi\hbar V H}{4mc_0^2k^2}}$$

Thus we have,

$$\delta\tilde{\phi}_k(s_{new}) = s_{new}\sqrt{\frac{\pi\hbar V H}{4mc_0^2k^2}}[J_1(s_{new}) + iY_1(s_{new})] = h_k(s_{new}) \tag{3.25}$$

From this, we can write the mode expansion for the phase fluctuation as:

$$\delta\hat{\tilde{\phi}}(\mathbf{x}, \tau) = \sum_{\mathbf{k}} \hat{a_k} f_k^{(0)}(\mathbf{x}, \tau) + \sum_{\mathbf{k}} \hat{a_k}^\dagger f_k^{(0)*}(\mathbf{x}, \tau) \tag{3.26}$$

where $\hat{a_k}$ and $\hat{a_k}^\dagger$ are time independent creation and annihilation operators obeying the commutation relation, $[\hat{a_k}, \hat{a_{k'}}^\dagger] = \delta^{(2)}_{\mathbf{k},\mathbf{k'}}$. Also the mode function, $f_k^{(0)}(\mathbf{x}, \tau)$ is written as:

$$f_k^{(0)}(\mathbf{x}, \tau) = \frac{1}{V} h_k(s_{new}) e^{i\mathbf{k}.\mathbf{x}} \tag{3.27}$$

Now using the definition of the mode expansion of phase fluctuation, let us calculate the correlation function, ξ.

$$\begin{aligned}\xi(\mathbf{x}-\mathbf{y}) &= \langle 0|\delta\hat{\tilde{\phi}}(\mathbf{x},\tau)\delta\hat{\tilde{\phi}}(\mathbf{y},\tau)|0\rangle \\ &= \frac{1}{V}\sum_{\mathbf{k}}\frac{|h_k|^2}{V}e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})} \end{aligned} \tag{3.28}$$

Power spectrum is defined as the fourier transform of the correlation function. Hence from the above definition of correlation function, we can identify that $|h_k|^2/V$ is the power spectrum, $P(k)$ i.e:

$$P(k) = \frac{|h_k|^2}{V} \tag{3.29}$$

We want to calculate the power spectrum at late times i.e at $s_{new} \to 0$ (or $\tau \to \infty$), hence we want to know h_k as $s_{new} \to 0$ [9, 10].

$$\lim_{s_{new}\to\ 0} h_k = -i\sqrt{\frac{\hbar V H}{\pi m c_0^2}\frac{1}{k}} \tag{3.30}$$

Thus we get the power spectrum, $P(k)$, to be:

$$P(k) = \frac{\hbar H}{\pi m c_0^2}\frac{1}{k^2} \tag{3.31}$$

Previously, we had defined the Scale Invariant Power Spectrum (SIPS), $\Delta^2(k)$, in the following way:

$$\Delta^2(k) = k^2 P(k) \tag{3.32}$$

Putting back the expression for $P(k)$ from (3.31), we see that $\Delta^2(k)$ is independent of k and thus, still remains scale invariant.

3.3 Multiple Scale Analysis

In the previous section, we had ignored the terms having $\frac{\hbar^2 k^2}{4ma^2 M\rho_0}$ in the long wavelength limit. But now, continuing in the same limit, let us keep those terms and try to solve (3.6) perturbatively. In order to do this, we will be using the approach of multiple-scale analysis [13]. Before we begin the multiple-scale approach, we have to first identify a dimensionless time independent small parameter λ, which can be treated perturbatively. Even though quantity $\frac{\hbar^2 k^2}{4ma^2 M\rho_0}$ is small in the long wavelength limit, it has time dependence. To absorb this time dependence, let us define an other dimensionless quantity, Z. Then we can write,

$$\frac{\hbar^2 k^2}{4ma^2 M\rho_0} = \frac{\lambda}{Z} \tag{3.33}$$

such that,

$$\lambda = \frac{\hbar^2 k^2}{4m g^{2D}_{eff,0}\rho_0} \tag{3.34}$$

$$Z = \frac{a^2 M}{g^{2D}_{eff,0}} \tag{3.35}$$

Now using these definitions, we can re-write equation (3.6) in the following way:

$$\delta\ddot{\tilde{\phi}}_k + \Big(\lambda\frac{2}{Z}\frac{\dot{a}}{a} + \lambda\frac{\dot{M}}{ZM} - \frac{\dot{M}}{M}\Big)\delta\dot{\tilde{\phi}}_k + \frac{\rho_0 M}{m}k^2\Big(1 + \lambda\frac{1}{Z}\Big)\delta\tilde{\phi}_k = 0 \tag{3.36}$$

In the multiple scale approach, we will use two time scales, τ and τ' $(= \lambda\tau)$, and treat each time scale independently i.e:

$$\frac{d\mathcal{O}(\tau,\tau')}{d\tau} = \frac{\partial\mathcal{O}(\tau,\tau')}{\partial\tau} + \frac{d\tau'}{d\tau}\,\frac{\partial\mathcal{O}(\tau,\tau')}{\partial\tau'}$$

$$\frac{d\mathcal{O}(\tau,\tau')}{d\tau} = \frac{\partial\mathcal{O}(\tau,\tau')}{\partial\tau} + \lambda\frac{\partial\mathcal{O}(\tau,\tau')}{\partial\tau'} \tag{3.37}$$

But before we do that, we must write $\delta\tilde{\phi_k}$ in its perturbative expansion form.

$$\delta\tilde{\phi_k}(\tau,\tau') = \delta\tilde{\phi_k^0}(\tau,\tau') + \lambda\delta\tilde{\phi_k^1}(\tau,\tau') \tag{3.38}$$

where,

$$\begin{aligned} \delta\tilde{\phi_k^0}(\tau,\tau') &= \mathcal{F}_0(\tau')\ \delta\tilde{\phi_k^0}(\tau) \\ \delta\tilde{\phi_k^1}(\tau,\tau') &= \mathcal{F}_1(\tau')\ \delta\tilde{\phi_k^1}(\tau) \end{aligned}$$

($\delta\tilde{\phi_k^0}(\tau)$ is the solution (3.25), which solves the zeroth order equation (3.9))

Now we can put the equation (3.38) into (3.36), and use the definition prescribed in equation (3.37). Then, once we separate the terms zero and first order in λ, we have,

(From here, the overdot indicates the partial τ derivative and not the full τ derivative. The partial τ' derivative, wherever present, will be explicitly shown.)

Terms Zeroth Order in λ:

$$\mathcal{F}_0(\tau')\Big(\delta\ddot{\tilde{\phi_k^0}}(\tau)\ -\frac{\dot{M}}{M}\delta\dot{\tilde{\phi_k^0}}(\tau)\ +\frac{\rho_0 M}{m}k^2\delta\tilde{\phi_k^0}(\tau)\Big) = 0 \tag{3.39}$$

Terms First Order in λ:

$$\begin{aligned} -\mathcal{F}_1(\tau')\Big(\delta\ddot{\tilde{\phi_k^1}}(\tau)\ -\frac{\dot{M}}{M}\delta\dot{\tilde{\phi_k^1}}(\tau)\ +\frac{\rho_0 M}{m}k^2\delta\tilde{\phi_k^1}(\tau)\Big) & \\ =\Big(\mathcal{F}_0(\tau')\frac{2}{Z}\frac{\dot{a}}{a}\ +\mathcal{F}_0(\tau')\frac{\dot{M}}{ZM} + 2\frac{\partial\mathcal{F}_0(\tau')}{\partial\tau'}\ \Big)\delta\dot{\tilde{\phi_k^0}}(\tau) & \\ +\Big(\mathcal{F}_0(\tau')\frac{\rho_0 M}{mZ}k^2 - \frac{\partial\mathcal{F}_0(\tau')}{\partial\tau'}\frac{\dot{M}}{M}\Big)\delta\tilde{\phi_k^0}(\tau) & \end{aligned} \tag{3.40}$$

Since $\delta\tilde{\phi_k^0}(\tau)$ is the zeroth order solution (3.25), equation (3.39) is trivially solved. In order to solve equation (3.40), we have to ensure that co-efficient of secular terms on the right hand side are set to zero. This is needed because the secular term grows faster than the corresponding solution to the homogeneous equation. The secular term appears whenever the inhomogeneous term is itself a solution to the corresponding homogeneous equation. In this case, the associated homogeneous equation is

$$\delta\ddot{\tilde{\phi}}_k^1(\tau) \ - \frac{\dot{M}}{M}\delta\dot{\tilde{\phi}}_k^1(\tau) \ + \frac{\rho_0 M}{m}k^2\delta\tilde{\phi}_k^1(\tau) = 0$$

and $\delta\tilde{\phi}_k^0(\tau)$ would be its solution, so terms having a constant multiplied to $\delta\tilde{\phi}_k^0(\tau)$ would be the secular terms and thus their co-efficient must be set to zero. With this is mind, let us try to express $\delta\dot{\tilde{\phi}}_k^0(\tau)$ in terms of $\delta\tilde{\phi}_k^0(\tau)$ using the equations (3.18) and (3.25). If this can be done, it could contribute to more secular terms on the right hand side of equation (3.40).

$$\begin{aligned}
\delta\dot{\tilde{\phi}}_k^0(\tau) &= \frac{\partial(\delta\tilde{\phi}_k^0(\tau))}{\partial\tau} \\
&= \frac{\partial s_{new}}{\partial\tau}\ \frac{\partial(\delta\tilde{\phi}_k^0(s_{new}))}{\partial s_{new}} \\
&= -Hs_{new}\ \frac{\partial}{\partial s_{new}}\Big(s_{new}\sqrt{\frac{\pi\hbar VH}{4mc_0^2k^2}}[J_1(s_{new}) + iY_1(s_{new})]\Big) \\
&= -Hs_{new}\ \frac{\partial}{\partial s_{new}}\Big(s_{new}\sqrt{\frac{\pi\hbar VH}{4mc_0^2k^2}}\ \mathcal{H}_1^{(1)}(s_{new})\Big) \\
&\qquad (\mathcal{H}^{(1)}(s_{new}) \text{ is the Hankel function of the first kind}) \\
&= -Hs_{new}\sqrt{\frac{\pi\hbar VH}{4mc_0^2k^2}}\ (2\mathcal{H}_1^{(1)}(s_{new}) - s\mathcal{H}_2^{(1)}(s_{new})) \\
&= -2Hs_{new}\sqrt{\frac{\pi\hbar VH}{4mc_0^2k^2}}\ \mathcal{H}_1^{(1)}(s_{new}) + Hs^2\sqrt{\frac{\pi\hbar VH}{4mc_0^2k^2}}\ \mathcal{H}_2^{(1)}(s_{new})
\end{aligned}$$

$$\therefore\ \delta\dot{\tilde{\phi}}_k^0(\tau) = -2H\delta\tilde{\phi}_k^0(\tau) + D(\tau(s_{new})) \tag{3.41}$$

where,

$$D(\tau(s_{new})) = Hs_{new}^2\sqrt{\frac{\pi\hbar VH}{4mc_0^2k^2}}\ H_2^{(1)}(s_{new})$$

Now, substituting for $\delta\dot{\tilde{\phi}}_k^0(\tau)$ from the expression (3.41) into the equation (3.40), we have

$$\begin{aligned}
-\mathcal{F}_1(\tau')\Big(\delta\ddot{\tilde{\phi}}_k^1(\tau) - \frac{\dot{M}}{M}\delta\dot{\tilde{\phi}}_k^1(\tau) + \frac{\rho_0 M}{m}k^2\delta\tilde{\phi}_k^1(\tau)\Big) \\
= \Big(\mathcal{F}_0(\tau')\frac{2}{Z}\frac{\dot{a}}{a} + \mathcal{F}_0(\tau')\frac{\dot{M}}{ZM} + 2\frac{\partial\mathcal{F}_0(\tau')}{\partial\tau'}\Big)(-2H\delta\tilde{\phi}_k^0(\tau) + D(\tau)) \\
+ \Big(\mathcal{F}_0(\tau')\frac{\rho_0 M}{mZ}k^2 - \frac{\partial\mathcal{F}_0(\tau')}{\partial\tau'}\frac{\dot{M}}{M}\Big)\delta\tilde{\phi}_k^0(\tau)
\end{aligned} \tag{3.42}$$

Also, $\frac{\dot{M}}{M} = -2H$. Using this and re-arranging a few terms, we can write equation (3.42) as

$$\begin{aligned}
\mathcal{F}_1(\tau')\Big(\delta\ddot{\tilde{\phi}}_k^1(\tau) - \frac{\dot{M}}{M}\delta\dot{\tilde{\phi}}_k^1(\tau) + \frac{\rho_0 M}{m}k^2\delta\tilde{\phi}_k^1(\tau)\Big) \\
= \Big(2H\mathcal{F}_0(\tau')\frac{2}{Z}\frac{\dot{a}}{a} + 2H\mathcal{F}_0(\tau')\frac{\dot{M}}{ZM} + 2H\frac{\partial F_0(\tau')}{\partial\tau'} - \mathcal{F}_0(\tau')\frac{\rho_0 M}{mZ}k^2\Big)\delta\tilde{\phi}_k^0(\tau) \\
- \Big(\mathcal{F}_0(\tau')\frac{2}{Z}\frac{\dot{a}}{a} + \mathcal{F}_0(\tau')\frac{\dot{M}}{ZM} + 2\frac{\partial\mathcal{F}_0(\tau')}{\partial\tau'}\Big)D(\tau)
\end{aligned} \tag{3.43}$$

If we observe equation (3.43) carefully, we see that all the $\delta\tilde{\phi}_k^0(\tau)$ terms except $2H\frac{\partial\mathcal{F}_0(\tau')}{\partial\tau'}\delta\tilde{\phi}_k^0(\tau)$ have other factors which have τ dependence. So, $2H\frac{\partial\mathcal{F}_0(\tau')}{\partial\tau'}\delta\tilde{\phi}_k^0(\tau)$ is the only secular term and hence, let us set its co-efficient to zero.

$$2H\frac{\partial\mathcal{F}_0(\tau')}{\partial\tau'} = 0 \tag{3.44}$$

This tell us that $\mathcal{F}_0(\tau')$ must be a constant. Thus, the first order correction in $\delta\tilde{\phi}_k(\tau,\tau')$ due to amplitude is only by a constant. Hence, the resulting power spectrum ($\Delta^2(k) = k^2P(k)$) will also vary only by a constant and therefore still remains scale invariant. But in our work, we have not considered the first order correction due to $\delta\tilde{\phi}_k^1(\tau)$. Once this is considered, there is a good chance that it might introduce k dependence into the power spectrum, in which case the scale invariance will break down.

Chapter 4

Conclusion and Future Work

We introduced a Gaussian background density into the expanding BEC model, ignoring dipole-dipole interaction term and generated the phase fluctuation equation with this consideration. In the long wavelength limit, we dropped the terms having $\frac{\hbar^2 k^2}{4ma^2 M\rho_0}$ and proceeded to solve the phase fluctuation equation. The resulting power spectrum was found to be scale invariant. After this, we continued further by keeping the terms with $\frac{\hbar^2 k^2}{4ma^2 M\rho_0}$ in the phase fluctuation equation and solved the equation perturbatively using the multiple scale approach. The first order correction in solution due to the amplitude term, $\mathcal{F}_0(\tau')$, gave only a constant and hence the resulting the power spectrum continued to be scale invariant.

However in this work, the first order correction in the solution due to the $\delta\tilde{\phi}_k^1(\tau)$ term has not been considered. If this is considered, the resulting power spectrum might end up having a k dependence and this will result in the scale invariance of the power spectrum breaking down. If this is true, then we can show that the scale invariance of inflationary power spectrum breaks down, even in the absence of strong dipole-dipole interaction, once the co-ordinate dependence of the background state is taken.

Bibliography

[1] Seok-Yeong Chä and Uwe R. Fischer. "Probing the Scale Invariance of the Inflationary Power Spectrum in Expanding Quasi-Two-Dimensional Dipolar Condensates". In: *Phys. Rev. Lett.* 118 (13 Mar. 2017), p. 130404. DOI: 10.1103/PhysRevLett.118.130404. URL: https://link.aps.org/doi/10.1103/PhysRevLett.118.130404.

[2] Alan H. Guth. "Inflationary universe: A possible solution to the horizon and flatness problems". In: *Phys. Rev. D* 23 (2 Jan. 1981), pp. 347–356. DOI: 10.1103/PhysRevD.23.347. URL: https://link.aps.org/doi/10.1103/PhysRevD.23.347.

[3] Robert H Brandenberger and Jérôme Martin. "Trans-Planckian issues for inflationary cosmology". In: *Classical and Quantum Gravity* 30.11 (Apr. 2013), p. 113001. DOI: 10.1088/0264-9381/30/11/113001. URL: https://doi.org/10.1088%2F0264-9381%2F30%2F11%2F113001.

[4] John C. Baez. "Higher dimensional algebra and Planck scale physics". In: (1999), pp. 177–195. arXiv: gr-qc/9902017 [gr-qc].

[5] Piyush Jain et al. "Analogue model of a FRW universe in Bose-Einstein condensates: Application of the classical field method". In: *Phys. Rev.* A76 (2007), p. 033616. DOI: 10.1103/PhysRevA.76.033616. arXiv: 0705.2077 [cond-mat.other].

[6] L.P. Pitaevskii and S. Oxford University Press Stringari. *Bose-Einstein Condensation.* Comparative Pathobiology - Studies in the Postmodern Theory of Education. Clarendon Press, 2003. ISBN: 9780198507192. URL: https://books.google.co.in/books?id=rIobbOxC4j4C.

[7] Yu. Kagan, E. L. Surkov, and G. V. Shlyapnikov. "Evolution and Global Collapse of Trapped Bose Condensates under Variations of the Scattering Length". In: *Phys. Rev. Lett.* 79 (14 Oct. 1997), pp. 2604–2607. DOI: 10.1103/PhysRevLett.79.2604. URL: https://link.aps.org/doi/10.1103/PhysRevLett.79.2604.

[8] Yu. Kagan, E. L. Surkov, and G. V. Shlyapnikov. "Evolution of a Bose-condensed gas under variations of the confining potential". In: *Phys. Rev. A* 54 (3 Sept. 1996), R1753–R1756. DOI: 10.1103/PhysRevA.54.R1753. URL: https://link.aps.org/doi/10.1103/PhysRevA.54.R1753.

[9] Milton Abramowitz. *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables,* New York, NY, USA: Dover Publications, Inc., 1974. ISBN: 0486612724.

[10] G B Arfken, H J Weber, and Frank Harris. *Mathematical Methods for Physicists: A Comprehensive Guide*. Jan. 2012. ISBN: 9780123846549.

[11] Alexander Fetter and Christopher J. Foot. "Bose gas: Theory and Experiment". In: *Contemporary Concepts of Condensed Matter Science* 5 (Mar. 2012). DOI: 10.1016/B978-0-444-53857-4.00002-7.

[12] C. J. Pethick and H. Smith. *Bose–Einstein Condensation in Dilute Gases*. 2nd ed. Cambridge University Press, 2008. DOI: 10.1017/CBO9780511802850.

[13] Carl Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory.* Vol. 1. Jan. 1999. DOI: 10.1007/978-1-4757-3069-2.