Role Colouring Hereditary Graph Classes

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Sukanya Pandey

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

June, 2019

Supervisor: Prof. Venkatesh Raman

Certificate

This is to certify that this dissertation entitled Role Colouring Hereditary Graph Classes towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sukanya Pandey at the Institute of Mathematical Sciences, HBNI, Chennai under the supervision of Prof. Venkatesh Raman , Professor , Department of Computer Science , during the academic year 2018-2019.

Prof. Venkatesh Raman

Committee:

Prof. Venkatesh Raman

Dr. Soumen Maity

To Room 911,

You will be missed.

Declaration

I hereby declare that the matter embodied in the report entitled Role Colouring Hereditary Graph Classes are the results of the work carried out by me at the Department of Computer Science,, Institute of Mathematical Sciences, HBNI, Chennai, under the supervision of Prof. Venkatesh Raman and the same has not been submitted elsewhere for any other degree.

Sukanya Pandey

Sukanya Pandey

Acknowledgments

This thesis marks the completion of a long journey and along the way, there are several people I came across, who I wish to thank.

Foremost, I would like to express my heartfelt gratitude to my supervisor, Professor Venkatesh Raman, for patiently walking along with me at my pace. He let me have sufficient room for making mistakes and enough time to understand the things that I learned. It is through his encouragement, that I now confidently take a step forward towards becoming a researcher. In him, I have found a teacher who will be an inspiration for life.

Next, I would like to thank Dr Soumen Maity, who introduced me to the fascinating field of Parametrized complexity. He endured many a months of discussions and presentations that would go on for hours, and facilitated us to learn something new and exciting.

Third in this list are all the friendly people at the Institute of Mathematical Sciences, Chennai. I thank each and every one of them for making my stay here memorable. I specially would like to thank Dr Vibha Sahlot, for being available to help me whenever I got stuck with an unfamiliar or a difficult concept.

I am grateful to all my teachers in IISER Pune, who made me realize how beautiful mathematics is! To Dr Kaneenika Sinha, I shall be eternally grateful for being a role model as a brilliant mathematician, a supportive teacher and a wonderful human being.

Last but by no means the least, I thank my family and friends for being my anchor and keeping me grounded.

Abstract

We study a variant of graph colouring called Role colouring. Role colouring is an assignment of colours to the vertices of a graph, such that if two distinct vertices of the graph get the same colour, then their neighbourhoods are assigned the same set of colours. In the k-ROLE COLOURING problem, we are given a simple, undirected graph G and an integer k. We are required to determine if G can be role coloured in exactly k colours. k-ROLE COLOURING has been shown to be NP-Complete on arbitrary graphs, for all $k \geq 2$.

We explore the complexity of k-ROLE COLOURING on hereditary classes of graphs. In particular, we determine the complexity of the problem on the class of bipartite graphs and prove that it is NP-Complete, when $k \geq 3$. Furthermore, we give a polynomial time algorithm to 3-role colour bipartite chain graphs.

2-ROLE COLOURING is trivially solvable in polynomial time on bipartite graphs and split graphs. Let G be a graph that contains a set S of d vertices (or edges), d being a constant, such that $G \setminus S$ is either a bipartite or a split graph. We say that the graph G is d-away from a bipartite or a split graph. We show that even for a small constant d, 2-ROLE COLOURING is NP-Hard on G. In the realm of parametrized complexity, we say that on graphs that are d-away from bipartite or split graphs, 2-ROLE COLOURING is para-NP-Hard, when parametrized by d.

Contents

Chapter 1

Introduction

1.1 Context and Motivation

Graphs have long been used in social sciences, to model social networks. Typically, each node of a graph represents an individual and the edges represent the relationship between them. Interesting mathematical questions arise regarding the structure of such networks- the most natural question being the definition of social roles.

It is axiomatic that "Individuals play the same roles if they relate in the same way to individuals playing counterpart roles." White and Reitz [20] formalized this notion in terms of graph homomorphisms and vertex partitions, and called it *regular equivalence*. In 1991, Borgatti and Everett [13] used the idea of vertex colouring in graphs, to give an alternate definition of the concept. They called it *role colouring*.

Role colouring is an assignment of colours to the vertices of a graph such that if two distinct vertices are assigned the same colour then their neighbourhood get the same set of colours. Roberts and Sheng [16] proved that deciding if an arbitrary graph G has a 2-role colouring is NP-Complete. Later Fiala and Paulusma [7] showed that k-ROLE COLOURING, $k \geq 3$, is NP-Complete.

The following dichotomy exists for the problem of k-role colouring a graph: Given a graph G, deciding if it has a k-role colouring is polynomial time solvable when k= 1 or $|V(G)|$, and NP-Complete for $2 \leq k < |V(G)|$. Hence, it is interesting to look for smaller classes of graphs on which the problem may be tractable.

Purcell and Rombach [15] claimed that $2K_2$-free graphs are always 2-role colourable. In fact, they claimed to have proved the following Lemma:

Lemma 1. *If G is a connected $2K_2$-free graph which contains a maximal independent set I of size at least 2, then the induced subgraph on $G \setminus I$ is connected.*

We discovered a mistake in the proof of this lemma. In fact we found an example, namely, a complete bipartite graph, which shows that the Lemma is not true. If G is a complete bipartite graph, then it is $2K_2$-free. The maximal independent sets in G comprise each of the two partitions. However, $G \setminus I$ has no edges and is disconnected.

On split graphs, which are $(2K_2, C_4, C_5)$-free, k-ROLE COLOURING is NP-Complete for $k \geq 4$. Hence it is NP-Complete on the superclass of split graphs, that is $2K_2$-free graphs. However, the complexity of the problem is unresolved when $k = 2$ and $k = 3$.

While investigating these open problems, we showed that $2K_2$-free bipartite graphs (bipartite chain graphs) are always 3-role colourable. This led us to study the complexity of 3-ROLE COLOURING on another superclass of such graphs, that is, the class of bipartite graphs.

Thereafter, we proved that given a bipartite graph, k-ROLE COLOURING is NP-Complete when $k \geq 3$. Moreover, we showed that if a graph has a set of d vertices (or edges) that on deletion yields a bipartite graph, d being a constant, 2-ROLE COLOURING becomes NP-Complete on such a graph even for a small constant d.

This thesis comprises a survey of known results about the complexity of k-ROLE COLOURING on some hereditary classes of graphs, as well as, detailed proofs of our results on bipartite graphs, chordal graphs, bipartite chain graphs and graphs that are a constant number vertices (or edges) away from bipartite and split graphs.

1.2 Organization

In Chapter 2, we include definitions of important concepts used throughout the thesis. These concepts range over a variety of topics like Graph theory, Hyper-graph theory and Complexity theory.

Chapter 3 is a primer on role colouring with formal definitions and properties.

In Chapter 4, we survey the known results on complexity of the k-role colouring problem on classes of graphs like Trees, Cographs, Chordal graphs, Proper Interval graphs and Split graphs.

Chapter 5 discusses in detail the proof of NP-Completeness of the k-role colouring problem on bipartite graphs. The proof is divided into four parts. We separately prove the claim for the cases $k = 3, k = 4, k = 5$ and $k > 5$.

We slightly modify our proofs in Chapter 5, to give an alternate proof of NP-Completeness of the k-role colouring problem, $k \geq 4$ on Chordal graphs. We discuss this proof in Chapter 6.

Finally, in Chapter 7 we study graphs that are a constant number of vertices (or edges) away from bipartite graphs and split graphs, respectively. We show that 2-role colouring which is easy on both bipartite and split graphs becomes NP-Complete on such graphs.

Chapter 2

Preliminaries

2.1 Graph theoretic terminology

All the definitions in this section have been taken from [4].

Definition 2.1.1. *An undirected simple graph is a pair, $G = (V, E)$, where $V \cap E = \phi$ and $E \subseteq V^2$.*

All graphs considered in this paper are simple and undirected, unless specified otherwise. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. An edge between vertices u and v is denoted as uv or vu or (u, v).

Definition 2.1.2. *Two vertices u and v are* adjacent *or* neighbours *if uv is an edge of G.*

For a vertex $v \in V(G)$, its *neighborhood* $N_G(v)$ is the set of all vertices adjacent to it and its *closed neighborhood* $N_G[v]$ is the set $N_G(v) \cup \{v\}$. This notation is extended to subsets of vertices as $N_G[S] = \bigcup_{v \in S} N_G[v]$ and $N_G(S) = N_G[S] \setminus S$ where $S \subseteq V(G)$. An *independent set* is a set of pairwise non-adjacent vertices and a *clique* is a set of pairwise adjacent vertices. A complete graph on q vertices is denoted by K_q.

Definition 2.1.3. *The* degree *of a vertex $v \in V(G)$, denoted by $deg_G(v)$, is the size of $N_G(v)$.*

A *pendant vertex* is a vertex $v \in V(G)$ with $deg_G(v) = 1$.

Definition 2.1.4. *A* bipartite graph $G(U, V, E)$ *is a graph whose vertex set can be partitioned into two independent sets* U *and* V*.*

Definition 2.1.5. *A* split graph $G(C, I, E)$ *is a graph whose vertex set can be partitioned an induced clique* C *and an induced independent set* I*.*

Definition 2.1.6. *A* path *in a simple graph is a sequence of distinct vertices with an edge between every pair of consecutive vertices.*

The *length of a path* is the number of vertices in a path. We denote a path on n vertices by P_n. If $P = x_0, ..., x_{k-1}$ is a path and $k \geq 3$, then the graph $C = P + x_{k-1}x_0$ is called a *cycle*. A graph is *connected* if there exists a path between any pair of vertices in the graph.

Definition 2.1.7. *A graph that does not contain any cycle is called a* forest*. A connected forest is called a* tree*.*

Definition 2.1.8. *A class of graphs that is closed under isomorphism is called a* graph property*.*

Definition 2.1.9. *A graph* G' *is a* subgraph *of* G *if* $V(G') \subseteq V(G)$ *and* $E(G') \subseteq E(G)$*. We say that* G *contains* G'*, or* $G' \subseteq G$*. If* $G' \neq G$ *then,* G' *is a* proper subgraph *of* G*.*

Definition 2.1.10. *An* induced subgraph *of* G *is a subgraph* G' *such that* $E(G') = \{(x, y)|\forall x, y \in V(G')\} \cap E(G)$*. Hence, if* $U \subseteq V(G)$ *then the induced graph on* U*, denoted by* $G[U]$*, contains all those edges of* G *that have both their end points in* U*.*

Definition 2.1.11. *A* hereditary class *of graphs is a collection of graphs that is closed under vertex deletion. Therefore, if a graph* G *belongs to a hereditary class, all its induced subgraphs also belong to it.*

For further details on graphs, refer [4].

2.2 Colouring of graphs

Definition 2.2.1. *A* vertex colouring *of a graph* G*, is a map* $f : V(G) \longrightarrow C$*, where* C *is the set of available colours, such that* $f(u) \neq f(v)$ *if* $uv \in E(G)$*.*

Definition 2.2.2. *A homomorphism between two graphs G and H is a map $\phi : V(G) \longrightarrow V(H)$ such that if $uv \in E(G)$, then $\phi(u)\phi(v) \in E(H)$.*

Let f be a map from set A to set B, and let $A' \subseteq A$. We denote by $f(A')$ the set $\{f(a)|\forall a \in A'\}$.

2.3 Hyper-graphs

Definition 2.3.1. *A* hyper-graph *is a pair $H = (Q, S)$ where Q is a finite set of vertices and S is a collection of non-empty subsets of Q called hyper-edges.*

The canonical *incidence graph* of a a hyper-graph $H = (Q, S)$ is a bipartite graph $G(Q, S, E)$ with Q and S as the two parts of the bipartition, and for all $q \in Q$ and $s \in S$, $(q, s) \in E(G)$ if and only if q belongs to the hyper-edge in H corresponding to the vertex $s \in S$.

In Hyper-graph 2-coloring, we are given a hyper-graph $H = (Q, S)$. The goal is to determine whether the vertices in Q of H can be coloured with two colors such that in each hyper-edge, there is a vertex of either colour. Hyper-graph 2-coloring is known to be NP-Complete, [9].

2.4 Computational Complexity

For a background in computational complexity, refer [2]. Here, we give a few important definitions.

Definition 2.4.1. *An algorithm $\mathcal{A}$* accepts *a string $x \in \Sigma^*$, if given input x, the algorithm's output, $\mathcal{A}(x) = 1$.*

Definition 2.4.2. *A language L is* accepted *by an algorithm $\mathcal{A}$ if for all $x \in L$, $\mathcal{A}(x) = 1$. A language L is* decided *by $\mathcal{A}$, if for all $x \in L$, $\mathcal{A}(x) = 1$ and for all $y \notin L$, $\mathcal{A}(y) = 0$.*

Definition 2.4.3. *The complexity class* P *is the set of all languages* L*, such that there exists an algorithm* $\mathcal{A}$ *that decides* L *in time bounded by* $\mathcal{O}(n^c)$*, where* n *is the length of the input string and* c *is a constant.*

Definition 2.4.4. *A* verification algorithm $\mathcal{A}$*, is a two-argument algorithm, where the first argument is the input string* x *and the second argument is a certificate* y*. A language* L *is verified by* $\mathcal{A}$*, if for any input string* $x \in L$*, there exists a binary string* y*, such that* $\mathcal{A}(x, y) = 1$.

Definition 2.4.5. *A language* L *belongs to the class* NP *if for all* $x \in L$*, there exists a certificate* y*, where* $|y| = \mathcal{O}(|x|^c)$*, such that* $\mathcal{A}(x, y) = 1$*. We say that the language* L *is verified in polynomial time.*

Definition 2.4.6. *A language* L_1 *is* polynomial-time reducible *to a language* L_2*, if there exists a polynomial-time computable function* $f : \{0,1\}^* \longrightarrow \{0,1\}^*$*, such that for all* $x \in \{0,1\}^*$*,* $x \in L_1$ *if and only if* $f(x) \in L_2$*. The polynomial-time algorithm* F *that computes the function* f *is called a* reduction algorithm*.*

Definition 2.4.7. *A language* $L \subseteq \{0,1\}^*$ *is* NP*-Hard if for all* $L' \in \mathsf{NP}$*,* L' *is polynomial-time reducible to* L*. In addition, if* $L \in \mathsf{NP}$*, then* L *is NP-Complete.*

Chapter 3

Role colouring

3.1 Introduction

Let G=(V, E) be an undirected graph without multiple edges or loops. Role colouring is an assignment f of colours, say, $\{1,2...k\}$ to the vertices of G such that

$$\forall u, v \in V, f(u) = f(v) \implies f(N(u)) = f(N(v))$$

It is to be noted that if G admits a k-role colouring, then vertices of the same colour need not form an independent set in G. In that, role colouring is different from proper colouring of graphs, where any two adjacent vertices must receive different colours.

From the algorithmic viewpoint, the notion of role colouring poses an interesting question, that is, given a graph G and an integer k, does there exist a role colouring such that vertices of G can be role coloured in exactly k colours? Determining the minimum or maximum number k to k-role colour an arbitrary graph G is trivial. We can colour all the vertices in just one colour, so the minimum number of colours is needed is 1. Similarly, we can colour all the n vertices with n different colours and the maximum number of colours to role colour G is $|V| = n$.

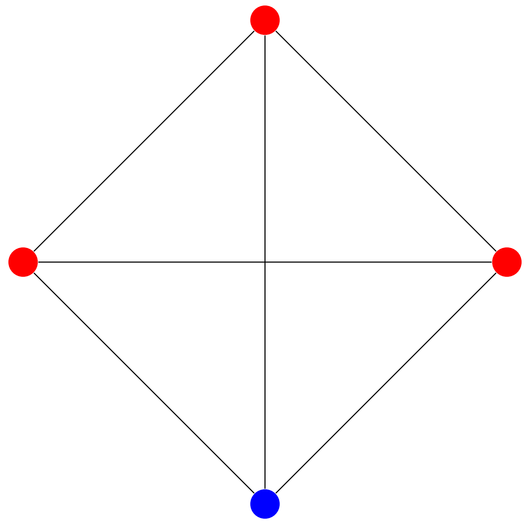

Figure 3.1: G

Figure 3.2: R_1

3.2 Role Graph and its properties

Given a role colouring assignment $f : V(G) \longrightarrow \{1, 2, ..., k\}$ the role graph R of G, corresponding to f is a graph such that $V(R) = \{1, 2..., k\}$ and $E(R) = \{(i, j) | \exists (u, v) \in E(G); f(u) = i, f(v) = j\}$

For example, consider the graph G below:

G has a valid 2-role colouring, as the blue vertex is adjacent to the red vertices and the red vertices are all adjacent to atleast one red and one blue vertex. For this role assignment to the vertices of G, the role graph R_1 is:

3.2.1 Properties of the role graph R

Notice that $deg_G(u)= |N_G(u)| \geq |f(N_G(u))| = |N_R(f(u))|= deg_R(f(u))$, where f is the role assignment. It is easily seen that $\delta(R) \leq \delta(G)$ and $\Delta(R) \leq \Delta(G)$.

Lemma 2. *([7]) If G is a connected graph with role assignment f, then the corresponding role graph R is also connected.*

Proof. Suppose that R is not connected. Then there must be a pair of vertices $i, j \in V(R)$ such that there does not exist a path between i and j. Assume that there exist vertices $u, v \in V(G)$ such that $f(u) = i$ and $f(v) = j$. Since G is connected, we know that there exists a path between u and v. Let $\{u, x_1, x_2, ..., x_k, v\}$ be a path in G. Then, there must exist an edge between $f(u)$ and $f(x_1)$ in R. Similarly, there must exist an edge between $f(x_n)$ and $f(x_{n+1})$ in R $\forall$ $1 \leq n \leq k$, as well as an edge between $f(x_k)$and $f(v) = j$. This gives us a path in R, namely, $\{i, f(x_1), f(x_2), ..f(x_k), j\}$. This contradicts the assumption that there is no path between i and j. Since, i, j are arbitrary vertices in R , we can conclude that every pair of vertices in R is connected by a path. Therefore, R is connected. □

Lemma 3. *([14]) Suppose that G is a graph isomorphic to P_n, that is, a path on n vertices with $n-1$ edges. If G is k-role colourable with the role assignment f, then the corresponding role graph R is isomorphic to P_k with at most one self loop.*

Proof. To show that the role graph R is a path, we will first show that no vertex in R has degree greater than 2. This would imply that $R \in \{C_k, P_k, P_k^*, P_k^{**}\}$ where P_k^*, P_k^{**} are paths on k vertices with a self loop on one leaf and self loops on each of the two leaves, respectively. We will then show that R has at least one leaf, which will prove the lemma.

Suppose that there exist vertices u in G and j in R, such that $f(u) = j$. Let $deg_R(j) \geq 3$. As $deg_R(j) \leq deg_G(u)$, we have that $deg_G(u) \geq 3$. This contradicts the fact that G is a path. Therefore, $\Delta(R) \leq 2$. Hence, $R \in \{C_k, P_k, P_k^*, P_k^{**}\}$. Now, suppose that R has no leaves. Then the degree of all the vertices in R is equal to 2. However, G has exactly 2 vertices of degree equal to 1, as G is a path. So, the colour assigned to the leaves of G must be of degree equal to 1. Hence, $R \in \{P_k, P_k^*\}$. This completes the proof. □

Lemma 4. *([5]) The distance between two vertices in the graph G is at least the distance between their corresponding colours in the role graph R.*

Proof. Let $i, j \in R$. Also, let the distance between i, j, $d(i, j) = k$. So, there exists a shortest path in R, $\{i, y_1, y_2, ..., y_{k-2}, j\}$. Let $u, v \in G$ such that $f(u) = i$ and $f(v) = j$. Then u must have a neighbour of colour y_1 and v must have a neighbour of colour y_{k-2}. We call these neighbours x_1 and x_{k-2}, respectively. By similar reasoning, x_1 must have at least one neighbour, say x_2 of colour y_2. Suppose for some n, such that $2 \leq n \leq k-3$, x_{n-1} has a neighbour x_n of colour y_n. Then, as y_n has a neighbour y_{n+1}, x_n must have a neighbour, x_{n+1} of colour y_{n+1}. Hence, we have proved by induction that there exists a path in G, $\{u, x_1, x_2, ...x_{k-2}, v\}$. We now need to show that does not exist a shorter path between u and v in G. Suppose, $\{u, z_1, z_2, ..., z_{k-3}, v\}$ is a path in G. Starting with vertex u, its neighbour should clearly have the colour y_1, as the shortest path in R between i, j goes through y_1. Following the same argument we have that each z_n, $1 \leq n \leq k-4$ has a neighbour z_{n+1} of colour y_{n+1}. So, $f(z_{k-3}) = y_{k-3}$. Now, since z_{k-3} is adjacent to v, we have that y_{k-3} is adjacent to j. However, this is a contradiction. So, the minimum length of a path between u and v must be k. □

Lemma 5. *([14]) The role graph of a k-role colourable tree T, is a tree T_R on k vertices with at most one loop.*

Proof. By Lemma 2, we know that the role graph T_R of T, is connected. Also, by the property that $deg_R(f(v)) \leq deg_G(v)$ $\forall$ $v \in V(G)$, we know that T_R must have at least one leaf. (It has exactly one leaf when all the leaves of T are monochromatic). Therefore, T_R cannot be isomorphic to C_k. Suppose T_R has a cycle of length $n < k$, say , $\{c_1, c_2, ..., c_n\}$. Then, there must be a path in T, say, P= $\{x_1, x_2, ...x_j\}$ such that its vertices have the colour scheme $c_1, c_2..., c_n, c_1, ...$ Since, the degree of each c_i, $1 \leq i \leq n$, is equal to 2, the degree of each vertex of P is at least 2. Suppose $f(x_j) = c_i$, then x_j must have at least one neighbour of colour c_{i-1} and one of colour c_{i+1}. Since, T has no cycle, it would mean that there exists an infinite sequence of vertices in T having the aforesaid colouring scheme. This is a contradiction as T has a finite number of vertices. □

Lemma 6. *([5]) Let G be a connected graph and f be a k-role assignment of G. Then, for every $v \in V(G)$ there is a set $S \subseteq V(G)$ containing v, such that $|S|$=k, $f(u) \neq f(u')$ $\forall$ u, u' $\in S$ and $G[S]$ is connected.*

Proof. To construct S, pick a vertex $v \in V(G)$, add it to S and mark it. While there are vertices unmarked in $N(v)$ $\forall u \in N(v)$, if u is unmarked and $f(u) \in N(f(v))$ such that S does not contain a vertex with role $f(u)$, add u to S.

We need to show that S is the desired set. $G[S]$ is clearly connected, as at every stage we added a vertex which was adjacent to the vertex previously added. Also $|S| \leq k$ because no colour is repeated in S. For a contradiction,assume that $|S| < k$. By Lemma 1.1, we know that the role graph of G, say R is connected. There exists a pair of vertices x, x' in $V(R)$ such that there exists $w \in V(G)$ such that $f(w) = x$ belongs to S but there does not belong any vertex of G in S with role x'. But if we consider the iteration of the above algorithm when w was added to S, in the next iteration, we would come across a vertex of role x' unmarked by S and hence it would be added to S. This is a contradiction to the fact that $|S| < k$. □

3.3 The k-Role colouring problem

k-Role colouring is the problem where given an undirected and simple graph G and an integer k, we want to decide if there exists a colouring assignment $\alpha : V(G) \longrightarrow \{1, 2, ..., k\}$, satisfying the property $\alpha(u) = \alpha(v)$ implies $\alpha(N(u)) = \alpha(N(v))$ for all u and v in $V(G)$.

Roberts and Sheng ([16]) proved that 2-Role colouring was NP-Complete on arbitrary graphs. Later, Paulusma et al. ([7]) showed that on arbitrary graphs, k-Role colouring is NP-Complete when $k \geq 3$. Hence, this problem is NP-Complete on the class of all graphs, for all $k \geq 2$.

Hence, it is interesting to determine how hard it is on smaller graph classes. The only classes of graphs where k-Role colouring is known to be polynomial time solvable, for all k, are trees, cographs and proper interval graphs ([1], [14], [12]).

3.4 The R-Role colouring problem

Another variant of the role colouring problem is R-Role colouring. In this problem, given graphs G and R, we want to decide if G has a role colouring assignment $\alpha : V(G) \longrightarrow V(R)$, such that for all u in $V(G)$, $\alpha(N_G(u)) = N_R(\alpha(u))$.

This problem is equivalent to deciding if there exists a *locally surjective* homomorphism between G and R. This problem has been proved to be NP-Complete when R is a connected graph on at least 3 vertices ([7]).

The following theorem exists due to Paulusma et al.

Theorem 3.4.1. *([7]) R-Role colouring is in P if and only if:*

- *R has no edges, or*
- *one of the components of R is a single loop incident vertex, or*
- *R is bipartite and has at least one component isomorphic to a K_2.*

In every other case, it is NP-Complete.

Chapter 4

Complexity of k-Role colouring on some graph classes

In the previous chapter we saw that k-role colouring is NP-Complete on the class of all graphs, when $k \geq 2$. However, when restricted to some smaller graph classes the problem may be tractable. Therefore, an interesting goal is to characterize the graph classes on which the problem may be solvable in polynomial time.

In this chapter, we discuss some known results on k-role colouring restricted to specific hereditary classes of graphs.

4.1 Trees

Definition 4.1.1. *([3]) A* tree decomposition *of a graph G is a pair $T = (T; \{X\}_{t \in V(T)})$, where T is a tree whose every node t is assigned a vertex subset $X_t \subseteq V(G)$, called a bag, such that the following three conditions hold:*

1 $\bigcup_{t \in V(T)} X_t = V(G)$. In other words, every vertex of G is in at least one bag.

2 For every $uv \in E(G)$, there exists a node t of T such that bag X_t contains both u and v.

3 For every $u \in V(G)$, the set $T_u = \{t \in V(T) : u \in X_t\}$, i.e., the set of nodes whose corresponding bags contain u, induces a connected subtree of T.

The *width* of tree decomposition $T = (T; \{X_t\}_{t \in V(T)})$ equals $max_{t \in V(T)} |X_t| - 1$, that is, the maximum size of its bag minus 1. The *treewidth* of a graph G, denoted by $tw(G)$, is the minimum possible width of a tree decomposition of G.

Chaplick et al. [1] showed that on graphs of bounded treewidth, R-ROLE COLOURING is polynomial time solvable if either the graph G or R has bounded degree. As a corollary to this theorem, we know that the problem is polynomial time solvable on trees, as trees have a treewidth equal to 1. k-ROLE COLOURING can be solved on trees in polynomial time if either k or $n - k$ is a constant [14].

4.2 Cographs

Definition 4.2.1. *A cograph is a simple graph that satisfies the following criteria:*

- *K_1 is a cograph.*
- *If G is a cograph, then so is $\bar{G}$*
- *If X and Y are cographs, then so is $X \cup Y$*

Cographs are the smallest family of graphs that include K_1. A family of cographs is also characterised as P_4-free class of graphs [17].

Theorem 4.2.1. *[14] Any cograph G with $|V(G)| \geq k$ is k-role colourable and k-ROLE COLOURING is in P, for $k \geq 2$.*

4.3 Chordal Graphs

Definition 4.3.1. *A chordal graph is a simple graph in which every cycle C_m, of length $m \geq 4$, has a chord, that is, an edge between non-consecutive vertices of the cycle.*

Chordal graphs are also called *triangulated graphs*.

A *simplical vertex* is one whose neighbourhood induces a clique. In other words, if $v \in V(G)$ is a simplical vertex, then $G[N(v)]$ is a complete graph. Dirac [1961], and Lekkerkerker and Boland [1962] proved that every chordal graph has a simplical vertex. In fact, if the chordal graph is not a complete graph, it has two simplical vertices [10]. In 1965, Fulkerson and Gross exploited this hereditary property of chordal graphs, to give a recognition algorithm for such graphs. For further details on the algorithm, refer [10].

An ordering $\sigma = [v_1, ..., v_n]$ on the vertices of an undirected graph G, is called a *perfect elimination scheme* if for all i, the graph induced on $X_i = \{v_j \in N(v_i) | j > i\}$ is a complete graph. Any simplical vertex can start a perfect elimination scheme.

Using the hereditary property that the vertices of chordal graphs can be ordered in a perfect elimination scheme, Sheng [18] gave an elegant greedy algorithm to 2-role colour chordal graphs.

Theorem 4.3.1. *[18] If G is a connected, non-bipartite chordal graph with at most one pendant vertex and no isolated vertices, then G is 2-role colorable if and only if G is R_1-role colourable, R_1 being isomorphic to the graph in Figure 3.2.*

Later, Paulusma et al.[19] resolved the complexity of 2-ROLE COLOURING on chordal graphs and proved the following:

Theorem 4.3.2. *[19]* 2-ROLE COLOURING *can be solved in linear time on the class of chordal graphs.*

Theorem 4.3.3. *[19] For $k \geq 3$, k-*ROLE COLOURING *is NP-Complete on chordal graphs.*

4.4 Proper Interval graphs

A graph is an *interval graph* if the intervals of a real line can be mapped onto its vertices and two vertices are adjacent if and only if the corresponding intervals overlap. Interval graphs are a subclass of chordal graphs. Several equivalent characterizations of interval graphs exist [10] due to Lekkerkerker and Boland [1962], Gilmore and Hoffman [1964], and Fulkerson and Gross [1965]. We shall consider the following definition:

Figure 4.1: R_2

Definition 4.4.1. *A chordal graph is an interval graph if and only if it does not contain any asteroidal triples.*

Asteroidal triples are any three vertices of a graph that *cannot* be ordered in a way such that a path from the first to the third vertex passes through a neighbour of the second vertex.

Definition 4.4.2. *A proper interval graph is an interval graph if and only if none of its induced subgraphs is isomorphic to a claw, that is, a* $K_{1,3}$.

Theorem 4.4.1. *[12] Given a proper interval graph* G *with* n *vertices and* m *edges and a graph* R *with* c_R *connected components,* R-ROLE COLOURING *can be solved on* G *in time* $\mathcal{O}(c_R.(n+m))$.

4.5 Split Graphs

Definition 4.5.1. *Let* $G = (V, E)$ *be a simple graph.* G *is a* split graph *if its vertex set* V *can be partitioned into two disjoint sets* I *and* C *such that the graph induced on* I *is an independent set in* G *and the one induced on* C *is a clique in* G.

2-ROLE COLOURING can be trivially solved on split graphs by assigning one colour to the set I and the other to the set C. However, R_2-ROLE COLOURING, where R_2 is isomorphic to the graph in Figure 4.1, is NP-Complete on such graphs.

It was shown by M.C Dourado [5] that 3-ROLE COLOURING is polynomial time solvable on split graphs, whereas k-ROLE COLOURING is NP-Complete, for $k \geq 4$. The author also gave a characterization of split graphs that admitted a 3-role colouring.

Definition 4.5.2. *[5] A good vertex of a split graph is a vertex* $v \in C$, $N(v) \subseteq (C \setminus twins(v))$ *and every vertex of* $(C \setminus twins(v))$ *has a neighbour in* $I \setminus N_I(v)$.

Theorem 4.5.1. *[5] A split graph G has a 3-role colouring if and only if at least one of these conditions holds:*

- *G is disconnected*
- *G has a good vertex*
- *G has no pendant vertices*
- *G admits a unique split partition (C, I), and $|N_I(v)| \geq 2$, for all $v \in C$.*

Corollary 4.5.2. *[5]* 3-role colouring *can be solved in linear time on split graphs.*

Chapter 5

Role colouring bipartite graphs

We restrict our input graph G to the class of bipartite graphs. R-Role colouring is known to be NP-Complete on bipartite graphs when

- $|E_R| \geq 2$ and
- R does not have a component isomorphic to a single loop-incident vertex, and
- R is not bipartite with at least one component isomorphic to K_2 [7].

We build on this result and focus on the more general problem of k-Role colouring bipartite graphs. Our primary result is that k-Role colouring is NP-Complete on the class of bipartite graphs for all $k \geq 3$.

Note that the NP-completeness of k-Role colouring does not follow from the NP-completeness of R-Role colouring where R is a graph on k vertices satisfying the characterization in [7]. To exemplify this, let us consider the two variants of the problem with respect to the class of split graphs. We have seen in Section 4.5 that R_2-Role colouring is NP-Complete on split graphs. However, 2-Role colouring is trivially solvable in polynomial time on split graphs with the corresponding role graph isomorphic to R_1.

5.1 NP-Completeness of k-Role colouring, $k \geq 3$, on bipartite graphs

In this section we prove that k-Role colouring is NP-Complete on connected bipartite graphs, when $k \geq 3$. We show a reduction from Hyper-graph 2-colouring (which is known to be NP-Complete [9]).

In that endeavour, we first address the problem of R-Role colouring on bipartite graphs with pendants in only one partition, where $R = P_3$, an induced path on three vertices.

Lemma 7. *Let G be a bipartite graph where exactly one partition of the vertex set has pendant vertices. Then P_3-Role colouring is NP-Complete on such graphs.*

Proof. We show a reduction from Hyper-graph 2-colouring. Let $H = (Q, S)$ be an instance of Hyper-graph 2-colouring. We assume that H has no hyper-edges with less than two vertices, else H would trivially be a *n*o instance of the problem.

Let $G(Q, S, E)$ be the canonical incidence graph of the hyper-graph H, with bipartition (Q, S). Then for all $q \in Q$ and $s \in S$, $(q, s) \in E(G)$ if and only if q belongs to the hyper-edge in H corresponding to vertex $s \in S$. Since every hyper-edge in S contains at least two vertices, no vertex in S can be a pendant vertex. We construct the instance G' from G by adding an edge (a, b), and joining it to some $q \in Q$ via the edge (b, q) (refer to figure 5.1). Therefore, G' has pendant vertices in exactly one partition. Moreover, G and G' are connected, else we could consider the problem on a connected component of G.

We claim that H is a *y*es instance of Hyper-graph 2-colouring if and only if G' is a *y*es instance of P_3-Role colouring. First, we suppose that H is a *y*es instance. Let $\beta : Q \to \{1, 2\}$ be an assignment of colors to the vertices in Q. Then, the role assignment $\alpha : V(G') \to \{1, 2, 3\}$ can be defined as follows:
$\alpha(u) = \beta(u)$ for all $u \in Q$; $\alpha(s) = 3$, for every $s \in S$; $\alpha(a) = 2$, if $\alpha(q) = 1$ and $\alpha(a) = 1$ otherwise ; $\alpha(b) = 3$.

Let $R(V_R, E_R)$ be the role graph corresponding to α. We claim that
$E_R = \{(1, 3), (2, 3)\}$. To prove that α indeed respects R, notice that the colour 3 is assigned only to the vertices in S and the vertex b. As H is a *y*es instance of Hyper-graph 2-colouring and $\beta(u) = \alpha(u)$, for all $u \in Q$, every vertex coloured 3 is adjacent to some

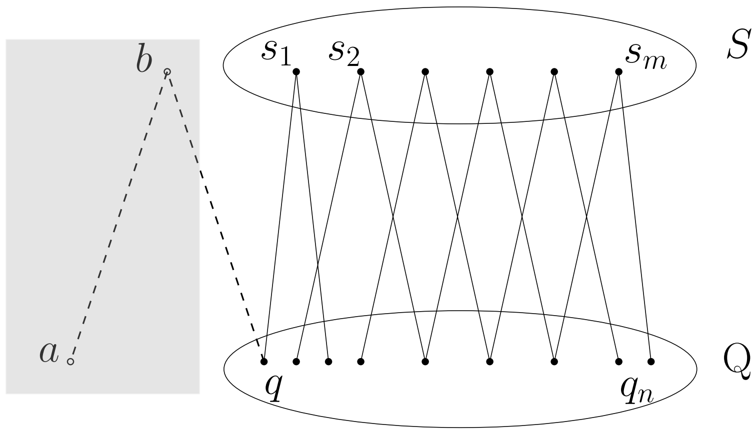

Figure 5.1: Construction in the proof of Lemma 7

vertex coloured 1 and some vertex coloured 2. Also, every vertex in Q is adjacent to a vertex in S and a is adjacent to b. Thus, every vertex coloured 1 or 2 is adjacent to a vertex coloured 3. As R is isomorphic to P_3, G' is a *y*es instance of P_3-Role colouring.

Conversely, suppose that G' is a *y*es instance of P_3-Role colouring with the role assignment α. Let $V(P_3) = \{1, 2, 3\}$ and $E(P_3) = \{(1, 3), (2, 3)\}$. Since the $deg_{G'}(a) = 1$, $\alpha(a) \in \{1, 2\}$. Without loss of generality, suppose that $\alpha(a) = 1$. Then the neighbour of a, that is, b must be coloured 3. The other neighbour of b, which is q must be coloured 2 as per the P_3-role colouring assignment of G'. Now consider a path P in G' that starts at q and ends at some $s \in S$. As P starts in one partition of G' and ends in the other, it must be of even length as G' is bipartite. Since $\alpha(q) = 2$, its neighbour can only be coloured 3. This implies that every even numbered vertex in P gets the colour 3, because any vertex of colour 3 must be followed by a vertex coloured either 1 or 2, which in turn, must be followed by some vertex coloured 3. This ensures that every vertex $s \in S$ is assigned the colour 3. Therefore, the vertices in G' corresponding to the vertices of Q can only be coloured either 1 or 2. Moreover, since α is a P_3-role colouring of G', we know that every vertex coloured 3 is adjacent to at least one vertex each of colors 1 and 2. Now, we define the following colouring for the hyper-graph H. Let $\beta : Q \to \{1, 2\}$ such that $\beta(u) = \alpha(u)$ for all $u \in Q$. No hyper-edge of H is monochromatic as the corresponding vertices in G' have at least one neighbour of each colour. Hence, H is a *y*es instance of Hyper-graph 2-colouring. □

Theorem 5.1.1. *Given a connected bipartite graph G,* 3-Role colouring *is NP-Complete on G.*

Proof. We show a reduction from P_3-Role colouring on bipartite graphs. By Lemma 7, P_3-Role colouring is NP-Complete on the class of bipartite graphs with pendants in only one partition.

Let $G(X, Y, E)$ be a connected bipartite graph with pendant vertices occurring in Y. We add to each vertex $y \in Y$, a path on 2 vertices called p^y, such that the sub-graph induced on $y \cup p^y$ is a path of length 3 (refer to figure 5.2 in Section **??** of the appendix). We call these paths "pendant paths" and the new graph thus obtained G'. We claim that G is a *y*es instance of P_3-Role colouring if and only if G' is a *y*es instance of 3-Role colouring.

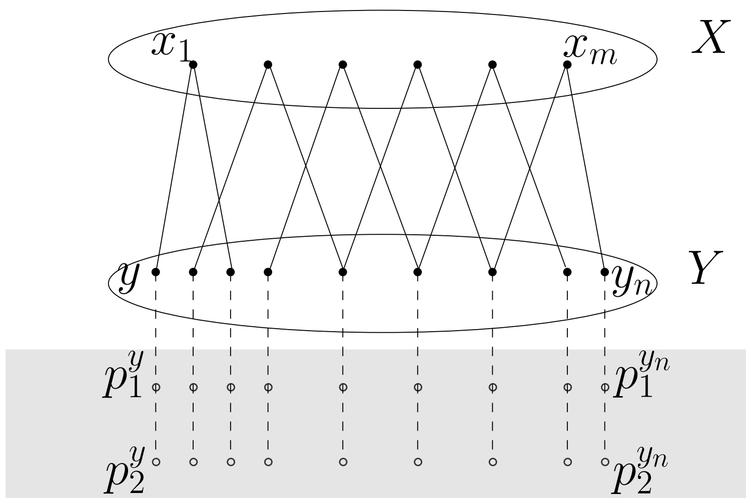

Figure 5.2: Construction in the proof of Theorem 5.1.1

To prove the forward direction, let α be the P_3-role colouring assignment of G. Let the corresponding role graph $R(V_R, E_R)$ be defined as $V_R = \{1, 2, 3\}$ and $E_R = \{(1, 2), (2, 3)\}$. We show that α can be extended to vertices of G'. By the construction in Lemma 7, we know that X is monochromatic with $\alpha(x) = 2$ for all $x \in X$. This ensures that Y does not have any vertex coloured 2, else there would be a vertex in G with role colour 2 that has no neighbours with role colour 1 and 3, which is not possible.

Now, we can easily extend the role assignment in G to G', as follows. Let $p^y = \{{p_1}^y, {p_2}^y\}$ be the pendant path attached to $y \in Y$, with an edge between y and ${p_1}^y$. Let $\alpha({p_1}^y) = 2$; $\alpha({p_2}^y) = 1$ if $\alpha(y) = 3$, and $\alpha({p_2}^y) = 3$ if $\alpha(y) = 1$, for all $y \in Y$. Therefore, α is a P_3 role colouring of G'.

Conversely, suppose that G' has a 3-role colouring. Then we prove that G has a P_3-role colouring. Let us consider the path p^y attached to $y \in Y$. Let the vertex with degree 1 be ${p_2}^y$ and without loss of generality, let $\alpha({p_2}^y) = 1$. Then, if the neighbour of ${p_2}^y$, that is ${p_1}^y$ gets the role colour 1, then in the role graph, 1 is adjacent to itself only. Thus, all the vertices in G' must be role coloured 1, as G' is connected. This is not possible as α is a 3-role colouring of G'. Without loss of generality, assume $\alpha({p_1}^y) = 2$. In the role graph R, 2 must be adjacent to 3, or else R would be disconnected, which is not possible as G is connected. So, $\alpha(y) = 3$. Suppose that y had a neighbour $x \in X$, such that $\alpha(x) = 3$. Then, x must have had a neighbour $y' \in Y$ such that $\alpha(y') = 2$. However, since $N_R(2) = \{1, 3\}$, $\alpha({p_1}^{y'}) \in \{1, 3\}$. This in turn would mean that $\alpha({p_2}^{y'}) = 2$, which is not possible as $deg_{G'}({p_2}^{y'}) = 1$ but $deg_R(2) = 2$. Therefore, $N_R(1) = N_R(3) = \{2\}$ and the role graph corresponding to α is isomorphic to P_3. □

Theorem 5.1.2. *Let G be a connected bipartite graph. Then,* 4-ROLE COLOURING *is NP-Complete on G.*

Proof. Once again, we show a reduction from HYPER-GRAPH 2-COLOURING. Let $H = (Q, S)$ be an instance of HYPER-GRAPH 2-COLOURING. The incidence graph $G(Q, S, E)$ is a connected bipartite graph, with the bipartition (Q, S). To construct the graph G' from G, we add a pendant vertex p^s to every vertex $s \in S$ (refer to Figure 5.3). Let $P^s = \{p^s | s \in S\}$.

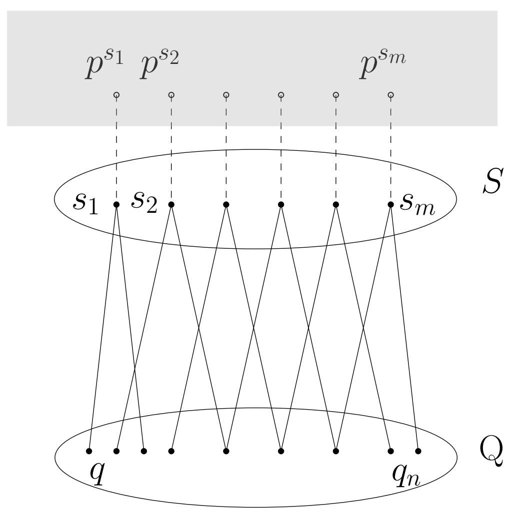

Figure 5.3: Construction in Theorem 5.1.2

We now claim that the hyper-graph $H = (Q, S)$ is 2-colourable if and only if G' is 4-role colourable. Suppose that H has a 2-colouring, $\beta : Q \to \{1, 2\}$. Then the role assignment for G' is defined as: For each $q \in Q$, $\alpha(q)$=$\beta(q)$; for all $s \in S$, $\alpha(s) = 3$ and $\alpha(p^s) = 4$. Corresponding to α, the role graph R is defined as $V(R) = \{1, 2, 3, 4\}$ and $E(R) = \{(1, 3), (2, 3), (3, 4)\}$. As H is a *y*es instance of HYPER-GRAPH 2-COLOURING, every vertex $s \in S$ coloured 3 is adjacent to vertices coloured 1 and 2 in Q and the vertex p^s coloured 4. Thus, α is consistent with R.

Conversely, suppose that G' has a 4-role colouring $\alpha : V(G) \longrightarrow \{1, 2, 3, 4\}$.

Case 1. Q is monochromatic. Without loss of generality, let $\alpha(q) = 1$, for each $q \in Q$.

Claim. No vertex in S is assigned a colour same as that of the vertices in Q.

Proof. Towards a contradiction, suppose that there exists $s \in S$ coloured 1. Then, 1 has a self-loop incident on it in the role graph R. If the pendant attached to s gets coloured 1, then by connectivity of G', all the vertices of G' get assigned the colour 1, which is a contradiction to α being a 4-role colouring of G'. Therefore, suppose that $\alpha(p^s) = 2$. So, $N_R(2) = \{1\}$. As the neighbour of s in Q, say q, is a vertex coloured 1, it must have another neighbour coloured 2. So, every vertex in Q is adjacent to some vertex coloured 1 and some vertex coloured 2. As vertices of Q have neighbours only in S, there must exist a vertex $s' \in S$ such that $\alpha(s') = 2$. This in turn would force $\alpha(p^{s'})$ to be equal to 1, which is not possible, as $deg_R(1) = 2$. Hence, S cannot have a vertex with the same colour as that assigned to the vertices of Q. □

Claim. None of the pendant vertices in P^s gets the colour 1.

Proof. Suppose that there exists a vertex p^s of P^s coloured 1. By the previous claim, no vertex in S is coloured 1. Without loss of generality, suppose that $\alpha(s) = 2$. Since the vertices coloured 1 are only adjacent to vertices coloured 2, S must be monochromatic with the colour 2, as Q is monochromatic with the colour 1. Therefore, P^s must be monochromatic with the colour 1 assigned to its vertices, a contradiction to α being a 4-role colouring of G'. Therefore, it follows that $1 \notin \alpha(P^s \cup S)$. □

Claim: P^s must be monochromatic.

Proof. Towards a contradiction, suppose that there exist vertices p^s and $p^{s'}$ coloured 2 and 3, respectively. Then $\alpha(s) \neq 2$ and $\alpha(s') \neq 3$, else every vertex coloured 2 or 3 could only have neighbours of the same colour. As every $s \in S$ has a neighbour in Q coloured 1, we know that is not possible. This implies that some vertices in S are adjacent to vertices coloured 1 and 2, whereas the others have neighbours coloured 1 and 3. Therefore, there is no colour that can be assigned to the vertices of S, which is a contradiction to G' being 4-role colourable. Hence, P^s must be monochromatic. □

Let $\alpha(P^s)$=$\{2\}$. That forces S to be monochromatic with a third colour, say 3, which again contradicts the assumption that G' is 4-role colourable. Therefore, when Q is monochromatic, G' is not 4-role colourable, which contradicts our assumption.

Case 2: Q is bi-chromatic. Without loss of generality, let the colors assigned to vertices in Q by α be 1 and 2.

Claim: The vertices in S are not assigned the colors 1 and 2.

Proof. We assume the contrary. Suppose that there exists $s \in S$ such that $\alpha(s) = 1$. If $\alpha(p^s)$ $= 1$, then 1 would only be adjacent to itself and the role graph corresponding to α would be disconnected, which is not possible as G' is connected. Suppose that $\alpha(p^s) = 2$. Then $N_R(2) = \{1\}$. For the role graph to be connected, 1 must be adjacent to a third colour, say 3. However, the neighbours of s in Q are coloured either 1 or 2, which is a contradiction. Similarly, no $s \in S$ gets coloured 2. □

Claim: The pendants in P^s are not assigned any colour same as the vertices in Q.

Proof. Suppose that there exists a pendant vertex p^s coloured 1. Let its neighbour s in S have role 3 (from the previous claim we know that it can get neither role 1 nor role 2). Also, 3 must be adjacent to a different colour, say 2, otherwise the role graph would be disconnected with 1 and 3 forming an induced K_2. This means that no pendant receives the colour 3 as the degree of 3 is at least equal to 2. If there exists a pendant coloured 2, then its neighbour must be coloured 3 as well. So, $N_R(1) = N_R(2) = \{3\}$. Hence, S is monochromatic with the role 3, all the vertices in Q are adjacent to some vertex in S. This is a contradiction to α being a 4-role assignment. This implies that $1 \notin \alpha(P^s)$ and $2 \notin \alpha(P^s)$. This proves our claim that $\alpha(Q) \cap \alpha(P^s) = \phi$. □

Therefore, the only possible role assignment in this case is : $\alpha(Q) \in \{1, 2\}$, $\alpha(S) = \{3\}$, and $\alpha(P^s) = \{4\}$. So, every vertex in S is forced to be adjacent to at least one vertex of role 1 and at least one vertex of role 2. The corresponding hyper-graph vertices in Q are coloured as follows: $\beta(q) = \alpha(q)$ for all $q \in Q$.

Having dealt with the cases when Q is monochromatic and bi-chromatic respectively, we now show that $|\alpha(Q)| = 2$.

Claim: If G' has a 4-role colouring, $|\alpha(Q)| = 2$.

Proof. For $|\alpha(Q)| = 1$, we have already seen that no 4-role colouring exists for G'. Suppose that $|\alpha(Q)| \geq 3$. By the previous claims $\alpha(Q) \cap \alpha(\{P^s \cup S\}) = \phi$. Therefore, the set $P^s \cup S$ is monochromatic. That is not possible because every vertex in S is adjacent to vertices of at least 2 different role colors, whereas the vertices in P^s can only be adjacent to one vertex. Therefore, $|\alpha(Q)| = 2$.

□

□

Theorem 5.1.3. *Let G be a connected bipartite graph.* 5-ROLE COLOURING *is NP-Complete on G.*

Proof. We again show a reduction from HYPER-GRAPH 2-COLOURING.The construction is mostly the same as in the proof of Theorem 5.1.2. To the bipartite graph $G'(Q \cup P^s, S, E)$, we add a vertex u and make it adjacent to all the vertices in Q.The resulting graph G'' is bipartite, as u has no neighbours in S (refer to Figure 5.4).

We now show that $H = (Q, S)$ has a 2-colouring if and only if G'' has a 5-role colouring. Suppose that the hyper-graph $H = (Q, S)$ has a 2-colouring assignment, $\beta : Q \longrightarrow \{1, 2\}$. Let the role assignment for G'' be the following:

- $\alpha(q)$=$\beta(q)$, for every $q \in Q$.
- $\alpha(u)$= 3

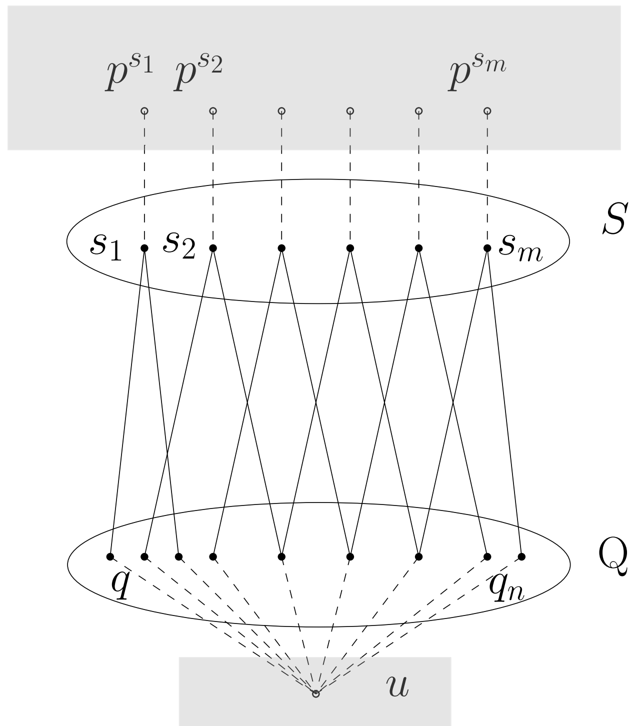

Figure 5.4: Construction in Theorem 5.1.3

- $\alpha(s)= 4$, for all $s \in S$
- $\alpha(p^s)= 5$, for all $p^s \in P^s$

The role graph $R(V_R, E_R)$, corresponding to the 5-role colouring α, is defined as: $V_R = \{1, 2, 3, 4, 5\}$ and $E_R = \{(1, 3), (1, 4), (2, 3), (2, 4), (4, 5)\}$.

Conversely, suppose that G'' has a 5-role colouring, $\alpha : V(G'') \longrightarrow \{1, 2, 3, 4, 5\}$. We first argue that the set $X \subseteq V(G)$ can not be monochromatic.

Case 1. Q is monochromatic. Without loss of generality, let $\alpha(Q) = 1$.

Claim: The vertex u must not be assigned the colour 1.

Proof. If $\alpha(u) = 1$, then every vertex with role 1 can only be adjacent to vertices coloured 1. This would mean that the set S and the pendant vertices would be given role 1 as G'' is connected, which contradicts the hypothesis that α is a 5-role colouring. $\square$

Without loss of generality, let $\alpha(u)$=2. So, $N_R(2) = \{1\}$.

Claim: Neither the vertices in S nor the pendants in P^s are assigned the colors 1 and 2.

Proof. Suppose that there exists $s \in S$, such that $\alpha(s) = 1$. Then s is adjacent to vertices coloured 1 in Q as well as a pendant vertex p^s. Then, $\alpha(p^s)$= 2, as 1 is adjacent to 1 and 2, but all the other neighbours of s are coloured 1. This however implies that all the vertices in the set S must get coloured 1 because $N(Q)$= $S \cup \{u\}$ and u is coloured 2. This forces every p^s to be coloured 2. This is a contradiction to the hypothesis that α is a 5-role colouring of G''.

Since each $s \in S$ is adjacent to a pendant vertex, $\alpha(s) \neq 2$, else $\alpha(p^s) = 1$. Since 1 is adjacent to both 1 and 2 but $deg_{G''}(p^s) = 1$, this is not possible. So, the pendant vertices cannot receive the role 1. Therefore, each element of $S \cup P^s$ must be assigned one of the three colors, 3, 4 and 5. $\square$

Claim: P^s is monochromatic.

Proof. Suppose that P^s has vertices with role 3 and 4. Then no vertex of S gets the colors 3 and 4 as these vertices are adjacent to vertices in Q coloured 1, whereas the pendants are not. This leaves us with the role 5 for S but, some vertices of S are adjacent to pendants coloured 3 whereas some others to pendants coloured 4. Thus, there is no role that can be given to S which contradicts the assumption that G'' has a 5-role colouring. $\square$

Hence, we assume that P^s is monochromatic with the role 3. However, this forces S to be monochromatic with the role 4. This again contradicts α being a 5-role colouring. Therefore, when X is monochromatic, G'' does not have any 5-role colouring.

Case 2. Q is bi-chromatic. Without loss of generality, assume that $\alpha(Q) = \{1, 2\}$.

Claim: $\alpha(u) \notin \{1, 2\}$.

Proof. Suppose that $\alpha(u) = 1$. Then, $N_R(1) = \{1, 2\}$. This implies that the vertices in Q coloured 1 must be adjacent to some vertex coloured 2. That is only possible if some vertices in S receive the colour 2. Let Q_1 be the set of vertices in Q coloured 1 and S_2 be the set of vertices in S coloured 2. Then, vertices in $P^s \cap N(S_2)$ cannot be coloured 1 ($deg_R(1)$= 2). It cannot be coloured 2 either, as 2 must be adjacent to some vertex coloured 1. Suppose that these vertices get coloured 3. The vertices in $Q \setminus Q_1$ must be adjacent to some vertex with role 3, as well. This would imply that some vertices in S get the role 3. However, this is not possible, as 3 is only adjacent to 2 but the pendants attached to these vertices cannot be given the role 2. Therefore, $\alpha(u) \neq 1$. Similarly, $\alpha(u) \neq 2$. □

Therefore, let $\alpha(u) = 3$.

Claim: $\alpha(Q \cup \{u\}) \cap \alpha(S \cup P^s) = \phi$.

Proof. Suppose that there exists $s \in S$ coloured 1. Then s must be adjacent to a vertex coloured 3 as $3 \in N_R(1)$. However, the neighbours of s include some vertices in Q, which have roles 1 or 2, as well as the pendant vertex p^s, which cannot be given role 3 because $deg_R(3) = 2$. Similarly, no vertex of S can be given the role 2. Suppose that some $s' \in S$ gets the role 3, then as 3 is only adjacent to 1 and 2, the pendant $p^{s'}$ must get assigned the colors 1 or 2. This would force all the vertices in S to be coloured 3 and all the vertices of P^s to get the role $\alpha(p^{s'})$. This is a contradiction to α being a 5-role colouring. □

Now, the vertices in $S \cup P^s$ can be coloured with 4 and 5. Suppose that p^s gets the role 4. Then, $\alpha(s) \neq 4$ because s is adjacent to vertices of role 1 and 2 whereas p^s is not. So, $\alpha(s) = 5$. This implies that no pendant vertex gets the role 5, so P^s is monochromatic with the role 4 and S is monochromatic with role 5. Also, each vertex of S must be adjacent to at least one vertex with role 1 and one vertex of role 2.

In this case, we assign colors to the vertices of the hyper-graph as $\beta(q)$= $\alpha(q)$ for all $q \in Q$.

Claim: If G'' is 5-role colourable, then $|\alpha(Q)| = 2$.

Proof. We have seen that when Q is monochromatic, there is no possible 5-role assignment for G''. Suppose that $|\alpha(S)| \geq 3$. By the previous claim, $\alpha(Q \cup \{u\}) \cap \alpha(S \cup P^s) = \phi$. That would leave us with at most one role colour for the vertices in $S \cup P^s$. However, we have also shown that $S \cup P^s$ cannot be monochromatic. So, G'' does not have a 5-role colouring, which is a contradiction. Hence, $|\alpha(Q)| = 2$. □

□

Corollary 5.1.4. *k-*ROLE COLOURING *is NP-Complete on connected bipartite graphs, for all $k > 5$.*

Proof. We use the same reduction as above. Consider the graph G''. In G'', we tweak the set P^s to include "*pendant paths*" instead of vertices, defined as $P^s = \{[{p^s}_1, ..., {p^s}_{k-4}] : \text{for all } s \in S\}$. Each path in P^s is attached to the corresponding s through the edge $({p^s}_{k-4}, s)$. Refer to Figure 5.5

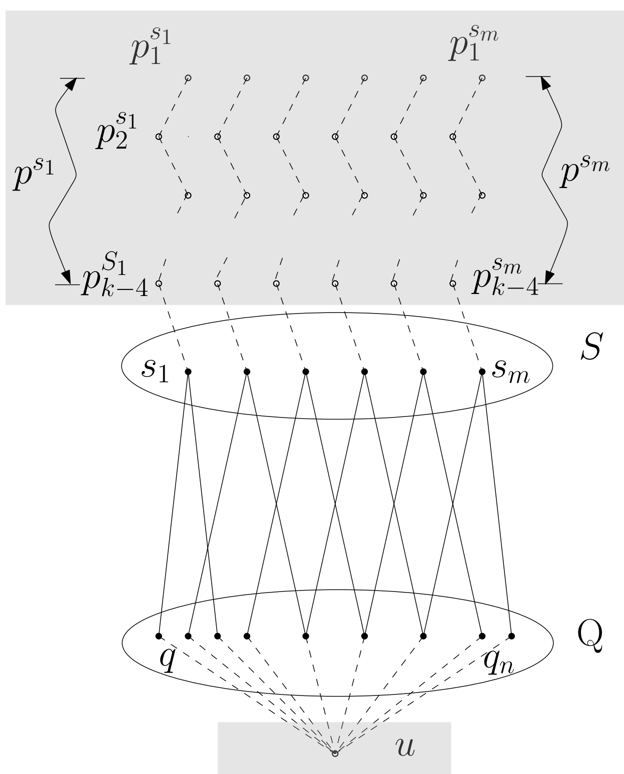

Figure 5.5: Construction in Theorem 5.1.4

Claim: $H = (Q, S)$ has a 2-colouring assignment if and only if G'' has a k-role colouring.

Proof. Suppose that the hyper-graph $H = Q, S$ has a 2-colouring, $\beta : Q \longrightarrow \{1, 2\}$. Let the role assignment for G'' be the following:

- $\alpha(q) = \beta(q)$, for each $q \in Q$,
- $\alpha(u) = 3$,
- $\alpha(s) = 4$, for all $s \in S$
- $\alpha({p^s}_{k-4}) = 5$, for all $s \in S$
- $\alpha(p^s_j) = k - j + 1$, where $1 \leq j \leq k - 5$

□

This gives us a k-role colouring of G'', α with the corresponding role-graph $R(V_R, E_R)$ defined as: $V_R = \{1, 2, \dots, k\}$ and
$E_R = \{(1, 3), (2, 3), (1, 4), (2, 4), (4, 5), (j, j + 1)\}$, for all $5 \leq j \leq k - 1$.

To prove the converse, we have already shown that Q cannot be monochromatic and that $|\alpha(Q)| = 2$. We have also shown that $\alpha(Q \cup \{u\}) \cap \alpha(S \cup P^s) = \phi$. Hence, the only thing left to show is that $\alpha(Q \cup S \cup \{u\}) \cap \alpha(P^s) = \phi$.

Claim: $\alpha(Q \cup S \cup \{u\}) \cap \alpha(P^s) = \phi$.

Proof. To see this, consider the path attached to s. For convenience, we remove the superscript and denote the path as $[p_1, ..., p_{k-4}]$. Towards a contradiction, suppose that $\alpha(p_{k-4}) = \alpha(s) = 4$. Then, $N_R(4) = \{1, 2, 4\}$. However, the degree of p_{k-4} is 2, hence it cannot be assigned a colour with degree greater than or equal to 3.

$\alpha(p_{k-4}) \neq 3$ as 3 is not adjacent to 4. Its colour cannot be equal to 1 or 2 because that would force p_{k-3} to get role 3, contradicting G'' having a k-role assignment, $k > 5$. So, let $\alpha(p_{k-4}) = 5$. Since, 1, 2 and 3 are not adjacent to 5, $\alpha(p_{k-5}) \notin \{1, 2, 3\}$. Since 4 is adjacent to 1, 2 and 5 in R, no vertex in the path can be assigned the role 4 , as these vertices have degree = 2 which is less than the degree of 4 in the role graph. If $\alpha(p_{k-5}) = 5$ then $\alpha(p_{k-6})$ is forced to be 4, which is not possible. Hence, p_{k-5} must receive a fresh colour, say 6. By induction on j, we can show that each p_{k-j} receives a fresh colour j, $j \geq 4$. This completes the proof. □

□

5.2 3-role colouring bipartite chain graphs

Definition 5.2.1. *([11], [21]) A bipartite graph is a* chain graph *if and only if the neighbourhoods of the vertices of each partition can be linearly ordered with respect to inclusion. Equivalently, a bipartite graph is a chain graph if and only if it is $2K_2$-free.*

A *pendant* is a vertex of degree 1. A *universal vertex*[1] is a vertex in one partition of a bipartite graph that is adjacent to every vertex in the other partition.

Theorem 5.2.1. *A bipartite chain graph is 3-role colourable.*

Proof. Let $G = (X, Y, E)$ be a connected bipartite chain graph and let $|X| \geq 3$, $|Y| \geq 3$. P_3 and C_4 are trivially 3-role colourable whereas K_2 and P_4 are not 3-role colourable. Hence, our assumption makes sense. We demonstrate a 3-role colouring for each of the following cases:

Case 1. The graph G has no pendant vertex.

As there are no isolated vertices in G, the linear ordering on the neighbourhoods of vertices ensures that the largest neighborhood contains all the vertices in the partition. Hence, each partition has at least one universal vertex. Let u be a universal vertex of G. We define a role colouring $\alpha : V(G) \longrightarrow \{1, 2, 3\}$ as, $\alpha(u) = 1$, $\alpha(v) = 2$ for all $v \in N(u)$ and $\alpha(w) = 3$ for all $w \in G \setminus N[u]$.

We claim that α is a valid role-colouring of G. We need to show that every vertex coloured 2 is adjacent to some vertex of colour 1 and some vertex coloured 3. As, we have coloured only the neighbours of u with the colour 2, each of them has a neighbour of role 1. Since there are no pendants in G, each of them has at least one other neighbour in the partition containing u. This ensures that each vertex coloured 2 has a neighbour coloured 3.

Case 2. Pendants occur in exactly one partition.

[1]Our definition of "*universal vertex*" is different from the standard definition.

Let X be the partition containing the pendants and Y be the other partition.

Observation 1. The unique neighbour of a pendant must be a universal vertex.
Proof. Let v be a pendant and let u be its unique neighbour. If u is not a universal vertex, then there exists a vertex x in X, that is not adjacent to u. However, since no isolated vertices exist in G, x must be adjacent to some other vertex in G, say y. Therefore, (x,y) and (u, v) induce a $2K_2$ which contradicts the assumption that G is a bipartite chain graph.

Observation 2. There is exactly one universal vertex in Y.
Proof. Suppose there were 2 universal vertices in Y, then every vertex in X would have degree at least 2. This contradicts our assumption that X contains pendants.

Now, we define the role colouring of vertices in G as follows: $\alpha : V(G) \longrightarrow \{1, 2, 3\}$, $\alpha(v) = 1$, for all v in Y; $\alpha(w) = 2$, where w is a universal vertex in X and $\alpha(X \backslash \{w\}) = 3$. Every vertex coloured 1 is adjacent to the vertex coloured 2 (that is w) and a vertex coloured 3. The edges in the role graph are $E_R = \{(1, 2), (1, 3)\}$.

Case 3. Pendants occur in both the partitions.

Let u and v be the universal vertices in X and Y respectively. We first assume that $N(u) \cup N(v)$ does not induce an independent set.

Observation 1. Each partition X and Y has a unique universal vertex, namely, u and v.
Proof. Follows by Observation 2 of Case 2.

In this case, we colour all the pendants with colour 1. The universal vertices adjacent to the pendants can be coloured 2. The remaining vertices are given the role 3. Therefore, every vertex coloured 1 is adjacent only to vertices coloured 2. The two vertices coloured 2 are adjacent to all the vertices in each partition and hence adjacent to both 1 and 3. Since the neighbourhoods of u and v do not form an independent set, we have that at least one vertex in each partition is not a pendant. Hence, there are vertices coloured 3 which are adjacent to each other and the universal vertices.

Now, suppose that $N(u) \cup N(v)$ is an independent set. Then, $|N(u)| \geq 2$ and $|N(v)| \geq 2$, as we assumed that each partition has at least 3 vertices. So, colour u and v with the colour 2. As none of the neighbours of u and v are adjacent to each other, all of them are pendants. Colour any one neighbour of both u and v with colour 1 and another neighbour with the colour 3. The remaining vertices can be arbitrarily given the colors 1 or 3. Hence, the role graph is defined by: $E(R) = \{(1,2),(2,3)\}$. $\square$

Chapter 6

Role colouring chordal graphs

Paulusma et al. [19] had first given a proof of NP-Completeness of k-Role colouring on chordal graphs for $k \geq 3$. With slight modification to our proof of Theorem 5.1.2, we were able to give an alternate, simpler proof for the case $k \geq 4$.

6.1 NP-Completeness of k-Role colouring on chordal graphs

Theorem 6.1.1. k-Role colouring *is NP-Complete on chordal graphs, when* $k \geq 4$.

Proof. **Case 1.** $k = 4$.

Once again, we show a reduction from Hyper-graph 2-colouring. Let G be the graph proposed in Theorem 5.1.2. We construct G' from G by making $Q \subseteq V(G)$ a clique by adding the required edges.

We show that $H = (Q, S)$ has a 2 colouring if and only if $G'(Q, S, E)$ has a 4-role colouring. Suppose the hyper-graph H has a 2-colouring, $\beta : Q \longrightarrow \{1, 2\}$. Let the role assignment for G' be the following:

- $\alpha(q) = \beta(q)$ for all $q \in Q$

- $\alpha(s) = 3$, for all $s \in S$
- $\alpha(p^s) = 4$, for all $s \in S$

Hence the role graph $R(V_R, E_R)$ corresponding to α can be described as: $V_R = \{1,2,3,4\}$ and $E_R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,4)\}$.

Conversely, suppose G' has a 4-role colouring.

Case 1.a. Q is monochromatic. Without loss of generality, assume $\alpha(Q) = 1$.

Claim: No vertex in S is coloured 1.

Proof. Suppose the contrary. Let there exists $s \in S$ such that $\alpha(s) = 1$. Then, $\alpha(p^s) \neq 1$, otherwise 1 would only be adjacent to itself in the role graph. As G' is connected, every vertex would have to be coloured 1. This is a contradiction to G' having a 4-role colouring. Suppose that $\alpha(p^s) = 2$. Then $N_R(2) = \{1\}$. Every vertex coloured 1 in the clique Q must have a neighbour coloured 2. This implies that there exists some vertex s in S coloured 2. However, since 2 is adjacent only to 1, p^s must be coloured 1. This is not possible as the degree of 1 in R is 2. Therefore, S has no vertex with role 1. $\square$

Claim: There is no vertex in P^s coloured 1.

Proof. Q is a monochromatic clique with vertices coloured 1. This implies 1 is adjacent to itself in R. If a pendant vertex was coloured 1, 1 would have only been adjacent to itself. That would force S and P^s to be monochromatic with role 1, a contradiction to α being a 4-role colouring. $\square$

Claim: P^s is monochromatic.

Proof. We assume the contrary. Let $\alpha(p^s) = 2$ and $\alpha(p^{s'}) = 3$. Then, no vertex in S gets coloured 2 or 3 as the vertices in S have a neighbourhood coloured with at least 2 colors, but 2 and 3 are vertices of degree equal to 1 in the role graph, R. So, S must be monochromatic and coloured 4. This, however, is not possible as some vertices of S have neighbours with coloured 1 and 2, whereas the others have neighbours coloured 1 and 3. Hence, P^s is monochromatic. ☐

Without loss of generality, let $\alpha(P^s) = 2$. Since P^s is monochromatic, S is forced to be monochromatic with a colour different from 1 and 2. Therefore, let $\alpha(S) = 3$. We have shown that when Q is monochromatic, G' cannot have a 4-role colouring.

Case 1.b. Q is bi-chromatic. Without loss of generality, assume $\alpha(Q) = \{1, 2\}$.

Claim. $\alpha(Q) \cap \alpha(P^s \cup S) = \phi$.

Proof. Since both 1 and 2 have $deg = 2$ in the role graph, clearly the pendant vertices do not get coloured 1 or 2. Suppose that there exists $s \in S$, such that $\alpha(s) = 1$. Then s is adjacent to vertices coloured 1 and 2 in Q, as well as a pendant vertex p^s. Suppose that $\alpha(p^s) = 3$. Then the vertices in Q coloured 1 are not adjacent to any vertex coloured 3 but s is, which is not possible. Hence, no vertex in S gets coloured 1. Similarly, no vertex in S gets coloured 2. ☐

Since we have shown that vertices in S and P^s cannot be assigned the same colors, we must have that each of these sets is monochromatic with a different colour. Let $\alpha(S) = 3$ and $\alpha(P^s) = 4$ The corresponding colouring assignment for the hyper-graph H in this case is : $\beta(q) = \alpha(q)$ for all $q \in Q$.

Claim: If G' has a 4-role colouring, $|\alpha(Q)| = 2$.

Proof. Follows from the previous claims and the observation that the set $P^s \cup S$ cannot be monochromatic. ☐

Case 2 $k > 4$.

Consider the same graph G' as above. In G', we tweak the set P^s to include "pendant paths" instead of vertices. Let P^s be the set of paths $\{{p^s}_1, ..., {p^s}_{k-4}\}$ for all $s \in S$. Each path in P^s is attached to the corresponding s through the edge $({p^s}_{k-4}, s)$.

In order to prove that H has a 2 colouring if and only if G' has a k-role colouring, we argue on the same lines as above. Suppose that the hyper-graph H has a 2-colouring, $\beta : Q \longrightarrow \{1, 2\}$. Let the role assignment for G' be the following:

- $\alpha(q)= \beta(q)$, for all $q \in Q$
- $\alpha(s)= 3$, for all $s \in S$
- $\alpha({p^s}_{k-4}) = 4$, $s \in S$
- $\alpha(p^s_j) = k - j + 1$ for all $1 \leq j \leq k - 5$ and $s \in S$

This gives us a k-role colouring of G', with the role graph $R(V_R, E_R)$ defined as: $V_R = \{1, 2, ..., k\}$ and $E_R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 4), (j, j + 1)\}$ for all $4 \leq j \leq k - 1$.

To prove the converse, we argue as in Case 1 above. The only thing left to show is that the paths in the set P^s do not get assigned any of the 3 colors assigned to Q and S.

Claim: $\alpha(Q \cup S) \cap \alpha(P^s) = \phi$.

Proof. To see this, consider the path attached to s. For convenience, we remove the superscript and denote the path as $[p_1, ..., p_{k-4}]$. None of the path vertices get coloured 1 or 2 because both 1 and 2 are adjacent to 3 different colors in the role graph, whereas the degree of the path vertices is equal to 2. Towards a contradiction, suppose that $\alpha(p_{k-4})= \alpha(s) = 3$. This means that 3 is adjacent to 1, 2 and 3. However, this is a contradiction as the path vertices cannot have neighbors of 3 different colors. □

□

Chapter 7

Role colouring almost bipartite and almost split graphs

An "*almost bipartite*" graph is a graph that has a set of at most d vertices or d edges, d being a constant, which on deletion yield a bipartite graph. Similarly, an "*almost split*" graph is one that is d-away from being a split graph. Here, we prove that 2-Role colouring is NP-Complete on the class of almost bipartite graphs and almost split graphs, even if d is a small constant.

7.1 Almost Bipartite Graphs

Theorem 7.1.1. *2-*Role colouring *is NP-Complete on the class of almost bipartite graphs.*

Proof. We know from [7] that given a connected bipartite graph G and a role graph R isomorphic to an edge with a self loop incident on one of its end points, deciding if G is 2-role colourable, is NP-Complete. Thus, we show a reduction from this problem to 2-Role colouring on almost bipartite graphs with d= 2.

Construction Let $R = (V_R, E_R)$ be defined as $V_R = \{1, 2\}$ and $E_R = \{(1, 1), (1, 2)\}$. Let G be a connected bipartite graph and let $x \in V(G)$. We construct

G' from G by adding to it 2 vertices a and b and the edges $(a, b), (x, a), (x, b)$. G' is not bipartite as it has a triangle, namely, abx. It is 2 vertices and 3 edges away from the graph G. Refer to Figure 7.1.

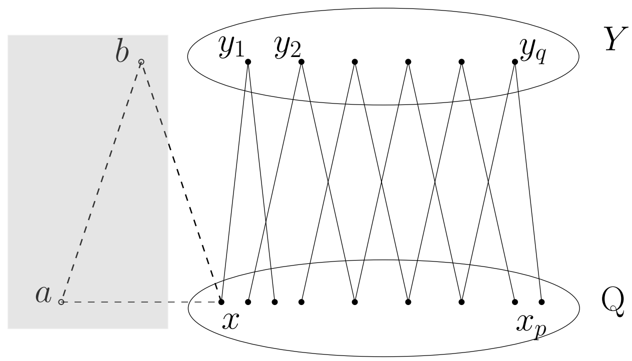

Figure 7.1: Construction in Theorem 7.1.1

Suppose that G is a *y*es instance. Then, it has a role assignment α, with the role graph being isomorphic to R. Then we can extend α to the vertices of G' as follows: If $\alpha(x) = 1$, then colour a with 1 and b with 2. If $\alpha(x) = 2$ then colour both a and b with 1. Thus, G' has an R-role colouring and therefore a 2-role colouring.

Conversely, suppose that G' has a 2-role colouring α'. First assume that $\alpha'(a) = 1$. If both x and b received the colour 1, then the role graph would be disconnected, as 1 could only be adjacent to itself. Hence, at least one of them gets the colour 2. The colour of a forces the role graph to be isomorphic to R. Similarly, if $\alpha'(a) = 2$, the role graph is again isomorphic to R. Hence, if G' has a 2-role colouring it has an R-role colouring and therefore G is R-role colourable. □

7.2 Almost Split Graphs

Theorem 7.2.1. 2-Role colouring *is NP-Complete on almost split graphs.*

Proof. It was shown in [19] that R-Role colouring is NP-Complete on split graphs when R is isomorphic to an edge with self loops incident on both its end points. We show a reduction from this problem to the problem 2-Role colouring on graphs that are d vertices away from split graphs, where d= 7.

Construction. Let R be the graph defined as $R = (V_R, E_R)$, $V_R = \{1, 2\}$ and $E_R = \{(1, 1), (1, 2), (2, 2)\}$. Let G be a split graph with the split partition (C, I). We may also assume that the minimum degree in G is greater than or equal to 2. If there exists a vertex of degree 1 in G, then G is a *no* instance of R-ROLE COLOURING. Let $x \in C$ be a vertex of G. We construct G' by adding to G a C_7, $C' = \{p, q, s, t, u, v, w\}$ and the edge (p, x). Refer to Figure 7.2.

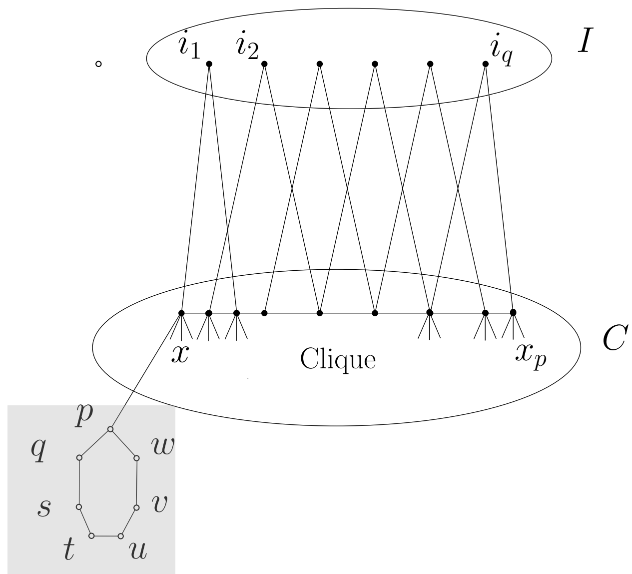

Figure 7.2: Construction in Theorem 7.2.1

Suppose that G is a *yes* instance of R-ROLE COLOURING. Let $\alpha : V(G) \longrightarrow \{1, 2\}$ be the role colouring of G. Since 1 and 2 are symmetrical in R, without loss of generality, let $\alpha(x) = 1$. We extend α to the vertices of C' as follows: $\alpha(p) = 1$, $\alpha(q) = 2$, $\alpha(s) = 2$, $\alpha(t) = 1$, $\alpha(u) = 1$, $\alpha(v) = 2$, $\alpha(w) = 2$. Since every vertex of C coloured 1 is adjacent to some vertex coloured 1 and some vertex coloured 2 and vice-versa, α is an R-ROLE COLOURING of G'. Hence, G' is a *yes*-instance of 2-ROLE COLOURING.

Conversely, suppose that G' is a *yes* instance of 2-ROLE COLOURING and α' is the role colouring of G'. Let us first consider the vertices of C'. Without loss of generality, let $\alpha(t) = 1$. Then, either s or u gets coloured 2, else the role graph would be disconnected. Suppose that $\alpha'(s) = 1$ and $\alpha'(u) = 2$. Then $\alpha'(q) = 2$, as s must have a neighbour coloured 2. We claim that p must be coloured 2. Towards a contradiction, suppose that $\alpha'(p) = 1$. Then $\alpha'(w) = 1$, as p needs a neighbour coloured 1. However, v must be coloured 1, as its neighbour u is coloured 2, which is only adjacent to 1. But, w needs a neighbour coloured

2, so v must be coloured 2. Hence, we run into a contradiction. Thus $\alpha'(p) = 2$ and G' is R-role colourable. Alternatively, suppose that $\alpha'(s) = r'(u) = 2$. Then neither q nor v can be coloured 1 or else the role graph would be isomorphic to a K_2. We know from [7] that a graph is K_2-role colourable if and only if it is bipartite. Hence, $\alpha'(q) = r'(v) = 2$. Both q and v need a neighbour coloured 1, hence $\alpha'(p)= \alpha'(w)= 1$, which is a contradiction as t does not have a neighbour coloured 1, hence, 1 cannot be adjacent to itself in the role graph. Therefore $\alpha'(s) \neq \alpha'(u)$.

What remains to show is that G is a *y*es instance of R-Role colouring. Suppose that $\alpha'(x) = 1$, then x already has a neighbour coloured 2, namely, p. Suppose that its neighbours in G are all coloured 1. Then, the whole clique C is coloured 1, which implies that no vertex in I is adjacent to a vertex coloured 2. This is a contradiction to α' being an R-role colouring of G'. So, x must have a neighbour in G coloured 1 and a neighbour coloured 2. As 1 and 2 are symmetric in R, the same argument holds good if $\alpha'(x) = 2$. Thus we have shown that G is a *y*es instance of R-Role colouring.

□

7.3 Para-NP-Hardness

A *parametrized problem* is a language $L \subseteq \Sigma^* \times \mathbb{N}$. For a problem instance $I = (x, k) \in \Sigma^* \times \mathbb{N}$, k is referred to as the parameter.

A parametrized problem is *fixed parameter tractable* (FPT), if there exists an algorithm $\mathcal{A}$, a computable function f, and a constant c such that given an instance $I = (x, k)$, $\mathcal{A}$ correctly decides if $I \in L$ in time bounded by $f(k).|x|^c$.

A parametrized problem is in XP if there exists an algorithm $\mathcal{A}$, computable functions f and g, such that given an instance $I = (x, k)$, $\mathcal{A}$ correctly decides whether $I \in L$, in time bounded by $f(k).|x|^{g(k)}$

In order to classify parametrized problems the W-hierarchy is defined as:

$$FPT \subseteq W[1] \subseteq W[2] \subseteq ... \subseteq XP$$

It is believed that all the subset relations in this sequence are strict and a parametrized problem that is hard for some complexity class above FPT in this hierarchy, is said to be *fixed parameter intractable*.

A parametrized problem with instances of the form $I = (x, k)$ is para-NP-Hard if for a fixed constant value of k, it is NP-Hard. Unless P = NP, para-NP-Hard problems do not belong to the class XP and therefore, cannot be FPT.

For a background in parametrized complexity, refer [3], [6], [8].

Clearly, on general graphs k-ROLE COLOURING is para-NP-Hard when parametrized by k. In the previous sections, we had explored the complexity of the problem using a different parameter, that is, distance from a graph class. We slightly modify the definition of almost bipartite and almost split graphs for the parametrized version of 2-ROLE COLOURING. An almost bipartite graph G is one that has a set S of at most d vertices (or edges) such that $G \setminus S$ is a bipartite graph. We define almost split graphs similarly.

Theorem 7.3.1. *On the class of almost bipartite graphs and almost split graphs,* 2-ROLE COLOURING *is para-NP-Hard, when parametrized by d.*

Proof. Follows from Theorems 7.1.1 and 7.2.1. □

Chapter 8

Conclusion

Theorems 5.1.1, 5.1.2, 5.1.3 along with Corollary 5.1.4 completely resolve the complexity of k-ROLE COLOURING on bipartite graphs. Theorem 6.1.1 reiterates the NP-Completeness of the problem on chordal graphs for all $k \geq 4$. The underlying idea in our proofs of these theorems was that we can exploit the structure of the given graph to pin down its possible role assignments. Special features of a graph like pendants, induced paths, cycles, cliques, etc are specially helpful in restricting the degrees of the roles assigned to their vertices. Consequently, by such analysis we can limit our attention to typically a constant number of possible role graphs, as opposed to all possible role graphs on k vertices.

Fiala and Paulusma [1] had shown that R-ROLE COLOURING is para-NP-Hard when parametrized by either the treewidth of the given graph G or its maximum degree. Due to Theorems 7.1.1 and 7.2.1, we have shown that 2-ROLE COLOURING is para-NP-Hard when parametrized by the distance of G from bipartite and split graphs, respectively. We have, therefore, shown another parameter for which the problem remains para-NP-Hard on a smaller graph class. An interesting question that remains to be solved is whether 3-ROLE COLOURING is para-NP-Hard on graphs d-away from split graphs using the same parameter. Along similar lines, we ask if k-ROLE COLOURING is in XP if the input graph G is d-away from a tree or a proper interval graph, when parametrized by d.

Bibliography

[1] Steven Chaplick, Jirí Fiala, Pim van 't Hof, Daniël Paulusma, and Marek Tesar. Locally constrained homomorphisms on graphs of bounded treewidth and bounded degree. *Theor. Comput. Sci.*, 590:86–95, 2015.

[2] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms, 3rd Edition.* MIT Press, 2009.

[3] M. Cygan, F. V. Fomin, K. Łukasz, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms.* Springer, 2015.

[4] R. Diestel. *Graph Theory.* Springer-Verlag, 2005.

[5] Mitre Costa Dourado. Computing role assignments of split graphs. *Theor. Comput. Sci.*, 635:74–84, 2016.

[6] R. G. Downey and M. R. Fellows. *Fundamentals of Parameterized Complexity.* Springer-Verlag, 2013.

[7] Jirí Fiala and Daniël Paulusma. A complete complexity classification of the role assignment problem. *Theor. Comput. Sci.*, 349(1):67–81, 2005.

[8] J. Flum and M. Grohe. *Parameterized Complexity Theory.* Springer-Verlag, 2006.

[9] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W.H.Freeman and Company, 1979.

[10] M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs, Second Edition.* Elsevier Science B.V., 2004.

[11] Pinar Heggernes and Dieter Kratsch. Linear-time certifying recognition algorithms and forbidden induced subgraphs. *Nordic J. of Computing*, 14(1):87–108, jan 2007.

[12] Pinar Heggernes, Pim van 't Hof, and Daniël Paulusma. Computing role assignments of proper interval graphs in polynomial time. *J. Discrete Algorithms*, 14:173–188, 2012.

[13] Steve Borgatti Martin G. Everett. Role colouring a graph. *Mathematical Social Sciences*, pages 183–188, 1991.

[14] Christopher Purcell and M. Puck Rombach. On the complexity of role colouring planar graphs, trees and cographs. *J. Discrete Algorithms*, 35:1–8, 2015.

[15] Christopher Purcell and M. Puck Rombach. Role colouring graphs in hereditary classes. *CoRR*, abs/1802.10180, 2018.

[16] Fred S. Roberts and Li Sheng. How hard is it to determine if a graph has a 2-role assignment? *Networks*, 37:67–73, 2001.

[17] D Seinsche. On a property of the class of n-colorable graphs. *Journal of Combinatorial Theory, Series B*, 16(2):191 – 193, 1974.

[18] Li Sheng. 2-role assignments on triangulated graphs. *Theor. Comput. Sci.*, 304(1-3):201–214, 2003.

[19] Pim van 't Hof, Daniël Paulusma, and Johan M. M. van Rooij. Computing role assignments of chordal graphs. *Theor. Comput. Sci.*, 411(40-42):3601–3613, 2010.

[20] Douglas R. White and Karl P. Reitz. Graph and semigroup homomorphisms on networks of relations. *Social Networks*, 5(2):193 – 234, 1983.

[21] Mihalis Yannakakis. The complexity of the partial order dimension problem. *SIAM J. Algebraic Discrete Methods*, 3:351–358, 1982.