Modular Symmetry in Conformal Field Theories

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Certificate

This is to certify that this dissertation entitled "Modular Symmetry in Conformal Field Theories" towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents work carried out by A. Ramesh Chandra at Indian Institute of Science Education and Research under the supervision of Professor Sunil Mukhi, during the academic year 2018-2019.

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Declaration

I hereby declare that the matter embodied in the report entitled "Modular Symmetry in Conformal Field Theories " are the results of the work carried out by me at the IISER, Pune, under the supervision of Professor Sunil Mukhi and the same has not been submitted elsewhere for any other degree.

A. Ramesh Chandra

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Permissions

This thesis is based on the following work done by the author and supervisor:

- Towards a Classification of Two-Character Rational Conformal Field Theories, A.R. Chandra and S. Mukhi, arXiv:1810.09472, *submitted for publication*
- Curiosities above c = 24, A.R. Chandra and S. Mukhi, arXiv:1812.05109, *submitted for publication*

Abstract

The classification of conformal field theories is an important ongoing research area with applications ranging from high-energy physics to condensed-matter systems. Modular invariance has been an invaluable tool in the study of two-dimensional conformal field theories. We review a method of classification using modular-invariant linear differential equations which is based on two parameters: n, the number of characters and ℓ, the number of zeroes of the Wronskian of the differential equation. Previously, this method has been successful for theories with a small number of characters and when $\ell < 6$. We provide new results giving a simple and complete construction of consistent solutions for all values of $\ell \geq 6$ in the case of two-character theories. We further illustrate our method in the specific case of $\ell = 6$ where we realise some new theories as novel cosets of meromorphic conformal field theories.

Contents

Chapter 1

Conformal Field Theory in two dimensions

1 Introduction

Conformal field theories (CFT's) are a distinguished class of field theories whose Poincaré invariance is enhanced to conformal invariance. $2d$ CFT's have their origins in statistical physics and in string theory. For statistical models in $2d$, the continuum description of second order critical phenomenon is described by a CFT. In string theory, the worldsheet quantum field theory description of a string is a CFT. In this chapter we will review the basics of 2d CFT's and define the main objects of our interest – characters and the torus partition function of a CFT. In the subsequent chapters, we will exploit the fact that these partition functions are invariant under modular transformations and in turn use that to classify them.

In dimensions $d \geq 3$ the number of generators of conformal transformations is finite: d translations, $d(d-1)/2$ rotations, 1 dilation and d special conformal transformations which add upto $(d+2)(d+1)/2$. Computing the commutation relations of the generators we see that the algebra satisfied by these generators is $\mathfrak{so}(d,2)$ in a Lorentzian space-time or $\mathfrak{so}(d+1,1)$ in a Euclidean space. Our main focus will be on conformal field theory in two-dimensional flat Euclidean space. In two dimensions, it is convenient to work with complex coordinates $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$. Then, all holomorphic or anti-holomorphic functions of these complex variables are locally angle preserving and hence are conformal transformations. Since these holomorphic maps can be written as infinite Laurent series (allowing for possible singularities at zero or infinity), we see that the number of independent generators of conformal transformations is infinite, one for each term in the series. Thus, the conformal algebra in two dimensions is infinite dimensional and contains $\mathfrak{so}(3,1) \simeq \mathfrak{sl}(2,\mathbb{C})$ as the subgroup of global conformal transformations.

We can look at how fields $\phi(z,\bar{z})$ transform under these conformal transformations. The fields having the simplest transformation properties are called as primary fields and change as follows when $z \to f(z)$

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(z,\bar{z}) \tag{1.1}$$

Then the field is said to have conformal dimensions $(h,\bar{h})$. One of the most important fields is the energy-momentum tensor $T_{\mu\nu}$. A key feature of conformal invariance in any dimension is that $T_{\mu\nu}$ is not only conserved, but is also traceless, i.e. $T^{\mu}_{\mu} = 0$. In our 2d complex coordinates we find that tracelessness implies $T_{z\bar{z}} = 0$, and conservation implies that $\partial_{\bar{z}} T_{zz} = 0$. In other words, the energy-momentum tensor is a holomorphic field. Such fields which only depend on z or only on $\bar{z}$ are called chiral fields and anti-chiral fields respectively.

Even in the Euclidean setting, the formalism of operators and a Hilbert space is very useful. This operator formalism distinguishes a time direction, and here, the most useful choice is the radial quantisation [1] in which time is along the radial direction. Integrals over space are now contour integrals around the origin and time ordered products are radially ordered products. The conserved charge of the current associated to conformal symmetry can be written as

$$Q = \frac{1}{2\pi i} \oint \left(\mathrm{d}z\, \varepsilon(z) T(z) + \mathrm{d}\bar{z}\, \bar{\varepsilon}(\bar{z}) \bar{T}(\bar{z})\right) \tag{1.2}$$

The above equation allows us to compute the infinitesimal change in the field $\phi(z,\bar{z})$ generated by a conserved charge by $[Q,\phi]$. Comparing that with the infinitesimal form of equation Eq. (1.1), we can derive an operator product expansion (OPE) [1] between $T(z)$ and $\phi(z,\bar{z})$ of weight $(h,\bar{h})$ as:

$$\begin{aligned} T(z)\phi(w,\bar{w}) &= \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{(z-w)}\partial_w \phi(w,\bar{w}) + \ldots, \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{(\bar{z}-\bar{w})}\partial_w \phi(w,\bar{w}) + \ldots \end{aligned} \tag{1.3}$$

Here, it is implicitly understood that $|z| > |w|$, and only the non-singular terms are written. Thus, the OPE defines a product structure on the space of operators. Considering the OPE of the energy-momentum tensor with itself reveals an important property of the theory. Since the dimension of $T(z)$ is 2, we would naively expect an OPE similar to the above having double and single poles with $h = 2$. In fact, the most general OPE would also involve another term and is given as:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \ldots \tag{1.4}$$

The first term implies that $T(z)$ is not a primary operator. The numerical constant c is a parameter of the theory and is called the central charge. It cannot be derived just using symmetry arguments and depends on the details of the underlying theory. Similar OPE holds for $\bar{T}(\bar{z})$ with itself, and the OPE of $T(z)$ with $\bar{T}(\bar{w})$ is trivial. We mode expand the operator $T(z)$ as follows

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \\ L_n &= \frac{1}{2\pi i}\oint \mathrm{d}z\, z^{n+1} T(z) \end{aligned} \tag{1.5}$$

The modes $L_n, \bar{L}_n$ are the generators of local conformal transformations on the Hilbert space. The algebra of these modes can be computed using the above OPE of the energy-momentum tensor with itself and is given by the *Virasoro algebra*:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{1.6}$$

The subalgebra of the modes L_{-1}, L_0, L_1 can be identified as the global conformal algebra of $\mathfrak{sl}(2,\mathbb{C})$. Here, c is again the central charge of the CFT, which commutes with all the other operators of the algebra. The presence of a non-zero c doesn't affect the $\mathfrak{sl}(2,\mathbb{C})$ subalgebra of L_{-1}, L_0, L_1. There is another copy of the same algebra for the anti-holomorphic modes $\bar{L}_n$'s with a possibly different $\bar{c}$. Thus, the symmetry algebra involves two copies of the Virasoro algebra which commute with each other. The operator $L_0 + \bar{L}_0$ generates dilations $(z,\bar{z}) \to \lambda(z,\bar{z})$ and hence is the generator of time translations in radial quantisation. In other words, $L_0 + \bar{L}_0$ is proportional to the Hamiltonian of the system. Similarly, $i(L_0 - \bar{L}_0)$ generates rotations and would be proportional to the angular momentum.

CFT Hilbert Space

It is natural to expect that the energy eigenstates of a conformal field theory fall into the irreducible representations of the Virasoro algebra. We will look for highest weight representations of the Virasoro algebra.

The vacuum $|0\rangle$ is the unique state that is invariant under global conformal transformations. Requiring that $T(z)|0\rangle$ be well-defined as z approached zero implies that

$$L_n|0\rangle = 0 \quad \forall n \geq -1 \tag{1.7}$$

Denote by $|h\rangle$ an eigenstate of L_0. From the Virasoro algebra, we can see that L_n's act like an infinite set of raising and lowering operators:

$$\begin{aligned} L_0|h\rangle &= h|h\rangle \\ [L_0, L_n] &= -nL_n \\ L_0 L_n|h\rangle &= (h-n)L_n|h\rangle \end{aligned} \tag{1.8}$$

A highest weight representation is one that contains a state with a smallest L_0 eigenvalue. A highest weight state is a state that is annihilated by all the lowering operators

$$L_n|h\rangle = 0 \quad \forall n \geq 1 \tag{1.9}$$

The highest weight state $|h\rangle$ is used to generate the so called descendant states by acting on $|h\rangle$ by the raising operators L_{-n} for $n \geq 0$. The representation space, consisting of all such descendant states and their linear combinations is called a Verma module. A basis for the set of states is obtained by applying the raising operators in all possible ways:

$$L_{-k_1} L_{-k_2} \ldots L_{-k_n}|h\rangle \ \ (1 \leq k_1 \leq \ldots \leq k_n) \tag{1.10}$$

This would be state with L_0 eigenvalue $h' = h + \sum k_i = h + N$. Here N is called the level of the state. Here, we are just dealing with the holomorphic generators L_n's, since the anti-holomorphic generators $\bar{L}_n$'s have a completely analogous result. If we denote by $V(c,h)$ and $\bar{V}(c,\bar{h})$ the Verma modules generated from a highest weights h and $\bar{h}$ of a CFT with central charge c, then the energy eigenstates would belong to a tensor product $V \otimes \bar{V}$. In general, the the Hilbert space is a sum over all the conformal dimensions $(h, \bar{h})$ of the theory:

$$\mathscr{H} = \sum_{h,\bar{h}} V(c,h) \otimes \bar{V}(c,\bar{h}) \tag{1.11}$$

To a particular Verma module $V(c,h)$ we can associate a generating function $\chi_{(c,h)}(q)$, which counts the number of linearly independent states at each level n:

$$\chi_{(c,h)}(q) = \mathrm{Tr} q^{L_0 - c/24} \quad = \sum_{n=0}^{\infty} \dim(h+n) q^{h+n-c/24} \tag{1.12}$$

$\dim(h+n)$ denotes the number of linearly independent states of level n. The definition will make more sense when we look at modular invariance in CFT's. From equation (10), we can see that the number of states at each level n is just the number of ways to partition n, $p(n)$. Thus we can write

$$\begin{aligned}\chi_{(c,h)}(q) &= q^{h-c/24}\sum_{n=0}^{\infty} p(n)q^n \\ &= q^{h+(1-c)/24}/\eta(q) \\ \eta(q) &= q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\end{aligned} \tag{1.13}$$

Here $\eta(q)$ is the Dedekind eta function, and the generating function $\chi_{(c,h)}(q)$ is called a Virasoro character for a primary of non-zero dimension h. In case of the the identity primary field with $h = 0$, the Virasoro character is different, with the product in Eq. (1.13) starting from $n = 2$ instead of $n = 1$.

2 Modular Invariance and Rational CFT's

The study of CFT's defined on a torus arises quite naturally in many situations. In finite temperature CFT's, the partition function is given as a path integral performed over a torus. Often, while studying statistical systems periodic boundary conditions are chosen in both the directions on the $2d$ plane. This too leads to the topology of a torus. Also, in string theory, CFT's on a torus describes the one-loop scattering amplitudes of strings. In fact, one considers CFT's defined on a Riemann surface of arbitrary genus $g \geq 0$ in string theory to calculate multi-loop amplitudes.

Tori and modular invariance

A torus (equipped with a complex structure) is equivalent to the complex plane under the identification that any two points that differ by a integer linear combination of lattice vectors are considered the same. This is just imposing periodic boundary conditions along the directions of the two lattice vectors. Let (ω_1, ω_2) be two linearly independent complex numbers defining the lattice on the complex plane. Then, it is easy to see that another pair of lattice vectors (ω_1', ω_2') are related to (ω_1, ω_2) by the following also describes the same lattice, and hence the same torus:

$$\begin{pmatrix}\omega_1' \\ \omega_2'\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix} \quad \forall a,b,c,d \in \mathbb{Z},\ ad-bc = 1 \tag{1.1}$$

This transformation takes linear combinations of the old lattice vectors (ω_1, ω_2) over the integers, and the condition that $ad - bc$ is unity is to preserve the area of the unit cell of the lattice. The parameter $\tau = \omega_2/\omega_1$ is called the modular parameter of a torus. Since the absolute orientation of the lattice doesn't matter, the relevant parameter is just the ratio τ. Under the above transformation, the modular parameter changes as:

$$\tau \to \tau' = \frac{a\tau+b}{c\tau+d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \tag{1.2}$$

The above transformation of τ is called a modular transformation and the group of all such transformations, $SL(2,\mathbb{Z})/\mathbb{Z}_2 \cong PSL(2,\mathbb{Z}) = \Gamma$ is called the modular group. We choose the convention that $\mathrm{Im}(\tau) > 0$, i.e. τ lives on the upper half complex plane, $\mathbb{H}$. (Note that we could have chosen $\mathrm{Im}(\tau)$ to be either positive or negative, the above transformation doesn't change this property). The modular group has two generators T and S:

$$\begin{gathered} \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \tau \to \tau + 1 \\ \mathrm{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \tau \to -1/\tau \\ PSL(2,\mathbb{Z}) = \langle \mathrm{T}, \mathrm{S} | (\mathrm{S})^2 = (\mathrm{ST})^3 = 1 \rangle \end{gathered} \tag{1.3}$$

It would be useful to briefly review modular forms, which are objects behaving in a particularly nice fashion under these modular transformations. A good reference for detailed definitions, theorems and proofs is [2]. A modular form of weight $2k$ is a holomorphic function $f(\tau)$ on the upper half plane $\mathbb{H}$ that transforms as follows under the modular group

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k} f(\tau) \tag{1.4}$$

From the definition it follows that these are periodic under $\tau \to \tau+1$ and have a Fourier series expansion in the variable $q \doteq e^{2\pi i \tau}$. The modular forms that we will encounter most frequently are the Eisenstein series, E_{2k} for $k \geq 2$. The first two Eisenstein series defined in terms of their q-expansions are

$$\begin{aligned} E_4(\tau) &= 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = 1 + 240q + 2160q^2 + \cdots \\ E_6(\tau) &= 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n} = 1 = 504q - 16632q^2 - \cdots \end{aligned} \tag{1.5}$$

The Eisenstein series $E_2(\tau)$ is does not transform as a modular form, but will still play a major role later. It can similarly be defined by its q-expansion

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \tag{1.6}$$

Under modular transformation this transforms as

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) - \frac{6}{\pi} ic(c\tau+d) \tag{1.7}$$

The ring of all holomorphic modular forms is generated by E_4 and E_6. The derivatives of Eisenstein series with respect to τ are given by the Ramanujan identities as

$$\begin{aligned} E_4' &= \frac{E_2E_4 - E_6}{3} \\ E_6' &= \frac{E_2E_6 - E_4^2}{2} \end{aligned} \tag{1.8}$$

Finally, the modular invariant Klein j-function is defined as

$$j(\tau) = 1728 \frac{E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)} = q^{-1} + 744 + 196884q + \cdots \tag{1.9}$$

Torus partition function

The transition from the plane to the torus is best done via the cylinder. The complex plane can be conformally mapped to the cylinder using the transformation $z = e^w$. In fact, this cylinder picture is very useful in radial quantisation and also for the state-operator correspondence in CFT's. Now, the torus can be obtained from cutting a finite part of this infinite cylinder and applying periodic boundary conditions. The torus partition function can be calculated using a path integral. Let H and P be the Hamiltonian and momentum operators on the cylinder. Due to the periodic boundary conditions, the path integral can be written as the following trace

$$\begin{aligned} Z(\tau_1, \tau_2) &= \mathrm{Tr}_{\mathscr{H}}\, e^{-2\pi\tau_2 H} e^{2\pi i \tau_1 P} \\ &= \mathrm{Tr}_{\mathscr{H}}\, e^{2\pi i(\tau_1 + i\tau_2)(H+P)} e^{-2\pi i(\tau_1 - i\tau_2)(H-P)} \end{aligned} \tag{1.10}$$

Writing H and P in terms of L_0 and $\bar{L}_0$ we get

$$\begin{aligned} Z(\tau,\bar{\tau}) &= \mathrm{Tr}_{\mathscr{H}}\, e^{2\pi i\tau(L_0-c/24)} e^{-2\pi i\bar{\tau}(\bar{L}_0-c/24)} \\ &= \mathrm{Tr}_{\mathscr{H}}\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24} \end{aligned} \tag{1.11}$$

We have used the common notation $q = e^{2\pi i\tau}$, and the factors of $c/24$ are present due to the transformation from the plane to the cylinder. The partition function depends on the moduli of the torus, and is invariant under all the modular transformations. Thus the constraint imposed by putting the CFT on a torus is the invariance of the partition function $Z(\tau,\bar{\tau})$ under modular transformations. Here the trace is over all the physical states of the Hilbert space $\mathscr{H}$. Following from the decomposition of the Hilbert space into the sum of tensor products of Verma modules, we can write the partition function as

$$\begin{aligned} Z(\tau,\bar{\tau}) &= \sum_{i,j} M_{ij}\chi_i(\tau)\bar{\chi}_j(\bar{\tau}) \\ \chi_{h_i}(\tau) &= \mathrm{Tr}_{h_i} q^{L_0-c/24} = q^{h_i-c/24}\sum_{n=0}^{\infty} d(n)q^n \end{aligned} \tag{1.12}$$

Here each χ_i is a character of a particular representation, encoding data about a specific primary of weight h_i. The trace is over all physical states contained in the Verma module associated to h_i. The $d(n)$'s are the degeneracies in the Verma modules at level n and are bounded by $p(n)$. The M_{ij}'s are non negative integers giving the multiplicity of the ij^{th} representation, corresponding to the state with weights $(h_i,\bar{h}_j)$. The uniqueness of the vacuum state imposes $M_{00} = 1$. All the other values of M_{ij} are constrained by the modular invariance of $Z(\tau,\bar{\tau})$. The characters χ_i's transform in the following way

$$\begin{aligned} \chi_i(\tau+1) &= \mathrm{T}_{ij}\chi_j(\tau) = e^{2\pi i(h_i-c/24)}\delta_{ij}\chi_j(\tau) \\ \chi_i(-1/\tau) &= \mathrm{S}_{ij}\chi_j(\tau) \end{aligned} \tag{1.13}$$

Here T_{ij} and S_{ij} are unitary matrices acting on the characters χ_i. Modular invariance imposes the condition that these matrices must commute with the M_{ij} matrix.

Rational Conformal Field Theories

For Virasoro minimal models [1] , which have $c < 1$, the sum in Eq. (1.12) is performed only over a finite number of representations of the Virasoro algebra. These minimal models thus have a finite number of primary fields which have all rational conformal dimensions. More generally, whenever the Hilbert space can be factorised into a sum over a finite number of representations of any extended symmetry algebra (which includes the Virasoro algebra as a subalgebra), then it defines a rational conformal field theory (RCFT). Even this case, the central charge and dimensions of all primary operators are rational numbers [3]. Such extended symmetry algebras are generated due to the presence of conserved chiral fields of integer or half-integer spins, and are often refereed to as the chiral algebra of a CFT. Prime examples of theories having such an extended symmetry algebra are the Wess-Zumino-Witten models, whose Hilbert spaces are a finite sum of integrable representations of a Kac-Moody algebra. Just like the Virasoro algebra being generated by the modes of the energy-momentum $T(z)$, in WZW models we have a set of conserved spin-1 currents $J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J^a_n$ whose modes generate the Kac-Moody algebra given by

$$[J^a_n, J^b_m] = i f^{ab}_c J^c_{n+m} + km\delta^{ab}\delta_{n+m,0} \tag{1.14}$$

The zero modes of the currents form a Lie algebra $\mathfrak{g}$ with f^{ab}_c as it's structure constants, and the constant k, called the level of the algebra is the analogue of central charge. This algebra is often denoted as $\mathfrak{g}_k$, and is also reffered to as an affine Lie algebra. For the path integral of a WZW theory to be well defined, the level needs to be restricted to positive integral values. The energy-momentum tensor and central charge of such a theory is given by the Sugawara construction [1] as follows

$$\begin{aligned} T(z) &= \frac{1}{2(k+h^\vee)} \sum_a J^a(z) J^a(z) \\ c &= \frac{k \dim\mathfrak{g}}{k+h^\vee} \end{aligned} \tag{1.15}$$

Here $h^\vee$ is the dual Coxeter number defined by the quadratic Casimir of the adjoint representation $f^{ab}_c f^{bc}_d = 2h^\vee \delta^{ad}$. As was with the case of Virasoro algebra, the Hilbert space now can be factorised into a sum of highest weight representations of a Kac-Moody algebra. Apart from considering the minimal series of such Kac-Moody algebras, they can also be used to generate new theories by the coset construction method. This involves two algebras $\mathfrak{g}_k$ and a subalgebra $\mathfrak{h}_{k'} \subset \mathfrak{g}_k$. Then the new coset theory is constructed by all the chiral fields of $\mathfrak{g}_k$ which have a trivial OPE with the fields of $\mathfrak{h}_{k'}$. The Sugawara energy momentum tensor and central charge of the coset theory are given by $T^{\mathfrak{g}} - T^{\mathfrak{h}}$ and $c^{\mathfrak{g}} - c^{\mathfrak{h}}$ respectively. Then standard example of such a coset construction is unitary series of Virasoro minimal models given by the coset

$$\frac{su(2)_k \oplus su(2)_1}{su(2)_{k+1}} \tag{1.16}$$

In the later chapters, we will use a generalised coset construction to describe some new class of interesting theories. Returning to the discussion on modular invariance, we note that the modular transformations of characters of a RCFT determine the fusion rules of the theory. These fusion rules are selection rules telling us which fields are allowed to appear in a given OPE expansion. Very similar to the tensor product of group representations, the fusion product of two primary fields can be decomposed into a sum of contributions from other primary fields which appear in the OPE expansion. This product gives a commutative and associative algebra on the space of primary field, called the fusion algebra. The structure constants or fusion coefficients of this algebra are given by the famous Verlinde formula. The entries of modular S matrix in Eq. (1.13) determine these fusion coefficients (which are all non-negative integers) as

$$N_{ij}^{k} = \sum_{m} \frac{S_{im} S_{jm} S^{*}_{mk}}{S_{0m}} \tag{1.17}$$

With this quick review of rational conformal field theories and modular invariance, let us now proceed to study the classification of RCFT's using modular linear differential equations.

Chapter 2

Modular Linear Differential Equations

1 Classification of Rational Conformal Field Theories

As seen in the previous chapter, rational conformal field theories are characterised by two important algebraic structures: their chiral algebras and their fusion algebras. The traditional classification of rational conformal field theories is based on chiral algebras. Given a particular chiral algebra, the structure of its null vectors allows one to write down a minimal series of CFT's all of which furnish realisations of the given algebra. In the simplest case, the chiral algebra is just the Virasoro algebra, and this procedure leads to the Virasoro minimal models. Among possible chiral algebras, Kac-Moody algebras have played an important role in the classification of RCFT's. Using WZW models, which are the minimal series of a given Kac-Moody algebra, one can construct coset models. This coset construction generates a vast supply of RCFT's including the various minimal series already obtained by null vector methods.

However, this approach towards classification is not complete and there are several interesting and simple RCFT's that do not fall within this classification method. There are several such RCFT's with a small number of characters which do not belong to any minimal series, but are still extremely interesting like the three-character baby monster module. A complementary approach would be to use the fusion algebras to classify the theories. In this approach, one would classify all possible theories with a given number of primaries (with respect to any chiral algebra). A classification method in this spirit was proposed long ago in [4]. This method was based on the following facts: the torus partition function $Z(\tau, \bar{\tau})$ is modular invariant but it is not a holomorphic function, while the characters $\chi_i(\tau)$ are holomorphic within the moduli space, but are not modular invariant. They are in fact modular covariant and transform as a vector under modular transformations. However, one can write these characters as solutions of a differential equation that is both modular invariant and holomorphic. These two properties together are very constraining and can be used to classify those modular linear differential equations which give rise to characters of rational conformal field theories. A complete classification of the simplest models (with a small number of primaries) could be very useful both in physics and mathematics.

To achieve this, we write down the most general differential equation that is invariant under $SL(2,\mathbb{Z})$ transformations. Such an equation can be written as follows

$$\left(f_{2\ell}D^n + f_{2\ell+2}D^{n-1} + \cdots + f_{2(n-1+\ell)}D + f_{2(n+\ell)}\right)\chi = 0 \tag{2.1}$$

Here the coefficient functions f_k's are holomorphic modular forms of weight k, and D is the modular covariant derivative taking modular forms of weight k to weight $k+2$

$$\begin{aligned} D &= \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{k}{12}E_2(\tau) \\ D^n &= D_{(k+2n-2)} \circ \cdots \circ D_{(k+2)} \circ D_{(k)} \end{aligned} \tag{2.2}$$

These differential equations are labelled by two integers, n and ℓ. The role of n is very clear, it is just the number of *independent* characters of an RCFT. Note that the MLDE is not written in a monic form above, and it is often useful to divide throughout by the first coefficient making the MLDE monic.

$$\left(D^n + \phi_{n-1}D^{n-1} + \cdots + \phi_1 D + \phi_0\right)\chi = 0 \tag{2.3}$$

The coefficients ϕ_k's will now be meromorphic modular forms, and can be expressed in terms of the n linearly independent solutions using the Wronskian determinants as

$$\phi_k = (-1)^{n-k}\frac{W_k}{W}$$

$$W_k = \begin{vmatrix} \chi_0 & \chi_1 & \cdots & \chi_{n-1} \\ D\chi_0 & \chi_1 & \cdots & D\chi_{n-1} \\ \vdots & & & \vdots \\ D^{k-1}\chi_0 & D^{k-1}\chi_1 & \ldots & D^{k-1}\chi_{n-1} \\ D^{k+1}\chi_0 & D^{k+1}\chi_1 & \ldots & D^{k+1}\chi_{n-1} \\ \vdots & & & \vdots \\ D^n\chi_0 & D^n\chi_1 & \ldots & D^n\chi_{n-1} \end{vmatrix} \tag{2.4}$$

and $W = W_n$. In fact the number of zeroes of W in the fundamental domain is equal to the number of zeroes of $f_{2\ell}$ which is exactly $\frac{l}{6}$. Since there is no holomorphic modular form of weight 2, ℓ

takes the values $0, 2, 3, \cdots$. The fractional nature of zeroes indicates the fact that the torus moduli space has two orbifold points, one at $\tau = e^{2i\pi/3}$ of order $1/3$ and the other at $\tau = i$ of order $1/2$. More importantly, ℓ satisfies a relation with the conformal dimensions of the primary fields as follows. Let the leading behaviour of the n independent characters be q^{α_i}. Then noting from (2.4) that the Wronskian is modular, and of weight $n(n-1)$, we have the relation

$$\sum_i \alpha_i + \frac{\ell}{6} = \frac{n(n-1)}{12} \tag{2.5}$$

Since this relation follows from the Riemann-Roch equation on the fundamental domain, we will refer to Eq. (2.5) as the Riemann-Roch equation.

The classification strategy is as follows. We start with a fixed number of primaries n and also fix the integer ℓ. This will fix the form of all the coefficients $f_{2(\ell+k)}$. These functions are polynomials of the Eisenstein series E_4, E_6 with arbitrary coefficients. This leaves us with a order n differential equation with a finite number of arbitrary parameters which we need to fix. To do this, we insert the following ansatz for the solutions

$$\chi_i(\tau) = \sum_{n=0}^{\infty} a_n^{(i)} q^{\alpha_i + n} \tag{2.6}$$

We can now rewrite the covariant derivatives in terms of ordinary derivatives, and solve the differential equation by Frobenius method. This will determine the unknowns $\alpha_i, a_n^{(i)}$, order by order, in terms of the arbitrary parameters in the MLDE. Since we want the solutions of the MLDE to be interpreted as characters, the coefficients $a_n^{(i)}$ should be positive integers representing the degeneracies of states above a given primary of dimension α_i. We call such solutions which are not only modular invariant, but also have all the coefficients $a_n^{(i)}$ as positive integers as *admissible* characters. Before we discuss this procedure in more detail, let us look at the simple case when $n = 1$.

2 Meromorphic Conformal Field Theories

While classifying conformal field theories using their fusion algebras, it would be simplest to start with those theories having trivial fusion. This happens when the only primary is the vacuum state, and every other state is a descendant of the vacuum. The torus partition function in this case is

$$Z(\tau, \bar{\tau}) = |\chi(\tau)|^2 \tag{2.1}$$

The single vacuum character $\chi(\tau) = q^{-\frac{c}{24}} \sum_{n=0}^{\infty} a_n q^n$, with $a_0 = 1$ satisfies a first order MLDE as follows

$$(f_{2\ell} D + f_{2+2\ell})\chi = 0 \tag{2.2}$$

Since $n = 1$ and $\alpha_0 = -\frac{c}{24}$, the Riemann-Roch equation (2.5) relates the central charge of the theory c to the integer ℓ, given by

$$c = 4\ell \tag{2.3}$$

We can now build solutions using the strategy discussed in the previous section. For $\ell = 0$, we have $f_0 = 1$ and $f_2 = 0$. So we only have the trivial solution $\chi = 1$. Note that in this first order case, D just reduces to the ordinary derivative in τ. Next, for $\ell = 2$, we have $f_4 = E_4$ and $f_6 = E_6$, so the most general equation now is

$$\left(D + \mu \frac{E_6}{E_4}\right)\chi = 0 \tag{2.4}$$

From (2.3), we see that if a theory exists it should have a central charge $c = 8$. Since we know the q-series of Eisenstein series, we can solve the differential equation order by order in q. At the first two orders, this will lead to $\mu = \frac{c}{24} = \frac{1}{3}$ and $a_1(\mu) = 744\mu = 248$. The subsequent orders will determine the higher coefficients $a_n(\mu)$. All of these will turn out to be positive integers, indicating that there really exists a CFT here. In fact, this is nothing but the Kac-Moody theory based on E_8 at level 1. The number of spin-1 generators is given by a_1 is equal to $\dim E_8 = 248$. The character can be written as a function of the Klein j-invariant as

$$\chi^{\ell=2} = \chi_{E_{8,1}} = j^{\frac{1}{3}} = q^{-\frac{1}{3}}(1 + 248q + 4124q^2 + \cdots) \tag{2.5}$$

The above character itself is not modular invariant, but acquires with a phase $e^{\frac{2\pi i}{3}}$ under the S transformation. The partition function however is modular invariant. Moving on to $\ell = 3$, we see that the MLDE is

$$\left(D + \mu \frac{E_4^2}{E_6}\right)\chi = 0 \tag{2.6}$$

Repeating the analysis, we find this time, $c = 12$, $\mu = \frac{1}{2}$ and $a_1(\mu) = -984\mu = -492$. We see that in this case at the first level above the vacuum, the "degeneracy" is a negative integer. Computing to higher orders, one easily finds that all coefficients (except $a_0 = 1$) are negative integers. Thus, the only solution at $\ell = 3$ fails to be an admissible character. Again, this solution can be expressed in terms of the j-function as follows

$$\chi^{\ell=3} = \sqrt{j-1728} = q^{-\frac{1}{2}}(1-492q-22590q^2-\cdots) \tag{2.7}$$

Though the degeneracies are negative, all of them are integers. In the next chapter, we will study such solutions, which are not admissible, but whose coefficients are all still integers. We similarly can find out the characters for all higher ℓ. But using the well known fact that $j(\tau)$ is the only modular invariant function for $SL(2,\mathbb{Z})$, we can express the characters for arbitrary ℓ in terms of j as

$$\chi(\tau) = j^{w_\rho}(j-1728)^{w_i}P_{w_\tau}(j) \tag{2.8}$$

Here, $w_\rho \in \{0, \frac{1}{3}, \frac{2}{3}\}$, $w_i \in \{0, \frac{1}{2}\}, w_\tau \in \mathbb{Z}$ and $P_{w_\tau}(j)$ is a polynomial of degree w_τ in $j(\tau)$. The equation (2.3) is now

$$c = 24(w_\rho + w_i + w_\tau) = 4\ell \tag{2.9}$$

Since $(j-1728)^{\frac{1}{2}}$ has infinitely many negative coefficients, whenever $w_i \neq 0$ the solution will not be admissible. This implies that ℓ must always be even for the solution to be an admissible character. We list the next few admissible solutions in the below.

ℓ	2	4	6	8
c	8	16	24	32
$\chi(\tau)$	$j(\tau)^{\frac{1}{3}}$	$j(\tau)^{\frac{2}{3}}$	$j(\tau)+\mathcal{N}$	$j(\tau)^{\frac{1}{3}}(j(\tau)+\mathcal{N})$

Table 2.1: single character theories

At $\ell = 4, c = 16$ we have two theories - $E_{8,1}^2$ and $D_{16,1}$ Kac-Moody theories (though their characters are identical). Contrary to the case of $\ell < 6$ where there were finitely many admissible characters, for $\ell \geq 6$ there are infinitely many admissible characters depending on arbitrary integers. For example, in the case of $c = 24$, the character takes the form

$$\chi(\tau) = j(\tau) + \mathcal{N} = q^{-1}(1+(\mathcal{N}+744)q+196884q^2+\cdots) \tag{2.10}$$

where $\mathcal{N} \geq -744$ could be any integer. However, we know from Schellekens' classification of these theories in [5] that only for 71 values of $\mathcal{N}$ actually correspond to genuine theories. Also, note that at $\mathcal{N} = -744$ we have the Monster CFT. To summarise the one-character scenario, we found that there do exist solutions which are admissible characters for all even ℓ. And, when $\ell \geq 6$, there are infinitely many admissible characters depending on arbitrary integers, though we expect only a finite subset of these to be realised as CFT's.

3 Two Characters: Complete classification for $\ell < 6$

The two-character case is much more interesting from the MLDE point of view due to their non-trivial yet simple fusion rules. This is the case when the conformal field theory has a single fractional critical exponent. The general second order MLDE is of the following form

$$\left(f_{2l}D^2 + f_{2l+2}D + f_{2l+4}\right)\chi = 0 \tag{2.1}$$

Again, the functions f_k's are holomorphic modular forms weight $2k$, generated by E_4 and E_6. The free parameters in the differential equation come as coefficients to these modular forms. Since the dimension of the vector spaces of modular forms is known, the number of free parameters can be computed, and is given by

$$\begin{aligned}\sharp(l) &= \mathrm{dim}\mathsf{M}_{2l}(\Gamma) + \mathrm{dim}\mathsf{M}_{2l+2}(\Gamma) + \mathrm{dim}\mathsf{M}_{2l+4}(\Gamma) - 1 \\ &= \left[\frac{\ell}{2}\right]\end{aligned} \tag{2.2}$$

Here [.] denotes the integer part. The total number is one less than the sum of dimensions as an overall factor can always be eliminated. Thus, the number of free parameters grows linearly with ℓ. We can proceed to solve the differential equation by making the following ansatz for the two solutions

$$\begin{aligned}\chi_0 &= q^{\alpha_0}\left(1 + a_1^{(0)}q + a_2^{(0)}q^2 + \cdots\right) \\ \chi_1 &= \mathrm{D}\,q^{\alpha_1}\left(1 + a_1^{(1)}q + a_2^{(1)}q^2 + \cdots\right)\end{aligned} \tag{2.3}$$

Here, D is the degeneracy of non-vacuum primary, and $\alpha_0 = -\frac{c}{24}$ and $\alpha_1 = h - \frac{c}{24}$ are the critical exponents. The Riemann-Roch equation now tells us that

$$\alpha_0 + \alpha_1 = -\frac{c}{12} + h = \frac{1-\ell}{6} \tag{2.4}$$

Once we have found the solutions, we could in principle compute the S transformation

$$\chi_i\left(-\frac{1}{\tau}\right) = \sum_{j=0}^{1} S_{ij}\chi_j(\tau) \tag{2.5}$$

If this matrix turns out to be unitary then the partition function:

$$Z(\tau,\bar{\tau}) = \overline{\chi}_0(\bar{\tau})\chi_0(\tau) + \overline{\chi}_1(\bar{\tau})\chi_1(\tau) \tag{2.6}$$

will be modular invariant. It is also possible that the non-vacuum character has a multiplicity M, corresponding to the fact that a single character corresponds to more than one primary. Such will be the case when there is some discrete symmetry such as complex conjugation that causes the primaries to appear in pairs or larger multiples. In this case, the modular-invariant partition function will be:

$$Z(\tau,\bar{\tau}) = \overline{\chi}_0(\bar{\tau})\chi_0(\tau) + \mathrm{M}\,\overline{\chi}_1(\bar{\tau})\chi_1(\tau) \tag{2.7}$$

It is important to distinguish the degeneracy D from the multiplicity M. The former tells us only that a particular primary is degenerate, the latter says that there are M different primaries, each with degeneracy D, whose character is the same. In the presence of a multiplicity, the matrix S_{ij} will not be unitary but one can enlarge the matrix to make it unitary. Before we actually solve the MLDE and find the admissible pairs of characters, it would be very useful to consider a change of co-ordinates on the moduli space of the torus.

Modular Re-parametrisation

To analyse some general properties of the MLDE (2.1), it would be useful to change the co-ordinates from τ to $j(\tau)$. This changes the domain of the differential equation from the fundamental domain $\Gamma/\mathbb{H}\cup\infty$ to $\mathbb{C}\cup\infty$. If we use θ_x to denote the differential operator $x\frac{d}{dx}$, then the covariant derivatives take the following form in terms of the new variable $j(\tau)$

$$\begin{aligned} D &= \theta_q = -\left(\frac{E_6}{E_4}\right)\theta_j \\ D^2 &= (\theta_q - E_2/6)\theta_q = \left(\frac{E_6}{E_4}\right)^2\left(\theta_j^2 - \frac{2(12)^3 + j}{6(12^3 - j)}\theta_j\right) \end{aligned} \tag{2.8}$$

Both the above equations follow from the Ramanujan identities. Dividing by the overall factor of $\left(\frac{E_6}{E_4}\right)^2$, one sees that the MLDE is mapped to an ordinary differential equation with rational function coefficients in terms of $j(\tau)$. When the zeroes of $f_{2\ell}$ are only at the orbifold points $\tau = i, e^{\frac{2\pi i}{3}} = \rho$, this differential equation has singularities only at $j = 0, 1728, \infty$ corresponding to $\tau = \rho, i, i\infty$ respectively. This equation with only three singular points is just the hypergeometric equation, and the solutions can be expressed as hypergeometric functions of j.

Suppose the characters behave near the points $\tau = \rho, i, i\infty$ where $j \to 0, j \to 1728, j \to \infty$ as:

$$\begin{aligned} \chi_0(\tau) &\sim j^{r_\rho}, \quad \sim (j-1728)^{r_i}, \quad j^{-\alpha_0} \\ \chi_1(\tau) &\sim j^{s_\rho}, \quad \sim (s-1728)^{s_i}, \quad j^{-\alpha_1} \end{aligned} \tag{2.9}$$

where $r_\rho, s_\rho \geq 0$ and $\in \mathbb{Z}/3$, while $r_i, s_i \geq 0$ and $\in \mathbb{Z}/2$. Further defining,

$$\begin{aligned} w_\rho &= r_\rho + s_\rho - \tfrac{1}{3}, \qquad w_i = r_i + s_i - \tfrac{1}{2}, \\ t_\rho &= s_\rho - r_\rho, \qquad t_i = s_i - r_i \end{aligned} \tag{2.10}$$

it can be shown [6] that:

$$\begin{aligned} &\frac{\ell}{6} = w_\rho + w_i, \\ &t_\rho, t_i > 0,\ t_\rho, t_i \notin \mathbb{Z} \end{aligned} \tag{2.11}$$

The characters expressed in terms of j are:

$$\begin{aligned} \chi_0(j) &= j^{\frac{c}{24}} \left(1 - \frac{1728}{j}\right)^{r_i} {}_2F_1\left(\alpha, \beta, \gamma; \frac{1728}{j}\right) \\ \chi_1(j) &= \sqrt{m}\, j^{\frac{c}{24}-h} \left(1 - \frac{1728}{j}\right)^{r_i} {}_2F_1\left(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; \frac{1728}{j}\right) \end{aligned} \tag{2.12}$$

where:

$$\alpha = -\tfrac{1}{2}(h + t_\rho + t_i - 1), \quad \beta = -\tfrac{1}{2}(h - t_\rho + t_i - 1), \quad \gamma = 1 - h \tag{2.13}$$

and

$$m = (1728)^{2h} \frac{\sin\pi\alpha\, \sin\pi\beta}{\sin\pi(\alpha-\gamma)\, \sin\pi(\gamma-\beta)} \left(\frac{\Gamma(\gamma)\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(2-\gamma)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\right)^2 \tag{2.14}$$

This complicated looking formula for the numerical factor m is equal to MD^2, where M and D are the multiplicity and degeneracy of the non-vacuum primary. The formula involves computing the Schwarzian triangle function, and returns an exact integer whenever the solution corresponds to an admissible character.

The case of $\ell = 1, 3, 5$

The case of $\ell = 1$ is trivial since f_2, leading coefficient of the differential equation would vanish in Eq. (2.1), making it a first order equation. When the solutions of a second order MLDE are expanded about the singular points $\tau = \rho, i$, we again need to have two independent solutions.

This imposes $t_\rho, t_i \notin \mathbb{Z}$, which, using (2.11), leads to $w_i \in \mathbb{Z}$ and $\ell \in 2\mathbb{Z}$. This rules out two-character solutions for $\ell = 3,5$ and all higher odd values of ℓ. Instead, at these odd values of ℓ, the one-character theories reappear as solutions to a second order equation. Note that this argument is valid in general, not only when the singularities are at the orbifold points.

The case of $\ell = 0$

In this case, the MLDE takes the form

$$\Big(D^2 + \mu E_4(\tau)\Big)\chi = 0 \tag{2.15}$$

This was first studied in [4], and will be referred to as the MMS equation. All the possible admissible solutions were classified in [4]. The general solution given in terms of hypergeometric functions is

$$\begin{aligned}\chi_0(\tau) &= j^{\frac{c}{24}}\,{}_2F_1\left(-\frac{c}{24}, \frac{1}{3}-\frac{c}{24}; \frac{5}{6}-\frac{c}{12}; \frac{1728}{j}\right)\\ \chi_1(\tau) &= \sqrt{m}\, j^{-\frac{1}{6}-\frac{c}{24}}\,{}_2F_1\left(\frac{1}{6}+\frac{c}{24}, \frac{1}{2}+\frac{c}{24}; \frac{7}{6}+\frac{c}{12}; \frac{1728}{j}\right)\end{aligned} \tag{2.16}$$

where, the central charge c is related to the free parameter μ as

$$\mu = -\frac{c(c+4)}{576} \tag{2.17}$$

From the Riemann-Roch equation, we know that $h = \frac{c}{12} + \frac{1}{6}$. Using the equations above, one can generate the q-expansions and check if the solutions are admissible. For generic rational values of μ, the q-coefficients of the solutions will be arbitrary rational numbers with increasing denominators. Only for very special values of μ, the coefficients will turn out to be positive integers (or positive rationals with a bounded denominator). In this case, exactly ten such values of μ were found which make all the q-coefficients positive leading to admissible characters. Though this procedure of finding the special values of μ is empirical, we can prove that all the q-coefficients are indeed positive to all orders. We list the details of the 10 solutions with their identified theories in table 2.2. Here $a_1^{(0)}$ denotes the number of states at level-1 above the vacuum and D, M denote degeneracy and multiplicities of the non-vacuum primary. The most straightforward ones in this list are entries 2-8, which are WZW models at level-1. Entry 1 is the non-unitary Lee-Yang minimal model (with the roles of vacuum and non-vacuum primaries interchanged) and entry 9

No.	μ	c	h	$a_1^{(0)}$	D	M	Theory
1	$-\frac{11}{3600}$	$\frac{2}{5}$	$\frac{1}{5}$	1	1	1	Lee-Yang
2	$-\frac{5}{576}$	1	$\frac{1}{4}$	3	2	1	$A_{1,1}$
3	$-\frac{1}{48}$	2	$\frac{1}{3}$	8	3	2	$A_{2,1}$
4	$-\frac{119}{3600}$	$\frac{14}{5}$	$\frac{2}{5}$	14	7	1	$G_{2,1}$
5	$-\frac{2}{36}$	4	$\frac{1}{2}$	28	8	3	$D_{4,1}$
6	$-\frac{299}{3600}$	$\frac{26}{5}$	$\frac{3}{5}$	52	26	1	$F_{4,1}$
7	$-\frac{5}{48}$	6	$\frac{2}{3}$	78	27	2	$E_{6,1}$
8	$-\frac{77}{576}$	7	$\frac{3}{4}$	133	56	1	$E_{7,1}$
9	$-\frac{551}{3600}$	$\frac{38}{5}$	$\frac{4}{5}$	190	57	1	$E_{7.5,1}$
10	$-\frac{1}{6}$	8	$\frac{5}{6}$	248	–	0	$E_{8,1}$

Table 2.2: The MMS Series.

corresponds to an "intermediate vertex operator algebra" (IVOA). This notion is inspired from that of an intermediate Lie algebra, of which $E_{7.5}$ is a prime example, lying between E_7 and E_8. Finally, entry 10 is the familiar $E_{8,1}$ one-character theory reappearing as a modular invariant solution for a second order MLDE. The S modular transformation can be computed from the monodromy of the differential equation around the point $\tau = i$, from which the fusion rules can be derived using the Verlinde formula. For the two-character case, exactly four distinct fusion classes are observed: the Lee-Yang (LY) class which includes entries 1,4,6,9, the A_1 class including entries 2,8, the A_2 class including entries 3,7 and the D_4 class including entry 5. Note the pattern between the central charges and conformal dimensions of theories that lie within the same fusion class. In the LY class, the two WZW models based on G_2 and F_4 have positive fusion rules. The Lee-Yang theory itself, presented in this form with $c = \frac{2}{5}$ has negative fusion rules, since the role of vacuum and non-vacuum have been interchanged. When we reverse these roles, we get the familiar non-unitary model of Lee-Yang with $c = -\frac{22}{5}$. Similarly, the fusion rules in the $E_{7.5,1}$ case are also negative. But unlike in the previous case, reversing the roles of the two primaries will make the vacuum 57-fold degenerate. Nevertheless, this theory still has an interpretation in terms of an IVOA [7], and we will use the terminology "IVOA type" for other solutions of MLDE's with this property of having a multiply degenerate vacuum.

The case of $\ell = 2$

The $\ell = 2$ equation written in monic form is given below

$$\left(D^2 + \frac{1}{3}\frac{E_6}{E_4}D + \mu E_4\right)\chi = 0 \tag{2.18}$$

The coefficient $\frac{1}{3}$ in front of the second term was determined from the indicial equation about $\tau = i\infty$. The two solutions in terms of hypergeometric functions are

$$\begin{aligned}\chi_0(\tau) &= j^{\frac{c}{24}}{}_2F_1\left(-\frac{c}{24}, \frac{2}{3} - \frac{c}{24}; \frac{7}{6} - \frac{c}{12}; \frac{1728}{j}\right)\\ \chi_1(\tau) &= \sqrt{m}\, j^{\frac{1}{6}-\frac{c}{24}}{}_2F_1\left(-\frac{1}{6} + \frac{c}{24}, \frac{1}{2} + \frac{c}{24}; \frac{5}{6} + \frac{c}{12}; \frac{1728}{j}\right)\end{aligned} \tag{2.19}$$

where again, the parameter μ is related to the central charge via

$$\mu = -\frac{c(c-4)}{576} \tag{2.20}$$

This time, we have $h = \frac{c}{12} - \frac{1}{6}$. Expanding in a q-series, we search for admissible characters. Exactly 10 solutions were found again in [6, 8], and these are shown in table 2.3.

No.	μ	c	h	m_1	D	M
1	$-\frac{1}{3}$	16	$\frac{7}{6}$	496	–	0
2	$-\frac{1271}{3600}$	$\frac{82}{5}$	$\frac{6}{5}$	410	902	1
3	$-\frac{221}{576}$	17	$\frac{5}{4}$	323	1632	1
4	$-\frac{7}{16}$	18	$\frac{4}{3}$	234	2187	2
5	$-\frac{1739}{3600}$	$\frac{94}{5}$	$\frac{7}{5}$	188	4794	1
6	$-\frac{5}{9}$	20	$\frac{3}{2}$	140	5120	3
7	$-\frac{2279}{3600}$	$\frac{106}{5}$	$\frac{8}{5}$	106	15847	1
8	$-\frac{11}{16}$	22	$\frac{5}{3}$	88	16038	2
9	$-\frac{437}{576}$	23	$\frac{7}{4}$	69	32384	1
10	$-\frac{2891}{3600}$	$\frac{118}{5}$	$\frac{9}{5}$	59	32509	1

Table 2.3: Consistent CFT's with $\ell = 2$.

These admissible characters were identified with actual theories using a novel coset construction in [9]. The entries 3-9 in the table correspond to very special non-diagonal invariants for certain direct-sum Kac-Moody algebras. All of them are cosets of meromorphic theories of [5] by one of the original $\ell = 0$ MMS series, namely those appearing in entries 2-8 of Table 2.2. It can be verified that the entries 3-9 of Table 2.3 pair up (in reverse order) with entries 2-8 of Table 2.2 in such a way that

$$c + \tilde{c} = 24, \qquad h + \tilde{h} = 2 \tag{2.21}$$

where $(c,h),(\tilde{c},\tilde{h})$ are the central charge and conformal dimension in Tables 2.2 and 2.3 respectively. The coset relation also extends to a holomorphic bilinear relation satisfied by the characters

$$\chi_0(\tau)\tilde{\chi}_0(\tau) + \chi_1(\tau)\tilde{\chi}_1(\tau) = j(\tau) - 744 + \mathscr{N} \tag{2.22}$$

where $\mathscr{N}$ is a non-negative integer that varies from case to case, such that $j(\tau) - 744 + \mathscr{N}$ is the character of Schelleken's $c = 24$ meromorphic CFT.

The case of $\ell = 4$

Here we shall complete the classification of all two character theories with $l = 4$ started in [6, 10, 8]. In this case the MLDE is:

$$\left(D^2 + \frac{2}{3}\frac{E_6}{E_4}D + \mu E_4 - 384\frac{\Delta}{E_4^2}\right)\chi = 0 \tag{2.23}$$

The function Δ is the unique weight-12 cusp form equal to $(E_4^3 - E_6^2)/1728$. Again, the coefficient of the second term is determined by the indicial equation (in fact it is always equal to $\frac{\ell}{6}$). The coefficient of the last term is determined by considering the indicial equation around $\tau = \rho$. That leaves one free parameter, μ, related to the central charge by

$$\mu = -\frac{c(c-12)}{576} \tag{2.24}$$

We definitely expect to find some admissible characters, namely products of the form $\chi_{E_8}\chi_i$ where χ_i have $\ell = 0$. These products all have $\ell = 4$ as one can see from the formula for the ℓ-value of the tensor product of two theories with (p, p') characters and ℓ-values (ℓ, ℓ') respectively [8]:

$$\tilde{\ell} = \frac{1}{2}pp'(p-1)(p'-1) + \ell' p + \ell p' \tag{2.25}$$

The question is whether there are any additional admissible characters that are not of tensor-product form. The value $l = 4$ corresponds to the Wronskian having a double zero at the orbifold point $\rho = e^{\frac{2\pi i}{3}}$, and no zeros elsewhere in the moduli space. The characters written as functions of $j(\tau)$ take the following form (up to normalisation):

$$\begin{aligned}\chi_0(\tau) &= j^{\frac{c}{24}}\,{}_2F_1\left(\frac{1}{3}-\frac{c}{24},\frac{2}{3}-\frac{c}{24};\frac{3}{2}-\frac{c}{12};\frac{1728}{j}\right)\\ \chi_1(\tau) &= j^{\frac{1}{2}-\frac{c}{24}}\,{}_2F_1\left(-\frac{1}{6}+\frac{c}{24},\frac{1}{6}+\frac{c}{24};\frac{1}{2}+\frac{c}{12};\frac{1728}{j}\right)\end{aligned} \tag{2.26}$$

We have chosen c as the free parameter above, and from the Riemann-Roch formula we know that $h = \frac{c}{12} - \frac{1}{2}$. We will elaborate the procedure here in a little more detail. Doing a q-expansion of the above characters, we can derive the following expression

$$a_1^{(0)} = \frac{-5c^2+306c-4608}{c-18} \tag{2.27}$$

which gives:

$$(5c)^2 + 5c(a_1^{(0)} - 306) = 90(a_1^{(0)} - 256) \tag{2.28}$$

Thus, we see that $5c$ is an integer, such that the discriminant of the above quadratic equation is a perfect square. This restricts c to 34 rational values. Next we calculate $a_i^{(0)}$ for $i \geq 2$ in terms of c. For $i = 2$ we have:

$$a_2^{(0)} = \frac{25c^4 - 3105c^3 + 117324c^2 - 1355292c + 6635520}{2(c-30)(c-18)} \tag{2.29}$$

Requiring that $a_i^{(0)}$ for $i \geq 2$ be a non-negative integer further cuts down the list. This leaves us with 15 possibilities, shown in Table 2.4.

As indicated, entries No. 3-12 in the table are the expected tensor products. Hence we may focus our attention on the remaining five cases. Entry No. 2 is a re-appearance of the one-character theory for $E_{8,1}$. Entry No. 1 has $h = 0$ and corresponds to a logarithmic solution of the differential equation. This happens whenever h is an integer, which means $c = 12m + 6$, $m \in \mathbb{Z}$ using the Riemann-Roch formula for $\ell = 4$. Similar logarithmic solutions also appear in the $\ell = 0$ case for $c = 2(6m+5)$ and in the $\ell = 2$ case for $c = 2(6m+1)$, as can be seen by the formulae below Eq. (2.17) and Eq. (2.20).

This leaves, as the interesting cases, entries No. 13-15 which have been labelled "A". All three were missed in [6], while No. 13 and 14 (but not 15) appeared implicitly in [10], and No. 14 and

No.	c	h	$a_1^{(0)}$	D	M	Remarks
1	6	0	246	1		logarithmic
2	8	$\frac{1}{6}$	248	1	0	one-character $E_{8,1}$
3	$\frac{42}{5}$	$\frac{1}{5}$	249	1	1	$E_{8,1} \otimes YL$
4	9	$\frac{1}{4}$	251	2	1	$E_{8,1} \otimes A_{1,1}$
5	10	$\frac{1}{3}$	256	3	2	$E_{8,1} \otimes A_{2,1}$
6	$\frac{54}{5}$	$\frac{2}{5}$	262	7	1	$E_{8,1} \otimes G_{2,1}$
7	12	$\frac{1}{2}$	276	8	3	$E_{8,1} \otimes D_{4,1}$
8	$\frac{66}{5}$	$\frac{3}{5}$	300	26	1	$E_{8,1} \otimes F_{4,1}$
9	14	$\frac{2}{3}$	326	27	2	$E_{8,1} \otimes E_{6,1}$
10	15	$\frac{3}{4}$	344	56	1	$E_{8,1} \otimes E_{7,1}$
11	$\frac{78}{5}$	$\frac{4}{5}$	438	57	1	$E_{8,1} \otimes E_{7.5,1}$
12	16	$\frac{5}{6}$	496	1	0	one-character $E_{8,1} \otimes E_{8,1}$
13	$\frac{162}{5}$	$\frac{11}{5}$	4	310124	1	A
14	33	$\frac{9}{4}$	3	565760	1	A
15	34	$\frac{7}{3}$	1	767637	2	A

Table 2.4: Potentially consistent CFT's with $\ell = 4$.

15 (but not 13) have appeared in the mathematics literature [11]. No. 13 with $c = \frac{162}{5}$ is in the same fusion-rule category as the Lee-Yang theory and hence, in the given unitary presentation, has negative fusion rules. Upon reversing the roles of the characters, one finds a non-unitary theory with a 310124-fold degenerate identity character, thus in our language it is a candidate theory of IVOA type. We claim that Nos. 13-15 form the complete set of admissible $\ell = 4$ characters that are not tensor products of other admissible characters.

Using the same strategy to study MLDE's with $\ell \geq 6$ becomes increasingly difficult. In the following chapter, we will explore a novel approach to classify solutions for the cases corresponding to $\ell \geq 6$.

Chapter 3

Quasi-characters

In the previous chapter we obtained a complete classification for the two-character cases corresponding to $\ell < 6$. While searching for admissible characters, we only retained solutions whose q-coefficients were all positive integers and rejected solutions if they had any negative coefficients. However, among these rejected set of solutions, there was a very interesting subclass of solutions which will help us in the classification for the $\ell \geq 6$ case. These were sets of characters which have one or more negative coefficients, but whcih are still all integers or rationals with a bounded denominator. In the following, we define such solutions and give a complete list of them in the two-character case based on previous work by [12, 13] and new coset dual generalisations by us.

Definition: Quasi-characters are solutions to an MLDE whose q-coefficients are all integers or rational numbers with a bounded denominator which are not necessarily positive. Quasi-characters include admissible characters as a special case. While there are only finitely many admissible character solutions (as given in tables 2.2, 2.3, 2.4), there are infinitely many solutions once we drop the restrictive criterion that *all* q-coefficients need to be negative. In other words there are infinitely many quasi-characters. We find that quasi-characters come in two distinct types, denoted as type I and type II. In the case of type I there are a finite number of negative integer coefficients in the identity character after normalising the ground state to be positive. One encounters all of the negative coefficients below a certain level, above which all the coefficients are positive integers. Meanwhile the other character has all positive coefficients. In the case of type II there are a finite number of positive coefficients in the identity character when the ground state is again normalised to be positive. If we denote the identity character as:

$$\chi_0 = q^{-\frac{c}{24}}(a_0 + a_1 q + \cdots a_n q^n + \cdots) \tag{3.1}$$

with $a_0 > 0$, then for the type I some of $a_1, a_2, \cdots a_n$ are ≤ 0, while all of the $a_m > 0$ for $m > n$. For type II some of $a_1, a_2, \cdots a_n \geq 0$ while all of the $a_m < 0$ for $m > n$. The second character is entirely non-negative in both the cases. We did not find any examples with a random or alternating collection of plus and minus signs all the way to infinity (which would have not come under either type I or II). We also find that negative signs occur only in the vacuum character while the other character has a completely non-negative q-series. The number of negative signs in the case of type I (and postive signs in the case of type II) quais-characters was proportional to the central charge parameter c of the solution. After discussing the modular transformations for quasi-characters, we will prove these results about asymptotic positivity or negativity of q-coefficients. Let us first look at a parametrisation of the MLDE introduced in [14] which will be useful to enlist the quasi-characters.

1 Kaneko-Zagier parametrisation

$\ell = 0$ **case**

In 1998, Kaneko and Zagier [14] studied a variant of the MMS differential equation in the context of supersingular elliptic curves and their invariants. Their equation was for weight-k (rather than weight-0) modular forms, and took the form:

$$\left(D_{(k)}^2 - \frac{k(k+2)}{144}E_4(\tau)\right) f_{(k)}(\tau) = 0 \tag{3.1}$$

This is easily transformed to MMS form by the substitution:

$$f_{(k)}(\tau) = \eta(\tau)^{2k}\chi(\tau) \tag{3.2}$$

This results in:

$$\left(D^2 - \frac{k(k+2)}{144}E_4(\tau)\right)\chi(\tau) = 0 \tag{3.3}$$

which, as we see, is the MMS equation Eq. (2.15) with

$$\mu = -\frac{k(k+2)}{144} \tag{3.4}$$

Since we had previously noted that $\mu = -\frac{c(c+4)}{576}$, it follows that $c = 2k$. Solutions of Eq. (3.1) were first studied for integral and half-integral weights and then later for another fractional weight.

Kaneko and Koike [12] considered the above equation for the following sets of values: $k = 3n + \frac{1}{2}$, $k = 2n+1, n \neq 2 \bmod 3$, and $k = 4n+2$. Thereafter, Kaneko [13], considered $k = \frac{6n+1}{5}, n \neq 4 \bmod 5$. These sets will turn out to be of crucial importance below, and will correspond to different fusion classes:

$$\begin{aligned}
\text{Lee-Yang}: \ k &= \frac{6n+1}{5},\ n \neq 4 \bmod 5 \quad \rightarrow \quad k = \frac{1}{5}, \frac{7}{5}, \frac{13}{5}, \frac{19}{5} \\
A_1: \ k &= 3n + \frac{1}{2} \quad \rightarrow \quad k = \frac{1}{2}, \frac{7}{2} \\
A_2: \ k &= 2n+1,\ n \neq 2 \bmod 3 \quad \rightarrow \quad k = 1, 3 \\
D_4: \ k &= 6n+2 \quad \rightarrow \quad k = 2
\end{aligned} \tag{3.5}$$

The values of k that are being excluded, for example $k \neq -1, 5$, are exactly those at which we expect logarithmic solutions (as discussed towards the end of the previous chapter). In the range $-1 < k < 5$ this is a complete set of k values for admissible characters in each of the four categories. Each case labels a distinct fusion-rule class. For values of k outside this range, the above series do not give admissible RCFT characters, but remarkably they all turn out to provide quasi-characters.

$\ell = 2$ case: "dual" parametrisation

While the Kaneko-Zagier equation leads, via Eq. (3.2), to $\ell = 0$ quasi-characters, an analogous equation leading to $\ell = 2$ quasi-characters does not appear to have been studied in the literature. In fact the desired equation is:

$$\left(D_{(k)}^2 + \frac{1}{3}\frac{E_6}{E_4} D_{(k)} - \frac{k(k-2)}{144} E_4(\tau) \right) f_{(k)}(\tau) = 0 \tag{3.6}$$

We will refer to this as the "dual Kaneko-Zagier equation". Scaling the solutions as in Eq. (3.2) to obtain weight-0 (quasi-)characters, one gets the equation:

$$\left(D^2 + \frac{1}{3}\frac{E_6}{E_4} D - \frac{k(k-2)}{144} \right) \chi = 0 \tag{3.7}$$

which is a special subfamily of Eq. (2.18) obtained by setting $\mu = -\frac{k(k-2)}{144}$. We see again that the associated central charge is $c = 2k$. We study the solutions of Eq. (3.7) for the following ranges of the parameter k, which will again correspond to different fusion classes:

$$\begin{aligned}
\text{Lee-Yang}: \; & k = \frac{6n-1}{5}, \; n \neq 1 \bmod 5 \;\; \rightarrow \;\; k = \frac{41}{5}, \frac{47}{5}, \frac{53}{5}, \frac{59}{5} \\
A_1: \; & k = 3n - \frac{1}{2} \;\; \rightarrow \;\; k = \frac{17}{2}, \frac{23}{2} \\
A_2: \; & k = 2n-1, \; n \neq 1 \bmod 3 \;\; \rightarrow \;\; k = 9, 11 \\
D_4: \; & k = 6n-2 \;\; \rightarrow \;\; k = 10
\end{aligned} \tag{3.8}$$

The values of k that are being excluded give logarithmic solutions, for example at $k = 7, 13$. As in the $\ell = 0$ case, the k values listed above are a complete set for the given four series in a particular range, namely $7 < k < 13$. If now we allow k to take values outside this range, then we find that the corresponding solutions to Eq. (3.7) are quasi-characters in every case.

Thus, the complete set of quasi-characters both for the $\ell = 0$ and $\ell = 2$ cases can be summarised in the table below.

Class	c	h	n values	Class	c	h	n values
LY	$\frac{2}{5}(6n+1)$	$\frac{n+1}{5}$	$n \neq 4 \bmod 5$	$\widetilde{\text{LY}}$	$\frac{2}{5}(6n-1)$	$\frac{n-1}{5}$	$n \neq 1 \bmod 5$
A_1	$6n+1$	$\frac{2n+1}{4}$	$n \in \mathbb{N}$	$\tilde{A}_1$	$6n-1$	$\frac{2n-1}{4}$	$n \in \mathbb{N}$
A_2	$4n+2$	$\frac{n+1}{3}$	$n \neq 2 \bmod 3$	$\tilde{A}_2$	$4n-2$	$\frac{n-1}{3}$	$n \neq 1 \bmod 3$
D_4	$12n+4$	$\frac{2n+1}{2}$	$n \in \mathbb{N}$	$\tilde{D}_4$	$12n-4$	$\frac{2n-1}{2}$	$n \in \mathbb{N}$

Table 3.1: Quasi-characters with $\ell = 0$, $\ell = 2$.

The fact that these are indeed quasi-characters whenever k takes the form given in table 3.1 was proved in [15]. Using results of [12, 13], we showed that all the quasi-characters are in fact just polynomials of admissible characters, proving that they have integer q-coefficients. The results can also be verified by computing explicit solutions as q-series, several examples of which can be found in the appendix B of [15], and also a few examples in the following chapter. Note that the explicit formulae in terms of hypergeometric functions for characters in Eq. (2.16), Eq. (2.19) continue to be valid for quasi-characters.

Note that the fusion-rule classification [16, 17] for two-character theories admits conformal dimensions which can only be multiples of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$. Now the range of k considered in the various series above exhausts all possible such values, making this a complete list of quasi-characters. Finally we note that an infinite set of $\ell = 4$ quasi-characters can be obtained by simply multiplying those for $\ell = 0$ by $j^{\frac{1}{3}}$.

2 Modular transformations for quasi-characters

An important operation that we will consider in the following chapter is the addition of quasi-characters. When suitably done, this can remove negative signs and convert quasi-characters into admissible characters. However, the process of addition must commute with modular transformations. For this to be true, the different quasi-characters being added must all have the same modular S and T matrices. These have been computed for admissible characters in [6] in terms of trigonometric functions, and in [18] in terms of Γ-functions. Because the solutions of the $\ell = 0$ differential equation have universal expressions in terms of hypergeometric functions that depend only on k, the modular transformation matrices likewise can be expressed for all cases as a function of k by simply extending these formulae to all k. The result can be written:

$$S = \begin{pmatrix} \frac{1}{\sin\frac{\pi(k+1)}{6}} & \left(\mathrm{M}\frac{1-4\sin^2\frac{\pi(k+1)}{6}}{4\sin^2\frac{\pi(k+1)}{6}}\right)^{\frac{1}{2}} \\ \left(\frac{1}{\mathrm{M}}\frac{1-4\sin^2\frac{\pi(k+1)}{6}}{4\sin^2\frac{\pi(k+1)}{6}}\right)^{\frac{1}{2}} & -\frac{1}{\sin\frac{\pi(k+1)}{6}} \end{pmatrix} \tag{3.1}$$

Here M is the multiplicity of the non-trivial primary. We see that the result is periodic under the shift $k \to k+12$, but not any smaller shift. This formula has a nice property under the inversion $k \to -2-k$. This shift preserves the Kaneko-Zagier parametrisation and is known to interchange the characters. One can show that under this shift, the above formula transforms by an exchange of the two rows and the two columns.

As an explicit example, for the Lee-Yang class of quasi-characters one finds:

$$\begin{aligned}
n = 1,2 \bmod 10: \;\; S &= \begin{pmatrix} \sqrt{\frac{2}{5+\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \\ \sqrt{\frac{2}{5-\sqrt{5}}} & -\sqrt{\frac{2}{5+\sqrt{5}}} \end{pmatrix} \\
n = 0,3 \bmod 10: \;\; S &= \begin{pmatrix} \sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}} \\ \sqrt{\frac{2}{5+\sqrt{5}}} & -\sqrt{\frac{2}{5-\sqrt{5}}} \end{pmatrix} \\
n = 6,7 \bmod 10: \;\; S &= \begin{pmatrix} -\sqrt{\frac{2}{5+\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \\ \sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}} \end{pmatrix} \\
n = 5,8 \bmod 10: \;\; S &= \begin{pmatrix} -\sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}} \\ \sqrt{\frac{2}{5+\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \end{pmatrix}
\end{aligned} \tag{3.2}$$

The shift $k \to -k-2$ discussed above acts on the label n of this case as $n \to -2-n$ and we can see that the expected pattern holds. For example if we consider $n = -10$, this will have an S given by the second line in the above equation. Under inversion it becomes $n = 8$ which corresponds to the fourth line. Now if we take the matrix in the fourth line and exchange the role of identity and non-identity character (which is done by exchanging the rows and then exchanging the columns) then we get back the same matrix as in the second row.

An important application of the above modular S matrix is to determine the asymptotic behaviour of the q-coefficients in quasi-characters. This can be achieved using a generalised Rademacher series for Fourier coefficients of vector-valued modular forms. Detailed explanations and proofs, as well as original references, can be found for example in [19]. For a vector-valued modular function, $\chi_i = q^{\alpha_i} \sum_{n=0}^{\infty} a_i(n) q^n$, the coefficients are given by

$$a_j(n) = \sum_{i=0,1} \sum_{m+\alpha_i<0} \mathscr{K}_{n,j;m,i}\, a_i(m) \tag{3.3}$$

The sum on the right side involves coefficients only of the singular (or polar) part. The infinite $\times$ finite matrix $\mathscr{K}_{n,j;m,i}$ encodes the modular properties of χ_i. The above series can be written using generalised Kloosterman sums. In our case of weight zero modular functions, this reduces to:

$$\mathscr{K}_{n,j;m,i} = (2\pi)^2 \sum_{0\le -D/C\le 1} C^{-2} e^{\frac{2\pi i}{C}\left[(n+\alpha_j)D+(m+\alpha_i)A\right]} M(\gamma)^{-1}_{ji} |m+\alpha_i|\, \tilde{I}_1\left[\frac{4\pi}{C}\sqrt{|m+\alpha_i|(n+\alpha_j)}\right] \tag{3.4}$$

Here the summation is over all coprime numbers C, D and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{Z})$. The matrix $M(\gamma)$ is the modular transformation of χ_i corresponding to γ and $\tilde{I}_1$ is a modified Bessel function. Since we want to analyse the asymptotic behaviour of $a_j(n)$, we can consider just the leading term in the series corresponding to $C = 1, D = 0$, and also approximate $\tilde{I}_1$ with its asymptotic expansion. In this case, $M(\gamma)$ will just be the S-transformation matrix of χ_i. Now, the above expression reduces to

$$a_j(n) = \sum_{i=0,1} \sum_{m+\alpha_i<0} \left(S^{-1}_{ji}\, e^{4\pi\sqrt{|m+\alpha_i|(n+\alpha_j)}} + \cdots \right) a_i(m) \tag{3.5}$$

For two character theories only the vacuum character is singular, having $m+\alpha_0 < 0$, while the other character always has $m+\alpha_1 > 0$ (note that m, being the argument of the q-coefficient $a_i(m)$, starts from 0). Thus, the leading behaviour in the sum arises from the $i = 0$ term and when $m+\alpha_0 < 0$

is most negative, i.e, $m=0$ and we have:

$$a_j(n)=S^{-1}_{j0}\,a_0(0)\,e^{4\pi\sqrt{\frac{c}{24}(n+\alpha_j)}}+\cdots \tag{3.6}$$

where we have replaced α_0 with $-\frac{c}{24}$. In the asymptotic large n limit all the corrections are subleading, and thus do not affect the sign of $a_j(n)$. Thus the asymptotic sign of the coefficients is determined by the sign of $S^{-1}_{j0}\,a_0(0)$. From this, we can immediately read off several facts about quasi-characters. Since $a_0(0)$ is always normalised to be positive, we only need to check the sign of S^{-1}_{j0}, i.e, the entries of the first column in modular S matrix. The type I or type II behaviours, in which the coefficients have asymptotically positive or negative signs respectively is determined by the sign of S^{-1}_{00}. From Eq. (3.1) we find:

$$\begin{aligned}
0<\frac{k+1}{6}<1 \quad \text{mod } 2: \ \text{type I}\\
1<\frac{k+1}{6}<2 \quad \text{mod } 2: \ \text{type II}
\end{aligned} \tag{3.7}$$

For example, in the Lee-Yang class, type I would occur for $n=0,1,2,3$ and type II for $n=5,6,7,8$. Also, since S^{-1}_{10} is always positive, the non-vacuum character will always be asymptotically positive.

3 Coset relations among quasi-characters

If we compare the quasi-characters for $\ell=0$ and $\ell=2$, we find that they are related as follows. Consider the former at some value of k and the latter at a value of $\tilde{k}$ such that $k+\tilde{k}=12$. From Riemann-Roch it also follows that the conformal dimensions $h,\tilde{h}$ associated to the two sets satisfy $h+\tilde{h}=2$. Using $c=2k$, these are the same coset relations obeyed by the admissible characters at $\ell=0$ and $\ell=2$, as shown in [9]. Following the previous discussion, it should not be surprising that this coset (or more properly, bilinear) relation continues to hold for quasi-characters. Combining Eqs.(2.16,2.19) it can be shown that:

$$\chi_0^k\tilde{\chi}_0^{\tilde{k}}+\chi_1^k\tilde{\chi}_1^{\tilde{k}}=j(q)+f_k \tag{3.1}$$

where we have denoted the $\ell=2$ quasi-characters as $\tilde{\chi}_i$, and $f_k=-\frac{1728(k-4)}{2(k-5)}$. The above equation is basically the same as Eq.(4.10) of [9].

It is pleasing to see that this relation, once thought to hold only for a finite set of admissible characters, holds for an infinite set of quasi-characters. One should not be surprised to see that the additive constant is, in general, a fraction. The reason is that the characters listed in Eqs.(2.16,2.19) are normalised so that their q-series starts with 1. However, with this normalisation, quasi-characters are not always integral, but have bounded denominators. Thus upon multiplication with a suitable integer they become integral, but now must be thought of as being in the IVOA category. It is clear that the same normalisation that makes each quasi-character integral will also make the constant on the RHS of Eq. (3.1) integral.

Chapter 4

Generating characters for arbitrary $\ell \geq 6$

In this chapter we will use quasi-characters to generate admissible characters for all $\ell \geq 6$. Though quasi-characters are themselves inadmissible due the presence of negative integer coefficients, we will demonstrate two methods to eliminate these negative coefficients – the multiplicative and additive methods. Out of these two, we will see that the additive method gives us a complete set of solutions for all $\ell \geq 6$ MLDE's.

1 Multiplicative method

The first observation is that products of quasi characters of type I (having a finite number of negative coefficients) with a sufficiently high degree polynomial in $\chi_{E_{8,1}} = j^{\frac{1}{3}}$ can eliminate all of the negative signs in the former (this was correctly anticipated in [10]). In this process, the vector-valued modular transformations of the quasi-characters are preserved up to a phase, since $j^{\frac{1}{3}}$ is modular invariant up to a phase all by itself. The resulting characters, if shown to correspond to a consistent RCFT, will correspond to new theories and not tensor products of old ones. The simple reason is that the original quasi-character, being of type I, has a certain number of negative coefficients in its q-expansion and does not qualify to be a CFT – only its product with a power of $j^{\frac{1}{3}}$ qualifies.

It is easy to verify that if we tensor $j^{\frac{r}{3}}$ with a pair of quasi-characters having a particular value of ℓ, the result has:

$$\ell' = \ell + 4r \tag{4.1}$$

In many cases the vacuum of the quasi character is degenerate, hence the resulting set of new characters will be of "IVOA type".

The first non-trivial example of this kind is provided by the three admissible characters that we found with $\ell = 4$ in Subsection 3. These correspond to the following values of c, h:

$$(c,h) = (\tfrac{162}{5}, \tfrac{11}{5}),\ (33, \tfrac{9}{4}),\ (34, \tfrac{7}{3}) \tag{4.2}$$

It is easy to verify that each of these character pairs corresponds to the product of $\chi_{E_{8,1}} = j^{\frac{1}{3}}$ with the $\ell = 0$ quasi-characters having:

$$k = \tfrac{61}{5}, \tfrac{25}{2}, 13 \tag{4.3}$$

in Eq. (3.3). Notice that the h-values are preserved in the process: the above quasi-characters and the final admissible characters both have $h = \frac{11}{5}, \frac{9}{4}, \frac{7}{3}$. Also note that the k-values in Eq. (4.3) belong to the series of quasi-characters $k = \frac{6n+1}{5}, n = 10; k = 3n + \frac{1}{2}, n = 4; k = 2n+1, n = 6$ respectively, which fall in the Lee-Yang, A_1 and A_2 classes respectively. Each of these has only a single negative coefficient $a_1^{(0)}$ at first level above the vacuum state in the identity character. Multiplication by $j^{\frac{1}{3}} \sim q^{-\frac{1}{3}}(1 + 248q + \cdots)$ will add 248 to this coefficient, so if $a_1^{(0)} \geq -248$ then the final character after multiplication will have all positive integral coefficients and thereby become admissible. Inspection of the three examples reveals that their values of $a_1^{(0)}$ are -244, -245 and -247 respectively, satisfying the requirement. Within the space of $\ell = 0$ quasi-characters, it is easy to verify that these are the only ones satisfying $a_1^{(0)} \geq -248$ which explains why there are precisely three new admissible characters with $\ell = 4$.

In general, whenever we start with quasi-characters at $\ell = 0$ we find that the products made by tensoring with $j^{\frac{r}{3}}$ have $\ell = 4r$. If we start with dual quasi-characters at $\ell = 2$, we will find theories with $\ell = 4r + 2$. This exhausts all even values of ℓ.

2 Additive method

In this section we discuss how linear combinations of type I quasi-characters can be used to create admissible characters with higher values of ℓ. In this approach, we add different q-series, each corresponding to a quasi-character, such that the sum is an admissible character with only non-negative q-coefficients. Of course, arbitrary sums of characters destroy modular invariance so these sums need to be taken carefully. To preserve modular invariance we can only add two characters which transform in the same way under modular transformations. Moreover we should use rational coefficients when adding them, and if necessary normalise the result to integers. As we now show, the resulting theory will have the ℓ-value increased in multiples of 6. Moreover we will show in the following section that this process leads to the *complete* set of admissible characters for all ℓ.

Let us start with some illuminating examples. Take the $\ell = 0$ quasi-characters in the Lee-Yang family corresponding to $n = 0$ and 10. From Eq. (3.2) these have the same modular transformations and so the result will also transform in the same way. Introducing an integer N_1, we consider the family of characters:

$$\chi_i^{n=10} + N_1 \chi_i^{n=0} \tag{4.1}$$

For the identity character $i = 0$, the q-expansion of this sum is:

$$\begin{aligned} & q^{-\frac{61}{60}}(1 - 244q + 169641q^2 + 19869896q^3 + \cdots) + N_1 q^{-\frac{1}{60}}(1 + q + q^2 + \cdots) \\ =& q^{-\frac{61}{60}}(1 + (N_1 - 244)q + (N_1 + 169641)q^2 + (N_1 + 19869896)q^3 + \cdots) \end{aligned} \tag{4.2}$$

while for $i = 1$ (the non-identity character) the q-expansion is:

$$\begin{aligned} & q^{\frac{71}{60}}(310124 + 27523505q + 1012864984q^2 + \cdots) + N_1 q^{\frac{11}{60}}(1 + q^2 + q^3 + \cdots) \\ =& q^{\frac{11}{60}}(N_1 + 310124q + (N_1 + 27523505)q^2 + (N_1 + 1012864984)q^3 + \cdots) \end{aligned} \tag{4.3}$$

Now if we choose $N_1 \geq 244$, we have eliminated all negative signs and the resulting characters are admissible. To find out more about the potential CFT that they could describe, notice that for the identity character, the leading exponent is that of the quasi-character of higher central charge ($n = 10$ in this example) while for the non-identity character the situation is reversed: the exponent is that of the quasi-character of lower central charge ($n = 0$ in this example). It follows that the CFT would have $c = \frac{122}{5}$ and $h = \frac{6}{5}$, from which we find that $\ell = 6$. Thus by simply adding two quasi-characters with $\ell = 0$, we have found an infinite set of admissible characters with $\ell = 6$, one for each integer $N_1 \geq 244$. While we do not necessarily expect there to be an RCFT for each of these cases, this achieves the goal of generating large classes of admissible characters.

The next example is striking because, for some choices, it allows us to convert a quasi-character with a degenerate identity field (what we called IVOA type) to an admissible character with a non-degenerate identity. We add quasi-characters, again for the Lee-Yang series, but for the values $n = 11$ and $n = 1$. The result, for the identity character, is:

$$\begin{aligned} \chi_0 = q^{-\frac{67}{60}}\big(& 7 + (-1742 + N_1)q + (722729 + 14N_1)q^2 + (133716590 + 42N_1)q^3 \\ & + (7374239425 + 140N_1)q^4 + (220691372762 + 350N_1)q^5 \\ & + (4460548657432 + 840N_1)q^6 + (68133599246580 + 1827N_1)q^7 + \cdots\big) \end{aligned} \tag{4.4}$$

Like the previous case, this one again has $\ell = 6$. In this case the original identity character for $n = 11$ was of IVOA type, and in fact 7-fold degenerate as we see from the 7 multiplying the leading term. This means the higher degeneracies were not divisible by 7, if they had been so then we could have normalised the character to have a non-degenerate vacuum state. However after adding characters as above, a miracle takes place when $N_1 = 1742$, its lowest allowed value. In this case the first level degeneracy above the identity vanishes, but all the higher degeneracies become multiples of 7. Thus we find:

$$\begin{aligned}\chi_0 = 7q^{-\frac{67}{60}}(&1 + 106731q^2 + 19112822q^3 + 1053497615q^4 \\ &+ 31527426066q^5 + 637221445816q^6 + 9733371775602q^7 + \cdots)\end{aligned} \tag{4.5}$$

We are now in a position to drop the leading 7 and find an admissible character with a non-degenerate vacuum. We see the encouraging fact that quasi-characters of IVOA type (which are the generic type) can lead to regular non-degenerate admissible characters upon being added to each other. By examining numerous examples we have found that this generically seems to happen at least for the minimal allowed value of the integer constant.

The above example shows that one should in general consider rational, rather than integer, linear combinations of quasi-characters. For example to achieve the correctly normalised admissible character in the above equation (after the overall 7 has been dropped) one would need to add $\frac{1}{7}$ of one quasi-character to $\frac{N_1}{7}$ times the other one.

In general one can take sums (with rational coefficients) of any number of quasi-characters that all have the same modular transformations and, for suitable choices of the coefficients, generate large sets of admissible characters. The question is then, what is the ℓ-value of the result. We can provide a general formula for this. Consider a set of $\ell = 0$ quasi-characters all lying in the same class (but not necessarily the Lee-Yang class). Label them by the parameter k. As we have seen, the shift $k \to k + 12$ leaves the modular transformations invariant. Thus, we may consider sums of the form:

$$\sum_{p=0}^{p_{\max}} N_p\, \chi_i^{k-12p}, \quad N_0 = 1 \tag{4.6}$$

Assuming $N_{p_{\max}} \neq 0$, the critical exponents of the resulting characters are:

$$\alpha_0 = -\frac{k}{12}, \quad \alpha_1 = \frac{k}{12} + \frac{1}{6} - p_{\max} \tag{4.7}$$

from which it follows that $\ell = 6p_{\max}$. This agrees with our previous examples where $p_{\max}$ was 1 and we found $\ell = 6$. Thus we see that it is no problem to generate infinite sets of admissible characters with arbitrarily large values of ℓ just by adding a number of quasi-characters, and choosing the integers N_p to ensure that all minus signs are removed.

The above was for $\ell = 0$ quasi-characters. Had we instead started with $\ell = 2$ quasi-characters, we would end up with $\ell = 6p_{\text{max}} + 2$. Finally, we have seen that all $\ell = 4$ quasi-characters are products of $j^{\frac{1}{3}}$ with $\ell = 0$ quasi-characters. By adding these to each other as above, we can generate admissible characters with $\ell = 6p_{\text{max}} + 4$. Thus we have shown how to generate infinitely many admissible characters for all even values of ℓ.

The specific examples considered so far involved only Type I quasi-characters with a single negative coefficient. However the procedure works equally well with more negative coefficients. For example one can consider the quasi-character in the A_1 case at $\ell = 0$ and $n = 12$, which has three negative coefficients in front of q, q^3 and q^5. On taking an arbitrary linear combination of this with the $n = 8$ and $n = 4$ quasi-characters as well as the well-known $n = 0$ characters (which correspond to the $SU_{2,1}$ WZW theory), we find that the identity character goes as:

$$\begin{aligned} q^{-\frac{73}{24}} \Big(& 119 + (-53363 + 13N_1)q + (14459256 - 4361N_1 + N_2)q^2 \\ & + (-3364790387 + 1024492N_1 - 245N_2 + N_3)q^3 \\ & + (842188593869 - 284433485N_1 + 142640N_2 + 3N_3)q^4 \\ & + (-303881533638137 + 296843797565N_1 + 18615395N_2 + 4N_3)q^5 \\ & + (461207383305660887 + 84306237909803N_1 + 837384535N_2 + 7N_3)q^6 + \cdots \Big) \end{aligned} \tag{4.8}$$

Notice that, because of the addition process, there are now minus signs at all orders from q to q^5. However for suitable choices of the integers N_1, N_2, N_3 one can easily ensure that all these terms become non-negative. The subsequent terms in the above character from $\mathcal{O}(q^6)$ onwards are all positive linear combinations of the N_i, so they will remain positive if we choose all N_i non-negative, and it is clear that this allows for infinitely many choices. It may further be possible to choose the N_i to be rational and even negative, yet obtaining admissible characters after taking the sum. But our aim here is only to show that there are infinitely many solutions to the requirement of admissibility, and that they are easily constructed.

In the previous subsection we saw how to get admissible characters by multiplying quasi-characters by $j^{\frac{r}{3}}$. One may wonder whether this approach is exhaustive, generating all admissible characters with $\ell \geq 4$. In such a situation, potentially that method would yield identical results to the one explained in this subsection. However it is easy to see that this is not the case. For example, starting from a given quasi-character with $\ell = 2$, we can get a single (potentially admissible) character at $\ell = 6$ upon multiplying by $j^{\frac{1}{3}}$. However the methods of the present section allow for infinitely many characters, all with the same central charge and conformal dimension, at $\ell = 6$. Thus the method of adding quasi-characters is more powerful.

Nonetheless the method of the previous subsection is essential. If we are interested in admissible characters with $\ell = 4 \bmod 6$ (for example $\ell = 10$) we can only get them by the addition method if we start with $\ell = 4$ quasi-characters. But these, in turn, can be generated by multiplying $\ell = 0$ quasi-characters by $j^{\frac{1}{3}}$. Thus it appears that the "seed" quasi-characters that could potentially generate all admissible characters, are those which we have described based on the Kaneko-Zagier parametrisation and its dual ($\ell = 0$ and $\ell = 2$ respectively) as well as $j^{\frac{1}{3}}$ times the $\ell = 0$ quasi-characters. Just using these three sets, one can generate infinitely many admissible characters for all even $\ell \geq 6$ by adding quasi-characters. The remarkable thing is that this process generates the complete set of admissible characters for all allowed (i.e. even) ℓ. We will prove this below.

3 Completeness of the additive method

In this subsection we show that by adding suitable $\ell = 0$ Type I quasi-characters to each other with chosen rational coefficients, one can generate *every* admissible character with $\ell = 6m$ for all positive integers m. Our strategy will be to work the other way: if we are given a pair of admissible characters with $\ell = 6m$, we will show that one can add quasi-characters with suitably chosen rational coefficients in such a way as to *reduce* the ℓ-value to $6m-1$[1]. Repeating sequentially, one is able to reduce the given pair to a linear combination of $\ell = 0$ quasi-characters.

Thus let us start by considering a pair of admissible characters, assumed to be given, having $\ell = 6m$ and exponents $\alpha_0 = -\frac{c}{24}, \alpha_1 = -\frac{c}{24} + h$:

$$\alpha_0 + \alpha_1 = \frac{1-\ell}{6} = \frac{1}{6} - m \tag{4.1}$$

Hence they have expansions of the form:

$$\begin{aligned} \chi_0 &= q^{\alpha_0}(a_0^0 + a_1^0 q + a_2^0 q^2 + \cdots) \\ \chi_1 &= q^{\alpha_1}(a_0^1 + a_1^1 q + a_2^1 q^2 + \cdots) \end{aligned} \tag{4.2}$$

Let us now find an $\ell = 0$ pair of quasi-characters $\hat{\chi}$ in the same fusion class which, when added to the above, gives a quasi-character with the value of ℓ reduced by 6. From the Riemann-Roch theorem, this will happen if $\alpha_0 + \alpha_1$ increases by one unit. This in turn can be done by increasing only α_0 or only α_1 by a single unit, or varying both such that the sum increases by one unit. Let us try to keep α_0 fixed and increase α_1.

[1] We are grateful to Ashoke Sen for suggesting this strategy.

For this, we start with a pair of $\ell = 0$ quasi-characters with exponents $\hat{\alpha}_0, \hat{\alpha}_1$ and choose $\hat{\alpha}_1 = \alpha_1$. It follows that $\hat{\alpha}_0 = \frac{1}{6} - \alpha_1 = \alpha_0 + m$. Thus the quasi-characters are:

$$\begin{aligned} \hat{\chi}_0 &= q^{\alpha_0+m}(\hat{a}^0_0 + \hat{a}^0_1 q + \hat{a}^0_2 q^2 + \cdots) \\ \hat{\chi}_1 &= q^{\alpha_1}(\hat{a}^1_0 + \hat{a}^1_1 q + \hat{a}^1_2 q^2 + \cdots) \end{aligned} \tag{4.3}$$

Now consider the new quasi-characters defined by:

$$\tilde{\chi}_i = \hat{a}^1_0 \chi_i - a^1_0 \hat{\chi}_i \tag{4.4}$$

These have the following q-expansions:

$$\begin{aligned} \tilde{\chi}_0 &= \hat{a}^1_0 \chi_0 - a^1_0 \hat{\chi}_0 \\ &= q^{\alpha_0} \hat{a}^1_0 (a^0_0 + a^0_1 q + a^0_2 q^2 + \cdots) - q^{\alpha_0+m} a^1_0 (\hat{a}^0_0 + \hat{a}^0_1 q + \hat{a}^0_2 q^2 + \cdots) \\ &= q^{\alpha_0}(b^0_0 + b^0_1 q + b^0_2 q^2 + \cdots) \\ \tilde{\chi}_1 &= q^{\alpha_1}\Big[\hat{a}^1_0(a^1_0 + a^1_1 q + a^1_2 q^2 + \cdots) - a^1_0(\hat{a}^1_0 + \hat{a}^1_1 q + \hat{a}^1_2 q^2 + \cdots)\Big] \\ &= q^{\alpha_1+1}(b^1_0 + b^1_1 q + b^1_2 q^2 + \cdots) \end{aligned} \tag{4.5}$$

Thus the new quasi-characters $\tilde{\chi}_0, \tilde{\chi}_1$ have exponents $\alpha_0, \alpha_1 + 1$ as desired. It follows that their ℓ-value relative to the original characters is $\tilde{\ell} = \ell - 6$.

We can invert the relation to write the original characters as:

$$\chi_i = \frac{1}{\hat{a}^1_0}\Big(\tilde{\chi}_i + a^1_0 \hat{\chi}_i\Big) \tag{4.6}$$

Both objects on the RHS are quasi-characters, the first has the ℓ-value $\ell - 6$ while the second has $\ell = 0$. If $\ell = 6$ we are done, otherwise we can repeat the procedure to express the first term as a sum of terms with ℓ-value $\ell - 12$ and $\ell = 0$. Continuing in this way we will find that our original characters are written:

$$\chi_i = \sum_{s=0}^{s_{\max}} r_s \hat{\chi}^{(s)}_i = \frac{1}{A}\sum_{s=0}^{s_{\max}} n_s \hat{\chi}^{(s)}_i \tag{4.7}$$

where r_s are rational numbers, which we have also expressed in terms of integers A, n_s. The characters on the RHS all have $\ell = 0$.

This proves that rational linear combinations of quasi-characters generate all admissible characters. Alternatively one can take integer linear combinations up to a single overall normalisation. The same method can be easily applied to extend the theorem to admissible characters with

$\ell=6m+2, 6m+4$, expressing them as rational sums of quasi-characters with $\ell=2,4$.

4 Relation to Hecke images

In [20], Harvey and Wu introduced novel Hecke operators that act on vector-valued modular forms which occur as RCFT characters and give rise to new sets of potential characters. These Hecke images generically have increasing values of ℓ, and it was shown that under certain conditions they are admissible in the sense we have used in this paper. In light of the present work, we can interpret the more general Hecke images as quasi-characters. Let us briefly review their construction as it applies to the two-character case. Consider the q-expansions of the characters of a particular RCFT:

$$\chi_i = \sum_{n=0}^{\infty} c_i(n) q^{n+\alpha_i} = \sum_{m=n_i}^{\infty} b_i(m) q^{\frac{m}{N}} \tag{4.1}$$

In the second expression above, N is the common denominator of the original exponents α_i, which are therefore written as $\frac{n_i}{N}$, and the summation is now over $m=n_i+nN$, with

$$b_i(m) = c_i\left(\tfrac{m-n_i}{N}\right) \tag{4.2}$$

The integer N is called the conductor of the CFT. Then, the Hecke image T_p of the above characters is defined to be the new q-series:

$$(T_p\chi)_i = \sum_{n=0}^{\infty} c_i^{(p)}(n) q^{n+\alpha_i^{(p)}} = \sum_{m=n_i^{(p)}}^{\infty} b_i^{(p)}(m) q^{\frac{m}{N}} \tag{4.3}$$

Here, $p\in\mathbb{N}$ is a prime and is relatively prime to N. The new exponents $\alpha_i^{(p)}=\frac{n_i^{(p)}}{N}$ are defined as:

$$\begin{aligned}
\alpha_0^{(p)} &= p\alpha_0 \\
\alpha_1^{(p)} &= p\alpha_1 \bmod 1 = p\alpha_1 - \lfloor p\alpha_1 \rfloor
\end{aligned} \tag{4.4}$$

and the new q-coefficients $b_i^{(p)}(m)$ are defined in terms of the old q-coefficients as follows:

$$
\begin{aligned}
b_i^{(p)}(n) &= \begin{cases} p b_i(pn) & p \nmid n \\ p b_i(pn) + \rho_{ij} b_j(n/p) & p \mid n \end{cases} \\
&= \begin{cases} p c_i(\frac{pn-n_i}{N}) & p \nmid n \\ p c_i(\frac{pn-n_i}{N}) + \rho_{ij} c_j(\frac{n-pn_j}{pN}) & p \mid n \end{cases}
\end{aligned}
\tag{4.5}
$$

for a certain matrix ρ_{ij} which is completely determined by p. Finally the q-expansion can be recast in a standard form using coefficients $c_i^{(p)}$, obtained from $b_i^{(p)}$ using Eq. (4.2).

The Hecke images $(T_p\chi)_i$ will be admissible according to our definition for certain values of p which ensure that the $c_i^{(p)}$ determined by the above procedure are non-negative. For Hecke images of $\ell = 0$ theories, the case predominantly considered in [20], the ℓ-value of the resulting characters is found by applying the Riemann-Roch theorem:

$$
\ell^{(p)} = 1 - 6\left(\alpha_0^{(p)} + \alpha_1^{(p)}\right) \tag{4.6}
$$

Inserting Eq. (4.4), one finds:

$$
\ell^{(p)} = 1 + 6\lfloor p\alpha_1 \rfloor - p \tag{4.7}
$$

From this one can derive that:

$$
\begin{aligned}
\ell^{(p)} &= 0 \text{ mod } 6 \;\rightarrow\; p = 1 \text{ mod } 6 \\
\ell^{(p)} &= 2 \text{ mod } 6 \;\rightarrow\; p = 5 \text{ mod } 6
\end{aligned}
\tag{4.8}
$$

The main virtue of the Hecke image procedure is that we can easily generate infinite sets of admissible characters with arbitrarily large values of ℓ as well as c.

However this procedure has a limitation. If we want to find $\ell^{(p)} = 4$ mod 6 starting from an $\ell = 0$ theory, then we must have $p = 3$ mod 6. But this is never realised, because p has to be co-prime to the conductor N, and in two-character theories the conductor is always a multiple of 3. Therefore in particular one can never get an $\ell = 4$ theory as a Hecke image of something with $\ell = 0$. This point was noted in [20] but incorrectly attributed to the fact that [8] did not find any such theory. As we have seen in Section 3, there are indeed admissible characters with $\ell = 4$ that escaped the notice of several previous works. Yet, they are not Hecke images of $\ell = 0$ objects.

This leaves open the possibility that for $\ell = 0, 2$ mod 6, Hecke images of a finite set of $\ell = 0$ admissible characters, together with their linear combinations, span the space of quasi-characters.

No.	Description	c	h
1	$T_{61}\,\chi_{LY}$	$\frac{122}{5}$	$\frac{6}{5}$
2	$T_{67}\,\chi_{LY}$	$\frac{134}{5}$	$\frac{7}{5}$
3	$T_{73}\,\chi_{LY}$	$\frac{146}{5}$	$\frac{8}{5}$
4	$T_{79}\,\chi_{LY}$	$\frac{158}{5}$	$\frac{9}{5}$
5	$T_{25}\,\chi_{A_{1,1}}$	25	$\frac{5}{4}$
6	$T_{31}\,\chi_{A_{1,1}}$	31	$\frac{7}{4}$
7	$T_{13}\,\chi_{A_{2,1}}$	26	$\frac{4}{3}$
8	$T_{7}\,\chi_{D_{4,1}}$	28	$\frac{3}{2}$

Table 4.1: Admissible Hecke images with $\ell = 6$

Let us examine this for the special case of $\ell = 6$. We start by classifying all primes p such that starting from an $\ell = 0$ CFT, we generate $\ell = 6$. Starting from the $\ell = 0$ Lee-Yang CFT, the only primes p that generate an admissible $\ell = 6$ Hecke image are $p = 61, 67, 73, 79$[2]. If we start with the $\ell = 0$ $A_{1,1}$ theory, the primes are $p = 25, 31$. On the $A_{2,1}$ theory one can use the prime $p = 13$. Finally on $D_{4,1}$ one can use $p = 7$. These exhaust all the cases leading to admissible $\ell = 6$ characters using Hecke images. The central charge and conformal dimension of these characters are given in Table 4.1.

We will now show that there is a sum of $\ell = 0$ quasi-characters that reproduces each of these cases (this is a special case of the general theorem proved in Subsection 3). In fact, all one has to do is set $k = \frac{c}{2}$ for each entry in the table, and then consider the sum:

$$\chi^{k} + N_1 \chi^{k-12} \tag{4.9}$$

We have verified that for some value of N_1 in each case, this set precisely reproduces all the Hecke images in Table 4.1. Moreover, as explained in [20], one is allowed to consider sums of Hecke images. Likewise, we can consider varying the coefficient N_1. One finds that the two procedures agree. For example, for the first entry in Table 4.1 one has:

$$T_{61}\,\chi_{LY} + (N_1 - 244)\,\chi_{LY} = \chi^{k=\frac{61}{5}} + N_1\,\chi_{LY} \tag{4.10}$$

[2]Two of these, 61 and 79, do not obey the more stringent criteria in [20] because the first has non-unitary fusion rules and the second is of IVOA type. However we have consistently included both types of cases within our broader definition of admissibility.

where of course χ_{LY} is the same as $\chi^{k=\frac{1}{5}}$.

The LHS is the sum of a Hecke image and a character, while the RHS should be seen as the sum of two quasi-characters, the second one being in fact an admissible character. For the remaining entries of the table, one has the general result:

$$\chi^k + N_1\chi^{k-12} = T_p\chi + N_1'\chi \tag{4.11}$$

for suitably chosen N_1', where on the right-hand side, one has to adjust p according to the value of k (this can be read off from the table). We conclude that the space of (sums of) Hecke images at $\ell = 6$ is precisely equal to the space of sums of quasi-characters. We conjecture that this equivalence, between sums of Hecke images and sums of quasi-characters, is true more generally for $\ell = 0, 2$ mod 6.

The relation between Hecke images and linear sums of quasi-characters, for two-character theories, is reminiscent of a well-known phenomenon in the study of Hecke images of modular functions. If we act with a Hecke operator on $j(\tau)$, the result is a sum of j-functions evaluated at (shifted) multiples of τ of the form $n\tau$ and $\frac{\tau+i}{n}$. On the other hand, the result of this action can be written as a polynomial in $j(\tau)$ using meromorphy of j. Comparing coefficients on both sides, one finds that linear combinations of the q-coefficients of j are equated to sums of powers of the same coefficients. For the present case, the Hecke image of a particular character χ as defined in [20] provides an analogue of the left-hand side of this relation, since its q-coefficients are linear combinations of the q-coefficients of χ. Meanwhile, our quasi-characters can be written as polynomials of admissible characters [15]. Thus on the RHS we encounter powers of coefficients of the same character χ. Our conjectured equivalence then becomes a nice analogue of the famous result for modular functions.

Chapter 5

Novel coset constructions for $\ell = 6$

We have seen that the additive method produces a complete list of characters for all $\ell \geq 6$. It is amusing that the $\ell \geq 6$ MLDE's themselves played no role in this classification. Recall that the number of free parameters in these second order MLDE's grows with ℓ as $[\frac{\ell}{2}]$, making the strategy used in the cases for $\ell < 6$ increasingly unrealistic for larger values ℓ. Our process of adding quasi-characters with $\ell = 0$ automatically augments the value of ℓ while preserving the modular transformations and integrality. In view of the general completeness proof, this procedure exhausts all admissible $\ell = 6$ characters. We will find it useful to explicitly exhibit how this completeness operates in the specific class of examples of interest here, namely $\ell = 6$. The discussion in this chapter will closely follow [21].

1 Admissible characters and $\ell = 6$ MLDE

The steps in the proof are as follows. The fusion categories for two-character theories are completely classified [16, 17], and in [15] we have found, in particular, $\ell = 0$ quasi-characters for every allowed value of the central charge compatible with these fusion rules (see Table 3.1). But in fact the fusion category classification applies to all values of ℓ since it only uses the fact of having two characters. Thus, the allowed central charges for $\ell = 6$ must lie in the same list. Now adding $\ell = 0$ quasi-characters always augments ℓ by multiples of 6. Thus, the set of $\ell = 0$ quasi-characters can be thought of as a basis for the characters with any value of ℓ that is divisible by 6.

Next, by looking at the q-series, it is easily verified that the only way to produce $\ell = 6$ solutions using this basis is to add precisely two quasi-characters – and the values of their central charge must differ by 24. Additionally if the result is to be admissible, then any negative signs in the quasi-characters being added must turn positive after addition. Now suppose the sum is of the form

No.	c	h	Character sum
1	$\frac{122}{5}$	$\frac{6}{5}$	$\chi_{LY}^{n=10}+N_1\chi_{LY}^{n=0}$
2	25	$\frac{5}{4}$	$\chi_{A_1}^{n=4}+N_1\chi_{A_1}^{n=0}$
3	26	$\frac{4}{3}$	$\chi_{A_2}^{n=6}+N_1\chi_{A_2}^{n=0}$
4	$\frac{134}{5}$	$\frac{7}{5}$	$\chi_{LY}^{n=11}+N_1\chi_{LY}^{n=1}$
5	28	$\frac{3}{2}$	$\chi_{D_4}^{n=2}+N_1\chi_{D_4}^{n=0}$
6	$\frac{146}{5}$	$\frac{8}{5}$	$\chi_{LY}^{n=12}+N_1\chi_{LY}^{n=2}$
7	30	$\frac{5}{3}$	$\chi_{A_2}^{n=7}+N_1\chi_{A_2}^{n=1}$
8	31	$\frac{7}{4}$	$\chi_{A_1}^{n=5}+N_1\chi_{A_1}^{n=1}$
9	$\frac{158}{5}$	$\frac{9}{5}$	$\chi_{LY}^{n=13}+N_1\chi_{LY}^{n=3}$

Table 5.1: $\ell=6$ pairs obtained by addition of quasi-characters

$\chi^{c+24}+N_1\chi^c$. Let us focus on the negative signs in the individual terms in this sum. Suppose first that χ^c is admissible, thus it has all non-negative terms and also $0<c\le 8$. In that case χ^{c+24} has a central charge in the range $24<c\le 32$. In [15] we have noted that Type I quasi-characters in this range have a single negative sign, which moreover occurs at the first level above the ground state in the identity character, i.e. in the term of order $q^{-\frac{c}{24}}$. In the sum, the leading term of χ^c contributes precisely to the same power of q. Therefore a suitable choice of N_1 will make the sum admissible.

Finally, supposing χ^c is not itself admissible, then both χ^c and χ^{c+24} contain negative terms in their q-series. One can verify from the q-coefficients that no value of N_1 will turn all the negative terms positive. Thus, as claimed, the above classification of $\ell=6$ admissible characters is complete.

As a confirmation, let us note that the MLDE for $\ell=6$ initially has four free parameters. It can be parametrised as follows:

$$\left(D^2+\mu_2\frac{E_4^2E_6}{E_4^3+\mu_1\Delta}D+\frac{(\mu_3E_4^3+\mu_4\Delta)E_4}{E_4^3+\mu_1\Delta}\right)\chi=0 \tag{5.1}$$

and we see that the coefficient functions have a "movable" pole at $E_4^3+\mu_1\Delta=0$. Clearly the location of this pole is determined by μ_1. Now the Riemann-Roch theorem fixes $\mu_2=1$, and μ_3 is determined by the central charge. This leaves the parameters μ_1,μ_4. Next we require that the solution is not logarithmic around the free pole, which turns out to relate μ_1 and μ_4. That finally leaves one free parameter in addition to the central charge. A sum of the form $\chi^{c+24}+N_1\chi^c$ also has one free parameter, N_1, in addition to the central charge (for purposes of this argument we can treat N_1 as a real number rather than an integer, since the sum solves the MLDE for any real N_1). Thus the number of parameters in our proposed general solution is equal to the number in the MLDE, consistent with our solutions being complete. Such a parameter count can actually be done for higher values of ℓ that are multiples of 6, but we leave that for a future investigation.

The question to which we now turn is, how do we identify some (or all) of these admissible characters with actual CFT's?

2 Coset construction

In order to identify CFT's for these $\ell = 6$ characters, we will use the novel coset construction first used in [9] to identify $\ell = 2$ CFT's. Let us briefly recall this construction. Say we have a meromorphic theory $\mathscr{H}$ having a Kac-Moody algebra, as well as possible higher-spin chiral algebras. If $\mathscr{D}$ is an affine theory (i.e. a diagonal invariant) of a Kac-Moody algebra which in turn is a direct summand of the algebra of $\mathscr{H}$, then we can construct the coset $\mathscr{C} = \mathscr{H}/\mathscr{D}$ as explained in [9]. The decomposition of the character of $\mathscr{H}$ in terms of the characters of $\mathscr{D}$ determines the characters of $\mathscr{C}$ via the following relation:

$$\chi_0^{\mathscr{H}} = \chi_0^{\mathscr{D}} \cdot \chi_0^{\mathscr{C}} + \sum_{i=1}^{n-1} \mathrm{M}_i \, \chi_i^{\mathscr{D}} \cdot \chi_i^{\mathscr{C}} \tag{5.1}$$

where the integers M_i are multiplicities. Notice that this bilinear relation is completely holomorphic. From this we immediately have relations among central charges and conformal dimensions: $c^{\mathscr{H}} = c^{\mathscr{D}} + c^{\mathscr{C}}$ and $h_i^{\mathscr{C}} + h_i^{\mathscr{D}} \in \mathbb{N}$.

The central charges, conformal dimensions and ℓ-values of a coset pair are known [9] to satisfy:

$$\ell + \tilde{\ell} = 2 + \tfrac{1}{2}(c+\tilde{c}) - 6(h+\tilde{h}) \tag{5.2}$$

From this we see that $\ell = 0$ and $\tilde{\ell} = 6$ characters pair up such that $c+\tilde{c} = 32$ and $h+\tilde{h} = 2$. Moreover from their modular properties we find they satisfy the bilinear relation:

$$\chi_0(\tau)\tilde{\chi}_0(\tau) + \mathrm{M}\chi_1(\tau)\tilde{\chi}_1(\tau) = j^{\frac{1}{3}}(\tau)\left(j(\tau)+\mathscr{N}\right) \tag{5.3}$$

The integer M counts the multiplicity with which the non-identity primary occurs. On the RHS, we have the character of a potential $c = 32$ meromorphic theory, which depends on an integer parameter $\mathcal{N}$. From the q-expansion of the RHS:

$$j^{\frac{1}{3}}(\tau)\,(j(\tau)+\mathcal{N}) = q^{-\frac{4}{3}}(1+(\mathcal{N}+992)q+\cdots) \tag{5.4}$$

we see that such a theory, if it exists, has $\mathcal{N}+992$ spin-1 currents. This imposes a bound $\mathcal{N} \geq -992$. Since the spin-1 currents form a Kac-Moody algebra, which contributes to the central charge via the Sugawara construction, $\mathcal{N}$ cannot be arbitrarily large or else c would exceed 32. The upper bound on $\mathcal{N}$ is achieved when the currents form a $D_{32,1}$ Kac-Moody algebra, for which $\mathcal{N}+992 = 2016$. Thus, $-992 \leq \mathcal{N} \leq 1024$. Since the $\mathcal{N}+992$ currents come from the currents of the $\ell = 0$ and $\ell = 6$ characters that form a coset pair, one can relate $\mathcal{N}$ to the integer N_1 appearing in Table 5.1, placing bounds on the latter (the precise bound for N_1 will vary by fusion category).

We have established that our $\ell = 6$ characters are cosets of $c = 32$ meromorphic characters by $\ell = 0$ theories. If now we can show that the meromorphic character in question really corresponds to a CFT, then it follows that the $\ell = 6$ characters also describe a genuine CFT. Thus we need to identify $c = 32$ meromorphic theories whose chiral algebra contains any of the Kac-Moody algebras arising in $\ell = 0$ theories as a direct summand.

Meromorphic CFT's with $c = 32$ are far from being classified, unlike the cases of $c = 8, 16$ and 24 where they are completely classified. As noted above, the simplest constructions for such CFT's are based on even unimodular lattices. Such lattices have three important properties in dimensions 8, 16, 24 which were crucial for the classification problem in these dimensions [22, 23, 24]:

1. The root system (set of points of norm 2) of these lattices is either empty, or has rank equal to the dimension of the lattice. Moreover there is a unique lattice for each root system.

2. If the root lattice is a sum of several irreducible components, then all the components have the same Coxeter number.

3. The number of lattices for dimension ≤ 24 is small, namely $(1, 2, 24)$ for dimension $(8, 16, 24)$ respectively. This property actually follows from the two above.

Lattices whose root systems have rank equal to the dimension of the lattice are said to have a *complete root system*. Thus, all even unimodular lattices with dimension less than 24 have a complete root system, except for the Leech lattice which has none.

The above three properties can be translated into properties of meromorphic CFT's with $c \leq 24$. For a lattice CFT, spin-1 currents arise from the roots of the lattice as well as Cartan generators of the form ∂X^i where i runs over the dimension of the lattice. The first property above says that either there are no roots, in which case the abelian currents form a $U(1)^c$ algebra, or there are roots which combine with the Cartan generators to form a semi-simple Kac-Moody algebra (direct sum of non-abelian factors) with a Sugawara central charge c. The second case will be referred to as a *complete Kac-Moody algebra* because in this case the structure of the non-abelian algebra (integrable primaries, null vectors etc.) determines the CFT. In the first case the situation is less clear, as the abelian algebra alone does not tell us enough about the theory.

Going beyond lattice theories, the situation becomes more complex. For example, we encounter non-simply-laced factors in the Kac-Moody algebra and the total rank of this algebra can be < 24. Nonetheless, we refer to such Kac-Moody algebras as complete if they are semi-simple and their central charge is equal to the total central charge of the theory. With this definition, the only incomplete Kac-Moody algebras at $c = 24$ are the Leech lattice CFT with $\mathrm{U}(1)^{24}$ and the Monster CFT, obtained by orbifolding the Leech lattice CFT to remove the 24 abelian currents.

The second property of such lattices listed above, applied to a meromorphic CFT with $c \leq 24$, says that if its Kac-Moody algebra is a direct sum of irreducible components, then the dual Coxeter number $\check{h}$ is the same for each component. If we go beyond lattice CFT's then a more general version of the result holds, namely the ratio of $\check{h}$ to the level k is the same for each of the components [5].

The third property listed above for lattices in dimension ≤ 24 – that their number is small – is also related to, though does not immediately imply, a comparably small number of meromorphic CFT's with $c \leq 24$. The actual number turns out to be $(1,2,71)$ for $c = (8,16,24)$.

None of these restrictive properties is applicable once we go above $c = 24$, making the classification there very difficult. To start with, in 32 dimensions the lower bound on the number of even unimodular lattices is itself of order 10^9, as shown in [25]. Quite contrary to the cases in ≤ 24 dimensions, the root systems of these 32 dimensional lattices have all possible ranks, ranging from 0 (empty root system), 1, 2, $\cdots$, 31, 32 (complete root system). In fact, the vast majority of these lattices have incomplete root systems. If the rank of the root system is $r < 32$, the corresponding CFT has an additional $\mathrm{U}(1)^{32-r}$ factor in its spin-1 algebra and by our definition its Kac-Moody algebra is incomplete. If we go beyond lattices and construct more general $c = 32$ meromorphic CFT by methods parallel to those of [5], the total rank of abelian and non-abelian algebras together will typically fall below 32 and one will encounter both complete and incomplete cases.

Things become much more manageable if we start out by restricting ourselves to lattices with complete root systems. There are only 132 (out of more than a billion!) such indecomposable lattices, and they were classified by Kervaire in [26]. There are 119 distinct rank 32 root systems, all simply laced, corresponding to these lattices (unlike in $c \leq 24$, a few inequivalent lattices have the same root system). The simplest examples of $c = 32$ meromorphic CFT's can be now constructed from these 132 lattices, and all of them will have a complete Kac-Moody algebra at level 1 with rank 32 and a Sugawara central charge equal to 32.

We can now return to our initial problem. We pick any of the above CFT's which have a Kac-Moody algebra containing an $\ell = 0$ affine theory as a direct summand, and take the coset to get an $\ell = 6$ theory. Since all the Kervaire lattices have simply laced root systems, the Kac-Moody algebras only have simply laced Lie components. Hence these cases have very similar properties to the cosets considered in [9][1]. Thus, for a sizeable number of 32-dimensional lattices with complete root systems, the coset theory is completely well-defined as a CFT.

The list of $\ell = 6$ CFT's obtained as cosets of these theories is given below in Table 5.2. As was the case for $\ell = 2$ theories [9], here too we see that there are several distinct CFT's with different Kac-Moody algebras at each value of the central charge. The list of Kac-Moody algebras for the cosets can be found in the appendix of [21].

	$\ell = 0$			$\tilde{\ell} = 6$	
No.	c	h	Algebra	$\tilde{c}$	$\tilde{h}$
2	1	$\frac{1}{4}$	$A_{1,1}$	31	$\frac{7}{4}$
3	2	$\frac{1}{3}$	$A_{2,1}$	30	$\frac{5}{3}$
5	4	$\frac{1}{2}$	$D_{4,1}$	28	$\frac{3}{2}$
7	6	$\frac{2}{3}$	$E_{6,1}$	26	$\frac{4}{3}$
8	7	$\frac{3}{4}$	$E_{7,1}$	25	$\frac{5}{4}$

Table 5.2: $\ell = 6$ coset duals for simply laced algebras

An example in detail

Let us illustrate the construction of our $\ell = 6$ two-character CFT's via the above coset construction in some detail using a simple example. Consider a 32-dimensional lattice having the complete root system A_2^{16}. The root lattice of A_2^{16} is itself not unimodular, but one can construct an even unimodular lattice which contains this as a sublattice. To do this we need to add in a few vectors from the dual lattice of A_2^{16} such that one obtains a unimodular lattice. This filling (or "gluing")

[1]This is certainly the case when the lattice is unique for a given root system. The few cases where it is not unique may require additional information.

set, as given in [26] is the span of eight vectors which form the rows of the 8×16 matrix (I_8, H_8), where I_8 is the 8×8 identity matrix and H_8 is a certain 8×8 Hadamard matrix. The resulting lattice is unique, coming from a unique self-dual ternary code. In turn this lattice defines a unique $c=32$ meromorphic CFT which has $A_{2,1}^{16}$ as its Kac-Moody algebra. The number of spin-1 currents is simply the dimension of the algebra, which is 128. From the q-expansion of the single character Eq. (5.4), we see that $\mathcal{N}=-864$.

We can write the single character of this theory as a non-diagonal modular invariant combination of the affine characters of $A_{2,1}^{16}$. These are of the form $\chi_0^p\chi_1^{16-p}$ where χ_0, χ_1 are the $A_{2,1}$ characters. They have conformal dimensions in the range $0, \frac{1}{3}, \frac{2}{3}, 1, \cdots, \frac{14}{3}, 5, \frac{16}{3}$. Denote these by χ_{m_i} where m_i take the above values. The modular invariant (upto a phase) combination of these characters is easily found to be:

$$\begin{aligned}\chi(\tau) &= \chi_0 + 224\chi_2 + 2720\chi_3 + 3360\chi_4 + 256\chi_5 \\ &= j(\tau)^{\frac{1}{3}}(j(\tau)-864)\end{aligned} \tag{5.5}$$

In fact, the weight enumerator polynomial of the self-dual code constructed from the Hadamard matrix H_8 is $(1+224y^6+2720y^9+3360y^{12}+256y^{15})$. This exemplifies a more general phenomenon: the weight enumerator of a self-dual code determines the Kac-Moody character expansion for the CFT based on the associated lattice [24].

Since this $c=32$ meromorphic theory has $A_{2,1}$ as one of its direct summands, we can coset it by the $\ell=0$ two-character $A_{2,1}$ affine theory, to get a new $\ell=6$ two-character CFT. The bilinear pairing tells us that $c+\tilde{c}=32$, and $h+\tilde{h}=2$. Since $c=2$, and $h=\frac{1}{3}$, we find that the $\ell=6$ theory has $\tilde{c}=30$ and conformal dimension $\tilde{h}=\frac{5}{3}$. We can say more, since we know that the new theory has a Kac-Moody algebra $A_{2,1}^{15}$. This determines the integer N_1 in line 7 of Table 5.1 to be 378. Using the known q-expansions of the quasi-characters, the characters of our theory are:

$$\begin{aligned}\tilde{\chi}_0(\tau) &= (\chi_{A_2}^{n=7})_0 + 378(\chi_{A_2}^{n=1})_0 \\ &= q^{-\frac{5}{4}}(1+120q+109035q^2+32870380q^3+2612623965q^4+\cdots) \\ \tilde{\chi}_1(\tau) &= (\chi_{A_2}^{n=7})_1 + 378(\chi_{A_2}^{n=1})_1 \\ &= q^{\frac{5}{12}}(10206+5988735q+669491730q^2+32140359765q^3+\cdots)\end{aligned} \tag{5.6}$$

These characters can be expressed in terms of the characters of $A_{2,1}^{15}$ as follows. The latter are of the form $\chi_0^p\chi_1^{15-p}$ and have conformal dimensions $0, \frac{1}{3}, \frac{2}{3}, 1, \cdots, \frac{14}{3}, 5$. Analogous to what we did previously, we now label these as χ_{m_i} where the m_i take the above values (to avoid confusion, we stress that these χ_{m_i} are not the same as the ones in Eq. (5.5)). It is then easily verified that:

$$\begin{aligned}\tilde{\chi}_0(\tau) &= \chi_0 + 140\chi_2 + 1190\chi_3 + 840\chi_4 + 16\chi_5 \\ \tilde{\chi}_1(\tau) &= 42\chi_{\frac{5}{3}} + 765\chi_{\frac{8}{3}} + 1260\chi_{\frac{11}{3}} + 120\chi_{\frac{14}{3}}\end{aligned} \tag{5.7}$$

In this way the coset theory is precisely established as a non-diagonal Kac-Moody invariant.

The $c = 30$ two-character CFT constructed here is unique. However one can construct other two-character $c = 30$ CFT's having different Kac-Moody algebras by starting with a different Kervaire lattice. For example we can find an $\ell = 6$ CFT with algebra $A_{2,1}^{12} \oplus E_{6,1}$ having the same central charge 30 and conformal dimension $\frac{5}{3}$ as the previous one. Restricting just to cosets of lattice meromorphic CFT, this is the full list of possible algebras. However there will surely be more general (non-lattice) meromorphic CFT, still having complete Kac-Moody algebras.

2.1 CFT's with an incomplete Kac-Moody algebra

Here we consider meromorphic $c = 32$ CFT with an incomplete Kac-Moody algebra and discuss the possibility of taking their cosets. Because of the difficulty of this problem, our discussion will be briefer and less conclusive than the previous section. Let us first classify the possible types of situations. Leaving out lattice theories with complete root systems, which we have already discussed, the landscape of the remaining even, self-dual lattices is as follows. It includes lattices with root systems of every rank from 0 to 31. Of these, the number of lattices in rank 0 alone is bounded below by 1.096×10^7 [25]. For rank $1 \leq r \leq 31$, there is a total of 13,099 distinct root systems. Each one can have a very large number of distinct lattices associated to it.

Given such a daunting number of cases, we cannot carry out a general discussion but will instead try to highlight a few interesting examples. The most extreme example of an incomplete root system is to have no root system at all. A famous lattice with this property is the Barnes-Wall lattice BW_{32}, which has an automorphism group of order $2^{31}.3^5.5^2.7.17.31$. The fact that it has no root system is simply the statement that the minimum (length)^2 of any vector in the lattice is greater than 2. Thus, as we have seen, there can be no non-abelian currents, but there are 32 U(1) currents of the form $\partial X^i, i = 1,2,\cdots 32$. Because it has a very large automorphism group, this lattice can be thought of as a close analogue of the 24-dimensional Leech lattice, whose automorphisms form the Conway group Co_0 of order $2^{22}.3^9.5^4.7^2.11.13.23$. Moreover, there is an orbifold of the CFT based on BW_{32} that removes even the abelian current algebra, and the resulting VOA has a larger automorphism group studied in [27]. We may think of this as being analogous in some ways to the Monster CFT at $c = 24$. Following the mathematical literature we will refer to any CFT (or even admissible character) having no Kac-Moody algebra as being of "OZ type" where OZ stands for "one zero" and denotes that the level-1 degeneracy for the identity character is zero [28]. In this notation, the Monster CFT and the CFT of [27] are meromorphic theories of OZ type.

In [29] the possibility of OZ-type coset pairs was considered. The cases considered there had two or three characters and low values of ℓ, and the coset pairs combined to give the unique $c=24$ meromorphic CFT of OZ type, namely the Monster CFT. Unfortunately for the case of two characters the coset dual of the $c=-\frac{22}{5}$ minimal was actually not admissible, indeed it had mostly negative coefficients – thus falling in the category of "type II quasi-character" [15]. However, an admissible example was uncovered in the three-character case: the Baby Monster CFT, dual to the Ising model. Due to the OZ nature of the numerator (Monster) and denominator (Ising), one has no Kac-Moody algebra to help in defining the coset. Nevertheless, the existence of the coset dual as a VOA has been established by other means [30, 31] and consistency of its correlation functions was shown in [32]. Very recently [33] other OZ coset pairs have been found, and the duals have large sporadic groups as their automorphisms.

Encouraged by this, we may wonder if there is an $\ell=6$ two-character CFT obtained by taking the coset of the BW_{32} orbifold by some $\ell=0$ CFT of OZ type. Unfortunately this does not work, for the same reason as in [29]. For example, an $\ell=6$ dual of the Lee-Yang minimal model would have $c=\frac{182}{5}$. But this is not in Table 5.1, and we have verified that it is a quasi-character of type II. We may instead start with admissible OZ-type $\ell=6$ characters and look for their $\ell=0$ duals, for example consider the character in line 1 of Table 5.1 which has $(c,h)=(\frac{122}{5},\frac{6}{5})$, and choose $N_1=244$, the value that removes the degeneracy of the first state above the identity. The $\ell=0$ theory that pairs up with it $c=\frac{38}{5}, h=\frac{4}{5}$ which is precisely the $E_{7.5}$ theory, identified in [7] as an intermediate vertex operator algebra (IVOA). However, this latter theory is not OZ, as it has a spin-1 algebra of dimension 190, a number that sits between 133 and 248 (the dimensions of E_7 and E_8) and famously fills a gap in the Deligne series [34]. We have verified that no OZ coset pairs of two-character theories with $\ell=0$ and $\ell=6$ exist. Past experience strongly suggests, however, that such pairs may exist from three characters onwards.

Better examples are found by considering each of the entries in Table 5.1 and first choosing N_1 so that they become of OZ type. As explained above, their $\ell=0$ coset duals are not of OZ type, in fact they are Deligne series CFT's having a simple level-1 Kac-Moody algebra. This suggests that we look for $c=32$ meromorphic theories with a simple level-1 Kac-Moody algebra, and coset them by a Deligne series CFT. From Table 1 of [25] we see that there are indeed lattices having A_1, A_2, D_4, E_6 as their root systems (but curiously not E_7). The CFT on these lattices will have an extra $\mathrm{U}(1)^{32-r}$ symmetry. Assuming this can be removed by orbifolding, one would find the right kind of meromorphic CFT such that, when quotiented by a simply-laced CFT in the Deligne series (and excluding E_7), we will recover our desired $\ell = 6$ CFT of OZ type.

We have looked at just a few special cases of cosets of meromorphic CFT with incomplete Kac-Moody algebras. We identified a few concrete possible examples but did not give a precise proof of the existence of any of these coset CFT's. It should be possible to construct them using VOA techniques, as was done for the Baby Monster in [30, 31]. Also many more examples can be found, and we leave this subject for future investigation.

Chapter 6

Summary and Outlook

The main premise of this thesis was to utilise the technology of modular linear differential equations (MLDE's) to discover and classify simple rational conformal field theories. The classification is based on two integer parameters, the order n of the MLDE and the paramter ℓ which counts the number of zeroes of the Wronskian. This method is especially successful when the order n of the differential equation is small. We focussed solely on the case of second order MLDE's, classifying solutions for *all* values of the parameter ℓ. We started by reviewing the cases with $\ell < 6$. The admissible solutions with $\ell = 0$ were familiar Wess-Zumino-Witten models based on the Deligne exceptional series of Lie algebras. The solutions for $\ell = 2$ corresponded to new interesting theories which in an earlier work were shown to be novel cosets of meromorphic CFT's with the $\ell = 0$ theories.

The number of free parameters in the MLDE grows with ℓ making it harder to find admissible solutions in the cases with a higher value of ℓ. Previously, the status of CFT's with an arbitrary value of ℓ was unclear. Taking tensor products of CFT's increases the value of ℓ, and apart from this trivial example of tensor product CFT's no new two-character examples with a higher ℓ were known. We first completed the classification for the case of $\ell = 4$, showing that there exist three new non-trivial admissible solutions. These admissible solutions have not yet been identified with new CFT's, and have interesting features such as a highly degenerate primary and a surprisingly high central charge given that the number of spin-1 currents is very small.

To tackle the difficult cases of $\ell \geq 6$ we proposed a novel approach by defining and utilising quasi-characters. These, in contrast to admissible characters, are allowed to have negative integer degeneracies of states. These were completely classified for the case of two-character theories and are given in table 3.1. We proposed two methods, the multiplicative and additive methods, to generate admissible solutions for all the cases of $\ell \geq 6$, and showed that these generate the complete set of solutions. We compared our solutions with Hecke images of $\ell = 0$ theories and argued that they are in correspondence with each other. In the case of $\ell = 6$, we identified certain admissible solutions as cosets of $c = 32$ lattice theories based on 32-dimensional Kervaire lattices. Just in the $\ell = 6$ case alone, we realised that there exist a large number of two-character CFT's. This is very similar to the situation of meromorphic theories which occur in huge numbers once we have $c > 24$. The fact that such huge number of CFT's makes the classification out of question should not be seen as a negative outcome since this method still allows us to discover exotic or special theories among the huge number of possibilities.

Finally, we note some ideas to be pursued in the future. Since the fusion rules of three-character theories are well studied and their characters can be written as generalised hypergeometric functions, a similar classification quasi-characters and admissible characters should be well within reach. The situation is trickier when the number of characters gets large, but we must look for efficient and novel ways to generate vector-valued modular forms as the number of characters gets larger. The addition method involves free integer parameters and allows us to control the gap in the spectrum of the CFT. More speculatively, theories with large n as well as large ℓ could be interesting, as they will have a large central charge and the gap in their spectrum can also be made very large. It is also very interesting to study modular differential equations based on subgroups of the modular group, such as the theta subgroup in the case of superconformal theories.

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