Astrophysical Probes of Exotic Particles

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

PVS Pavan Chandra

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2019

Supervisor: Dr. Arun Thalapillil

Certificate

This is to certify that this dissertation entitled Astrophysical Probes of Exotic Particles towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by PVS Pavan Chandra at Indian Institute of Science Education and Research under the supervision of Dr. Arun Thalapillil, Assistant Professor, Department of Physics , during the academic year 2018-2019.

Dr. Arun Thalapillil

Committee:

Dr. Arun Thalapillil

Prof. Sunil Mukhi

This thesis is dedicated to my parents

Declaration

I hereby declare that the matter embodied in the report entitled Astrophysical Probes of Exotic Particles are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Arun Thalapillil and the same has not been submitted elsewhere for any other degree.

Pavan

PVS Pavan Chandra

Acknowledgments

I am deeply grateful to my adviser, Dr. Arun Thalapillil, for his constant support, generosity and guidance. The innumerable discussions we had benefited me immensely allowing me to discover the excitement of research in physics and I owe him my deepest gratitude.

I would like to thank Prof. Sunil Mukhi for agreeing to be on my thesis advisory committee and carefully evaluating my progress throughout the project.

A huge thanks to my friends for the great emotional support throughout the stay at IISER. I will never forget the intense debates, academic and otherwise, the late night discussions, and the random shenanigans we used to pull. My sincerest gratitude for all the unforgettable memories we have had over the past five years. I may have never had a moment from myself had you not been as important a part of my life, my consciousness, as you are.

Lastly, the support I have received from my parents has been so integral throughout my life that it doesn't need a mention here, but most certainly deserves one.

Abstract

Dark matter has been one of the most elusive puzzles in our understanding of the cosmos for over seventy years now. In this document, we explore the effects of the pair production of certain exotic particle states called *millimagnetically charged particles* (mmCPs) on the gravitational waves generated by a magnetar. We explicitly calculate the difference in the time evolution of the gravitational wave (GW) amplitude which, when the waves are detected in the future, could serve as a signature of the presence of said mmCPs.

In this enterprise, we first present the necessary background on gravitational waves and then look at the existing literature on the gravitational waves in the context of isolated neutron stars. Due to the presence of multiple ideas on neutron star magnetic fields, we choose the ideas which we believe are the closest to reality and proceed to calculate the deformation to the star which generates a non-zero quadrupolar ellipticity and thus, gravitational waves. The amplitude of the gravitational waves is directly affected by the strength of the magnetic field. We compare the GW amplitude by evolving the magnetic field with and without the presence of mmCPs and find that there is a difference.

In the last chapter, we explore the application of an interesting idea regarding worldline instantons that recently appeared in the literature. We wish to see the potential arenas this new idea may open up in this subfield. We also apply the technique to two different situations and find that the solution matches the known solution.

Contents

Abstract **xi**

1 Introduction **1**

2 Gravitational Waves **5**

2.1 Linearized gravity 5

2.2 Effective Energy Momentum tensor 8

2.3 Pulsars as a source of gravity waves 13

3 Neutron stars **17**

3.1 Effect of magnetic field on the shape of the neutron star 17

3.2 Models of the magnetic fields 27

3.3 GWs from a neutron star with magnetic field induced distortion 38

4 Magnetic Monopoles and Millicharges **45**

4.1 Dirac quantization condition 46

4.2 Millimagnetically charged particles 47

4.3 mmCP-SPP in Neutron Stars 51

4.4 Incorporating thermal corrections 53

4.5 Results and Implications 58

5 Worldline Deformations **63**

5.1 Deformations in scalar fields 64

5.2 Our saddle point equations 67

5.3 Applications 69

6 Conclusion and future work **73**

Bibliography **74**

Chapter 1

Introduction

In this chapter, we introduce the main players in our work, the elusive *dark matter* and the ever intriguing *neutron stars*.

Dark Matter

The extensive presence of *dark matter* is an intriguing puzzle with a large number of ideas in the literature [1]. The first evidence of dark matter was pointed out by Zwicky in 1933 [2]. He saw that the nebulae in the Coma cluster had unexpectedly high velocities. Further evidence was provided by the observations of Rubin *et al* [3]. They observed that the rotation velocities of the stars at the edge of the galaxy was not agreeing with the values predicted by Newtonian gravity. In fact, the rotation curves increase linearly up to a certain distance and then flat out. According to Newtonian gravity, however, if our galaxy was the only mass in the surrounding area, the velocities should fall off as the square root of the distance after the linear increase. A resolution to this conundrum was provided by Ostriker and Peebles' [4] proposal of the presence of a *dark matter halo*, which they had used to account for instabilities in the galactic disc models. The presence of dark matter also plays an important role in large scale structure structure formation in cosmology, as it the density perturbations generated by dark matter that are believed to have lead to the large scale structures we see today [5]. In addition, it is of incredible interest to understand the particle nature of dark matter due to it's role in big bang nucleosynthesis [6]. Among the possible candidates for dark matter are exotic states called *millicharged particles* (mCPs) [7] and *millimagnetically charged particles* (mmCPs) [8].

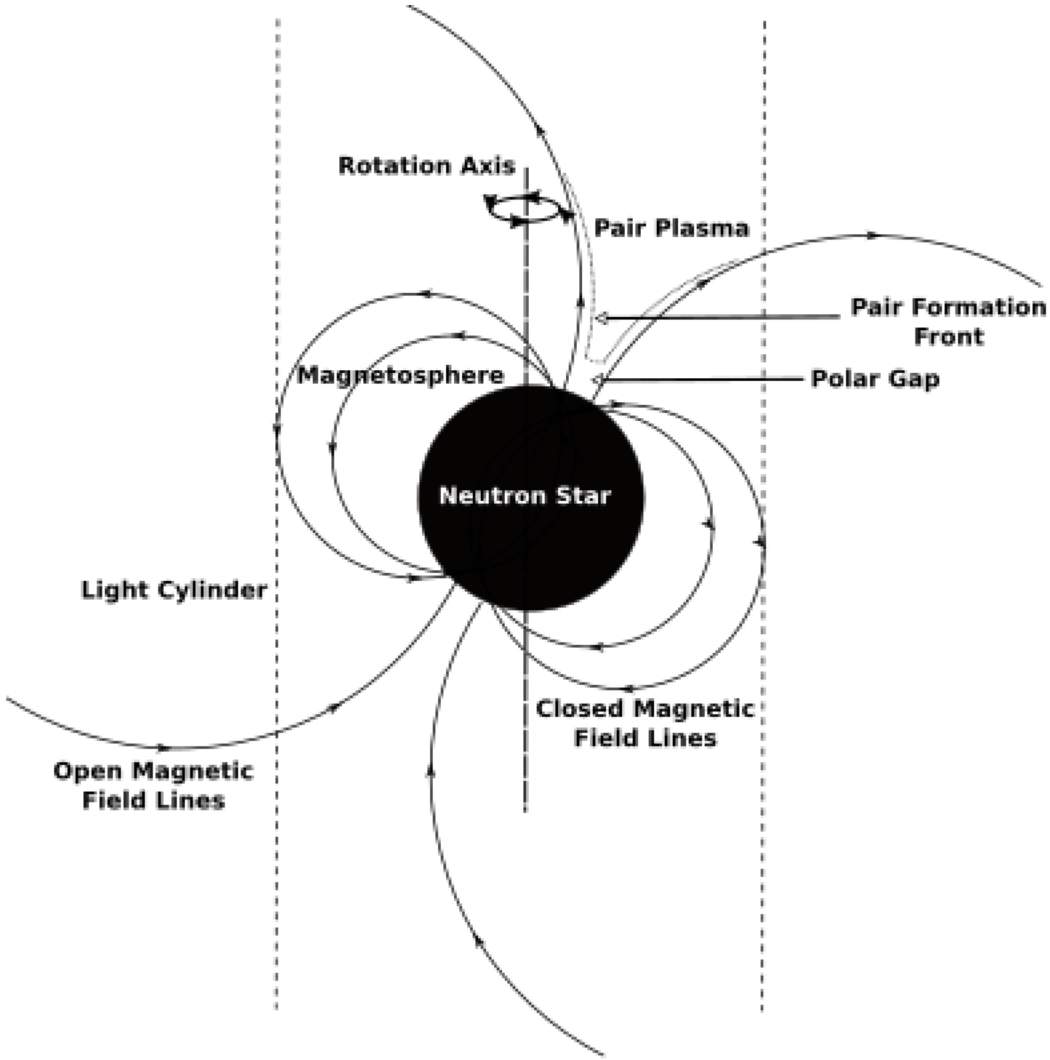


Figure 1.1: A schematic of a neutron star [14]

Neutron Stars and Magnetars

Neutron stars are one of the end stages of stellar evolution, along with black holes and white dwarfs. They form due the gravitational collapse of a massive star under their own mass. Historically, they were thought to have a mass between the Chandrasekhar limit of 1.4 solar masses and 3 solar masses. However, the GW170817 event where the result of a binary neutron star merger collapsed into a black hole [9], suggests a limit closer to the final mass of the merger, $\simeq 2.17$ solar masses [10, 11, 12, 13]. A schematic of a typical neutron star is given in Fig. 1.1 [14].

Magnetars are neutron stars with extremely high magnetic fields [15]. On an average, their magnetic fields are as high as 10^{15} G [16, 15], making them the most magnetic objects in the universe. Originally, they were proposed as likely candidates for soft gamma repeaters (SGRs) and anomalous X-ray pulsars (AXPs) [17]. Our interest in magnetars stems from the fact that due to the presence of such high magnetic fields, we can expect the non-perturbative quantum field theoretic process of Schwinger pair production of mmCPs which can potentially affect the amplitude of the gravitational waves from the star. Note that this type of gravitational wave is called a "continuous" gravitational wave in contrast to a "burst" type gravitational wave that are generated by binary neutron star mergers or binary black hole mergers that the Laser Interferometer Gravitational-Wave Observatory (LIGO) is already observing [18, 19, 20]. These type of GWs are expected to be detected once the LIGO sensitivity becomes better and future telescopes like the Einstein Telescope begin operations.

The outline of this thesis is as follows. In Chapter 2, we will understand the basic theory behind the generation of gravitational waves and rederive important results that will be indispensable in later exercises. In Chapter 3 we will look specifically at our system of neutron stars and rederive the important formulae, gain intuition and calculate a few quantities that will be directly applicable in the meat of the thesis, which is Chapter 4. In this chapter, we shall motivate mmCPs and look at them in the context of neutron stars. We will use the results obtained in the previous chapters to obtain the effect of the Schwinger pair production (SPP) of mmCPs on the gravitational waves from the star. Following this, in Chapter 5, we endeavour on a different enterprise of trying to calculate the SPP formula for different field configurations. In this context, we look at a few recent papers and lay down a path for a future project.

Chapter 2

Gravitational Waves

In this chapter, we will understand the general theory behind the generation and propagation of gravitational waves (GWs). At the end of the chapter, we will see how pulsars, which are rapidly rotating neutron stars can act as a source of GWs.

2.1 Linearized gravity

To begin discussing gravitational waves, we start with linearized gravity, i.e., there exists a coordinate system in which the metric is given by $g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}$, where $\overline{g}_{\mu\nu}$ is the background and $h_{\mu\nu}$ is considered fluctuation over the background. This notion will be made much clearer later. For now, we take $\overline{g}_{\mu\nu} = \eta_{\mu\nu}$. We are going to follow the discussion in [18] and [20] The action for the theory is given by

$$S_g = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g}\mathcal{R} \tag{2.1}$$

where $\mathcal{R}$ is the *Ricci scalar*. Working upto $\mathcal{O}(h^2)$, we get expressions for the Riemann tensor, Ricci tensor and the Ricci scalar. Note that the Einstein field equations, $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ are 10 in number(symmetric tensors), constrained by the *Bianchi identities* $G^{\mu\nu}_{\ ;\nu} = 0$ which are 4 in number, so we solve for 6 equations in total. Note also that GR is invariant under $x^\mu \rightarrow x'^\mu(x)$, under which $g_{\mu\nu} \rightarrow g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}$. By choosing a particular coordinate system in which $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we are using up this freedom. We still have a residual gauge freedom $x^\mu \rightarrow x'^\mu(x) = x^\mu + \xi^\mu$, with $|\partial_\mu \xi_\nu| \ll |h_{\mu\nu}|$, under which the metric transforms as $g_{\mu\nu} \rightarrow g'_{\mu\nu} = \eta_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + h_{\mu\nu} + \mathcal{O}(\partial\xi^2)$. Renaming $-\partial_\mu \xi_\nu - \partial_\nu \xi_\mu + h_{\mu\nu}$ as $h'_{\mu\nu}$, we get $g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}$

with $|h_{\mu\nu}| \ll 1$, i.e., this is indeed a symmetry of the (linearized) theory. Under finite global Lorentz transformations, $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$, the metric transforms as $g_{\mu\nu} \to g'_{\mu\nu} = \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu} g_{\rho\sigma}$ which reduces to

$$\begin{aligned} g'_{\mu\nu} &= \eta_{\mu\nu} + \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu} h_{\rho\sigma} \\ &= \eta_{\mu\nu} + h'_{\mu\nu} \end{aligned}$$

therefore, $h_{\mu\nu}$ is a tensor under Lorentz transformations. Similarly, under global space-time translations, $x'^{\mu} = x^{\mu} - a^{\mu}$, $g'_{\mu\nu} = \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} g_{\rho\sigma} = g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Therefore, $h_{\mu\nu}$ is invariant under translations. Thus, we conclude that $h_{\mu\nu}$ is invariant under the Poincare group. In the next subsection, we shall see the form the field equations take in the linear regime.

2.1.1 Einstein's equations in the linear regime

For the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the Riemann tensor turns out to be [18]:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \left(\partial_{\rho\nu} h_{\sigma\mu} + \partial_{\sigma\mu} h_{\rho\nu} - \partial_{\rho\mu} h_{\nu\sigma} - \partial_{\sigma\nu} h_{\rho\mu} \right)$$

and it is clearly invariant under $h_{\mu\nu} \to h'_{\mu\nu} = -\partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} + h_{\mu\nu}$, as expected since this transformation is a symmetry of the theory. We now introduce the *trace-reversed metric tensor*, $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ (so called since h^{μ}_{μ} = trace of $h_{\mu\nu} = -\overline{h}^{\mu}_{\mu}$), and putting this back in Einstein's equations gives

$$\Box \overline{h}_{\nu\sigma} + \eta_{\nu\sigma} \partial^{\rho\lambda} \overline{h}_{\rho\lambda} - \partial^{\rho}_{\nu} \overline{h}_{\rho\sigma} - \partial^{\rho}_{\sigma} \overline{h}_{\rho\nu} + \mathcal{O}(h^2) = -\frac{16\pi G}{c^4} T_{\nu\sigma}$$

which, analogous to electromagnetism, using the *Lorentz gauge* $\partial_{\mu} \overline{h}^{\mu\nu} = 0$ and neglecting higher order terms reduces to

$$\Box \overline{h}_{\nu\sigma} = -\frac{16\pi G}{c^4} T_{\nu\sigma} \tag{2.2}$$

where the $T_{\mu\nu}$ is the energy momentum tensor and is a conserved current of space time translations, which amounts to saying $\partial_{\mu} T^{\mu\nu} = 0$ in the linear theory (in the full theory, this becomes $\nabla_{\mu} T^{\mu\nu} = 0$). Setting the condition that $T^{\mu\nu} = 0$,

$$\Box \overline{h}_{\nu\sigma} = 0 \tag{2.3}$$

we obtain the wave equation for vacuum propagation.

Our metric is a symmetric rank 2 tensor and thus has 10 independent components. Using the Lorentz gauge, we have 4 constraints, thus $10 - 4 = 6$ independent components. Observing that Eq.(2.3) is invariant under $\overline{h}_{\mu\nu} \to \overline{h}_{\mu\nu} + \xi_{\mu\nu}$ with $\xi_{\mu\nu} = \eta_{\mu\nu}\partial_\rho \xi^\rho - \xi_{\mu,\nu} - \xi_{\nu,\mu}$ with $\Box \xi_\mu = 0$, we have 4 further constraints for a total of 2 independent components. We can make use of this gauge freedom by choosing ξ_0 such that the trace of $\overline{h}_{\mu\nu}$ becomes 0. Then, we can choose ξ_i such that $\overline{h}_{0i} = 0$. Also, once trace becomes 0, $\overline{h}_{\mu\nu} = h_{\mu\nu}$. Therefore, until now, we have used up 4 degrees of freedom, $h^\mu_\mu = 0$ and $h_{0i} = 0$. Now, using Lorentz gauge's $\nu = 0$ equation, we get $\partial_0 h^{00} + \partial_i h^{0i} = \partial_0 h^{00} = 0$ meaning that h_{00} is time independent. We can thus choose it to be 0 such that we now have $h_{0\mu} = 0$. Including the trace condition, we have used up 5 degrees of freedom. The Lorentz condition now reads $\partial_i h^{ij} = 0$ (these are called transverse conditions - analogous to electromagnetism's polarization condition). Thus, this gauge is called **transverse traceless gauge** or **TT gauge** [20]. To summarize, the components are given by

$$\Box h_{\mu\nu} = 0 \tag{2.4}$$

with the gauge conditions

$$h_{0\mu} = 0, \quad h^\mu_\mu = h^i_i = 0 \quad \text{and} \quad \partial_i h^{ij} = 0 \tag{2.5}$$

We call this metric h^{TT}_{ij}. For a single plane wave, with direction of propagation along $\hat{n}$, the gauge condition reduces to $n^i h^{TT}_{ij} = 0$. Assuming $\hat{n} = \hat{z}$, the solution of the wave equation can be written as

$$h^{TT}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\omega(t - z/c) \tag{2.6}$$

where h_+ and $h_\times$ are the polarization states, the two independent components in the metric.

Now, let us define two operators called *projection operator* and Λ - *operator* as:

$$P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j, \text{i.e., } P_{ij} = P_{ji},\ n^i P_{ij} = 0,\ P_{ij}P^{jk} = P^k_i, and\ P^{ii} = 2 \tag{2.7}$$

$$\Lambda_{ij;kl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \tag{2.8}$$

and observing that $\Lambda_{ii;kl} = 0$, we say $h^{TT}_{ij} = \Lambda_{ij;kl}h_{kl}$, where h_{ij} is in the Lorentz gauge. This above relation will be very useful in later obtaining a solution to the full wave equation with a source term.

Graviton On a side note, we can connect some important properties about gravity and semi classical physics by obtaining the spin of "graviton". For this, we rotate the 2×2 matrix of the metric in the plane perpendicular to $\hat{n}$ by an angle ϕ about $\hat{n}$. This is mathematically achieved by multiplying with rotation matrix R from one side and R^T, the transpose, from the other to obtain the new h'_+ and $h'_\times$ as:

$$h'_+ = \cos\phi h_+ - \sin\phi h_\times \tag{2.9}$$

$$h'_\times = \sin\phi h_+ + \cos\phi h_\times \tag{2.10}$$

Looking at $h_\times \pm ih_+$, after rotation, it becomes:

$$h_\times \pm ih_+ \to h'_\times \pm ih'_+ = e^{\mp 2i\phi}(h_\times \pm ih_+) \tag{2.11}$$

which can be interpreted, from helicity concepts of particle physics, as graviton having spin 2. This conclusion is discussed below

Helicity and Group Theory Helicity, h, in particle physics is defined the eigenvalue of the operator defined as:

$$\mathcal{H} = \mathbf{S}\cdot\hat{p}$$

where $\mathbf{S}$ is the spin operator and $\hat{p}$ is the momentum direction. A helicity state is an eigenstate of $\mathcal{H}$. Noting, in addition, that $R(\hat{p},\phi) = e^{i\mathbf{S}\cdot\hat{p}\phi} = e^{i\mathcal{H}\phi}$, we have:

$$\mathcal{H}|\psi\rangle = h|\psi\rangle \implies e^{\mathcal{H}\phi}|\psi\rangle = e^{h\phi}|\psi\rangle.$$

Using this argument above with $h_\times \pm ih_+$ as the helicity state, we have the helicity $h = 2$. This means that the value of $|\mathbf{S}| = 2$, i.e., spin is 2.

2.2 Effective Energy Momentum tensor

In this section, we generalize gravitational wave propagation to the case when the background is not necessarily flat, following the discussion from [18]. The metric is given by $g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. If we think of GW's as plane waves, we have a scale for comparison. This scale is frequency when talking about GW propagation in time, idea coming from $e^{i\omega t}$. Similarly, the scale is wave vector when talking about GW propagation in space, as in e^{ikx}. The background varies much slower than the fluctuations, by definition and hence the values k_B and ω_B for the background are much smaller than those of the fluctuation

$$k_B \ll k \quad \text{and} \quad \omega_B \ll \omega.$$

We choose one of the above conditions to Taylor expand the relevant quantities and though the mathematics is the same, there is a subtlety of time and space. Thus, expanding $\mathcal{R}_{\mu\nu}$ to $\mathcal{O}(h^2)$, we get

$$\mathcal{R}_{\mu\nu} = \overline{\mathcal{R}}_{\mu\nu} + \mathcal{R}_{\mu\nu}{}^{(1)} + \mathcal{R}_{\mu\nu}{}^{(2)} + \mathcal{O}(h^3) \tag{2.12}$$

where the superscript 1 and 2 represent the order of h and $\overline{\mathcal{R}}_{\mu\nu}$ is calculated purely from the background, i.e., 0th order in h. We will now integrate out the higher frequency modes. In order to identify them, note that $\overline{\mathcal{R}}_{\mu\nu}$ is low frequency, by default, and $\mathcal{R}_{\mu\nu}{}^{(1)}$ is high frequency (since Ricci tensor contains derivatives of $h \sim e^{ikx}$ and k is high here as previously discussed), and $\mathcal{R}_{\mu\nu}{}^{(2)}$ has both high and low (since $h \cdot h \sim e^{i(k_1+k_2)x}$ and $h \cdot h \sim e^{i(k_1-k_2)x}$ are both possible, the first term gives the high frequency term, the second gives the low frequency term) [20]. Putting this $\mathcal{R}_{\mu\nu}$ in the trace reversed Einstein equations, and taking only the low frequency modes, we get

$$\overline{\mathcal{R}}_{\mu\nu} = -\left[\mathcal{R}_{\mu\nu}{}^{(2)}\right]^{low freq.} + \frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]^{low freq.} \tag{2.13}$$

introducing a length scale $\frac{1}{k} \ll l \ll \frac{1}{k_B}$ and averaging, we get

$$\overline{\mathcal{R}}_{\mu\nu} = -\left\langle \mathcal{R}_{\mu\nu}{}^{(2)} \right\rangle + \frac{8\pi G}{c^4}\left[\langle T_{\mu\nu}\rangle - \frac{1}{2}g_{\mu\nu}\langle T\rangle\right] \equiv -\left\langle \mathcal{R}_{\mu\nu}{}^{(2)} \right\rangle + \frac{8\pi G}{c^4}\left[\overline{T}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{T}\right] \tag{2.14}$$

where $\langle\cdot\rangle$ represents an average over a spatial volume l^3 and thus, the high frequency modes average out to 0. Taking the trace of this equation, we get

$$\begin{aligned}\overline{\mathcal{R}} &= -\left\langle \mathcal{R}^{(2)} \right\rangle + \frac{8\pi G}{c^4}\left[\overline{T} - \frac{1}{2}4\overline{T}\right] \\ &= -\left\langle \mathcal{R}^{(2)} \right\rangle - \frac{8\pi G}{c^4}\overline{T}.\end{aligned} \tag{2.15}$$

Multiplying (12) by $\frac{1}{2}\overline{g}_{\mu\nu}$ and subtracting from (11), we get

$$\begin{aligned}\overline{\mathcal{R}}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{\mathcal{R}} &= -\left\langle \mathcal{R}_{\mu\nu}{}^{(2)} - \frac{1}{2}\overline{g}_{\mu\nu}\mathcal{R}^{(2)} \right\rangle + \frac{8\pi G}{c^4}\overline{T}_{\mu\nu} \\ &= \frac{8\pi G}{c^4}\left(\overline{T}_{\mu\nu} + t_{\mu\nu}\right),\end{aligned} \tag{2.16}$$

where we have defined $t_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle \mathcal{R}_{\mu\nu}{}^{(2)} - \frac{1}{2}\overline{g}_{\mu\nu}\mathcal{R}^{(2)}\right\rangle$ as the *effective energy momentum tensor.* Explicit calculation gives $t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right\rangle$.

The 00th component of the energy momentum tensor gives the energy density. Writing the metric in the TT gauge, we have

$$t_{00} = \frac{c^4}{32\pi G}\left\langle \partial_0 h_{ij}\partial_0 h^{ij}\right\rangle = \frac{c^4}{16\pi G}\left\langle \dot{h}_+{}^2 + \dot{h}_\times{}^2\right\rangle. \tag{2.17}$$

Now, we can write energy density t_{00} as $\frac{dE}{dV} = \frac{dE}{dldA}$ and thus the energy flux per unit area as

$$\boxed{\frac{dE}{dtdA} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+{}^2 + \dot{h}_\times{}^2\right\rangle.} \tag{2.18}$$

2.2.1 Solving the linearized Einstein equations

In this section, we will solve the Einstein equations given by,

$$\Box\overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \quad \partial_\mu\overline{h}^{\mu\nu} = 0, \quad \text{and} \quad \partial_\mu T^{\mu\nu} = 0 \tag{2.19}$$

in the following limits:

1. In weak field (obviously, since we are using the wave equation which is valid only in weak field)
2. Slow internal motion(that is, the internal motions of the object are non-relativistic), and
3. Negligible self-gravity(i.e., the dynamics are determined by non-gravitational forces)

Expanding a bit more on point no. 3, for self gravitational systems, we can use the virial theorem, $K.E. = -\frac{1}{2}P.E.$, to get [20]

$$\frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r} \tag{2.20}$$

that is,

$$\frac{v^2}{c^2} = \frac{2GM}{c^2}\frac{1}{2r} = \frac{R_S}{2r} \tag{2.21}$$

where m is the reduced mass of the system, M is the total mass of the system, R_S is the Schwarzschild radius of a mass M. For self-gravitational systems, weak field (i.e., $\frac{R_S}{2r} \ll 1$) implies, through Eq.(2.21) that internal motions are non-relativistic. For systems with negligible self-gravity though, the weak field expansion and the internal velocity $\frac{v}{c}$ can be treated independently. We are going to use this in the following discussion. Using the Green's function method to solve the PDE, introduce

$$G(x-x') = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\delta(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c} - t') \tag{2.22}$$

which satisfies $\Box_x G(x-x') = \delta^{(4)}(x-x')$ and the solution can thus be written as:

$$\overline{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4}\int d^4x' G(x-x')T_{\mu\nu}(x').$$

Using the Λ-operator (with $\hat{n} = \frac{\mathbf{x}}{|\mathbf{x}|}$), we can go into the TT-gauge and substituting for the Green's function, we get:

$$\begin{aligned} h_{ij}^{TT} &= \Lambda_{ij;kl}(\hat{n})\frac{4G}{c^4}\int d^4x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\delta(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c} - t')T_{kl}(t', \mathbf{x}') \\ &= \Lambda_{ij;kl}(\hat{n})\frac{4G}{c^4}\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}T_{kl}(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}') \\ &\simeq \Lambda_{ij;kl}(\hat{n})\frac{4G}{c^4}\int d^3x' \frac{1}{r}T_{kl}(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{n}}{c}, \mathbf{x}') \end{aligned}$$

where in the last line, the approximation $|\mathbf{x}-\mathbf{x}'| \simeq |\mathbf{x}|(1-\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}^2|}) = r - \mathbf{x}'\cdot\hat{n}$, where $|\mathbf{x}| = r \gg d$(the typical size of the object), is used. Next, using Fourier transform of $T_{kl}(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{n}}{c}, \mathbf{x}')$, to get

$$T_{kl}(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{n}}{c}, \mathbf{x}') = \int \frac{d^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega, \mathbf{k})e^{-i\omega(t-\frac{r}{c}+\frac{\mathbf{x}'\cdot\hat{n}}{c})+i\mathbf{k}\cdot\mathbf{x}'}$$

and using $\omega\frac{\mathbf{x}'\cdot\hat{n}}{c} \sim \frac{\omega d}{c} \sim \frac{v}{c}$ and assuming $\frac{v}{c} \ll 1$, we can expand the exponential in a power series as follows

$$\begin{aligned} T_{kl}(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\hat{n}}{c}, \mathbf{x}') &= \int \frac{d^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega, \mathbf{k})e^{-i\omega(t-\frac{r}{c})+i\mathbf{k}\cdot\mathbf{x}'}(1 - i\omega\frac{\mathbf{x}'\cdot\hat{n}}{c} - \omega^2\frac{\mathbf{x}'\cdot\hat{n}^2}{c^2} + \ldots) \\ &= T_{kl}(t - \frac{r}{c}, \mathbf{x}') + \frac{n_a}{c}\frac{\partial}{\partial t}T_{kl}(t - \frac{r}{c}, \mathbf{x}')x'_a + \\ &\quad \frac{n_a n_b}{c^2}\frac{\partial^2}{\partial t^2}T_{kl}(t - \frac{r}{c}, \mathbf{x}')x'_a x'_b + \ldots \end{aligned} \tag{2.23}$$

where the derivatives are all evaluated at retarded time $t - \frac{r}{c}$. We can recast the above expressions using the mass moments. They are defined as follows:

$$M = \frac{1}{c^2}\int d^3x T_{00}(t, \mathbf{x}) \tag{2.24}$$

$$M_i = \frac{1}{c^2}\int d^3x T_{00}(t,\mathbf{x})x_i \tag{2.25}$$

$$M_{ij} = \frac{1}{c^2}\int d^3x T_{00}(t,\mathbf{x})x_i x_j \tag{2.26}$$

and using $\partial_\mu T^{\mu\nu} = 0$, we can prove that $\dot{M} = \ddot{M} = 0$, $\ddot{M}_i = 0$ and that $\ddot{M}_{ij} = 2\int d^3x T_{ij}$, i.e., we can rewrite the leading order contribution to the amplitude h_{ij}^{TT} in terms of $\ddot{M}_{ij}$ as

$$\boxed{h_{ij}^{TT} = \frac{1}{r}\Lambda_{ij;kl}(\hat{n})\frac{2G}{c^4}\ddot{M}_{kl}(t-\frac{r}{c})}. \tag{2.27}$$

2.2.2 The self-gravity issue

In the above derivation, we neglected self-gravity and this point is reflected in the treatment of the weak field and $\frac{v}{c}$ expansions and also in the Green's function($T_{\mu\nu}$ is only the matter part of energy momentum). Once the self-gravity becomes significant, we need to consider the contribution of the gravitational field itself to the energy and momentum on the RHS of Einstein's equations. This discussion is followed from the sections on energy-momentum pseudo-tensor from Landau and Lifshitz [21] and Maggiore [20]. We start by deriving the relaxed Einstein equations. Define

$$\mathsf{h}^{\mu\nu} = -\eta^{\mu\nu} + (-g)^{1/2}g^{\mu\nu} \tag{2.28}$$

This definition is *exact*. $\mathsf{h}^{\mu\nu}$ is not a small quantity, Eq.(2.28) gives the precise definition of this quantity. In the limit of weak field, we have $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$ and thus, the determinant $g = -1 - h$(h being the trace of the perturbation) and $g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu}$. Thus, Eq.(2.28) becomes

$$\begin{aligned}\mathsf{h}^{\mu\nu} &= -\eta^{\mu\nu} + (1+h)^{1/2}(\eta^{\mu\nu} - h^{\mu\nu}) \\ &= -(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h) \\ &= -(\bar{h}^{\mu\nu}).\end{aligned}$$

Thus, in the limit of weak field, h becomes our familiar $\bar{h}$. Now, imposing the harmonic gauge or the de Donder gauge, $\partial_\mu \mathsf{h}^{\mu\nu} = 0$, we get the Einstein equations in the form:

$$\boxed{\Box \mathsf{h}^{\mu\nu} = \frac{16\pi G}{c^4}((-g)T^{\mu\nu} + \tau^{\mu\nu})} \tag{2.29}$$

where $\tau^{\mu\nu}$ is a pseudo-tensor with no matter contribution whatsoever. This pseudo-tensor is completely written in terms of the metric. Note that the $\Box = -\partial_0^2 + \nabla^2$ is the d'Alembertian in flat space. This $\tau^{\mu\nu}$ can be further split into a Landau-Lifshitz part and a divergence as follows:

$$\begin{aligned}\tau^{\mu\nu} &= (-g)t_{LL}^{\mu\nu} + \frac{c^4}{16\pi G}\left(\partial_\beta \mathsf{h}^{\mu\alpha}\partial_\alpha \mathsf{h}^{\nu\beta} - \mathsf{h}^{\alpha\beta}\partial_\alpha\partial_\beta \mathsf{h}^{\mu\nu}\right) \\ &= (-g)t_{LL}^{\mu\nu} + \frac{c^4}{16\pi G}\partial_\alpha\partial_\beta\left(\mathsf{h}^{\mu\alpha}\mathsf{h}^{\nu\beta} - \mathsf{h}^{\mu\nu}\mathsf{h}^{\alpha\beta}\right) \quad \text{(using the gauge condition)} \\ &= (-g)t_{LL}^{\mu\nu} + \partial_\alpha\partial_\beta\chi^{\mu\nu\alpha\beta} \end{aligned} \tag{2.30}$$

where $\chi^{\mu\nu\alpha\beta} = \frac{c^4}{16\pi G}\left(\mathsf{h}^{\mu\alpha}\mathsf{h}^{\nu\beta} - \mathsf{h}^{\mu\nu}\mathsf{h}^{\alpha\beta}\right)$ and thus write the Einstein equations as:

$$\Box\mathsf{h}^{\mu\nu} = \frac{16\pi G}{c^4}\left((-g)(T^{\mu\nu} + t_{LL}^{\mu\nu}) + \partial_\alpha\partial_\beta\chi^{\mu\nu\alpha\beta}\right). \tag{2.31}$$

From Eq.(2.29), we can see that the gauge condition imposed force that

$$\partial_\mu(((-g)T^{\mu\nu} + \tau^{\mu\nu})) = 0. \tag{2.32}$$

Thus, this is the energy-momentum conservation equation in this formulation.

2.3 Pulsars as a source of gravity waves

Pulsars are fast rotating neutron stars, one of the end stages of stellar evolution that result when the leftover mass is above the Chandrasekhar limit, but less than 2-3 solar masses. We will deal with the quadrupolar ellipticity of these objects as discussed in the Introduction. Quadrupolar ellipticity results from the time varying quadrupole moment of the neutron star. We consider the moment of inertia in two frames associated with a **rigid, rotating body**, the *body-fixed frame* and *inertial frame*. Let the axes of the body frame lie along the principal axes of inertia, labelled as (x', y', z') and the axes of the inertial frame labelled as (x, y, z) with z coinciding with z'. In this scenario, the inertia tensor in the body frame is diagonal and is related to the inertial frame by a time-dependent rotation matrix R_{ij}. The relation goes as $I'_{ij} = R_{ik}R_{jl}I_{kl} = R_{ik}I_{kl}R^T_{lj} \Rightarrow I' = RIR^T$, multiplying from the left by R^T and from the right by R, we get $I = R^T I' R$. The explicit form of the rotation matrix is, assuming angular velocity ω,

$$R = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and thus, obtain

$$I(t) = \begin{bmatrix} \frac{I_1+I_2}{2} + \frac{1}{2}(I_1 - I_2)\cos 2\omega t & \frac{1}{2}(I_1 - I_2)\sin 2\omega t & 0 \\ \frac{1}{2}(I_1 - I_2)\sin 2\omega t & \frac{I_1+I_2}{2} - \frac{1}{2}(I_1 - I_2)\cos 2\omega t & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{2.33}$$

Inspecting the fact that the general definition of the moment of inertia is $I_{ij} = \int d^3x\rho(R^2\delta_{ij} + x_ix_j) = M_{ij} + \int d^3x\rho R^2\delta_{ij}$, where M_{ij} is the second mass moment as previously defined. Recalling that in the formula for leading order contribution to the GW signal (i.e., the perturbation to the background metric), we have the second time derivative of $M_{ij}(t)$, we try out

$$\begin{aligned} \frac{2G}{c^4r}\Lambda_{ij;kl}(\hat{n})\ddot{I}_{kl} &= \frac{2G}{c^4r}\Lambda_{ij;kl}(\hat{n})\left(\delta_{kl}\int d^3x\ddot{\rho}R^2 + \ddot{M}_{kl}\right) \\ &= \frac{2G}{c^4r}\Lambda_{ij;kl}(\hat{n})\ddot{M}_{kl} \\ &= h_{ij}^{TT}(t, \mathbf{x}) \end{aligned}$$

where, in the second line, we have used the fact that $\Lambda_{ij;kl}\delta_{kl} = 0$. We can now use all the moment of inertia formulae that we have obtained in this section to find explicit expressions for h_+ and $h_\times$, for an observer at an angle θ to x_3 axis (x_1 and x_2 oriented such that $\phi = 0$) as follows:

$$\boxed{\begin{aligned} h_+ &= \frac{4G\omega^2}{rc^4}\epsilon I_3\frac{(1+\cos^2\theta)}{2}\cos 2\omega t & (2.34) \\ h_\times &= \frac{4G\omega^2}{rc^4}\epsilon I_3\cos\theta\sin 2\omega t. & (2.35) \end{aligned}}$$

Using Eq.(2.18) and writing $\frac{dE}{dtdA} = \frac{dP}{dA} = \frac{dP}{r^2d\Omega}$, and substituting the above formulae and integrating, we get

$$P = -\frac{dE}{dt} = \frac{32G}{5c^5}\epsilon^2 I_3^2\omega^6 \tag{2.36}$$

which is the rate of **energy loss** of the neutron star. Next, using the expression for rotational energy from classical mechanics, $E = \frac{1}{2}I\omega^2$, we get,

$$\frac{dE}{dt} = I\omega\dot{\omega} = -\frac{32G}{5c^5}\epsilon^2 I_3^2\omega^6$$

that is,

$$\boxed{\dot{\omega} = -\frac{32G}{5c^5}\epsilon^2 I_3\omega^5} \tag{2.37}$$

is the rate of decrease of the angular velocity of the star.

2.3.1 Applicability of the above derived amplitudes

In this section, we will check/justify the applicability of Eq.(2.27) and thus, Eqs.(2.34) and (2.35) to the case of neutron stars. In general, pulsars are rapidly rotating neutron stars-the most compact directly observable objects in the Universe. Clearly, these objects are strong field objects and the formulae we derived using weak field limits may not be applicable. For such relativistic objects, Ipser [22] showed that the leading order contribution to the amplitude for a slow moving strong field object is structurally identical to the Eq.(2.27) with some subtle differences:the mass quadrupole moment has to be the coefficient of $\frac{1}{r^3}(r = x_i x^i)$ in the expansion of the 00th component of the metric in a family of coordinate systems known as *asymptotically Cartesian and mass centred*(ACMC) coordinate systems [23], of which the harmonic coordinate system we previously alluded to is an example. The space surrounding a neutron star with mass M, radius R, and angular velocity ω can be divided into three regions [23]:

1. the strong field regime with $r \lesssim \frac{2GM}{c^2}$
2. the weak field near zone with $max(R, \frac{2GM}{c^2}) \lesssim r \lesssim \frac{c}{\omega}$
3. the wave zone with $r > \frac{c}{\omega}$

We know that even for extreme cases, the weak field near zone is well-defined [24]. Now, to use the formula developed by Ipser, we need to know how ACMC coordinate systems are defined. An *asymptotically Cartesian and mass centred*(ACMC) coordinate system is a coordinate system (x^0, x^1, x^2, x^3) in which the metric has the following $1/r$ expansion in the weak field near zone:

$$g_{00} = -1 + \frac{2M}{r} + \frac{\alpha_1}{r^2} + \frac{1}{r^3}\left[3\mathcal{I}_{ij}\frac{x^i x^j}{r^2} + \beta_{1i}\frac{x^i}{r} + \alpha_2\right] + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{2.38}$$

$$g_{0i} = -4\epsilon_{ikl}J^k\frac{x^l}{r^3} + \mathcal{O}\left(\frac{1}{r^3}\right) \tag{2.39}$$

$$g_{ij} = \delta_{ij} + \frac{\alpha_{3ij}}{r} + \frac{1}{r^2}\left[\beta_{2ijk}\frac{x^k}{r} + \alpha_{4ij}\right] + \mathcal{O}\left(\frac{1}{r^3}\right) \tag{2.40}$$

where $\alpha_1, \alpha_2, \alpha_{3ij}, \alpha_{4ij}, \beta_{1i}$, and β_{2ijk} are some constants. According to [24] and [23], the quantity $\mathcal{I}_{ij}$ is a tensor that resides in flat space. This means that it's indices can be manipulated using the flat space metric. It is also symmetric and traceless. In the Newtonian limit, it can be written in terms of the moment of inertia tensor as:

$$\mathcal{I}_{ij} = -I_{ij} + \frac{1}{3}I^k_k\delta_{ij}$$

with the moment of inertia tensor I_{ij} defined by

$$I_{ij} = \int d^3x \, \rho(\mathbf{x})(x_k x^k \delta_{ij} - x_i x_j). \tag{2.41}$$

Finally, the GW amplitude is given by [24], [22] and [23]:

$$\boxed{h_{ij}^{TT} = \frac{1}{r}\Lambda_{ij;kl}(\hat{n})\frac{2G}{c^4}\ddot{\mathcal{I}}_{kl}\left(t - \frac{r}{c}\right)} \tag{2.42}$$

Chapter 3

Neutron stars

In this chapter, we will first understand what the effect of the magnetic field is on the shape of the neutron star. We will prove that it is similar to that of a rotation, in that it flattens at the poles [25]. Following this, we will explicitly calculate the deformations due to the magnetic field [26].

3.1 Effect of magnetic field on the shape of the neutron star

In this section, we will check the stability of the shape of neutron stars with magnetic fields. We will model the neutron star as an incompressible fluid with a uniform magnetic field inside and a dipole magnetic field outside. We follow the discussion from section IV of [25]. Using the above cited model for the magnetic field of a neutron star, using spherical polar coordinates, we have:

$$B_r^{int} = B_0 \cos\theta \quad \text{and} \quad B_\theta^{int} = -B_0 \sin\theta \tag{3.1}$$

$$B_r^{ext} = B_0 \left(\frac{R}{r}\right)^3 \cos\theta \quad \text{and} \quad B_\theta^{ext} = \frac{1}{2} B_0 \left(\frac{R}{r}\right)^3 \sin\theta \tag{3.2}$$

where B_r^{int} and B_θ^{int} are the components of the internal magnetic field and B_r^{ext} and B_θ^{ext} are components of the external dipole magnetic field. Next, calculating the energy stored in the magnetic field, we get

$$\begin{aligned}
E^{int} &= \frac{1}{2\mu_0}\int d\tau \mathbf{B}^{int^2} \\
&= \frac{1}{8\pi}\int_{r<R} r^2 \sin\theta dr d\theta d\phi((B_r^{int})^2 + (B_\theta^{int})^2) \\
&= \frac{1}{6}B_0^2 R^3 \qquad (3.3)
\end{aligned}$$

where we have used natural units in which $\mu_0 = 4\pi$ and the integration limits are $0 < r < R, 0 \leq \theta < \pi$, and $0 \leq \phi < 2\pi$. For the energy stored in the external field,

$$\begin{aligned}
E^{ext} &= \frac{1}{2\mu_0}\int d\tau \mathbf{B}^{ext^2} \\
&= \frac{1}{8\pi}\int_{r>R} r^2 \sin\theta dr d\theta d\phi((B_r^{ext})^2 + (B_\theta^{ext})^2) \\
&= \frac{1}{12}B_0^2 R^3 \qquad (3.4)
\end{aligned}$$

with integration limits for r changed to $R < r < \infty$ and the other two remaining the same. Therefore, total energy $E = E^{int} + E^{ext} = \frac{1}{4}B_0^2 R^3$.

Now that we have established the initial configuration of the system, let us deform the neutron star. The main idea is that we measure the energy change due to the deformation and:

1. if it is **negative**, then the deformed configuration is **more** stable
2. if it is **positive**, then the deformed configuration is **less** stable

Using these ideas, we deform the sphere with a $P_l(\cos\theta)$ deformation:

$$r(\cos\theta) = R + \epsilon P_l(\cos\theta) \quad (\epsilon \ll R) \qquad (3.5)$$

and substitute $c_\theta = \cos\theta$, such that $\sin\theta = (1 - c_\theta^2)^{1/2}$ and $\frac{\partial}{\partial\theta} \equiv -(1 - c_\theta^2)^{1/2}\frac{\partial}{\partial c_\theta}$ and $P_l(\cos\theta) \equiv P_l(c_\theta)$. This kind of deformation of the star can be thought of as a displacement vector ξ applied at each point. This is called a *deformation vector*. Due to the assumption of an incompressible fluid, we have $\nabla \cdot \xi = 0$. We also assume irrotationality and thus have:

$$\xi = \nabla\psi \implies \nabla^2\psi = 0. \qquad (3.6)$$

This is nothing but Laplace's equation. The solutions for axially symmetric systems is given by a linear combination of Legendre polynomials. For a P_l-deformation, the solution is given by:

$$\psi = Ar^l P_l(c_\theta) \quad (\because \text{there is no deformation at the centre}). \tag{3.7}$$

This in turn, implies that $\xi_r = \frac{\partial\psi}{\partial r} = Alr^{l-1}P_l(c_\theta)$ and $\xi_\theta = \frac{\partial\psi}{r\partial\theta} = -Ar^{l-1}(1-c_\theta^2)^{\frac{1}{2}}P_l'(c_\theta)$(and $\xi_\phi = 0$). Here the prime denotes differentiation w.r.t c_θ. But, we know that the deformation at $r = R$ is given by Eq.(3.5) and thus,

$$\begin{aligned} \xi_r &= \epsilon P_l(c_\theta) \quad \text{at} \quad r = R \\ \implies AlR^{l-1}P_l(c_\theta) &= \epsilon P_l(c_\theta) \\ \implies A &= \frac{\epsilon}{lR^{l-1}} \end{aligned} \tag{3.8}$$

and putting this value of A back in the expressions for ξ_r and ξ_θ, we obtain:

$$\xi_r = \frac{\epsilon}{R^{l-1}}r^{l-1}P_l(c_\theta) \quad \text{and} \quad \xi_\theta = -\frac{\epsilon}{lR^{l-1}}r^{l-1}(1-c_\theta^2)^{1/2}P_l'(c_\theta). \tag{3.9}$$

Now that we have the deformation vector, we can find its effect on the magnetic field and thus the effect it has on the energy of the system. Denote this change in magnetic field by $\delta\mathbf{B}$.

3.1.1 Inside the star

We assume that inside the star, we have infinite conductivity i.e., $\sigma = \infty$. Physically, this means that a change in the existing magnetic field can only be obtained by physically pushing aside the field lines(because by Alfven's theorem the field lines are "frozen" into the fluid [27]). Then, to obtain the mathematical form for this "pushing", using Ohm's law($\frac{\mathbf{J}}{\sigma} = \mathbf{E} + \mathbf{v}\times\mathbf{B}$) and that the deformation happens continuously in time in which case, the velocity vector is given by $\mathbf{u} = \frac{\partial\xi}{\partial t}$ and thus, the electric field due to the changing magnetic field and hence the change in the magnetic field is given by:

$$\begin{aligned} \delta\mathbf{E} = -\mathbf{u}\times\mathbf{B} \quad \text{and} \quad \nabla\times\delta\mathbf{E} &= -\frac{\partial}{\partial t}(\mathbf{B}+\delta\mathbf{B}) \\ \implies \nabla\times\Big(-\frac{\partial\xi}{\partial t}\times\mathbf{B}\Big) &= -\frac{\partial\delta\mathbf{B}}{\partial t} \end{aligned}$$

$$\begin{aligned} &\implies \nabla\times(\xi\times\mathbf{B}) = \delta\mathbf{B} \\ &\implies \delta\mathbf{B} = (\mathbf{B}\cdot\nabla)\xi - (\xi\cdot\nabla)\mathbf{B} \quad (\because \nabla\cdot\mathbf{B} = 0 = \nabla\cdot\xi) \end{aligned}$$

Since the magnetic field inside the star to begin with was homogeneous, we have $(\xi \cdot \nabla)\mathbf{B} = 0$ and using spherical polar coordinates, we have:

$$\delta B_r^{int} = B_r^{int} \frac{\partial \xi_r}{\partial r} + \frac{B_\theta^{int}}{r} \frac{\partial \xi_r}{\partial \theta} - \frac{B_\theta^{int} \xi_\theta}{r} \quad (\because \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}) \tag{3.10}$$

$$\delta B_\theta^{int} = B_r^{int} \frac{\partial \xi_\theta}{\partial r} + \frac{B_\theta^{int}}{r} \frac{\partial \xi_\theta}{\partial \theta} + \frac{B_\theta^{int} \xi_r}{r} \quad (\because \frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}). \tag{3.11}$$

We have thus obtained the expressions for the change in the magnetic field - the next step is to calculate the change in energy due to this. But before that let's put the expressions (3.9) back in the above equations to obtain equations such as:

$$\begin{aligned}\delta B_r^{int} &= \epsilon B_0 \frac{r^{l-2}}{R^{l-1}} (l-1) \left[c_\theta P_l(c_\theta) + \frac{(1-c_\theta^2)}{l} P_l'(c_\theta) \right] \\ &= \epsilon B_0 \frac{r^{l-2}}{R^{l-1}} (l-1) P_{l-1}(c_\theta)\end{aligned} \tag{3.12}$$

where we have used the identity $xP_l(x) + \frac{(1-x^2)^{1/2}}{l}\frac{\partial P_l(x)}{\partial x} = P_{l-1}(x)$, and

$$\begin{aligned}\delta B_\theta^{int} &= -\epsilon B_0 \frac{r^{l-2}}{R^{l-1}} (1-c_\theta^2)^{1/2} \left[c_\theta \frac{l-2}{l} P_l'(c_\theta) + \frac{(1-c_\theta^2)}{l} P_l''(c_\theta) + P_l(c_\theta) \right] \\ &= -\epsilon B_0 \frac{r^{l-2}}{R^{l-1}} (1-c_\theta^2)^{1/2} P_{l-1}'(c_\theta)\end{aligned} \tag{3.13}$$

where the last line can be obtained by differentiating the previously mentioned identity. Now, the change in energy density is given by

$$\begin{aligned}\delta \mathcal{E}^{int} &= \delta \Big(\frac{\mathbf{B}^2}{8\pi} \Big) \\ &= \frac{1}{4\pi} \mathbf{B} \cdot \delta \mathbf{B} \\ &= \epsilon \frac{B_0^2}{4\pi} \frac{r^{l-2}}{R^{l-1}} [(l-1) c_\theta P_l(c_\theta) + (1-c_\theta^2) P_l'(c_\theta)]\end{aligned} \tag{3.14}$$

and averaging this over all directions, and using $P_0(c_\theta) = 1$ and the orthogonality properties of Legendre polynomials, $\int_{-1}^{+1} dx P_l(x) P_l'(x) = \frac{2}{2l+1}\delta_{ll'}$, we get:

$$\begin{aligned}
\langle \delta\mathcal{E}^{int} \rangle &= \frac{3\epsilon(l-1)B_0^2}{8\pi R^3}\frac{R^2}{l+1}\int_{-1}^{+1} dc_\theta P_{l-2}(c_\theta)P_0(c_\theta) \\
&= \frac{3\epsilon B_0^2}{8\pi R}\frac{l-1}{l+1}\delta_{l-2,0}\frac{2}{2\cdot 0+1} \\
&= \frac{\epsilon B_0^2}{4\pi R}
\end{aligned} \tag{3.15}$$

and thus multiplying by volume, we get the energy change as

$$\boxed{\delta E^{int} = \langle \delta\mathcal{E}^{int} \rangle \times \frac{4\pi}{3}R^3 = \frac{\epsilon}{3}B_0^2 R^2}. \tag{3.16}$$

3.1.2 The external field energy

The change in the energy of the magnetic field outside the star due to a P_l-deformation will be the subject of this section. We can't use the conductivity arguments here and will thus have to use a different method to obtain the changes in the magnetic field. Let us write the new magnetic field components as:

$$B_r^{ext} = B_0\Big(\frac{R}{r}\Big)^3 c_\theta + \delta B_r^{ext} \quad \text{and} \quad B_\theta^{ext} = \frac{1}{2}B_0\Big(\frac{R}{r}\Big)^3(1-c_\theta^2)^{1/2} + \delta B_\theta^{ext}. \tag{3.17}$$

Using the assumption that $\nabla \times \delta\mathbf{B}^{ext} = 0$, we may write $\delta\mathbf{B}^{ext} = \nabla\delta\phi^{ext}$. Thus, $\nabla\cdot\delta\mathbf{B}^{ext} = \nabla^2\delta\phi^{ext} = 0$. Once again, we are dealing with the Laplace equation and for spherical symmetry, we have the solutions as

$$\delta\phi^{ext}(r,\theta) = \Sigma_l A_l r^l P_l(\cos\theta) + \frac{B_l}{r^{l+1}}P_(\cos\theta) \tag{3.18}$$

for $r \geq R$. But we will only take the second set of terms because we want the perturbations to die out at infinity. Thus, writing $\delta\phi^{ext}$ as above with a slight modification, we have:

$$\delta\phi^{ext} = -\epsilon B_0\Big[\frac{l-1}{l}\Big(\frac{R}{r}\Big)^l P_{l-1}(c_\theta) + \Sigma_j A_j\Big(\frac{R}{r}\Big)^{j+1}P_j(c_\theta)\Big] \tag{3.19}$$

and taking the gradient in spherical polar coordinates, we get the expressions for δB_r^{ext} and δB_θ^{ext} as follows:

$$\delta B_r^{ext} = \epsilon B_0 \left[(l-1)\frac{R^l}{r^{l+1}}P_{l-1}(c_\theta) + \Sigma_j A_j (j+1)\frac{R^{j+1}}{r^{j+2}}P_j(c_\theta)\right] \tag{3.20}$$

$$\begin{aligned}\delta B_\theta^{ext} &= \epsilon B_0 \left[\frac{(l-1)}{l}\frac{R^l}{r^{l+1}}(1-c_\theta^2)^{1/2}P'_{l-1}(c_\theta) + \Sigma_j A_j \frac{R^{j+1}}{r^{j+2}}(1-c_\theta^2)^{1/2}P'_j(c_\theta)\right] \\ &= \epsilon B_0 \left[\frac{(l-1)}{l}\frac{R^l}{r^{l+1}}P^1_{l-1}(c_\theta) + \Sigma_j A_j \frac{R^{j+1}}{r^{j+2}}P^1_j(c_\theta)\right].\end{aligned} \tag{3.21}$$

Our aim now is to find the values of all the coefficients A_j. For this purpose, we will use the boundary condition that the component of the magnetic field normal to the surface should be continuous. The surface that we have is $r = R + \epsilon P_l(c_\theta)$ or $f(r,\theta) = r - R - \epsilon P_l(c_\theta) = 0$. From basic vector calculus, the normal to a surface $f(x,y,z) = constant$ is given by ∇f, the gradient. Using the same logic here, the normal vector at $r = R + \epsilon P_l(c_\theta)$ is obtained, to first order in ϵ, as $\overrightarrow{n} = \hat{r} - \frac{\epsilon}{R}\frac{\partial P_l(c_\theta)}{\partial\theta}\hat{\theta}$. Thus, for the continuity condition, we have:

$$\begin{aligned}&\mathbf{B}^{ext}\cdot\overrightarrow{n} = \mathbf{B}^{int}\cdot\overrightarrow{n} \\ \Longrightarrow\ & B_r^{ext}|_{R+\epsilon P_l(c_\theta)} + B_\theta^{ext}|_R \frac{\epsilon}{R}(1-c_\theta^2)^{1/2}\frac{\partial P_l(c_\theta)}{\partial c_\theta} \\ &\quad = B_r^{int}|_{R+\epsilon P_l(c_\theta)} + B_\theta^{int}|_R \frac{\epsilon}{R}(1-c_\theta^2)^{1/2}\frac{\partial P_l(c_\theta)}{\partial c_\theta}\end{aligned} \tag{3.22}$$

where in the second line, B_θ's are evaluated at R only because $R + \epsilon P_l(c_\theta)$ would anyway make the correction second order in ϵ. Next, we will evaluate the LHS and RHS separately and equate them in the end. First, the LHS is given, from Eqs.(3.17), as:

$$\begin{aligned}\text{LHS} &= c_\theta B_0 \frac{R^3}{R^3\left(1+\frac{\epsilon P_l}{R}\right)^3} + \epsilon B_0 \Big[(l-1)\frac{R^l}{R^{l+1}\left(1+\frac{\epsilon P_l}{R}\right)^{l+1}}P_{l-1}(c_\theta) \\ &\quad + \Sigma_j A_j (j+1)\frac{R^{j+1}}{R^{j+2}\left(1+\frac{\epsilon P_l}{R}\right)^{j+2}}P_j(c_\theta)\Big] + \Big\{\frac{1}{2}(1-c_\theta^2)^{1/2}B_0 \\ &\quad + \epsilon\frac{B_0}{R}\left[\frac{(l-1)}{l}P^1_{l-1}(c_\theta) + \Sigma_j A_j P^1_j(c_\theta)\right]\Big\}\frac{\epsilon}{R}(1-c_\theta^2)^{1/2}\frac{\partial P_l(c_\theta)}{\partial c_\theta} \\ &\simeq c_\theta B_0\left(1 - \frac{3\epsilon P_l}{R}\right) + \frac{\epsilon B_0}{R}\Big[(l-1)(1-\mathcal{O}(\epsilon))P_{l-1} \\ &\quad + \Sigma_j A_j (j+1)(1-\mathcal{O}(\epsilon))P_j\Big] + \frac{1}{2}B_0(1-c_\theta^2)\frac{\epsilon}{R}P'_l + \mathcal{O}(\epsilon^2)\end{aligned}$$

$$= c_\theta B_0 \left(1 - \frac{3\epsilon P_l}{R}\right) + \frac{\epsilon B_0}{R}\Big[(l-1)P_{l-1} + \Sigma_j A_j (j+1) P_j\Big]$$
$$+ \frac{1}{2} B_0 (1 - c_\theta^2) \frac{\epsilon}{R} P_l' \tag{3.23}$$

and moving on to the RHS, using Eqs.(3.1), (3.12) and (3.13), we obtain:

$$\text{RHS} = c_\theta B_0 + \epsilon B_0 (l-1) P_{l-1}(c_\theta) \frac{R^{l-2}}{R^{l-1}} \Big(1 + (l-2)\frac{\epsilon P_l}{R}\Big)$$
$$+ \Big[- B_0 (1 - c_\theta^2)^{1/2} - \mathcal{O}(\epsilon)\Big] \frac{\epsilon}{R} (1 - c_\theta^2)^{1/2} P_l'$$
$$= c_\theta B_0 + \frac{\epsilon B_0}{R}(l-1) P_{l-1}(c_\theta) - \frac{\epsilon B_0}{R}(1 - c_\theta^2) P_l' \tag{3.24}$$

where, in both LHS and RHS, we have used the fact that we require the expressions only till $\mathcal{O}(\epsilon)$. Now, using LHS = RHS, we obtain the following

$$\Sigma_j A_j (j+1) P_j = 3 c_\theta P_l - \frac{3}{2}(1 - c_\theta^2) P_l'. \tag{3.25}$$

Using the identities

$$(2l+1) x P_l(x) = (l+1) P_{l+1}(x) + l P_{l-1}(x), \text{and}$$
$$P_{l-1}(x) = x P_l(x) + \frac{(1-x^2)^{1/2}}{l} \frac{\partial P_l}{\partial x}$$

we can simplify the above written equation to obtain the following

$$\Sigma_j A_j (j+1) P_j = \frac{3}{2(2l+1)} [(l+1)(l+2) P_{l+1} - l(l-1) P_{l-1}] \tag{3.26}$$

from which we may conclude:

$$A_{l-1} = -\frac{3(l-1)}{2(2l+1)}, \quad A_{l+1} = \frac{3(l+1)}{2(2l+1)}, \quad \text{and} \quad A_j = 0 \;\; \forall j \neq l-1, l+1. \tag{3.27}$$

Substituting the above obtained values for the coefficients back into Eqs.(3.20) and (3.21), we have our changes to the external field as:

$$\delta B_r^{ext} = \epsilon B_0 \Big[\frac{3(l+1)(l+2)}{2(2l+1)} \frac{R^{l+2}}{r^{l+3}} P_{l+1} + \frac{(l-1)(l+2)}{2(2l+1)} \frac{R^l}{r^{l+1}} P_{l-1}\Big] \tag{3.28}$$

$$\delta B_\theta^{ext} = \epsilon B_0 \Big[\frac{3(l+1)}{2(2l+1)} \frac{R^{l+2}}{r^{l+3}} P^1_{l+1} + \frac{(l-1)(l+2)}{2l(2l+1)} \frac{R^l}{r^{l+1}} P^1_{l-1}\Big] \tag{3.29}$$

from which we can now calculate the change in the energy stored in the magnetic field. The energy for the deformed configuration is given by:

$$E^{ext}_{new} = E^{ext} + \delta E^{ext} = \frac{1}{8\pi}\iiint_{R+\epsilon P_l}^{\infty} d\tau(\mathbf{B}^{ext})^2, \quad (\mathbf{B}^{ext} = B^{ext}_r\hat{r} + B^{ext}_\theta\hat{\theta}) \tag{3.30}$$

where B^{ext}_r and B^{ext}_θ are given by Eqs.(3.17). If we represent the original dipole magnetic field by $\mathbf{B}^{ext}_0$, we may write $(\mathbf{B}^{ext})^2 = (\mathbf{B}^{ext}_0)^2 + 2\mathbf{B}^{ext}_0 \cdot \delta\mathbf{B}^{ext} + \mathcal{O}(\epsilon^2)$. Also note that the integration limits should signify the exterior of the star and hence start from $R+\epsilon P_l$ and not R. We already know that $E^{ext} = \iiint_R^\infty d\tau(\mathbf{B}^{ext}_0)^2$ and hence, paying close attention to the limits, we obtain the change in energy of the magnetic field as:

$$\delta E^{ext} = -\frac{1}{8\pi}\iiint_R^{R+\epsilon P_l} r^2 dr dc_\theta d\phi \, (\mathbf{B}^{ext}_0)^2 + \frac{1}{8\pi}\iiint_{R+\epsilon P_l}^{\infty} r^2 dr dc_\theta d\phi \, 2\mathbf{B}^{ext}_0 \cdot \delta\mathbf{B}^{ext} \tag{3.31}$$

where in the second integral we may replace the lower limit by R, since the correction will be of second order in ϵ. Thus, putting in the expressions for the magnetic field in the above equation and using $P_1(c_\theta) = c_\theta$, $P_2(c_\theta) = \frac{1}{2}(3c_\theta^2 - 1)$ and $P^1_1(c_\theta) = (1-c_\theta^2)^{1/2}$, we obtain:

$$\begin{aligned}
\delta E^{ext} &= -\frac{1}{4}\int_{-1}^{1}\int_R^{R+\epsilon P_l} r^2 dr dc_\theta B_0^2\left(\frac{R}{r}\right)^6\left[c_\theta^2 + \frac{1}{4}(1-c_\theta^2)\right] \\
&\qquad + \frac{1}{4}\int_{-1}^{1}\int_R^{\infty} r^2 dr dc_\theta B_0\left(\frac{R}{r}\right)^3\left[2c_\theta\delta B^{ext}_r + 2\frac{1}{2}(1-c_\theta^2)^{1/2}\delta B^{ext}_\theta\right] \\
&= -\frac{1}{4}\epsilon B_0^2 R^2\int_{-1}^{1} dc_\theta P_l(c_\theta)\left[\frac{1}{2}P_2(c_\theta) + \frac{1}{2}\right] \\
&\quad + \frac{1}{4}\int_{-1}^{1}\int_R^{\infty} r^2 dr dc_\theta B_0\left(\frac{R}{r}\right)^3\left[2P_1(c_\theta)\delta B^{ext}_r + P^1_1(c_\theta)\delta B^{ext}_\theta\right]
\end{aligned}$$

and now using Eqs.(3.28) and (3.29) in the above expressions, we have the following:

$$\begin{aligned}
\delta E^{ext} &= -\frac{1}{4}\epsilon B_0^2 R^2\int_{-1}^{1} dc_\theta P_l(c_\theta)\left[\frac{1}{2}P_2(c_\theta) + \frac{1}{2}\right] \\
&\quad + \frac{1}{2}\epsilon B_0^2\int_{-1}^{1}\int_R^{\infty} r^2 dr dc_\theta\left(\frac{R}{r}\right)^3 P_1(c_\theta)\Big[\frac{3(l+1)(l+2)}{2(2l+1)}\frac{R^{l+2}}{r^{l+3}}P_{l+1} + \frac{(l-1)(l+2)}{2(2l+1)}\frac{R^l}{r^{l+1}}P_{l-1}\Big] \\
&\quad + \frac{1}{4}\epsilon B_0^2\int_{-1}^{1}\int_R^{\infty} r^2 dr dc_\theta\left(\frac{R}{r}\right)^3 P^1_1(c_\theta)\Big[\frac{3(l+1)}{2(2l+1)}\frac{R^{l+2}}{r^{l+3}}P^1_{l+1} + \frac{(l-1)(l+2)}{2l(2l+1)}\frac{R^l}{r^{l+1}}P^1_{l-1}\Big].
\end{aligned}$$

In the above expressions, we can use orthogonality of Legendre polynomials and associated Legendre polynomials to conclude that only $l = 0$ and $l = 2$ are possibly non zero. Now, noticing the l in the denominator of the 2nd term of the 3rd integral, we have $l = 0$ not viable. Hence, the change in the magnetic energy is non zero to first order in ϵ only for a P_2-deformation. Thus, for a P_2-deformation, we have the change in the external magnetic energy given by

$$\begin{aligned}
\delta E^{ext} &= -\frac{1}{4}\epsilon B_0^2 R^2 \int_{-1}^{1} dc_\theta \frac{1}{2}[P_2(c_\theta)]^2 + \frac{1}{2}\epsilon B_0^2 R^5 \int_{-1}^{1}\int_{R}^{\infty} dr dc_\theta [P_1(c_\theta)]^2 \frac{2}{5}\frac{1}{r^4} \\
&\quad + \frac{1}{4}\epsilon B_0^2 R^5 \int_{-1}^{1}\int_{R}^{\infty} dr dc_\theta [P_1^1(c_\theta)]^2 \frac{1}{5}\frac{1}{r^4} \\
&= \epsilon B_0^2 R^2 \Big(-\frac{1}{20} + \frac{2}{45} + \frac{1}{45}\Big) \\
\implies & \boxed{\delta E^{ext} = \frac{1}{60}\epsilon B_0^2 R^2},
\end{aligned} \tag{3.32}$$

where we have used

$$\int_{-1}^{1} dx [P_l(x)]^2 = \frac{2}{2l+1} \quad \text{and} \quad \int_{-1}^{1} dx [P_l^m()x)]^2 = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}.$$

Thus, the total change in energy of the system, to $\mathcal{O}(\epsilon)$, is, from Eqs. (3.16) and (3.32),

$$\begin{aligned}
\delta E &= \delta E^{int} + \delta E^{ext} \\
&= \frac{1}{3}\epsilon B_0^2 R^2 + \frac{1}{60}\epsilon B_0^2 R^2 \\
&= \frac{7}{20}\epsilon B_0^2 R^2
\end{aligned} \tag{3.33}$$

which is first order in ϵ. This δE is

1. > 0 if $\epsilon > 0$ or *prolate* shape
2. < 0 if $\epsilon < 0$ or *oblate* shape.

The change in the gravitational potential energy for a P_l-deformation is given by

$$\delta U = \frac{3(l-1)}{(2l+1)^2}\left(\frac{\epsilon}{R}\right)^2 \frac{GM^2}{R}, \tag{3.34}$$

clearly second order in ϵ and is thus always positive. The total change in energy is, therefore,

$$\delta \mathbb{U} = \delta U + \delta E$$

and putting in the values obtained in Eqs.(3.33) and (3.34), for a P_2-deformation, we obtain:

$$\delta\mathbb{U} = \frac{3}{25}\left(\frac{\epsilon}{R}\right)^2 \frac{GM^2}{R} + \frac{7}{20}\epsilon B_0^2 R^2$$

In the above equation, since δU is second order in ϵ and δE is first order in ϵ, the sign of change in energy of the system is determined by the sign of ϵ for small ϵ. Energy previously was $\mathbb{U} = U + E$, after the deformation, it becomes $\mathbb{U} + \delta\mathbb{U} = U + E + (\delta U + \delta E)$. In order to obtain the most stable configuration, we need to minimize this change and if it is negative the configuration thus obtained will be stable. Thus, minimizing the previously obtained $\delta\mathbb{U}$, we have:

$$\frac{\epsilon}{R} = -\frac{35}{24}\frac{B_0^2 R^4}{GM^2}. \tag{3.35}$$

Using the virial theorem, we can do a simple calculation(cf. section I of [25]) wherein we obtain that for dynamical stability $(|U| - E) > 0$ when $\gamma > \frac{4}{3}$. Thus, setting $E = |U|$, with $U = -\frac{3}{5}\frac{GM^2}{R}$ and $E = \frac{1}{4}B_*^2 R^3$, we have the value of B_*^2 to be

$$B_*^2 = \frac{12}{5}\frac{GM^2}{R^4}, \tag{3.36}$$

putting it back in Eq.(3.35), we obtain:

$$\boxed{\frac{\epsilon}{R} = -\frac{7}{2}\left(\frac{B_0}{B_*}\right)^2}$$

where B_* is the "limit" set by the virial theorem [25].

3.1.3 Quadrupolar ellipticity vs surface ellipticity

At this point, we may introduce some terminology. Following [28], we define *surface ellipticity* and *quadrupolar ellipticity*. When we talk of surface ellipticity, we are talking of:

$$\boxed{\epsilon_{surf} = \frac{R_{equatorial} - R_{polar}}{R_{polar}}}$$

and by quadrupolar ellipticity, we mean:

$$\boxed{\epsilon_{quad} = -\frac{Q}{I} = \frac{I_{zz} - I_{yy}}{I_{zz}}}$$

where Q is the mass energy quadrupole moment and I is the mean moment of inertial of the tensor I_{ij}. The second equality, though, can only be written in the limit of a weak field. More details on this terminology can be found in [28] and references therein.

In order to generate gravitational waves, we saw that the variation of the quadrupole moment with time is essential. On this note, let us look at what the status is on the time dependence of quadrupolar ellipticity. In [29], evidence is obtained that the quadrupolar ellipticity evolves in time. To see this, the authors consider two models for the magnetic fields of neutron stars [29], called AL and BL models. In the AL model, the magnetic field is confined to the crust and in the BL model, there is an internal magnetic field. Following the results of magneto-thermal simulations (taken from [29] and the references therein), the ellipticity of the magnetar is seen to change in magnitude from $\sim 10^{-11}$ at 0 yrs to $\sim 2.5 \times 10^{-6}$ after 10^6 yrs [29] for BL models. Same procedure for AL models yields values of ellipticity to be $\sim 1.09 \times 10^{-8}$ at 0 yr which after a million years evolves to $\sim 9.93 \times 10^{-10}$. From this, it is clear that the quadrupolar ellipticity varies with time and thus, we also have a variation in gravitational wave signal over this time. (It can also be seen that the BL model predicts higher ellipticities than the AL model). Now that we have obtained some rough values for the ellipticities, we may use Eqs.(2.34) and (2.35) for the GW amplitudes. We see that the GW amplitude is directly proportional to the ellipticity ϵ and the square of the frequency of rotation of the neutron star. Based on the proportionality to ϵ alone, magnetars seem to be good sources of GWs, with $\epsilon \sim 10^{-6}$. But magnetars also have the frequency of rotation to be around 0.1 Hz which makes the GW amplitude quite small in magnitude [29] which is very tough to detect, especially because thermal and quantum noise reduce the sensitivity of the detectors when the frequency is below 10 Hz. For this reason, proto-magnetars are far more promising candidates for GW signals than magnetars [29] due to having higher frequencies of rotation. Hence, the best candidates for possible experimental realization of our current proposition, the effect of SPP of mmcps on the GW signals from neutron stars, seem to be proto-magnetars.

3.2 Models of the magnetic fields

Now that we have a basic understanding of the effect of the magnetic field on the shape of the neutron star, we can now tackle the problem of emission of GW's from such a star. We require the magnetic axis and to be at an angle to the rotation axis so that we have a non-zero quadrupole moment(due to the flattening discussed above) and thus, a gravitational wave signal. For this purpose, we need to be able to calculate the deformation due to the magnetic field. In the following discussion, we will consider different simple models for the neutron star (with an external dipolar field), and using the equations of hydromagnetic equilibrium, solve for the internal magnetic field and the consequent deformation [26]. We shall assume that the magnetic field can be treated as a perturbation. The equations of hydromagnetic equilibrium for a non-rotating star are given by:

$$\frac{\nabla p}{\rho} + \nabla\Phi = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0 \rho} \tag{3.37}$$

where p, ρ, Φ, and $\mathbf{B}$ are pressure, density, gravitational potential, and magnetic field vector respectively and Maxwell's equations for current density have been used. We may replace μ_0 by 4π(i.e., working in natural units). Now, the gravitational potential satisfies the Poisson equation $\nabla^2\Phi = 4\pi G\rho$ and the magnetic field, from Maxwell's equations, satisfies $\nabla \cdot \mathbf{B} = 0$. Taking the curl of Eq.(3.37) in addition to the assumption that the equation of state is given by $\rho = \rho(p)$, we obtain:

$$\nabla \times \left(\frac{\nabla \times \mathbf{B}) \times \mathbf{B}}{\rho}\right) = 0 \tag{3.38}$$

wherein we have used $\nabla\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\nabla\rho = -\frac{1}{\rho^2}\frac{\partial\rho}{\partial p}\nabla p(\because \rho = \rho(p))$ and then that the curl of a gradient is identically zero. There are ways/justifications around the above constraint (3.38), but they are harder problems [26] with more subtleties popping up and we shall thus impose this constraint in our problem. As previously stated, assume that the magnetic field effects are a perturbation of the spherically symmetric background. We can thus expand all our variables as:

$$f(r,\theta) = f_0(r,\theta) + f_1(r,\theta)P_l(c_\theta)$$

where P_l are the Legendre polynomials and $f_1 \sim \mathcal{O}(B^2)$. We are ultimately interested in the gravitational wave emission due to these deformations and thus, the most important contribution for us is $l = 2$ quadrupole deformation [26]. Another reason is as discussed in the previous section, only the $l = 2$ deformation is the significant one at first order in deformation. We shall now delve into calculating the internal magnetic field using all the equations we have written above and by assuming a form for the equation of state. We shall, in particular, consider two configurations:

1. The star has uniform density throughout

2. The fluid inside the star is a polytrope with $n = 1$

Using the concept of a stream function $S(r,\theta)$ and using $\nabla \cdot \mathbf{B} = 0$, we have:

$$\frac{\partial}{\partial r}(r^2 \sin\theta B_r) = \frac{\partial}{\partial\theta}(-r\sin\theta B_\theta)$$

and now, using the concepts of exact differentials(and equality of mixed partial derivatives), we may write an exact differential dS, where S is the stream function, as:

$$dS = r^2 \sin\theta B_r d\theta - r\sin\theta B_\theta dr. \tag{3.39}$$

Using the above equation, $\frac{\partial S}{\partial \theta} = r^2 \sin\theta B_r$ and $\frac{\partial S}{\partial r} = -r\sin\theta B_\theta$. Inverting the foregoing relations, we have:

$$B_r = \frac{1}{r^2 \sin\theta}\frac{\partial S}{\partial \theta} \quad \text{and} \quad B_\theta = -\frac{1}{r\sin\theta}\frac{\partial S}{\partial r}. \tag{3.40}$$

Since we have axial symmetry in our problem, using Eq.(3.37), we have

$$[\mathbf{B} \times (\nabla \times \mathbf{B})]_\phi = 0. \tag{3.41}$$

This, along with $\nabla \cdot \mathbf{B} = 0 \nabla \cdot \mathbf{B}_p$, gives:

$$\mathbf{B}_p \cdot \nabla(r \sin\theta B_\phi) = 0. \tag{3.42}$$

Here, $\mathbf{B}_p = (B_r, B_\theta, 0)$ is the poloidal field and $\mathbf{B}_t = (0, 0, B_\phi)$ is the toroidal field. The above equation may be obtained by expanding the curl in spherical coordinates and looking at the ϕ-component. Using the above mentioned concept of stream function, we may write the ϕ-component of the magnetic field as:

$$r\sin\theta B_\phi = \beta(S) \implies B_\phi = \frac{\beta(S)}{r\sin\theta}.$$

Now, we may expand Eq.(3.38) to obtain a large differential equation for S and $\beta(S)$ which, to be solved, requires us to assume a form for the equation of state. As mentioned above, we will be dealing with the two cases of uniform density and $n = 1$ polytrope.

Constant density fluid

For a constant density fluid, Eq.(3.38) reduces to $\nabla \times (\mathbf{B} \times (\nabla \times \mathbf{B})) = 0$. Assume a form for the magnetic field $\mathbf{B}$ as:

$$\mathbf{B} = \hat{r}[W(r)\cos\theta] + \hat{\theta}[X(r)\sin\theta] + \hat{\phi}[iZ(r)\sin\theta]$$

so that $\nabla \cdot \mathbf{B} = 0$ becomes

$$rW'(r) + 2[W(r) + X(r)] = 0. \tag{3.43}$$

In our constant density case, putting Eq.(3.2) into Eq.(3.38), we obtain the following three equations:

$$W(rZ)' + 2ZX = 0 \tag{3.44}$$
$$W(r^2Z'' - 2Z) + 2Z(rX' - X) = 0 \tag{3.45}$$
$$W[r^2X'' - 4(W + X)] + 2Z(rZ' - Z) = 0 \tag{3.46}$$

To go further, we may work with three types of magnetic fields:

1. Purely poloidal $\mathbf{B} = \mathbf{B}_p = (B_r, B_\theta, 0)$
2. Purely toroidal $\mathbf{B} = \mathbf{B}_t = (0, 0, B_\phi)$
3. A general mixed case

Among these, the poloidal case is the most important to us and we shall thus solve it.

1. Pure poloidal In the pure poloidal case, $Z = 0$. Thus, only Eq.(3.43) and Eq.(3.46) remain pertinent. Using these two equations, we can write a differential equation in terms of only X and one interms of both W and X as follows:

$$\begin{aligned}
r^2X''' + 4rX'' - 4X' = 0 &\implies X = \frac{A}{r^3} + Cr^2 + D \\
rW'(r) + 2[W(r) + X(r)] = 0 &\implies rW'(r) + 2[W(r) + Cr^2 + D] = 0 \\
&\implies W(r) = -\frac{1}{2}Cr^2 - D
\end{aligned}$$

where $A, C,$ and D are constants of integration and we have set $A = 0$ because we want field solutions that are regular at the centre $r = 0$. We now have to evaluate the constants C and D. For this step, following [26], we shall solve for the magnetic field through the stream function method as well and match the solutions. Following [26] and section 3 of [30], let us assume that the stream function $S(r, \theta)$ is of the form $A(r)\sin^2\theta$ and $\beta(S) = 0$. The stream function solves the equation [30], [26]:

$$\nabla^2 S = \frac{Br^2}{R^2}\sin^2\theta \tag{3.47}$$

where B is a constant parameter and we have assumed a current density with $J_\phi = \frac{B}{R^2}r^3\sin^3\theta$. With a stream function of this form, the magnetic field looks like:

$$\mathbf{B} = \Big(\frac{2A}{r^2}\cos\theta, -\frac{A'\sin\theta}{r}, 0\Big). \tag{3.48}$$

Matching this with an external dipole field at the surface,

$$\mathbf{B}_{ext} = B_0\left(\frac{R}{r}\right)^3\left(\cos\theta\hat{r} + \frac{1}{2}\sin\theta\hat{\theta}\right)$$

we obtain a condition for A. This is done as follows:

$$\begin{aligned}\frac{2A}{r^2} &= B_0, \text{ and } -\frac{A'}{r} = \frac{1}{2}B_0 \\ \implies \frac{A}{r} &+ A' = 0 \text{ at the surface}\end{aligned} \tag{3.49}$$

Also making sure that the magnetic field is regular at the centre, we have the solution for $S(r,\theta)$ as:

$$S(r,\theta) = \frac{Br^2}{2R^2}\left(\frac{r^2}{5} - \frac{R^2}{3}\right)\sin^2\theta \tag{3.50}$$

and using Eqs.(3.40), we have:

$$\boxed{\begin{aligned}B_r &= -B\left(\frac{1}{3} - \frac{1}{5}\left(\frac{r}{R}\right)^2\right)\cos\theta \\ B_\theta &= B\left(\frac{1}{3} - \frac{2}{5}\left(\frac{r}{R}\right)^2\right)\sin\theta\end{aligned}} \tag{3.51}$$

where, these components are also matched to the external field through Eq.(3.49).

2. Other cases As mentioned above, there are other field configurations which one may explore. In the toroidal case, one would take $W = 0 = X$, with Z non-zero. Consistency with the surface conditions demands even this component to be 0 thus making the toroidal field solution trivial [26]. In the mixed case however, the solution which for which neither $Z = 0$ nor $W = 0$ turns out to have $Z = arW$. This means that this solution can't be matched to an external dipole($\because Z = 0$ there). Furthermore, trying to match the θ component as well to a general solution for the external field which is regular at infinity demands the external field to be trivially 0 throughout the exterior [26]. A few thoughts on this matter may be found in section 2 of [26].

$n = 1$ **polytrope**

Now that we have dealt with a constant density fluid, we may look at a more realistic $n = 1$ polytrope for the interior of the neutron star. A polytropic relation is one of the form

$$p = K\rho^{1+\frac{1}{n}} \tag{3.52}$$

which holds throughout the star, where p, ρ are the pressure and density with K being a proportionality constant and n being the polytropic index. To connect with something more familiar, consider an isothermal gas. In this case, $p = K\rho$, i.e., this is a $n = \infty$ polytrope. In our particular case, we shall look at a $n = 1$ polytrope. The form for the density of a polytrope is obtained by solving the Lane-Emdem equation with boundary conditions as:

1. ρ is regular at $r = 0$.

2. ρ becomes 0 at the surface for the first time.

In the case of $n = 1$ polytrope, the solution for ρ is of the form

$$\rho = \rho_c \frac{\sin y}{y}, \text{ where } y = \frac{\pi r}{R}$$

where ρ_c is the value of ρ at the centre. Using this form for ρ in Eq.(3.38), we may solve for the magnetic field. We may write Eq.(3.38) interms of the stream function $S(r,\theta)$ so that the equation looks like [26]:

$$\begin{aligned}\frac{\partial}{\partial r}\Big[\frac{1}{\rho r\sin\theta}\frac{\partial S}{\partial\theta}\Big\{\frac{1}{r\sin\theta}\frac{\partial^2 S}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta}\frac{\partial S}{\partial\theta}\Big)\Big\} + \frac{\beta}{\rho r^2\sin^2\theta}\frac{\partial B}{\partial\theta}\Big] \\ -\frac{\partial}{\partial\theta}\Big[\frac{1}{\rho r\sin\theta}\frac{\partial S}{\partial r}\Big\{\frac{\partial^2 S}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta}\frac{\partial S}{\partial\theta}\Big)\Big\} + \frac{\beta}{\rho r^2\sin^2\theta}\frac{\partial B}{\partial r}\Big] = 0.\end{aligned} \tag{3.53}$$

1. Poloidal fields Now putting in $\rho(y)$ and solving for S, with $\beta(S) = 0$, we obtain S to be of the form [31, 26]:

$$S(r,\theta) = \frac{2\sin^2\theta}{3y}[y^3 + 3(y^2 - 2)\sin y + 6y\cos y]$$

and thus have the expressions for B_r and B_θ, using Eqs.(3.40), as:

$$B_r = \frac{\pi^2}{R^2y^2}\frac{4\cos\theta}{3y}[y^3 + 3(y^2-2)\sin y + 6y\cos y]$$
$$B_\theta = \frac{2\pi^2}{3y^3R^2}\sin\theta[-2y^3 + 3(y^2-2)(\sin y - y\cos y)]$$

which now have to satisfy the condition of being regular at $r = 0$ as well as match the external dipole(Eq.(3.2)) at the surface. With B_0 representing the polar magnetic field strength, we obtain the poloidal to be of the form:

$$\boxed{\begin{aligned} B_r &= \frac{B_0\cos\theta}{\pi^3 - 6\pi}[y^3 + 3(y^2-2)\sin y + 6y\cos y] \\ B_\theta &= \frac{1}{2}\frac{B_0\sin\theta}{\pi^3 - 6\pi}[-2y^3 + 3(y^2-2)(\sin y - y\cos y)] \end{aligned}} \qquad \begin{aligned}(3.54)\\(3.55)\end{aligned}$$

Note: Only matching the magnetic field to the external dipole and not imposing regularity, will give expressions of the form:

$$B_r = \frac{\pi^2}{y^3}\frac{B_0\cos\theta}{\pi^2 - 6}[y^3 + 3(y^2-2)\sin y + 6y\cos y]$$
$$B_\theta = \frac{\pi^2}{y^3}\frac{1}{2}\frac{B_0\sin\theta}{\pi^2 - 6}[-2y^3 + 3(y^2-2)(\sin y - y\cos y)]$$

which are clearly not regular at $r = y = 0$. But the solutions given in Eqs.(3.54) and (3.55) match the boundary conditions and the regularity at centre and are thus valid physical solutions.

2. Other cases Following the same line of thought as in the constant density case, we may consider other cases where the magnetic field is pure toroidal or mixed or even a case where the magnetic field is confined to the crust [26]. Unlike the constant density case, there is a non-trivial toroidal field solution for the $n = 1$ polytrope. The mixed case can also be solved, as in [26, 32], by considering $\beta(S) = \frac{\pi\lambda}{R}S$, where λ is a parameter that can be varied and $S = A(r)\sin^2\theta$. A more exotic case is where we may consider the core of the neutron star to be a Type I superconductor. In this case, the magnetic field will be expelled from the core and will be confined to the crust. See [26] and references therein for the details on the solution and implications of this configuration.

3.2.1 Deformations due to the above obtained field configurations

In the preceding couple of sections, we assumed that the magnetic field is poloidal and solved Eq.(3.38) for the cases of a uniform density fluid and the case of a $n = 1$ polytrope. Now, we would like to calculate the effect of such field configurations on the shape of the star. For this, we need to solve Eqs.(3.37) along with the Poisson eqation, $\nabla^2\Phi = 4\pi G\rho$ after perturbing the current system and obtain the new equilibrium configuration. The perturbed surface of the star takes the form:

$$x_S = R[1 + \epsilon(R)P_l(c_\theta)]$$

where $\epsilon(R)$ is a small quantity and P_l are the Legendre polynomials. The pressure, potential and density are perturbed as:

$$p(r,\theta) = p(r) + \delta p(r)P_l(c_\theta) \tag{3.56}$$

$$\Phi(r,\theta) = \Phi(r) + \delta\Phi(r)P_l(c_\theta) \tag{3.57}$$

$$\rho(r,\theta) = \rho(r) + \delta\rho(r)P_l(c_\theta) \tag{3.58}$$

The Eqs.(3.37) at $\mathcal{O}(B^2)$ take the form:

$$\nabla[\delta p(r)P_l(c_\theta)] + \rho\nabla[\delta\Phi(r)P_l(c_\theta)] + \delta\rho P_l\nabla\Phi(r) + \mathcal{O}(B^4) = \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{\mu_0} \tag{3.59}$$

which, after neglecting the $\mathcal{O}(B^4)$ term, becomes:

$$P_l\frac{d\delta p}{dr} + \rho\frac{d\delta\Phi}{dr}P_l + \delta\rho P_l\frac{d\Phi}{dr} = \frac{[(\nabla\times\mathbf{B})\times\mathbf{B}]_r}{\mu_0} \tag{3.60}$$

$$\frac{1}{r}(\delta p + \rho\delta\Phi)\frac{dP_l}{d\theta} = \frac{[(\nabla\times\mathbf{B})\times\mathbf{B}]_\theta}{\mu_0}. \tag{3.61}$$

We now have to solve these equations along with the perturbed Poisson equation:

$$\begin{aligned}&\nabla^2(\Phi + \delta\Phi P_l) = 4\pi G(\rho + \delta\rho P_l)\\ \implies &\nabla^2(\delta\Phi P_l) = 4\pi G\delta\rho P_l\end{aligned} \tag{3.62}$$

which, once expanded after putting $l = 2$, becomes:

$$\begin{aligned}
&P_2(c_\theta)\nabla^2\delta\Phi(r) + \delta\Phi\nabla^2 P_2(c_\theta) = 4\pi G\delta\rho P_2 \\
\implies & P_2\Big(\frac{d^2\delta\Phi}{dr^2} + \frac{2}{r}\frac{d\delta\Phi}{dr}\Big) + \delta\Phi\frac{(-6)}{r^2}P_2 = 4\pi G\delta\rho P_2 \\
\implies & \frac{d^2\delta\Phi}{dr^2} + \frac{2}{r}\frac{d\delta\Phi}{dr} - \frac{6}{r^2}\delta\Phi = 4\pi G\delta\rho
\end{aligned} \tag{3.63}$$

Now, we need to choose a configuration of the magnetic field from the above solutions we obtained and solve Eqs.(3.60) and (3.61) along with the Poisson equation to ultimately calculate the effect of the magnetic field. The configurations we will be using are:

1. A uniform density equation of state with poloidal configuration with magnetic field given by Eqs.(3.51).

2. A $n = 1$ polytropic equation of state with poloidal configuration with magnetic field components given by Eqs.(3.54) and (3.55).

1. Uniform density star with poloidal field We shall assume the fluid to be incompressible which means that the surface deformation x_S can't be an $l = 0$ deformation. We shall only consider $l = 2$ deformation, the quadrupolar deformation. The $\delta\Phi$ then takes the form [26]:

$$\delta\Phi(r) = -\frac{4\pi}{5}G\epsilon(R)r^2\rho. \tag{3.64}$$

Using the magnetic field in Eqs.(3.51), we obtain:

$$[(\nabla\times\mathbf{B})\times\mathbf{B}]_r = \frac{2}{3}\Big(\frac{B}{R^2}\Big)^2\Big(\frac{R^2 r}{3} - \frac{2r^3}{5}\Big)(1 - P_2) \tag{3.65}$$

$$[(\nabla\times\mathbf{B})\times\mathbf{B}]_\theta = -\frac{1}{3}\Big(\frac{B}{R^2}\Big)^2\Big(\frac{R^2 r}{3} - \frac{r^3}{5}\Big)\frac{dP_2}{d\theta}. \tag{3.66}$$

Putting these back in Eq.(3.61) along with the expression for $\delta\Phi$ and substituting $r = R$, we obtain $\delta p(R)$ to be

$$\delta p(R) = \frac{4\pi}{5}G\epsilon(R)R^2\rho^2 - \frac{B^2}{90\pi}. \tag{3.67}$$

Equating the $\mathcal{O}(B^2)$ terms of pressure, we have:

$$\begin{aligned}
&\delta p(x) = \delta p(r) + r\epsilon(r)\frac{dp}{dr}(r) \\
\implies &\delta p(x_S) = \delta p(R) + R\epsilon(R)\frac{dp}{dr}(R) \text{ (putting } r = R) \\
\implies &\delta p(R) = -R\epsilon(R)\frac{dp}{dr}(R) \; (\because \delta p(x_S) = 0, \text{i.e., pressure at the surface}) \qquad (3.68)
\end{aligned}$$

which after putting in the background pressure p to be [26]

$$p(r) = \frac{2\pi}{3}\rho^2 G(R^2 - r^2)$$

becomes

$$\delta p(R) = R\epsilon(R)\frac{4\pi G}{3}\rho^2 \times (R).$$

Equating the above equation to Eq.(3.67), we obtain $\epsilon(R)$ to be of the form:

$$\epsilon(R) = -\frac{1}{48}\frac{B^2}{\pi^2 G\rho^2 R^2}. \qquad (3.69)$$

Previously, we had defined something called the quadrupolar ellipticity. In the Newtonian limit, the expression reads:

$$\epsilon_Q = \frac{I_{33} - I_{11}}{I_0} \qquad (3.70)$$

where $I_{ij} = \int d^3x\rho(\mathbf{x})(r^2\delta_{ij} - x_i x_j)$ and $I_0 = \frac{2}{5}MR^2$, the moment of inertia of a solid sphere. Taking $\rho = const.$ and the limits for r to be from 0 to $x_S = R(1 + \epsilon(R)P_l)$ with $0 \leq \theta < \pi$ and $0 \leq \phi < 2\pi$, we can calculate the above expression to give

$$\epsilon_Q = -\frac{3}{2}\epsilon \qquad (3.71)$$

Note, during the calculation of the above integrals, that the upper limit of r has θ-dependence in the P_ls and we need to use the orthogonality properties of the Legendre polynomials to calculate them. Thus, the quadrupolar ellipticity takes the form

$$\boxed{\epsilon_Q = \frac{1}{32}\frac{B^2}{\pi^2 G\rho^2 R^2} = \frac{1}{18}\frac{B^2 R^4}{GM^2}} \qquad (3.72)$$

which, after casting into a convenient numerical form, looks like:

$$\epsilon_Q \sim 10^{-12}\Big(\frac{R}{10 \text{ km}}\Big)^4\Big(\frac{1.4M_{sun}}{M}\Big)^2\Big(\frac{B}{10^{13} \text{ G}}\Big)^2$$

2. $n = 1$ **polytrope with poloidal field** Now that we have quantified the deformation for the uniform density equation of state, we turn to the more realistic $n = 1$ polytropic equation of state. The components on the RHS of Eqs.(3.60) and (3.61) can be obtained from Eqs.(3.54) and (3.55) to be of the form:

$$[(\nabla \times \mathbf{B}) \times \mathbf{B}]_r = \frac{\pi^3 B_0^2 R}{2(\pi^2-6)y}[-2y^3 + 3(y^2-2)(\sin y - y\cos y)]\sin y P_2(c_\theta) \tag{3.73}$$

$$[(\nabla \times \mathbf{B}) \times \mathbf{B}]_\theta = -\frac{\pi^3 B_0^2 R}{2(\pi^2-6)y}[y^3 + 3(y^2-2)\sin y + 6y\cos y]\sin y \frac{dP_2}{d\theta} \tag{3.74}$$

where $y = \frac{\pi r}{R}$ as previously defined and B_0 is the strength of the magnetic field at the poles, also as previously defined. Then, following [26], we have $\delta\rho$, from Eqs.(3.60) and (3.61), as:

$$\delta\rho = -\frac{\pi^5 B_0^2}{8(\pi^2-6)^2 G R^2 \rho_c y}\Big[y^3 + 3(y^2-2)\sin y + 6y\cos y + \frac{2(\pi^2-6)^2\rho_c}{\pi^4 B_0^2} y\delta\Phi\Big] \tag{3.75}$$

and $\delta\Phi$ as

$$\delta\Phi = \frac{\pi^4 B_0^2}{4(\pi^2-6)^2\rho_c y^3}\Big[-2y^5 - 3(y^4 + 4\pi^2 y^2 - 12\pi^2)\sin y + y(y^4 - 36\pi^2)\cos y\Big] \tag{3.76}$$

Using a similar $\mathcal{O}(B^2)$ argument as in Eq.(3.68) in the case of density, we have $\delta\rho(R)$ given by

$$\delta\rho(R) = -\epsilon(R) R \frac{d\rho}{dr}\Big|_{r=R}.$$

Thus, we have the surface deformation ϵ given by:

$$\epsilon(R) = -\frac{\pi^5 B_0^2(\pi^2-24)}{16(\pi^2-6)^2 G R^2 \rho_c^2} \tag{3.77}$$

where we have used that the density taked the form $\rho(y) = \rho_c \frac{\sin y}{y}$. From Eq.(3.70), we have the quadrupolar ellipticity to be given by:

$$\boxed{\epsilon_Q = -\frac{3\pi^5(\pi^2-12)}{(\pi^2-6)^3}\frac{R^4 B_0^2}{GM^2}.} \tag{3.78}$$

Casting this above formula into a numerically revealing form, we get the following form:

$$\epsilon_Q \sim 2 \times 10^{-10}\Big(\frac{R}{10\text{ km}}\Big)^4\Big(\frac{1.4M_{sun}}{M}\Big)^2\Big(\frac{B}{10^{13}\text{ G}}\Big)^2,$$

a 2 orders of magnitude increase in the ellipticity and thus, in the detection amplitude.

3.3 GWs from a neutron star with magnetic field induced distortion

Following the discussion in [24], in this section, we shall explore how gravitational waves are generated by neutron stars with a non-zero *wobble angle*(the angle between the magnetic axis and the rotation axis). As previously discussed in section 2.3.1, the formula for the amplitude of GW in the strong field limit is structurally identical to the one the weak field limit provided we are in the ACMC family of coordinate systems and is given by Eq.(2.42). Now, consider a neutron star spinning about an axis with its magnetic axis aligned at an angle α to this axis. Let (x, y, z) represent the coordinates in the *fixed* frame, with the rotation axis aligned with the $z-$axis. Also define $(\tilde{x}, \tilde{y}, \tilde{z})$ to be the *inertial* frame with the magnetic axis along $\tilde{z}$. Thus, we have the angle between z and $\tilde{z}$ as α. The inertial frame is rotating with rotation frequency Ω_{rot}. The angle between $\hat{x}$ and $\hat{\tilde{x}}$ is then given by $\Phi = \Omega_{rot}t$. Thus, for transforming from the body frame's (x, y, z) to the inertial frame's $(\tilde{x}, \tilde{y}, \tilde{z})$, we need to use two rotation matrices, $R_1(\Phi)$ about z and $R_2(\alpha)$ about $\tilde{x}$. Thus, the net rotation matrix is given as:

$$R = \begin{bmatrix} \cos \Omega_{rot}t & -\sin \Omega_{rot}t & 0 \\ \sin \Omega_{rot}t & \cos \Omega_{rot}t & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \tag{3.79}$$

We shall assume that the mass quadrupole moment $\mathcal{I}_{ij}$ can be decomposed as $\mathcal{I}_{ij} = \mathcal{I}_{ij}^{rot} + \mathcal{I}_{ij}^{dist}$, i.e., a rotational contribution and a distortion contribution. Due to the absence of precession, $\exists$ an ACMC coordinate system (t, x, y, z) such that $\mathcal{I}_{ij}^{rot}$ is time independent [24] and is diagonal, with a form given by:

$$\mathcal{I}_i^{rot} j = \text{diag}(-\mathcal{I}^{rot}/2, -\mathcal{I}^{rot}/2, \mathcal{I}^{rot}). \tag{3.80}$$

We shall also make an assumption that $\exists$ an inertial frame $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ in which $\mathcal{I}_{ij}^{dist}$ may be expressed in the following form [24]:

$$\tilde{\mathcal{I}}_{ij}^{dist} = \text{diag}(-\tilde{\mathcal{I}}_{33}/2, -\tilde{\mathcal{I}}_{33}/2, \tilde{\mathcal{I}}_{33}) \tag{3.81}$$

Note that this inertial frame need not necessarily be ACMC. Thus, to transform $\tilde{\mathcal{I}}^{dist}$ back to the ACMC coordinates, we shall use the rotation matrix from Eq.(3.79), i.e., $\mathcal{I}^{dist} = R\tilde{\mathcal{I}}^{dist}R^T$. This quadrupole moment tensor $\mathcal{I}_{ij}^{dist}$ can now be used in the expression for GW amplitude, Eq.(2.42). As physically expected, there is no contribution from the $\mathcal{I}_{ij}^{rot}$ because, in this ACMC coordinate system, it is time independent, as previously stated [24]. The path ahead is now clear: take the second time derivative of $\mathcal{I}_{ij}^{dist}$ and write the h_{ij}^{TT} in terms of $\tilde{\mathcal{I}}_{33}, \alpha, \Omega_{rot}$ and time t as:

$$\mathcal{I} = \frac{1}{2}\tilde{\mathcal{I}}_{33}^{dist} \begin{bmatrix} 3\sin^2\alpha\sin^2\phi(t) - 1 & -\frac{3}{2}\sin^2\alpha\sin 2\phi(t) & -3\sin\alpha\cos\alpha\sin\phi(t) \\ -\frac{3}{2}\sin^2\alpha\sin 2\phi(t) & 3\sin^2\alpha\cos^2\phi(t) - 1 & 3\sin\alpha\cos\alpha\cos\phi(t) \\ -3\sin\alpha\cos\alpha\sin\phi(t) & 3\sin\alpha\cos\alpha\cos\phi(t) & 3\cos^2\alpha - 1 \end{bmatrix} \tag{3.82}$$

with its second time derivative as:

$$\ddot{\mathcal{I}} = \frac{3}{2}\tilde{\mathcal{I}}_{33}^{dist}\sin\alpha \begin{bmatrix} 2\sin\alpha\cos 2\phi(t) & 2\sin\alpha\sin 2\phi(t) & \cos\alpha\sin\phi(t) \\ 2\sin\alpha\sin 2\phi(t) & -2\sin\alpha\cos 2\phi(t) & -\cos\alpha\cos\phi(t) \\ \cos\alpha\sin\phi(t) & -\cos\alpha\cos\phi(t) & 0 \end{bmatrix}. \tag{3.83}$$

Now, assuming that the direction to the observer, $\hat{n}$, is in the $y-z$ plane and that the angle between $\hat{z}$ and $\hat{n}$ is Θ, i.e., $\hat{n} = (0, \sin\Theta, \cos\Theta)$, we have the projection operator as:

$$\begin{aligned} P_{ij} &= \delta_{ij} - n_i n_j \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\Theta & -\sin\Theta\cos\Theta \\ 0 & -\sin\Theta\cos\Theta & \sin^2\Theta \end{bmatrix}. \end{aligned} \tag{3.84}$$

Thus, we finally have h_{ij}^{TT} as [24]:

$$h_{ij}^{TT} = h_+ e_{ij}^+ + h_\times e_{ij}^\times \tag{3.85}$$

where e_{ij}^+ and $e_{ij}^\times$ are the polarization tensors given by [24]:

$$e_{ij}^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos^2\Theta & \sin\Theta\cos\Theta \\ 0 & \sin\Theta\cos\Theta & -\sin^2\Theta \end{bmatrix} \tag{3.86}$$

$$e_{ij}^\times = \begin{bmatrix} 0 & \cos\Theta & -\sin\Theta \\ \cos\Theta & 0 & 0 \\ -\sin\Theta & 0 & 0 \end{bmatrix} \tag{3.87}$$

and the polarizations h_+ and $h_\times$ are given by [24]:

$$\boxed{\begin{aligned} h_+ &= h_0 \sin\alpha \Big[\frac{1}{2}\cos\alpha\sin\Theta\cos\Theta\cos\Omega_{rot}t_r \\ &\qquad - \sin\alpha\frac{1+\cos^2\Theta}{2}\cos 2\Omega_{rot}t_r\Big] && (3.88) \\ h_\times &= h_0 \sin\alpha\Big[\frac{1}{2}\cos\alpha\sin\Theta\sin\Omega_{rot}t_r - \sin\alpha\cos\Theta\sin 2\Omega_{rot}t_r\Big] && (3.89) \end{aligned}}$$

with $h_0 = -\frac{6G}{c^4}\tilde{\mathcal{I}}_{33}^{dist}\frac{\Omega_{rot}^2}{r}$ and $t_r = t - \frac{r}{c}$ is the retarded time.

Ellipticity and quadrupole moment In the weak field limit, the quadrupolar ellipticity is given by:

$$\epsilon_Q = \frac{I_3 - I_1}{I_3}$$

Using the definition of $\mathcal{I}_{ij}$ from section 2.3.1, the ellipticity can be written in terms of $\tilde{\mathcal{I}}_{33}$ after the following steps:

$$\begin{aligned} \tilde{\mathcal{I}}_{33} &= \frac{2}{3}(I_1 - I_3) \\ &= -\frac{2}{3}\epsilon_Q I_3 \\ \implies \epsilon_Q &= -\frac{3}{2}\frac{\tilde{\mathcal{I}}_{33}}{I_3} \end{aligned} \tag{3.90}$$

where I_3 is the moment of inertia about the rotation axis in the diagonal form. For various numerical estimates, look at section 2.5 of [24]. The typical order of magnitude for h_0 is estimated to be $\sim 10^{-27}$ for a variety of pulsars. For a numerical estimate, h_0 may be written as:

$$h_0 = 4.21 \times 10^{-24}\left[\frac{\mathrm{ms}}{P}\right]^2\left[\frac{\mathrm{kpc}}{r}\right]\left[\frac{I}{10^{38}\ \mathrm{kg\ m}^2}\right]\left[\frac{\epsilon_Q}{10^{-6}}\right]. \tag{3.91}$$

An incompressible, magnetized fluid Next, consider a star with an incompressible fluid with a uniform internal magnetic field and a dipole magnetic field outside the star. This star is going to be ellipsoid in shape, with the equation of the ellipsoid given by

$$(\delta_{ij} + a_{ij})X^i X^j = R^2, \tag{3.92}$$

where R is the mean radius of the ellipsoid and, using the expressions for potentials given in [33, 34, 24], the a_{ij} are given by:

$$a_{ij} = \frac{15}{2}\frac{\Omega^i_{rot}\Omega^j_{rot}}{\omega_J^2} + \frac{45\mu_0}{32\pi^2}\frac{\mathcal{M}_i\mathcal{M}_j}{R^8\rho\omega_J^2} \tag{3.93}$$

where ρ is the constant mass density, $\omega_J = \sqrt{4\pi G\rho}$ is the Jean's frequency, $\Omega^i_{rot} \equiv (0, 0, \Omega_{rot})$ is the angular frequency vector in X^i coordinates and $\mathcal{M}_i \equiv (0, \mathcal{M}\sin\alpha, \mathcal{M}\cos\alpha)$ is the magnetic dipole moment vector, also in X^i coordinates. Diagonalizing a_{ij} to find the principal axes, we may calculate the moment of inertia tensor I_{ij}, from which $\mathcal{I}_{ij}$ is readily obtained. Separating $\mathcal{I}_{ij} = \mathcal{I}^{rot}_{ij} + \mathcal{I}^{dist}_{ij}$ as previously discussed, we have $\mathcal{I}^r_{ij}ot$ given by Eq.(3.80) with

$$\mathcal{I}^{rot}_{33} = -\frac{R^5\Omega^2_{rot}}{3G} \tag{3.94}$$

and $\mathcal{I}^{dist}_{ij}$ given by Eq.(3.82), with the $\tilde{\mathcal{I}}^{dist}_{33}$ given by:

$$\tilde{\mathcal{I}}^{dist}_{33} = -\frac{\mu_0\mathcal{M}^2}{16\pi^2 G\rho R^3}. \tag{3.95}$$

Using the fact that $I = \frac{8\pi}{15}\rho R^5$ for a sphere and Eq.(3.90), we have the quadrupolar ellipticity given by:

$$\epsilon_Q = \frac{45\mu_0\mathcal{M}^2}{256\pi^3 G\rho^2 R^8} = \frac{45 B^2_{pole}}{64\pi\mu_0 G\rho^2 R^2} \tag{3.96}$$

where $B_{pole} = \frac{\mu_0}{4\pi}\frac{2\mathcal{M}}{R^3}$ is the strength of the magnetic field at the (North) pole.

Magnetic field induced deformations

In this section, we will calculate a few quantities important for deriving the amplitude of the gravitational waves. Following the intuition gathered from the derivation of Eq.(3.96), we can write the quadrupolar ellipticity to be proportional to B^2 in the following form:

$$\epsilon_Q = \mathfrak{D}\frac{B^2}{B_*^2} \tag{3.97}$$

where B_* is a quantity with the dimensions of magnetic field so that the *magnetic deformation factor*, $\mathfrak{D}$, is dimensionless. We shall make the particular choice of $B_*^2 = \frac{12}{5}\frac{GM^2}{R^4}$, in sync with our choice in Eq.(3.36). In the above considered choice of an incompressible, magnetized fluid, neatly summarized by Eq.(3.96), we have the distortion factor, $\mathfrak{D} = \frac{3}{4}\frac{4\pi}{\mu_0}$. We make the choice of units to be $\mu_0 = \frac{4\pi}{c^2}$ and $c = 1$, so that $\mathfrak{D} = \frac{3}{4}$ in this configuration. We also have the form for the quadrupolar ellipticity as:

$$\epsilon_Q = \mathfrak{D}\frac{5}{12}\frac{B^2 R^4}{GM^2} = \frac{3}{4}\frac{B^2}{B_*^2} \tag{3.98}$$

The *braking indices* for pulsars being closer to 3 than to 5 seems to indicate that the primary spin down of pulsars is due to magnetic dipole radiation [24]. The following spin down equation may thus be written:

$$\begin{aligned} \dot{\Omega}_{rot} &= -\frac{4\pi}{\mu_0}\frac{1}{6c^3}\Omega_{rot}^3\frac{B^2R^6}{I}\sin^2\alpha \\ \implies B^2 &= -\frac{\mu_0}{4\pi}\frac{6c^3 I}{R^6\sin^2\alpha}\frac{\dot{\Omega}_{rot}}{\Omega_{rot}^3} = -\frac{6I}{R^6\sin^2\alpha}\frac{\dot{\Omega}_{rot}}{\Omega_{rot}^3} \end{aligned} \tag{3.99}$$

where we have invoked the choice of our units for the last equality. We are now in a position to obtain a form for the ϵ_Q to be:

$$\epsilon_Q = \mathfrak{D}\frac{\dot{\Omega}_{rot}}{\Omega_{rot}^3}\frac{1}{GM\sin^2\alpha} \tag{3.100}$$

The same physics is entailed in the corresponding expressions in [24] although, in order to see it manifestly, some change of notation is in order. Our distortion factor, $\mathfrak{D}$, is related to the β in [24] through the following relation:

$$\beta = \mathfrak{D}\frac{4}{15} \tag{3.101}$$

which may be easily obtained by a few algebraic manipulations. Using Eqs.(3.90), (3.88) and (3.89), we may write h_0 as

$$h_0 = \frac{4}{r}I\epsilon_Q\Omega_{rot}^2 = 16\pi^2\frac{I\epsilon_Q}{P^2 r} \tag{3.102}$$

where $P = \frac{2\pi}{\Omega_{rot}}$ is the period of rotation. We write the formula in this form because, observationally, P is what we measure. Now, using Eqs.(3.98) and (3.99) in the above expression, we have the following form for h_0 [24]:

$$h_0 = \frac{8}{5}\mathfrak{D}\frac{R^2}{r}\frac{\dot{P}}{P}\frac{1}{\sin^2\alpha}. \tag{3.103}$$

Looking back at Eqs.(3.88) and (3.89), the amplitude of the $2\Omega_{rot}$ rotational frequency part is given by:

$$\begin{aligned} h_0^{2\Omega_{rot},+} &= h_0 \sin^2\alpha \frac{1+\cos^2\Theta}{2} \quad (\text{multiplying } \cos 2\Omega_{rot} t_r) \\ &= \frac{8}{5}\mathfrak{D}\frac{R^2}{r}\frac{\dot{P}}{P}\frac{1+\cos^2\Theta}{2} \end{aligned} \tag{3.104}$$

which is independent of the wobble angle. This form is only dependent on the observables, $\dot{P}$ and P along with a dependence on Θ, which may be ignored for an order of magnitude estimate. A similar line of action for the $\times$ polarization gives:

$$\begin{aligned} h_0^{2\Omega_{rot},\times} &= h_0 \sin^2\alpha \cos\Theta \quad (\text{multiplying } \sin 2\Omega_{rot} t_r) \\ &= \frac{8}{5}\mathfrak{D}\frac{R^2}{cr}\frac{\dot{P}}{P}\cos\Theta \end{aligned} \tag{3.105}$$

where the exact same reasoning for an order of magnitude estimate may be applied, due to which we shall only look at the order of magnitude of $h_0^{2\Omega_{rot},+}$. Transforming the expression to a different units system, having put $\mu_0 = \frac{4\pi}{c^2}$ and $c = 1$, we have [24]:

$$h_0^{2\Omega_{rot},+} = 1.729 \times 10^{-31}\mathfrak{D}\left(\frac{R}{10\text{km}}\right)^2\left(\frac{\text{kpc}}{r}\right)\left(\frac{\text{s}}{P}\right)\left(\frac{\dot{P}}{10^{-11}}\right) \tag{3.106}$$

(To compare with the amplitude in [24], note that the units for P are s here in contrast to ms and the order for $\dot{P}$ is 10^{-11} in contrast to 10^{-13}. If we convert to the units in [24], we will have an extra 10^{-3} in the denominator, coming from the millisecond, and an extra 10^{-2} in the numerator, coming from the 10^{-13}, for an effective $\times 10$.) Putting in the values for Crab pulsar, the value of $h_0^{2\Omega_{rot},+} \sim 10^{-31}$. For contrast, the LIGO sensitivity, at 100 Hz, is at its peak and the value is of the order 10^{-22}. This clearly shows that in order for LIGO to detect the signal at frequency $2\Omega_{rot}$, we need to have a large $\mathfrak{D}$. Following the results from [24], the values of $\beta = \mathfrak{D}\frac{4}{15}$ for the regular case of having both internal and external field configurations are not significantly high. The prospects improve significantly once one introduces the assumption of superconductivity of the core. In the cases under this assumption, the values of β improve by 2 orders of magnitude. For a deeper discussion of the simulations performed, please look at [24].

We had previously derived the quadrupolar ellipticity in section 3.2.1, which we may now put in the above discussed language. From Eq.(3.72), we have the ellipticity for a *constant density* *fluid* given by:

$$\epsilon_Q = \frac{1}{32}\frac{B^2}{\pi^2 G\rho^2 R^2} = \frac{1}{18}\frac{B^2R^4}{GM^2} = \frac{2}{15}\frac{B^2}{B_*^2}$$
$$\Longrightarrow \boxed{\mathfrak{D} = \frac{2}{15}} \tag{3.107}$$

and from Eq.(3.78), for an $n = 1$ *polytrope*, we have:

$$\epsilon_Q = -\frac{3\pi^5(\pi^2-12)}{(\pi^2-6)^3}\frac{R^4B^2}{GM^2} = -\frac{36\pi^5(\pi^2-12)}{5(\pi^2-6)^3}\frac{B^2}{B_*^2}$$
$$\Longrightarrow \boxed{\mathfrak{D} = -\frac{36\pi^5(\pi^2-12)}{5(\pi^2-6)^3}} \tag{3.108}$$

Chapter 4

Magnetic Monopoles and Millicharges

In this chapter, we will initially discuss the theoretical mechanisms through which magnetic monopoles arise in QFTs. Following that we will address how this discussion fits into our picture for mmCPs. In the later parts of this chapter, we shall use the techniques from the previous chapters to calculate the effect of the pair production of these mmCPs on the gravitational wave amplitudes.

Classically, Maxwell's equation, $\nabla \cdot \mathbf{B} = 0$ forbids the existence of magnetic monopoles... Maxwell's equations in vacuum, in covariant form, are:

$$\partial_\mu F^{\mu\nu} = 0 \quad \text{and} \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{4.1}$$

where the $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the Hodge dual of the field tensor $F^{\mu\nu}$. With the standard definition $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, the second equation is a *Bianchi identity* and is automatically satisfied. The vacuum equations thus have a symmetry under the following transformation:

$$F^{\mu\nu} \to \tilde{F}^{\mu\nu} \quad \text{and} \quad \tilde{F}^{\mu\nu} \to -F^{\mu\nu}$$

However, once we introduce a source term J^ν on the RHS of the first equation, this symmetry no longer exists. In order to restore the symmetry, we add a magnetic current term K^ν on the RHS of the second equation, such that the equations become [35]:

$$\partial_\mu F^{\mu\nu} = -J^\nu \quad \text{and} \quad \partial_\mu \tilde{F}^{\mu\nu} = -K^\nu \tag{4.2}$$

which are clearly symmetric under the transformations:

$$F^{\mu\nu} \to \tilde{F}^{\mu\nu} \quad \text{and} \quad \tilde{F}^{\mu\nu} \to -F^{\mu\nu}$$
$$J^{\nu} \to K^{\nu} \quad \text{and} \quad K^{\nu} \to -J^{\nu}$$

The existence of such a K^{ν} current, however, requires that magnetic monopoles exist in nature. All equations, unless otherwise mentioned, are in natural units, $\hbar = 1 = c$.

4.1 Dirac quantization condition

For the existence of monopoles to be consistent with quantum mechanics, the following condition due to Dirac must be satisfied [35]:

$$\frac{qg}{4\pi} = \frac{n}{2}, \quad n \in \mathbb{Z}^{+} \tag{4.3}$$

where g is the magnetic charge and q is our regular electric charge. This condition can be derived heuristically from the condition of anglular momentum quantization as described below.

Heuristic derivation of the Dirac quantization condition

The Lorentz force on a charge q moving at a velocity $\mathbf{v} = \dot{\mathbf{r}}$ in a magnetic field $\mathbf{B}$ is given by

$$m\ddot{\mathbf{r}} = q(\dot{\mathbf{r}} \times \mathbf{B}) \tag{4.4}$$

where the magnetic field is assumed to be due to a magnetic monopole of strength g, with field strength given by:

$$\mathbf{B} = \frac{g}{4\pi r^2}\hat{\mathbf{r}} \tag{4.5}$$

The orbital angular momentum of such a charge is given by:

$$
\begin{aligned}
\mathbf{L} &= \mathbf{r} \times m\dot{\mathbf{r}} \\
\implies \frac{d\mathbf{L}}{dt} &= \mathbf{r} \times m\ddot{\mathbf{r}} \\
&= \mathbf{r} \times (\dot{\mathbf{r}} \times \frac{qg}{4\pi r^2}\hat{\mathbf{r}}) \\
&= \frac{d}{dt}(\frac{qg}{4\pi}\hat{\mathbf{r}}) \qquad (4.6) \\
& \qquad (4.7)
\end{aligned}
$$

The last line in the previous series of equations tells us that the *total* angular momentum, $\mathbf{J}$, is conserved and is given by,

$$\mathbf{J} = \mathbf{L} - \frac{qg}{4\pi}\hat{\mathbf{r}}. \qquad (4.8)$$

Imposing the regular commutation relations between the components of the angular momentum, we have the quantization of eigenvalue of angular momenta J_i as *half-integers* (the reason the orbital angular momenta eigenvalues are *integers* is due to the fact that L_is are the generators of $SO(3)$, whereas the total angluar momentum components are generators of $SU(2)$). Thus, for the total angular momentum to have half-integer eigenvalues, we need to have:

$$\frac{qg}{4\pi} = \frac{n}{2}, \quad n \in \mathbb{Z}^+$$

which is precisely the Dirac quantization condition in Eq.(4.3).

4.2 Millimagnetically charged particles

Now that we have looked at how monopoles arise and are motivated, let us look at how mmCPs arise in field theories. They arise in field theories with an additional $U(1)$ symmetry wherein a kinetic mixing term added to the Lagrangian, so that it has the following form:

$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}F_D^{\mu\nu}F_{D\mu\nu} + \frac{\epsilon}{2}F^{\mu\nu}F_{D\mu\nu} + \frac{1}{2}m_D^2A_D^2. \qquad (4.9)$$

Here, the subscript D represents gauge fields in the additional "dark" sector. The dark gauge field is massive because otherwise, the electron and the mmCP don't interact [8]. But, let us first understand what this mmCP is. For this purpose, we will be looking at the equations of motion for the above Lagrangian:

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} - \epsilon \partial_\mu F_D^{\mu\nu} &= eJ^\nu \\
\partial_\mu F_D^{\mu\nu} - \epsilon \partial_\mu F^{\mu\nu} &= m_D^2 A_D^\nu + e_D J_D^\nu \\
\partial_\mu \tilde{F}^{\mu\nu} &= 0 \\
\partial_\mu \tilde{F}_D^{\mu\nu} &= g_D K_D^\nu
\end{aligned}
\tag{4.10}
$$

Under a field redefinition of the form $A \to A + \epsilon A_D$, we are supposed to have:

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= eJ^\nu \\
\partial_\mu F_D^{\mu\nu} &= m_D^2 A_D^\nu + e_D J_D^\nu + \epsilon e J^\nu \\
\partial_\mu \tilde{F}^{\mu\nu} &= -\epsilon g_D K_D^\nu \\
\partial_\mu \tilde{F}_D^{\mu\nu} &= g_D K_D^\nu
\end{aligned}
\tag{4.11}
$$

where, in the third equation, we see the dark monopoles become ϵ magnetically charged under our gauge field.

4.2.1 Incorporating monopoles into ordinary electromagnetism

In ordinary electromagnetism, we have the following Lagrangian density:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu$$

and the equations of motion thus to be of the form:

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= J^\nu \\
\partial_\mu \tilde{F}^{\mu\nu} &= 0.
\end{aligned}
$$

But, if we now want to describe a theory with magnetic monopoles, the second equation has to be modified as follows:

$$\partial_\mu \tilde{F}^{\mu\nu} = K^\nu \tag{4.12}$$

In order to do this $\mathcal{L}$ has to be modified. The problem of Lagrangin density for dual electromagnetism has been addressed previously [36, 37]. We will take inspiration from these and write down our Lagrangian. Consider the following form for $\mathcal{L}$:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A_\mu J^\mu + B_\mu K^\mu \tag{4.13}$$

where $F^{\mu\nu} = (\partial \wedge A)^{\mu\nu} - \epsilon^{\mu\nu\rho\lambda}\partial_\rho B_\lambda$ is the field tensor and B_μ is called the dual photon. J^μ is the electric current and K^μ is the magnetic current. Equation of motion obtained by varying w.r.t. A_μ are:

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{4.14}$$

and varying w.r.t. B_μ yields:

$$\partial_\mu \tilde{F}^{\mu\nu} = K^\nu \tag{4.15}$$

which are precisely the equations of motion in the presence of a magnetic current.

4.2.2 Incorporating the dark sector into dual EM

In our model of mmCPs, we do *not* have monopoles in our sector, but we assume that there are monopoles in the dark sector. For incorporating these monopoles into the dark sector of the Lagrangian density, we use the theory presented in the previous section. The mmCPs are actually these dark monopoles that gain a small magnetic charge under our gauge field.

Thus, with kinetic mixing and without monopoles in our sector, the full Lagrangian density is given by:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}F_D^{\mu\nu}F_{D\mu\nu} + g_D B_{D\nu}K_D^\nu - eA_\nu J^\nu - e_D A_{D\nu}J_D^\nu + \frac{1}{2}m_D^2 A_{D\mu}^2 + \frac{\epsilon}{2}FF_D \tag{4.16}$$

where $F = (\partial \wedge A)$ and $F_D^{\mu\nu} = (\partial \wedge A_D)^{\mu\nu} - \epsilon^{\mu\nu\rho\lambda}\partial_\rho B_{D\lambda}$. Under a field redefinition of the form $A \to A + \epsilon A_D$, we have:

$$\begin{aligned} F \to & F + \epsilon(\partial \wedge A_D) \\ &= F + \epsilon(F_D + \epsilon^{\mu\nu\rho\lambda}\partial_\rho B_{D\rho\lambda}) \end{aligned}$$

and we thus have F^2 written as

$$F^2 \to F^2 + 2\epsilon F(F_D + \epsilon^{\mu\nu\rho\lambda}\partial_\rho B_{D\lambda}) + O(\epsilon^2)$$

in $\mathcal{L}$. Although this cancels the kinetic mixing term, it gives another term of the form

$$
\begin{aligned}
F_{\mu\nu}(\epsilon^{\mu\nu\rho\lambda}\partial_\rho B_{D\rho\lambda}) &= (\partial\wedge A)_{\mu\nu}\epsilon^{\mu\nu\rho\lambda}\partial_\rho B_{D\rho\lambda} - \epsilon_{\mu\nu\sigma\delta}\partial^\sigma B^\delta \epsilon^{\mu\nu\rho\lambda}\partial_\rho B_{D\lambda} \\
&= 0 + 2\delta^{\rho\lambda}_{\sigma\delta}\partial^\sigma B^\delta \partial_\rho B_{D\lambda}
\end{aligned} \tag{4.17}
$$

where, in the last line, we performed a partial integration over the first two terms as follows:

$$
\begin{aligned}
\partial_\mu A_\nu \epsilon^{\mu\nu\rho\lambda}\partial_\rho B_{D\lambda} &\equiv -A_\nu \epsilon^{\mu\nu\rho\lambda}\partial_\mu \partial_\rho B_{D\lambda} \\
&= 0
\end{aligned}
$$

since it is a product of a symmetric and an anti-symmetric tensors and also used $\epsilon^{\mu\nu\rho\lambda}\epsilon_{\mu\nu\sigma\delta} = -2\delta^{\rho\lambda}_{\sigma\delta}$. With this in mind, consider the field redefinition $B_D \to B_D + \epsilon B$. In this case,

$$
F_D \to F_D - \epsilon\epsilon^{\mu\nu\rho\lambda}\partial_\rho B_\lambda
$$

and thus, F_D^2 can be written as:

$$
\begin{aligned}
F_D^2 \to & F_D^2 - 2\epsilon F_{D\mu\nu}\epsilon^{\mu\nu\rho\lambda}\partial_\rho B_\lambda \\
&= F_D^2 - 4\epsilon\delta^{\rho\lambda}_{\sigma\delta}\partial_\rho B_\lambda \partial^\sigma B_D^\delta
\end{aligned}
$$

using the same trick as that used in Eq.(4.17), which exactly cancels out the extra term written down in Eq.(4.17). Thus, after the field redefinitions

$$
A \to A + \epsilon A_D \quad \text{and} \quad B_D \to B_D + \epsilon B, \tag{4.18}
$$

we have the Lagrangian density taking the form:

$$
\begin{aligned}
\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}F_D^{\mu\nu}F_{D\mu\nu} + g_D B_{D\nu}K_D^\nu + \epsilon g_D B_\nu K_D^\nu \\
- eA_\nu J^\nu - A_{D\nu}(e_D J_D^\nu + \epsilon e J^\nu) + \frac{1}{2}m_D^2 A_{D\mu}^2
\end{aligned} \tag{4.19}
$$

which exactly reproduces the equations of motion given in Eq.(4.11), rewritten here for clarity:

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= eJ^\nu \\
\partial_\mu F_D^{\mu\nu} &= m_D^2 A_D^\nu + e_D J_D^\nu + \epsilon e J^\nu \\
\partial_\mu \tilde{F}^{\mu\nu} &= -\epsilon g_D K_D^\nu \\
\partial_\mu \tilde{F}_D^{\mu\nu} &= g_D K_D^\nu
\end{aligned}
$$

Let us define a new quantity $Q_m = \epsilon \frac{g_D}{g}$. Thus, the charge of a single mmCP is $Q_m g$

4.3 mmCP-SPP in Neutron Stars

In this section, we begin by presenting how the evolution of the magnetic field of the neutron star and its rotation frequency are modelled. For the rotation frequency, we have contributions coming from both the electromagnetic dipole braking and that from gravitational waves [18, 38, 39], and we have the energy evolution given by:

$$\left.\frac{dE}{dt}\right|_{rot.} = \frac{1}{6}B^2R^6\Omega^4 - \frac{32}{5}GI^2\epsilon_Q^2\Omega^6 \tag{4.20}$$

Using the standard rotational energy expression $E = \frac{1}{2}I\Omega^2$ and then approximating the neutron star by a sphere so that $I = \frac{2}{5}MR^2$, we have the following equation for the time evolution of the rotational frequency:

$$\frac{d\Omega}{dt} = -\frac{5}{12}\frac{B^2R^4}{M}\Omega^3 - \frac{64}{25}GMR^2\epsilon_Q^2\Omega^5 \tag{4.21}$$

In this spin down equation, we have assumed that the magnetic axis is orthogonal to the rotation axis, i.e., $\alpha = \pi/2$ [38]. Our next step is to model the decay of the magnetic field. The mmCP-SPP takes energy from the NS magnetic fields and thus contributes to its decay. The total energy contained in the magnetic field (as was derived in chapter 3) is given by:

$$E_{mag.} = \frac{1}{4}B^2R^3 \implies \frac{dE_{mag.}}{dt} = \frac{1}{2}B\dot{B}R^3. \tag{4.22}$$

The energy extracted from the magnetic field for driving an mmCP-anti mmCP pair a distance l from each other is given by $Q_m gBl$ (analogous to ordinary EM where we would write eEl). Thus, for a pair production rate of Γ in an active volume V_{AV}, the energy extracted per second is:

$$\frac{dE_{lost}}{dt} = -\Gamma Q_m gBlV_{AV} \tag{4.23}$$

or, in terms of the magnetic field,

$$\frac{dB}{dt} = -\frac{2}{R^3}\Gamma Q_m glV_{AV} \tag{4.24}$$

Realistically, the magnetic field evolution is studied using complex magnetothermal simulations [40, 41, 42]. The salient features, however, are captured if we use an equations of the following form [40]:

$$\frac{dB(t)}{dt} = \frac{B(t)}{2\tau_D}e^{-t/2\tau_D} - \frac{B(t)}{\tau_O} - \frac{B^2(t)}{B(0)\tau_H} - \frac{2}{R^3}\Gamma Q_m g l V_{AV} \tag{4.25}$$

where the first term is the dynamo term, the second is called the Ohmic term, the third is called the Hall drift term and the fourth is our new mmCP-SPP term. τ_O is modelled as having a value of 10^6 yrs and $\tau_H = 10^4$ yrs. The value of τ_D depends on whether we are modelling magnetars or pulsars. Magnetars typically have a dynamo for about $\tau_D = 10$ s, whereas pulsars have much longer active dynamos.

Using the fact that the magnetic field of a magnetar has remained at the current value, $B_0 \sim 10^15$ G, through its lifetime, $t_{life} \sim 10^4$ yrs, we can put a bound on the SPP contribution to the magnetic field decay and thus, a bound on the charge Q_m [8]. This equation is as follows:

$$\Gamma Q_m g l B_{eff} \frac{4\pi}{3} R_{AV}^3 - \frac{B_0^2}{4t_{life}} R^3 = 0 \tag{4.26}$$

Using the SPP formula at zero temperature [43],

$$\Gamma_{T=0} = \frac{Q_m^2 g^2 B_{eff}^2}{3\pi^2} e^{-\frac{\pi m^2}{Q_m g B_{eff}}}, \tag{4.27}$$

where $B_{eff} = B(1 - e^{-m_D l})$ is the effective magnetic field felt by the mmCP [8], we can use Eq.(4.26) to get a bound on Q_m of:

$$Q_m \simeq 3.82 \times 10^{-19}$$

which different from that obtianed in [8] by a small factor since while we have considered the average SPP over a volume in Eq.(4.26), the volume was not considered in [8]. We feel this is a better energetics argument since our entire analysis till that point has considered the active volume.

Now that we have Eqs.(4.25) and (4.21), all that is left is to substitute the SPP expression and obtain the evolution of the gravitational amplitude using the following formula (as derived in chapter 3, Eqs. (3.102) and (3.97)):

$$h_0 = \frac{4}{r} I \epsilon_Q \Omega^2, \text{ where } \epsilon_Q = \mathfrak{D}\frac{B^2}{B_*^2}. \tag{4.28}$$

The solution is obtained numerically in the last section of this chapter. However, there are some subtleties involved in SPP term regarding the thermal expression (since our system is *not* at zero temperature) and we shall address it in the following section.

4.4 Incorporating thermal corrections

The system that we are dealing with is at some finite temperature, hence the correct SPP term is the thermal Schwinger pair production formula for Γ [44, 45]. It is not that simple to integrate this into Eq. (4.25) since the prefactor in this formula is plagued by singularities which should go away when the next order term is taken into consideration. But this is a conundrum since calculation of this term is extremely difficult even in the zero temperature case, where the exact expression is unknown. Due to these difficulties, there have been other methods developed which might help in dodging the singularities. We will come to those shortly, after we see what the thermal formula looks like.

The full pair production rate for thermal SPP in the case of a pure electric field in QED is given by [44, 45]:

$$\begin{aligned}\Gamma_{T\neq 0} &= \Gamma_{T=0} + \Gamma_T \\ &= \sum_{p=1}^{\infty} \frac{e^2E^2}{4\pi^3 p^2} e^{\frac{-\pi p m^2}{eE}} \\ &\quad + \sum_{p=0}^{\infty}\sum_{n=1}^{n_{max}} \frac{(-1)^{p+1}4e^2E^2}{(2\pi)^{3/2}(n\beta m)^{1/2}\Theta^2}\left[1-\left(\frac{nT_c}{T}\right)^2\right]^{-\frac{1}{4}} \exp\left(-\frac{m^2}{2eE}\Theta + \frac{nm}{2T}\sqrt{1-\frac{n^2T_c^2}{T^2}}\right)\end{aligned} \tag{4.29}$$

where $\beta = \frac{1}{T}$ is the inverse temperature and T_c is the critical temperature given by $T_c = \frac{eE}{2m}$. The $\Theta = 2\pi(p+1) - \sin^{-1}\left(\frac{nT_c}{T}\right)$. Clearly, the contribution from Γ_T is non-zero only if $T > T_c$. For $T < T_c$, the contribution to $\Gamma_{T\neq 0}$ comes only from $\Gamma_{T=0}$. For our case of mmCPs, we only need to make the following replacements in the above formulae:

$$e \to Q_m g \quad \text{and} \quad E \to B$$

and the rest is the same. The critical temperature thus changes to

$$T_c(m, Q_m) = \frac{Q_m g B}{2m}. \tag{4.30}$$

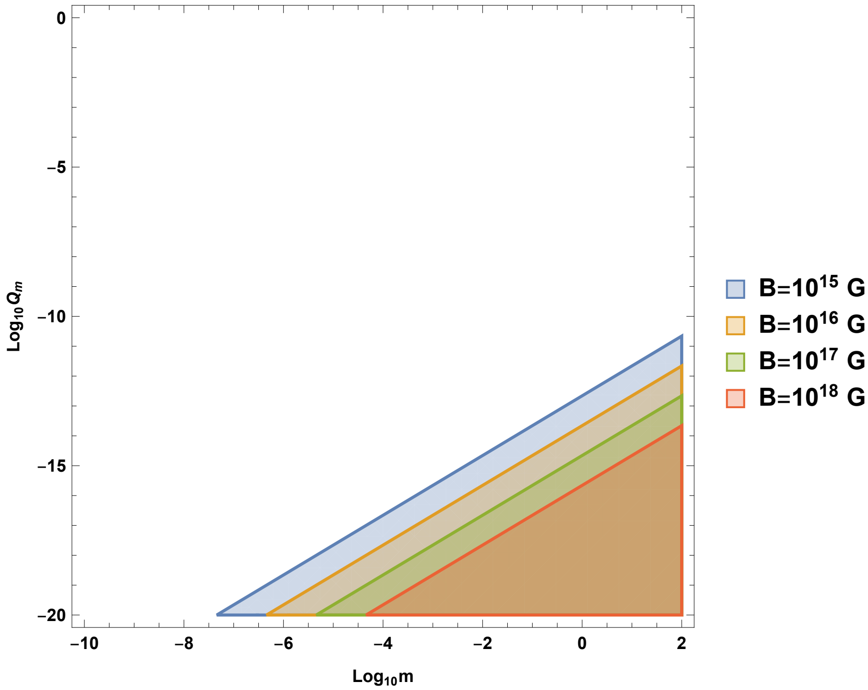


Figure 4.1: The regions are shaded in different colors for different values of magnetic field, as labelled. Inside these regions, the temperature $T > T_c$ for the given value of the magnetic field.

Thus, in the (m, Q_m) plane, there is a region where we have $T > T_c$. For a magnetar, $T \sim 10^6$ K ~ 86.25 eV, this region where $T > T_c$ is given in Fig. 4.1 for different values of magnetic field, ranging from 10^{15} G to 10^{18} G.

The singularities we talked about before are due to the factor

$$\left[1 - \left(\frac{nT_c}{T}\right)^2\right]^{-\frac{1}{4}}$$

where, at $T = nT_c$, we have Γ_T blowing up. There is a way to soften the singularities though, within the "hard thermal loop"(HTL) framework [44]. In this procedure, we replace m by $m - i\gamma$, where $\gamma \sim 0.04e^2T$ in QED. This is tantamount to the following replacement:

$$\left[1 - \left(\frac{nT_c}{T}\right)^2\right]^{-\frac{1}{4}} \to \left[\frac{1 - \left(\frac{nT_c}{T}\right)^2}{\left(1 - \left(\frac{nT_c}{T}\right)^2\right)^2 + \left(\frac{2\gamma}{m}\left(\frac{nT_c}{T}\right)^2\right)^2}\right]^{\frac{1}{4}} \tag{4.31}$$

thus, removing the singularities. We may now use this in the pair production rate formula and see how the magnetic field and the GW amplitude evolve in time. To this end, we need to (numerically) solve the following differential equation, assuming that the dynamo has already

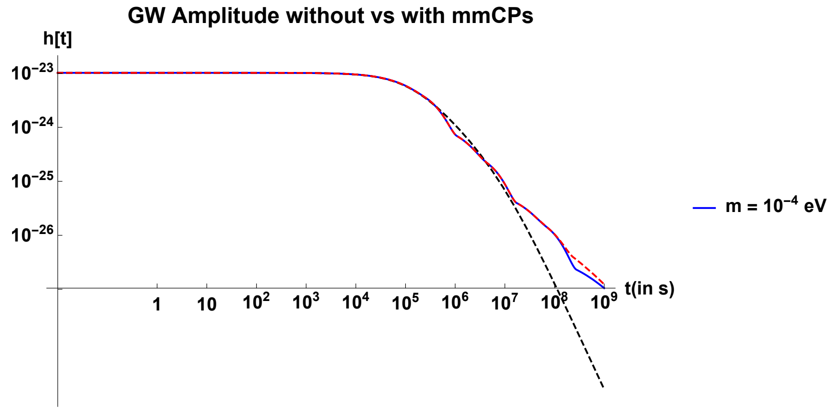


Figure 4.2: The GW amplitude when $B_{initial} = 10^{15}$ G, $m = 10^{-5}$ eV and $Q_m = 3.81 \times 10^{-19}$. In this figure, the blue line is the amplitude evolution not considering mmCP-SPP, red line considers mmCP-SPP, but not the thermal corrections, and the black dashed line consideres the full mmCP-SPP.

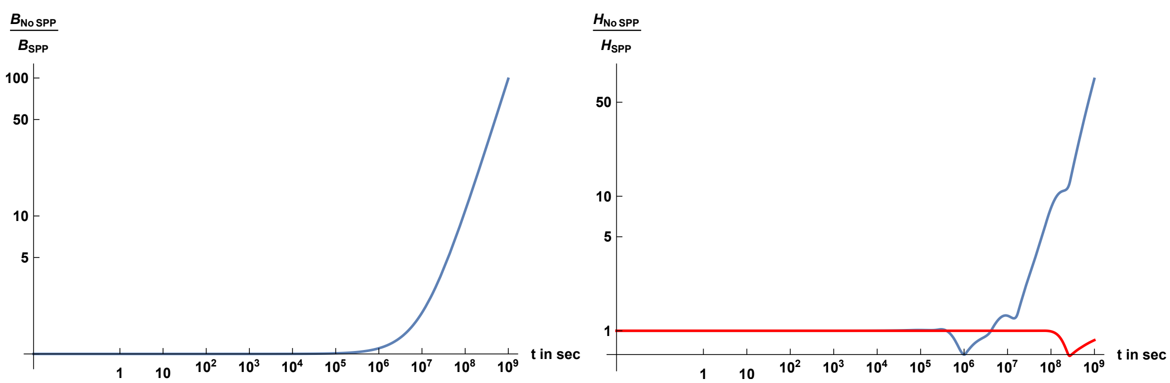


Figure 4.3: *Left*: The ratio of magnetic fields without SPP and with SPP. *Right*: The ratio of GW amplitude without SPP vs with SPP. Change can be observed after ~ 0.41 years or ~ 5 months. The red line doesn't take into account the thermal corrections to the mmCP-SPP rate and the blue line includes the thermal corrections to the SPP rate

ramped up the magnetic field to 10^{15} G:

$$\frac{dB(t)}{dt} = -\frac{B(t)}{\tau_O} - \frac{B^2(t)}{B(0)\tau_H} - \Gamma_{T\neq 0}(B(t))\frac{2}{R^2} \times Q_m g l V_{AV} \tag{4.32}$$

where $\Gamma_{T\neq 0}$ is given by Eq.(4.29). Solving this equation using the initial value of the magnetic field to be 10^{15} G, we obtain Figs. 4.2 and 4.3. Before we progress any further though, we need to understand how and where the HTL "trick" can be used. The rate is plotted against the temperature in Fig. 4.4. As can be seen, the singularities are not yet completely eliminated, so we need to be really careful about where the full thermal SPP formula can be used. One line of progress is to use the formula only when the ratio $\frac{T}{T_c}$ is not "close" to an integer. Quantifying

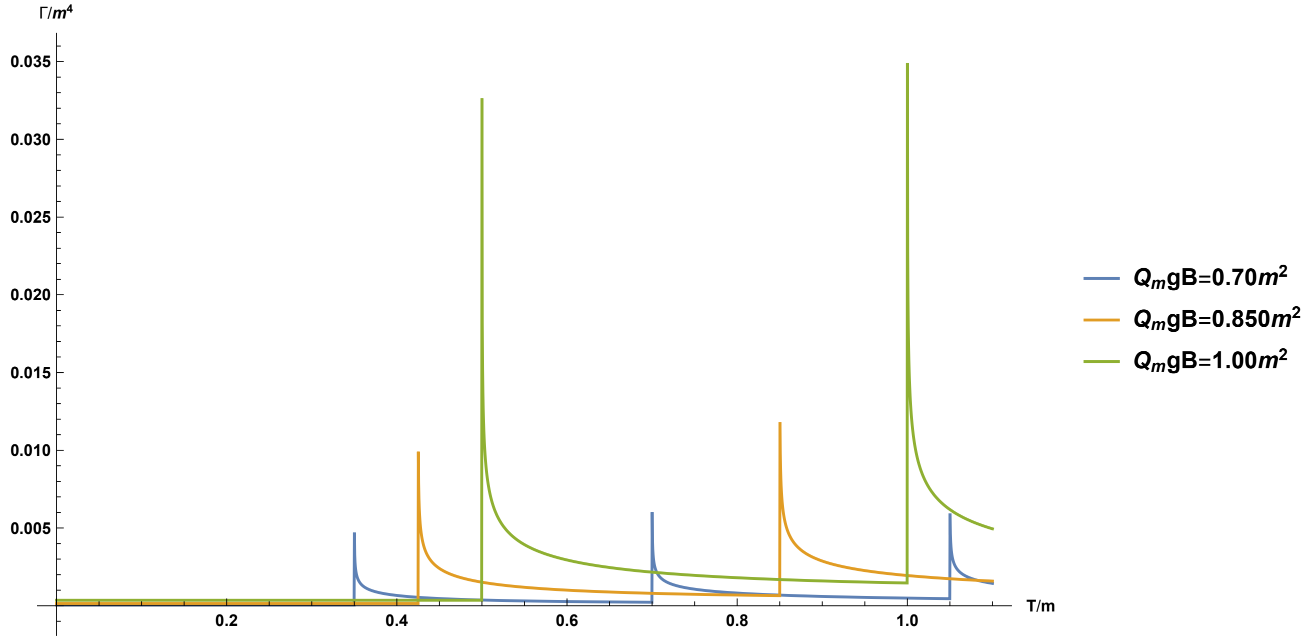


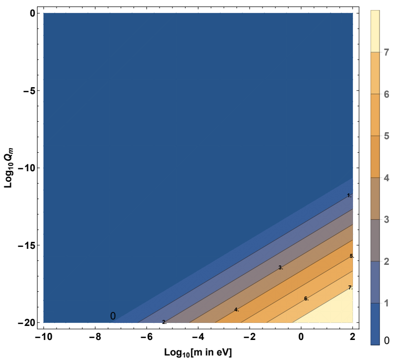
Figure 4.4: The HTL fudge that we incorporate in Eq.(4.31) is not a solution though. As can be clearly seen from the above plot, it only makes the singularities finite, doesn't eliminate them.

this, we could use a condition like:

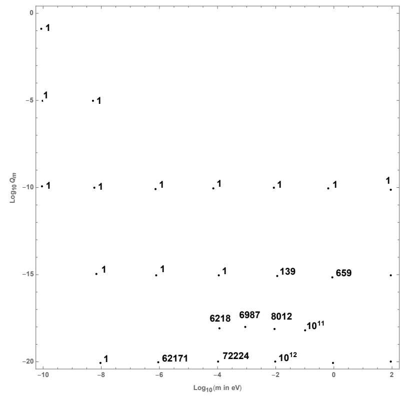
$$\left|\frac{T}{T_c} - \left[\frac{T}{T_c}\right]\right| > 0.2 \tag{4.33}$$

to use the formula.

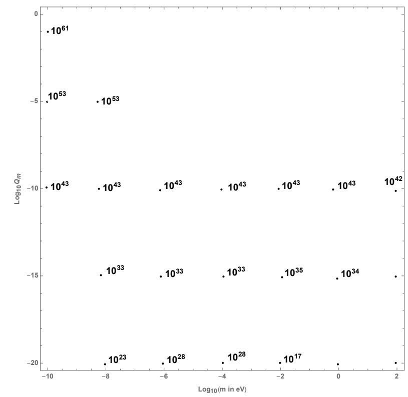
The thermal SPP is expected to give an increase in the pair production rate by some orders of magnitude, as can be confirmed by looking at Figs. 4.5, 4.6, 4.7, 4.8 and 4.9. With such an increase in the pair production rate, we can expect to see its effect in the GW amplitude.



(a) The contour plot of the logarithm of $n_{max} = \left[\frac{T}{T_c}\right]$.

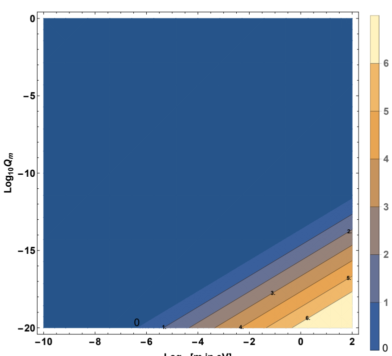


(b) The ratio of the total SPP rate including the thermal contribution to the zero temperature SPP rate for a magnetic field of 10^{15} G

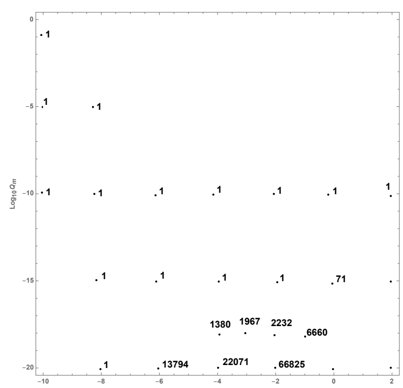


(c) The full SPP rate in units of $m^{-3}s^{-1}$

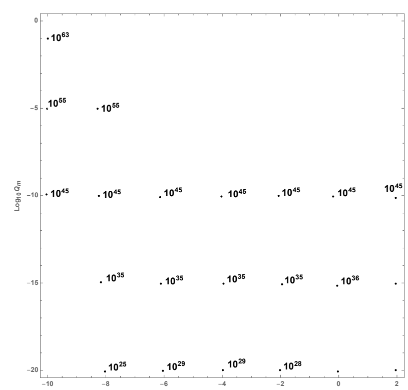
Figure 4.5: For a magnetic field of 10^{15} G



(a) The contour plot of the logarithm of $n_{max} = \left[\frac{T}{T_c}\right]$.

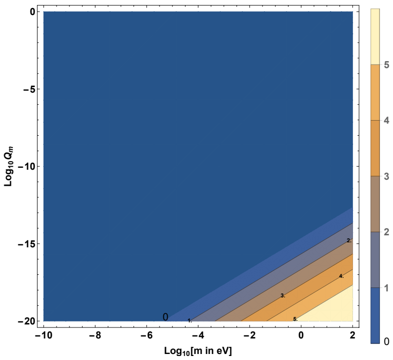


(b) The ratio of the total SPP rate including the thermal contribution to the zero temperature SPP rate for a magnetic field of 10^{16} G

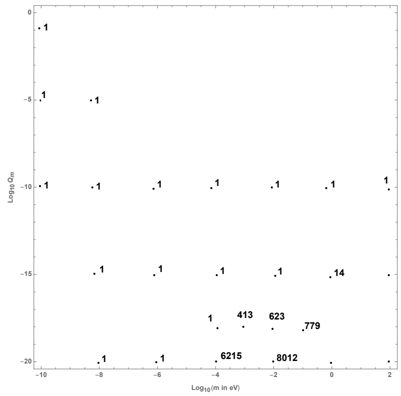


(c) The full SPP rate in units of $m^{-3}s^{-1}$

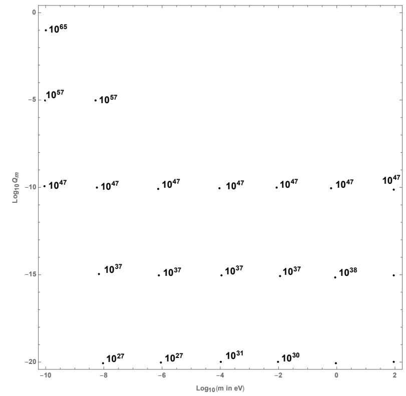
Figure 4.6: For a magnetic field of 10^{16} G



(a) The contour plot of the logarithm of $n_{max} = \left[\frac{T}{T_c}\right]$.

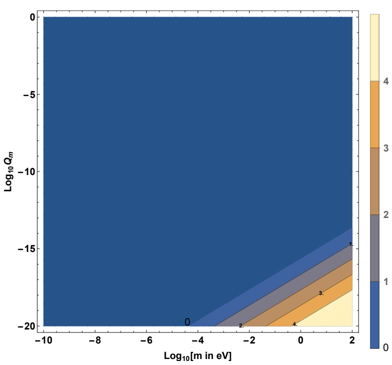


(b) The ratio of the total SPP rate including the thermal contribution to the zero temperature SPP rate for a magnetic field of 10^{17} G

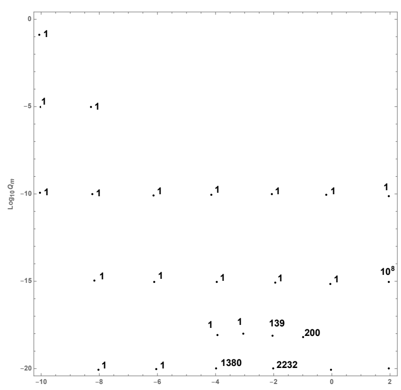


(c) The full SPP rate in units of $\mathrm{m}^{-3}\mathrm{s}^{-1}$

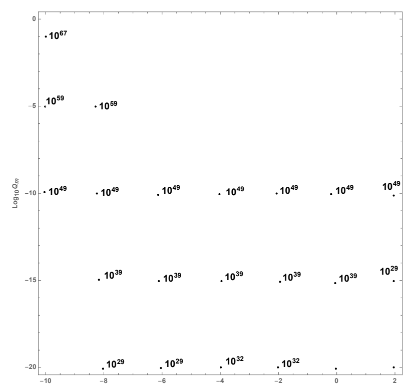
Figure 4.7: For a magnetic field of 10^{17} G



(a) The contour plot of the logarithm of $n_{max} = \left[\frac{T}{T_c}\right]$.



(b) The ratio of the total SPP rate including the thermal contribution to the zero temperature SPP rate for a magnetic field of 10^{18} G

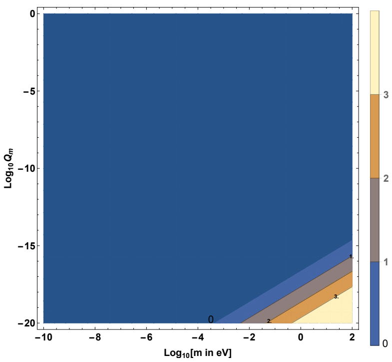


(c) The full SPP rate in units of $\mathrm{m}^{-3}\mathrm{s}^{-1}$

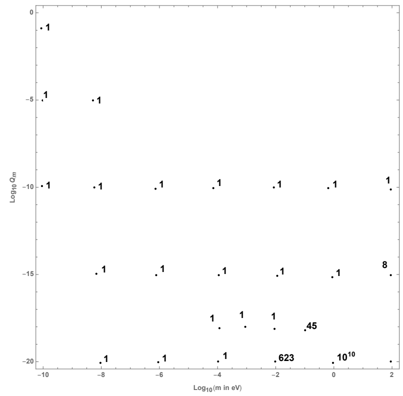
Figure 4.8: For a magnetic field of 10^{18} G

4.5 Results and Implications

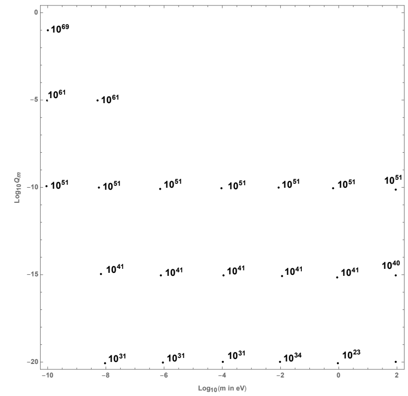
In this section, we numerically solve Eq.(4.32) using the expression for $\Gamma_{T\neq0}$ given in Eq.(4.29). The plots for the gravitational wave amplitude are the most important for us and thus, will be presented in this section. The condition (4.33) has been checked for each of the following plots. From the plots in Figs. 4.2, 4.3, 4.10, 4.11 and 4.12, we see that the "sweet spot" for



(a) The contour plot of the logarithm of $n_{max} = \left[\frac{T}{T_c}\right]$.



(b) The ratio of the total SPP rate including the thermal contribution to the zero temperature SPP rate for a magnetic field of 10^{19} G

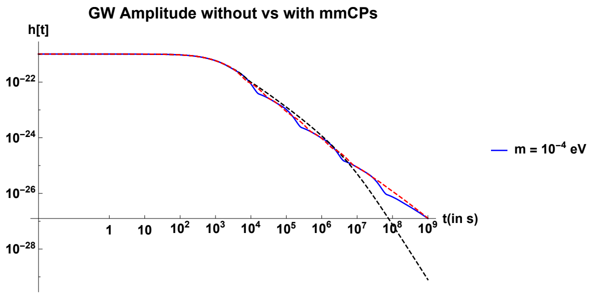


(c) The full SPP rate in units of $\mathrm{m}^{-3}\mathrm{s}^{-1}$

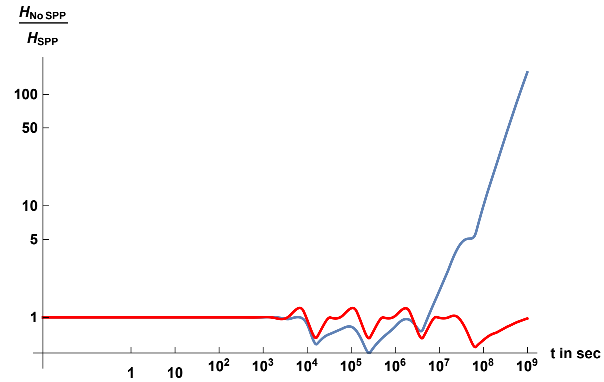
Figure 4.9: For a magnetic field of 10^{19} G

endeavour is when the initial magnetic field is around $10^{15} - 10^{16}$ G. This is because as the magnetic field increases, the n_{max} decreases which, in turn, means that the deviation from the case when mmCP-SPP is not considered will be lesser. In addition, notice that for plots using an initial magnetic field of 10^{17} and 10^{18} G, the mass of the mmCP has been increased to 10^{-2} eV, as is consistent with Fig.4.1.

In the future, when we do start detecting gravitational waves from neutron stars or magnetars with amplitudes $\sim 10^{-25}$, we should see one of the curves shown in the above referred figures and thus infer whether millimagnetically charged particles are present or not. These are continuous gravitational waves, in that they are not bursts, like the ones LIGO has already detected. Thus, taking the data over months (or even years) will help deciphering the mystery of dark matter.

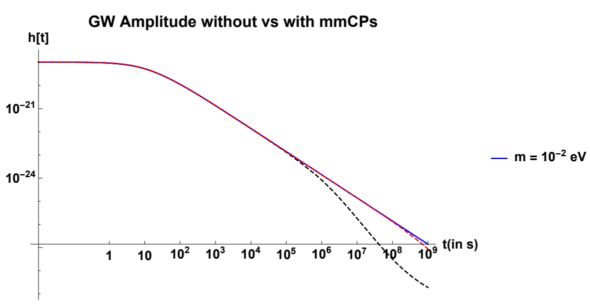


(a) The GW amplitude for an initial magnetic field of 10^{16} G and an mmCP mass of 10^{-4} eV. In this figure, the blue line is the amplitude evolution not considering mmCP-SPP, red line considers mmCP-SPP, but not the thermal corrections, and the black dashed line consideres the full mmCP-SPP.

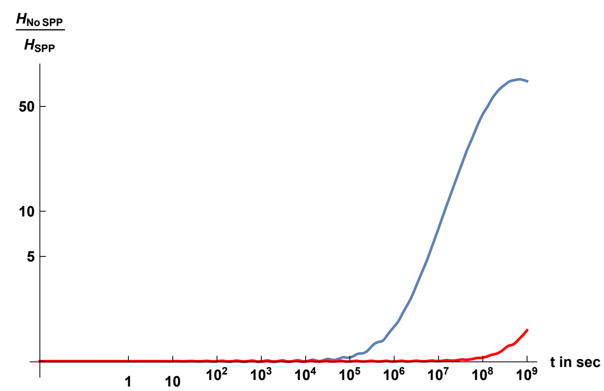


(b) The red line doesn't take into account the thermal corrections to the mmCP-SPP rate and the blue line includes the thermal corrections to the SPP rate

Figure 4.10: $B_0 = 10^{16}$ G, $m = 10^{-4}$ eV, $T = 86.25$ eV

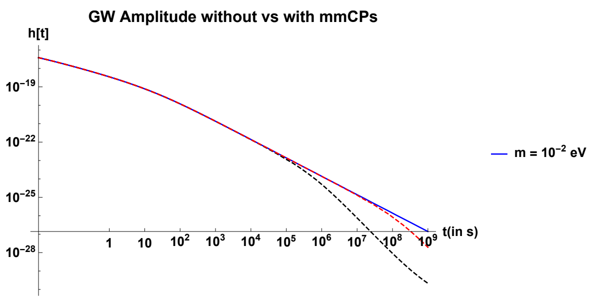


(a) The GW amplitude for an initial magnetic field of 10^{17} G and an mmCP mass of 10^{-2} eV. In this figure, the blue line is the amplitude evolution not considering mmCP-SPP, red line considers mmCP-SPP, but not the thermal corrections, and the black dashed line consideres the full mmCP-SPP.

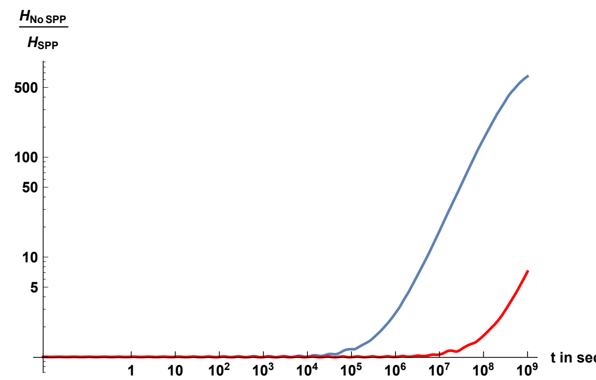


(b) The red line doesn't take into account the thermal corrections to the mmCP-SPP rate and the blue line includes the thermal corrections to the SPP rate

Figure 4.11: $B_0 = 10^{17}$ G, $m = 10^{-2}$ eV, $T = 86.25$ eV



(a) The GW amplitude for an initial magnetic field of 10^{18} G and an mmCP mass of 10^{-2} eV. In this figure, the blue line is the amplitude evolution not considering mmCP-SPP, red line considers mmCP-SPP, but not the thermal corrections, and the black dashed line consideres the full mmCP-SPP.



(b) The red line doesn't take into account the thermal corrections to the mmCP-SPP rate and the blue line includes the thermal corrections to the SPP rate

Figure 4.12: $B_0 = 10^{18}$ G, $m = 10^{-2}$ eV, $T = 86.25$ eV

Chapter 5

Worldline Deformations

Let us now change gears and look at the method to obtain the exact worldline instanton solutions for a given model in this chapter. For this, let us look at the basic premise of the *Worldline Instanton* formalism. In this formalism, we have the probability of pair production is given by [44]

$$\begin{aligned}\text{Prob[pair prod.]} = 1 - e^{i(W[A]-W^*[A])} &= 1 - e^{-2\text{Im}(W[A])} \\ &\simeq 2\text{Im}(W[A]) \end{aligned} \tag{5.1}$$

in the Minkowski picture, where $W[A]$ is the effective action. After performing a Wick rotation to go into the Euclidean picture, we can calculate the Euclidean effective action to be

$$W^{\mathbb{E}}[A] = -\int \frac{dT}{T} e^{-m^2 T} \oint \mathcal{D}x e^{-\int_0^T d\tau \left[\frac{\dot{x}^2}{4} + iA_\mu \dot{x}^\mu\right]} \tag{5.2}$$

where $\mathcal{D}x$ is the path integral measure over the closed worldlines. The next step is to perform a saddle point analysis by treating the exponent of the exponential as an action S. The action can be obtained to be:

$$S \simeq ma + i\int_0^1 du A_\mu \dot{x}^\mu \tag{5.3}$$

with $x_\mu(0) = x_\mu(1)$ and $\dot{x}^2 = a^2$. Obtaining the EOM's for this action, we have

$$\ddot{x}^\mu(u) = ia\mathcal{F}^{\mu\nu}\dot{x}^\nu(u) \tag{5.4}$$

where $\mathcal{F}^{\mu\nu}$ is the field tensor for the gauge field A_μ and $x^\mu(u)$ are the instantons.

This method has been used to obtain instanton solutions for homogeneous fields and we wish to see what the solutions would look like for some inhomogeneous configurations [46]. We shall thus take the vector potential to be:

$$A_3 = -iEF(x_4) \tag{5.5}$$

where E is the electric field and $F(x_4)$ is some function of the Euclidean time x_4. With this gauge choice, the instanton equations thus become [46]:

$$\ddot{x}_3 = -\frac{aE}{m}F'(x_4)\dot{x}_4 \tag{5.6}$$

$$\ddot{x}_4 = \frac{aE}{m}F'(x_4)\dot{x}_3 \tag{5.7}$$

with x_1 and x_2 being trivial.

5.1 Deformations in scalar fields

Let us now look at a different problem of a real scalar field in (1+1)D. For this, consider the model:

$$\mathcal{L}_\chi = \frac{1}{2}(\partial_\mu \chi)^2 - V_\chi(\chi) \tag{5.8}$$

where $V_\chi(\chi)$ is a given potential. The equation of motion for the this Lagrangian are given by

$$\partial_\mu \partial^\mu \chi + V'_\chi(\chi) = 0$$

where if we are looking for static solutions, i.e., $\chi = \chi(u)$, not dependent on time, then the equation reduces to [47]:

$$\frac{d^2\chi}{du^2} = \frac{dV_\chi}{d\chi} \tag{5.9}$$

which can be manipulated as below:

$$\begin{aligned} \frac{d}{du}(\chi') &= \frac{du}{d\chi}\frac{dV_\chi}{du} \\ \implies \frac{d}{du}(\chi'^2) &= \frac{d}{du}(2V_\chi) \\ \implies \chi'^2 &= 2V_\chi(\chi) + const. \end{aligned}$$

where we impose the boundary conditions

$$\chi \to \bar{\chi} \;\; \& \;\; \frac{d\chi}{du} \to 0 \text{ as } u \to -\infty$$

where $\bar{\chi}$ is the critical point for the potential, i.e., $V_\chi(\bar{\chi}) = 0$, in order to get the *const.* to be 0. We thus have the full EOM as:

$$\frac{d^2\chi}{du^2} = \frac{dV_\chi}{d\chi} \tag{5.10}$$

$$\chi'^2 = 2V_\chi(\chi) \tag{5.11}$$

Consider a similar model for a different scalar field ϕ with a different potential $V_\phi(\phi) \neq V_\chi(\chi)$. We have the following EOM for the static solution from a similar analysis:

$$\frac{d^2\phi}{du^2} = \frac{dV_\phi}{d\phi} \tag{5.12}$$

$$\phi'^2 = 2V_\phi(\phi) \tag{5.13}$$

Now, let $\mathcal{L}_\chi$ be exactly solvable and let it also support finite energy defects (see section 5.1.1). Then, it is possible to write [46]:

$$V_\phi(\phi) = \frac{V_\chi(\chi \to f(\phi))}{(f'(\phi))^2} \tag{5.14}$$

where f is a real isomorphism. This deformation implies that the solution in the deformed $\mathcal{L}_\phi$ is given exactly by $\phi = f^{-1}(\chi)$, where χ is that value which exactly solves the static equations in the original $\mathcal{L}_\chi$ theory.

5.1.1 Defects

In this section, we will see some basic theory on defects, enough so that we can understand the technique outlined in [46]. For this purpose, consider a theory described by the following Lagrangian density [48]:

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi)^2 - V_\phi(\phi) \tag{5.15}$$

in (1+1)D. Let the potential have the following properties:

1. $V(\phi) = 0$ at a set of n critical points $\{\bar{\phi}_1, ..., \bar{\phi}_n\}$.

2. $V(\bar{\phi}_i) = 0 \; \forall i$.

3. $V'(\bar{\phi}_i) = 0 \; \forall i$.

The static EOM we derived in the previous section is still applicable, rewritten here for convenience:

$$\frac{d^2\phi}{du^2} = \frac{dV_\phi(\phi)}{d\phi} \tag{5.16}$$

As before, we can integrate this equation to obtain

$$\phi'^2 = 2V_\phi(\phi) + C \tag{5.17}$$

And in order to determine the value of C, we will make use of the existence of defects. The energy density for the static solutions is given by:

$$\mathcal{H} = \frac{1}{2}\left(\frac{d\phi}{du}\right)^2 + V_\phi(\phi) \tag{5.18}$$

where the following condition must be satisfied for the finiteness of the kinetic portion of the energy:

$$\frac{d\phi}{du} \to 0 \text{ as } u \to \pm\infty. \tag{5.19}$$

Using the critical points, we can distinguish between two types of defect structures. One type is called topological or kink-like defect structures which connect two different, adjacent critical points, like so:

$$\lim_{u\to-\infty} \phi(u) = \bar{\phi}_i \qquad \& \qquad \lim_{u\to+\infty} \phi(u) = \bar{\phi}_{i+1} \tag{5.20}$$

and the other type, called non-topological or lump-like defects mostly require only a single critical point, like so:

$$\lim_{u\to\pm\infty} \phi(u) = \bar{\phi}_i. \tag{5.21}$$

It was using these boundary conditions that we obtained the Eqs.(5.10) and (5.11). Now that we have a basic understanding of defects, let us go back to our main issue.

5.2 Our saddle point equations

In this section, we come back to our main problem after that slight digression. We are going to apply the technique in Eq.(5.14) to our instanton equations (5.6) and (5.7). For that, we will recast the equations to a convenient form. Integrating Eq.(5.6), we get

$$\dot{x}_3 = -\frac{aE}{m} F(x_4)$$

which when put into Eq.(5.7) immediately gives:

$$\frac{d^2 x_4}{du^2} = -\frac{d}{dx_4}\left[\frac{aE}{\sqrt{2}m} F(x_4)\right]^2 \tag{5.22}$$

Let us now define a potential

$$V(x_4) = -\left[\frac{aE}{\sqrt{2}m} F(x_4)\right]^2 = -\frac{\dot{x}_3^2}{2} \tag{5.23}$$

Now, using the condition $\dot{x}^2 = \dot{x}_3^2 + \dot{x}_4^2 = a^2$, in addition to the above equation, we have:

$$\frac{d^2 x_4}{du^2} = \frac{dV(x_4)}{dx_4} \tag{5.24}$$

$$\dot{x}_4^2 = a^2 + 2V(x_4) \tag{5.25}$$

which, barring the RHS of the second equation, is very similar to Eqs.(5.10) and (5.11). This prompts us to proceed in a similar fashion, by introducing a "world line deformation". For convenience of notation, let us rename x_4 as χ. The equations then become

$$\frac{d^2 \chi}{du^2} = \frac{dV_\chi(\chi)}{d\chi} \tag{5.26}$$

$$\dot{\chi}^2 = a_\chi^2 + 2V_\chi(\chi) \tag{5.27}$$

where a_χ is the constant. Consider $x_4 = \phi$ with a different potential $V_\phi(\phi)$. In this model, once again, the EOM are:

$$\frac{d^2 \phi}{du^2} = \frac{dV_\phi(\phi)}{d\phi} \tag{5.28}$$

$$\dot{\phi}^2 = a_\phi^2 + 2V_\phi(\phi). \tag{5.29}$$

Let us now apply the technique outlined in Eq.(5.14) to the above χ and ϕ models. This involves imposing [46]:

$$a_\phi^2 + 2V_\phi(\phi) = \frac{a_\chi^2 + 2V_\chi(\chi \to f(\phi))}{f'(\phi)^2} \tag{5.30}$$

and then finding ϕ using $\phi = f^{-1}(\chi)$, where χ solves Eqs.(5.26) and (5.27). Therefore, if the exact worldline instantons in the χ model are known, we can obtain the exact worldline instantons in the ϕ model. Next, let us prove that $\phi = f^{-1}(\chi)$ is indeed the exact deformed solution.

Claim. $\phi = f^{-1}(\chi)$ *is the exact deformed solution of the Eqs.(5.28) and (5.29).*

Proof. Using Eq.(5.30) in Eq.(5.28), we obtain

$$\frac{d^2\phi}{du^2} = \frac{V_\chi'(f(\phi))}{f'(\phi)} - \frac{f''(\phi)}{f'(\phi)^3}[a_\chi^2 + 2V_\chi(f(\phi))] \tag{5.31}$$

In order to prove the claim, it is sufficient to prove that direct differentiation of $\phi = f^{-1}(\chi)$ gives the above equation. For that, we use the fact that $(f^{-1})'(\chi) = \frac{1}{f'(\phi)}$ and obtain

$$\begin{aligned} \frac{d^2\phi}{du^2} &= \frac{1}{f'(\phi)}\left[\frac{d^2\chi}{du^2}\right] - \frac{f''(\phi)}{f'(\phi)^3}\left[\frac{d\chi}{du}\right]^2 \\ &= \frac{V_\chi'(f(\phi))}{f'(\phi)} - \frac{f''(\phi)}{f'(\phi)^3}[a_\chi^2 + 2V_\chi(f(\phi))] \end{aligned} \tag{5.32}$$

where, in the last line, we have used Eqs.(5.26) and (5.27). □ □

That was the basic outline of the deformation technique. Let us now apply this technique to some examples, following the prescriptions defined in [46]. Let us first redefine certain parameters, as below:

$$\tilde{F}(\chi) = \omega F(\chi) \quad \text{and} \quad \gamma = \frac{\omega m}{E} \tag{5.33}$$

where ω is the frequency and γ is called the Keldysh parameter. Thus, the potential can be written as:

$$V(\chi) = -\frac{1}{2}\frac{a^2\tilde{F}^2(\chi)}{\gamma^2} \tag{5.34}$$

Plugging this form of the potential into Eq.(5.30) and making $f'(\phi)$ the subject of the equation, we have the following differential equation for $f(\phi)$:

$$\left(\frac{df(\phi)}{d\phi}\right)^2 = \frac{a_\chi^2}{a_\phi^2}\left(\frac{\gamma^2 - \tilde{F}_\chi^2(f(\phi))}{\gamma^2 - \tilde{F}_\phi^2(\phi)}\right) \tag{5.35}$$

Equivalently, we also have a differential equation for $f^{-1}(\chi)$, given by:

$$\left(\frac{df^{-1}(\chi)}{d\chi}\right)^{-2} = \frac{a_\chi^2}{a_\phi^2}\left(\frac{\gamma^2 - \tilde{F}_\chi^2(\chi)}{\gamma^2 - \tilde{F}_\phi^2(f^{-1}(\chi))}\right) \tag{5.36}$$

Inverting these equations, given the potentials, we can get the deformation function and thus obtain the exact instanton solution in one model if the exact instanton solution in the other model is known. This, however, is extremely difficult due to the sheer mathematical complexity. Another way to proceed is to actually go the other way. Choose the function f to go from one model to another and apply f consecutively to produce a tower of exactly solvable models by just differentiating the deformation function and obtaining the potential [46]. The following schematic clearly illustrates this process:

$$(V_\chi, \chi) \underset{\text{differentiating } f}{\overset{\text{integrating } V}{\rightleftharpoons}} (V_\phi, \phi) \underset{\text{differentiating } f}{\overset{\text{integrating } V}{\rightleftharpoons}} (V_\psi, \psi) \underset{\text{differentiating } f}{\overset{\text{integrating } V}{\rightleftharpoons}} \ldots \tag{5.37}$$

We can generate a different tower by using another $\tilde{f}$ and applying it iteratively. In most situations, however, we are given the potential and wish to find the worldline instantons, so let us look at a few applications of the technique that establish it's authenticity in the next section.

5.3 Applications

In this section we shall solve the Eqs. (5.35) and (5.36) for a few handpicked field configurations. We will solve for the cases when the electric field is sinusoidal in time and another case where the electric field is "single-pulse time-dependent" [49]. Before we begin, however, let us see what the instanton solutions are in the case of a constant electric field background, i.e., $F(x_4) = x_4$. In this case, the solutions turn out to be:

$$x_3(u) = \frac{m}{E}\cos(2\pi n u) = \frac{\gamma}{\omega}\cos(2\pi n u) \tag{5.38}$$

$$x_4(u) = \frac{m}{E}\sin(2\pi n u) = \frac{\gamma}{\omega}\sin(2\pi n u). \tag{5.39}$$

In the following calculations, this x_4 solution will serve as our χ model with the invariant a given by:

$$a = 2\pi n \frac{\gamma}{\omega}. \tag{5.40}$$

5.3.1 Sinusoidal time-dependent electric background

For this case, the electric field is given by $\mathbf{E}(t) = E\cos\omega t\hat{x}_3$ [46, 49]. From Eq.(5.5) and $t = -ix_4$, we have $F(x_4) = \frac{\sinh\omega x_4}{\omega}$. To put it into the current context of the deformation technique, we take this $F(x_4)$ to be the ϕ model and the constant electric background case to be the χ model. In that spirit, let us rewrite F and x_4 written earlier as $F_\phi(\phi)$ and ϕ respectively. Then, according to our notation, we have the following information available:

$$\begin{aligned}
F_\chi(\chi) &= \chi \implies \tilde{F}_\chi(\chi) = \omega\chi \\
F_\phi(\phi) &= \frac{\sinh\omega\phi}{\omega} \implies \tilde{F}_\phi(\phi) = \sinh\omega\phi \\
\chi(u) &= \frac{\gamma}{\omega}\sin(2\pi n u) \\
a_\chi &= 2\pi n\frac{\gamma}{\omega}.
\end{aligned}$$

Given this information, we need to find $\phi(u)$ and a_ϕ (through this, x_3 for the ϕ model is automatically determined using the relation between the three quantities). The way forward is now clear. We solve the ODE's (5.35) and (5.36) and obtain $f(\phi)$ and $f^{-1}(\chi)$ in terms of χ, a_χ, ϕ and a_ϕ. We then substitute χ and a_χ from Eqs.(5.39) and (5.40) and simultaneously solve for $\phi(u)$ and a_ϕ.

Solving Eq.(5.35) gives $f(\phi)$ to be [46]:

$$f(\phi) = \frac{\gamma}{\omega}\frac{\tan\left(\frac{ia_\chi}{\gamma a_\phi}\mathbf{F}\left(i\phi\omega\middle| -\frac{1}{\gamma^2}\right)\right)}{\sqrt{1+\tan^2\left(\frac{ia_\chi}{\gamma a_\phi}\mathbf{F}\left(i\phi\omega| -\frac{1}{\gamma^2}\right)\right)}} \tag{5.41}$$

where $\mathbf{F}(\cdot)$ represents the incomplete elliptical integral of the first kind, given by

$$\mathbf{F}(\phi|m) = \int_0^\phi d\theta \frac{1}{\sqrt{1-m\sin^2\theta}}$$

Solving Eq.(5.36), on the other hand, gives the inverse $f^{-1}(\chi)$ to be:

$$f^{-1}(\chi) = \frac{i}{\omega}\mathbf{am}\left(\frac{\gamma a_\phi}{a_\chi}\tan^{-1}\left(\frac{\omega\chi}{\sqrt{\gamma^2-\omega^2\chi^2}}\right)\middle| -\frac{1}{\gamma^2}\right) \tag{5.42}$$

where $\mathbf{am}(\cdot)$ is the Jacobi amplitude, defined in the following way. If $u = \mathbf{F}(\phi|m)$, then, $\phi = \mathbf{am}(u|m)$ [50]. Substituting χ and a_χ as mentioned at the start of this subsection, we solve for ϕ and a_ϕ and obtain the following:

$$\phi(u) = \frac{i}{\omega}\mathbf{am}\left(\frac{2\gamma\tan^{-1}\left(\frac{\sin 2\pi nu}{\sqrt{\cos^2 2\pi nu}}\right)\mathbf{K}\left(\frac{\gamma^2}{1+\gamma^2}\right)}{i\pi\sqrt{1+\gamma^2}}\middle| -\frac{1}{\gamma^2}\right) \tag{5.43}$$

$$a_\phi = \frac{4\gamma n\mathbf{K}\left(\frac{\gamma^2}{1+\gamma^2}\right)}{\omega\sqrt{1+\gamma^2}} \tag{5.44}$$

where $\mathbf{K}(\cdot)$ is the complete elliptical elliptical integral of the first kind, given by $\mathbf{K}(m) = \mathbf{F}(\frac{\pi}{2}|m)$. Using the identities in [50], this $\phi(u)$ can be transformed into the following form:

$$\phi(u) = \sin^{-1}\left(-i\left(\frac{\gamma^2}{1+\gamma^2}\right)^{\frac{1}{2}}\mathrm{sd}\left(4nu\mathbf{K}\left(\frac{\gamma^2}{1+\gamma^2}\right)\right)\middle|\frac{\gamma^2}{1+\gamma^2}\right) \tag{5.45}$$

where $\mathrm{sd}(u|m) = \frac{\sin\psi}{\sqrt{1-m\sin^2\psi}}$, and $\psi = \mathbf{am}(u|m)$. (We use the identity $\mathrm{sn}(u|-m) = \left(\frac{1}{1+m}\right)^{\frac{1}{2}}\mathrm{sd}(u(1+m)^{\frac{1}{2}}\left|\frac{m}{1+m}\right.)$, where $\mathrm{sn}(u|m) = \sin(\mathbf{am}(u|m))$.) Now, x_3 can be obtained using

$$\frac{dx_3}{du} = -\frac{a}{\gamma}\tilde{F}_\phi(\phi(u)) \tag{5.46}$$

to be

$$x_3(u) = \sin^{-1}\left(\left(\frac{\gamma^2}{1+\gamma^2}\right)^{\frac{1}{2}}\mathrm{cd}\left(4nu\mathbf{K}\left(\frac{\gamma^2}{1+\gamma^2}\right)\right)\middle|\frac{\gamma^2}{1+\gamma^2}\right) \tag{5.47}$$

where $\mathrm{cd}(u|m) = \frac{\cos\psi}{\sqrt{1-m\sin^2\psi}}$, and $\psi = \mathbf{am}(u|m)$. This can be cross verified to be correct through the solution in [49].

5.3.2 Single-pulse time-dependent electric background

In this scenario, the initial steps are the same as in the previous subsection with the replacement that the electric field is now $\mathbf{E}(t) = E\text{sech}^2\omega t\hat{x}_3$ [49] and thus $\tilde{F}_\phi(\phi) = \tan(\omega\phi)$. We once again solve Eq.(5.35) to obtain $f(\phi)$ to be:

$$f(\phi) = \frac{\gamma}{\omega}\sin\left[\frac{a_\chi}{a_\phi\sqrt{1+\gamma^2}}\sin^{-1}\left\{\frac{\sqrt{1+\gamma^2}\sin\omega\phi}{\gamma}\right\}\right] \tag{5.48}$$

and solve Eq.(5.36) to obtain $f^{-1}(\chi)$ to be of the form:

$$f^{-1}(\chi) = \frac{1}{\omega}\sin^{-1}\left[\frac{\gamma}{\sqrt{1+\gamma^2}}\sin\left\{\frac{a_\phi\sqrt{1+\gamma^2}}{a_\chi}\sin^{-1}\left(\frac{\omega\chi}{\gamma}\right)\right\}\right]. \tag{5.49}$$

Now, solving for ϕ and a_ϕ after substituting the constant electric background instanton solution for χ and a_χ, we find that the following solution set for $\phi(u)$ and a_ϕ satisfies the above equations:

$$\phi(u) = \frac{1}{\omega}\sin^{-1}\left[\frac{\gamma}{\sqrt{1+\gamma^2}}\sin(2\pi nu)\right] \tag{5.50}$$

$$a_\phi = \frac{\gamma}{\omega}\frac{2\pi n}{\sqrt{1+\gamma^2}}. \tag{5.51}$$

Of course, $x_3(u)$ for the single-pulse background can also be obtained using the above two solutions in the ODE:

$$\frac{dx_3}{du} = -\frac{a}{\gamma}\tilde{F}_\phi(\phi(u)) \tag{5.52}$$

to obtain

$$x_3(u) = \frac{1}{\omega\sqrt{1+\gamma^2}}\sinh^{-1}[\gamma\cos(2\pi nu)] \tag{5.53}$$

Thus, the worldline instantons for the single-pulse time-dependent electric field are given by Eqs.(5.50) and (5.53). Once again, this solution can be verified with the one presented in [49]

Chapter 6

Conclusion and future work

Millimagnetically charged particles (mmCPs) are intriguing due to their status as a viable candidate for the particle nature of dark matter. In this thesis, we first look at a way in which these exotic states may arise in quantum field theory and then propose a possible detection strategy. By looking at the gravitational wave amplitudes from magnetars, where the Schwinger pair production (SPP) of mmCPs is believed to occur, in two different scenarios of mmCP-SPP occurring and not occurring, we can infer the presence of these exotic states. While the current sensitivity of the Laser Interferometer Gravitational-Wave Observatory is not yet viable for detecting the continuous gravitational waves where the difference is likely to occur, future detectors such as the Einstein telescope have great hope. Einstein telescope is projected to be able to detect gravitational wave amplitudes of up to 10^{-26} [51, 52, 53] which, according to Fig. 4.2, is very good news. The result of the experiment will provide some progress in our understanding of the particle nature of one of the greatest mysteries of the universe, dark matter.

In Chapter 5, we look at a novel technique to calculate exact worldline instantons applicable to a variety of field configurations [46]. This will, of course, serve as an important step in calculating the SPP formula for that configuration. This enterprise of calculating worldline instantons, until now, has been very difficult to undertake and has only been done for a handful of cases [49] such as a single-pulse time dependent electric field. We are currently trying to fully understand the implications of such a result and it would potentially be a very rewarding endeavour to see its application to a challenging problem.

Bibliography

[1] Gianfranco Bertone and Dan Hooper. “History of dark matter”. In: *Rev. Mod. Phys.* 90.4 (2018), p. 045002. DOI: 10.1103/RevModPhys.90.045002. arXiv: 1605.04909 [astro-ph.CO].

[2] F. Zwicky. “Die Rotverschiebung von extragalaktischen Nebeln”. In: *Helvetica Physica Acta* 6 (1933), pp. 110–127.

[3] V. C. Rubin, W. K. Ford Jr., and N. Thonnard. “Extended rotation curves of high-luminosity spiral galaxies. IV - Systematic dynamical properties, SA through SC”. In: *Astrophys. J.* 225 (Nov. 1978), pp. L107–L111. DOI: 10.1086/182804.

[4] J. P. Ostriker and P. J. E. Peebles. “A Numerical Study of the Stability of Flattened Galaxies: or, can Cold Galaxies Survive?” In: *Astrophys. J.* 186 (Dec. 1973), pp. 467–480. DOI: 10.1086/152513.

[5] Joel R. Primack. “Dark matter and structure formation”. In: *Midrasha Mathematicae in Jerusalem: Winter School in Dynamical Systems Jerusalem, Israel, January 12-17, 1997.* 1997. arXiv: astro-ph/9707285 [astro-ph].

[6] Karsten Jedamzik and Maxim Pospelov. “Big Bang Nucleosynthesis and Particle Dark Matter”. In: *New J. Phys.* 11 (2009), p. 105028. DOI: 10.1088/1367-2630/11/10/105028. arXiv: 0906.2087 [hep-ph].

[7] Bob Holdom. “Two U(1)’s and Epsilon Charge Shifts”. In: *Phys. Lett.* 166B (1986), pp. 196–198. DOI: 10.1016/0370-2693(86)91377-8.

[8] Anson Hook and Junwu Huang. “Bounding millimagnetically charged particles with magnetars”. In: *Phys. Rev.* D96.5 (2017), p. 055010. DOI: 10.1103/PhysRevD.96.055010. arXiv: 1705.01107 [hep-ph].

[9] David Pooley et al. “GW170817 Most Likely Made a Black Hole”. In: *The Astrophysical Journal* 859.2 (2018), p. L23. DOI: 10.3847/2041-8213/aac3d6. URL: https://doi.org/10.3847%2F2041-8213%2Faac3d6.

[10] Milton Ruiz, Stuart L. Shapiro, and Antonios Tsokaros. "GW170817, general relativistic magnetohydrodynamic simulations, and the neutron star maximum mass". In: *Phys. Rev. D* 97 (2 Jan. 2018), p. 021501. DOI: 10.1103/PhysRevD.97.021501. URL: https://link.aps.org/doi/10.1103/PhysRevD.97.021501.

[11] Masaru Shibata et al. "Modeling GW170817 based on numerical relativity and its implications". In: *Phys. Rev. D* 96 (12 Dec. 2017), p. 123012. DOI: 10.1103/PhysRevD.96.123012. URL: https://link.aps.org/doi/10.1103/PhysRevD.96.123012.

[12] Luciano Rezzolla, Elias R. Most, and Lukas R. Weih. "Using Gravitational-wave Observations and Quasi-universal Relations to Constrain the Maximum Mass of Neutron Stars". In: *The Astrophysical Journal* 852.2 (Jan. 2018), p. L25. DOI: 10.3847/2041-8213/aaa401. URL: https://doi.org/10.3847%2F2041-8213%2Faaa401.

[13] Ben Margalit and Brian D. Metzger. "Constraining the Maximum Mass of Neutron Stars from Multi-messenger Observations of GW170817". In: *The Astrophysical Journal* 850.2 (Nov. 2017), p. L19. DOI: 10.3847/2041-8213/aa991c. URL: https://doi.org/10.3847%2F2041-8213%2Faa991c.

[14] Mrunal Korwar and Arun M. Thalapillil. "Novel Astrophysical Probes of Light Millicharged Fermions through Schwinger Pair Production". In: (2017). arXiv: 1709.07888 [hep-ph].

[15] Sandro Mereghetti. "The strongest cosmic magnets: Soft Gamma-ray Repeaters and Anomalous X-ray Pulsars". In: *Astron. Astrophys. Rev.* 15 (2008), pp. 225–287. DOI: 10.1007/s00159-008-0011-z. arXiv: 0804.0250 [astro-ph].

[16] Victoria M. Kaspi and Andrei M. Beloborodov. "Magnetars". In: *Annual Review of Astronomy and Astrophysics* 55.1 (2017), pp. 261–301. DOI: 10.1146/annurev-astro-081915-023329. eprint: https://doi.org/10.1146/annurev-astro-081915-023329. URL: https://doi.org/10.1146/annurev-astro-081915-023329.

[17] R. C. Duncan and C. Thompson. "Formation of very strongly magnetized neutron stars - Implications for gamma-ray bursts". In: *Astrophys. J. Let.* 392 (June 1992), pp. L9–L13. DOI: 10.1086/186413.

[18] Alessandra Buonanno. "Gravitational waves". In: *Les Houches Summer School - Session 86: Particle Physics and Cosmology: The Fabric of Spacetime Les Houches, France, July 31-August 25, 2006*. 2007. arXiv: 0709.4682 [gr-qc]. URL: https://inspirehep.net/record/762437/files/arXiv:0709.4682.pdf.

[19] B. P. Abbott et al. "Observation of Gravitational Waves from a Binary Black Hole Merger". In: *Phys. Rev. Lett.* 116 (6 Feb. 2016), p. 061102. DOI: 10.1103/PhysRevLett.116.061102. URL: https://link.aps.org/doi/10.1103/PhysRevLett.116.061102.

[20] Michele Maggiore. *Gravitational Waves. Vol. 1: Theory and Experiments*. Oxford Master Series in Physics. Oxford University Press, 2007. ISBN: 9780198570745, 9780198520740. URL: http://www.oup.com/uk/catalogue/?ci=9780198570745.

[21] L.D. Landau, L.D. Landau, and M. Hamermesh. *The Classical Theory of Fields*. Course of theoretical physics. Butterworth-Heinemann, 1975. ISBN: 9780750627689. URL: https://books.google.co.in/books?id=X18PF4oKyrUC.

[22] J. R. Ipser. “Gravitational Radiation from Slowly Rotating, Fully Relativistic Stars”. In: *Astrophys. J.* 166 (May 1971), p. 175. DOI: 10.1086/150948.

[23] Kip S. Thorne. “Multipole expansions of gravitational radiation”. In: *Rev. Mod. Phys.* 52 (2 Apr. 1980), pp. 299–339. DOI: 10.1103/RevModPhys.52.299. URL: https://link.aps.org/doi/10.1103/RevModPhys.52.299.

[24] S. Bonazzola and E. Gourgoulhon. “Gravitational waves from pulsars: Emission by the magnetic field induced distortion”. In: *Astron. Astrophys.* 312 (1996), p. 675. arXiv: astro-ph/9602107 [astro-ph].

[25] S. Chandrasekhar and E. Fermi. “Problems of Gravitational Stability in the Presence of a Magnetic Field.” In: *Astrophys. J.* 118 (July 1953), p. 116. DOI: 10.1086/145732.

[26] B. Haskell et al. “Modelling magnetically deformed neutron stars”. In: *Mon. Not. Roy. Astron. Soc.* 385 (2008), pp. 531–542. DOI: 10.1111/j.1365-2966.2008.12861.x. arXiv: 0705.1780 [astro-ph].

[27] H. C. Spruit. “Essential Magnetohydrodynamics for Astrophysics”. In: *ArXiv e-prints* (Jan. 2013). arXiv: 1301.5572 [astro-ph.IM].

[28] A. Colaiuda et al. “Relativistic models of magnetars: structure and deformations”. In: *Mon. Not. Roy. Astron. Soc.* 385 (2008), pp. 2080–2096. DOI: 10.1111/j.1365-2966.2008.12966.x. arXiv: 0712.2162 [astro-ph].

[29] Arthur George Suvorov, Alpha Mastrano, and Ulrich Geppert. “Gravitational radiation from neutron stars deformed by crustal Hall drift”. In: *Mon. Not. Roy. Astron. Soc.* 459.3 (2016), pp. 3407–3418. DOI: 10.1093/mnras/stw909. arXiv: 1604.04305 [astro-ph.HE].

[30] V. C. A. Ferraro. “On the Equilibrium of Magnetic Stars.” In: *Astrophys. J.* 119 (Mar. 1954), p. 407. DOI: 10.1086/145838.

[31] J. J. Monaghan. “Magnetic fields in steller bodies III Distortion of polytropes”. In: *Mon. Not. Roy. Astron. Soc.* 134 (1966), p. 275. DOI: 10.1093/mnras/134.3.275.

[32] I. W. Roxburgh. “Magnetostatic equilibrium of polytropes”. In: *Mon. Not. Roy. Astron. Soc.* 132 (1966), p. 347. DOI: 10.1093/mnras/132.2.347.

[33] S. Chandrasekhar. *Ellipsoidal figures of equilibrium*. 1969.

[34] D. V. Gal'Tsov and V. P. Tsvetkov. "On the gravitational radiation of an oblique rotator". In: *Physics Letters A* 103 (July 1984), pp. 193–196. DOI: 10.1016/0375-9601(84)90249-4.

[35] T.P. Cheng and L.F. Li. *Gauge Theory of Elementary Particle Physics*. Oxford science publications. Clarendon Press, 1984. ISBN: 9780198519614. URL: https://books.google.co.in/books?id=lk8GEzVNb10C.

[36] Daniel Zwanziger. "Local Lagrangian quantum field theory of electric and magnetic charges". In: *Phys. Rev.* D3 (1971), p. 880. DOI: 10.1103/PhysRevD.3.880.

[37] M. Blagojevic and P. Senjanovic. "The Quantum Field Theory of Electric and Magnetic Charge". In: *Phys. Rept.* 157 (1988), p. 233. DOI: 10.1016/0370-1573(88)90098-1.

[38] Simone Dall'Osso et al. "Gravitational Waves from Massive Magnetars Formed in Binary Neutron Star Mergers". In: *The Astrophysical Journal* 798.1 (2015), p. 25. URL: http://stacks.iop.org/0004-637X/798/i=1/a=25.

[39] Alessandra Corsi and Peter Mészáros. "GAMMA-RAY BURST AFTERGLOW PLATEAUS AND GRAVITATIONAL WAVES: MULTI-MESSENGER SIGNATURE OF A MILLISECOND MAGNETAR?" In: *The Astrophysical Journal* 702.2 (Aug. 2009), pp. 1171–1178. DOI: 10.1088/0004-637x/702/2/1171. URL: https://doi.org/10.1088%2F0004-637x%2F702%2F2%2F1171.

[40] Deborah N. Aguilera, Jose A. Pons, and Juan A. Miralles. "2D Cooling of Magnetized Neutron Stars". In: *Astron. Astrophys.* 486 (2008), pp. 255–271. DOI: 10.1051/0004-6361:20078786. arXiv: 0710.0854 [astro-ph].

[41] Daniele Vigan et al. "Unifying the observational diversity of isolated neutron stars via magneto-thermal evolution models". In: *Mon. Not. Roy. Astron. Soc.* 434 (2013), p. 123. DOI: 10.1093/mnras/stt1008. arXiv: 1306.2156 [astro-ph.SR].

[42] Jose A. Pons and U. Geppert. "Magnetic field dissipation in neutron star crusts: From magnetars to isolated neutron stars". In: *Astron. Astrophys.* 470 (2007), p. 303. DOI: 10.1051/0004-6361:20077456. arXiv: astro-ph/0703267 [ASTRO-PH].

[43] Ian K. Affleck and Nicholas S. Manton. "Monopole Pair Production in a Magnetic Field". In: *Nucl. Phys.* B194 (1982), pp. 38–64. DOI: 10.1016/0550-3213(82)90511-9.

[44] Leandro Medina and Michael C. Ogilvie. "Schwinger pair production at finite temperature". In: *Phys. Rev. D* 95 (5 2017), p. 056006. DOI: 10.1103/PhysRevD.95.056006. URL: https://link.aps.org/doi/10.1103/PhysRevD.95.056006.

[45] Mrunal Korwar and Arun M. Thalapillil. “Finite temperature Schwinger pair production in coexistent electric and magnetic fields”. In: *Phys. Rev. D* 98 (7 2018), p. 076016. DOI: 10.1103/PhysRevD.98.076016. URL: https://link.aps.org/doi/10.1103/PhysRevD.98.076016.

[46] Ibrahim Akal. “Exact instantons via worldline deformations”. In: (2018). arXiv: 1808.06868 [hep-th].

[47] D. Bazeia, L. Losano, and J. M. C. Malbouisson. “Deformed defects”. In: *Phys. Rev. D* 66 (10 2002), p. 101701. DOI: 10.1103/PhysRevD.66.101701. URL: https://link.aps.org/doi/10.1103/PhysRevD.66.101701.

[48] A. T. Avelar et al. “New Lump-like Structures in Scalar-field Models”. In: *Eur. Phys. J.* C55 (2008), pp. 133–143. DOI: 10.1140/epjc/s10052-008-0578-6. arXiv: 0711.4721 [hep-th].

[49] Gerald V. Dunne and Christian Schubert. “Worldline instantons and pair production in inhomogenous fields”. In: *Phys. Rev. D* 72 (10 2005), p. 105004. DOI: 10.1103/PhysRevD.72.105004. URL: https://link.aps.org/doi/10.1103/PhysRevD.72.105004.

[50] Milton Abramowitz and Irene A. Stegun, eds. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.* New York: Dover Publications, Inc., 1965.

[51] S. Kroker and R. Nawrodt. “The Einstein Telescope”. In: *2014 IEEE Metrology for Aerospace (MetroAeroSpace)*. 2014, pp. 288–292. DOI: 10.1109/MetroAeroSpace.2014.6865936.

[52] S. Hild et al. “Sensitivity Studies for Third-Generation Gravitational Wave Observatories”. In: *Class. Quant. Grav.* 28 (2011), p. 094013. DOI: 10.1088/0264-9381/28/9/094013. arXiv: 1012.0908 [gr-qc].

[53] Kostas Glampedakis and Leonardo Gualtieri. “Gravitational waves from single neutron stars: an advanced detector era survey”. In: (2017). arXiv: 1709.07049 [astro-ph.HE].