Entanglement in Chern Simons theory and edge states in Haldane spin chains

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Certificate

This is to certify that this dissertation entitled Entanglement in Chern Simons theory and edge states in Haldane spin chains towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Aayush Vijayvargia at Indian Institute of Science Education and Research under the supervision of Sandip Trivedi, Professor, Department of Theoretical Physics , during the academic year 2018-2019.

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This thesis is dedicated to Yiter and my parents

Declaration

I hereby declare that the matter embodied in the report entitled Entanglement in Chern Simons theory and edge states in Haldane spin chains are the results of the work carried out by me at the Department of Theoretical Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Sandip Trivedi and the same has not been submitted elsewhere for any other degree.

Aayush Vijayvargia

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Abstract

In the first part of this thesis, we review Chern Simons theory and study its topological entanglement entropy using the replica trick. We review the quantisation of Chern Simons theory and see how this seemingly trivial theory has a rich structure and a deep connection with chiral conformal field theory which lets us calculate partition function on one manifold from one on another manifold. We see why this calculation is difficult to do for Chern Simons theory coupled with matter. In the second part, we look at 1 D antiferromagnetic spin chains and their edge states and see how their low energy physics can be mapped to the $O(3)$ quantum rotor model. We look at an exactly solvable model, the AKLT model of spin 1 particles and discuss a pictorial way to see how edge states emerge. We exactly diagonalize $S = 1$ Heisenberg spin chain to look at the Haldane gap and see how the parity of length of the spin chain decides the ground state. We support these numerical results with an analytical study of the effective Hamiltonian between the edge spins and by deriving an effective Hamiltonian between them.

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Chapter 1

Introduction

Topological phases are states of matter in two dimensions which display ground states that do not exhibit spontaneous symmetry breaking. The degeneracy of their ground states depend exclusively on the genus of the manifold they are on. This degeneracy is robust and isn't lifted by any local perturbation. At low energy limits, their physics is described by a topological quantum field theory.

It was shown[1],[2] that for topological phases, the entanglement entropy of a simply connected region A is of the form $S_A = \alpha L - \gamma + ...$ here L refers to the size of the boundary of the entangling regions, α is some non universal coefficient and gamma a universal constant. Here ... refers to the contributions which go to zero as size of A goes to infinity.

Chern Simons (CS) theories are topological quantum field theories which describe the low energy physics of Fractional Quantum Hall Effect system which exhibits topological order (same as topological phase). Since CS theory is topological, it doesn't contain any local degrees of freedom so we expect only the topological part of the entanglement entropy to be predicted correctly. This would help us understand the long range entanglement in FQHE states.

We will see that the edge states of a Chern Simons can be observed by doing a gauge transformation in the bulk and the gauge degrees of freedom, after choosing an appropriate boundary conditions, become physical degrees of freedom on the boundary. Similarly, in a S=1 antiferromagnetic Heisenberg open spin chain, if one does a rotation by 360 degrees on

one of the ends, a negative sign is observed, suggesting the presence of spin 1/2 degrees of freedom on the edge.

These edge states in the antiferromagnetic spin 1 chain has interesting characters which we explore in the later part of this thesis. It is shown that the antiferromagnetic Heisenberg spin chain can be mapped to the O(3) quantum rotor model and using this mapping we calculate the effective interaction Hamiltonian between the two spin 1/2 particles on the either sides of the chain and see that they form a singlet and triplet. The interaction is ferromagnetic for odd length chain and antiferromagnetic for even length of chain and the coupling decays exponentially as the length of the chain. And we do a numerical calculation to support our result.

Chapter 2

Chern Simons theory

Let us start with a review of Chern-Simons theory. Chern Simons theories are a kind of quantum field theories which do not have any local degrees of freedom, i.e. topological quantum field theories. It is a gauge theory, very different from the familiar Maxwell theory. We would be working with Chern-Simons theory in 2+1 dimensions. The action of this theory is devoid of the metric, so the theory is diffeomorphism invariant. This means that one cannot write down a non-trivial quantity in this theory which is a function of the position. Because you can always do a diffeomorphism and change positions, relative distances or more generally the local geometry, but the theory would be invariant and all the measurable quantities will be invariant.

It can be seen more clearly why the theory is diffeomorphism invariant by looking at the Action, which we will introduce of the Abelian version now.

2.1 Abelian Chern-Simons Theory

The familiar 3+1 dimensional Maxwell Lagrangian, coupled with charges and currents from which Maxwell's equations are derived is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_{mu}J^{\nu} \tag{2.1}$$

If we reduce the spatial dimensions from 3 to 2, we have another allowed gauge invariant term, which leads to a richer structure. We can have an additional term, the Chern-Simons term which is gauge invariant, Poincare invariant and local.

$$\mathcal{L} = \frac{k}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \tag{2.2}$$

There is no metric $g_{\mu\nu}$ present in the action as it was in the previous Lagrangian, used to lower the indices. The Levi-Civita symbol is used here to contract the indices. This is the reason that the theory is manifestly invariant under coordinate transformations. Under first glance, the presence of the vector field itself in the Lagrangian makes it look like it won't be gauge invariant. Let's check that for a gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda$:

$$\delta\mathcal{L} = \frac{k}{2}\partial_\mu(\lambda\epsilon^{\mu\nu\rho}\partial_\nu A_\rho) \tag{2.3}$$

So we have got a total derivative term which we can ignore if the manifold is compact. If the manifold has boundaries, the above term leads to some interesting physics which we will see later.

Even though the Chern Simons Lagrangian, which has only first order quadratic terms, looks trivial, having equations of motion $F_{\mu\nu} = 0$. It can be made non-trivial by various ways, of which the one we are interested in is putting it on a manifold with non-trivial topology.

A very useful feature of Chern Simons theory is if you couple some source fields, it attaches some flux to them. We can see this classically in the following way, by writing one of the Maxwell's equation from the 3 dimensional electromagnetism Lagrangian including the Chern-Simons term:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} + kB = \rho \tag{2.4}$$

By integrating this equation over the whole space, we obtain the following:

$$\int d^2\boldsymbol{x}\boldsymbol{\nabla}\cdot\boldsymbol{E} + k\int d^2\boldsymbol{x}B = \int d^2\boldsymbol{x}\rho \tag{2.5}$$

The first term, by Stoke's theorem is just the line integral of a curve which can be made arbitrarily far from the sources, where the electric field is small, hence making this term zero. The next term involving be gives the total flux, ϕ and right hand side of the equation

is the total charge, q. So we have,

$$\phi = \frac{q}{k} \tag{2.6}$$

Just to be clear, this magnetic flux is not the usual magnetic flux that is obtained from magnetic dipoles rather the Chern-Simons theory is attaching a flux to every charged particle. These particles' spin also get changed and hence their statistics also get modified making them anyons.

2.2 Non Abelian Chern-Simons Theory

Just like in Maxwell's theory, we can have Non-Abelian vector fields, we can have the same in Chern-Simons too, which gives rise to very interesting physics that is not present in the Abelian case. The Non-Abelian case is based on a Lie group G and its elements are generally non-commuting. If T^a are the generators of the group elements, then they follow the Lie algebra:

$$[T^a, T^b] = if^{abc}T^c \tag{2.7}$$

Where f^{abc} are the structure constants of the Lie group. They follow the following trace condition

$$tr(T^aT^b) = \frac{1}{2}\delta^{ab} \tag{2.8}$$

The vector fields are defined in the following way, making them vectors with matrix components in the euclidean space M.

$$A_\mu(x) = A^a_\mu(x)T^a \tag{2.9}$$

The action, is written usually given by:

$$S[A] = \frac{k}{4\pi}\int_M tr(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A) \tag{2.10}$$

Written component wise for more clarity,

$$S[A] = \frac{k}{8\pi}\int_M d^3x\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho - \frac{1}{3}f^{abc}A^a_\mu A^b_\nu A^c_\rho) \tag{2.11}$$

For consistency check, we can put $f^{abc} = 0$ and obtain the previous abelian Chern Simons action with a redefinition of k to include the 4π.

The promised interesting physics will be seen now in this action's test for gauge invariance.

Unlike the simple gauge transformations allowed in the abelian case, one can have gauge transformations like:

$$A_\mu \to A_\mu^g = g^\dagger A_\mu g - i g^\dagger \partial \mu g \tag{2.12}$$

Here, $g \in G$. We can see that apart from an additive term, the transformation also transform the vector field itself like a rotation of sorts. Substituting the transformation in the action:

$$S[A^g] = S[A] + i\frac{k}{4\pi}\int_M d^3x \epsilon^{\mu\nu\rho}\partial_\mu tr(\partial_\nu g g^\dagger A_\rho) + \frac{k}{12\pi}\int_M d^3x \epsilon^{\mu\nu\rho} tr(g^\dagger \partial_\mu g g^\dagger \partial_\nu g g^\dagger \partial_\rho g) \tag{2.13}$$

The total derivative term on the right hand side can again be ignored for a compact manifold but the other term is something you can't get rid of. So, it seems that the action is not gauge invariant, which would mean that Abelian Chern Simons is not a legitimate gauge theory. But looking closely at the non-vanishing troublesome term, we find, it is proportional to the winding number ω:

$$\omega[g] = \frac{1}{24\pi^2}\int_M d^3x \epsilon^{\mu\nu\rho} tr(g^\dagger \partial_\mu g g^\dagger \partial_\nu g g^\dagger \partial_\rho g) \tag{2.14}$$

It is an integer which counts the number of times the gauge transformation spans the group it belongs to. To see this, let us look at $g(x)$ which is a function of the euclidean space. If $g(x)$ is a permissible gauge transformation, its value at infinity should be specified. We can take it to be identity.

We see that g(x) is defined in the whole real space including the point at infinity, so we can treat it as an object living in S^3. $g(x)$ is a mapping from S^3 to the parametric space of the group, in the case when the group is SU(2), this mapping is from S^3 to S^3, in this case, it is clear that this mapping will be labelled by a winding number which is an integer, this fact is true for general non-abelian compact gauge groups. It is invariant under small deformations of $g(x)$, which is clear from the definition of ω because it doesn't involve the metric, hence we can call it a topological quantity. So, this action is invariant under small gauge transformations whose winding number is zero. But it is not invariant under large gauge transformations (eq. 2.12):

$$S \to S + 2\pi k N \tag{2.15}$$

For any physically relevant gauge theory, we need the measurable quantities of the theory to be gauge invariant. Action is not a measurable quantity, but appears in the path integrals as $exp(iS)$, for this quantity to remain invariant, we need k to be an integer. So we find that non-abelian Chern-Simons theories only exist when $k \in \mathbb{Z}$ and is called the level of the theory.

2.3 Chern-Simons theory with a boundary

In the last section, we had ignored the boundary terms in the variation of the Chern-Simons action because we assumed the manifold Σ to be compact. But in the cases when the manifold Σ is not compact and has a boundary $\partial\Sigma$, new interesting physics can be seen at the edges. Namely, we find an infinite dimensional Hilbert space of the chiral currents of a CFT defined on $\partial\Sigma \times \mathbb{R}$. Keeping the boundary term, the variation in the action is:

$$\delta S = \frac{k}{4\pi}\int_{\Sigma} d^3x \epsilon^{\mu\nu\rho} tr(\delta A_\mu F_{\nu\rho}) + \frac{k}{4\pi}\int_{\Sigma} d^3x \partial_\mu[\epsilon^{\mu\nu\rho} A_\nu \delta A_\rho] \tag{2.16}$$

From this we can see that the appropriate boundary condition would be the one in which integral of $A\delta A$ on the boundary is zero. Choosing the boundary condition $A_0 = 0$. The local symmetry which is remaining corresponds to the gauge transformations that are identity on the boundary and the rest, they have to be time independent transformations (which are global) to be compatible with the boundary condition.

With this boundary condition, we have the action:

$$S = -\frac{k}{4\pi}\int_{\Sigma\times\mathbb{R}} d^3x \epsilon^{ij} tr(A_i \partial_0 A_j) + \frac{k}{12\pi}\int_{\Sigma\times\mathbb{R}} d^3x \epsilon^{ij} tr(A_0 F_{ij}) \tag{2.17}$$

Now performing the variation with respect to A_0 which acts as a Lagrange multiplier, gives us the equation of motion, which is $F_{ij} = 0$

The solution to this is, pure gauge. that means the degrees of freedom in the bulk are flat connections:

$$A_i = g^{-1}\partial_i g \tag{2.18}$$

Substituting it back in the action, we get:

$$tr(\epsilon^{ij}A_i\partial_0 A_j) = tr(\epsilon^{ij}\partial_i g g^{-1}\partial_0\partial_j g g^{-1}) - tr(\epsilon^{ij}\partial_i g g^{-1}\partial_j g g^{-1}\partial_0 g g^{-1}) \tag{2.19}$$

We see that the first term in the above equation is a total derivative because $g^{-1}\partial_\mu g g^{-1} = -\partial_\mu g^{-1}$ so we have:

$$tr(\epsilon^{ij}\partial_i g g^{-1}\partial_0\partial_j g g^{-1}) = -\partial_j(tr(\epsilon^{ij}\partial_i g^{-1}\partial_0 g)) \tag{2.20}$$

Explicitly writing component wise in polar coordinates:

$$\partial_r tr(\partial_\theta g^{-1}\partial_0 g) - \partial_\theta tr(\partial_r g^{-1}\partial_0 g) \tag{2.21}$$

Integrating the second term gives zero because of the single-valudeness of g. So, the boundary integral term:

$$-\frac{k}{4\pi}\int_\Sigma tr(\epsilon^{ij}\partial_i g g^{-1}\partial_0\partial_j g g^{-1}) = \frac{k}{4\pi}\int_{\partial\Sigma} d\theta dt\ \ tr(g^{-1}\partial_\theta g g^{-1}\partial_0 g) \tag{2.22}$$

Finally, we get

$$S = -\frac{k}{4\pi}\int_{\partial\Sigma\times\mathbb{R}} d\theta dt\ \ tr(g^{-1}\partial_\theta g g^{-1}\partial_0 g) + \frac{k}{12\pi}\int_{\Sigma\times\mathbb{R}} \epsilon^{\mu\nu\rho} tr(g^{-1}\partial_\mu g g^{-1}\partial_\nu g^{-1}\partial_\rho g) \tag{2.23}$$

This is the WZW action, quantising it, we get a chiral current algebra of the same gauge group, with the boundary values of the connection, $A_\theta = \partial_\theta g g^{-1}$ being identified with the chiral Kac-Moody currents. So, the Hilbert space of the theory on the boundary is the trivial representation of the Kac-Moody algebra, with the highest weight state dual to the identity operator. In other words, the Hilbert space is the descendents of the the identity operator of the corresponding WZW model.

2.4 Quantising Chern Simons theory on a compact manifold

Witten's famous paper "Quantum Field theory and Jones Polynomial"[5], solves the pure Chern Simons theory on compact manifolds. It was shown that the physical Hilbert space obtained by canonically quantizing CS theory of a gauge group at level k can be interpreted as the space of conformal blocks of a WZW model with the corresponding Kac-Moody algebra at level k. A consequence of this result is that for CS theory on a Riemann surface of the form $\Sigma \times \mathbb{R}$, with no insertion of a Wilson line, the Hilbert space is one-dimensional because no insertions correspond to the vacuum block of the CFT.

Then the paper introduces a way to use the results obtained above to calculate the partition function of CS theories on compact manifolds, while introducing a method called surgery which can be used to calculate the partition function on one manifold from another.

The result relevant to us was the calculation of partition function on the manifolds $S^2 \times S^1$ and on S^3. Partition function on $S^2 \times S^1$ is easily calculated, since the Hilbert space is one-dimensional, Partition function will be of the form $Z(S^2 \times S^1) = e^{-\beta H}$ but since the Hamiltonian is vanishing, $Z(S^2 \times S^1) = 1$

Now, surgery, which involves removing a tubular manifold from a manifold M and then a diffeomorphism is made on the boundary and then the tubular manifold is glued back which results into a new manifold N which differs from M. A general equation is obtained:

$$Z(N) = \sum_j K_0^j Z(M; R_j)$$

Where R_j represents a Wilson line insertion in the manifold M and K represents the diffeomorphism matrix.

This equation is used to calculate $Z(S^3)$, the diffeomorphism here being a modular transformation. We can see why a modular transform is the relevant diffeomorphism connecting S^3 and $S^2 \times S^1$ by noting that $S^2 \times S^1$ can be constructed using two solid tori by gluing contractible cycles of torus 1 with contractible cycles of torus 2, because the non-contractible cycle formed this way would correspond to S^1 and the rest of it gluing together two disks at each point of solid torus along the S^1, which would give a S^2, hence obtaining a $S^2 \times S^1$.

Now if we instead glue together non-contractible cycles to the contractible cycles, we would obtain an object which topologically is a S^3. Exchanging the cycles that have to be glued is equivalent to performing a modular transform on one of the torus, so we have:

$$Z(S^3) = \sum_j S_0^j Z(S^2 \times S^1; R_j)$$

Since with no Wilson line insertion, $Z(S^2 \times S^1) = 1$, we get that $Z(S^3) = S_0^0$

Chapter 3

Entanglement Entropy

Entanglement of a region in a quantum system with its complement is a non local quantum correlation between the two regions. Such correlations are not seen in classical systems, the reason for this can be traced down to the possibility of superposition of states which is not allowed in classical systems. More precisely, when the wavefunction of a system cannot be written in the form of a product of a region, say A and its complement B, the regions are entangled.

$$|\psi\rangle \neq |\psi_A\rangle\,|\psi_B\rangle \tag{3.1}$$

But in this above definition we have assumed that the system is in a pure state. Generally, if the system is in a mixed state in a Hilbert state $\mathcal{H}_A \otimes \mathcal{H}_B$, the criteria for a state being entangled is the existence of a decomposition of the state ρ in the following form:

$$\rho = \sum_j p_j \rho_A^{(j)} \otimes \rho_B^{(j)}$$

where $\sum p_j = 1$ and $\rho_A^{(j)}$ and $\rho_B^{(j)}$ are density matrices. So if such a decomposition is found then the state is separable and not entangled and if such a decomposition is not found then the test is inconclusive as there is no prescription available to find the possible decomposition.

These tests just tell if the state is entangled or not. But in recent years, entanglement has come to be recognized as a new kind of resource which can be used to perform different tasks which are inefficient or impossible to do in the classical realm. But for that, one needs to be able to quantify entanglement. For this, a number of different measures have been

proposed which follow some basic rules. Some examples are: Entanglement entropy (Von Neumann entropy), Renyi entropy, Entanglement Negativity, Concurrence etc. We wish to calculate in a Quantum field theory, so let's look at a general prescription to do so Von Neumann entropy is, where ρ_A is the reduced density matrix:

$$S = Tr(\rho_A \ln(\rho_A))$$

In a quantum field theory where one deals with infinite degrees of freedom, this density matrix is infinite dimensional and the logarithm of such a matrix is a bigger problem. We instead we use the form:

$$\begin{aligned} S(\rho) &= \lim_{n\to 1} \frac{\partial}{\partial n} Tr(\rho^n) \\ &= \lim_{n\to 1} \frac{\partial}{\partial n} \sum_i \lambda_i^n \\ &= \lim_{n\to 1} \sum_i \lambda_i^n \ln(\lambda_i) \\ &= Tr(\rho \ln(\rho)) \end{aligned} \tag{3.2}$$

3.1 The replica trick

We have to calculate the trace of the nth-power of the density matrix. For this, Cardy and Calabrese invented a novel way called the replica trick, it was an old trick used in condensed matter but calculating euclidean path integrals on an n-sheeted Riemann surface to calculate $Tr\rho^n$ was proposed first by Cardy and Calabrese in 2004.

First we need to write down the vacuum density matrix. To do that, consider a scalar field $\hat{\phi}(x,t)$ and use the basis formed by this operator at t = 0 as:

$$\hat{\phi}(x,0)\,|\alpha\rangle = \alpha(x)\,|\alpha\rangle$$

The vacuum wavefunction where N is a normalisation

$$\langle 0|\alpha\rangle = N^{-\frac{1}{2}} \int_{\phi(x,0)=\alpha(x)}^{\phi(x,-\infty)=0} \mathcal{D}\phi\; e^{-S_E[\phi]} \tag{3.3}$$

The functional integral is over the lower half time ($-\infty$ to 0) to select the ground state.

We can write down the vacuum density matrix is:

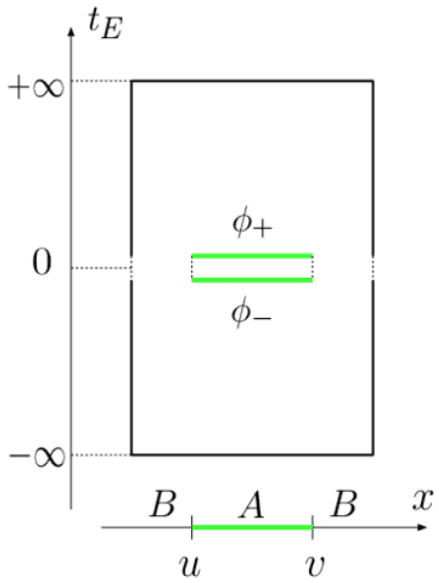
$$\rho(\alpha, \alpha') = \langle \alpha | 0 \rangle \langle 0 | \alpha' \rangle$$

$$= N^{-1} \int_{\phi(x,-\infty)=0}^{\phi(x,0^-)=\alpha(x)} \mathcal{D}\phi e^{-S_E(\phi)} \int_{\phi(x,0^+)=\alpha'}^{\phi(x,0)=0} \mathcal{D}\phi e^{-S_E[\phi]}$$

This can be written in a different form which is probably more illuminating and also it is clear that the normalization N is the partition function Z because when we integrate over α , we get Z so that trace becomes 1.

$$\frac{1}{Z} \int \mathcal{D}\phi \prod_x \delta(\phi(x, 0^+) = \alpha'(x)) \delta(\phi(x, 0^-) = \alpha) e^{-S_E[\phi]} \tag{3.4}$$

Below is a figure showing the path integral. Here $\phi_+ = \alpha$; and $\phi_- = \alpha$



Now, it is much more easy to write the reduced density matrix: just restrict the product to the region of space, $A\ \epsilon(u, v)$ whose complement we want to trace over. This essentially means that we are integrating over field configurations in the region $B = A^c$

$$\rho_A = \frac{1}{Z}\int \mathcal{D}\phi \prod_{x\,\epsilon A} \delta(\phi(x,0^+) = \alpha'(x))\delta(\phi(x,0^-) = \alpha)e^{-S_E[\phi]} \tag{3.5}$$

Now we need to calculate the nth power of the matrix we have got, then take the trace and then take the limit. To do that, we need to join the multiply the matrices and equate the indices between them which in our case translate to sewing n Riemann sheets together and identifying the field configurations between two consecutive sheets along the region A. Then to take the trace, we have to equate the field configuration on the 1st and the last sheets and then integrate over all the fields. So we have:

$$Tr(\rho_A^n) = \frac{Z_n}{Z^n} \tag{3.6}$$

Where Z_n is the partition function on the n sheeted Riemann surface and and Z is the regular path integral on the single plane which is there to ensure that the in the limit n goes to 1, the trace is 1.

3.2 Entanglement in Chern Simons theory

Now we use the above general prescription to calculate the topological entanglement entropy of Chern-Simons theory for a general compact gauge group at level k while giving exact results for $SU(2)_k$ and $U(1)_k$. This way, we would have calculated the topological entanglement entropy of some Fractional Quantum Hall Effect systems, the other non-topological part of their entanglement entropy cannot be obtained by the entanglement calculated in the topological effective field theories as these terms, like the area terms become zero here because of the topological nature of these theories.

Now let's set up the replica trick calculation for the Chern-Simons theory[4]. The first issue, that we face is that the ground state is not necessarily unique here. In fact, for every Hilbert space that is not 1 dimensional, all the states have the same energy, i.e. zero, because the Hamiltonian is zero. So, the above procedure just suggests that we will work with a glued geometry here, but we can't borrow the whole discussion for the case of Chern-Simons theory. We can formally write down the glued geometry path integral by using the Chern Simons wave functional, that we calculate in a later section. Assuming holomorphic factorization,

we have the following WZW path integral (the measure of the integral contains the Kahler potential term):

$$\left\langle \overline{\beta_i} \right| \left\langle \overline{\alpha_i} | \Psi \right\rangle \sim \int [dg_{\alpha,i} dg_{\beta,i}] exp(-kW_{\alpha(g_{\alpha,i})} - kW_{\beta(g_{\beta,i})}) - \frac{k}{2\pi} \int_{\Sigma_\alpha} tr(\overline{\alpha_i} g_{\alpha,i}^{-1} \partial g_{\alpha,i}) - \frac{k}{2\pi} \int_{\Sigma_\beta} tr(\overline{\beta_i} g_{\beta,i}^{-1} \partial g_{\beta,i}) \tag{3.7}$$

Where W(g) represents the WZW action. And α, β correspond to the different regions in the space between which we are calculating the entanglement.

Since the above expression is a wave functional, so to formally write down ρ^n we need to glue together 2n copies of the above quantity:

$$\int \prod_{k=1}^{n} [d\mu(\alpha_k) d\mu(\beta_k)] \left\langle \overline{\beta}_1 \right| \left\langle \overline{\alpha}_1 | \Psi \right\rangle \left\langle \Psi | \beta_1 \right\rangle |\alpha_2\rangle \left\langle \overline{\beta}_2 \right| \left\langle \overline{\alpha}_2 | \Psi \right\rangle \left\langle \Psi | \beta_2 \right\rangle |\alpha_3\rangle \ldots \left\langle \overline{\beta}_n \right| \left\langle \overline{\alpha}_n | \Psi \right\rangle \left\langle \Psi | \beta_n \right\rangle |\alpha_1\rangle \tag{3.8}$$

Now this is the glued geometry equivalent for the Chern-Simons theory.

Before moving forward, we need to review some ideas briefly which will be required in the future:

Modular S matrix and the quantum dimension: For a set of characters of a CFT, we have:

$$\chi(-1/\tau) = \mathcal{S} \chi(\tau) \tag{3.9}$$

The characters are labelled by quantum numbers which label the representations. For $SU(2)_k$ the characters' representations are labelled by half integers $j = 0, 1/2, ..., k/2$:

$$\chi_j^{(k)}(-1/\tau) = \sum_{j'} \mathcal{S}_j^{(k)\ j'} \chi_{j'}^{(k)}(\tau) \tag{3.10}$$

Where

$$\mathcal{S}_j^{(k)\ j'} = \sqrt{\frac{2}{k+2}} \quad \sin[\frac{\pi(2j+1)(2j'+1)}{k+2}] \tag{3.11}$$

This S matrix is usually unitary. We define the quantum dimension in the following way:

$$d_j = \frac{\mathcal{S}_0^j}{\mathcal{S}_0^0} \tag{3.12}$$

Noting that the quantum dimension for $SU(2)_k$ is:

$$d_j = \frac{sin[\frac{\pi(2j+1)}{k+2}]}{sin[\frac{\pi}{k+2}]} \tag{3.13}$$

Using the unitarity of S matrix, we can multiply eq 4.8 by its conjugate and sum over k to get:

$$(\mathcal{S}_0^0)^{-1} = \sqrt{\sum_k \|d_k\|^2} = \mathcal{D} \tag{3.14}$$

Now let's begin with a concrete problem: We want to compute the entanglement entropy between the two hemispheres of a spatial S^2 slice of a Chern Simons theory. We have seen in the previous section that the Hilbert space is one dimensional, so there is no ambiguity of choosing a state. Regions A and B are connected and these hemispheres can be thought of s being disks. The wavefunctional corresponds to the solid 3 ball shown in the figure while the spatial slice in the boundary S^2. We need to construct the glued geometry, so it is

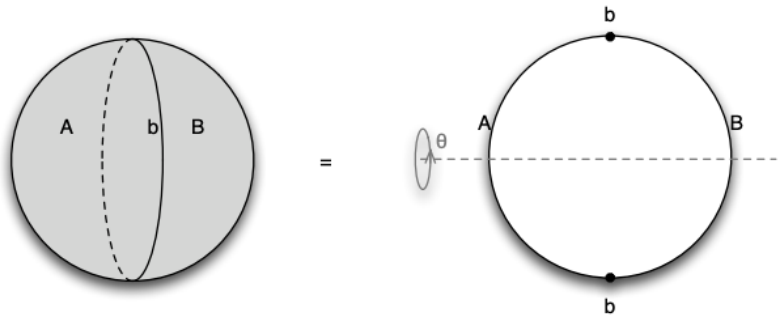


Figure 3.1: Representation of a wavfunctional. Source: [4]

instructive to think of the 3 ball as the rotation of a disk about any axis passing through the origin, as shown in the above figure. Each 3 ball represents a wavefunctional, so we would need to glue together 2n copies of these balls to construct $tr(\rho_A^n)$.

In the figure below, it is shown, how to do the gluing for $n = 2$. The numbers 1, 2, 3 and 4 represents $|\psi_1\rangle, \langle\psi_1|, |\psi_2\rangle and \langle\psi_2|$ respectively. So these rotated disks which are , when glued together gives a rotated S^2 which has a S^3 topology. So, $tr(\rho_A)$ which was a path integral on S^3 is same as $tr(\rho_A^n)$ which is again now a path integral on S^3, this is the power of a Topological QFT.

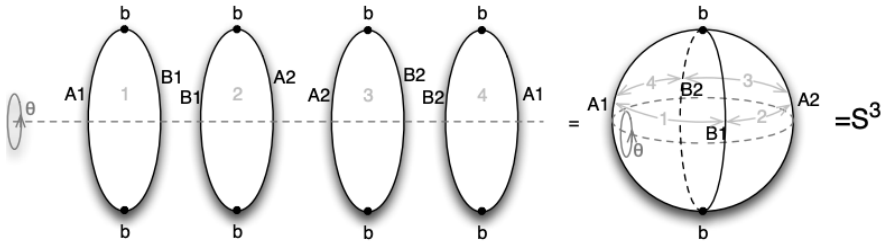


Figure 3.2: Representation of $tr(\rho^2$. Source: [4]

The same geometry will result, if one asks the question, what is the entanglement entropy of a disk on a spatial $\mathbb{R}^2$ slice with the rest of the plane. This is because there is a conformal map between the plane and S^2 and the entanglement between disks can again be thought of as entanglement between hemispheres, which we have discussed above, because of diffeomorphism invariance.

Now, moving forward, because of the analysis is the 2n- glued object we have:

$$\frac{tr\rho_A^n}{(tr\rho_A)^n} = \frac{Z(S^3)}{Z(S^3)^n} = (Z(S^3))^{1-n} \tag{3.15}$$

We have seen in the previous section that the Path integral on S^3 is the object we introduced earlier: $\mathcal{S}_0^0$.

$$\frac{tr\rho_A^n}{(tr\rho_A)^n} = (S_0^0)^{1-n} \tag{3.16}$$

From this, calculating the entanglement entropy:

$$S_A = \lim_{n\to 1} \frac{\frac{d}{dn}(S_0^0)^{1-n}}{\frac{d}{dn}(1-n)} = \ln S_0^0 \tag{3.17}$$

From the definition of the total quantum dimension, this is same as the topological entanglement entropy that was derived by Kitaev and Preskill, Levin and Wen.

$$S_A = -\ln \mathcal{D} \tag{3.18}$$

Let us look at some specific cases now:

For $U(1)_k$, the S matrix is well known:

$$\mathcal{S}_0^0 = \frac{1}{\sqrt{\sum_j \|d_j\|^2}} = \frac{1}{\sqrt{k}} \tag{3.19}$$

Hence, the entanglement entropy between two hemispheres of a spatial slice S^2 for a $U(1)_k$ Chern-Simons theory is:

$$S_A = -\ln\sqrt{m} \tag{3.20}$$

Now for $SU(2)_k$ also, from eq (4.7), we have:

$$S_A = \ln\sqrt{\frac{2}{k+2}} sin[\frac{\pi}{k+2}] \tag{3.21}$$

Here, we are done with the problem of calculating the Chern Simons entanglement entropy. These calculations heavily use the fact that Chern Simons is a topological field theory. Even if one looks at an analogous calculation done for a CFT, it would be clear that the most simplifying steps of the calculation, like finding that the partition function on the n sheeted surface is the same as the partition function on the original manifold, will not hold true. The problem we had in mind was to perform this entanglement calculation for Chern Simons theory with matter, where the matter degrees of freedom are gapless hence the final QFT is still conformal.

But to generalize to a CFT, we first need to do a calculation in Pure Chern Simons theory in the most general way, without using any tricks that used the topological nature of the underlying QFT. So we cannot use surgery as well. In the next chapter, we try to set up the straightforward calculation without using any tricks by first evaluating the wave functional on S^2.

Chapter 4

Wavefunctional calculation on two-sphere for abelian CS theory

In this section, we will calculate the wavefunction of abelian Chern Simons on two sphere in two ways. First, using canonical quantization, second, by a path integral way.

We consider Chern Simons theory at level k on a three manifold M given by path integration over U(1) gauge field A with measure e^{iS}, where the action is given by [1]

$$S = -\frac{k}{4\pi}\int_M A \wedge dA \tag{4.1}$$

We take $M = \mathbb{R} \times \Sigma$. $\mathbb{R}$ is assumed to be direction of time and Σ is spatial surface at constant time on which we construct Hilbert space of the theory in what follows next.

[1] We normalise length of the U(1) circle to 2π. We stress that action is independent of the metric on the manifold.

4.1 Cannonical Quantization

Here we are concentrating on the simplest case of Σ being S^2. We work with conformal Cartesian coordinates (x^1, x^2) on the sphere. Following are the conventions that we adopt

$$\epsilon^{012} = 1, d^2x = dx^1 \wedge dx^2, d^3x = dx^0 \wedge d^2x \tag{4.2}$$

In $A_0 = 0$ gauge action given in equation (5.1) becomes

$$S = -\frac{k}{4\pi}\int_M d^3x\epsilon^{\mu 0\rho}A_\mu\partial_0 A_\rho \tag{4.3}$$

$$= \frac{k}{4\pi}\int_M d^3x\epsilon^{\mu\rho}A_\mu\partial_0 A_\rho$$

At this point we emphasize that above form (5.3) of the action is valid in any coordinate system as long as ϵ appearing is the tensorial transformation of the Cartesian one. As an example we note that for

$$z = \frac{1}{\sqrt{2}}(x^1 + ix^2), \overline{z} = \frac{1}{\sqrt{2}}(x^1 - ix^2) \tag{4.4}$$

We have

$$\epsilon^{z\overline{z}} = -i, d^2Z \equiv dz \wedge d\overline{z} = \epsilon^{z\overline{z}}dx^1 \wedge dx^2 \tag{4.5}$$

Collectively we denote $w = (z, \overline{z}$ on Σ. IN Cartesian coordinates canonical conjugate momentum to A_μ is [2]

$$\Pi^\mu = -\frac{ik}{2\pi}\epsilon^{\mu\rho}A_\rho \tag{4.6}$$

Commutation relations becomes

$$[A_\mu(t, x^1, x^2), \Pi^\nu(t, x'^1, x'^2)]_- = i\delta^\mu_\nu\delta(x^1 - x'^1)\delta(x^2 - x'^2) \tag{4.7}$$

$$\implies [A_\mu(t, x^1, x^2), A_\nu(t, x'^1, x'^2)]_- = \frac{2\pi}{k}\epsilon^{-1}_{\mu\nu}\delta(x^1 - x'^1)\delta(x^2 - x'^2)$$

Here we have defined

$$\epsilon^{\mu\nu}\epsilon\nu\rho^{-1} = \delta^\mu_\rho \tag{4.8}$$

[2]We have integrated by parts as needed

In general coordinates covariant version of above commutation is [3]

$$[A_\mu(t,w_1),A_\nu(t,w_2)]_- = \frac{2\pi}{k}\epsilon\mu\nu^{-1}\delta^{(2)})(w_1-w_2)$$

$$[A_z(t,w_1),A_{\overline{z}}(t,w_2)]_- = \frac{2\pi}{k}\epsilon z\overline{z}\delta^{(2)})(w_1-w_2) \tag{4.9}$$

$$[A_z(t,w_1),A_{\overline{z}}(t,w_2)]_- = -i\frac{2\pi}{k}\delta^{(2)})(w_1-w_2)$$

Where

$$\int d^2z\delta^{(2)}(w) = 1 \tag{4.10}$$

Since A_z, $A_{\overline{z}}$ are conjugates of each other in a quantum theory we cannot specify both of them together. We choose to work in holomorphic polarization, i.e., wave function ψ is a functional of A_z only. $A_{\overline{z}}$ is represented as

$$A_{\overline{z}}(w) = -i\hbar\frac{\delta}{\delta A_z(w)}, \hbar = -\frac{2\pi}{k}, \frac{\delta}{\delta A_z(w)}A_z(w') = \delta^{(2)}(w-w') \tag{4.11}$$

Next we impose Gauss law as an operator equation to see the space of gauge invariant wavefunctions.

$$F_{z\overline{z}}\psi[A_z] = 0 \implies [-i\hbar\partial_z\frac{\delta}{\delta A_z(w)} - \partial_{\overline{z}}A_z(w)]\psi[A_z] = 0 \tag{4.12}$$

This implies upto normalization factor

$$\psi[A_z] = \exp\left\{\frac{i}{2\hbar}\int_{\partial M} d^2z' A_z(w')\partial_{\overline{z}'}\int^{w'} A_z(w'')dz''\right\} \tag{4.13}$$

To see that this is indeed the case, we plug back it to the equation (2.12) above and note that

[3] We stress here ϵ is the covariant one defined above

$$
\begin{aligned}
&\frac{\delta}{\delta A_z(w)} \int_{\partial M} d^2z' A_z(w') \partial_{\overline{z}'} \int^{w'} A_z(w'') dz'' \\
&= \int_{\partial M} d^2z' \delta^{(2)}(w-w') \partial_{\overline{z}'} \int^{w'} A_z(w'') dz'' + \int_{\partial M} d^2z' A_z(w') \partial_{\overline{z}'} \int^{w'} \delta^{(2)}(w-w'') dz'' \\
&= \int_{\partial M} d^2z' (-\partial_{\overline{z}'}) \delta^{(2)}(w-w') \int^{w'} A_z(w'') dz'' + \int_{\partial M} d^2z' (-\partial_{\overline{z}'} A_z(w')) \delta(\overline{z}-\overline{z}') \\
&= \int_{\partial M} d^2z' \partial_{\overline{z}} \delta^{(2)}(w-w') \int^{w'} A_z(w'') dz'' + \int_{\partial M} d^2z' A_z(w') \partial_{\overline{z}'} \delta(\overline{z}-\overline{z}') \\
&= \partial_{\overline{z}} \int_{\partial M} d^2z' \delta^{(2)}(w-w') \int^{w'} A_z(w'') dz'' + \int_{\partial M} d^2z' A_z(w') (-\partial_{\overline{z}}) \delta(\overline{z}-\overline{z}') \\
&= \partial_{\overline{z}} \int^{w'} A_z(w'') dz'' + \partial_{\overline{z}} \int_{\partial M} d^2z' A_z(w') \delta(\overline{z}-\overline{z}') \\
&= \partial_{\overline{z}} \int^{w'} A_z(w'') dz'' + \partial_{\overline{z}} \int^{w} dz' A_z(w') \\
&= 2\partial_{\overline{z}} \int^{w} dz' A_z(w')
\end{aligned}
\tag{4.14}
$$

In above manipulations one has to be very careful that there is no differential with respect to the variable which we are integrating over with a delta function.

4.2 Path integral

Next we turn to path integral approach. To get wave function on t = 0 surface, we path integrate from $t = \infty$ to $t = 0$ surface where boundary conditions are specified. At this point we note that action given by (3.3) requires a boundary term to define well posed variational problem with our boundary condition of specifying A_z on $t = 0$ slice. To see this we vary action

$$
\delta S = -\frac{k}{4\pi} \int_M d^3x \epsilon^{\mu\nu\rho} [2\delta A_\mu \partial_\nu A_\rho + \partial_\nu (A_\mu \delta A_\rho)] \tag{4.15}
$$

In $A_0 = 0$ gauge to specify A_z on the boundary the term that we need to add to action (3.3) to get rid of the boundary variation is

$$
S_b = \frac{k}{4\pi} \int_M d^3x \epsilon^{z0\overline{z}} [\partial_0 (A_z A_{\overline{z}}] = -\frac{k}{4\pi} \int_{\partial M} d^2x \epsilon^{z\overline{z}} (A_z A_{\overline{z}}) \tag{4.16}
$$

Now we proceed to evaluate the wavefunction on t=0 slice

$$\psi[A_z] = \int \mathcal{D}A e^{i(S+S_b)} \tag{4.17}$$

To evaluate it we gauge fix to $A_0 = 0$ and take into account corresponding Gauss law (equation of motion)

$$F_{z\overline{z}} = \partial_z A_{\overline{z}} - \partial_{\overline{z}} A_z = 0 \tag{4.18}$$

This equation is easily solved by

$$A_z = \partial_z \chi, A_{\overline{z}} = \partial_{\overline{z}} \chi \tag{4.19}$$

On $\Sigma = S^2$ only A_z is specified. This determines χ upto a constant [4]. Faddeev Popov gauge fixing procedure gives

$$\psi[A_z] = \int \mathcal{D}A e^{i(S+S_b)|_{A_z=\partial_z\chi, A_{\overline{z}}=\partial_{\overline{z}}\chi}} \tag{4.20}$$

First we simplify S term

$$S|_{A_z=\partial_z\chi, A_{\overline{z}}} = -\frac{k}{4\pi}\int_{\partial M} d^2x \epsilon^{z0\overline{z}}[\partial_z\chi\partial_0\partial_{\overline{z}}\chi - \partial_{\overline{z}}\chi\partial_0\partial_z\chi] \tag{4.21}$$

$$= \frac{k}{4\pi}\int_{\partial M} d^2x \epsilon^{z0\overline{z}}[\partial_{\overline{z}}\partial_z\chi\partial_0\chi - \partial_z\partial_{\overline{z}}\chi\partial_0\chi] = 0 \tag{4.22}$$

Here we used the fact that partial integration over compact co-ordinates $z, \overline{z}$ does not produce any boundary term.

$$S_b|_{A_z=\partial_z\chi, A_{\overline{z}}} = -\frac{k}{4\pi}\int_{\partial M} d^2x \epsilon^{z\overline{z}}\partial_z\partial_{\overline{z}}\chi \tag{4.23}$$

Putting these things together we get

$$\psi[A_z = \partial_z\chi] = e^{-\frac{ik}{2\pi}\int_{\partial M} d^2z\partial_z\partial_{\overline{z}}\chi} \tag{4.24}$$

To convince ourselves that wavefunction that we are computing is really independent of the choice of gauge, we redo the calculations in $A_z = 0$ gauge below. Note that in this gauge we dont need to add any boundary term to the action. Arguing in same manner as before one

[4]This follows from the fact that on $\mathbb{R}^2$ any anti-holomorphic function except constant blows up at infinity and only entire function is a constant. On S^2 point at infinity on $\mathbb{R}^2$ maps to an ordinary point and we expect regularity at that point.

gets

$$\psi[A_z] = \int \mathcal{D}A e^{iS|_{A_z=\partial_z\chi, A_0=\partial_0\chi}} \tag{4.25}$$

Now we simplify

$$\begin{aligned}
S|_{A_z=\partial_z\chi, A_0=\partial_0\chi} &= -\frac{k}{4\pi}\int_M d^3x\epsilon^{z\bar{z}0}[A_z\partial_{\bar{z}}A_0 - A_0\partial_{\bar{z}}A_z] \\
&= -\frac{k}{4\pi}\int_M d^3x\epsilon^{z\bar{z}}[\partial_z\chi\partial_{\bar{z}}\partial_0\chi - \partial_0\chi\partial_{\bar{z}}\partial_z\chi] \\
&= -\frac{k}{4\pi}\int_M d^3x\epsilon^{z\bar{z}}[\partial_z\chi\partial_{\bar{z}}\partial_0\chi - \partial_0(\chi\partial_{\bar{z}}\partial_z\chi) + \chi\partial_0\partial_{\bar{z}}\partial_z\chi] \\
&= -\frac{k}{4\pi}\int_M d^3x\epsilon^{z\bar{z}}[\partial_z\chi\partial_{\bar{z}}\partial_0\chi - \partial_0(\chi\partial_{\bar{z}}\partial_z\chi) - \partial_z\chi\partial_0\partial_{\bar{z}}\chi] \\
&= -\frac{k}{4\pi}\int_M d^3x\epsilon^{z\bar{z}}[-\partial_0(\chi\partial_{\bar{z}}\partial_z\chi)] \\
&= -\frac{k}{4\pi}\int_{\partial M} d^2x\epsilon^{z\bar{z}}[\partial_z\chi\partial_{\bar{z}}\chi] \\
&= -\frac{k}{4\pi}\int_{\partial M} d^2z[\partial_z\chi\partial_{\bar{z}}\chi]
\end{aligned}$$

And as we can see, this agrees with our previous answer. Now we have to write down the density matrix and regulate it to get the entanglement spectrum and entanglement entropy, but before that, we need to normalize this wavefunction which seems pure phase in first glance, but χ is a complex valued field, so it isn't pure phase. We keep in mind, the calculation done here was on the manifold $S^2 \times \mathbb{R}$, to get the result $S_A = \ln S_0^0$, we need to derive the result at a finite temperature. We faced troubles in normalizing the wave function and moving forward from here because that involved a path integral on S^3 which we struggled with. We plan to look at it again and this work is still in progress.

Chapter 5

Topological phases and SPT phases

In the last few chapters, we focused on calculating the topological entanglement entropy in the Chern Simons theory, it is an artifact of the long-range entanglement phase that the system is in. By long range entanglement we informally mean that the entanglement isn't just coming from neighbouring degrees of freedom, which would mean that the apart from the area term that is usually seen in entanglement entropy for QFTs, we will have an additional universal term which indicates the presence of topological order. A more proper definition of long range entanglement[8] is if a state can't be transformed to a direct product state by the action of local unitary evolution operators, then it has long range entanglement. We saw that the topological phases, whose low energy physics is described by Chern-Simons theory, have some edge states which are the degrees of freedom through which transport phenomena happen as the bulk is gapped. So, topological phases have a characteristic edge behaviour.

But there are other models that we see in nature, that have such characteristic edge behaviour. These are not "topological phases" and they don't have long range entanglement, but are called symmetry protected topological phases. One such phase that we will look at is the Haldane phase, the thought of which germinated from the Haldane's conjecture[13], which was given by Duncan Haldane in 1983, which says: Anti-ferromagnetic Heisenberg spin chains of integer spins have a gap in their energy spectrum and the spin-spin correlations die down exponentially. While anti-ferromagnetic Heisenberg spin chains of half integer spins don't have such a gap in their energy spectrum and spin-spin correlations are power laws rather than exponential.

Here by anti-ferromagnetic Heisenberg spin chains we mean Hamiltonian of spins on a 1 D lattice which have this following term:

$$\mathcal{H} = J \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j \tag{5.1}$$

There have been a lot of experimental and theoretical studies of this conjecture, one of them being the exactly solvable model proposed by Affleck, Kennedy, Lieb and Tasaki, which shows the validity of the conjecture for integer spins, by calculating the gap and the correlation functions. Also, there is a beautiful pictorial way to see that this model has characteristic edge states.

5.1 The AKLT Model

This exactly solvable model displays all the properties of the Haldane phase very well and agrees with the Haldane's conjecture. The AKLT model[12] is $S = 1$, 1 D model with nearest-neighbour interactions. Before writing down the Hamiltonian, let's see what happens when you have 2 $S = 1$ particles and you add their spins. They can add to give $S_{tot} = 2, 1, 0$. The Hamiltonian is the sum of projectors acting on neighbouring sites which project them onto the $S = 2$ subspace.

$$\mathcal{H} = \sum_i \mathcal{P}_2^i (S_i + S_{i+1}) \tag{5.2}$$

Using the Casimir, one can work out the projector, so the Hamiltonian os:

$$\mathcal{H} = \sum_i [\frac{1}{2} S_i \cdot \S_{i+1} + \frac{1}{6}(S_i \cdot S_{i+1})^2 + \frac{1}{3} \tag{5.3}$$

From the form of the Hamiltonian, we can say that the ground state would be the one having no adjacent spins having total $S = 2$. To construct such a ground state, we break up each $S = 1$ at each site into two $S = 1/2$ particles. The spin half particles belonging to the same site are projected to the $S_{tot} = 1$ subspace and the adjacent spin half particles belonging to different sites are made into a spin singlet, i.e.

$$|\psi\rangle_- = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \tag{5.4}$$

So, the complete ground state for periodic boundary condition is:

$$|GS\rangle = (\otimes_{k=1}^{N} \mathcal{P}_{k\overline{k}}) \, |\psi\rangle_{-}^{\overline{1}2} \, |\psi\rangle_{-}^{\overline{2}3} \ldots |\psi\rangle_{-}^{\overline{N-1}N} \, |\psi\rangle_{-}^{\overline{N}1} \tag{5.5}$$

Where k in the sum labels the site and k and $\overline{k}$ represents the 2 spin halves at each site.

If we have ope boundary conditions, we immediately see that two end spin halves will have no neighbour to form a spin singlet with. The two dangling spin halves are the edge states in this model, it is one of the simplest example of a model where one observes edge states. These spin halves have a 2 dimensional Hilbert space, so the ground state is 4-fold degenerate. These edge states are characteristics of the Haldane phase.

Figure 5.1: A size 7 spin chain with open boundary condition. We can see the dangling spin halves are labelled by α and β, giving rise to a four fold degeneracy.

This model is exactly solvable and spin spin correlations were calculated in this and was found out to be exponentially decaying with distance. Again according to the Haldane's conjecture.

One can also calculate the entanglement entropy of a block of spins with the rest and in this model and it comes to be[7] $2 \ln 2$ in the large block size limit. This is understandable because if one forgets about the spin 1 projectors, the groundstate is a product of spin singlets, so the two cuts made to pick the spin block, will contribute $\ln 2$ each. For half integer spin chains, it is found that the entanglement goes as logarithm of the block size.

These two facts together, add to the Haldane's conjecture that for integer spin chains, the entanglement is a constant term for large block sizes and for half integer spin chains, it goes as the logarithm of the block size.

We have seen that edge states exist in the AKLT model in a beautiful pictorial way. But to see the same edge states in different models which are in the Haldane phase is not an easy task. In the next section, we will develop some machinery to display the existence of edge states in general 1 D antiferromagnets.

Also, due to finite size effects, one would expect some interaction between between the

two edge spins, which would make the 4 fold degenerate groundspace split. We would work out this interaction between the two edge spins for the spin 1 Heisenberg antiferromagnet in the next section.

Chapter 6

Effective theory for quantum anti-ferromagnets

Let us consider a spin system with spin 1/2s on a square lattice with antiferromagnetic nearest neighbour interaction of the following form:

$$\mathcal{H} = J \sum_{<ij>} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \tag{6.1}$$

Here, $J > 0$ that means the coupling is anti-ferromagnetic. The interaction Hamiltonian is invariant under lattice symmetries and SU(2) rotational symmetry. Although this is a very generic looking model, it is actually an effective Hamiltonian for Mott insulators.

The classical lowest energy state that we associate with such a Hamiltonian is the Neel state which is basically the alternating spins state, i.e. the classical groundstate is the state where $m = S_z = +1/2$ on sublattice A and $m = -1/2$ on the complement sublattice B. The order parameter of Neel state, which is called the staggered magnetization, doesn't commute with the Hamiltonian. This means that this classical groundstate is not the actual groundstate of the quantum problem. So it is interesting to see if the Neel order atleast appears in a long range in the quantum state or not. The real reason for Neel state not bein the quantum groundstate is quantum fluctuations which decrease as the the value of S increases. It was shown rigorously for above type of Hamiltonian on a hypercubic lattice at $d \geqslant 3$ for all values of S, the groundstate has Neel order. Same goes with $d = 2$ but with

$S \geqslant 1$. The case which is never in Neel order is spin halves on a square lattice. The Neel state is labelled by a Neel vector, $\vec{N}$, which is the order parameter and is used to define staggered magnetization:

$$\vec{S}_i = \eta_i \vec{N}_r \tag{6.2}$$

Where η is +1 on sublattice A and -1 on the complement sublattice B. Neel state is marked by $\left\langle \vec{N}_i \right\rangle \neq 0$ and the excited states in the spectrum are spin waves which are gapless.

After this generic discussion, our main aim is to show a mapping between Neel phase (or light deviation from it) and the quantum O(3) rotor model, first in the continuum limit, which is also called the O(3) non linear sigma model.

To do that, first we need to develop path integral for quantum spins

6.1 Path Integral for Quantum Spins

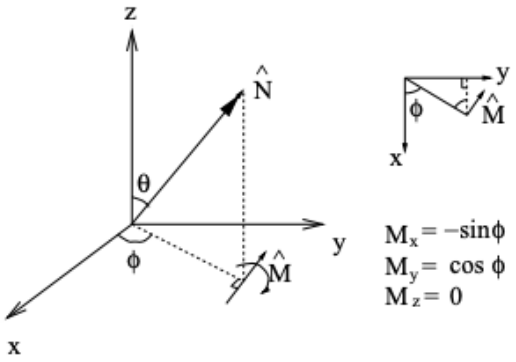
We will now set up the partition function calculation of spins interacting with the Heisenberg Hamiltonian using path integrals. For the current purposes, we will consider the case of spin half in detail. The final generalized expressions will also be mentioned later.

6.1.1 Spin Coherent basis

We begin by pointing out that the $S_z = 0, 1$ basis won't work for a path integral, we need to come up with a basis which has continuous parameters. So we go over to an over-complete basis $\left|\vec{N}\right\rangle$ which is defined as follows for a spin half.

$$\vec{S} \cdot \vec{N} \left|\vec{N}\right\rangle = \frac{1}{2} \left|\vec{N}\right\rangle \tag{6.3}$$

The vector $\vec{N}$ actually is fixing a direction on the unit sphere and just like the Bloch sphere, one can fix the north pole to be the state $|\uparrow\rangle$. Then one can define $\vec{N}$ in terms of the north pole by a rotation: $\left|\vec{N}\right\rangle = exp(-i\theta \vec{M} \cdot \vec{S}) \left|\uparrow\right\rangle$, we have defined the unit vector $\vec{M}$ in the figure and $\vec{S}$ are the spin operators (half multiplied with Pauli matrices: We can also write



$$\left|\vec{N}\right\rangle = \cos\frac{\theta}{2}\left|\uparrow\right\rangle + \sin\frac{\theta}{2}\left|\downarrow\right\rangle \tag{6.4}$$

Now if one calculates the norm, $\left\|\left\langle\vec{N}\middle|\vec{N'}\right\rangle\right\|^2 = (1 + \vec{N}\vec{N'})/2$. So, the basis is over-complete. Let us write down the resolution of identity in this basis:

$$\int \frac{d\vec{N}}{2\pi}\left|\vec{N}\right\rangle\left\langle\vec{N}\right| = \int \frac{d(\cos\theta)d\phi}{2\pi}\left|\vec{N}\right\rangle\left\langle\vec{N}\right| = \mathbb{I} \tag{6.5}$$

We also have

$$\left\langle\vec{N}\middle|\,\vec{S}\,\middle|\vec{N}\right\rangle = \frac{1}{2}\vec{N} \tag{6.6}$$

Now we get to the path integral calculation of the partition function Z. Starting with the case of a single spin:

$$Z = \sum_{\alpha}\langle\alpha|\, exp(-\beta H)\,|\alpha\rangle \tag{6.7}$$

$$= \int \mathcal{D}\vec{N}(0)\left\langle\vec{N}(0)\middle|\, exp(-\beta H)\,\middle|\vec{N}(0)\right\rangle \tag{6.8}$$

We now break-up the exponential into a large number of infinitesimal parts:

$$Z = \int \mathcal{D}\vec{N}(\tau_0)\mathcal{D}\vec{N}(\tau_1)...\mathcal{D}\vec{N}(\tau_n)\prod_{i=0}^{n}\left\langle\vec{N}(\tau_i+\epsilon)\middle|\, exp(-\epsilon H)\,\middle|\vec{N}(\tau_i)\right\rangle \tag{6.9}$$

Where we use periodic boundary condition ($n\epsilon = \beta$) and $\left|\vec{N}(\tau_0 + n\epsilon)\right\rangle = \left|\vec{N}(\tau_0)\right\rangle$. Now we

need to calculate this matrix element in the limit $\epsilon \to 0$

$$\left\langle \vec{N}(\tau_i + \epsilon) \right| exp(-\epsilon H) \left| \vec{N}(\tau_i) \right\rangle = exp[-\epsilon(\left\langle \vec{N} \right| \frac{d\vec{N}}{d\tau} \left\rangle + H(S\vec{N}))] \tag{6.10}$$

Now we take the limit $\epsilon \to 0$ keeping $n\epsilon$ constant and get the partition function, with the boundary condition $\left| \vec{N}(0) \right\rangle = \left| \vec{N}(\beta) \right\rangle$

$$Z = \int \mathcal{D}\vec{N}(\tau) exp(-\int_0^\beta d\tau(\left\langle \vec{N} \right| \frac{d\vec{N}}{d\tau} \left\rangle + H(S\vec{N}))) \tag{6.11}$$

We can see that $\int_0^\beta d\tau(\left\langle \vec{N} \right| \frac{d\vec{N}}{d\tau} \left\rangle)$ is a pure phase term (which we would pay more attention to in the next section) it would lead to interesting geometric interpretations and some elegant physics. To see this:

The derivation of the coherent state path integral for a single spin is over, generalizing it to interacting d dimensional systems is not so difficult for a class of Hamiltonians which are multilinear in spin operators, that means in a coherent state $\left| \vec{N}_1(\tau), \vec{N}_2(\tau) ... \vec{N}_{L^d}(\tau) \right\rangle$, the matrix element element of any multilinear Hamiltonian $H(\vec{S}_1, \vec{S}_2 ... \vec{S}_{L^d}$ can be written as:

$$\left\langle \vec{N}_1(\tau), \vec{N}_2(\tau) ... \vec{N}_{L^d}(\tau) \right| H(\vec{S}_1, \vec{S}_2 ... \vec{S}_{L^d}) \left| \vec{N}_1(\tau), \vec{N}_2(\tau) ... \vec{N}_{L^d}(\tau) \right\rangle = H(S\vec{N}_1(\tau), S\vec{N}_2(\tau) ... S\vec{N}_{L^d}(\tau)) \tag{6.12}$$

So we can carry forward the above derivation without any significant change. Thus, for a Heisenberg antiferromagnet in d dimensions with the following Hamiltonian

$$H = J \sum_{<ij>} \vec{S}(\vec{r}_i) \cdot \vec{S}(\vec{r}_j) \tag{6.13}$$

we have the partition function:

$$\begin{aligned} Z = \int_{\vec{N}(\vec{r}_i,\beta)=\vec{N}(\vec{r}_i,0)} \mathcal{D}\vec{N}(\vec{r}_i,\tau) exp(-\int_0^\beta d\tau[\sum_i \left\langle \vec{N}(\vec{r}_i,\tau) \right| \frac{d}{d\tau} \vec{N}(\vec{r}_i,\tau) \left\rangle + \right. \\ \left. JS^2 \sum_{<ij>} \vec{N}(\vec{r}_i,\tau) \cdot \vec{N}(\vec{r}_j,\tau)]) \end{aligned} \tag{6.14}$$

We see that apart from the first term which is the overlap of $\vec{N}(\tau)$ with its own derivative

the other term is same as the Boltzmann weight that we would have in the expression of the partition function if we were solving the classical problem. This is a great advantage because it allows us to carry forward all the classical intuition we have because this term would play a significant role in determining the paths that are dominated in the path integral of the quantum problem.

We yet have to analyze the first term, which essentially bears all the quantum dynamics. so to continue our quantum anti-ferromagnet discussion, we will now analyze this term which is also called the berry phase.

6.1.2 The Berry phase term

The first term in the final formula of the partition function in the last section is an example of a Berry phase:

$$S_B = -\int_0^\beta d\tau \sum_i \left\langle \vec{N}(\vec{r}_i,\tau)\middle|\frac{d}{d\tau}\vec{N}(\vec{r}_i,\tau)\right\rangle \tag{6.15}$$

This is a purely imaginary term, as we can see

$$S_B^* = -\int_0^\beta \sum_i \left\langle \frac{d}{d\tau}\vec{N}(\vec{r}_i,\tau)\middle|\vec{N}(\vec{r}_i,\tau)\right\rangle \tag{6.16}$$

Now using

$$\frac{d}{d\tau}(\left\langle \vec{N}(\tau)\middle|\vec{N}(\tau)\right\rangle) = 0 \tag{6.17}$$

We have

$$S_B^* = +\int_0^\beta d\tau \sum_i \left\langle \vec{N}(\vec{r}_i,\tau)\middle|\frac{d}{d\tau}\vec{N}(\vec{r}_i,\tau)\right\rangle = -S_B \tag{6.18}$$

Since this is a pure phase, it is necessary to understand in more detail how this term adds a phase factor to every path.

First we consider again the case of a single spin and define $\vec{M}(\tau)$ again through the following relation: $\left|\vec{N}(\tau)\right\rangle = exp(-i\theta(\tau)\vec{M}(\tau)\cdot\vec{S})\left|\uparrow\right\rangle$ We also introduce additional notation by introducing a parameter u,

$$\left|\vec{N}(u,\tau)\right\rangle = exp(-iu\theta(\tau)\vec{M}(\tau)\cdot\vec{S})\left|\uparrow\right\rangle$$

Where u is in the interval $[0,1]$. Thus $\left|\vec{N}(0,\tau)\right\rangle = |\uparrow\rangle$ and $\left|\vec{N}(1,\tau\right\rangle = \left|\vec{N}(\tau)\right\rangle$. THe evolution of u from 0 to 1 is rotating our vector from the north pole to the final vector on the unit sphere along the circle of a constant u. Now, we have $\vec{M}(\tau)\cdot\vec{N}(u,\tau)=0$, using this we get:

$$\left\langle \vec{N}(\tau)\middle|\frac{d}{d\tau}\vec{N}(\tau)\right\rangle = iS\int_0^1 du\theta(\tau)\vec{M}(\tau)\cdot\frac{d\vec{N}(u,\tau)}{d\tau} \tag{6.19}$$

To simplify it further, we will use the following identity:

$$\vec{N}(u,\tau)\times\frac{\partial\vec{N}(u,\tau)}{\partial u} = \theta(\tau)\vec{M}(\tau) \tag{6.20}$$

We finally obtain:

$$\int_0^\beta d\tau\left\langle \vec{N}(\tau)\middle|\frac{d}{d\tau}\vec{N}(\tau)\right\rangle = iS\int_0^\beta d\tau\int_0^1 du\vec{N}(u,\tau)\cdot(\frac{\partial\vec{N}(u,\tau)}{\partial u}\times\frac{\partial\vec{N}(u,\tau)}{\partial\tau}) \tag{6.21}$$

This quantity has a beautiful geometric interpretation: It is iS times the area $\mathcal{A}_z$ of the spherical cap like area swept by the periodic path $\vec{N}(\tau)$ we have put a subscript z to signify that the north-pole is always included in this area as is visible from the figure:

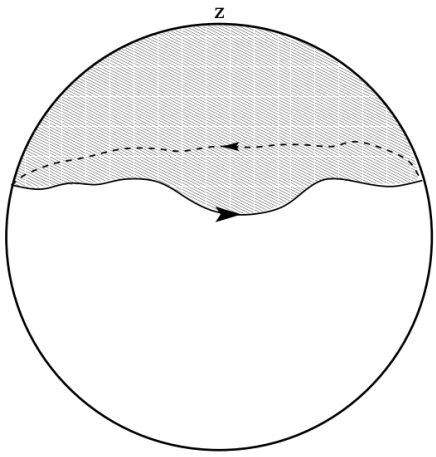


Figure 6.1: The arrowed path is the path inscribed by $\vec{N}(\tau)$ and the shaded area is the area that appears in the Berry phase term. Source:

Finally writing the full expression of the path integral for antiferromagnetic Heisenberg

Hamiltonian:

$$Z = \int \mathcal{D}\vec{N}(\vec{r}_i,\tau) exp(-\int_0^\beta d\tau [\sum_i iS \int_0^\beta d\tau \int_0^1 du \vec{N}(u,\tau)\cdot(\frac{\partial \vec{N}(u,\tau)}{\partial u}\times\frac{\partial \vec{N}(u,\tau)}{\partial \tau})+ JS^2 \sum_{<ij>} \vec{N}(\vec{r}_i,\tau)\cdot\vec{N}(\vec{r}_j,\tau)]) \tag{6.22}$$

With the condition $\vec{N}(\vec{r}_i,\beta) = \vec{N}(\vec{r}_i,0)$

6.2 Long-wavelength expansion in the Neel state

Now we will come to our main aim of developing an effective theory for the anti-ferromagnetic Heisenberg spin chain[6]. The ground state of the classical Heisenberg Hamiltonian will be the Neel state in which spins are either parallel or anti-parallel so the ordering is collinear. Let us look at the case of quantum antiferromagnets which in their ground state have Neel ordering that is present at least over short distances.

If we assume the survival of Neel ordering even for a few lattice spacings, we can convert our problem to a continuum problem of two new fields $\vec{L}$ *and* $\hat{n}$. Where $\hat{n}$ is the local Neel order parameter while $\vec{L}$ corresponds to the fluctuations from the Neel state. In terms of these fields, we write:

$$\vec{N}_i(r_i,\tau) = \eta_i \hat{n}(r_i,\tau)(1-(\frac{a^d}{S})^2\vec{L}^2(r_i,\tau))^{1/2} + \frac{a^d}{S}\vec{L}(r_i,\tau) \tag{6.23}$$

where d is the dimension of the lattice and a is the spacing. We also have $\hat{n}_i\cdot\hat{n}_i = 1$. Now since $\vec{L}_i$ is fluctuations about the uniform vector $\hat{n}$. Let's handle the Berry phase term later and first focus on the enegy term:

$$JS^2 \int_0^\beta d\tau \sum_{<ij>} \vec{N}(r_i,\tau)\cdot\vec{N}(r_j,\tau) \tag{6.24}$$

Now we substitute from eq (6.22) and using the fact that $\hat{n}$ and $\vec{L}$ are orthogonal at each site, we see that we will have three terms. First would be from the $\hat{n}$ piece of each $\vec{N}$, the other from $\vec{L}$ piece of each $\vec{N}$ and the third part is the cross term between $\hat{n}$ and $\vec{L}$ part of

the two different $\vec{N}$s. Now for the first contribution we do an expansion of the square root to leading order and use the fact that for nearest neighbours the product of η would always be -1. We get

$$JS^2 \sum_{<ij>} (\frac{(\hat{n}(r_i,\tau) - \hat{n}(r_j,\tau))^2}{2} - 1)[1 - \frac{1}{2}\frac{a^{2d}}{S^2}\vec{L}(r_i,\tau) - \frac{1}{2}\frac{a^{2d}}{S^2}\vec{L}(r_j,\tau)] \tag{6.25}$$

$$\approx \frac{JS^2}{2} \sum_{\mu=x,y} \sum_j (\Delta_\mu \hat{n})^2(r_j,\tau) + \frac{2dJa^{2d}}{2} \sum_j \vec{L}^2(r_j,\tau) \tag{6.26}$$

Above we have introduced a convenient notation $(\Delta_\mu \hat{n})^2) = \sum_{\mu,\alpha} (\partial_\mu n_\alpha)^2$.

For the second contribution, we use the fact that $\vec{L}$ is slowly changing and hence we ignore its spatial dependence to leading order. So product of cross $\vec{L}_i$ is $\vec{L}_i^2$.

For the third contribution, we see that there is an oscillatory spatial dependence because of the sublattice factor, $\eta(r_i)$ and since we have slowly varying $\hat{n}$ and small $\vec{L}$, this term would be very small, hence we ignore it at leading order.

Taking the continuum limit by replacing finite differences with derivatives and summations with integrals, we get:

$$\begin{aligned} &JS^2 \int_0^\beta d\tau \sum_{<ij>} \vec{N}(r_i,\tau) \cdot \vec{N}(r_j,\tau) = \\ &\frac{1}{2} \int_\Lambda d^d r \int_0^\beta d\tau [\rho_s (\nabla_\mu \hat{n})^2(r,\tau) + \frac{1}{\chi_\perp} \vec{L}^2(r,\tau)] \end{aligned} \tag{6.27}$$

Where we introduced the notation $\rho_s \approx JS^2 a^{2-d}; \chi_\perp^{-1} \approx 4dJa^d$ The subscript Λ in the integral denotes that although we have taken a continuum limit, we are actually working with variables which are defined on a course grained lattice. The coefficient ρ_s can be interpreted as a stiffness parameter which controls the spatial fluctuation from the anti-ferromagnetic ordering but we see that there is no term that penalizes the temporal fluctuation. This is obvious because all the information about the quantum dynamics is stored in the berry phase term. We also note the fact that $\vec{L}(r_i,\tau)$ can be interpreted as the total angular momentum density because if we sum over $S\vec{N}_i$, the oscillatory term gets cancelled and we are left with an integral over $\vec{L}$ in the continuum limit.

Let us consider the left out berry phase terms now. We again substitute the parametrization of $\vec{N}$ term and just retaining terms till first order of $\vec{L}$

$$\begin{aligned}
S_B &= iS\sum_i \int_0^\beta d\tau \int_0^1 du\vec{N}(u,\tau)\cdot(\frac{\partial\vec{N}(u,\tau)}{\partial u}\times\frac{\partial\vec{N}(u,\tau)}{\partial\tau}) \\
&= iS\sum_i \epsilon_i \int_0^\beta d\tau \int_0^1 du\hat{n}\cdot(\frac{\partial\hat{n}}{\partial u}\times\frac{\partial\hat{n}}{\partial\tau}) \\
&+ i\int d^dx \int_0^\beta d\tau\int_0^1 du[\hat{n}\cdot(\frac{\partial\hat{n}}{\partial u}\times\frac{\partial\hat{n}}{\partial\tau})+\hat{n}\cdot(\frac{\partial\hat{n}}{\partial u}\times\frac{\partial\vec{L}}{\partial\tau})+\vec{L}\cdot(\frac{\partial\hat{n}}{\partial u}\times\frac{\partial\hat{n}}{\partial\tau})]
\end{aligned}$$

Now we will try to eliminate some terms, for that we note, $\vec{L}$, $\frac{\partial\hat{n}}{\partial\tau}$ and $\frac{\partial\hat{n}}{\partial u}$ are perpendicular to $\hat{n}$ hence they lie in the same plane and also the last term is zero in the above equation. Also we note that

$$\hat{n}\cdot(\frac{\partial\hat{n}}{\partial u}\times\frac{\partial\vec{L}}{\partial\tau})+\vec{L}\cdot(\frac{\partial\hat{n}}{\partial u}\times\frac{\partial\hat{n}}{\partial\tau}) = \frac{\partial}{\partial\tau}[\hat{n}\cdot(\frac{\partial\hat{n}}{\partial u}\times\vec{L})]+\frac{\partial}{\partial u}[\vec{L}\times\frac{\partial\hat{n}}{\partial\tau}] \tag{6.28}$$

Now we can perform a surface integral over τ and u in the two terms, we see that the first term vanishes because of the periodic nature of $\hat{n}$ and $\vec{L}$. The u integral won't vanish but the $u=0$ term will vanish. So the final berry phase term is:

$$S_B = iS\sum_i \epsilon_i\int_0^\beta d\tau\int_0^1 du\hat{n}\cdot(\frac{\partial\hat{n}}{\partial u}\times\frac{\partial\hat{n}}{\partial\tau})-\int d^dx\int_0^\beta d\tau\vec{L}\cdot(\hat{n}\times\frac{\partial\hat{n}}{\partial\tau}) \tag{6.29}$$

We see that the first term here is again the area swept by the $\hat{n}(\tau)$, we will call it S'_B. We write down the total partition function as:

$$Z=\int_{\hat{n}^2=1}\mathcal{D}\hat{n}\mathcal{D}\vec{L}\delta(\vec{L}\cdot\hat{n})exp(S'_B(\hat{n}))-\int_0^\beta d\tau\int d^dx[\frac{\rho_s}{2}(\nabla_\mu\hat{n})^2(r,\tau)+\frac{\vec{L}^2(r,\tau)}{2\chi_\perp}+i\vec{L}\cdot(\hat{n}\times\frac{\partial\hat{n}}{\partial\tau})] \tag{6.30}$$

Now that we have the expression of the partition function, let us for now forget the presence of the $S'_B(\hat{n}$ term and try to link this partition function to some other system. To move forward, we consider a spatial lattice with lattice spacing as the inverse of our coarse graining scale. At each lattice point, we have a unit vector $\hat{n}$ and the position vectors of neighbouring sites interact with the strength given by ρ_s, so the nearest neighbours like to

be aligned just like in the classical ferromagnetic O(3) model. The kinetic energy of these terms bring the quantum mechanics in the system. If we think of the position vectors as position of a particle of mass $\chi_\perp$. So we need to find the corresponding angular momentum and write the kinetic energy down

$$\vec{L} = \chi_\perp(\hat{n} \times \frac{\partial \hat{n}}{\partial \tau}); \quad KE = \frac{1}{2\chi_\perp}\vec{L}^2 \tag{6.31}$$

So, if we ignore the berry phase term, the phase space path integral of such a system would be exactly of the same form as our partition function above. So we can make the statement that without the S'_B term, our expression describes the partition function of O(3) quantum rotors having moment of inertia $\chi_\perp$ and aligning interactions' strength is given by ρ_s.

Now we can write down the coordinate-space path integral for this quantum rotor model by integrating over the $\vec{L}$ in the previous phase space integral, this can be done by completing the square and the final answer, which actually is the low energy effective theory of an antiferromagnetic order displaying system ,is:

$$Z \propto \int_{\hat{n}^2=1} \mathcal{D}\hat{n} exp(S'_B(\hat{n}) - \frac{\rho_s}{2}\int_0^\beta d\tau \int d^d x((\nabla \hat{n})^2 + \frac{1}{c^2}(\frac{\partial \hat{n}}{\partial \tau})^2)) \tag{6.32}$$

Where the velocity $c = \sqrt{\rho_s/\chi_\perp} \sim JSa$. Now a;though we have written down the theory, we do not know the value of these parameters and also the course graining parameter is not fixed and its not a measurable quantity as it depends on the choice of the procedure taken. But these can be re-expressed in terms of the gap in the theory which in turn can be measured in the lab.

Also we need to know the dictionary between the microscopic observables and the quantity in the effective field theory. This is actually clear from our derivation itself. The Fourier components $\vec{S}(\vec{k})$ of the spin density operator at $\vec{k} = \boldsymbol{Q} + \vec{q}$ where q is small are represented by the corresponding Fourier components $\hat{n}(\vec{q})$ of the $\hat{n}$ field times S. And the Fourier component $\vec{S}(\vec{q})$ of the spin density operator for small q is represented by the Fourier component $\vec{L}(\vec{q})$ of the $\vec{L}$ field:

$$\vec{S}(\boldsymbol{Q} \vec{+} \vec{q}) \sim S\hat{n}(\vec{q}) \tag{6.33}$$

$$\vec{S}(\vec{q}) = \vec{L}(\vec{q}) \tag{6.34}$$

Where $\boldsymbol{Q}$ is the antiferromagnetic ordering wave vector where Fourier transform of the oscillating variable $\eta(\vec{r})$ is concentrated. So if one looks at the position basis variables, for each spin variable at site x, we have a sign associated to the corresponding $\hat{n}$ variable which is just $(-1)^x$.

Now we will do the final step, analyzing the Berry phase (S'_B) term. We will do it for the d=1 case only since it is the easiest and also relevant to us. We consider a chain of L sites with periodic boundary condition (assume L is even), for this case, we can write S'_B in the followning way which has a good interpretation.

$$S'_B = iS \sum_{j=1}^{L/2} (\mathcal{A}_z(\hat{n}(r_{2j}, \tau)) - \mathcal{A}_z(\hat{n}(r_{2j-1}, \tau)) \tag{6.35}$$

with PBC ($\hat{n}(L, \tau) = \hat{n}(0, \tau)$). Now the geometric interpretation of this term is that it is the area of the strip made by subtracting the area swept by two neighbouring site $\hat{n}_i$. We need to sum up the areas of all such strips. This geometric interpretation helps us to get rid of the internal coordinate u. So now we can write

$$iS \sum_{j=1}^{L/2} (\mathcal{A}_z(\hat{n}(r_{2j}, \tau)) - \mathcal{A}_z(\hat{n}(r_{2j-1}, \tau)) = \frac{iS}{2}(2a) \sum_{j=1}^{L/2} \int_0^\beta d\tau \hat{n} \cdot (\frac{\Delta \hat{n}(r_{2j}, \tau)}{a} \times \frac{\partial \hat{n}}{\partial \tau}(r_{2j}, \tau)) \tag{6.36}$$

where $\Delta \hat{n}(r_{2j}, \tau) = \hat{n}(r_{2j}, \tau) - \hat{n}(r_{2j-1}, \tau)$. We have also multiplied and divided by 2a to go to the continuum limit which is valid if the variation of $\hat{n}$ is slow:

$$\frac{iS}{2} \int_0^L dx \int_0^\beta d\tau \hat{n} \cdot (\frac{\partial \hat{n}}{\partial x} \times \frac{\partial \hat{n}}{\partial \tau}) \tag{6.37}$$

The integral in the above equation actually counts the number of time the path $\hat{n}(x, \tau)$ wraps itself around the unit sphere. Therefore, this integral is equal to $4\pi m$ where m is an integer.

So, we have

$$S'_B = \frac{iS}{2} 4\pi m = 2\pi i S m \tag{6.38}$$

So the quantity which is relevant to us: $exp(S'_B) = (-1)^{2mS}$. We see that for integer spins, this factor is just 1 for all configurations.

So we make the statement that the low energy effective field theory for integer antiferro-

magnetic spin chains is just the quantum rotor model. But if $S = 1/2, 3/2...$ then the Berry phase factor dramatically changes the physics because some configurations will have a -1 sign to them and some will have +1 depending on the covering number of the configuration.

This is actually the reason why integer and half integer chains have such striking differences and these lead to the famous Haldane's conjecture which we saw in a previous chapter.

6.3 Edge states of antiferromagnetic spin chains

The derivation in the previous section was done for a spin chain with periodic boundary condition. As we saw in the previous chapter, the spin 1 AKLT model had edge degrees of freedom which were spin 1/2. This remarkable feature is not just limited to the AKLT model, it is true for a class of models called the Haldane spin chains, they are characterized by a gap and edge states also they have something called a string order parameter.

Now it was easy to see the edge states in the AKLT chain because of the beautiful pictorial way of describing the ground state. For general Heisenberg antiferromagnets which are not exactly solvable, it is difficult to see that they have edge states. But using the above derivation of the Berry phase expression for Heisenberg chains, we can see (not rigorously) how the edge states emerge[10].

Starting from the Berry phase expression:

$$S'_B = S\sum_i (-1)^i \mathcal{A}_z(x_i) \sim \frac{S}{2}\int \frac{\partial \mathcal{A}_z(x)}{\partial x} dx \tag{6.39}$$

This would be equal to the result from the periodic boundary condition case plus the contributions from the edge

$$S'_B = \frac{S}{2}[\mathcal{A}_z(L) - \mathcal{A}_z(0) + 4\pi m] \tag{6.40}$$

Where m is the number of times the path of spin configuration $\hat{n}(\tau)$ covers the unit sphere. For integer chains since Sm is always an integer, so the third term has no effect on the path integral and can be dropped. If there was periodic boundary condition, the Berry phase from the two ends would cancel each other. But for open chains, the edge corrections would

survive. Looking at them, we can interpret them as Berry phase of two spins of magnitude $S/2$ located at the edges of the chain.

We have established the presence of these edge states. They also have been observed experimentally. We can go one step ahead and ask the question, is there any interaction between these two spins for a chain of size L? Let's try to think about this question in the AKLT model's case. We saw that in the AKLT case, the ground state is 4 fold degenerate because of the 2 dangling spin 1/2s on the either side. Now the spin spin correlation function decays exponentially in the AKLT model so we should expect an interaction between the two spin 1/2s which also decays exponentially with the length of the chain. We expect the same effect to be carried forward in the antiferromagnetic Heisenberg spin chain.

It is known in the literature that the coupling between the edge states of the antiferromagnetic Heisenberg spin chain goes as[9] $J \sim e^{-L/\xi}$ and there is a striking difference between the edge properties of chains with odd length vs the ones with even length. The ground state for a S=1 spin chain of even length is a singlet state, on the other hand the ground state for a odd length chain is a spin triplet. We will derive these two above results analytically in the next section by computing the effective Hamiltonian by perturbation theory in the next section and it will be of the form noted above. Also we will run an exact diagonalization program to diagonalize the S=1 antiferromagnetic Heisenberg spin chain to observe the interaction between the edge spins and to observe the odd-even effect.

Chapter 7

Interaction between the edge spins

7.1 Schrieffer Wolff perturbation theory

Schrieffer Wolff method gives a way to calculate low energy effective Hamiltonians by constructing a Unitary transformation which decouples the low energy and high energy subspaces. It can be thought of as a degenerate perturbation theory method[11].

We begin by considering a Hamiltonian H_0 which describes an unperturbed system. Then we specify a low-energy subspace of our interest P_0 which is invariant under the action of H_0. We define Q_0 to be the rest of the subspace. Also we introduce a perturbation λV which allows transitions from P_0. We want an effective Hamiltonian which acts only on P_0 but it still produces the spectrum of the perturbed Hamiltonian which is in the subspace of interest P_0. This SW transformation is actually generated by W such that the transformed Hamiltonian $e^{-iW}(H_0 + \lambda V)e^{iW}$ acts only on and is invariant in P_0. So the effective Hamiltonian is

$$H_{eff} = P_0 \tilde{H} P_0 = P_0 e^{-iW} H e^{iW} P_0 \tag{7.1}$$

Expanding in Taylor series

$$\tilde{H} = H + i[W, H] + \frac{i^2}{2!}[W, [W, H]] + ... \tag{7.2}$$

Now we also imagine a Taylor expansion of W

$$W = \lambda W_1 + \lambda^2 W_2 + ... \tag{7.3}$$

We point out that the transformation generated by W won't be unique. We can see it in the following way: W decouples the high and low energy subspaces, but after that, one can rotate among the decoupled subspaces to get different Ws, all of which perform the necessary work.

To remove such a large ambiguity, we constrain W to connect only the different decoupled blocks

$$P_0 W P_0 = 0;\ Q_0 W Q_0 = 0 \tag{7.4}$$

Now substituting the expansion of W in the expression of $\tilde{H}$ we obtain order by order:

$$\begin{aligned}
&\lambda^0 : H_0 \\
&\lambda^1 : V + [iW_1, H_0] \\
&\lambda^2 : [iW_2, H_0] + [iW_1, V] + \frac{1}{2}[iW_1, [iW_1, H_0]
\end{aligned}$$

and so on

From all this, we can write expressions of the effective Hamiltonian order by order after a little algebra. At first order, it will be $H_{eff}^{(1)} = P_0 V P_0 = 0$ by definition of the perturbation term.

For the second order term we need to solve for W_1. The equation to be solved are:

$$\begin{aligned}
P_0[V + [iW_1, H_0]]Q_0 = 0 \\
Q_0[V + [iW_1, H_0]]P_0 = 0
\end{aligned} \tag{7.5}$$

After doing a little algebra, we get

$$H_{eff}^{(2)} = P_0 V R_0 V P_0 \tag{7.6}$$

Where $R_0 = Q_0/(E_0 - H_0)$. The denominator is the difference between the ground state and the excited state energy in Q_0.

7.2 Effective Hamiltonian for edge spins of a Heisenberg antiferromagnet

We will now calculate the 2^{nd} order effective Hamiltonian between the the two edge spins of a $S = 1$ anti-ferromagnetic Heisenberg spin chain. For that, first we need to use the mapping between integer spin AFHC and the O(3) quantum rotor model. The quantum rotor model will behave like the bulk and we will by hand couple 2 spin halves on both the ends of the spin chain. These spin halves will be coupled to the first and the last n and L in a way such that the full Hamiltonian reads:

$$\mathcal{H} = \frac{1}{2I}\sum_{i=1}^{M} \boldsymbol{L_i^2} - J\sum_{<ij>} \vec{\boldsymbol{n_i}}\cdot\vec{\boldsymbol{n_j}} + \lambda(\boldsymbol{\sigma_1}\cdot\boldsymbol{L_1} + \boldsymbol{\sigma_1}\cdot\vec{\boldsymbol{n_1}} + \boldsymbol{\sigma_M}\cdot\boldsymbol{L_M} + (-1)^{M+1}\boldsymbol{\sigma_M}\cdot\vec{\boldsymbol{n_M}}) \quad (7.7)$$

What we need to do is to use the spectrum of the O(3) quantum rotor model and in second order in perturbation, calculate the effective Hamiltonian using the Schrieffer Wolff perturbation method.

For that, we need to specify the following:

- The sub space of the Hilbert Space in which we want to calculate the effective Hamiltonian, that is P_0
- From the rest of the Hilbert space, the subspace that is relevant to us (which will have nonzero matrix elements at second order) will be called Q_0
- The perturbation term V
- The energy difference between the ground state and the excited states in Q_0, labelled by $\epsilon_{k,m}$, where k labels the momentum and m labels the S_z component

Using these symbols, the effective Hamiltonian is:

$$\mathcal{H}^{(2)}_{eff} = \frac{P_0 V Q_0 V P_0}{-\epsilon_{k,m}} \quad (7.8)$$

Let us specify them one by one:

P_0 is the ground state projector of the bulk O(3) quantum rotor model tensor product with the 4 dimensional Hilbert space of 2 spins:

$$P_0 = \prod_{i=0}^{M} |l_i = 0, m_i = 0\rangle\langle l_i = 0, m_i = 0| \otimes (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)(\langle\uparrow\uparrow| + \langle\downarrow\downarrow| + \langle\uparrow\downarrow| + \langle\downarrow\uparrow|) \tag{7.9}$$

Which we will write in a shorthand as:

$$P_0 = |0, ud\rangle\langle 0, ud| \tag{7.10}$$

V, the perturbation term is the following:

$$V = (\boldsymbol{\sigma_1} \cdot \boldsymbol{L_1} + \boldsymbol{\sigma_1} \cdot \hat{\boldsymbol{n_1}} + \boldsymbol{\sigma_M} \cdot \boldsymbol{L_M} + (-1)^{M+1} \boldsymbol{\sigma_M} \cdot \hat{\boldsymbol{n_M}}) \tag{7.11}$$

As we can see from the form of $\mathcal{H}$ and V, we would require only those states in Q_0 which have non zero matrix elements with the ground states, which are the first excited states, tensored with the 4 dimensional space, so Q_0 is:

$$Q_0 = \sum_j |l_j = 1, m\rangle \prod_{i=0}^{M} |l_i = 0, m_i = 0\rangle\langle l_i = 0, m_i = 0| \otimes (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)(\langle\uparrow\uparrow| + \langle\downarrow\downarrow| + \langle\uparrow\downarrow| + \langle\downarrow\uparrow|) \tag{7.12}$$

Which we would write in a shorthand as:

$$Q_0 = \sum_j |j, ud\rangle\langle j, ud| \tag{7.13}$$

Now we need to calculate the eigenvalue of the first excited state. To do that, first we need to workout the matrix elements of $\hat{\boldsymbol{n_i}}$ between the ground state and the first excited state:

$$\begin{aligned} \hat{\boldsymbol{n}} |l, m\rangle &= \hat{\boldsymbol{n}} \int d\hat{\boldsymbol{n}} |\hat{\boldsymbol{n}}\rangle\langle\hat{\boldsymbol{n}}|l, m\rangle \\ &= \sum_{i=x,y,z} \hat{n_i} \int d\hat{\boldsymbol{n}} |\hat{\boldsymbol{n}}\rangle Y_m^l(\hat{\boldsymbol{n}}) \end{aligned}$$

Just calculating the z component for now, since $\hat{\boldsymbol{n}}$ is a unit position vector, we have $\hat{n_z} = \cos\theta$

, for $l = 1, m = 0$ and pre-multiplying with the ground state then taking the complex conjugate:

$$\langle l = 1, m = 0| \cos\theta \, |l = 0, m = 0\rangle = \int\int \cos\theta \left\langle \hat{\boldsymbol{n}} \middle| \hat{\boldsymbol{n'}} \right\rangle Y_0^1 Y_0^0 d\hat{\boldsymbol{n}} d\hat{\boldsymbol{n'}} \tag{7.14}$$

$$= \int \cos\theta \sin\theta Y_0^1 Y_0^0 d\theta$$

Performing this integral with normalized Y_m^L we get:

$$\langle l = 1, m = 0| \cos\theta \, |l = 0, m = 0\rangle = \frac{1}{\sqrt{3}} \tag{7.15}$$

Now, redefining $\boldsymbol{\sigma}$ *and* $\hat{\boldsymbol{n}}$ in the following way:

$$\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y)$$
$$\sigma^- = \frac{1}{2}(\sigma^x - i\sigma^y)$$
$$n^+ = n^x + in^y$$
$$n^- = n^x - in^y$$

$$\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} = \sigma^z n^z + n^+\sigma^- + n^-\sigma^+ \tag{7.16}$$

Performing all the integrals to calculate the matrix elements, we find the following non-zero matrix elements (matrix elements for the spin raising and lowering operators is 1), notation being $|l \; m\rangle$:

$$\langle 1 \; 0| \, n^z \, |0 \; 0\rangle = \frac{1}{3} \tag{7.17}$$

$$\langle 1 \; 1| \, n^+ \, |0 \; 0\rangle = -\sqrt{\frac{2}{3}} \tag{7.18}$$

$$\langle 1 \; -1| \, n^- \, |0 \; 0\rangle = \sqrt{\frac{2}{3}} \tag{7.19}$$

We will calculate the first excited state in the momentum basis. We have noticed above that the first excited state, which is three-fold degenerate, has one of the l=1. Now, the $n_i n_{i+1}$ term acts like a hopping term and hops the l=1 excitations to neighbouring sites. Let's

calculate the matrix element for that, the notation would be as following: $|l_i\ m_i,\ l_{i+1}\ m_{i+1}\rangle$

$$\langle 1\ 0, 0\ 0|\, n_i^z n_j^z + n_i^+ n_j^- + n_i^- n_j^+ \,|0\ 0, 1\ 0\rangle = \langle 1\ 0, 0\ 0|\, n_i^z \,|0\ 0, 0\ 0\rangle \langle 10\ 0, 0\ 0|\, n_j^z \,|0\ 0, 1\ 0\rangle = \frac{1}{3} \tag{7.20}$$

We could insert the ground state in the middle because even if we insert the whole basis, only these matrix elements would be non zero. And we could ignore the raising and lowering operators because m = 0 in initial and final state.

Similarly, we can calculate for other bands (m = 1, -1) but the answer is the same.

So we have for every band:

$$\langle i| - J n_i \cdot n_{i+1} \,|i+1\rangle = -\frac{J}{3} \tag{7.21}$$

We calculate the difference between ground state and first excited state now by first writing the states in the momentum basis:

$$|k\rangle = \frac{1}{\sqrt{M}} \sum_i e^{ikx_i} \,|i\rangle \tag{7.22}$$

Now, we calculate ($H = \mathcal{H} - V$) at first excited level:

$$\langle k|\, H \,|k\rangle = \frac{1}{I} + \frac{1}{M} \sum_{i,j} e^{-ikx_j} \langle j| - J n_i \cdot n_{i+1} \,|i\rangle\, e^{ikx_i} \tag{7.23}$$

This matrix element would be non zero only for neighbouring i and j:

$$\langle k|\, H \,|k\rangle = \frac{1}{I} + \frac{1}{M} \sum_i (e^{-ikx_i} \langle i| - J n_i \cdot n_{i+1} \,|i+1\rangle\, e^{ikx_{i+1}} + e^{-ikx_{i+1}} \langle i+1| - J n_i \cdot n_{i+1} \,|i\rangle\, e^{ikx_i} \tag{7.24}$$

$$= \frac{1}{I} - \frac{J}{3}(e^{ik} + e^{-ik}) \tag{7.25}$$

$$= \frac{1}{I} - \frac{2J}{3} \cos(k) \tag{7.26}$$

Now we have all the ingredients for the calculation and we can now, calculate the final

effective Hamiltonian

$$\mathcal{H}^{(2)}_{eff} = \frac{P_0 V Q_0 V P_0}{-\epsilon_{k,m}} \tag{7.27}$$

This Hamiltonian would be a 4×4 matrix. Let us calculate a diagonal term first and let us omit $(-1)^M$ (not M+1 because we will add another site to enforce a boundary condition) for clarity, we will put it back in the end:

$$\sum_k \langle 0 \uparrow\uparrow | \, \sigma_1^z n_1^z + \sigma_1^+ n_1^- + \sigma_1^- n_1^+ + \sigma_M^z n_M^z + \sigma_M^- n_M^+ + \sigma_M^+ n_M^- \, | k \uparrow\uparrow \rangle \times$$
$$\langle k \uparrow\uparrow | \, \sigma_1^z n_1^z + \sigma_1^+ n_1^- + \sigma_1^- n_1^+ + \sigma_M^z n_M^z + \sigma_M^- n_M^+ + \sigma_M^+ n_M^- \, | 0 \uparrow\uparrow \rangle$$

Here we have dropped L_i terms because matrix elements of it connecting ground state and excited state is zero. Now analysing the different kinds of terms we can have here, we notice that since it is a diagonal element, only $\sigma^z n^z$ terms will be non-zero. Also, the terms which are product of matrix elements of the same site, will just give a constant term in the Hamiltonian which we can forget about. So we need to calculate the elements which connect the two ends. This is how the two spins will talk to each other, via the excited rotor state on the either side talking to the one on the other end. Such a term is:

$$\sum_k \langle 0 \uparrow\uparrow | \, \sigma_1^z n_1^z \, | k \uparrow\uparrow \rangle \times \langle k \uparrow\uparrow | \, \sigma_M^z n_M^z \, | 0 \uparrow\uparrow \rangle \tag{7.28}$$

Before moving forward, let us write down the momentum space wavefuntion with the boundary condition taken care of. Let us imagine, that instead of having sites labelled from 1 to M, we had sites from 0 to M. Now we will enforce the boundary condition that the wavefunction at this site is zero. Writing the first excited state in momentum basis

$$|k\rangle = c_k \sum_i e^{ikx_i} \, |i\rangle \tag{7.29}$$

Just like the particle in a box calculation, we can write it as:

$$|k\rangle = \sum_{i=0}^{M} (A_k \sin(kx_i) + B_k \cos(kx_i)) \, |i\rangle \tag{7.30}$$

Now, the boundary condition is $\psi(0) = 0$. So,

$$|k\rangle = A_k(\sin(k*0) + B_k\cos(k*0))\,|0\rangle + \sum_{i=1}^{M}(A_k\sin(kx_i) + B_k\cos(kx_i))\,|i\rangle \tag{7.31}$$

Now, since the site 0 wavefuntion should be zero, we see that B_k also should be zero. That means now we have the following expansion of a momentum basis state after normalization:

$$|k\rangle = \frac{1}{\sqrt{L}}\sum_i \sin kx_i\,|i\rangle \tag{7.32}$$

Going back to eq. (6.22) we insert the full basis $|i\rangle\langle i|$ with the states $|k\rangle$ and only that state corresponding to the n_i will survive:

$$\sum_k \langle 0\uparrow\uparrow|\,\sigma_1^z n_1^z\,|k\uparrow\uparrow\rangle \times \langle k\uparrow\uparrow|\,\sigma_M^z n_M^z\,|0\uparrow\uparrow\rangle = \sum_k \langle 0\uparrow\uparrow|\,\sigma_1^z n_1^z\,\langle 1|k\uparrow\uparrow\rangle \times |M\rangle\langle k\uparrow\uparrow|\,\sigma_M^z n_M^z\,|0\uparrow\uparrow\rangle$$

After putting the appropriate denominator and using the matrix elements evaluated above and including the previously ommited $(-1)^M$:

$$= (-1)^M\frac{1}{M}\sum_k \frac{\frac{1}{3}\sin(k)\times\sin(-k(1+(M-1)))}{-\epsilon_{k,m}} = (-1)^M\frac{1}{M}\sum_k \frac{1}{3}\frac{\sin(k)\sin(kM)}{\frac{1}{I}-\frac{2J}{3}\cos(k)} \tag{7.33}$$

Now let's calculate an off diagonal term:

$$\sum_k \langle 0\uparrow\downarrow|\,\sigma_1^z n_1^z + \sigma_1^+ n_1^- + \sigma_1^- n_1^+ + \sigma_M^z n_M^z + \sigma_M^- n_M^+ + \sigma_M^+ n_M^-\,|k\downarrow\downarrow\rangle \times$$
$$\langle k\downarrow\downarrow|\,\sigma_1^z n_1^z + \sigma_1^+ n_1^- + \sigma_1^- n_1^+ + \sigma_M^z n_M^z + \sigma_M^- n_M^+ + \sigma_M^+ n_M^-\,|0\downarrow\uparrow\rangle$$

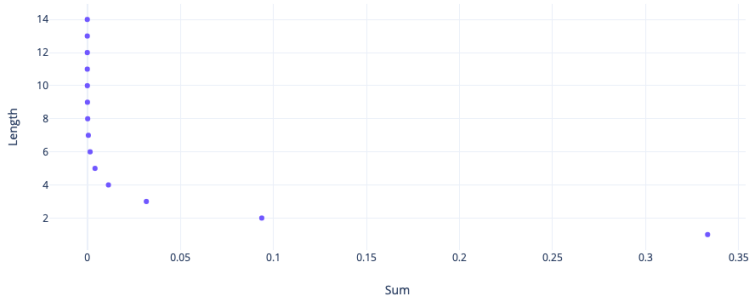
Evaluating in the same way as before and using the appropriate matrix elements:

$$(-1)^M\sum_k \langle 0\uparrow\downarrow|\,\sigma_1^+ n_1^-\,|k\downarrow\downarrow\rangle \times \langle k\downarrow\downarrow|\,\sigma_M^- n_M^+\,|0\downarrow\uparrow\rangle = (-1)^M\frac{1}{M}\sum_k \frac{2}{3}\frac{\sin(k)\sin(kM)}{\frac{1}{I}-\frac{2J}{3}\cos(k)} \tag{7.34}$$

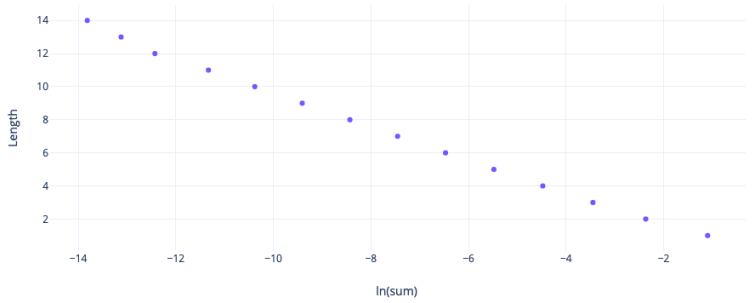
Although we haven't computed all the terms, but because of symmetry considerations, one can say that each term is a multiple of the final sum that we obtain. This sum runs from $k = 0$ *to* π in steps of $n\pi/(M+1)$ This sum can't be converted into an integral, because that would mean taking the large M limit. But the aim for this calculation was to obtain a coupling between the two edge spins for a finite length of chain.

Performing this sum for various values of L, we see that the expression is of the form $(-1)^M e^{-L/\xi}$. That means the coupling is anti-ferromagnetic for even length chains which would result in a singlet as it's ground state and for odd length chains, the coupling is ferromagnetic and the triplet state is the ground state. The exponential coupling makes sense because the spin-spin correlation function in anti-ferromagnetic S=1 Heisenberg spin chain is exponentially decaying too.

We plot the graph between the length of the chain and the absolute value of the sum in eq (6.27) and (6.28):



We make a semi-log plot to check if the above graph is exponential:



The semi-log plot is a straight line, so we can conclude that the coupling goes as the exponential of the negative length. Where the prefactor of L would be $\frac{I}{J}$

7.3 Numerical calculation

To support our analytical results, we wrote a program which calculates the smallest few eigenvalues of the $S = 1$ Heisenberg anti-ferromagnetic spin chain for various system sizes.

We used the Lanczos algorithm to obtain the eigenvalues. We see here also that for even length chains (length here denotes the number of sites) the singlet state is groundstate and for odd length chains, the triplet state is the groundstate.

To see which state is triplet or singlet, we diagonalize the Hamiltonian in different total S_z sectors. If a groundstate is in the total $S_z = 0$ sector as well as in the $S_z = 1$ sector, then it belongs to the triplet sector. If a groundstate is in $S_z = 0$ sector but is absent from the $S_z = 1$ sector, then it belongs to the singlet sector.

Apart from the odd-even effect, we see that the gap between ground state and first excited state is reducing as length is increasing, which is the exponential interaction we calculated and the gap between 1st excited state and 2nd excited state remains almost constant. This is the Haldane gap

l=8,s=0	l=8, s=1	l=8,s=2	l=9, s=0	l=9, s=1	l=9, s=2
-10.1246	-9.9227	-9.0491	-11.4329	-11.4329	-10.5110
-9.9227	-9.0491	-8.3015	-11.2202	-10.5110	-9.8427
-9.0491	-8.8978	-7.7629	-10.5110	-10.4854	-9.3231
-8.8978	-8.8043	-7.6505	-10.4854	-10.2158	-9.2665

l=10, s=0	l=10, s=1	l=10, s=2	l=11, s=0	l=11, s=1
-12.8945	-12.7562	-11.9907	-14.2303	-14.230
-12.7562	-11.9907	-11.3772	-14.0885	-13.4314
-11.9907	-11.8691	-10.8649	-13.4314	-13.4056
-11.8691	-11.8227	-10.8009	-13.4056	-13.2895

Chapter 8

Conclusion

We studied the calculation of entanglement entropy in pure Chern Simons theory for a spatial slice of S^2 and we saw that the methods used here won't carry forward to the case where Chern Simons theory is coupled to matter, we tried to do a path integral calculation to compute the partition function on S^3 but were unsuccessful although we plan to look at it in the future.

We studied the coherent basis representation of the path integral of a Heisenberg spin chain and saw the emergent edge states in case of open boundary conditions. We looked at relations between an $O(3)$ quantum rotor model and $S = 1$ Heisenberg spin chains at low energy and used it to calculate an effective Hamiltonian using Schrieffer Wolff perturbation theory between the two edge spins and showed that the interaction decays exponentially with the length of the chain, also we looked at the differences in the properties of this effective Hamiltonian between odd and even length chains.

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