Wages and Utilities in a Closed Economy- A Strategic Analysis

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by

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Certificate

This is to certify that this dissertation entitled Wages and Utilities in a Closed Economy-A Strategic Analysis towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sanyukta Deshpande at Indian Institute of technology, Bombay under the supervision of Milind Sohoni, Professor, Department of Computer Science and Engineering, during the academic year 2018-2019.

Milind Sohoni

Committee:

Milind Sohoni

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This thesis is dedicated to my parents and my brother

Declaration

I hereby declare that the matter embodied in the report entitled Wages and Utilities in a Closed Economy- A Strategic Analysis are the results of the work carried out by me at the Department of Computer Science and Engineering, Indian Institute of technology, Bombay under the supervision of Milind Sohoni and the same has not been submitted elsewhere for any other degree.

Sanyukta Deshpande

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Abstract

The broad objective of this work is to study the wage distribution and resource allocation through a mathematical model and analyse those as functions of choices of consumption and the technologies of production in an economy. A simple *Heterodox Model* is constructed, in which the consumption and production parts are separated and given using two global optimization functions. An iterative process, called the 'tatonnement' is proposed, which couples these functions and works similar to the Walrasian auction, computing the equilibrium under certain settings. The equilibria of this process correspond directly with those of a related Arrow-Debreu model, where existence of an equilibrium is proved. The formulation allows us to identify the combinatorial data which link parameters of the economic system with its equilibria, and in particular, the impact of consumer preferences on wages. The Heterodox model also allows the formulation and explicit construction of the *consumer choice game*, where individual utilities serve as the strategies with total or relative wages as the pay-offs. We illustrate the mathematical details of the consumer choice game through (2×2) markets and explore properties of the combinatorial data associated. We show that consumer preference, expressed through modified utility functions, does indeed percolate through the economy, and influences not only prices, but also production and wages. We further look at the geometric structures associated with the Heterodox model and propose a formulation which captures its equilibria at the vertices. Finally, we look at applications of the model on the trade theory, by explicitly deriving outcomes of trade and comparing those with the existing models.

Contents

Introduction

In an economy of a developing country like India, we see some characteristic concerns. These are great variances in wages, widespread unemployment across different classes of labour, and inefficiency in production and its allocation [4], [16]. On the other hand, increase in brand consciousness has resulted in lower demand for generic goods and thereby failure in sustainability and job creation in local industries [5].

A society, comprising of various classes, allocates its resources based on consumer preferences, technological constraints and the financial resources, with the last being somewhat dependent on the first two factors. All this accounts for the consumption of the society. For the production of goods and resources, what matters is suitable labour, capital as well as the profit margins and the available technology. Furthermore, there is a natural relationship between consumer choices and technological options as depending on the market demand, appropriate goods are produced in the society. With the help of a mathematical model, we aim to formulate the effects of these factors on resource allocations to explore the reasons for the above mentioned issues.

We propose a simple model called the *Heterodox model*, which couples the consumption and production processes and illustrates the interdependence of wages and utilities. The production is determined by a technology matrix T, which utilizes m labour classes and produces n goods and determines quantities of goods produced, labour utilization and wages. This part assumes prices as a given, i.e., which cannot be changed. The second part is the consumption part, which is modelled as a Fisher market and a utility matrix U. This part of the economy assumes the production part as given, i.e., wages and quantities of goods produced, and allocates goods based on wages (i.e., endowments or disposable incomes) held by each labour class, and determines prices.

Through this model, we show the connection between U, the utility matrix and the wages obtained by various labour classes, as implemented by T. In other words, it traces the connection between personal consumption choice, with prices of goods, their production and finally wages received. Next, it shows that having a "private and real" utility U_r, and posting or posturing a different U into the economy does indeed alter wages and has the potential to improve both the social welfare as well as the relative welfare for certain classes. This sets up the *consumer choice game*, where the manipulation of U is the strategy, and the relative or total welfare, as measured by the allocation of goods and their utilities according to U_r, are the pay-offs. Motivation of this work is [1], where Adsul et al. proved that it is possible to feign utility preferences in the consumption market to achieve a higher payoff. Under certain conditions, the market attains a Nash equilibrium when buyers strategize on consumer choices, while maximizing their payoffs.

Recently, much work has been done on computations of Fisher market and the existence of Nash equilibria in the Fisher market game [1], [2], [7], [11], [22]. There have been numerous attempts on solving the Arrow Debreu model as well [6], [13], [18], [20], [6], dealing with a few specific instances of the market. Owing to the connections of the Heterodox theory with the Fisher and AD market models, we further study its geometric structures and define optimality conditions for equilibria. We also propose a Linear Complimentarity Problem (LCP) formulation for the same.

The Heterodox model borrows from many existing models, specially so from the Arrow-Debreu model and its earlier cousin, the Fisher model, from Sraffa's accounting methods for calculating prices [8] and labour inventory using the theory of value, the marginal production principle for calculating wages, and finally the use of utility functions to compute allocation of goods. We also see that the 'Heterodox' model proposes a weak *tatonnement* process, similar to that of Walrasian auction [14].

The rest of the work is organized into 8 chapters as follows. In chapter-1, we describe the above mentioned market models, i.e., Fisher market and Arrow- Debreu market. We define the Fisher forest and discuss its two solutions for linear utilities, *viz.*, Eisenberg-Gale algorithm and a simplex like algorithm. We discuss continuity properties of the Fisher market and set up the Fisher market game. Next, we state the Arrow Debreu theorem and provide an LCP formulation for AD markets with separable piecewise linear concave (SPLC) utilities and polyhedral production sets. We look at a special case of AD markets-the exchange markets and their reduction to Fisher market. Further, we talk about the continuity of AD markets.

In Chapter 2, we describe the Heterodox Model $\mathcal{H}(\mathcal{C}, \mathcal{P})$ as composed of two interconnected systems, the consumption model $\mathcal{C}$ and the production model $\mathcal{P}$. We also define two global optimization functions $f_{\mathcal{C}}$ and $f_{\mathcal{P}}$, which couple $\mathcal{C}$ and $\mathcal{P}$. They set up the *tatonnement* as an iterative interaction between $\mathcal{C}$ and $\mathcal{P}$. We further analyse the tatonnement process and exhibit certain non-convergent trajectories.

In Chapter 3, we cast the Heterodox market $\mathcal{H}$ as an Arrow-Debreu market. We show the equivalence of solutions of the Heterodox model, i.e., its fixed points, with the equilibria of the corresponding A-D market. This proves the existence of an equilibrium in Heterodox model. Thus, it connects the two concepts and also gives an explicit description of the dependence of the A-D equilibria on the parameters of the economy.

In Chapter 4, we associate combinatorial structures (I, J, F) with equilibria and understand how they vary with the parameters of the economy, i.e., T, Y and U. We use these results to define the $\mathcal{CCG}$, the *consumer choice game*, where T, Y are fixed, and U is the strategy space. We do this over a collection of open sets and show that explicit description of the game is obtained over these open sets. We use the above results to illustrate a particular market of three labour classes and three goods, and examine the vicinity of a particular fixed point $\mathcal{H}_3$. We use the combinatorial data associated with the fixed point, *viz.*, the Fisher solution forest, to explicitly construct the consumer choice game, i.e., the use of U as strategies and the total utility as the pay-offs.

In Chapter 5, we analyse (2×2) economies and the combinatorial data arising from their fixed points. First, through a 2-player example, we show the decomposition of the strategy space, i.e., the U-space into various regions indexed by Fisher forests. This also leads to a correspondence $\mathcal{N}$ between the strategy space and the space of possible pay-offs. In other words, $\mathcal{N} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$. We show that $\mathcal{N}$ is largely a 2-dimensional manifold. We also give a few conjectures commenting on the general $(m \times n)$ markets.

In Chapter 6, we give a geometric formulation using market constraints and define its feasible points and vertices. We define 'fair' vertices which capture the Heterodox equilibria. We also look at some vertices, edges and transitions of the structure. Further, we derive a partial LCP formulation which captures the set of Heterodox equilibria, and discuss its computability using the Lemke algorithm.

In Chapter 7, we look at some applications of the theory developed here. We describe the classical theory of trade - Ricardo's theory of comparative advantage. We then consider two cases of utilities - linear and piecewise linear concave (plc) and analyse the trade outcomes of Ricardo's model and the Heterodox model. Next, we look at the basic difference between the models by analysing their optimization programs.

Finally, Chapter 8, we conclude by pointing out what was achieved, its economic significance, and possible future directions.

Chapter 1

Review of Market Models

In this chapter, we discuss the two most fundamental market models - Fisher market and the Arrow-Debrue model, which form the building blocks of the theory described later. We give their mathematical formulations and analyse some of their algorithmic and analytical solutions. We describe the Fisher market game and discuss continuity properties of both the models.

1.1 Fisher Market

The development of general equilibrium theory started with Walras (1874) modelling the economy to compute prices and investigate the existence of equilibrium and conditions for its stability and uniqueness. He proposed an iterative process called 'tatonnement' to reach an equilibrium in the economy. In this process, an auctioneer calls for a price p and the market computes demand and supply which are supposed to equal each other. If those are not, the auctioneer gives a new price based on the deficit of demand and supply, and the process terminates when an equilibrium is attained. However, convergence of this process could not be established and the existence of an equilibrium in general economies was one of the most sought after questions in Economics.

The existence of equilibrium was only proved in 1951 by Arrow and Debreu though a celebrated result, which gives a non-constructive proof of existence in a production and consumption economy. The result also demonstrated the first theorem of Welfare Economics- If preferences are locally non-satiated, and (p, X) is a price equilibrium, then the allocation X is Pareto efficient and Walras's law- Value of excess demand and supply always equals zero, whether or not the economy is in general equilibrium. Prior to Arrow and Debreu's work, Fisher had modelled the consumption part of an economy, which can be given as a special case of the Arrow Debreu market. Fisher market determines prices and allocations of goods to the buyers, based on their utility preferences. Though AD market is more general, Fisher market has its own importance due its simplicity and ease of computation. Here, we first explore the Fisher market, and then move to the AD market.

The input to a typical Fisher market is a set of buyers $\mathcal{B}$ s.t. $|\mathcal{B}| = m$, a set of goods $\mathcal{G}$ s.t. $|\mathcal{G}| = n$, a utility function $U_i(x) : \mathbb{R}^n \to \mathbb{R}$ for each buyer i, a quantity vector $q = (q_j)_{j\in\mathcal{G}}$ and a money vector $m = (m_i)_{i\in\mathcal{B}}$, where x is a set of goods consumed, q_j is the quantity of good j, and m_i is the money possessed by buyer i. Considering linear utilities, we assume that for every good j, there is a buyer i such that $u_{ij} > 0$ and for every buyer i, there is a good j such that $u_{ij} > 0$, otherwise we may discard those goods and buyers from the market. Solution of Fisher market is equilibrium prices $p = [p_j]_{j\in G}$ and allocations $X = [x_{ij}]_{i\in\mathcal{B}, j\in\mathcal{G}}$ such that they satisfy the following two constraints

Market Clearing: Allocations are such that all goods are completely sold and the money of all the buyers is exhausted, i.e.
$\forall j \in \mathcal{G}, \sum_{i\in\mathcal{B}} x_{ij} = q_j$ and $\forall i \in \mathcal{B}, \sum_{\in\mathcal{G}} p_j x_{ij} = m_i$

Optimal Goods: Each buyer buys only those goods which give her the maximum utility per unit of money i.e if $x_{ij} > 0$, then $\frac{u_{ij}}{p_j} = \max \frac{u_{ik}}{p_k}$

Example 1. *Consider a 2 buyers, 2 goods market with* $m_1 = 4, m_2 = 1, q_1 = 2, q_2 = 2$, $(u_{11}, u_{12}) = (1, 0.4)$ *and* $(u_{21}, u_{22}) = (0.5, 1)$. *Note that the utilities are linear here. The equilibrium prices of this market are* $(p_1, p_2) = (1.79, 0.71)$ *and the unique equilibrium allocation is* $(x_{11}, x_{12}, x_{21}, x_{22}) = (2, 0.6, 0, 1.4)$.

It is also possible to consider markets where we have piece-wise linear concave (plc) utilities [20]. In this case, we express the utilities with segments to indicate the points (pieces) where the utilities decrease.

Example 2. *Consider the above market with modified utilities -* $(u_{111}, u_{112}, u_{121}, u_{122})$ *=(1, 0.8, 0.4, 0.2) and* $(u_{211}, u_{212}, u_{221}, u_{222})$ *=(0.5, 0.35, 1, 0.75), where* u_{ijk} *means buyer i's utility for j'th good in the k'th segment. For each good and buyer, let the utilities decrease after 0.5 units are consumed. In this case, the equilibrium prices are* $(1.83, 0.67)$ *and the goods allocation is-* $(x_{11}, x_{12}, x_{21}, x_{22}) = (2, 0.5, 0, 1.5)$.

Fisher Forest

We note that when utilities are generic, i.e., satisfy no algebraic relation amongst themselves, the optimal solution to the Fisher market constitutes a unique weighted forest [1], which is defined as follows : Let V(H) $= G \cup B$. Let b_i be the node corresponding to the buyer i, and g_j be the node corresponding to good j. We place an edge between b_i and g_j, if $\frac{u_{ij}}{pj} = \max_k \frac{u_{ik}}{p_k}$ and call the edges of the solution graph as tight edges. Also, weight of a particular edge between nodes b_i and g_j is the amount of good j allocated to buyer i. For instance, the forest corresponding to Example 2 given above has four nodes (b_1, b_2, g_1, g_2) and three edges (e_{11}, e_{12}, e_{22}) with weights $(2, 0.5, 1.5)$ respectively i.e., the bipartite graph is a unique tree with three edges.

1.1.1 Solutions

Recently, many results on the algorithmic computability of Fisher markets have been obtained which include strongly polynomial time algorithms under various settings, primal dual type convex programs [22], a flow-network solution [17], a simplex-type algorithm [2] etc. We describe two such solutions, as we use the similar themes later.

Eisenberg-Gale Algorithm

The first Fisher market solution was given for linear utility functions by Eisenberg-Gale, via a non-linear convex program. The optimal solution of the optimization program or the primal dual variables capture the equilibrium prices and allocations. We now set up the conditions and give the program, along with the proof of existence of an equilibrium [17].
Let u_{ij} denote the utility buyer i derives after consuming one unit of good j. If the buyer i

is allocated x_{ij} amount of good $j \in \mathcal{G}$ for each j, her total utility is given by

$$\sum_j u_{ij} x_{ij}$$

Without loss of generality, we may assume that the number of goods produced is 1 in each case. Next, the important considerations for the program would be -

- Utilities for each buyer should be scalable, i.e., if a buyer multiplies each u_{ij} by a constant k, the solution should remain invariant.
- If buyer i's money is split into two buyers with same utilities, their total allocations should equal the initial allocation of buyer i.

Following is the optimization function which maximises the money weighted utilities and satisfies these conditions

$$\max \left(\prod_i u_i^{m_i} \right)^{\frac{1}{\sum_i m_i}}$$

Log of its equivalent is used in the Eisenberg Gale convex program

$$\begin{aligned} \max_X \ & m_i \log(\sum_j u_{ij} x_{ij}) \\ \text{subject to :} \ & \textstyle\sum_i x_{ij} \leq 1 \\ & x_{ij} \geq 0 \ \forall i, j \end{aligned} \tag{1.1}$$

We interpret the Lagrange variables associated with the first set of conditions as prices p_j. Karush-Kuhn-Tucker (KKT) conditions for the optimal solution of this program imply the following :

- Prices are non-negative, i.e., $\forall j : p_j \geq 0$
- If a good has positive price p_j, its supply is exhausted, i.e., $p_j > 0$ implies $\sum_{ij} x_{ij} = 1$
- For each buyer i, the total utility per total money is weakly larger than utility per buck from each good ji.e., $\frac{\sum_j u_{ij} x_{ij})}{m_i} \geq \frac{u_{ij}}{p_j}$
- If a player i buys good j, its bang per buck is maximized, i.e., $x_{ij} > 0$ implies $\frac{\sum_j u_{ij} x_{ij})}{m_i} = \frac{u_{ij}}{p_j}$.

These conditions imply that an optimal solution to the above program is indeed a market equilibrium. The KKT conditions extend to the following set of results :

Theorem 1.1.1. *For linear utility functions in the Fisher market,*

1. *If every good j has a potential buyer i, i.e., $u_{ij} > 0$, an equilibrium exists in the market.*
2. *The equilibrium utilities and prices are unique.*
3. *The set of equilibrium allocations is convex.*
4. *If u_{ij} and m_i are rational $\forall i, j$, then the equilibrium solutions are also rational.*

We now look at another formulation of Fisher market, which computes the equilibrium by traversing a simplex-like path.

Simplex-Like algorithm

Here, a new set of optimality conditions is given so that the solution, as guaranteed by the KKT conditions, lies on a vertex of the corresponding polytope defined. This optimal vertex can be reached by following a set of vertices, each time decreasing a cost function. In short, the program is explained as follows:

Let O be a convex polytope in $y - p - z$ space which refers to constraints (1.2-1.8) and O_{aux} be an auxiliary polytope in the $y - p$ space, with constraints (1.2-1.5).

$$\text{maximize :} \quad \max \sum_i m_i \log y_i$$

$$\text{s.t. : } \forall i \in B, j \in G \ : \ u_{ij} y_i \leq p_j \tag{1.2}$$

$$\sum_j p_j q_j \leq \sum_i m_i \tag{1.3}$$

$$\forall i, \quad y_i \geq 0 \tag{1.4}$$

$$\forall j, \quad p_j \geq 0 \tag{1.5}$$

$$\forall i, \quad \sum_j z_{ij} \leq m_i \tag{1.6}$$

$$\forall j, \quad \sum_i z_{ij} = p_j q_j \tag{1.7}$$

$$\forall i, j \quad , z_{ij} \geq 0 \tag{1.8}$$

Unlike the Eisenberg Gale algorithm, here prices are made as explicit variables and a new set of variables Z is introduced, which talk about the equilibrium money allocations. Also, y_i is used which refers to the inverse of the bang per buck ratio for buyer i. Now, let x_{ij}, q, μ_i, λ_j be the Lagrangian (dual) variables corresponding to the equations (1.2-1.5).
KKT Conditions for these equations are -

$$\frac{m_i}{y_i} = \sum_j u_{ij} x_{ij} - \mu_i \tag{1.9}$$

$$\forall i, j \quad (u_{ij} y_i - p_j) x_{ij} = 0 \tag{1.10}$$

$$(\sum_j p_j q_j - \sum_i m_i) q = 0 \tag{1.11}$$

$$\forall j \ , \quad -\sum_i x_{ij} - \lambda_j + q_j q = 0 \tag{1.12}$$

$$\forall i, j \qquad - \mu_i y_i = 0 \tag{1.13}$$

$$\forall i, j \qquad - p_j \lambda_j = 0 \tag{1.14}$$

$$\forall i, j \quad x_{ij}, \mu_i, \lambda_j, q \geq 0 \tag{1.15}$$

It is proved that the auxiliary polytope O_{aux} is a projection of the polytope in the whole space. If (y, p) is an optimal solution to the auxiliary program, then the KKT conditions imply that p is an equilibrium price vector. An optimal solution to this is uniquely projected back to the main polytope, which is our equilibrium solution. It is proved that this solution is a vertex in the $mn+m+n$ dimensional polytope. To compute the solution, special vertices are defined which capture the optimality conditions of utility for buying the goods. These vertices are such that each vertex is the Fisher solution with a different money endowment. It is then proved that a simplex like path exists, which reduces the cost function and surplus of each player when it starts with any special vertex and moves to another, finally reaching the optimal vertex. A strongly polynomial algorithm is given which follows one such path. For a detailed analysis, we refer to [2].

1.1.2 Properties

Continuity of Fisher markets

As we see later in the theory, continuity of Fisher markets is an important and desirable property. It is shown that in case of linear utilities, Fisher market prices are continuous with respect to initial endowments m_i and utilities u_{ij}. Moreover, the set of allocations X forms a correspondence which is upper hemicontinuous for specified U and m. A more general proof involving AD markets is given later in this chapter.

However, we note that Fisher markets are not always continuous. It is proved that for plc utilities, neither of the equilibrium prices, allocations or total utilities are upper hemicontinuous [10]. But, with spending constraints utilities, equilibrium prices are continuous and allocation and utilities are upper hemicontinuous. These results can be proved using the network-flow solution to the Fisher market [21].

The Fisher market game

We now set up the Fisher market game, where buyers strategize on utilities to achieve higher payoffs. We also list some results regarding Nash equilibria of this game.

This is a one shot non cooperative game, which has m players and the strategy set for i'th player given by the set of all possible utilities that she may pose, i.e $S_i = \{(s_{i1}, s_{i2}, ...s_{in}),\ s_{ij} \geq 0\ \forall j \in \mathcal{G}, \sum_j s_{ij} \neq 0\}$. The set of all possible strategies $\mathcal{S} = S_1 \times S_2 \times ...S_m$ constitutes a strategy matrix U. The Fisher market solution, i.e., equilibrium prices $p(U)$ and allocations $X(U)$ for an input of initial endowments are computed using U as the utility functions. However, the payoff functions are given by their true utility functions U_r and allocations $X(U)$. In other words, if player i with a true utility function U_{ri} gets allocation $X_i(U)$ in the Fisher market game, her payoff is given by $\sum_j u^r_{ij} X_{ij}(U)$. The following example gives the motivation and illustrates the definition.

Example 3. *Let us consider a 2 player Fisher market with* $m_1 = 1$, $m_2 = 1$ *and two goods with quantities* $q_1 = 1$ *and* $q_2 = 1$. *Let* U_r, *i.e., true utilities be given by* $(2, 1)$ *and* $(1, 2)$ *for player 1 and 2 respectively. The equilibrium prices for this Fisher market are* $p_1 = p_2 = 1$ *and the allocations are* $(1, 0, 0, 1)$. *Total utilities are* $(2, 2)$. *Now, the player 1 strategizes on* *her utilities as follows-* $U_1 = (0.8, 1)$*. The equilibrium prices are* $(\frac{8}{9}, \frac{10}{9})$ *and the allocations are* $(1, 1/10, 0, 9/10)$*. The total utilities are* $(2.1, 1.8)$*, where buyer-1 gets more utility after strategizing. Note that total utilities are still calculated using* U_r*, i.e., the true utilities.*

We now give a few definitions and state some of the key observations and results. A strategy profile is called a Nash equilibrium strategy profile (NESP) if no player has a strategy that gives a better payoff, when all other players have fixed strategies. A strategy is called conflict-free if there is an equilibrium allocation that gives maximum payoff to all players. A symmetric profile is one where all players play the same strategy.

- All NESPs are conflict free. Moreover, a symmetric profile is a NESP if and only if it is conflict free.
- All NESPs need not be symmetric and the payoff w.r.t. a NESP need not be Pareto optimal.
- For two player markets, all NESPs are symmetric. The set of NESPs form a piecewise linear concave (plc) curve and all these payoffs are Pareto optimal.
- For these markets, strategizing on utilities is same as differing initial endowments.
- There may exist NESPs, where social welfare (sum of all payoffs) is larger than that of the Fisher payoff.

For a detailed analysis and proofs, we refer to [1].

1.2 Arrow - Debreu Market

We see that Fisher market models only the consumption part of the economy, where it requires agents to have endowments of money and production amounts in the market. Arrow Debreu (AD) market generalizes the Fisher market where agents earn their incomes by producing and exchanging the goods. Arrow Debreu proved that under certain conditions like perfect competition, convex preferences and production etc., there must exist prices such that aggregate demand equals aggregate production [3]. The proof mainly uses Kakutani fixed point theorem and is non constructive. So far, there has been no computational way to solve for the most general case, though there are proposed solutions for simpler settings. Mainly, the existing literature focuses on the exchange economy, i.e., the Arrow Debreu market without production. Here, we describe the general model, its assumptions and solutions for some specific markets [15].

1. Let us consider a finite set of firms F, indexed $j = 1, ...l$. Let the possible production technologies be represented by a nonempty set $\mathcal{Y}_j$ where an element y in $\mathcal{Y}_j$ represents a technically possible combination of inputs (raw goods) and outputs (produced goods). We assume that $\mathcal{Y}_j$ is convex, bounded, closed and includes 0 for every j. The boundedness assumption is relaxed in the general model. We also assume that there can be no outputs without inputs and there does not exist any way to transform outputs to inputs.

2. We define S_j as the optimal supply function of firm j

$$S_j(p) = \{\ y^*|y^* \in \mathcal{Y}_j,\ p \cdot y^* \geq p \cdot y \quad \forall y \in Y_j\ \}$$

   Under the above assumptions, S_j is nonempty, convex, and upper hemicontinuous for all $p \in \mathbb{R}^n_+$, $p \neq 0$.

3. Let us consider a finite set of agents/households H, indexed $i = 1, ...m$. Let $X_i \in \mathbb{R}^n_+$ denote the set of possible consumption plans of agent i. We assume that X_i is closed, convex and unbounded above for each i.

4. The economy's initial endowment of resources is denoted as $r \in \mathbb{R}^n$. Each household has an initial endowment of goods given by r^i so that $\sum r^i = r$. Let $y \in \mathcal{Y} = \sum_j \mathcal{Y}_j$. The vector y is said to be attainable if $y + r \geq 0$.

5. Firm j's profit function is

$$\pi_j(p) = \max_{y \in Y_j}(p \cdot y) = p \cdot S_j(p)$$

   Each household i has shares in firm j's profit given by α_{ij} so that $\sum_i \alpha_{ij} = 1$.

6. Each agent is endowed with a convex preference quasi ordering $\geq_i$ on X_i. It is proved that the ordering can be represented by a continuous real-valued function u_i which is

the utility function for each i. We assume that there is always universal scarcity, i.e., for $x_i \in X_i$, there exists y_i so that agent i values it more than x_i. We also assume that the sets $A_i(x_0) = \{x | x \in X_i,\ x \geq_i x_0\}$ and $G_i(x_0) = \{x | x \in X_i,\ x_0 \geq_i x\}$ are closed.

7. Income of agent i is defined as-

$$M_i(p) = p \cdot r_i + \sum_{j \in F} \alpha_{ij} \pi_j(p)$$

   The artificially restricted convex budget set is defined as -

$$B_i(p) = \{x_i | x_i \in \mathbb{R}^n, p \cdot x_i \leq M_i(p), \quad |x| \leq c\}$$

   c is defined so that $|x| < c$ for all attainable X. Existence of such c can be proved using assumptions in 1.

8. Each agent has a convex demand set given by -

$$D_i(p) = \{x_i | x_i \in B_i(p) \cap X_i, \quad x_i \geq_i y \ \ \forall\, y \in B_i(p) \cap X_i\}$$

   $B_i(p) \cap X_i$ is continuous (lower and upper hemicontinuous), compact valued, and nonempty for all $p \in P$. Also, $D_i(p)$ is upper hemicontinuous, convex, nonempty, and compact for all $p \in P$.

9. We assume that for all i in H,

$$M_i(p) > \inf_{x \in X_i \cap \{x || x| \leq c\}} p \cdot x \ \forall\, p \in P$$

   For $x \in D_i(p)$, it can be proved that $p \cdot x \leq M_i(p)$. Moreover, if $p \cdot x < M_i(p)$, $|x| = c$.

10. The excess demand correspondence at prices p is defined as $Z(p) = D(p) - S(p) - r$. Under the assumptions given above, the weak Walrus's law states that $(p \cdot z) \leq 0$. If $(p \cdot z) < 0$, there exists $k = 1, 2, ...N$ such that $z_k > 0$. Furthermore, if $Z(p)$ is well defined, then $(p \cdot z) = 0$.

Arrow and Debreu proved that under the assumptions and specifications mentioned above, there always exists a competitive market equilibrium P. Precisely, the following four conditions should be satisfied for an equilibrium to exist. A set of vectors $(y_1, y_2, ...y_l, P_1, P_2, ...P_n, x_1, x_2, ...x_m,)$ is said to be an equilibrium if

1. y_j^* maximizes $y_j \cdot P$ over the set $\mathcal{Y}_j$, for each firm j. In other words, y_j^* belongs to $S_j(P)$ for each j.

2. x_i maximizes $u_i(x_i)$ over the set $B_i(P)$, i.e., $x_i \in D_i(P)$.

3. Prices are non negative, bounded and not all are zero. Without loss of generality, we can assume that $\sum_j p_j = 1$.

4. $Z(P) \leq 0$. Moreover, $p_k = 0$ when $Z_k(P) < 0$.

The results can be extended to a general economy with unbounded technology and demand-supply functions since the equilibrium of the restricted economy is an equilibrium in the unrestricted economy too, as shown using Walrus's law. We omit the proofs for brevity and refer to Chapters 11-18 from the book General Equilibrium Theory [15].

1.2.1 Solutions

For complete AD markets, a first polynomial time algorithm for nested CES (constant elasticity of substitution) utility and production functions was given by Codenotti et. al. [18]. It is proved that for SPLC (separable, piecewise linear concave) utility and production functions, computation of equilibrium is PPAD complete [20]. However, for a fixed number of either goods or agents, a polynomial time algorithm is proved [6]. Following is a polynomial time result for the most general case so far- for SPLC utilities and polyhedral production sets with constant number of goods [19].

Markets with production- LCP formulation

Here, we give the basic LCP formulation for AD markets with SPLC utilities and polyhedral production sets. Using Lemke algorithm, as given in Appendix A, it is then possible to solve for the equilibrium solution. Along with the AD ones, following are the additional assumptions for this setup-

- For each pair of agent i and good j, we have a non-decreasing, piece-wise linear concave utility function so that the utility i derives from a bundle of goods is additively separable (SPLC) over goods. There are $|U_i| \geq 2$ segments and the k'th segment is identified by u^i_{jk} = Slope c and l^i_{kj} = length $b-a$. Length of the last segment is infinity.

- Similar to the utility function, we assume that production is also SPLC. Every firm f produces exactly one good j_f using a set of goods as raw material. It is assumed that the initial endowment of each good is equal to one. Slope of the segments define the rate at which a good can be produced. We have slope $c = \alpha^f_{jk}$ and length $o^f_{jk} = b - a$.

We recall that agent i earns $r^i \cdot p$ from initial endowment r^i and $\sum_f \theta_{fi}\pi_f$ from profit shares in firms, where π_f is firm f's total profit. Optimal production plan of firm f is given by the following linear program, where x^f_{jk} denotes the amount of raw good j used by firm f.

$$\max_{j,k} \; x^f_{jk}(\alpha^f_{jk} p_{j_f} - p_j) \quad \text{Subject to} \quad 0 \leq x^f_{jk} \leq o^f_{jk}$$

Let β^f_{jk} denote the dual variables corresponding to the above inequalities, which capture the profit on segment (f, j, k). Letting $a^f_{jk} = x^f_{jk} p_j$ as the amount of money spent on raw goods, s^f_{jk} as the money spent for buying goods and R^f_{jk} as the total revenue of firm f, we have the following LCP formulation.

$$\forall (f,j,k): \quad \alpha^f_{jk} p_{j_f} - p_j \leq \beta^f_{jk} \quad \text{and} \quad a^f_{jk}(\alpha^f_{jk} p_{j_f} - p_j - \beta^f_{jk}) = 0 \tag{1.16}$$

$$\forall (f,j,k): \quad a^f_{jk} \leq o^f_{jk} p_j \quad \text{and} \quad \beta^f_{jk}(a^f_{jk} - o^f_{jk} p_j) = 0 \tag{1.17}$$

$$\forall (f,j,k): \quad R^f_{jk} = a^f_{jk} + o^f_{jk}\beta^f_{jk} \quad \text{and} \quad \forall\ \pi_f = \sum_{jk} o^f_{jk}\beta^f_{jk} \tag{1.18}$$

$$\forall j \in G: \; \sum_{i,k} s^i_{jk} + \sum_{f,k} a^f_{jk} \leq p_j + \sum_{f \in F(j), j', k} R^f_{jk} \quad \text{and} \quad p_j\Big(\sum_{i,k} s^i_{jk} + \sum_{f,k} a^f_{jk} - p_j - \sum_{f \in F(j), j', k} R^f_{jk}\Big) = 0 \tag{1.19}$$

$$\forall i \in H: \; \sum_{j,k} r^i_j p_j + \sum_f \theta^f_i \pi_f \leq \sum_{j,k} s^f_{jk} \quad \text{and} \quad \lambda_i\Big(\sum_{j,k} r^i_j p_j + \sum_f \theta^f_i \pi_f - \sum_{j,k} s^f_{jk}\Big) = 0 \tag{1.20}$$

$$\forall (i,j,k): \quad u^f_{jk}\lambda_i \leq p_j + \gamma^i_{jk} \quad \text{and} \quad s^i_{jk}(u^f_{jk}\lambda_i - p_j - \gamma^i_{jk}) = 0 \tag{1.21}$$

$$\forall (i,j,k): \quad s^i_{jk} \leq l^i_{jk} p_j \quad \text{and} \quad \gamma_i(s^i_{jk} - l^i_{jk} p_j) = 0 \tag{1.22}$$

We note that (1.19) and (1.20) are the market clearing constraints with $F(j)$ being the set of firms producing good j. The last two equations (1.21) and (1.22) characterize the optimal bundle for each player i. Given prices p, earnings of each agent is fixed. In case of splc utilities, optimal bundle for each buyer can be computed as follows: using the prices, bang per buck of every good is computed and all segments are sorted by decreasing bang per buck. Equivalence classes are formed and the buyer starts buying partitions in order, till she exhausts her money in say k'th partition. Class k is called flexible and λ_i is the inverse of bang per buck for that segment. Classes $1, ..k-1$ are called forced and γ^f_{jk} is introduced for those so that for segment (i, j, k), $1/\lambda_i = u^i_{jk}/(p_j + \gamma^i_{jk})$.

It can be proved that any equilibrium of market is a solution to the LCP so described. This is done by checking that the equilibrium variables and other defined quantities satisfy the above constraints. However, we note that the above formulation has two disadvantages, which are degeneracy and existence of solutions which are not equilibria. The Lemke algorithm which provides solutions for LCP does not account for degenerate formulations. Thus we modify the above equations by replacing p_j with $p^{'}_j + c_j$ so that the equations become non-homogeneous. It is then proved that Lemke's algorithm can be used to compute solutions, where it always attains an equilibrium. (Appendix A)

1.2.2 Properties

Exchange markets and Reduction to Fisher market

The most studied version of the AD markets is the exchange economy, i.e., markets without production. Exchange economy is similar to the Fisher market, except the buyers have initial endowments of goods instead of money. The economy achieves an equilibrium by deriving prices so that each buyer gets an optimal bundle of goods through exchange. Each buyer/agent i has endowment $(e_{i1}, e_{i2}, ...e_{in})$ of n goods and utility preferences $U_i(x_{i1}, x_{i2}...x_{in}) : \mathbb{R}^n \rightarrow \mathbb{R}$. No new goods are produced in the economy and hence, the total supply of goods is $(\sum_i e_{i1}, \sum_i e_{i2}, ... \sum_i e_{in})$. A solution to this market is prices p and allocations X such that (i) all goods are completely exhausted (ii) all agents maximize their payoffs, i.e., bang per buck conditions are satisfied. Thus, through this market, agents exchange their goods to determine optimal prices and allocations, thereby maximizing their own payoff. We now see how Fisher market can be modelled as a special case of these.

Let us consider a Fisher market $[m]$ as the set of agents and $[n]$ as the set of goods. Let $U_i : \mathbb{R}^n \to \mathbb{R}$ be the utility function for each agent i and m_i be the initial money endowment. Let q_j be the amount of good j available and $t = \sum_i m_i$ be the total money in the market. To see this as an exchange AD market, we add an agent $i = m + 1$ and a good $j = n + 1$ which is money so that there are $m+1$ agents and $n+1$ goods. In line with the new market, agent $m + 1$ has initial endowment $(q_1, q_2, ...q_n, 0)$, i.e., all goods and zero amount of money. Similarly, agent i's endowment is given by $(0, 0, ..0, m_i)$. For utility functions $U_i^{'}$, we see that for $i = 1, ..m$, utility for money is zero i.e $U_i^{'}(x_{i1}, x_{i2}...x_{in}, e_i) = U_i(x_{i1}, x_{i2}...x_{in})$ for any money allocation e_i. On the other hand, agent $m+1$ has nonzero utility only towards money, i.e., $U_{m+1}(x_{i1}, x_{i2}...x_{in}, x_{m+1}) = x_{m+1}$. This completes the specification of AD economy as an instance of the Fisher market.

It is easy to see that the solution to above market is a Fisher solution too. This is because after the market clears, agent $m+1$ gets all the money as she is the only agent with positive utility towards it. Similarly, all goods are exhausted as agent $m + 1$ has zero utility towards those. Moreover, each buyer maximizes the payoff given the consumption constraints.

There are multiple ways to find an equilibrium in the exchange economy. For a linear model of exchange, a finite time algorithm based on Lemke's scheme, a polynomial time algorithm using ellipsoid method and interior point method as well as a combinatorial type algorithm is given. For the general model, some combinatorial type algorithms are known, but no strongly polynomial time algorithm is known so far.

Continuity of AD markets

For AD markets without production, it is proved that equilibrium allocations and total utilities for the linear utilities case are upper hemicontinuous with respect to the utilities. However, equilibrium prices are not upper hemicontinuous. We give a short proof of the first result here [10].

We first consider the convex program given by Jain [9] which captures equilibrium allocations and utilities. Without loss of generality, we assume that each agent has an endowment of a unique good. Let G be the directed graph formed by the set of n agents as nodes and edges e_{ij} only if $u_{ij} > 0$. We define $w(i,j) = \frac{\sum_{0 \le t \le n} u_{it} x_{it}}{u_{ij}}$. Now, consider the convex program.

$$\begin{array}{c} \forall\ i,j\ :\ x_{ij} \ge 0 \\ \forall\ j:\ \sum_i x_{ij} = 1 \\ \text{For every cycle C in } G\ :\ \prod_{(i,j)\in C} w(i,j) \ge 1 \end{array} \tag{1.23}$$

It is proved that an allocation x is an equilibrium allocation if an only if it is a feasible point of this convex program. Using this result, we now prove that a small change in the utilities leads to a small enough change in allocations and total utilities.

Theorem 1.2.1. *In an exchange economy of the AD market with linear utilities, equilibrium allocation and utility are upper hemicontinuous.*

Proof. Let $\{U_k\}$ be a sequence of utilities such that $U_k \to U_0$. Let $x_k \in X(U_k)$ be a sequence of allocations such that $x_k \to x_0$. According to the definition of upper hemicontinuity, we have to prove that $x_0 \in X(U_0)$.

As stated earlier, x_k should be a feasible point in the convex program given above, for every k. Let G_k be the graph formed by all the edges where $u_{ij}^k > 0$. We now have that (1.23) holds for each k.

Now consider graph G_0. As $U_k \to U_0$, for large enough k, we have $u_{ij}^k > 0$ for every i,j such that $u_{ij}^0 > 0$. Therefore, if an edge (i,j) is present in G_0, then it is present in G_k as well. This implies that for large enough k, if G_0 has cycle C, G_k also has it. This ensures that the third condition above is satisfied. As $k \to \infty$, we see that the other two conditions are satisfied as well. Thus, we prove that $x_0 \in X(U_0)$, i.e., allocation is upper hemicontinuous. As the total utility is a linear function of X, we see that both equilibrium allocation and utility are upper hemicontinuous. □

As described earlier, this result also implies the continuity of Fisher markets.

Chapter 2

Heterodox Model Theory

In this chapter, we describe the Heterodox Model $\mathcal{H}(\mathcal{C}, \mathcal{P})$ as composed of two interconnected systems, the consumption model $\mathcal{C}$ and the production model $\mathcal{P}$, which are coupled through two optimization functions $f_{\mathcal{C}}$ and $f_{\mathcal{P}}$. We also set up the *tatonnement* as an iterative interaction between $\mathcal{C}$ and $\mathcal{P}$, and analyse some of its properties.

2.1 The Basic Notation and Assumptions

- A good can be both, a fixed unit of service or an output of a manufacturing plant, made available in a fixed time interval, called *epoch*, for example, a year. The set of goods is denoted by $\mathcal{G} = \{G_1, \ldots, G_m\}$. Also, q is a $(m \times 1)$ column vector with q_j is the amount of good G_j manufactured and p is an $1 \times m$ row vector with p_j being the price of each good G_j,

- The whole population in the economy consists of agents divided into distinct classes according to their training. Thus, let $\mathcal{L} = \{L_1, \ldots, L_k\}$ be these classes of labour, with each class L_i willing to devote Y_i units (e.g., person-years) of labour in every epoch. Thus, Y forms a $k \times 1$ vector.

- Each (manufactured) good G_i has exactly one production process or technology T_i and each T_i is a linear map $T_i : \mathcal{L} \to \mathbb{R}$. These tell us the amount of each labour-class required to produce each unit of good G_i. The class of all such technologies is denoted by $\mathcal{T}$, which is represented as a $k \times m$-matrix T with column T_j. Thus, in matrix T, T_{ij} is the the amount of labour L_i needed to produce G_i.

- For each labour class L_i , we have *utility* u_{ij} for each good G_j. We assume that utilities are same for all persons in one labour class and are measured in happiness per person per kilo units. For example, u_{ij} is the happiness derived by a person from class L_i by consuming a unit of good G_j. We denote the matrix formed by u_{ij} entries as U.
- We denote w_i as the wages received by a person from class L_i in one *epoch*. Thus, w_i is measured in Rs per person per *epoch* units.

Further, without loss of generality, we assume that for each labour class, there are technologies which utilize them. We also assume that the entries of T, Y and U are all in general position and satisfy no algebraic relation amongst themselves.

Remark 2.1.1. *The construction of the Technology matrix T is according to the Sraffa's theory of value, which argues that value of any good is ultimately equal to value of the labour gone into making it [8].*

2.2 Theory

2.2.1 The consumption space C

Consumption in our economy is modelled as a Fisher market. Recall that, in a Fisher market, there are k buyers ($\mathcal{L}$, as in our case), and m goods ($\mathcal{G}$). Each agent L_i is endowed with money m_i, and each good G_j has quantity q_j for sale.

Solution of Fisher market is equilibrium prices $p = [p_j]_{j \in G}$ and allocations $X = [x_{ij}]_{i \in B, j \in G}$ such that they satisfy the following two constraints

Market Clearing: Allocations are such that all goods are completely sold and the money of all the buyers is exhausted, i.e.
$\forall j \in G$, $\sum_{i \in \mathcal{L}} x_{ij} = q_j$ and $\forall i \in \mathcal{L}$, $\sum_{\in G} p_j x_{ij} = m_i$

The consumption space is defined as: $\mathcal{C}(m, q) = \{X, p | X, p$ satisfy market clearance$\}$. Next, on the set $\mathcal{C}(m, q)$ we define a solution to the Fisher market as one which satisfies:

Optimal Goods: Each buyer buys only those goods which give her the maximum utility per unit of money i.e if $x_{ij} > 0$, then $\frac{u_{ij}}{p_j} = \max \frac{u_{ik}}{p_k}$

2.2.2 The production space P

The production space $\mathcal{P}(p)$ is the collection of all wages w and quantities q such that:

$$Tq \leq Y \tag{2.1}$$

$$q_j \cdot (p_j - (wT)_j) \geq 0 \tag{2.2}$$

$$q \geq 0 \tag{2.3}$$

Note that the first condition states that the quantities of goods produced are limited by labour constraints, while the second says that unprofitable goods are not produced.

The global maximization function $f_{\mathcal{P}}$ is defined as $\sum_j p_j q_j$, i.e, revenue maximization. However, we must specify how wages w get decided. For this, we consider the following relaxation LP program:

$$\begin{array}{lrcll} \max\limits_{q} & p^T q & & & \\ \text{s.t.} & Tq & \leq & Y & \\ & q_j & \geq & 0 & \forall j \in G \end{array} \tag{2.4}$$

To find the wages, we consider its dual program -

$$\begin{array}{ll} \min\limits_{\lambda_1} & \lambda_1^T Y \\ \text{s.t.} & \lambda_1^T T \geq p^T \end{array} \tag{2.5}$$

Or

$$\begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} T \\ -I \end{bmatrix} = \begin{bmatrix} p \end{bmatrix}$$

where λ_1 and λ_2 correspond to the dual variables associated with the first and second inequalities respectively. Here, λ_1 and λ_2 are $1 \times m$ and $1 \times n$ vectors respectively.
We take $w = \lambda_1$ as the wages determined by $f_{\mathcal{P}}$. Karush-Kuhn-Tucker (KKT) conditions for the primal and dual program have the following implications-

1. If $q_i > 0$, then $\lambda_{2i} = 0$.

Therefore, it follows from (2.5) that $\lambda_{1i}T_i = p_i$, i.e., $wT_i = p_i$.

2. If $\lambda_{1i}T_i \neq p_i$, i.e., $wT_i \neq p_i$, then $q_i = 0$ as $\lambda_{2i} \neq 0$.

3. If $(Tq)_i < Y$, then $\lambda_{1i} = 0$, i.e., corresponding w_i is zero.

4. Similarly, if $w_i > 0$, then $(Tq)_i = Y_i$.

In particular, we see that the inequality in the dual program, *viz.* $w^T T \geq p^T$, is opposite of the required constraint. However, in accordance with (6.2), complementary slackness implies $q_j(p_j - (wT)_j) = 0$. The conditions reduce T to a matrix $T^{'}$ where all constraints are satisfied and tight for the active goods and classes. By 'active' goods and classes, we refer to the variables which are positive. Let the corresponding prices, production and wages be p', q', w'. We can find wages as the dual variables λ_1. We see that the tight equation $p' = w'T^{'}$ can also be derived through the marginal law of production as follows.

We see that the dual variables w actually correspond to the wages given by the marginal law of production. We determine the wages w in terms of p. For w_i, we find the marginal product valued at the prices p. In other words, let Δ_i be the extra production obtained by an additional unit of labour of type L_i. According to the marginal law of production, the wages w_i may then be equated to Δ_i valued at price p. Let e_i be the $|\mathcal{L}|$-sized vector with 1 for the i-th place and zero everywhere else. Clearly, $T(q_J + \Delta_i) = Y + e_i$, gives us $w_i \cdot T(e_i) = p$. Thus, each w_i is a specific linear combination of elements of p_J.

We also see that total money is conserved in the economy, i.e., $pq = wTq = wY$ holds true. This sets up a remarkable result which connects wages and production amounts as the dual variables of each other.

2.3 Tatonnement Process

We now set up the *tatonnement* process, in accordance with the Theory of General Equilibrium by Walrus [14]. In this process, an auctioneer calls for prices p and accordingly demand and supply states are computed. Based on these, the auctioneer increases or decreases the prices so as to attain the new states. The process terminates when an equilibrium is achieved, i.e., when demand equals supply. Similarly, in Heterodox settings, the basic objective of the tatonnement process is to arrive at an equilibrium $\eta = (p, q, w, X)$ such that (i) p, X are the outputs of the consumption side Fisher market if inputs are the money vector $w \cdot Y$, and quantities q, and (ii) q, w are the optimal solutions on the production side on input p. The process begins with a candidate η and checks first if η is indeed an equilibrium. If not, it updates alternately, the consumption side and the production side.

The detailed description of the iterator function is given below.

1. Input $\eta_0 = (p_0, q_0, w_0, X_0)$. Put $n = 0$.
2. We first check if the state (p_n, q_n, w_n, X_n) is an equilibrium. This is done by first checking if (q_n, w_n) is an optimal solution to $f_{\mathcal{P}}$ in the process $\mathcal{P}$ with input p_n. Next, we check if (p_n, X_n) satisfy the optimality conditions for the function $f_{\mathcal{C}}$ with inputs (W_n, q_n), where $W_n = w_n \cdot Y$, the total wages. If it does indeed satisfy both conditions, we declare the point η_n as a ***Heterodox equilibrium***.
3. If η_n is not in equilibrium, we follow the iterative steps below.
4. Using p_n, we first compute (q_{n+1}, w_{n+1}) by optimizing $f_{\mathcal{P}}$. This is the n-th production-side update.
5. We next find (p_{n+1}, X_{n+1}) through the process $f_{\mathcal{C}}$ using the input (q_{n+1}, W_{n+1}).
6. Note that p_{n+1} does not set prices for goods not produced. These are set, assuming that a small ϵ is indeed produced and predicting its price. Thus if $b_1, \ldots, b_k$ are the maximum bang per buck values for the players, then

   $$p_j = \max_i \frac{u_{ij}}{b_i}$$

   This tells us that when these Fisher-like prices are offered, for (at least) one player, the maximum bang per buck ratio equals the ratio these prices give, making the player buy the good. The computation of p as before and its modification is called $\mathcal{C}(n)$, i.e., the n-th consumption-side update.
7. This completes the definition of η_{n+1}. We go back to Step 2.

Remark 2.3.1. *The Tatonnement process so defined is in general not continuous as it involves optimising over two polytopes.*

We now illustrate two examples of equilibria obtained through the above iterative process.

Example 4. *Let us consider a 3 classes - 3 goods market with following specifications for technology, utility and labour availability.*

$$T = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 0.5 & 2.5 & 2 \end{bmatrix}; U = \begin{bmatrix} 1.5 & 0.41 & 0 \\ 0.58 & 1.1 & 0.2 \\ 0.5 & 1.4 & 0.6 \end{bmatrix}; Y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

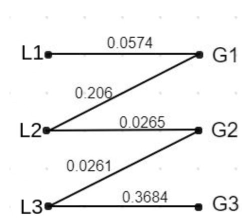
Starting with a random price vector $[0.7379, \quad 0.9379, \quad 0.3617]$*, the tatonnement process converges to an equilibrium point in 3 iterations, with the prices, production and wages in each iteration given by-*

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 2.0408 & 3.8706 & 0.7037 \\ 1.5961 & 3.027 & 1.2973 \\ 1.5099 & 2.8636 & 1.2273 \\ 1.5099 & 2.8636 & 1.2273 \end{bmatrix};$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0.2632 & 0.0526 & 0.3684 \\ 0 & 0.25 & 0.1875 \\ 0.2631 & 0.0526 & 0.3684 \\ 0.2631 & 0.0526 & 0.3684 \end{bmatrix};$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0.31636 & 0.52007 & 0.16357 \\ 0 & 0.68 & 0.32 \\ 0.086693 & 0.3865 & 0.52681 \\ 0.086693 & 0.3865 & 0.52681 \end{bmatrix}$$

And the allocations are given by the forest-



Example 5. *Let us now consider a* (2×2) *market with the following market specifications -*

$$T = \begin{bmatrix} 0.25 & 0 \\ 0.25 & 1 \end{bmatrix}; U = \begin{bmatrix} 1 & 0.81 \\ 1.234 & 1 \end{bmatrix}; Y = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

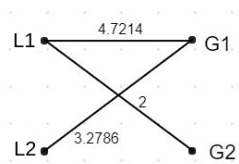
The tatonnement process converges to the following output of prices, production and wages, when it starts with $p_0 = [0.14196, \quad 0.75972]$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 8 & 2 \\ 8 & 2 \end{bmatrix}$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0.3085 & 0.25 \\ 0.10394 & 0.084232 \\ 0.10394 & 0.084232 \end{bmatrix}$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.66307 & 0.33693 \\ 0.66307 & 0.33693 \end{bmatrix}$$

The solution forest and the allocations are -



2.3.1 Analysis of the Tatonnement process

The *tatonnement* process does not always converge and it may alternate between two or more states. In each state, a different production set and/or set of active classes is chosen, though there may be overlaps of multiple active goods/classes. This happens when production and consumption do not agree on a common set of active goods and classes, but cyclically choose two or more states. We illustrate with two such examples.

Example 6. *Consider a 2 classes - 3 goods market with the following specifications.*

$$T = \begin{bmatrix} 0.62 & 0.75 & 0.49 \\ 0.36 & 0.41 & 0.69 \end{bmatrix}; \; U = \begin{bmatrix} 0.47 & 0.27 & 0.5 \\ 0.56 & 0.75 & 0.65 \end{bmatrix}; \; Y = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Here, both classes contribute in manufacturing the goods and have different utilities for those. The solution consists of two toggling states, each producing two goods. In both states, all *classes remain active. The production, prices and wages for these two states are-*

$$Prod = \begin{bmatrix} 0 & 6.29 & 10.65 \\ 8.01 & 0 & 10.2 \end{bmatrix}; \; Prices = \begin{bmatrix} 0.06 & 0.05 & 0.06 \\ 0.05 & 0.07 & 0.06 \end{bmatrix}; \; Wages = \begin{bmatrix} 0.68 & 0.32 \\ 0.64 & 0.36 \end{bmatrix}$$

Where first row represents first state and vice versa. For instance, in state 1, class-1 receives 0.68 and class-2 receives 0.32 wages. The total money in the economy is kept 1 and the values are reported to 2 significant digits. The results can be interpreted by computing the ensemble average of the system. It should be noted that the most produced goods result from their demand as well as their efficient production as compared to other goods. This is a consequence of the coupling of Utility and Technology inputs.

Example 7. *Let us now consider a 3 classes - 3 goods market with following specifications.*

$$T = \begin{bmatrix} 0.05 & 1 & 0.9 \\ 0.5 & 0.8 & 0.15 \\ 0.4 & 0.5 & 0.4 \end{bmatrix}; \; U = \begin{bmatrix} 0.2 & 0.3 & 0.8 \\ 0.9 & 0.2 & 0.4 \\ 0.25 & 0.85 & 0.33 \end{bmatrix}; \; Y = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

The production, prices and wages, as computed by the function described before, are-

$$Prod = \begin{bmatrix} 4.35 & 9.78 & 0 \\ 14.7 & 0 & 10.3 \end{bmatrix}; Prices = \begin{bmatrix} 0.11 & 0.05 & 0.14 \\ 0.04 & 0.12 & 0.05 \end{bmatrix}; Wages = \begin{bmatrix} 0.52 & 0.48 & 0 \\ 0.12 & 0 & 0.88 \end{bmatrix}$$

The above two states toggle cyclically. In this case it is not possible for a state to exist where all goods and classes are simultaneously active. The solution involves two states, where class-1 and good-1 are active in both of them and other goods and classes alternate between the two. In general, we see that $f_{\mathcal{P}}$ implies the following necessary condition for an equilibrium to exist with a given set of active goods and classes. For all goods (j) that are not active in the economy, we have

$$p_j = \max_i \frac{u_{ij}}{b_i} \text{ (U- Level)} < \sum_i T_{ij} w_i \text{ (T- Level)}$$

As we see earlier, if a good is not produced, it is allotted a Fisher like price, which is the U-level defined above. All produced goods have their prices greater than or equal to their U-levels. It is clear that if U-Level for an unproduced good j is more that its T-Level, i.e., $p_j > (w \cdot T)_j$, then it being profitable, that good becomes active in the next iteration, by perhaps pushing a less efficient good out of production.

As seen from the examples, this method does work similar to the *tatonnement* process given in Walrus' theory of general equilibrium [14]. It too starts with a price vector, computes production and wages and gives a next set of prices based on these market variables. It is clear that the process terminates if and only if it attains an equilibrium. From the above example, we see that the process may not always converge, and there may be toggling states. Moreover, it can be observed that equilibria whose Fisher forests are disconnected are unlikely to arise from the above process, even though they are fixed points. We thus make a distinction between a heterodox equilibrium or a fixed point and a limit point of the *tatonnement* process. We now make the following observation.

Theorem 2.3.1. *For a generic economy T, U, Y, the tatonnement process results in symmetric states.*

Proof. Let p be a generic price vector to start with. Using p, we find the optimal production and wages as linear combinations of the prices, thereby determining the active goods and classes. We recall the dual program of the optimization function $f_{\mathcal{P}}$-

$$\begin{array}{llllllllll} \max\limits_{q} & p^T q & & & & \min\limits_{w} & w^T Y & & & \\ \text{s.t.} & Tq & \leq & Y & & \text{s.t.} & wT & \geq & p & \\ & q_j & \geq & 0 & \forall j \in G & & w_i & \geq & 0 & \forall i \in B \end{array} \tag{2.6}$$

Using KKT conditions for the primal and dual program, we get-

1. If $q_i > 0$, then $wT_i = p_i$. If $wT_i \neq p_i$, then $q_i = 0$.
2. If $(Tq)_i < Y$, then corresponding w_i is zero.

These observations imply that that for all q_i that are positive, the same number of corresponding wages are positive, when p is generic. Thus, the conditions reduce T to a square matrix T' where all constraints are satisfied and tight for the active goods and classes. This means that the *tatonnement* results are always symmetric. □

Chapter 3

Existence of A Heterodox Equilibrium

In this chapter, we prove that the Heterodox model so described indeed has an equilibrium. We use the celebrated result of existence of equilibrium in A-D markets by Arrow and Debreu [3]. We construct an Arrow-Debreu (A-D) market $\bar{AD}$ from the heterodox model $\bar{M}(T, U, Y)$, and show the equivalence of the equilibrium points in the Heterodox model and the market equilibria in the A-D sense. We refer to the notation set up in Chapter 1.

3.1 Heterodox Model as An A-D instance

We shall now build a suitable A-D market, given the data T, U, Y for $\bar{M}$. We assume that T and U are $m \times n$, i.e., there are m labour classes and n processed goods, and Y is the $m \times 1$ vector of labour class size. Recall that, T_{ij} refers to the number of labour-units of type i required to produce one unit of good j. The labour availability is given as a vector Y. We now construct $\bar{AD}$ as follows.

- The set of firms in $\bar{AD}$ is $F = \{f_1, \ldots, f_n\}$. The total number of goods are $n + m$, viz., $\{g_1, \ldots, g_n, r_1, \ldots, r_m\}$, where r_i corresponds to the labour of class i. The number of agents is m, and each agent A_i begins with an endowment Y_i of good r_i above. We call labour inputs as 'raw' goods, i.e., the labour per hours of the producers are the raw goods that are needed to produce m goods.

- The supply function of f_j is $S(\bar{p})$ which arises from the column j of T. Define v_j as the $(m+n)$ vector $g_j - \sum_k T_{kj} r_k$ to represent that T_{kj} units of labour type k are used to make one unit of good g_j. Then $\mathcal{Y} = \{\lambda \cdot v_j | \lambda \in [0, L]\}$ where L is a large number. Thus firm f_j produces some multiple of v_j. As the production functions are linear, they satisfy all the assumptions specified by the A-D model.

- Agent A_i owns a fraction α_{ij} of the firm f_j. The exact numbers will be irrelevant since we will see that in equilibrium, the firms make zero profits. Hence, $M_i(\bar{p})$ is defined as $\bar{p} \cdot r_i = \bar{w}_i \times 1$(labour input of one agent)$= \bar{w}_i$, as production is decided so that all labour units are consumed.

- X_i is the set of consumption plans and is unbounded for every j. It can be easily seen that the sets $A_i(x_0)$ and $G_i(x_0)$ are closed for each $x_0 \in X_i$.

- The utility matrix serves to define the continuous real valued utility function u_i for each agent i. If $X_{ij} \geq 0$ is the amount of good j allocated to agent i, then $u_i = \sum_j u_{ij} X_{ij}$. Utilities are zero for labour units hours, i.e., $U_{ij} = 0$ for $j > n$, as it is only the firms which have any use for labour. Since the utilities are linear, it is clear that the principle of non-satiation holds.

- Budget set $B_i(\bar{p})$ and demand set $D_i(\bar{p})$ are defined appropriately so that each buyer buys according to their preferences and exhausts all the money available. $Z(\bar{p}) = D(\bar{p}) - S(\bar{p}) - r$ is defined as a difference of demand and supply.

This completes the specification of the A-D market $\bar{AD}$. An A-D equilibrium of the $\bar{AD}$ are prices $p_1, \ldots, p_n, p_{n+1} = w_1, p_{n+m} = w_m$, production $q_j \cdot v_j \in \mathcal{Y}_j$ and allocations X_{ij} such that (i) each firm maximizes profits $p_j q_j \cdot v_j$ under the given global labour constraints, (ii) each agent A_i maximizes her utility under the expenditure constraint of the wages received from endowments Y_i priced at wages w_i and (iii) $Z(P) \leq 0$. Moreover, $p_k = 0$ when $Z_k(P) < 0$, i.e., demand meets supply for each good only when the corresponding price is nonzero.

3.2 Heterodox Equilibrium as A Fixed Point in AD

For the market $\bar{M}(T, U, Y)$, we denote an Heterodox equilibrium as $(\bar{p}, \bar{q}, \bar{w}, \bar{X})$. Let us assume that in the $(m \times n)$ market, in the above equilibrium, it is the first m' labour classes

and the first n' goods which are active. We let (p, q, w) denote the prices, production and wages of active goods and labour, i.e., $\bar{p} = (p, p_M)$, $\bar{q} = (q, \bar{0})$ and $\bar{w} = (w, \bar{0})$. As defined in the Heterodox model, p_M refers to the modified prices of the unproduced goods.
By the feasibility of the equilibrium and the activity conditions, we have,

$$T\bar{q} \leq Y$$

$$\bar{w}T \geq \bar{p}$$

and

$$T^{'}q = Y^{'}$$

$$wT^{'} = p$$

where $T^{'}$ is the reduced technology matrix and $Y^{'}$ is the reduced Y vector in accordance with the active goods and classes. Let $\bar{Y} = (Y^{'}, \bar{0})$ be the amount of labour used. The variables are such that the production is optimal given p and prices are solutions to Fisher market. Also, by the price-setting mechanism of unproduced goods $p_{n'+1}, ...p_n$, and the choice of $\bar{w}$ as the dual variables, we have

$$p_j = \max_i \frac{u_{ij}}{b_i} \text{ (U- Level)} < \sum_i T_{ij} w_i \text{ (T- Level)}$$

where b_i refers to the bang per buck ratio of i's agent.
We note that the definition of $\mathcal{Y}_i$ includes all possible plans, even if those are unattainable. Here, the technology is linear for each good and thus, unbounded. We have artificially restricted it by considering an unattainable plan L which consumes more labour units than available, i.e., Y.

Proposition 3.2.1. *Let $\bar{M}(T, U, Y)$ be the market with a Heterodox equilibrium point $(\bar{p}, \bar{q}, \bar{w}, \bar{X})$ as described above. Let $P = (\bar{p}, \bar{w})$ and $Q = (\bar{q}, \bar{Y})$. Then, $(P, Q, \bar{X})$ is an equilibrium in the A-D market.*

Proof. By definition of A-D equilibrium, we need to prove that $y_j = q_j \cdot v_j \in S_i(P)$ for all firms, $\bar{X}_i \in D_i(p)$ for all labour classes and $Z(P) \leq 0$ and $P_k = 0$ for $Z_k(P) < 0$. For the last condition, as per the definition of $\bar{p}$, we want $Z_k \leq 0$ for $k = n' + 1, ..n$ and $Z_k = 0$ for $k = 1, 2, ..n'$. Similarly, by the definition of $\bar{w}$, $Z_k \leq 0$ for $k = n + m' + 1, ..n + m$ and $Z_k = 0$ for $k = n+1, ..n+m'$. This translates to saying that all 'produced' and 'raw' goods satisfying

$T'q = Y'$ should be exhausted and all 'raw' goods satisfying $T'q < Y'$ should receive zero wages.

- Let us first consider the active goods and classes, i.e., $k = 1, 2, ..n'$ and $k = n+1, ..n+m'$. We prove that each firm maximizes the profit, given the global constraints. For firm i, let $v_i^* = (q_i, -l_i)$ maximize the profit, given that it lies in the set of production possible technologies i.e $v^* \in S_i(p)$. Here, l_i is a vector of the amount of labour units consumed in making q_i amount of good i. For all i, v_i^* should satisfy

$$\begin{aligned} \max_{y} \quad & p_i q_i - w \cdot l_i = p_i q_i - (w \cdot T'_{*,i}) q_i \\ \text{s.t.} \quad & q_i, l_i \geq 0, \end{aligned} \tag{3.1}$$

  Using the Technology matrix, here we have l_i given by $q_i(T'_{*,i})$.
  We note that since $p = wT'$, the expression $p_i q_i - w \cdot l_i$ equals $p_i q_i - (w \cdot T'_{*,i}) q_i = 0$. This means that whenever $(q_i, -l_i) = (q_i, -q_i(T'_{*,i}))$, firm i gives an optimal production value, irrespective of q_i.

- We now consider $k > n'$. Since there is no production ($q_k = 0$) and consumption of labour, $p_k - (w \cdot T_{*,k}) < 0$ as given by the duality of q and w in the Heterodox model. Hence, the maximum occurs at $q_k = 0$, $l_k = 0$. Therefore, $S(\bar{p}) = \sum_i S_i(p)$ is a multivalued function and $(\bar{q}, -T\bar{q})$ given $T' q = Y'$ is an optimal point belonging to $S_i(\bar{p})$. This establishes that all firms maximize their profits.

- Next, we prove that each agent i finds an optimal consumption set $\bar{X}_i$ by maximizing her utility under the expenditure constraint . The optimization program given below exactly conveys this requirement. We define $D_i(\bar{p})$ for each agent i as a collection of x_i that satisfies-

$$\begin{aligned} &\max_{x_i} \sum_j x_{ij} u_{ij} \\ &\text{s.t.} \;\; \bar{p} \cdot x_i \leq M_i(\bar{p}) = w_i, \\ &0 < x_i \;\in\; X_i \end{aligned}$$

  Here, the first constraint requires X to belong to $B_i(p)$. KKT conditions for this program imply that the optimal point satisfies $(u_{i1}, u_{i2}, ...u_{in}) = \mu(p_1 - \lambda_1, p_2 - \lambda_2, ..., p_n - \lambda_n)$, where μ and λ_j are the Lagrange multipliers associated with constraint 1 and 2 respectively. This means that whenever x_{ij} is positive, $\lambda_j = 0$, and $p_j = \mu u_{ij}$. In other words, whenever agent i buys goods $j1, j2$, we have $\frac{u_{ij1}}{u_{ij2}} = \frac{p_{j1}}{p_{j2}}$, which is a Fisher condition.

Since the utility function is convex, we observe that the Heterodox output for allocations, i.e., $\bar{X}$ maximizes the above function, as given by the sufficiency of KKT. Moreover, the Heterodox output is such that all goods are completely exhausted, thereby making $\sum_i D_i(\bar{p})$ which is $\sum_i x_i$ equal to $D(\bar{p}) = (\bar{q}, 0)$. As the utility for raw goods is zero, $D_i(p)$ of those equals zero for each agent i.

Thus, we prove that all agents maximize their profits and $D_j(\bar{p}) = S_j(\bar{p})$ for the reduced set of producible goods , i.e., $Z_j(\bar{p}) = 0$ for $j = 1, 2, ..n$.

- We see that $T'q = Y'$ also implies that the raw 'used' goods are exhausted completely, i.e., $Z_j(p) = D_j(p) - S_j(p) - r_j = 0 - (-Y_j^{'}) - Y_j^{'} = 0$ for $j = n+1, ..n+m'$. On the other hand, $(Tq)_j \leq Y_j$ for raw 'unused' goods, i.e., $Z_j(p) = 0 - \sum_j T_{ij} q_j - Y_k \leq 0$ for $j = n+m'+1, ..n+m$. Hence, $Z_j(p) \leq 0$ for all raw goods, i.e., labour classes. This establishes that for raw goods $k = n+1, ...n+m :\ Z(P) \leq 0$ and, $p_k = 0$ when $Z_k(P) < 0$.

- Therefore, $Z_k(p) = S_k(p) - D_k(p) - r_k =\leq$ for all k and $Z_k(p) = 0$ for $p_k > 0$. This establishes that $(P, Q, \bar{X})$ so defined using the Heterodox model is an equilibrium point in the A-D market.

□

3.3 A-D Equilibrium as A Fixed Point In Heterodox Setup

Now, let $(P, Q, \bar{X})$ be an equilibrium point in the A-D market. We assume that m' classes and n' goods are active. Let $\bar{p}, \bar{w}$ be the corresponding prices and wages. We have $P = (\bar{p}, \bar{w})$ as the price vector so that $Z(P) \leq 0$, with $P_k = 0$ for $Z_k(P) < 0$. Also, the optimal production vector for each firm is $y_j = (q_j, l_j)$ so that the total output of firms is $Q = (\sum_j q_j = \bar{q}, \sum_j l_j = Y^{'}, \bar{0})$. Let $\bar{X}_{ij}$ give the consumption.

Proposition 3.3.1. *Let $\bar{AD}$ be the market described above with a A-D equilibrium point $(P, Q, \bar{X})$. Then, $(\bar{p}, \bar{q}, \bar{w}, \bar{X})$ is an equilibrium point in the Heterodox model.*

Proof. We let q, w be the 'active' vectors consisting of all positive entries from $\bar{q}, \bar{w}$. Similarly, let p denote the prices of active goods.

- We first look at the conditions $\bar{p}, \bar{q}, \bar{w}$ satisfy being a part of A-D equilibrium. For each firm j which is active, $S_j(P)$ equals $y_j = (q_j, -l_j)$ where $l_j \leq Y$. Using the definition of $S_j(P)$ as before, y_j has to maximize $p_j q_j - (w \cdot T_{*,i})q_j$ subject to the non-negativity constraints. We can now consider three cases -

  $$p_i - (w \cdot T_{*,i}) < 0$$

  $$p_i - (w \cdot T_{*,i}) = 0$$

  $$p_i - (w \cdot T_{*,i}) > 0$$

  Since p_j, w are given, we note that since firm j produces finite amount q_j, we can only have $p_i - (w \cdot T_{*,i}) \leq 0$. If this is not true, then any finite q_j cannot maximize $p_j q_j - (w \cdot T_{*,i})q_j$, which contradicts the definition. In other words, the function is strictly increasing as q_j increases, thus giving an unbounded solution. When we artificially put a bound on q_j, the optimal solution is at an unattainable production plan. Moreover, the fact that firm j is active, i.e., it is not making any losses, translates to the condition $p_j - (w \cdot T_{*,j}) = 0$. Therefore, we get that for all active firms/goods $p_j = w \cdot T_{*,j}$, or $p = w \cdot T'$ where T' represents the reduced T matrix corresponding to active goods. Continuing with the same concepts, for the inactive firms we must have $p_j - (w \cdot T_{*,j}) \leq 0$, for any feasible p_j.

- The analysis for $D(P)$ is exactly similar to that given in the earlier section, where the optimization program catches Fisher market conditions. Next, by the definition of an equilibrium point in AD, we know that $Z_k(P) = 0$ for all k such that $w_k > 0$. This means that $D_k(P) - S_k(P) - r_k = 0$. We have, $D_k(P) = 0$ for $k = n+1, ...m$ as there are no utilities for raw goods, i.e., labour hours. This forces the relation $-S_k(P) = r_k$. We thus get that $S_k(P) = \sum_{j \in F}(-l_j) = -r_k = -Y_k$. This confirms that $T'q = Y'$. For the inactive classes, we have $D_k(p) = 0$. Therefore, $Z_k(P)$ for those equals $S_k(p) - r_k = \sum_j T_{ij} q_j - Y_k$. As the equilibrium production plan is attainable, we have $T\bar{q} = \sum_j l_j \leq Y$. In all, we have that $T\bar{q} \leq Y$ for all labour classes and $T'q = Y'$ for all classes that are active. Similarly, we have that $\bar{p} \leq \bar{w}T$ for all goods, and $p = wT'$ for active goods. Along with these two conditions, we have that the allocations and prices follow Fisher market conditions, i.e., all goods and endowments are exhausted and every buyer maximises her utility and buys only those goods which give her maximum bang per buck value.

- Building from the observations, we see that $\bar{q}, \bar{w}$ are dual variables of each other and optimal for the following programs, as they satisfy the complementary slackness conditions.

$$\begin{array}{llll} \max_{\bar{q}} & \bar{p}\cdot(T\bar{q}) & & \\ \text{s.t.} & T\bar{q} & \leq & Y \\ & \bar{q}_j & \geq & 0 \end{array} \qquad \begin{array}{llll} \min_{\bar{w}} & \bar{w}\cdot Y & & \\ \text{s.t.} & \bar{w}T & \geq & \bar{p} \\ & \bar{w}_i & \geq & 0 \end{array} \tag{3.2}$$

  Moreover, we see that p_j for $j = n' + 1, ...n$ (unproduced goods) must satisfy

$$\max_i \frac{u_{ij}}{b_i} \text{ (U- Level)} \leq p_j < \sum_i T_{ij} w_i \text{ (T- Level)}$$

  If $p_j \geq$ T-Level, then it violates the constraint of optimality of production for firm j. Similarly, if $p_j <$ U-Level, we have $p_j < u_{ij}/b_i$ for some player i. This means that b_i is less than the bang per buck that good j offers, which contradicts the optimality condition of $D_i(p)$. Therefore, we see that the A-D model allows for a band for each p_j that corresponds to an unproduced good, which is valid for the Heterodox model as well. However, the tatonnement process fixes prices of such goods equal to their U-Levels, which belong to this band for every j. In short, Heterodox modifies the equilibrium prices of inactive goods in AD, while keeping all other variables and optimality conditions the same.

- Thus, we see that the A-D equilibrium point $(\bar{p}, \bar{q}, \bar{w}, X)$ satisfies all conditions for a fixed point in the Heterodox settings.

□

As described before, thus, Heterodox model can be considered as an instance of the A-D model, where the existence of equilibrium is proved. Using the proof given above, the equilibrium is a fixed point in the Heterodox model too.
This gives us the existence theorem.

Theorem 3.3.2. *Given any economy (U, T, Y), an equilibrium (p, q, w, X) always exists in the Heterodox model.*

Chapter 4

Equilibria and The Consumer Choice Game

In this chapter, we associate combinatorial structures associated with equilibria and understand how they vary with the parameters of the economy, i.e., T, Y and U. We define the $\mathcal{CCG}$, the *consumer choice game*, where T, Y are fixed, and U is the strategy space. We explore a few properties of the combinatorial structures and the open sets defining those. We then illustrate the theory through examples.

4.1 Generic Equilibrium and Combinatorial Data

We now associate a suitable combinatorial data with an equilibrium point $\eta = (p, q, w, X)$ for the parameters T, Y, U of the economy. Define $I(\eta) = \{i | w_i > 0\}$, $J(\eta) = \{j | q_j > 0\}$ and $F(\eta) = \{(i, j) | X_{ij} > 0\}$. The combinatorial data identify key features of the equilibrium, i.e., the labour classes with non-zero wages, the goods produced, and the Fisher forest, i.e., the price-determining consumption. We now define the notion of 'generic-ness', which allows us to construct the equilibrium from its combinatorial data, and to extend such equilibria at a point to its vicinity.

Definition 4.1.1. *We say that η is a generic equilibrium if (i) for $j \notin J(\eta)$, we have $(wT)_j > p_j$, and (ii) for $i \in I$ and $(i, j) \notin F(\eta)$, we have $u_{ij}/p_j < \max_k u_{ik}/p_k$.*

Let us now fix T, Y and vary U over $\mathcal{U} = \mathbb{R}^{m \times n}$. Given a $U \in \mathcal{U}$, and an equilibrium point η with the parameters T, Y, U, of the economy, we say that η sits over U, since it is for this element of $\mathcal{U}$, that η was observed. Theorem 4.1.1 relates to the existence of generic equilibria.

Theorem 4.1.1. *Let T, Y, U be matrices in general position, i.e., there be no algebraic relationship between the entries, with rational coefficients. Given a generic equilibrium η over U, with m wage-earning labour classes, i.e., $|I(\eta)| = m$ and n goods produced, i.e., $|J(\eta)| = n$, the number of connected components (k) of the solution Fisher forest $F(\eta)$ is at least $n - m + 1$.*

Proof. Let us consider an economy with m labour classes and n goods. Without loss of generality, we examine the existence of an equilibrium so that all labour classes and goods are active. Let us $n - m$ as the deficit 'def'. We first consider the case $k - 1 < def$ and prove that no equilibrium can exist in this case. We then consider the case $k - 1 \geq def$ and state the possibilities of an equilibrium by giving a system of solvable equations. We conclude the proof by giving an example of a 2×3 case.

4.1.1 $k - 1 < def$

Let us assume that there exist p, q, w such that $p = wT$, $Tq = Y$ and p is the Fisher market solution for the market set up by U, q and w, i.e., there exists a Heterodox equilibrium (p, q, w, X). Using the prices, it is possible to generate the Fisher forest. We assume that there are k trees in the forest so that $k - 1 < def$.
Setting up the profitability conditions, we have,

$$\begin{bmatrix} T_{11} & T_{21} & ... & T_{m1} & | & p_1 \\ T_{12} & T_{22} & ... & T_{m2} & | & p_2 \\ . & . & . & . & | & . \\ . & . & . & . & | & . \\ T_{1n} & T_{2n} & ... & T_{mn} & | & p_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ w_m \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$

We have $k \leq def$. Let us assume that the goods $g_1, \ldots, g_k$ are in distinct components. We construct the rows $R_1, \ldots, R_k \in \mathbb{R}^n$, where $R_i(i) = 1$, $R_i(j) = 0$ if g_j is not in the component of g_i, and $R_i(j) = \alpha_{ij}$ a monomial in the entries of U which relates the price of g_i with g_j. If we are to assume that $p_1, \ldots, p_k$ as temporarily known, then the equation $wT = p$ leads us to the equation:

$$\begin{bmatrix} p_1 & \ldots & p_k & -w \end{bmatrix} \begin{bmatrix} R_1 \\ \vdots \\ R_k \\ T \end{bmatrix} = 0 \tag{4.1}$$

Thus, in case of $k = def$, we can rearrange the matrix so that the determinant of the following matrix is zero.

$$n\left\{\underbrace{\begin{bmatrix} T_{11} & T_{21} & ... & T_{m1} & | & p_1 & . & . \\ T_{12} & T_{22} & ... & T_{m2} & | & \alpha p_1 & . & . \\ . & . & . & . & | & . & p_2 & . \\ . & . & . & . & | & . & \beta p_2 & . \\ . & . & . & . & | & . & . & . \\ . & . & . & . & | & . & . & . \\ T_{1n} & T_{2n} & ... & T_{mn} & | & . & . & \gamma p_k \end{bmatrix}}_{m+k}\right.$$

Here, $\alpha, \beta .., \gamma$ represent the algebraic relationships between prices of goods belonging to the same components of the forest. Each column in the right corresponds to a tree. We have rearranged p so that there are k columns. It is clear from the matrix that the zero determinant condition translates to an algebraic relationship between $\alpha, \beta .., \gamma$, i.e., ratio of entries of U and coefficients of T. This contradicts the assumption that U and T are independent. In case $k < def$, we can delete $def - k$ number of rows from the above matrix so that it becomes a $(m+k) \times (m+k)$ matrix. Determinant of this matrix is zero, which again contradicts the assumption. Thus, there does not exist any equilibrium when $k - 1 < def$.

4.1.2 $k - 1 \geq def$

We again consider (4.1), which is a system of n homogeneous equations in $m + k$ unknowns. Thus, if the first $m + k - n = k'$ prices were fixed, then all prices and wages are defined using this k' prices. Here, $k' \geq 1$.

Next, let us consider the conservation of money in each of the k components G_i and use these to solve for q. We of course have $wT = p$, whence $wY = wTq = pq$. Thus the overall conservation of money is already available and there are only $k-1$ independent equations. Next, we have the m equations $Tq = Y$. Now, for each component, the money available is given as $\sum_{L_j \in G_i} Y_j w_j$. On the goods side, we have $\sum_{g_j \in G_i} p_j q_j$. By the earlier argument, these translate into k algebraic equations in the variables $p_1, \ldots, p_{k'}$ and $q_1, \ldots, q_n$, which are homogeneous in p's. By dividing throughout by p_1, we get $k-1$ equations in $k'-1$ variables. This gives us a total of $m+k-1$ equations in $n+k'-1$ variables. Substituting $k' = m+k-n$ gives us $m+k-1$ equations in $m+k-1$ variables. These may be solved to obtain all quantities. Chapter 5 shows the existence of a disconnected (2×2) market, i.e., $k-1 = 1 > def = 0$, which confirms that $k-1 \geq def$.

In all, we have proved that no equilibrium exists if $k < def + 1$. Moreover, we can see that the condition $k-1 \geq def$ is necessary but not sufficient, as there are more conditions like nonnegativity of variables and optimality of prices. We also note that the condition of a generic equilibrium point is crucial. To see this, consider a non-generic equilibrium point with (i,j) satisfying the bang per buck condition for buyer i such that $x_{ij} = 0$, i.e., $(i,j) \notin F(\eta)$. This increases the number of actual connected components, thereby adding an extra tight constraint, which violates the generic economy argument. Thus, if there are z such zero weight edges, the number of connected components is at least $def + 1 + z$. We now complete the existence proof by giving an example of a 2×3 case.

Example 8. *Let us consider an economy with the following specifications.*

$$T = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1.5 & 0.9 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0.75 & 0.8 \\ 0.4 & 0.9 & 0.7 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Here, two labour classes, with one labour unit each, can produce three goods using the above technology constraints. We show that an equilibrium exists in this economy so that all goods and classes are active, as opposed to the conventional square matrix solution where the number of active classes and goods is the same.
Let us consider p, q, w as follows.

$$p = \begin{bmatrix} 0.5235 & 0.8324 & 0.6470 \end{bmatrix}$$

$$q^T = \begin{bmatrix} 0.5639 & 0.2426 & 0.3934 \end{bmatrix}$$

$$w = \begin{bmatrix} 0.2952 & 0.4565 \end{bmatrix}$$

We see that these variables follow the equations-

$$p = \begin{bmatrix} 0.5235 & 0.8324 & 0.6470 \end{bmatrix} = w \cdot T = \begin{bmatrix} 0.2952 & 0.4565 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1.5 & 0.9 \end{bmatrix}$$

And

$$Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = T \cdot q = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1.5 & 0.9 \end{bmatrix} \cdot \begin{bmatrix} 0.5639 \\ 0.2426 \\ 0.3934 \end{bmatrix}$$

According to the definition of equilibrium in the Heterodox model, we see that q and w are dual variables of each other and follow the market constraints. On the consumption part, when w and q are given as inputs to the Fisher market, solution is the price vector p with the following allocations-

$$Alloc = \begin{bmatrix} 0.5639 & 0 & 0 \\ 0 & 0.2426 & 0.3934 \end{bmatrix}$$

That is, class 1 only consumes good 1 while class 2 buys goods 2 and 3. This confirms the existence of an equilibrium in a general $m \times n$ economy with unequal number of active goods and classes.

□

In the following theorem, we examine the existence of generic and non-generic equilibrium points.

Theorem 4.1.2. *Let T, Y, U be matrices in general position, i.e., there be no algebraic relationship between the entries, with rational coefficients. Given an equilibrium ζ over U, there are arbitrarily close U' and equilibria η sitting over U' which are generic.*

Proof. Let us first consider the case where this equilibrium η is generic. We recall the definition of generic equilibrium points- we say, η is a generic equilibrium point if (i) for $j \notin J(\eta)$, we have $(wT)_j > p_j$, and (ii) for $i \in I$ and $(i,j) \notin F(\eta)$, we have $u_{ij}/p_j < \max_k u_{ik}/p_k$. In this case, the set of tight equations is the same as the set given by (I, J, F), i.e., $Tq = Y$, $wT = p$ and the maximum bang per buck and money conservation constraints given by the

Fisher market. As we have assumed that the matrices are in general position, any of the forest components cannot be cycles. This lets us choose a u_{ij} with $(i,j) \notin F(\eta)$. It satisfies the above generic equilibrium condition- $u_{ij} < p_j \times \max_k u_{ik}/p_k$. In other words, the above forest is valid for u_{ij} belonging to an open set $(0, p_j \times \max_k u_{ik}/p_k)$, as buyer i will buy good J only when its bang per buck condition is satisfied. Therefore, we can always choose a δ small enough so that $u_{ij} + \delta$ or $u_{ij} - \delta$ correspond to a generic equilibrium.
On the other hand, when we have a non-generic point, , i.e., $(wT)_j = p_j$ for $j \notin J(\eta)$ or $u_{ij}/p_j = \max_k u_{ik}/p_k$ for $(i,j) \notin F(\eta)$, the number of tight equations is more than those given by its I, J, F. For the first case, we reduce the p_j so that the following holds for the same U.

$$\max_{i \in I} \frac{u_{ij}}{b_i} \text{ (U- Level)} < p_j < \sum_i T_{ij} w_i \text{ (T- Level)} \tag{4.2}$$

In the second case, we decrease the corresponding u_{ij} so that (4.2) holds true. We thus have proved that given an equilibrium over U, there is an arbitrarily close U' which corresponds to a generic equilibrium. □

It is an important question if the data (I, J, F) does indeed determine η, the equilibrium. This is summarized in the next theorem.

Theorem 4.1.3. *Let T, U, Y be in general position and η be a generic equilibrium over U with the combinatorial data (I, J, F), then the parameters of η, viz., p, w, q are solutions of a fixed set of algebraic equations in the coefficients of U. For an open set of the parameter space of $\mathcal{U}$, the equilibria, as guaranteed by Prop. 3.2.1, 3.3.1, are generic and have the same combinatorial data as η.*

Proof. The combinatorial data does give us the relationships $w_I T_{I,J} = p_J$ and $T_{I,J} q_J = Y_I$. From this it follows that $|I| \leq |J|$ for otherwise there would be an algebraic relationship between T and Y. However, if $|I| = |J|$, and $k = 1$, then w is determined by p and q by Y. Since the forest F is connected, p is determined upto a scalar multiple and thus the whole system is solved. In summary, if $|I| = |J|$ and $k = 1$, there is a unique η sitting above this combinatorial data. However, in the general case, we must first append to the variables q_J, a suitable subset $\{p_1, \ldots, p_{k'}\}$, as in Theorem (4.1.1). The w's and the remaining p's are expressible as homogeneous linear combinations of these k' prices. Next, to the linear set of equations $T_{I,J} q_J = Y_I$ we add the $k-1$ independent money conservation equations to solve these simultaneously. Unfortunately, the conservation equations involve terms $p_i q_i$'s and are quadratic in the chosen variables with coefficients in the entries of U. Once these are solved, all other variables are known and the equilibrium point is reconstructed. Thus, over a given combinatorial data, we get an algebraic system with coefficients in U, but with finitely many solutions. By standard algebraic geometry results (Sard's theorem), other than a over a closed algebraic set, these solutions depend smoothly on the entries of U. □

4.2 The Consumer Choice Game

We now define the consumer choice game $\mathcal{CCG}(T, Y)$, which is parametrized by the technology matrix T and the labour inventory Y, which are henceforth assumed to be fixed. The players are the labour classes, i.e., $\mathcal{L} = \{L_1, \ldots, L_k\}$. The strategy space S_i for player i is the utility 'row' vector $(u_{i*}) \in \mathbb{R}^n$. These rows together constitute the matrix U. This strategy space is denoted by $\mathcal{U}$. We also assume that there is a 'real' utility matrix U_r which is used to measure outcomes.

Given a play U, the outcome is given by an $\eta(U) = (p, q, w, X)$, an equilibrium over U obtained in the Heterodox market. The payoffs, $\mathbb{U}_i(X) = \sum_j (U_r)_{ij} x_{ij}$, i.e., the equilibrium allocations evaluated by each player on their true utilities, define the preference relations for each player.

Let us now construct the pay-off functions in the vicinity of a generic equilibrium point $\eta(U)$ with the combinatorial data (I, J, F). We first see that there is an open set $O_{I,J,F} \subseteq \mathcal{U}$ containing U which has the same combinatorial data (I, J, F). The exact inequalities defining $O_{I,J,F}$ arise from the requirement that the Fisher forest F have non-negative flows in all edges of F, that the edge $(i, j) \notin F$ has an inferior bang-per-buck, and that $(wT)_j - p_j > 0$ for $j \notin J$. As an example, consider an edge $(i, j) \in F$, and the requirement that the flow in this edge be positive. Now, the flow in this edge is a suitable linear combination of the wages w's, prices p's and quantities q's. As we have argued before, these in turn, are smooth functions of the entries of U. Thus the condition that flow in the edge (i, j) be positive is the requirement that $f(U) > 0$ for a suitable smooth function f on U. We can obtain such conditions for each edge (i, j) which confirm that (I, J, F) remains valid.

Thus, there is indeed such an open set $O_{I,J,F}$, and the pay-off functions are solutions of algebraic equations in the entries of U, the coefficients of which depend on the combinatorial data (I, J, F). This gives us Theorem 4.2.1 below.

Theorem 4.2.1. *For a generic equilibrium point $\eta(U)$ with the combinatorial data (I, J, F), there is an open set $O_{I,J,F} \subseteq \mathcal{U}$ containing U and a smooth family $\eta'(U')$ of equilibria for each $U' \in O_{I,J,F}$ such that (i) $\eta'(U) = \eta(U)$ and (ii) the combinatorial data for $\eta'(U')$ is (I, J, F).*

The pay-off function in general is to be pieced together by such a collection of open sets, indexed by the combinatorics. On non-generic U', the equilibrium $\eta(U')$ will have multiple feasible allocations and this determines a correspondence between the strategy space $\mathcal{U}$ and $\mathbb{R}^k$, the pay-off space. Even for a generic U, there may be multiple equilibrium points, viz., $\eta_1, \ldots, \eta_k$, and each of these will define an analytic sheet of the correspondence over the generic open set. We now illustrate the theory describe so far through a (3×3) market.

4.3 An Example

In this section, we describe an economy with three labour classes and three goods, viz., $\mathcal{H}_3$ and construct the *consumer choice game* where two of the labour classes engage in strategic behaviour.

Let $I = J = \{1, 2, 3\}$. Let T, U, Y be as given below:

$$T = \begin{bmatrix} 1/10 & 0 & 0 \\ 1/10 & 1/5 & 0 \\ 0 & 1/5 & 1 \end{bmatrix} \; Y = \begin{bmatrix} 1 \\ 10 \\ 100 \end{bmatrix} \; U_r = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \; \mathbb{S} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & \beta & 1 \end{bmatrix}$$

There are 3 labour types, with numbers 1, 10 and 100 respectively. L_1 only prefers good G_1, L_2 only G_1, G_2 and L_3 prefers G_2 and G_3 as shown in the true utilities U_r. The example can be understood as an instance of a market with three socio-economic classes and a good such as footwear which is produced in three different ways or qualities. In such cases, the given utility matrix catches the general preference towards the goods produced by different classes. Here, T is such that each of good 1 and 2 are produced by two classes together. In terms of efficiency, good 1 is produced most efficiently. The classes have utilities defined for the goods they produce.

Let us consider labour class 2 and 3 as the players who exercise their strategies by choosing

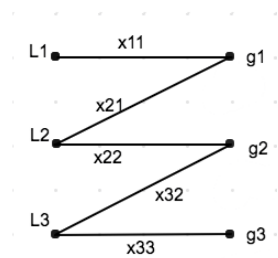
the variables α and β. This defines the strategy space $\mathbb{S}$ as shown above. Note that $U_r \in \mathbb{S}$. We compute (i) the dependence of the pay-offs on α and β, and (ii) the sub-domain of $\mathbb{S}$ over which the chosen forest F below is the equilibrium forest.
We first solve for the production part. We see that $Tq = Y$ gives:

$$q = T^{-1}Y = \begin{bmatrix} 10 & 0 & 0 \\ -5 & 5 & 0 \\ 1 & -1 & 1 \end{bmatrix} Y = \begin{bmatrix} 10 \\ 45 \\ 91 \end{bmatrix}$$

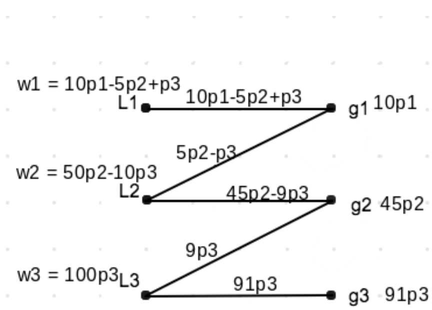
And wages are given as linear combinations of prices-

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 10p_1 - 5p_2 + p_3 \\ 50p_2 - 10p_3 \\ 100p_3 \end{bmatrix}$$

This describes wages in terms of prices. All this does not need the equilibrium forest F.
For the consumption and allocation processes, let us assume that the solution forest F is given by:



Note that this forest is motivated by the utility matrix given earlier. Conserving the total money while allotting the goods, the equations result in the following money flows:



The flows mentioned on edges are the amounts spent by classes on the corresponding goods. Using Fisher market constraints of optimum utility, it is easy to see that this market is feasible only if $\beta > \frac{1}{5}$ and $\alpha > 0.5 - \frac{1}{10\beta}$. Under these conditions, F will arise as the equilibrium forest.
Assuming total money in the economy as 1, we find class wages as functions of α and β

$$w_1 = \frac{10\alpha\beta - 5\beta + 1}{10\alpha\beta + 45\beta + 91}$$

$$w_2 = \frac{10(5\beta - 1)}{10\alpha\beta + 45\beta + 91}$$

$$w_3 = \frac{100}{10\alpha\beta + 45\beta + 91}$$

The forest has following implications on allocations and total utility-

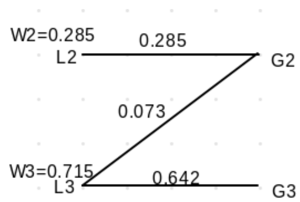
	Allocations	Total Utility
L1	$x_{11} = 10 - 5/\alpha + 1/\alpha\beta$	$U_1 = 10 - 5/\alpha + 1/\alpha\beta$
L2	$x_{21} = 5/\alpha - 1/\alpha\beta$, $x_{22} = 45 - 9/\beta$	$U_2 = 45 + 5/\alpha - 1/\alpha\beta - 9/\beta$
L3	$x_{32} = 9/\beta$, $x_{33} = 91$	$U_3 = 91 + 9/\beta$

Analysis

We note that the forest is valid in the open set defined by $\beta \in (1/5, \infty)$ and $\alpha \in (0, 0.5 - \frac{1}{10\beta})$. That is, the open set corresponds to the combinatorial data (I, J, F). Within this set, there is a clear dependence of utilities, i.e., α, β on wages and prices. However, the production vector q is independent of U. It remains so until U does not belong to the open set defined above. In other words, the dependence of q on U is explicit here so that q changes only if I, J change, where it equals another constant.

From the table, it is clear that decreasing α and β are the strategies for class-2 and 3. Moreover, we see that multiple equilibrium forests are possible here, depending on α and β, including the one given above. For each forest, the number of active classes may be different and thus the utility functions will vary. In fact, for a sufficiently small value of β, class-2 and thereby class-1 receive no wages. In this case, only good-3 is produced and its utility for class-3 is 100, which is smaller than the maximum achievable utility through the forest given above, implying class-3 is better off when other classes are producers too.

Posing a small value of α, class-2 may similarly squeeze class-1. For example, let $(\alpha, \beta) = (0.3, 1)$. Here, the optimal production is $q = (0, 50, 90)$ and the class wages are $(0, 0.285, 0.715)$, assuming total money equal to 1. The forest F' is given by-



We see that the open set given by $\alpha \in (0.5 - \frac{1}{10\beta}, \infty)$ and $(\beta \in (1/5, \infty))$ corresponds to the combinatorial data $(I = \{2, 3\}, J = \{2, 3\}, F')$.

This illustrates that the local combinatorial data is sufficiently explicit to enable the computation of the pay-off functions. Moreover, significant benefits may accrue to players if they utilize the freedom of posturing their utility functions. We give another example of a (2×2) case in Appendix B, where we give some numerical results illustrating the theory.

Chapter 5

2 × 2 Markets

We now consider a specific scenario involving 2 classes and 2 goods where T, Y are the inputs. We rigorously solve for markets, and compute the conditions for a forest to be in equilibrium. We argue that wages are continuous functions of utilities and payoffs and allocations are continuous for each forest. Moreover, we prove the existence of a correspondence between the strategy space and utilities and also that utilities on the boundaries are linear combinations of those of the forests on both sides. We also look at the necessary conditions for Nash equilibria to exist. Finally, we give a number of conjectures generalizing the (2×2) case. We discuss the general (2×2) economy in Appendix C, where we illustrate that the same results follow.

5.1 Market and Specifications

Let us consider a two class economy with the following specifications - T (Technology matrix), Y (Labour availability), U_r (True Utility matrix) and $\mathcal{U} = \mathbb{R}^{2\times 2}$ (Strategy matrix)

$$T = \begin{bmatrix} 0.25 & 0 \\ 0.25 & 1 \end{bmatrix} \; Y = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \; U_r = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \; U_2 = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

Since Fisher solutions do not change if the rows of U are scaled independently, we see that effectively, $\mathcal{U}$ is given by:

$$U_2 = \begin{bmatrix} \alpha & 1 \\ \beta & 1 \end{bmatrix}$$

We assume that $0 < \alpha, \beta < \infty$. Let us solve this case completely, i.e., decompose $\mathcal{U}$ into various zones by their combinatorial signatures. We also analyse the case when we transit from one zone to another, and finally, when one of the labour classes is shut out of the market.
Whenever both classes are active, q (production vector) is given by :

$$q = T^{-1}Y = \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix} Y = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

and

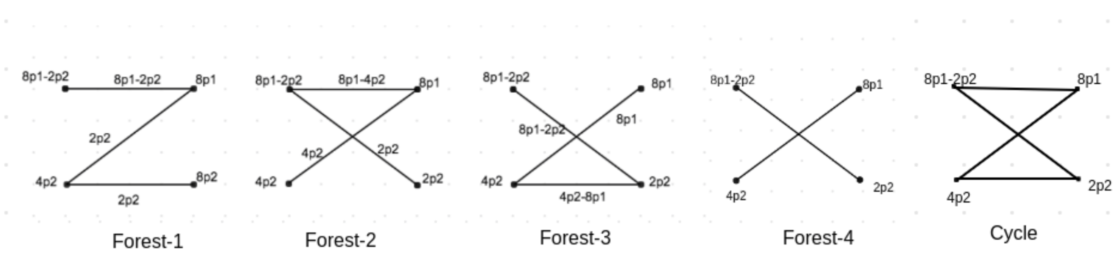
$$w = \begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \cdot T^{-1} = \begin{bmatrix} 4p_1 - p_2 \\ p_2 \end{bmatrix}$$

Therefore, class-wise wages may be obtained as -

$$m_1 = 2(4p_1 - p_2)$$

$$m_2 = 4p_2$$

In a general 2×2 economy, there are seven possible ways of allocating the produced goods are six forests and a cycle, where both classes participate, and two possible (1×1) graphs, where only one class is active. Out of these 9 graphs, the given set of inputs (T, Y) allows for six possible equilibrium solutions, each with a different allocation graph. Firstly, if both the classes are active in the equilibrium, we have these five possibilities-



It is easily observed that the conservation of money allows only these graphs. The combina-

torial data (I, J, F) and the actual equilibrium values (p, q, w, X) depend on the choice of α and β. As described in Section 3, for each forest, there is an open set, a 'zone', of the α-β space over which the forest is the Fisher forest of the equilibrium. The conditions defining this zone arise from positivity of wages and allocations and the maximum bang per buck conditions. For the four forests, these are tabulated in the table given below.
The sixth possible equilibrium state is when w_1 becomes zero. As we see later, it occurs when $\beta < 1/4$. In this case, labour class 2 produces good 2 ($q = 4$) and gets the whole share of economy. Note that for no set of prices, do we have $w_2 = 0$ in equilibrium.

5.2 Equilibrium Analysis

Here is a summary of the four forests given above. We analyse the condition $\alpha = \beta$, i.e., the cycle in the next section. Recall that $\alpha = \frac{u_{11}}{u_{12}}$ and $\beta = \frac{u_{21}}{u_{22}}$. Let $U1, U2$ denote the true utility functions of class-1 and 2 respectively. Also, $w_1 + w_2 = 1$.

	Forest-1	Forest-2	Forest-3	Forest-4
Prices	$\beta = \frac{p_1}{p_2} > 1/4$	$\alpha = \frac{p_1}{p_2} > 1/2$	$1/4 < \beta = \frac{p_1}{p_2} < 1/2$	$\frac{p_1}{p_2} = \frac{1}{2}$
Alloc - 1	$x_{11} = 8 - \frac{2}{\beta}$	$x_{11} = 8 - \frac{4}{\alpha}, \quad x_{12} = 2$	$x_{12} = 8\beta - 2$	$x_{12} = 2$
Alloc - 2	$x_{21} = \frac{2}{\beta}, \quad x_{22} = 2$	$x_{21} = \frac{4}{\alpha}$	$x_{21} = 8, \quad x_{22} = 4 - 8\beta$	$x_{21} = 8$
$U1$	$U1 = 8 - 2/\beta$	$U1 = 10 - 4/\alpha$	$U1 = 8\beta - 2$	$U1 = 2$
$U2$	$U2 = 2 + 2/\beta$	$U2 = 4/\alpha$	$U2 = 12 - 8\beta$	$U2 = 8$
Wages-1	$w_1 = 1 - \frac{2}{1+4\beta}$	$w_1 = 1 - \frac{2}{1+4\alpha}$	$w_1 = 1 - \frac{2}{1+4\beta}$	$w_1 = 1/3$
Zones	$\alpha \geq \beta > 1/4$	$\beta \geq \alpha > 1/2$	$1/4 < \beta < 1/2$ and $\beta \geq \alpha$	$\alpha \leq 1/2 \leq \beta$

In addition to the conditions for these forests, if $\beta < 1/4$, the equilibrium solution is $w_1 = 0$. In that case, $U1 = 0$ and $U2 = 4$ as the production amount is $(0, 4)$. We denote this by forest-5 or zone-5.

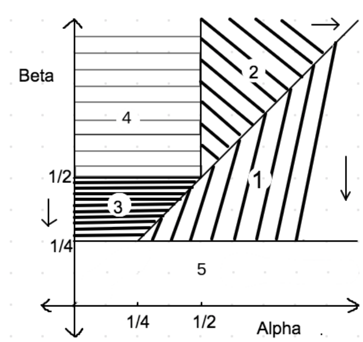
We also classify the generic equilibrium points here. Recall that η is a generic equilibrium if (i) for $j \notin J(\eta)$, we have $(wT)_j > p_j$, and (ii) for $i \in I$ and $(i, j) \notin F(\eta)$, we have

$u_{ij}/p_j < \max_k u_{ik}/p_k$. Therefore, we see that an equilibrium point belonging to, say zone-1, is generic if and only if $\alpha > \beta$. As proved in Chapter-4, we can define an open set for zone-1, *viz.* , $\alpha > \beta > 1/4$, comprising only of generic equilibrium points. Similarly, for zone-4, the open set would be $\alpha < 1/2 < \beta$. As we see in the figure given below, the interior of each zone is an open set consisting of generic points and the boundaries correspond to the non generic equilibrium points.

Remark 5.2.1. *As Theorem 4.1.3 states, for an open set of the parameter space $\mathcal{U}$, the equilibria are generic and have the same combinatorial data I, J, F. Here, we see that these open sets are given by interior of the zones marking each forest.*

5.3 The Space of Payoffs as A Manifold

We summarize the earlier section through a figure, where five zones are marked which correspond to the forests given above. The arrows indicate the direction of optimization for the dominant class, i.e., the class that has a control over the price ratio.



We now look at the points where two or more forests are feasible, i.e., at the boundaries. It is clear from the figure that by changing α or β it is possible to transit from one forest to another, by crossing the non-generic points where both forests are possible. It is shown earlier [21] that the set of allocations in Fisher market is hemicontinuous with respect to initial endowments and utility functions. Here, we show that though multiple allocations are possible at such points, utilities are bounded by the limits of utilities of forests at both sides. In other words, allocations and utilities at the transitions are convex combinations of the boundaries of those obtained in the adjoining zones. To make this precise, let us have the following definition.

Definition 5.3.1. *Let $x \in \mathcal{U}$ be a point on the boundary of two zones, say Zone A and Zone B and let $\eta = (p, q, w, X)$ be a typical point above x, i.e., η is an equilibrium for the parameter x. Let $\mathcal{X}(x)$ be the collection of all allocations of equilibria above x in the U-space. We say that x is a manifold point if the set $U_i(\mathcal{X}(x))$ is a bounded interval and its bounds are obtained as the limits $\lim_{q \to x} U_i(X(q))$ and $q \in$ Zone A and $q \in$ Zone B.*

Let us consider a point x on the line $\alpha = \beta$, and let $\alpha(x) = \beta(x) = \mu > 0.5$. Thus p sits on the transition between Zone 1 and Zone 2. Let $\eta(x) = (p, q, w, X)$ be a typical point above x. We see that (i) $I(x) = J(x) = \{1, 2\}$, and (ii) $F(p) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$, i.e., the cycle, since X has flows on all edges. In fact, there is a set $\mathcal{X}$ of allocations possible at $\eta(p)$ and the bounds on $U_i(\mathcal{X})$ are precisely those achieved as $\lim_{q \to x} U_i(X(q))$ for q in zone 1 and 2. Similarly, we see that for $1/4 < \alpha = \beta < 1/2$, the bounds on the corresponding $U_i(\mathcal{X})$ are obtained through zone 1 and 3. For the points on the boundaries with forest 4, however, we see that the limits $\lim_{q \to p} U_i(X(q))$ are equal from both zones, thus giving a unique allocation X. On the other hand, we see that the points on the boundary $\beta = 1/4$ are not manifold points. This is because for each point on this line, there is a unique allocation $\mathcal{X}$ leading to a unique $U_i(\mathcal{X})$ which does not equal $\lim_{q \to p} U_i(X(q))$ for q in zone 1 or 3.

Theorem 5.3.1. *There exists a correspondence $\mathcal{N}$ between the strategy space and utilities. Also, each space of payoffs forms a 2 dimensional manifold whose boundaries are given by those of the forests.*

Proof. We first classify the zones into interior points and the boundaries where two or more forests are possible. As stated earlier, the interior points refer to generic points which form an open set. Corresponding to the interior points, we have $U1, U2$ defined uniquely, which are continuous functions of α, β. Moreover, these are invertible functions on their restricted domain of α, β, which makes the sets of possible payoffs U_1^p, U_2^p open. It follows that the correspondences $\mathcal{N}_i = (\alpha, \beta, U_1^p, U_2^p)$ are open for each forest i. Moving further, we see that for $\alpha = \beta$, which stands for a cycle, corresponding U_1^p and U_2^p belong to open sets. For example, on the boundary of forest 1 and 3, the open set U_1^p is given by $1/4 < \alpha < 1/2$ and $(8\alpha - 2, 8 - 2/\alpha)$. The region looks like half of a parabola, bounded from all the sides. Thus, interior of every possible solution or zone is open and neighbourhood of each point is homeomorphic to open subsets of $\mathbb{R}^2$. For forests, the homeomorphism is given by the inverse of utility functions and for the cycle, the sets are open in $\mathbb{R}^2$.

We now claim that the boundaries of the forests and the cycle form 1 dimensional entities which serve as boundaries to the described 2 dimensional manifolds. Each boundary can be given by a unique equation in $\mathbb{R}^2$. On the line $\alpha = \beta$, there are two boundaries, one coming from $\alpha < \beta$ and another from $\alpha > \beta$. In between these two, a 2 dimensional plane is situated on on each part of the segment $\alpha = \beta$, i.e., on the boundary of forest-1 and 3 and forest 1 and 2. When considered closures of the open sets, we see that correspondences intersect along these boundaries.

We therefore establish that in the region $\beta > 1/4$, payoff function U_i is a 2 dimensional manifold with boundary, consisting of all 'manifold' points. Moreover, a correspondence $\mathcal{N} = (\alpha, \beta, U_1^p, U_2^p)$ can be defined between the strategy space and the payoffs space. A general version is dealt with in the Appendix C where we argue that the same results follow.

5.3.1 Strategic Analysis

Here, as we see from the figure, class-2 is dominant in the sense that it has a strategy to become the only active class in the economy, just by reducing β. However, it is not in its interest to completely drive out class-1. If the state is in any zone with 2 active classes, β can be decreased to reach zone 5, where class-2 gets utility 4. But, due to a discontinuity in the utility function, we see that for any α, class-2 achieves the highest payoff, which is arbitrarily close to 10, when β approaches 1/4, but is, strictly more than 1/4. Thus, the best strategy for class-2 is to keep class-1 active and pose β as close to, but greater than 1/4. Technically, we see that the discontinuity of the utility functions results in the non-existence of Nash equilibria.

Though this case rules out the possibility of a Nash equilibrium by making one class clearly dominant, the general scenario has a possibility of existence of Nash equilibria. Appendix C explores the necessary conditions for a Nash equilibrium to exist.

Taking insights from the (2×2) case, Theorem 4.1.3 and 4.2.1, and the continuity results of the Fisher and AD markets, we now have the following conjectures for a general economy (T, U, Y).

Conjecture 5.3.2. *Wages are continuous functions of the utilities. Moreover, within each forest F, allocations and payoffs are continuous.*

Conjecture 5.3.3. *For any economy characterized by a fixed (T, Y), the space of payoffs for each player is a m-dimensional manifold within the U space with the same combinatorial data I, J.*

Conjecture 5.3.4. *In any economy characterized by a fixed (T, Y), dominant classes are well defined. Moreover, the existence of Nash equilibria is subject to T, Y.*

Chapter 6

Geometric Formulation of Heterodox Model

As described earlier, the *tatonnement* process does not converge for all markets. To achieve this computability, we examine the geometric structure and explore the possibilities of finding an equilibrium geometrically.

In this chapter, we consider T, U, Y as the inputs given in an economy. We consider a real space $\mathbb{R}^{mn+2m+2n}$ and define 'feasible' points as those achievable by an economy, which lead to a polytope like region X. Under certain conditions, specific vertices correspond to Heterodox equilibrium points. We then explore the structure of X by analysing some of its vertices, edges and transitions. Further, we give a partial Linear Complementarity Problem (LCP) formulation and discuss its computability.

6.1 Feasibility Conditions

Let us consider the $X \subset \mathbb{R}^{mn+2m+2n}$ space consisting of variables m_i, y_i, with $i = 1, \ldots, m$ and p_j, q_j with $j = 1, \ldots, n$ and z_{ij} for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We assume that T and U are in "general position", i.e., there is no algebraic relations between the entries of T and U together, and that $U > 0$, i.e., $u_{ij} > 0$, for every i, j. Let us define the feasible set X as given by the following equations below.

$$\forall i \in [m], j \in [n] \; : \; y_i u_{ij} \leq p_j \tag{6.1}$$

$$\forall j \in [n] \quad p_j - \sum_i m_i T_{ij} \leq 0 \tag{6.2}$$

$$\forall i \in [m] \quad \sum_j T_{ij} q_j \leq 1 \tag{6.3}$$

$$\forall j \in [n], \quad \sum_i z_{ij} = p_j q_j \tag{6.4}$$

$$\forall i \in \{2, \ldots, m\}, \quad \sum_j z_{ij} = m_i (\sum_j T_{ij} q_j) \tag{6.5}$$

$$\sum_j p_j q_j = 1 \tag{6.6}$$

$$\forall i \in [m], j \in [n] \quad (p_j - y_i u_{ij}) z_{ij} = 0 \tag{6.7}$$

$$\forall j \in [n] \quad (p_j - \sum_i m_i T_{ij}) q_j = 0 \tag{6.8}$$

$$\forall i \in [m], j \in [n] \quad z_{ij}, m_i, y_i, p_j, q_j \geq 0 \tag{6.9}$$

Remark 6.1.1. *We note that the region given by the above constraints is bounded. This is because of the conditions (6.3- 6.6).*

As we will see, (6.1) captures the bang per buck ratio, thereby making up the consumption constraints along with (6.7), i.e., as in [2]. Conditions (6.4- 6.5) account for allocations and the remaining equations make up the feasibility conditions for an economy. Without loss of generality, as (6.3) suggests, we take $Y_i = 1$ for all i. We will use the auxiliary notation $e_i = \sum_j T_{ij} q_j$, which is the employment fraction for labour class i. Note that Eqns (6.5) does not include $i = 1$, and it would be useful to compute $\sum_j z_{1j}$ from the other inequalities. To this end, if we write $m_i(\sum_j T_{ij} q_j)$ as $m_i^* = m_i e_i$, the *shaded* wages, then we have:

$$1 = \sum_j p_j q_j = \sum_j \sum_i z_{ij} = \sum_j z_{1j} + \sum_{j=2}^{m} m_i^*$$

However, Eqns (6.2) tell us that

$$1 = \sum_j p_j q_j \geq \sum_{i=1}^{m} m_i^*$$

This tells us that:

$$\sum_j z_{1j} \geq m_1^* = m_1 e_1$$

Thus, the expenditure available to labour class 1 is more than its wages. Indeed, it is the special class to which all the *producer surplus* $s_j = q_j(p_j - c_j)$ accrues. We then have,

$$1 = \sum_j p_j q_j \;\; = \sum_{i=1}^{m} m_i$$

We note that this allows the possibility of a class i with $m_i > 0$ and $\sum_j T_{ij} q_j < 1$.

Remark 6.1.2. *We define this program to compute solutions for the Heterodox model, motivated by the similar structures for Fisher market and the AD market, as given in Chapter 1. Though we have established a correspondence between the equilibrium points of AD and Heterodox, here is why the Heterodox model is more complex than the most general case of AD market considered so far. To be precise, we illustrate why the LCP formulation (Chapter-1) cannot account for the Heterodox model.*

We recall the production function which is SPLC. For each good j and firm f which produces good j_f, we have a production function P_j^f which defines f's ability to produce good j_f as a function of the amount of good j. The overall production function is $P_j(x) = \sum_j P_j^f$ and K'th segment is defined by (f, j, k). The slope of a segment α_{jk}^f defines the rate at which good j_f can be produced from a unit of good j. Note that raw goods in our case refer to available labour hours for each labour class. Using the technology matrix defined earlier, we can invert the reduced matrix to express production q in terms of labour availability Y'. Here, Y' equals $(0, 0, .., 1(j), .., 0)$. In short, $\alpha_{jk}^f = T_{jf}^{-1}$.

Now, equation (1.16) requires $\alpha_{jk}^f p_{j_f} = p_j - \beta_{jk}^f$ whenever raw good j is used to produce j_f. Since we have only one segment for each good j, all segments which utilize j are active and $x_{jk}^f \leq o_{jk}^f$. The dual variables β_{jk}^f capture the profit on each segment so that the total profit made is $\sum_{j,k} \beta_{jk}^f$. If a firm makes zero profit, we have $\alpha_{jk}^f p_{j_f} = p_j$, i.e., $\beta_{jk}^f = 0$ for all segments. This means $T_{jf}^{-1} p_{j_f} = p_j$. In other words, $T_{jf}^{-1} p_j = w_f$, which gives a contradiction, as we have w_j given by linear combination of prices of all active goods.

Thus, the LCP formulation for the AD market cannot be considered for Heterodox.

6.2 Vertices and Optimality Conditions

Let us now examine the conditions which are satisfied by a feasible point $x^0 = (p^0, y^0, m^0, q^0, Z^0)$ such that it forms a vertex. Let us assume that exactly for $i \in I \subseteq [m]$, we have $m_i > 0$ and for $j \in J \subseteq [n]$, $q_j > 0$, and that $1 \in I$, i.e., the labour class 1 has positive wages. We make the following sequence of observations:

1. $z_{ij} = 0$ for all $i \notin I$ or for $j \notin J$. Thus, if $z_{ij} > 0$ it must be that $i \in I$ and $j \in J$. Here, unlike x_{ij}, which refer to goods allocations, z_{ij} are money allocations.

2. Let us build a bipartite graph with $V = [m]$, the set of labour classes and $W = [n]$, the set of goods. Let $E = \{(i,j) | y_i u_{ij} = p_j\}$ be the set of tight edges, and $F = \{(i,j) | z_{ij} > 0\}$ the set of positive edges. We see that $F \subseteq E$ and that E has no cycles by the general position of T, U. This is in accordance with the Fisher forest data that we have described before.

3. *Assumption A.* We may now assume that $e_i > 0$ for all $i \in I$ since otherwise we may scale m_i by any positive number and retain feasibility. This would imply that x^0 is not a vertex. Let us assume henceforth that for all $i \in I$, $e_i > 0$ as well.

4. Under assumption A, for every $i \in I$, there is a j such that $z_{ij} > 0$. This also means that $p_j q_j > 0$ for the same j.

5. Let C be a component of the forest E. We say that C is a zero-component if there is no edge $e = (i,j) \in C$ with $z_{ij} > 0$.

6. For a non-zero component $C = (V_C, W_C, E_C)$, there must be a non-zero z_{ij}, and hence a $j \in W_C$ such that $p_j, q_j > 0$. For any vertex $k \in W_C$, since there is a path from j to k of tight edges, we must have $p_k = p_j u^\alpha$, for some rational monomial u^α, in the variables of U. Similarly, for $a \in V_C$, we will have $y_a = p_j u^\beta$. Since every entry of U is positive, we must have $y_a > 0$ for all $a \in V_C$ and $p_k > 0$ for all $k \in W_C$.

7. In fact, for any $k \in [m]$, we have $p_k > 0$. This is because $p_k \geq y_i u_{ik}$, whence if there is an i such that $y_i > 0$, then for all k, $p_k > 0$.

8. For any zero component C, we must have $p_k > 0$ but $q_k = 0$ for all $k \in W_C$, and $m_i = 0$ for all $i \in V_C$.

9. By a suitable manipulation of x^0, we may assume that if $q_k = 0$, then p_k is suitably raised so that the vertex k is not in any non-zero component. This modification does not alter the feasibility of x^0. Thus, we may assume the goods in the union of all non-zero components exactly equals J.

10. Note that i such that $m_i = 0$ may well be members of non-zero components, i.e., they may need to work and yet receive no wages. In other words, the sets I and $I' = \{i | \sum_j T_{ij}q_j = 1\}$ may not equal.

Observations 1-10 imply that for a vertex with $i \in I \subseteq [m], m_i > 0$ and $j \in J \subseteq [n]$, $q_j > 0$, the corresponding variables would be

$$\begin{bmatrix} q_1 & q_2 & .. & q_{|J|} & 0 & .. & .. & 0 \end{bmatrix}$$

$$\begin{bmatrix} p_1 & p_2 & .. & .. & .. & .. & .. & p_n \end{bmatrix}$$

$$\begin{bmatrix} m_1 \\ m_2 \\ : \\ m_{|I|} \\ 0 \\ : \\ : \\ 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ : \\ : \\ : \\ : \\ : \\ y_m \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} & .. & z_{1,|J|} & 0 & .. & .. & 0 \\ z_{21} & z_{22} & .. & z_{2,|J|} & 0 & .. & .. & 0 \\ : & : & : & : & 0 & .. & .. & 0 \\ z_{|I|,1} & z_{|I|,2} & .. & z_{|I|,|J|} & 0 & .. & .. & 0 \\ 0 & 0 & 0 & 0 & 0 & .. & .. & 0 \\ : & : & : & : & 0 & .. & .. & 0 \\ : & : & : & : & 0 & .. & .. & 0 \\ 0 & 0 & 0 & 0 & 0 & .. & .. & 0 \end{bmatrix}$$

Let us define a point $v_{|I|,|J|}$ with $m_i > 0 \;\; \forall i \in I$ such that $I \subseteq [m]$ and $q_j > 0 \;\; \forall j \in J$ such that $J \subseteq [n]$. We want to set the variables in a way so that v becomes a vertex. Let us now list the important conditions required for a general case which can lead to the determination of all variables.

$$p_j - \sum_i m_i T_{ij} \geq 0$$

and $p_{j'} = \sum_i m_i T_{ij'} \;\; \forall j' \in J'$ such that $J \subseteq J'$ and $|J'| + a = |I|$. Also,

$$\sum_j T_{ij} q_j \leq 1 \;\; \forall i \in [m]$$

and $\sum_j T_{kj} q_j = 1 \ \ \forall k \in I^{'}$ such that $|I^{'}| + b = |J|$.
Note that $I^{'}$ may not be a subset of I. Let there be k connected components in the graph formed by E. Also, we assume that the number of edges in $E - F$ equals c, i.e., there are c number of z_{ij}'s which equal zero and for the same i, j, $y_i u_{ij} - p_j = 0$. Similarly, let d be the number of q_j's so that $q_j = 0$ and $p_j = \sum_i m_i T_{ij}$. We call such equations as double label. Under these settings, we prove the following.

Theorem 6.2.1. *If $a + b = c + d$, then a solution to $v_{|I|,|J|}$ defined above is a vertex.*

Proof. We first successively reduce the number of variables and then solve for a smaller system.

Without loss of generality we assume that there are no zero components, i.e., $|I| = m, |J| = n$ in the graph formed using E. Therefore, for k components, the number of edges equals $m + n - k$, i.e., the number of nonzero z_{ij} is at least $m + n - k$. As observation 6 notes, in each of the components, y_i's and p_j's can be expressed in terms of a single p. In other words, there are k number of variables (p_j) which determine the remaining $m + n - k$ variables (y_i, p_j). Thus, the remaining nonzero variables are $2m + 2n + mn - (m + n - k) - (mn - (m + n - k)) - (|E - F|) = 2m + 2n - (|E - F|)$, so that we have $p_j(k), m_i(m), q_j(n)$ and $Z_{ij}(m + n - k - (|E - F|))$.
Let us now list the tight inequalities concerning these variables. We have $m - a$ equations of type (6.2), i.e., $(p_j - \sum_i m_i T_{ij} = 0)$, $n - b$ equations of type (6.3), n equations of type (6.4), $m - 1$ equations of type (6.5) and the equation (6.6). In all, we have $2m + 2n - (a + b) = 2m + 2n - (|E - F|)$ tight equations.

We now prove the independence of (6.5), (6.6) which leads to the solution to z_{ij}'s, given m_i, q_j and p_j. Let us first observe that these equations individually hold true for each component of the forest formed by F. In other words, for each $i \in [m]$ and $j \in [n]$ belonging to a particular component K, $m_i(\sum_j T_{ij} q_j) = \sum_{j \in K} z_{ij}$ and $p_j q_j = \sum_{i \in K} Z_{ij}$ respectively. This can be easily seen as for any $k \in K$, $z_{kj} = z_{ik} = 0$ for all $i, j \notin K$. Noting the fact that the number of vertices in K is $m_K + n_K$, the number of nonzero z_{ij}'s is $m_K + n_K - 1$ as K is a tree. Now, from (6.5), (6.6), we have $m_K + n_K - 1$ equations in case m_1 belongs to K and $m_K + n_K$ equations otherwise. Note that because $\sum_j p_j q_j = 1$, both these cases are equivalent as it is possible to compute m_1 from this equation. Therefore, without loss of generality, we drop the corresponding equation for m_{K1} from (6.5). Since there are no cycles, no two vertices in K have the same set of edges. This means that $\sum_j z_{ij}$ and $\sum_i z_{ij}$

are independent for each i and j. Now, we solve $m_K + n_K - 1$ linear equations to find all z_{ij}'s. It can also be verified that the rank of above system is full.

Focusing on the remaining variables, we have p_j(k), q_j (n), and m_i(m) and equations (6.2), (6.3) along with $k+c$ conservation equations $\sum_i m_i(\sum_j T_{ij}q_j) = \sum_j p_j q_j$ for all $k+a+b$ connected components. Moreover, there are d equations which give additional constraints on p, m. Using the fact that T is in the general position, we invert the sub-matrix of T to express first $(n-b)$ q_j's in terms of last b q_j's. Similarly, we invert U to reduce the number of variables m_i to a by expressing in terms of p_k's and m_i. Note that if a or b equal zero, it is possible to invert U or T directly to find m, q. We're now left with $k+a+b$ variables and the same number of conservation equations, as $a+b=c+d$. We solve these for each component.

It can be easily seen that each of the equations can be rewritten as $\sum_{j\in b}\sum_{k\in K} h_{kj}p_k q_j + \sum_{j\in k} p_j e_j + \sum_{i\in a}\sum_{j\in b} l_{ij}m_i q_j + \sum_{i\in a} m_i f_i = 0$ for each component, where h_{ij}, l_{ij}, e_j, f_i are constants determined by T, U, K. These being quadratic equations, we note that multiple finite solutions are possible which satisfy the system, where each is a vertex. If b equals zero, there exists a unique solution where we determine the variables by solving linear equations. □

Remark 6.2.1. *If $|J'| + a = |I|$ and $|J'| + b = |J|$, it is crucial to have $c+d$ double label equations so that the desired number of equations is obtained. Note that if $c+d=0$, this is equivalent to the concept of generic points defined earlier.*

Moving further, we see that observation 10 given above allows us to define *fair* vertices where the classes with full employment are the ones which receive wages.

Definition 6.2.1. *A vertex $v_{|I|,|J|}$ is a fair vertex if and only if $I = \{i|\ m_i > 0\} = I' = \{i|\sum_j T_{ij}q_j = 1\}$.*

According to this definition, fair vertices actually correspond to Heterodox equilibrium points, as the following theorem shows.

Theorem 6.2.2. *Let $v_{|I|,|J|}$ be a vertex with $m_i > 0$ $\forall i \in I$ such that $I \subseteq [m]$ and $q_j > 0$ $\forall j \in J$ such that $J \subseteq [n]$. If $v_{|I|,|J|}$ is fair, then it is also a Heterodox equilibrium point.*

Proof. Let us recall the definition of a Heterodox equilibrium point. We want (p, q, w, X)

such that q, w are the optimal solutions of $f_{\mathcal{P}}$ and (p, X) is the Fisher optimal solutions, i.e., solutions to $f_{\mathcal{C}}$. In other words, along with the Fisher conditions, $Tq \leq Y$ and $p \leq wT$ should hold true such that equations are tight for all $i \in I,\ j \in J$.

Firstly, as observation 1 notes, $z_{ij} > 0$ only for I, J, which restricts the Fisher forest to active goods and classes. Then, as observations 2 and 6 state, (6.1), (6.7) account for the bang per buck conditions. Since $I = I'$, (6.5) reduce to $\sum_j z_{ij} = m_i\ \forall i \in I$. Then, (6.4- 6.6) satisfy the market clearance conditions, so that (p, X) is the solution to $f_{\mathcal{C}}$.

Moving further, we see that $wT \geq p$ and $Tq \leq 1$ are the defining equations of the region, as given by (6.2), (6.3). From the definition of *fair* vertices, it is easy to see that w and q are dual variables of each other and are the solutions to $f_{\mathcal{P}}$. We also observe that the conditions (6.1-6.2) define U-levels, T-levels and a range of feasible prices for goods with $q_j = 0$. Therefore, we establish that every *fair* vertex corresponds to a Heterodox equilibrium point. □

6.3 Vertices, Edges, and Transitions

In this section, we analyse the vertices and edges of the polytope-like region X defined above. We start with $(1, 1)$ vertices and subsequently analyse the $(1, k)$ and $(k, 1)$ cases by gradually increasing the number of goods and classes. We then take up a $(2, 2)$ vertex and examine its adjacencies. The motivation of the theory is clear, it is to reach an optimal vertex by traversing such vertices and edges of X.

6.3.1 $(1, 1)$ Case : A vertex

Let i be any row index and j a column index such that $\frac{T_{ij}}{u_{ij}} = \min_{j'} \frac{T_{ij'}}{u_{ij'}}$. We construct a feasible point $v_{i,j}$ with the above data with the property that $q_j > 0$ and $m_i > 0$ and but all other q's and m's are zero.

Let i^* be such that $T_{i^*,j}$ is the largest entry in that column. First, let us define $q_j = 1/T_{i^*,j}$, $p_j = 1/q_j = T_{i^*,j}$ and $m_i = p_j/T_{ij} = T_{i^*,j}/T_{ij}$. Next, let $y_i = p_j/u_{ij}$. We set $p_{j'} = y_i u_{ij'}$ for all $j' \neq j$. For $i' \neq i$, we choose $y_{i'}$ as the minimum of $p_k/u_{i'k}$ over all k. Finally, we put

$z_{i'j'} = 0$ for all i', j', except z_{ij} which equals 1.

Let us check the conditions under which $v_{i,j}$ so defined is feasible. Without loss of generality, we assume that $m_1 > 0$ and $q_1 > 0$. Let i^* be such that T_{i^*1} is the largest entry in column 1. Now, equations (6.9) are easily satisfied. Next, we see that the assignments of p_j's ensure that (6.1) are satisfied for $i = 1$. The cases $i = 2, 3, ...m$ are similar. As regards (6.2), (6.8), the definition of j and p_j imply the conditions. Equations (6.3) require that $T_{i1}/T_{i^*1} \leq 1$ for all i, which comes from the construction. Equations (6.4)-(6.6) are easily satisfied. Finally, in (6.7), since only $z_{11} > 0$, we check if $p_1 - y_1 u_{11} = 0$, and this is true. Thus $v_{1,1(i^*)}$ is indeed a feasible point when T_{i^*1} is the largest element in its column.

Let us now check if $v_{1,1(i^*)}$ is a vertex. Let us briefly recall the implications of the KKT conditions. Let $D \subset \mathbb{R}^M$ be a *domain* given by K inequalities $f_i(x) \leq 0$, Let $p \in D$ be a feasible point and let I_p be the tight-set of indices, i.e., $I_p = \{i | f_i(p) = 0\}$. We say that p is a *vertex* iff the vectors $V_p = \{(\nabla f_i)(p) | i \in I_p\}$ is full rank, i.e., $rank(V_p) = M$.

Let us now list all tight inequalities in this special case, and check if V_p is of full rank.

Equations

From (6.1), we see that the equations are tight for all $(1, j)$ pairs by the specific construction. Also, depending on the minimum of p_k/u_{ik} for each $i \neq 1$, a constraint is tight. Thus, $m+n-1$ equations are tight.
From (6.2), only one equation is tight, which is $p_j = m_i T_{ij}$.
Similarly, from (6.3), only one equation is tight, which is $T_{i^*1} q_1 = 1$.
From (6.9), p_j, y_i are positive $\forall i, j$. Similarly, $m_1, q_1, z_{11} > 0$. All remaining variables are zero. Thus, $mn + m + n - 3$ equations are tight. Also, (6.4-6.6) are obviously tight.

Vertex

As stated above, we have $2m + 2n + mn$ variables in the $\mathbb{R}^{mn+2m+2n}$ space. From these, we can drop $mn + m + n - 3$ variables which come from the tight equations of (6.8) as all these are equal to zero and their gradients are clearly independent of each other. After removing this subset, there are $m + n + 3$ variables left. From (6.1), by construction, y_1 is defined as p_1/u_{11} and all other prices are defined in terms of p_1 by $y_1 \times u_{1j}$. This gives n tight equations. Similarly, y_i are defined in terms of p_1. This tells us that these variables are functions of p_1

and can then be dropped. Thus, there are $m+n+3-(m-1)-n=4$ variables, which are m_1, p_1, q_1, z_{11}. Treating the dropped variables as constants, let us consider these four equations.

$$(p_1 - m_1 T_{11}) = 0$$

$$T_{i^*1} q_1 - 1 = 0$$

$$p_1 q_1 = 1$$

$$z_{11} = 1$$

We check that the set of gradients of these equations give rank 4. Following the order [m_1 p_1 q_1 z_{11}] while computing the gradients, we get,

$$D = \begin{bmatrix} T_{11} & 1 & 0 & 0 \\ 0 & 0 & T_{i^*1} & 0 \\ 0 & q_1 & p_1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It can be easily seen that the determinant of above matrix is nonzero which establishes that $v_{1,1(i^*)}$ is a vertex. In this case, we have $a=b=c=d=0$, and this is generic vertex.

Remark 6.3.1. *Note that the vertex so defined may not be fair. In fact, for each $i \in I$, we can define a unique $v_{1(i),1(j)}$ where j is determined through T_i and U_i. For 'fairness', i.e., Heterodox equilibrium, the required condition is $T_{ij} = \max_{j'} T_{ij'}$.*

6.3.2 $(1, k)$ Case

Having proved that $v_{1,1}$ is a vertex, we now see how equations vary as we make another q_j non-negative. In other words, we check if it is possible to determine all the variables when we consider one buyer and a production of two goods in a general $m \times n$ market. Let $m_i > 0$ and $q_{j1}, q_{j2} > 0$. If two q_j are positive, they must satisfy $p_j = m_i T_{ij}$. Moreover, (6.4) implies that $z_{ij1}, z_{ij2} > 0$, so that (6.7) implies $\frac{p_{j1}}{p_{j2}} = \frac{u_{ij1}}{u_{ij2}}$. This contradicts the generic T, U assumption. The same analysis is valid for a general $(1, k)$, $k > 1$ case as well. In particular, we see that Theorem (4.1.1) is valid for vertices of X as well. In this case, the least number of connected components required is 2, which is not feasible for the (1,2) case.
Next, we examine the $(k, 1)$ cases by successively increasing the number of non-negative buyers.

6.3.3 $(k,1)$ Case : An Edge

Let us now consider the case where there are two non-zero $m_i's$ and one non-zero q_j. We move from the (1,1) case with $m_1, q_1 > 0$ to this case by relaxing $m_i = 0$. Without loss of generality, let $m_2 > 0$. Later, we describe the necessary conditions on m_1, m_2. Let T_{i^*1} be the largest entry in its column so that $T_{i^*1}q_1 = 1$. When $m_2 > 0$, $z_{21} > 0$ and thereby $y_2 = p_1/u_{21}$. Let $p_1 = 1/q_1 = T_{i^*1}$ and $y_1 = p_1/u_{11}$. Let $p_j = \max(y_1u_{1j}, y_2u_{2j})$. Let us choose $y_i = \min_k \frac{p_k}{u_{ik}}$ for $k \neq 1,2$. We also have $m_1T_{11} + m_2T_{21} = p_1$. Finally, we put $z_{11} = 1 - m_2$, $z_{21} = m_2$ and all other $z_{ij} = 0$.

Let us now check the conditions under which this point is feasible. Constraints (6.1) are satisfied by the specific construction. From (6.2), we have $m_1T_{11} + m_2T_{21} = p_1 = T_{i^*1}$ and m_2 is such that m_1, m_2 satisfy-

$$\forall j' \neq 1, \quad p_{j'} = \max(y_1u_{1j'}, y_2u_{2j'}) < m_1T_{1j'} + m_2T_{2j'}$$

For (6.3), as per the definitions, $T_{i^*1}q_1 = 1$ and other inequalities are satisfied too. Since $p_1q_1 = 1$, it can be seen that the constraints (6.4), (6.5), (6.6) hold true. As in the last cases, (6.7-6.9) are easily satisfied. Thus, $v_{2,1}$ is indeed feasible. Next, we reduce the system by dropping dependent and zero variables to $[m_1, m_2, p_1, q_1, z_{11}, z_{21}]$. The five equations concerning these variables are-

$$(p_1 - T_{11}m_1 - T_{21}m_2) = 0$$

$$T_{i^*1}q_1 = 1$$

$$z_{11} + z_{21} = p_1q_1$$

$$z_{21} = m_2T_{21}q_1$$

$$p_1q_1 = 1$$

Gradients of these equations with the order $[m_1\ m_2\ p_1\ q_1\ z_{11}\ z_{21}]$ are -

$$D = \begin{bmatrix} -T_{11} & -T_{21} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_{i^*1} & 0 & 0 \\ 0 & 0 & -q_1 & -p_1 & 1 & 1 \\ 0 & -1 & 0 & -m_2\,T_{21} & 0 & 1 \\ 0 & 0 & q_1 & p_1 & 0 & 0 \end{bmatrix}$$

We see that rank of the above matrix is 5, which means that $v_{2,1}$ forms an edge. The total rank of the system is thus, $mn + 2m + 2n - 1$, which is one less than the full. This is also evident from the equation - $m_1T_{11} + m_2T_{21} = T_{i^*1}$, which forms a line.

Note that this case can also be generalized to the $(k, 1)$ case. Let us see that by introducing a nonzero m_3, which thereby adds a nonzero z_{31} too, according to (6.7). When we look at the governing equations given above, the first equation remains the same except that the term m_3T_{31} is added. z_{31} is defined in the same manner as z_{21}. Concluding as before, rank of the reduced system is 6, which is 2 less that the full rank, i.e., 8. This is because of the equation $p_1 = m_1T_{11} + m_2T_{21} + m_3T_{31}$, which involves three undetermined variables m_1, m_2, m_3. Building on this, we say that the rank of a general $(k, 1)$ case is $mn + 2m + 2n - k + 1$.

6.3.4 $(2, 2)$ Vertex and its adjacency

Next, we take up a $(2, 2)$ vertex and explore its adjacent vertices by relaxing appropriate edges. We start with defining $v_{2,2}$ and later look at its transformation to $v_{1,1}$, $v_{2*,2}$, an another $(2, 2)$ vertex, and $v_{3,3}$.

Without loss of generality, we let $m_1, m_2, q_1, q_2 > 0$ and $T_{i*1}m_1 + T_{ik*1}m_2 = 1$, $T_{i*2}m_1 + T_{k*2}m_2 = 1$. Let (6.1) and (6.2) be tight for $i, j = \{1, 2\}$. From (6.9), let us have $z_{11}, z_{21}, z_{22} > 0$. Let $p_{j'} = \max(y_1u_{1j'}, y_2u_{2j'})\ \forall j$ and $y_i = \min_k \frac{p_k}{u_{ik}}$. Let $\frac{p_1}{p_2} = \frac{u_{21}}{u_{22}} < \frac{u_{11}}{u_{12}}$ and m_1, m_2 satisfy

$$\forall j' \neq 1, 2, \quad p_{j'} = \max(y_1u_{1j'}, y_2u_{2j'}) < m_1T_{1j'} + m_2T_{2j'} \tag{6.10}$$

As done before, it can be checked that $v_{2,2}$ so defined is a generic vertex of X. Note that only when $\{i*, k*\} = \{1, 2\}$, the vertex is fair.
Firstly, we relax the equation $p_2 = m_1T_{12} - m_2T_{22}$. As a result, we see that $q_2 = z_{22} = 0$. Equations (6.3) are not tight. Dropping dependent and zero variables as before, we have [m_1 m_2 p_1 q_1 z_{11} z_{21}] variables and these equations-

$$(p_1 - m_1T_{11} - m_2T_{21}) = 0$$

$$z_{11} + z_{21} - p_1q_1 = 0$$

$$z_{21} = m_2(T_{21}q_1)$$

$$p_1q_1 = 1$$

It can be checked that the rank of gradient of the above system is 4, which gives a plane. Null vectors corresponding to this are,

$$\begin{matrix} m_1 \\ m_2 \\ p_1 \\ q_1 \\ z_{11} \\ z_{21} \end{matrix} \begin{bmatrix} -\frac{p_1 - m_2 T_{21}}{q_1 T_{11}} & -\frac{1}{q_1 T_{11}} \\ -\frac{m_2}{q_1} & \frac{1}{q_1 T_{21}} \\ -\frac{p_1}{q_1} & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$$

Now, following the first null vector, we reach an edge $(2, 1)$, when q_1 increases so that the equation $T_{i*1}q_1 = 1$ becomes tight, where $T_{i*1} = \max_i\{T_{i1}\}$. Here, rank of the gradient matrix is 5. Null vector of this is given by-

$$\begin{matrix} m_1 \\ m_2 \\ p_1 \\ q_1 \\ z_{11} \\ z_{21} \end{matrix} \begin{bmatrix} -\frac{1}{q_1 T_{11}} \\ \frac{1}{q_1 T_{21}} \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

which suggests that we reach a $(1, 1)$ vertex, when m_1 or m_2 become zero. Thus, depending on the entries of T and U, the edges and vertices of X are defined.

Next, we consider the $(2, 2)$ vertex again and relax the equation $T_{i*1}q_1 + T_{i*2}q_2 - 1 = 0$. Dropping dependent and zero variables, we have $[m_1\ m_2\ p_1\ p_2\ q_1\ q_2\ z_{11}\ z_{12}\ z_{21}\ ]$ variables and these equations-

$$(p_1 - m_1T_{11} - m_2T_{21}) = 0$$

$$(p_2 - m_1T_{12} - m_2T_{22}) = 0$$

$$\frac{p_1}{p_2} = \frac{u_{21}}{u_{22}} = \alpha$$

$$T_{k*1}q_1 + T_{k*2}q_2 - 1 = 0$$

$$z_{11} + z_{21} - p_1q_1 = 0$$

$$z_{22} - p_2q_2 = 0$$

$$z_{21} + z_{22} = m_2(T_{21}q_1 + T_{22}q_2)$$

$$p_1q_1 + p_2q_2 = 1$$

Note that we have 8 equations for 9 variables. We see that the rank of the gradients matrix is 8, which confirms that this is an edge. As the null vector of this gradient suggests, all variables of this system deviate while traversing the edge. We reach another vertex when an equation $T_{h*1}q_1 + T_{h*2}q_2 - 1 = 0$ becomes tight with (6.10) holding true. In this transformation, the combinatorial data $I = \{1, 2\}$ remains the same but $I' = \{i | \sum_j T_{ij}q_j = 1\}$ changes from $\{i*, k*\}$ to $\{h*, k*\}$.

Next, we relax the equation $m_i > 0$ for $i \neq \{1, 2\}$, i.e., we add an entry to the set of active buyers I. We see that (6.5) forces either of z_{31} or z_{32} to be positive. That makes either $y_3u_{31} = p_1$ or $y_3u_{32} = p_2$ tight. Note that the equations $y_1u_{11} = p_1, y_2u_{21} = p_1$ and $y_2u_{22} = p_2$ remain valid, i.e., $p_1 = p_2 \times \alpha$ holds true. Therefore, using the ratios p_1/p_2, u_{31}/u_{32}, it is easy to compute y_3. Without loss of generality, let us assume $y_3u_{32} = p_2$. Dropping the dependent and zero variables, we get the following variables $[m_1\ m_2\ m_3\ p_1\ p_2\ q_1\ q_2\ z_{11}\ z_{21}\ z_{22}\ z_{32}]$ and tight equations-

$$(p_1 - m_1T_{11} - m_2T_{21} - m_3T_{31}) = 0$$

$$(p_2 - m_1T_{12} - m_2T_{22} - m_3T_{32}) = 0$$

$$\frac{p_1}{p_2} = \frac{u_{21}}{u_{22}} = \alpha$$

$$T_{i*1}q_1 + T_{i*2}q_2 - 1 = 0$$

$$T_{j*1}q_1 + T_{j*2}q_2 - 1 = 0$$

$$z_{11} + z_{21} - p_1q_1 = 0$$

$$z_{22} + z_{32} - p_2q_2 = 0$$

$$z_{21} + z_{22} = m_2(T_{21}q_1 + T_{22}q_2)$$

$$z_{32} = m_3(T_{31}q_1 + T_{32}q_2)$$

$$p_1q_1 + p_2q_2 = 1$$

As done before, we compute gradient matrix of these equations. From the rank of the matrix, which is 10, we see that the equations form an edge. According to the null vector, p_1, p_2, q_1

and q_2 are constants, while other variables have directions of increase depending on the values of T, U. In particular, m_1 and m_2 have opposite directions of increase and similarly, each of z_{11}, z_{21} and z_{22}, z_{32} have different directions of increase.
There are a number of possibilities for vertices that this edge can connect to, including various $(2,2)$ and $(3,3)$ vertices. Here, we explore one such edge. Note that (6.2) holds true for all $j \in [n]$. While $m_i,\ i \in 1,2,3$ change along the edge, let $p_3 < \sum_i m_i T_{ij}$ become tight. At this point, we also have $q_3 = 0$. Then, according to Theorem 6.2.1, the resulting point is a vertex with $I = J = 1,2,3$ and $b = d = 1$. In other words, it forms a vertex, which is a non generic point. We denote such vertices as 'transition' vertices. Further, we may relax $q_3 = 0$ where we get an edge. It connects to a $(3,3)$ vertex when the corresponding equation from (6.3) becomes tight.

This analysis shows that vertices with different combinatorial data I, J are connected to each other through edges and it is possible to change the data by relaxing appropriate edges. We see that the $(2,2)$ vertex is connected to both $(1,1)$ and $(3,3)$ vertices. This way, we characterize the vertices and traverse the edges to gradually increase the combinatorial data. We may also reach 'fair' vertices by altering the combinatorial data I, as shown for the $(2,2)$ vertices.

6.4 A Partial LCP Formulation

In this section, we give a partial LCP formulation of the Heterodox model, which captures the set of Heterodox equilibria. Let us consider the following constraints and their complementary conditions-

$$\forall i \in [m], j \in [n] \ : \ y_i u_{ij} \leq p_j \quad \text{and} \quad (p_j - y_i u_{ij}) z_{ij} = 0 \tag{6.11}$$

$$\forall i \in [m], \ : \ \sum_j z_{ij} \leq m_i \quad \text{and} \quad (\sum_j z_{ij} - m_i) h_i = 0 \tag{6.12}$$

$$\forall j \in [n] \ : \ p_j - \sum_i m_i T_{ij} \leq 0 \quad \text{and} \quad (p_j - \sum_i m_i T_{ij}) q_j = 0 \tag{6.13}$$

$$\forall i \in [m] \ : \ \sum_j T_{ij} q_j \leq 1 \quad \text{and} \quad (\sum_j T_{ij} q_j - 1) m_i = 0 \tag{6.14}$$

For each fixed y_i, for $i \in I$, (6.11) define the optimality conditions for Fisher market as only for a particular set of j i.e $\{j|y_i = \frac{p_j}{u_{ij}}\}$, the allocations may be nonzero. We have the market feasibility (6.13) and labour constraints (6.14) here, and we let q_j, m_i be their dual variables, respectively. This serves two purposes. Firstly, it ensures that only optimal goods are produced and only fully employed classes earn wages. Secondly, $\forall j,\ \ p_j q_j = q_j(\sum_i m_i T_{ij})$, and $\forall i,\ m_i \sum_j T_{ij} q_j = m_i$, imply $\sum_i m_i = \sum_j p_j q_j$ which means that total money is conserved and (6.12) requires market clearance of money. We note that (6.14) is not considered while defining X and it captures the *fairness* condition.

We note that the equations can be brought to the LCP form, *e.g.*, for (6.11), we introduce new variables $v_{ij} = p_j - y_i u_{ij}$, which give the conditions $v_{ij}, z_{ij} \geq 0$ and $v_{ij} z_{ij} = 0$. Next, we have non-negativity conditions for all the variables.

$$\forall i \in [m], j \in [n] \quad x_{ij}, h_i, m_i, y_i, p_j, q_j \geq 0 \tag{6.15}$$

Let us call the LCP defined by the constraints (6.11- 6.15) as HET-LCP. It is easy to see that equilibrium points of the Heterodox model correspond to its solutions. Clearly, (6.13-6.14) hold true as these are the complimentary slackness conditions for $f_{\mathcal{P}}$. Further, y_i captures the bang per buck ratio for each $i \in I$. For the goods j which are not produced, U- levels are lower than the T-levels which imply that (6.11-6.13) hold true for all j.

We see that there are two shortcomings in HET-LCP, as regards its computability. Firstly, not all solutions to the LCP are equilibrium points. The formulation is homogeneous in h as (6.12) have the rhs constraints vector (denoted by q in Appendix A) as zero. As a consequence, solutions of the LCP include $(y = \bar{0}, p = \bar{0}, q, m = \bar{0}, X = \bar{0})$, where q satisfies (6.14). For the $h > 0$ space, however, nonzero solutions may be computed. Moreover, the non-linear constraints $\sum_i z_{ij} = p_j q_j$ may not be satisfied, though the total money is conserved. A heterodox solution is computed from the solution set to the LCP satisfying these equations, which implies that it may not be unique. Secondly, as we know, Lemke's algorithm can be used to compute a solution to the LCP, but it may terminate without finding an equilibrium. We see that an alternate LCP formulation can also be given consisting of (6.11), (6.13-6.15), where we solve for Z using (6.4-6.6). However, this formulation also suffers from homogeneity and the resultant Z_{ij}'s need not be non-negative.

Chapter 7

Applications - The Trade Theory

Theory of comparative advantage is a classical theory, developed in 1817 by David Ricardo to explain why countries should engage in trade even if one country is more efficient at producing every single good as compared to other countries [11]. Using the theory of Heterodox model and the tatonnement process described earlier, we look at the trade theory and model its various outputs such as production, prices and wages. In particular, we see how it is possible to derive Ricardo's results and determine other unsolved variables which are crucial to the economy.

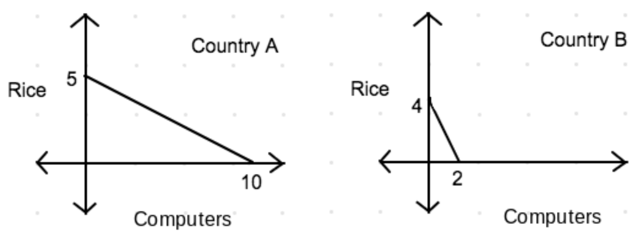
7.1 Ricardo's Theory of Comparative Advantage

In its basic form, Ricardo's theory considers two goods produced by two countries and shows that it is profitable to trade in terms of the goods produced and societal happiness achieved in both countries. Let us call this way of allocating resources as Algorithm-2, with the Heterodox model being Algorithm-1. As an illustration of Ricardo's theory, we refer to the following example.

Example 9. *Let us consider two countries A and B with 100 labour inputs each, that produce only two goods, computers and rice. Labour is homogeneous, i.e., there's no further division in terms of skills. Let both countries have different production technologies so that A is more* *efficient in producing both. Let the number of labour inputs needed by countries A and B to produce a unit of each good be given by-*

	Computers	**Rice**
Country A	10	20
Country B	50	25

Let us first look at the countries in autarky, where no trading takes place. Here are their production possibility frontiers (PPF), which give the possible tradeoff of producing goods in various combinations while keeping the labour and technology constraints -



In Algorithm-2, we see that the ratio of prices in autarky is decided as the inverse of the ratio of maximum possible production of each good. If the ratio equals any other value, the country would produce only the most profitable good, where the production does not meet with the demand for another good. This may be seen from the following optimization program for producers in country A.

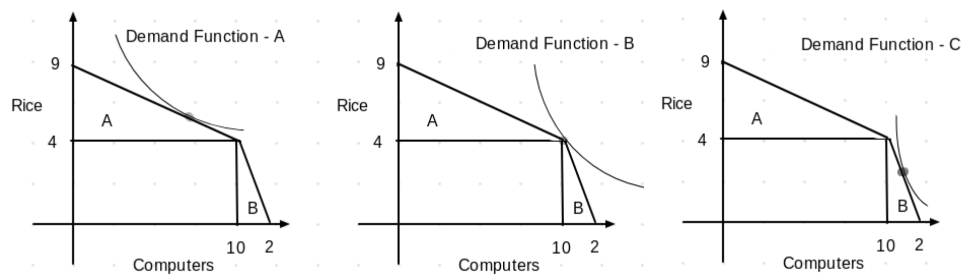
$$\begin{aligned} \max_{q} \quad & p_c q_c + p_r q_r \\ s.t. \quad & 10q_c + 20q_r = 100 \\ & q_c, q_r \geq 0 \end{aligned} \tag{7.1}$$

So, we have, for country A and B, the ratio of autarky prices as implied by the KKT conditions-

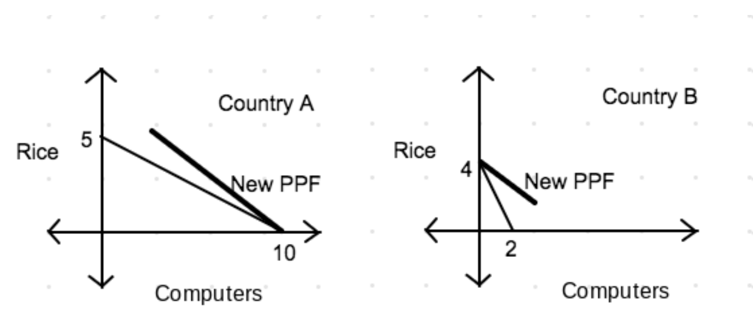
$$\left(\frac{p_c}{p_r}\right)_A = \frac{10}{20} = \frac{1}{2}, \quad \left(\frac{p_c}{p_r}\right)_B = \frac{50}{25} = \frac{2}{1}$$

Given the PPF, let us now decide the actual choice of production. This may be done by maximizing total social utility using a utility function. This can also be understood through demand functions. For both the countries, production lies on the PPF.

Let us now analyse the PPFs which get modified after the trading starts. Firstly, the optimal production is obtained through utility maximization on the combined PPF. Depending on the utility or demand functions, the KKT conditions imply that there are three possibilities *for optimal production, which also decides the relative ratio of prices. When both goods are produced, the relative ratio of prices, i.e., slope of the demand function at the optimal point is between 0.5 and 2, both inclusive.*



We see that with case-1, the ratio is 0.5, and B produces only Rice, and vice versa for Case-3. With the standard demand functions, let the solution be the second case. Now, with this price ratio, since it is greater than 0.5, country A will produce only computers, something it is comparatively better at. Similarly, Country B will produce only rice. Thus, with this new ratio of prices, PPFs will get modified-



Though Algorithm-2 doesn't explain absolute wages, it considers real wages or purchasing power. Let us see this by dividing the homogeneous labour classes from each country into two, each producing a good; for example, Rice producers and Computer makers. Real wage of a computer worker in terms of rice is given by $\frac{w_c}{p_r}$*, where* w_c *and* r_c *stand for wage of a computer worker and price of one unit of rice. [12] Before trading, each labour division has same real wages in terms of rice and computers; otherwise the labour would shift. After the trade begins, real wages of country 1 labour, i.e., computer workers increase in terms of rice and remain the same in terms of computers and vice versa. This is because the ratio of prices of rice to computers increases in A.*

With this new relative ratio, each country can buy more number of goods by exchanging the goods it is producing. For example, if the ratio is 1.5, country A will be able to sale one computer for 0.66 sacks of rice, as opposed to 0.5 before. Because of this outward shift in the PPFs, both countries are now better off. We see that with case 1 or 3 as well, at least one country benefits from trading. Hence, the theory of comparative advantage tells that trading makes countries produce goods that they are comparatively more efficient at, thereby increasing market's efficiency.

However, we note that this Algorithm-2 does not allow us to compute the wages or allocation of money and goods across the countries. In practical scenarios, though both countries are better off after trading, there's reallocation of money due to technological differences and free market, which remains unexplained. Also, the autarky wages do not give a satisfactory and accurate account of wages and allocations.

In case of prices, those can be found by various ways, perhaps relating to various theories of relative prices. One way could be to feed the production values in the Fisher market and compute prices which assumes initial money allocations. Another focuses on the capital-labour ratio, which is not considered here. In this case, another way is to assume same wages in a class and thus derive autarky prices by finding the technology ratio. Lastly, one can find prices by computing the production ratio as per the Ricardian theory. However, none of these give a notion of equilibrium wages, prices and allocations when two countries trade given their technological differences and utility preferences. The theory developed here focuses exactly on these aspects.

We first consider an example where production of a single country is computed using Algorithm-2, with piecewise linear concave (plc) utilities as the demand functions. There are some remarkable properties exhibited by such structure which reestablish the relation between KKT conditions for optimality and the Fisher market. We then consider two cases with different types of utilities- linear and plc. We assume that consumers in both countries are rational, i.e., they have same utilities towards the goods. In case of linear utilities, we prove that both Ricardo's model and the Heterodox model produce same results for productions. For plc utilities, we consider an example to illustrate the Ricardian theory and then analyse it using the Heterodox model.

Utilities as demand functions

Here we consider a country with two producible goods, *viz.* computers and rice, and compute its most efficient production using utilities as demand functions. We then also look at how Fisher prices are derived in case of plc markets [7]. Let us consider a market comprising of a labour class of 300 homogeneous units.

- Let the technology requirements be-

Computers	**Rice**
15	10

  That is, 20 labour units are needed to make one computer and 10 for a rice sack.

- The (plc) utilities in two segments are as follows:

$$\begin{array}{rcll} & & \text{C} \quad \text{R} & \\ U_1 & = & [0.5 \quad 0.8] & \\ U_2 & = & [0.4 \quad 0.2] & \end{array} \qquad \begin{array}{rcl} & & \text{C} \quad \text{R} \\ L_1 & = & [4 \quad 6] \\ L_2 & = & [10 \quad 15] \end{array}$$

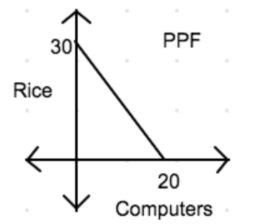
  Here, $U1$ and $U2$ denote the utilities in the first and second segment, respectively. Note that the utilities decrease from U_1 to U_2 after L_1 units of goods are consumed, to give a concave formulation. Also, instead of per person values, we consider per class values here, i.e., utility of computers decreases when the society consumes 4 units as a whole.

Now, we consider Algorithm-2 to determine optimal production, where x_1 and x_2 are computers and rice produced in A and x_3 and x_4 are the same for B:

$$\begin{aligned} \max_x \quad & U(x) = 0.5x_1 + 0.8x_2 + 0.4x_3 + 0.2x_4 \\ \text{s.t.} \quad & 15x_1 + 15x_3 + 10x_2 + 10x_4 = 300 \end{aligned} \tag{7.2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 6 \\ 10 \\ 15 \end{bmatrix}$$

Where we get our first constraint from the PPF



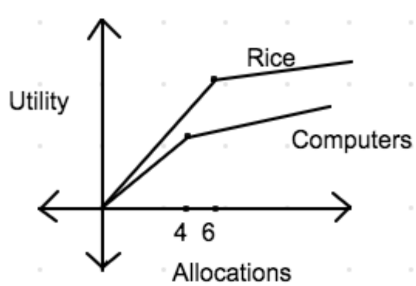
We can now solve the optimization problem to get the following results:

	Computers	**Rice**
Production	14	9

As explained before, Ricardian theory implies that the ratio of prices equals $\frac{20}{10} = 2$. This comes through the optimization program for producers. Thus, we have obtained the optimal production and price ratio using Algorithm-2.
We now consider an optimization program for consumers to find the optimal prices. Let us consider a consumer i with money m_i, utilities u_{ic}, u_{ir} and allocations x_{ic}, x_{ir}. The consumer maximizes her utility by spending at most m_i amount.

$$\begin{aligned} \max_x \quad & u_{ic}x_{ic} + u_{ir}x_{ir} \\ \text{s.t.} \quad & p_c x_{ic} + p_r x_{ir} \leq m_i \\ & q_c, q_r \geq 0 \end{aligned} \tag{7.3}$$

To find the prices, we see that KKT conditions and the Fisher market solution imply the same solutions.



Since 4 computers and 6 sacks of rice are used up in the first segment as required by L, the current allocation is in the second segment for both the goods. We know that utilities at the operating points are 0.4 and 0.2. Hence, when we maximize the total utility, KKT conditions imply that the ratio of prices is equal to the ratio of these utilities. This is also equal to the one given by Fisher market. We thus reestablish the connection between the KKT conditions and the Fisher market [2].

7.2 Ricardo and Heterodox

Firstly, we prove that the Heterodox way of finding optimal production indeed gives its output on the PPF.

Theorem 7.2.1. *Given a market comprising of two or more countries, corresponding technology matrix T, utility matrix U, and segments matrix L, the optimal production found using the process $\mathcal{P}$ lies on the PPF.*

Proof. To find the production, we use the following optimization program

$$\begin{aligned} P(X) = \max_{q} \quad & p_1q_1 + p_2q_2 + p_3q_3 + p_4q_4 \\ \text{s.t.} \quad & q_j \geq 0 \;\; \forall j \\ & T_{11}q_1 + T_{12}q_2 \leq Y_1 \\ & T_{21}q_3 + T_{22}q_4 \leq Y_2 \end{aligned} \tag{7.4}$$

Whereas, the (PPF) is a curve consisting of all maximum output possibilities for two goods, given resource constraints. In other words, for a point on PPF, production of one good can only increase if that of other good decreases. Let us assume that the optimal production given by the above program does not lie on the PPF.
In case of a generic vector p, the KKT conditions require that -

$$p_1 = T_{11}w_1 - \lambda_1$$

$$p_2 = T_{12}w_1 - \lambda_2$$

$$p_3 = T_{21}w_2 - \lambda_3$$

$$p_4 = T_{22}w_2 - \lambda_4$$

where λ correspond to the multipliers associated with $q_j \geq 0$.

It is evident that for positive q_j's, λ_j should equal zero, so p_j should be equal to $T_{ij}w_i$. Therefore, for a given vector p, either q_1 or q_2 is positive. Similarly, one of q_3 and q_4 is positive. Thus, it can be deduced that each country would specialise by producing exactly one good.

Since the optimal point does not lie on the PPF, without loss of generality, let the optimum point Q be such that $X = q_1 + q_3$ (Computers) can increase while $W = q_2 + q_4$ (Rice) remains constant. This means that at least one of q_1 or q_3 increases while $T_{11}q_1 + T_{12}q_2 \leq Y_1$ and $T_{21}q_3 + T_{22}q_4 \leq Y_2$ hold. This is possible only if the inequalities are not tight. This implies that $P(X)$ increases as X increases, which contradicts the optimality of Q. Therefore, the production lies on the PPF of both countries.
We note that this also implies that inequality constraints are tight, i.e., the amount that each country produces lies on that country's PPF. □

7.2.1 Linear utilities

In this section, we prove that both these models produce same output when the countries have linear utilities.

Theorem 7.2.2. *Given a market comprising of two countries, corresponding technology matrix T, and the linear utility matrix U, both the models,* viz. *Ricardo and Heterodox, produce the same output for production.*

Proof. In case of linear utilities, we have U as follows, where buyers in both countries are rational and have same utilities towards the goods.

$$\begin{bmatrix} C & R & C & R \\ u_1 & u_2 & u_1 & u_2 \end{bmatrix}$$

This has two implications. First, the Fisher Forest of allocations is a complete graph where both countries buy both the goods produced. Next, the iterative method gives the following output for prices, irrespective of the produced goods-

$$p = \begin{bmatrix} p_1 & p_2 = \frac{u_2}{u_1} \times p_1 & p_1 & p_2 = \frac{u_2}{u_1} \times p_1 \end{bmatrix}$$

Therefore, we have both optimization programs as-

$$\begin{array}{lcc lcc}
\max\limits_{q} & u_1q_1 + u_2q_2 + u_1q_3 + u_2q_4 & \qquad & \max\limits_{q} & p_1q_1 + p_2q_2 + p_1q_3 + p_2q_4 \\
\text{s.t.} & Tq \leq Y & & \text{s.t.} & Tq \leq Y \\
 & q_j \geq 0 & & & q_j \geq 0
\end{array} \tag{7.5}$$

It can be observed that both programs produce the same outputs for q since

$$\frac{u_1}{u_2} = \frac{p_1}{p_2}$$

We note that there exists only one stable state in case of linear utilities, which can be computed by solving the above program. □

Next, we illustrate the Ricardian market through an example, which considers piece-wise linear concave (plc) utilities. Later, we look at a general model and provide results for this example using the *tatonnement* process.

7.2.2 plc utilities

Here we have two countries, A with 200 homogeneous labour units and B with 50 homogeneous labour units. The countries produce two goods, Computers and Rice, with different technology inputs for each. We consider plc utilities here, by expressing the values in terms of segments. Here are the specifications of the market.

- Let the technology and utilities for both countries be as follows:

	C	R	C	R
A	20	10	0	0
B	0	0	5	10

$$\begin{array}{ccl cccc}
 & & & \text{C} & \text{R} & \text{C} & \text{R} \\
U_1 & = & [& 0.5 & 0.7 & 0.5 & 0.7\ ] \\
U_2 & = & [& 0.43 & 0.2 & 0.43 & 0.2\ ]
\end{array}$$

  We assume that two types of goods are produced in each country - Computers and Rice. The utilities are piece-wise linear, concave with U_1 and U_2 being the utility vectors in the first and second segments respectively.

- The L matrix $(2 \times 1 \times 2)$ (maximum amounts of all segments) is

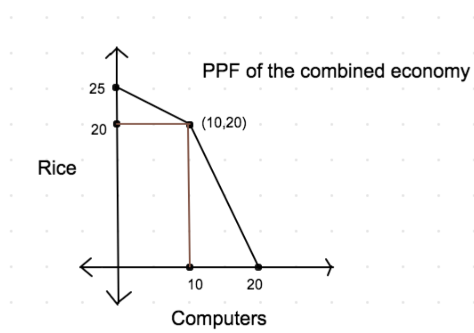
$$\begin{array}{rcl} & & \text{C} \quad\;\; \text{R} \\ L_1 & = & [0.02 \quad 0.3] \\ L_2 & = & [0.08 \quad 0.12] \end{array}$$

  where, values in these two segments are per person per good. For example, for a person in any of the countries, utility for Rice reduces after consuming 0.3 units of a Rice sack. Subsequently, for the combined economy of 250 people, the modified L is

$$\begin{array}{rcl} L_1 & = & [5 \quad 7.5] \\ L_2 & = & [20 \quad 30] \end{array}$$

  Note that the second segment values are superfluous in the sense that we consider only one decrement in the utilities and also that maximum possible production is [20, 25].

Now, PPF(Production Possibility Frontier) of the resultant economy is



We maximize the total utility to find optimal production, which lies on this PPF. Let z_1 and z_3 be the total number of computers consumed in the first and second segments respectively. z_2 and z_4 are for rice. Also, x_{ij} refers to computers produced by the ith country and consumed in the jth segment. Note that this does not talk about consumption by any country, it just says that out of the total computers produced in i, x_{ij} computers were consumed with corresponding utility by both the countries together. Similarly, y_{ij} is for rice. To find the production, the optimization program is -

$$
\begin{aligned}
\max_{z} \quad & U(z) = 0.5z_1 + 0.7z_2 + 0.43z_3 + 0.2z_4 \\
\text{s.t.} \quad & 20x_{11} + 20x_{12} + 10y_{11} + 10y_{12} = 200 \\
& 5x_{21} + 5x_{22} + 10y_{21} + 10y_{22} = 50
\end{aligned} \tag{7.6}
$$

$$
\begin{bmatrix} z_1 = x_{11} + x_{21} \\ z_2 = y_{11} + y_{21} \\ z_3 = x_{12} + x_{22} \\ z_4 = y_{12} + y_{22} \end{bmatrix} \leq \begin{bmatrix} 5 \\ 7.5 \\ 20 \\ 30 \end{bmatrix}
$$

Here, inequalities mean that consumption of each good is bounded by the L_1 values. Also, the equalities are the resource constraints.
Optimization results for the above program are :

	Computers	**Rice**
A	1.47 + 4.78 = 6.25	7.5 + 0 = 7.5
B	3.53 + 6.47 = 10	0 + 0 = 0

For example, $x_{11} + x_{12} = 1.47 + 4.78 = 6.25$ is the total number of computers produced by A. We see that at the optimal production, country A produces both the goods while country B produces only computers. Total production is [16.25 7.5] which lies on the PPF.

For prices, Algorithm-2 optimizes the producer's revenue i.e $p \cdot q$. As seen from the optimization program above, q lies on A's segment in the PPF. Therefore, from the allowed range of slopes $(1/2, 2)$, ratio of relative prices equals 2. This is in line with the fact that B produces only computers and A produces both the goods. We note that because of plc utilities, it is possible for the optimal production to lie on points other than the corner ones on the PPF. On the other hand, in case of linear utilities, the only possible optimal points are the cases where only rice or computers are produced or the point where one country produces rice and another produces computers.

As per the Ricardian theory, we can find the total production and relative prices. So now the question is to determine the wages in this combined economy. Also, it is important to see how these goods are allotted across these countries.

The Heterodox interpretation of Ricardo's model

Though an equilibrium exists as given by the existence theorems, the *tatonnement* process need not converge to it. We note that the existence theorems extend to plc utilities as well. Here, we look at how the *tatonnement* process works when the countries start trading with each other. We do not compute the equilibrium, but analyse the *tatonnement* solutions.

We first optimise the value of $p^T q$ to find the optimal production given the technology constraints, where q : production and p: input prices. Wages are the dual variables of this program. Using KKT conditions, we know that for a general vector p, only two entries of the q vector are nonzero. This is because we have the following type of feasibility conditions -

$$p_1 = T_{11} w_1 - \lambda_1$$

$$p_2 = T_{12} w_1 - \lambda_2$$

$$p_3 = T_{21} w_2 - \lambda_3$$

$$p_4 = T_{22} w_2 - \lambda_4$$

where λ correspond to the multipliers associated with $q_j \geq 0$. It is evident that for positive q_j's, λ_j should equal zero, so p_j should be equal to $T_{ij} w_i$. Thus, it can be deduced that each country would specialise by producing exactly one good. We can find this good in each state by looking at the following relation-
If

$$\frac{p_i}{p_j} > \frac{T_i}{T_j}$$

then, the country will produce good i, otherwise it will produce good j. Since T_i denotes the number of labour inputs required to produce good i, this equation makes perfect sense because it maximises the revenue the country can generate.
Once we know both the prices, wages can be computed as-

$$\frac{w_1}{w_2} = \frac{T_2}{T_1} * \frac{p_1}{p_2} = \frac{p_1/T_1}{p_2/T_2}$$

This can also be seen as a solution to $T^{-1}p$, where T is diagonal. Here, the subscript 1 denotes the first country. Thus, T_1 is the selected technology from the first country and so on.
Now, when we feed this data into the Fisher market, we get a new set of prices. The ratio

of these prices equals the ratio of appropriate utilities when the forest is connected. If the solution consists of only one state, i.e., if no switching is observed, then the next set of wages is again computed by the above mentioned formula. Once the Fisher forest is stable, all parameters can be found easily.

• Toggling states: As a part of our iterator function, every time we run the Fisher market, we attribute some prices to the unselected goods given by the formula-

$$P_{new} = u_{1new} \times \min_i \frac{p_i}{u_{2i}}$$

The states are stable unless the production changes. As stated above, those will change, if

$$\frac{p_i}{P_{1new}} < \frac{T_i}{T_{new}} \quad i.e., \quad \frac{u_{2i}}{u_{1new}} < \frac{T_i}{T_{new}}$$

for any good *new* that was unselected in the earlier iteration and good i that was produced. Because we have two segments here, we often see this kind of phenomenon happening, which results in toggling states. For example, when Fisher market is such that the following holds

$$\frac{p_1}{p_2} = \frac{u_{21}}{u_{22}} \tag{7.7}$$

and if,

$$\frac{u_{2i}}{u_{1j}} < \frac{T_i}{T_j} < \frac{u_{1i}}{u_{2j}}$$

then goods i and j are produced alternatively.

If good i gets over in the first segment itself and second good j is allotted to both the classes in the next segment, we see -

$$\frac{p_i}{p_j} = \frac{u_{1i}}{u_{2j}} \tag{7.8}$$

and the similar analysis holds for this case too.

Now, the results for example-2 are-

```
Wage Iterations-                Equilibrium Prices-

 A          B                  C-A        R-A        C-B        R-B
0.537      0.462               0.05       0.0814     0.05       0.0814
0.765      0.235               0.06       0.024      0.052      0.0843

Goods Iterations -               Total Allocations per person-
```

C-A	R-A	C-B	R-B
10	0	10	0
0	20	10	0

C-A	R-A	C-B	R-B
0.0537	0	0.185	0
0.0308	0.0925	0.0767	0.03

Thus, the stable system has two states, one in which both countries produce computers and the second in which A produces Rice and B produces Computers. The average production of these two states is -

	Computers	**Rice**
A	5	10
B	10	0

On the consumption part, Fisher market computes the prices. The unused goods are given Fisher-like prices in each iteration. Since the wages given here are per country wages, it can be computed that in first state, a person from A gets 0.0027 Rs and that from B gets 0.0092 Rs. While in the second, A gets 0.0038 Rs. and B gets 0.0047 Rs. Because B is more efficient as seen from the technology matrix, on average, a person from B receives more than that from A.

Here, we have looked at the outputs of the *tatonnement* process. In particular, note that this does not talk about the Heterodox equilibrium of the market.

7.3 The Basic Difference

Ricardo's theory is based on utility maximisation where, in a general sense, first the demand function is found using utilities and optimal production is decided by maximizing total utility. Thus, it maximises the total happiness a society can derive. Prices depend on the optimal production. On the other hand, as Eisenberg-Gale algorithm [13] suggests, an answer to Fisher's market is the solution to the following optimization problem-

$$\max_i \sum m_i \log(u_i) \tag{7.9}$$

Thus, Eisenberg-Gale approach gives a more concrete notion as it weighs utility of each person/class with the money available. In other words, because of more purchasing power of the higher classes, their utility gets a higher weightage as compared to the lower ones. An interesting interpretation of this result is that the multi-agent model derives global optimal while all agents mazimize their utilities. Also, it is interesting to see that prices and allocations emerge as a solution to this optimization problem. This is evident as while maximizing the utility, we have taken money considerations into account. Similarly, it is clear from the dual understanding of this problem, which is the determination of wages and production, that wages are nothing but prices attributed to the skills labourers have and their demand from the society.

Chapter 8

Conclusion

This work shows that consumer choice is indeed an important determinant of wage distribution in an economy. This connection provides an important tool for wage-earners to understand how they can adapt their consumption so as to support a more equitable distribution of wages. It does this by providing a modelling and analytic framework which allows us to explore concretely the thread between consumer choice, prices, production and wages.

The thesis also helps us understand pricing of many everyday items, e.g., smartphones, where two similar devices may have very different prices, and also that these prices may dramatically change based on a fluid consumer choice. It also suggests that preferring goods and services provided by small-branded and local/regional players, rather than buying the "best" may be a better strategy to ensure better wages.

Next, the key data required of the economy, viz., T and Y, is a *labour inventory* of the production processes of the economy and is part of some of the standard data sets of countries. Such an inventory could be used to develop a tool allowing each household to compute its *labour footprint*, i.e., an understanding of how their household consumption brings employment across the economy. One conjecture is that the consumption preferences of many wage-earners possibly do not support their own employment. Such an understanding may be useful to these very classes in modifying their personal consumption. Also note that the labour footprint, while very similar to the GDP calculation, does not need monetization. This is important in its own way.

The Heterodox model, may also be applied to the standard 'comparative advantage' arguments in Ricardian economics. Here it helps illustrate how the social benefits of trade actually transmit as wages across various classes.

Technically, the computation of an equilibrium, given a T, U, Y is an interesting problem. The A-D connection implies that already known efficient algorithms will shed some light. The algorithmic computability of fair vertices and the LCP formulation is an important question. The tatonnement process of this paper needs to be strengthened and its 'computing' power needs to be enhanced. A study of the pay-off correspondence and the existence of Nash equilibria in general economies needs to be undertaken.

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Appendix A

Lemke Algorithm

Lemke algorithm is a simplex-type vertex following algorithm, used to compute solutions to Linear complementarity problems. It has numerous applications in Game theory including the computation of Nash equilibria or market solutions. Here, we give a brief description of the problem and its solution [19].

An LCP problem can be posed as -

$$\begin{aligned} &\text{LCP(q,M)} \quad \text{Find : z} \\ &\text{s.t. } Mz - q \leq 0 \\ &z \geq 0 \\ &z(q - mz) = 0 \end{aligned} \tag{A.1}$$

We see that the problem has a nontrivial solution $z \neq 0$ only if $q \ngeq 0$. Now, we introduce slack variables $v = q - Mz$ so that the problem reduces to -

$$Mz + v = q, \quad z \geq 0, \quad v \geq 0, \quad z \cdot v = 0$$

Let P be the polyhedron formed by the first three conditions, and we assume that it is degenerate. This requires that any solution to (A.1) is a vertex, as it must satisfy $2n$ equalities. We now introduce a variable x and call this new system an augmented system.

$$Mz + v - x1 = q, \quad z \geq 0, \quad v \geq 0, \quad x \geq 0, \quad z \cdot v = 0$$

Let P' be the polyhedron formed by the first four conditions. Since any solutions to (A.1) must still satisfy $2n$ equalities, the set of solutions S will consists of vertices and edges of P'. Any solution to (A.1) also satisfies $x = 0$, and is a vertex of P'.

We now categorize the points of S into two. If those have either $z_i = 0$ or $v_i = 0$ for each i, those are termed as singles labels. If those have $z_i = v_i = 0$, those are said to have double label at i. These points are vertices of P' and have only one double label. There are exactly two ways of relaxing the double label, and thus, there are two edges of S incident on such vertices. It can be observed that a solution to (A.1) has $v = 0$ and is not a double label of P'. There is a unique edge from S incident on such vertices, which is obtained by relaxing $v = 0$.

We thus see that S consists of paths and cycles. Lemke's algorithm starts with a special path that either ends at the solution, i.e., a vertex with $v = 0$ or fails where, in most cases, the original system may not have a solution. An edge of S which connects a vertex in S with another which satisfies $v > 0$ is called a ray. A primary ray is special in the sense that it is incident on a vertex with $z = 0$. Lemke's algorithm starts with a primary ray and explores a path via pivoting. The algorithm cannot be used to find trivial solutions, however Lemke-Howson algorithm is applicable for these cases. As discussed above, Lemke's algorithm can be applied to any non-degerate LCP, but it may terminate without attaining an equilibrium. There are a few sufficient conditions for the processability of this algorithm under specific settings like semimontone LCPs.

Appendix B

Numerical Results

In this section, we give some numerical results for a 2×2 case. We look at the wages and total utilities of two classes as functions of their strategies, i.e., utilities.

Let us assume that we have two labour classes with T (Technology matrix), U (Utility matrix) and Y (Labour availability) given by-

$$T = \begin{bmatrix} 0.2501 & 0 \\ 0.25 & 1 \end{bmatrix} U = \begin{bmatrix} 1 & \beta \\ \alpha & 1 \end{bmatrix} Y = \begin{bmatrix} 2 \\ 4 \end{bmatrix} U_r = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Whenever both classes are active, q (production vector) is given by :

$$q = T^{-1}Y = \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix} Y = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

and

$$w = \begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \cdot T^{-1} = \begin{bmatrix} 4p_1 - p_2 \\ p_2 \end{bmatrix}$$

Therefore, class-wise wages may be obtained as -

$$m_1 = 2(4p_1 - p_2)$$

$$m_2 = 4p_2$$

In a general 2×2 economy, the seven possible ways of allocating the produced goods are six forests and a cycle, where both classes participate, and two possible (1×1) graphs, where only one class is active. The combinatorial data (I, J, F) and the actual equilibrium values (p, q, w, X) depend on the choice of α and β.

Consider the above market where, for simplicity, there are two labour classes producing two similar goods, with different production functions as given. They derive their wages only by producing these two goods. Assuming that the players are perfectly rational, their actual utility functions are given by U_r, comprising of all ones. For example, the footwear produced by a brand which employs both the classes and similar one produced by the lower class without any brand value should be valued equally by rational players. Now, depending on their consumption and utility matrix U, we compute their wages.

The following table (Table-1) shows the relationship between β, i.e., the utility preference of class-1 towards good-2 and its wage fraction (wages of class-1 divided by the total money) and total utility. Note that these are equilibrium values, i.e., given these α, β and U, T, Y as above, the market attains an equilibrium with wage fraction and utilities as given in the table. For example, when $\alpha = 0.75$ and $\beta = 0.81$, utility of class-1 is 5.33 and wage fraction is 0.5. As the number of goods produced is [8 , 2], the total utility of the society is 10 since we look at U_r while actually computing the utilities. Thus it follows that if utility of class-1 is x, then that of class-2 is $10 - x$.

We consider four different values for α; 0.75, 1.001, 1.5, 1.7; and for each of them, we study the effect of increase in β on total utility and wages of class-1. For each α, we see

	U	WF	U	WF	U	WF	U	WF
beta	α=0.75		α=1.001		α=1.5		α=1.7	
0.41	5.33	0.5	6.00	0.6	6.66	0.71	6.82	0.74
0.61	5.33	0.50	6.00	0.60	6.66	0.71	7.56	0.74
0.81	5.33	0.50	6.00	0.60	6.76	0.66	6.76	0.66
1.01	5.33	0.50	5.96	0.60	5.96	0.60	5.96	0.60
1.21	5.33	0.50	5.16	0.54	5.16	0.54	5.16	0.54
1.41	4.36	0.48	4.36	0.48	4.36	0.48	4.36	0.48
1.61	3.56	0.43	3.56	0.43	3.56	0.43	3.56	0.43
1.81	2.76	0.38	2.76	0.38	2.76	0.38	2.76	0.38

Table B.1: beta vs Utility and Wage fractions

that the wage fraction is decreasing w.r.t β. Also, U decreases when the utility for good-2 increases. This is because when β increases, total wages decrease, and thus purchasing power decreases, thereby lowering the total utility. The same results can be established for class-2. (Table 2)

	U	WF	U	WF	U	WF	U	WF
alpha	β=0.75		β=1.001		β=1.5		β=1.7	
0.61	5.28	0.42	5.28	0.42	5.28	0.42	5.28	0.42
0.81	4.47	0.53	4.47	0.53	4.47	0.53	6.00	0.45
1.01	3.98	0.60	3.98	0.60	4.01	0.60	6.00	0.45
1.21	3.65	0.66	3.65	0.66	4.01	0.60	6.00	0.45
1.41	3.42	0.70	3.00	0.68	4.01	0.60	6.00	0.45
1.61	3.24	0.73	3.00	0.68	4.01	0.60	6.00	0.45
1.81	3.11	0.76	3.00	0.68	4.01	0.60	6.00	0.45

Table B.2: alpha vs Utility and Wage fractions

We can see that as the wage fraction for class-1 increases, the same for class-2 decreases when α increases. Thus we conclude that a lower α value would help realize more payoff as well as wages. This clearly demonstrates the effects of consumption on wages and utilities or happiness derived.

Appendix C

(2×2) Market and Results

In this section, we generalize the results of a particular (2×2) market given in Chapter-5. We argue that wages are continuous functions of utilities and payoffs and allocations are continuous for each forest. Moreover, we prove the existence of a correspondence between the strategy space and utilities and also that utilities on the boundaries are linear combinations of those of the forests on both sides. We also look at the necessary conditions for Nash equilibria to exist.

Let us consider a 2×2 market with inputs T, Y, $\mathcal{U} = \mathbb{R}^{2\times 2}$ which is a strategy matrix and U_r as the true Utility matrix. Since Fisher solutions do not change if the rows of $\mathcal{U}$ are scaled independently, we see that effectively, $\mathcal{U}$ is given by:

$$\mathcal{U} = \begin{bmatrix} \alpha & 1 \\ \beta & 1 \end{bmatrix}$$

We assume that $0 < \alpha, \beta < \infty$. Let $p_1/p_2 = \alpha$ where α is the appropriate ratio of utilities when the forest is connected or the ratio of some technology inputs when the forest is disconnected. For example, if class-1 buys good 1 and class-2 buys good-2, we have $p_2 T_{21}^{-1} = p_1 T_{12}^{-1}$. Assuming total money in the economy constant and equal to 1, we can solve for q in terms of T, Y.

$$q_1 = T_{11}^{-1} Y_1 + T_{12}^{-1} Y_2$$

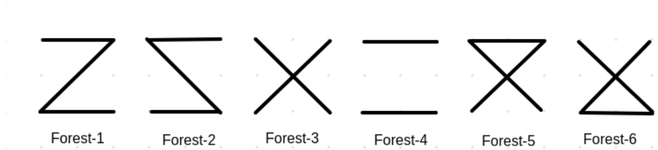
$$q_2 = T_{21}^{-1} Y_1 + T_{22}^{-1} Y_2$$

We assume that T and Y are such that q_1, q_2 are positive. Solving for wages, we get

$$w_1 = \frac{Y_1(\alpha T_{11}^{-1} + T_{21}^{-1})}{Y_1(\alpha T_{11}^{-1} + T_{21}^{-1}) + Y_2(\alpha T_{12}^{-1} + T_{22}^{-1})}$$

$$w_2 = \frac{Y_2(\alpha T_{12}^{-1} + T_{22}^{-1})}{Y_1(\alpha T_{11}^{-1} + T_{21}^{-1}) + Y_2(\alpha T_{12}^{-1} + T_{22}^{-1})}$$

Like the case described before, we know that for a 2×2 market, there are six possible forests and one cycle. Also, there are two possible (1×1) markets. The possible (2×2) forests are-



where each is a solution under specific conditions, which place it in a specific zone. For both classes to be active, we first require $\frac{p_1}{p_2} > \min\left(\frac{-T_{21}^{-1}}{T_{11}^{-1}}, \frac{-T_{22}^{-1}}{T_{12}^{-1}}\right) = \min\left(\frac{T_{21}}{T_{22}}, \frac{T_{11}}{T_{12}}\right)$. For connected forests such that class $i1$ buys only good $j1$ and class $i2$ buys $j1, j2$, we have $w_{i1} < p_{j1}q_{j1}$ and $w_{i2} > p_{j2}q_{j2}$. This ensures that all allocations are positive. Lastly, we require the bang per buck conditions to hold. All these conditions arise from the optimization functions $f_{\mathcal{P}}$, $f_{\mathcal{C}}$ and the Fisher market optimization function.

For example, consider the first graph given above, where $p_1/p_2 = u_{21}/u_{22} = \beta$. This graph constitutes a generic equilibrium if and only if,

1. $\frac{p_1}{p_2} = \beta > \frac{T_{21}}{T_{22}}, \frac{T_{11}}{T_{12}}$, i.e., wages are positive.

2. $\beta > \frac{T_{21}^{-1}Y_1}{T_{12}^{-1}Y_2}$, i.e., allocations are positive.
   $w_1 = Y_1(p_1T_{11}^{-1} + p_2T_{21}^{-1}) < p_1q_1 = p_1(T_{11}^{-1}Y_1 + T_{12}^{-1}Y_2)$ or
   $w_2 = Y_2(p_1T_{12}^{-1} + p_2T_{22}^{-1}) > p_2q_2 = p_2(T_{21}^{-1}Y_1 + T_{22}^{-1}Y_2)$

3. And, $\beta < \alpha = \frac{u_{11}}{u_{12}}$, i.e., class-1 gets more bang per buck from good-1 than good-2. $\left(\frac{u_{11}}{p_1} > \frac{u_{12}}{p_2}\right)$.

It can be seen that these conditions define an open set in the strategy space $\mathcal{U}$ which has a unique combinatorial data (I, J, F). In the above example, it is given by then

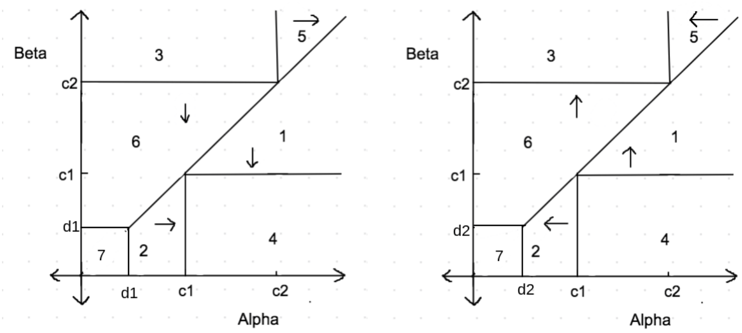
$I = \{1,2\}, J = \{1,2\}$ and F as forest-1. We note that the third condition can be relaxed, i.e., $\beta \leq \alpha$, which results in a non generic equilibrium at $\alpha = \beta$. This is because multiple allocations, including the forest-1 allocations are possible for this condition. We discuss the cycle, i.e., the set $\alpha = \beta$ in the next section.
In addition to the forests described before, we note that it is possible for wages of a particular class to become zero, i.e with a different combinatorial data (I', J'). When $p_1/p_2 \leq \min\left(\frac{T_{21}}{T_{22}}, \frac{T_{11}}{T_{12}}\right)$, only one class is active and it produces exactly one good. We refer to its corresponding domain in the U-space as Zone 7. In this case, if class i produces good j, it is possible to compute its production and the utility U_i that it gives. We note that U_i is smaller than the total utility of both classes combined, as there is more net production when both classes are active.

C.1 The Space of Payoffs as A Manifold

For the forests described above, the combinatorial data (I, J, K) is defined by the following conditions

Forest 1	Forest 2	Forest 3	Forest 4	Forest 5	Forest 6
$\alpha \geq \beta = \frac{p_1}{p_2}$	$\frac{p_1}{p_2} = \alpha \geq \beta$	$\beta \geq \frac{p_1}{p_2} = \frac{T_{22}^{-1}Y_2}{T_{11}^{-1}Y_1} \geq \alpha$	$\beta \leq \frac{p_1}{p_2} = \frac{T_{21}^{-1}Y_1}{T_{12}^{-1}Y_2} \leq \alpha$	$\beta \geq \alpha = \frac{p_1}{p_2}$	$\frac{p_1}{p_2} = \beta \geq \alpha$
$\beta > \frac{T_{21}}{T_{22}}, \frac{T_{11}}{T_{12}}$	$\alpha > \frac{T_{21}}{T_{22}}, \frac{T_{11}}{T_{12}}$	$\frac{T_{22}^{-1}}{T_{11}^{-1}} > \frac{T_{21}}{T_{22}}, \frac{T_{11}}{T_{12}}$	$\frac{T_{21}^{-1}}{T_{12}^{-1}} > \frac{T_{21}}{T_{22}}, \frac{T_{11}}{T_{12}}$	$\alpha > \frac{T_{21}}{T_{22}}, \frac{T_{11}}{T_{12}}$	$\beta > \frac{T_{21}}{T_{22}}, \frac{T_{11}}{T_{12}}$
$\beta > \frac{T_{21}^{-1}Y_1}{T_{12}^{-1}Y_2}$	$\alpha < \frac{T_{21}^{-1}Y_1}{T_{12}^{-1}Y_2}$			$\alpha > \frac{T_{22}^{-1}Y_2}{T_{11}^{-1}Y_1}$	$\beta < \frac{T_{22}^{-1}Y_2}{T_{11}^{-1}Y_1}$

We also graphically represent these six forests by marking their zones. We number these forests from 1 to 6, ordered as given above. 7th zone refers to the case where only one class is active and in equilibrium. Here, the arrows indicate the direction of optimization for the classes, considering U_r as the strategy matrix comprising of all ones.



Here, $c1 = \frac{T_{21}^{-1}Y_1}{T_{12}^{-1}Y_2}$ and $c2 = \frac{T_{22}^{-1}Y_1}{T_{11}^{-1}Y_1}$ and we have assumed $c2 > c1$. The first graph refers to the condition $\det(T) > 0$, i.e., $d1 = \frac{T_{11}}{T_{12}} > d2 = \frac{T_{21}}{T_{22}}$ and second refers to $\det(T) < 0$. There are four possible schemes corresponding to $c1, c2$ and $\det(T)$, of which we have considered two with $c2 > c1$. In the first scheme, class-1 always gets wages and vice versa. Note that the region for 7th zone can extend depending on $d1$. If $d1 > c1$, as we will see, continuity of wages implies zone 4 becomes a part of zone 7.

We see that there are clear ranges of α and β which define the solution forest. We claim that wages are always continuous.

Lemma C.1.1. *In a* (2×2) *market, equilibrium wages are always continuous functions of the utilities, i.e.,* α, β*. Moreover, allocations and utilities are continuous within each forest.*

Proof. First, we observe that wages are rational functions involving only $p1/p2$, i.e., the relevant α, β. In case the forest is connected, wages are always continuous within a forest. For disconnected forests, wages are constant. It is important to note that wages are bounded, $w_1 + w_2 = 1$. At the boundary of zone 4, $p1/p2$, i.e., α or β equals $c1$, which determines the wages. In other words, as α or β continuously approach $c1$, limit of w_i equals w_i for zone 4. Similarly, zone 3 allows for smooth transitions. It is clear that wages are continuous at the boundary $\alpha = \beta$ also, as w_i smoothly transitions. Similarly, in zone 7, wages for one class continuously converge to zero.

Next, we observe that allocations and utilities are continuous within each forest, where they are functions. It is because they are either linear or rational functions of α, β, and are well defined within each forest. For example, forest 1 has the following allocations-

$$x_{11} = w_1/p_1 = Y_1 T_{11}^{-1} + Y_1 T_{21}^{-1}/\beta$$

$$x_{21} = q_1 - x_{11}, \quad x_{22} = q_2$$

Though we have considered a special strategy matrix with all ones here, it can be proved that the result is general.

□

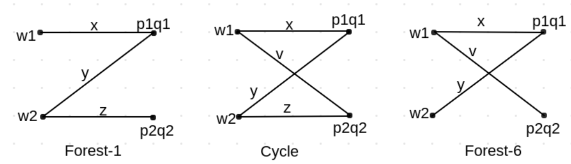
We now look at the points where two or more forests are feasible, i.e., at the boundaries. It is clear from the figure that by changing α or β it is possible to transit from one forest to another, by crossing the non-generic points where both forests are possible. It is shown earlier [21] that the set of allocations in Fisher market is hemicontinuous with respect to initial endowments and utility functions. Here, we show that though multiple allocations are possible at such points, utilities are bounded by the limits of utilities of forests at both sides. In other words, allocations and utilities at the transitions are convex combinations of the boundaries of those obtained in the adjoining zones. To make this precise, let us have the following definition.

Definition C.1.1. *Let $x \in \mathcal{U}$ be a point on the boundary of two zones, say Zone A and Zone B and let $\eta = (p, q, w, X)$ be a typical point above x, i.e., η is an equilibrium for the parameter x. Let $\mathcal{X}(x)$ be the collection of all allocations of equilibria above x in the U-space. We say that x is a manifold point if the set $U_i(\mathcal{X}(x))$ is a bounded interval and its bounds are obtained as the limits $\lim_{q \to x} U_i(X(q))$ and $q \in$ Zone A and $q \in$ Zone B.*

Lemma C.1.2. *Utilities on the boundaries are linear combinations of those of the forests on both sides, given that both forests have the same combinatorial data $I = J = \{1, 2\}$.*

Proof. Firstly, we note that except for the $\alpha = \beta$ line, other boundaries allow for a continuous function where for each pair of (α, β), a unique allocation/utility exists. Let us see this by changing α or β continuously to approach a forest/cycle from another. We see that when a tree is disconnected by making an edge weight zero, allocations and utilities continuously transform, according to the previous lemma. For example, in zone -1, when β approaches $c1$ from above, p_1/p_2 equals $c1$ where zone-4 starts, and x_{11} becomes q_1. Therefore, forest 3,4 include their boundaries while defining utility functions, i.e., boundaries have same utilities as the forests. In case of $\alpha = \beta$, first an edge is added to get a cycle, where multiple allocations are possible and then another edge is removed.

The governing equations for a connected forest, say forest 1 are : $w_1 = x < p_1q_1$, $y < p_1q_1$ and $z = p_2q_2$ along with $x + y = p_1q_1$. When we increase β to approach the cycle, the equations become $x, y < p_1q_1$, and $z, v < p_2q_2$ along with $x + y = p_1q_1$ and $z + v = p_2q_2$. Thus, the maximum value of z is p_2q_2 where zone 1 is overlapping and minimum value is 0 where $v = p_2q_2$ when zone 6 overlaps.



The utilities for forest 1 are-

$$(U1, U2) = \left(\frac{w_1}{p_1}, \quad q_1 + q_2 - \frac{w_1}{p_1}\right)$$

For forest-6-

$$(U1, U2) = \left(q_2 + q_1 - \frac{w_2}{p_1}, \quad \frac{w_2}{p_1}\right)$$

And for the cycle, these can be computed in terms of v, z-

$$(U1, U2) = \left(\frac{w_1 - v}{p_1} + \frac{v}{p_2}, \quad \frac{w_2 - x}{p_1} + \frac{z}{p_2}\right)$$

Rearranging the terms, this means -

$$(U1, U2) = \left(\frac{w_1}{p_1} + v\left(\frac{1}{p_2} - \frac{1}{p_1}\right), \quad \frac{w_2}{p_1} + z\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\right)$$

Firstly, we note that the ratio $1/p_2 - 1/p_1$ is a constant for any fixed α, β. Let this be a positive constant. Now, it is clear that when $v = 0$, $U1$ for the cycle is minimum which coincides with that of forest-1. Similarly, when $v = p_2q_2$, the maximum coincides with that of forest-6. This establishes that utilities at the cycle are linear combinations of those on the boundaries, so that for each α on the $\alpha = \beta$ line, utilities make a smooth transition. This makes us view the cycle as a linear combination of two forests depending on values of α and β. Though we have considered U_r made up of all ones, this result is valid for any matrix

U_r. ☐

Theorem C.1.3. *There exists a correspondence $\mathcal{N}$ between the strategy space and utilities. Also, each space of payoffs forms a 2 dimensional manifold whose boundaries are given by those of the forests.*

Proof. We classify the zones into interior points and the boundaries where two or more forests are possible. As stated earlier, the interior points refer to generic points which form an open set. Corresponding to the interior points, we have $U1, U2$ defined uniquely, which are continuous functions of α, β, using above lemma. Moreover, these are invertible functions on their restricted domain of α, β, which makes the sets of possible payoffs U_1^p, U_2^p open. It follows that the correspondences $\mathcal{N}_i = (\alpha, \beta, U_1^p, U_2^p)$ are open for each forest i. Thus, interior of every possible solution or zone is open and neighbourhood of each point is homeomorphic to open subsets of $\mathbb{R}^2$. For forests, the homeomorphism is given by the inverse of utility functions and for the cycle, the sets are open in $\mathbb{R}^2$.

We now claim that the boundaries of the forests and the cycle form 1 dimensional entities which serve as boundaries to the described 2 dimensional manifolds. Each boundary can be given by a unique equation in $\mathbb{R}^2$. On the line $\alpha = \beta$, there are two boundaries, one coming from $\alpha < \beta$ and another from $\alpha > \beta$. In between these two, a 2 dimensional plane is situated on each part of the segment $\alpha = \beta$, i.e., on the boundary of forest-1 and 6 and forest 6 and 2. When considered closures of the open sets, we see that correspondences intersect along these boundaries. ☐

We therefore establish that except for zone-7, payoff function U_i is a 2 dimensional manifold with boundary, consisting of all 'manifold' points. Moreover, a correspondence $\mathcal{N} = (\alpha, \beta, U_1^p, U_2^p)$ can be defined between the strategy space and the payoffs space. ☐

Next, we look at the conditions for a Nash equilibrium to exist in scheme-1. We note that same results hold for other schemes as well.

C.2 Strategic Analysis

Theorem C.2.1. *In a* (2×2) *market with true utility matrix* U_r *comprised of all ones, Nash equilibria exists subject to the condition that zone-4 exists.*

Proof. We first observe that allocations are functions of utilities, i.e., α, β, either linear or inversely related ($U_i = a + b\alpha$ or $c + d/\alpha$), when the forests are connected. In those cases, maximization can occur only at the boundaries, i.e., at the points where transition of graphs occurs. When the forests are disconnected, allocations and utilities are constants for a range of α, β. Therefore, if any Nash equilibrium exists, it should be at the transition points or forest 3 or 4.

Let us now analyse the first scheme. Looking at the arrows, we know that forest-3 cannot be an equilibrium, which leaves $\alpha = \beta$ and forest-4 as possibilities. Class-2 prefers forest-4 to forest-2, and similarly, class-1 prefers forest-4. For a point in forest-4 to be an equilibrium, $\beta > d1$ and $\alpha > c1$. In this region, given any α, class-2 prefers forest-4 to forest-1,3 and 5. We now have to find a region for β so that α maximises $U1$ in that region. Note that when $c1 > \beta > d1$, $U1$ is constant as a function of α in zone 6. When it enters zone-2, $U1$ increases until it is in zone 4. Thus, we prove Nash equilibria exist in the region $d1 < \beta < c1$ and $c1 < \alpha$. Similarly, it can be proved that Nash equilibria exists in scheme-2 in zone-3 for $d2 < \alpha < c1$ and $c2 < \beta$. This equilibrium is, however, subject to the condition that the forest exists, i.e., the region is feasible. For example, in scheme-1, forest 4 exists when $d1 < c1$, i.e., $\frac{T_{11}}{T_{12}} < \frac{T_{21}^{-1} Y_1}{T_{12}^{-1} Y_2}$ and $\beta < c1 < \alpha$. $\square$

Appendix D

Market Code Documentation

In this chapter, we describe two codes- Fisher market code and a closed loop code for the Tatonnement procedure. We explain the models, their inputs and outputs along with some numerical results.

D.1 Fisher Market Code - plc:

Fisher market is an economic model which determines prices and allocations of goods to the buyers, based on their utility preferences. The input to a typical Fisher market is a set of buyers B s.t. $|B| = l$, a set of goods G s.t. $|G| = n$, a utility function $U_i(x) : \mathbb{R}^n \to \mathbb{R}$ for each buyer i, a quantity vector $q = (q_j)_{j \in G}$ and a money vector $m = (m_i)_{i \in B}$, where x is a set of goods consumed, q_j is the quantity of good j, and m_i is the money possessed by buyer i. The market is solved in such a way that the money is completely spent and the market is cleared, i.e., all goods are sold.

Here, instead of individual buyers, we deal with labour classes as the basic units. We consider markets where we have linear as well as piecewise linear concave (plc) utilities. For plc, we give utility inputs in the form of linear segments. For each good, slope of the utility decreases successively which follows from the law of diminishing marginal utilities. The construction of inputs is illustrated below.

Inputs

1. U: Utility matrix $(l * n * k)$
   l: Number of labour classes
   n: Number of goods
   k: Number of segments

   Utility can be understood as happiness per person, per good, i.e., u_{ijk}= utility/happiness that a person from labour class i derives by consuming one unit of good j in segment k. Note that utilities are concave.
   For example, when $l = 3, g = 4, k = 2$;

$$U = \begin{bmatrix} 0.8 & 0.4 & 0.5 & 0.45 \\ 0.3 & 0.75 & 0.2 & 0.5 \\ 0.25 & 0.4 & 0.8 & 0.35 \end{bmatrix} ; \begin{bmatrix} 0.6 & 0.25 & 0.2 & 0.3 \\ 0.2 & 0.4 & 0.1 & 0.4 \\ 0.15 & 0.36 & 0.4 & 0.2 \end{bmatrix}$$

   Here, we have three labour classes, four goods and two segments, i.e., for each labour class, the utility of a good decreases to another value after consuming some fixed amount of it. e.g, $u_{141} = 0.45 =$ Utility of class 1 for good 4 in the first segment and, $u_{142} = 0.3 =$ utility for the same in the second segment.

2. L: Segments matrix $(l * n * k)$
   L denotes the maximum length of each segment. This is also per person, per good.
   $L_{ijk} =$ The maximum value of good j in the kth segment for which a person from class i has utility u_{ijk}. The length of the last segment is assumed to be infinity i.e the final utility value lasts till the good gets over, and thus, the last segment values are ignored.
   For example:

$$L = \begin{bmatrix} 3 & 4 & 7 & 4 \\ 2.5 & 3.8 & 6 & 0.5 \\ 9 & 6 & 1.5 & 3 \end{bmatrix} ; \begin{bmatrix} 6 & 4 & 3 & 3.5 \\ 2 & 5 & 8 & 4 \\ 8 & 5 & 4 & 8 \end{bmatrix}$$

   e.g. If we consider U, L, a person from class 2 has utility 0.3 until she has consumed 2.5 units of good 1, and later, her utility reduces to 0.2 and remains 0.2 till the end, irrespective of the given corresponding second segment entry.

3. q: The number of goods produced. $(1 * n)$
   For example,

$$q = \begin{bmatrix} 12 & 20 & 25 & 18 \end{bmatrix}$$

Here, four goods are produced in the above mentioned quantities.

4. w: The money vector $(l * 1)$ For example,

$$w = \begin{bmatrix} 16 \\ 7 \\ 10 \end{bmatrix}$$

Here, w shows the money endowed to each class as a whole.

Next, we illustrate a Fisher market and its solution.

Example 10. *Let us consider three labour classes, four goods and $U, L, tandm$ as follows:*

$$U = \begin{bmatrix} 0.8 & 0.4 & 0.5 & 0.45 \\ 0.3 & 0.75 & 0.2 & 0.5 \\ 0.25 & 0.4 & 0.8 & 0.35 \end{bmatrix} ; \begin{bmatrix} 0.6 & 0.25 & 0.2 & 0.3 \\ 0.2 & 0.4 & 0.1 & 0.4 \\ 0.15 & 0.36 & 0.4 & 0.2 \end{bmatrix}$$

$$L = \begin{bmatrix} 3 & 4 & 7 & 4 \\ 2.5 & 3.8 & 6 & 0.5 \\ 9 & 6 & 1.5 & 3 \end{bmatrix} ; \begin{bmatrix} 6 & 4 & 3 & 3.5 \\ 2 & 5 & 8 & 4 \\ 8 & 5 & 4 & 8 \end{bmatrix}$$

$$w = \begin{bmatrix} 16 \\ 7 \\ 10 \end{bmatrix} ; q = \begin{bmatrix} 12 & 20 & 25 & 18 \end{bmatrix}$$

We solve the Fisher market to find prices of these four goods and their allocations. Outputs are -

$$Money\ Allocations = \begin{bmatrix} 2.205 & 1.47 & 2.859 & 1.47 \\ 0 & 1.397 & 0 & 0.184 \\ 0 & 2.205 & 0.613 & 0 \end{bmatrix} ; \begin{bmatrix} 6.616 & 0 & 0 & 1.379 \\ 0 & 1.836 & 0 & 3.583 \\ 0 & 0.443 & 6.739 & 0 \end{bmatrix}$$

This matrix tells how each class has spent money on these four goods. For Ex, labour class 2 (row 2) has bought only two goods; good 2 and good 4, and it has spent 1.397 + 0.184 + 1.836 + 3.583 = 7 bucks in total.

Here, sum of rows = [16 7 10], i.e., each labour class has completely spent its money.

$$Goods\ Allocations = \begin{bmatrix} 3 & 4 & 7 & 4 \\ 0 & 3.8 & 0 & 0.5 \\ 0 & 6 & 1.5 & 0 \end{bmatrix}; \begin{bmatrix} 9 & 0 & 0 & 3.75 \\ 0 & 5 & 0 & 9.75 \\ 0 & 1.2 & 16.5 & 0 \end{bmatrix}$$

This displays the allocation of goods. For ex, total allocation of good 3 (column 3) = 7 + 1.5 + 16.5 = 25 units. Note how first segment entries match with the corresponding L ones. Here, sum of columns= [12 20 25 18] which equals the total production. Thus, all goods are sold completely.
And the prices are-

$$Prices = [0.735 \;\; 0.368 \;\; 0.408 \;\; 0.368]$$

Thus, $Prices * Production(t^T) = 33 =$ *Sum of m*$= 16 + 7 + 10$

D.2 Closed Loop Code - Trade

Inputs

As explained below, this program takes T, U, L, Y, p as inputs and outputs production, next set of prices, wages, Allocations etc. U, L are as before. As explained in the theory, it uses convex optimization tools to find production (q) using the initial set of prices. Further, wages (w) are computed. Then, using 'plc', and U, L, q, w as inputs, we can derive other outputs like prices and allocations.

1. T is the technology matrix. $(l * n)$
   T_{ij} is the amount of labour L_j needed to produce one unit of G_i. In simple terms, to produce one good, we have labour requirements as per its corresponding column.
   For example,
   $$T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$
   Thus, to produce one unit of good 2, we need one person from labour class 2 and one person from labour class 3.

2. p is the initial price vector. Using p and T, 'Trade' finds the production by optimization.

3. Y: The number of labour units available in each class.

Thus, using 'Trade', we have set up an iterator which computes desired market variables. As per the proposed method, given a $m \times n$ technology matrix, a sub matrix is chosen where the number of buyers and goods is same, and market variables are computed. In the next iteration, after modifying prices for the unused goods, the same procedure is repeated. All this is illustrated through the following examples.

Examples

Here we consider a market with three labour classes and three possible goods. We start with a price vector and see how various goods are produced and how classes derive wages from those. The following is the first iteration.

Example 11. *Let T, U, L, Y, p be as follows-*

$$T = \begin{bmatrix} 1 & 0.4 & 0.5 \\ 0.5 & 1.5 & 0.25 \\ 0.2 & 0.35 & 0.6 \end{bmatrix}$$

$$U = \begin{bmatrix} 0.85 & 0.5 & 0.4 \\ 0.4 & 0.9 & 0.45 \\ 0.55 & 0.4 & 0.8 \end{bmatrix} ; \begin{bmatrix} 0.6 & 0.2 & 0.17 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.35 & 0.6 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.33 & 0.40 & 0.1 \\ 0.3 & 0.4 & 0.2 \\ 0.93 & 0.6 & 0.7 \end{bmatrix} ; \begin{bmatrix} 0.58 & 0.4 & 0.2 \\ 0.20 & 0.5 & 0.08 \\ 0.84 & 0.5 & 0.45 \end{bmatrix}$$

$$Y = [10\ , 10\ , 10]; \quad p = [1\ , 1.2\ , 1.3]$$

And the outputs are (precise up to 3 decimal places)-

$$prices = [2.182,\ 1.284,\ 1.15]$$

$$Wages = [5.384,\ 2.831,\ 16]$$

Normalized prices and wages -

$$prices = [0.09,\ 0.053,\ 0.0475]$$

$$Wages = [0.222,\ 0.117,\ 0.66]$$

The production in units is (All the three goods are produced) -

$$q = \begin{bmatrix} 1.5 \\ 3.85 \\ 13.92 \end{bmatrix}$$

Edited L (according to Y)-

$$L = \begin{bmatrix} 3.3 & 4 & 1 \\ 3 & 4 & 2 \\ 9.3 & 6 & 7 \end{bmatrix} ; \begin{bmatrix} 5.8 & 4 & 2 \\ 2 & 5 & 0.8 \\ 8.4 & 5 & 4.5 \end{bmatrix}$$

Goods allocation-

$$Alloc = \begin{bmatrix} 1.5 & 1.645 & 0 \\ 0 & 2.205 & 0 \\ 0 & 0 & 7 \end{bmatrix} ; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6.92 \end{bmatrix}$$

Note that the total money is conserved-

$$p1 * q = 24.215 = sum(m) = 5.384 + 2.831 + 16$$

However, when above prices are used as input, we do not see the same market solution. In the third iteration, only good 1 and good 2 are produced and wages are allotted accordingly. The following tables summarize the results of 7 such iterations. It can be observed there is an underlying pattern in occurrence of these states. Even if the first input of prices is varied, the system stabilizes by showing the same pattern of states.

```
Active Goods =
(1 represents that the goods is produced and vice versa.)
    G-1   G-2   G-3
     1     1     1
     1     1     1
```

1	1	0
1	1	1
1	1	0
1	1	1
1	1	0

Wage Iterations (up to two significant digits) =

class-1	class-2	class-3
0.22	0.12	0.66
0.83	0.12	0.05
0.86	0.14	0
0.70	0.06	0.24
0.86	0.14	0
0.70	0.06	0.24
0.86	0.14	0

Example 12. *With T and U as given below and the same L, p and Y, we get the following output. Here, all goods are produced and all classes receive wages.*

$$T = \begin{bmatrix} 1.00 & 0.10 & 0.50 \\ 0.50 & 0.80 & 0.25 \\ 0.20 & 0.35 & 0.60 \end{bmatrix}$$

$$U = \begin{bmatrix} 0.85 & 0.3 & 0.4 \\ 0.4 & 0.9 & 0.35 \\ 0.3 & 0.4 & 0.80 \end{bmatrix} ; \begin{bmatrix} 0.6 & 0.2 & 0.17 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.35 & 0.6 \end{bmatrix}$$

Outputs:

Active Goods =

G-1	G-2	G-3
1	1	1
1	1	1
1	1	1

1	1	1
1	1	1

Wage Iterations (up to two significant digits) =

class-1	class-2	class-3
0.11	0.29	0.6
0.092	0.207	0.7
0.086	0.096	0.82
0.067	0.091	0.84
0.067	0.091	0.84

Price Iterations(in fractions) =

G-1	G-2	G-3
1.00	1.20	1.30
0.03	0.04	0.05
0.03	0.04	0.06
0.03	0.04	0.06