Models of Inflation in the light of Planck data 2013

IISER PUNE

A thesis submitted towards partial fulfillment of BS-MS dual degree program by

Sandesh Bhat J R

under the guidance of

Prof. Aninda Sinha
CHEP, IISc

Certificate

This is to certify that this thesis entitled "Models of Inflation in the light of Planck data 2013" submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by *J R Sandesh Bhat* at *Center for High Energy Physics, Indian Institute of Science, Bangalore*, under the supervision of *Prof. Aninda Sinha* during the academic year 2013-14.

Student
J R Sandesh Bhat

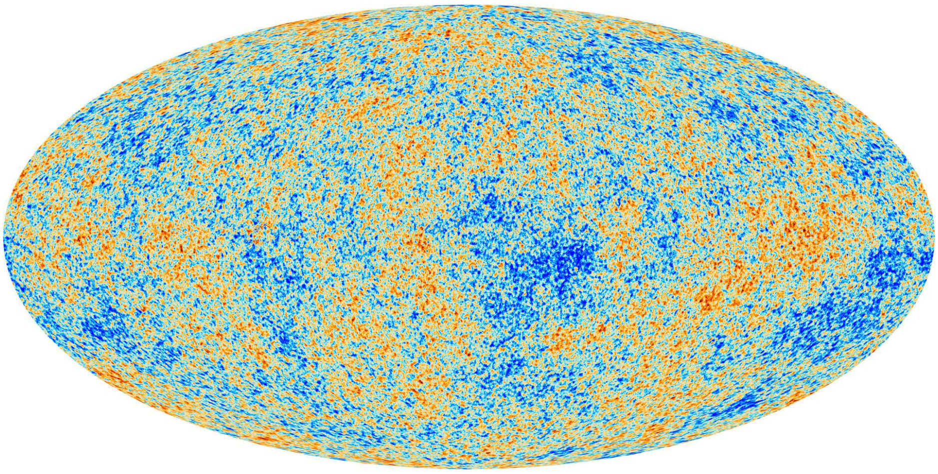
Supervisor
Prof. Aninda Sinha

Acknowledgements

I would like to thank my supervisor Prof. Aninda Sinha for his support and supervision throughout the whole project. I would also like to thank Aravind H V, M J Jimenez, and Shinjini Basu for interesting discussions, on the topic.

I thank Prof. Anil D Gangal for consenting to be my local advisor at IISER Pune, and helping me out in the process.

The INSPIRE grant was helpful in terms of financial support through the entire duration.



Cosmic Microwave Background
[Source: Planck data team ESA]

1 Abstract

On the frontiers of modern cosmology is quantum cosmology, which brings together two speculative ideas: inflation and quantum gravity. While, there are theories such as the ekpyrotic model, which do not make use of inflation, a vast majority of the theories support inflation. The real motivation for these speculative ideas lie in the fact that they can be observed as inhomogeneities in the Cosmic Microwave Background (CMB), and this has given us a window to the working of the early universe.

This was one of the key motivations underlying the Planck collaboration, which measured the CMB data at exceedingly high sensitivity ($\times 3$ compared to WMAP data). This has ushered in a new era of what one may call "precision cosmology".

With this increased sensitivity, Planck data has already disfavoured some of the previous inflationary models, and as new theories are being put forth- emphasis has shifted from just phenomenological models to theoretically-motivated models of inflation. The Lyth bound, and the actual bound on tensor-to-scalar ratio have not been very helpful in deciding the energy scale of inflation, but a theory incorporating a grand-unified theory or a theory of quantum gravity is well justified. It is in this context that string theory has contributed to some of the models. A form of heterotic string action has been studied in this thesis, at lower energy scales. The popular theories of inflation such as brane-inflation, trace anomaly driven inflation and Starobinsky inflation have been briefly discussed.

The research in this topic is a very active one, and in the next year, with the release of Planck data 2014 containing B-mode data, the interest in this field will only increase further. The recent announcement of BICEP2 dataset, supporting inflation at an energy scale higher than previously believed, is clearly going to increase interest in the new field of "String cosmology".

Table of Contents

1 Notation

A positive signature is used for the metric throughout.

Einstein summation index is implied everywhere, unless specified explicitly. Greek letters imply indices running over $0,1,2$ and 3; while Latin letters run over $1,2$ and 3. In the latter case, the zero component is indicated by t generally.

The convention for partial derivatives are as follows:

$$\partial_t \equiv \frac{\partial}{\partial t}, \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

Total derivative terms are grouped are written as $(b.t...)$

Covariant derivatives are represented by ∇, and the following convention is used:

$$\nabla^2 = \nabla^\mu \nabla_\mu; (\nabla\phi)^2 = (\nabla^\mu \phi)(\nabla_\mu \phi)$$

For convenience, we usually denote contractions of all tensor indices as squares, as follows:

$$A^{\nu_1\nu_2\ldots.\nu_i}_{\mu_1\mu_2\ldots.\mu_j} A^{\mu_1\mu_2\ldots.\mu_j}_{\nu_1\nu_2\ldots.\nu_i} \equiv \left(A^{\nu_1\nu_2\ldots.\nu_i}_{\mu_1\mu_2\ldots.\mu_j}\right)^2$$

Planck units $c = \frac{h}{2\pi} = 1$ are used in most places, unless specified.

2 Introduction

The Standard model of cosmology, called the CDM-Λ model, is a well accepted theory of the universe in cosmology. The theory in itself, however, involves a lot of open problems. It needs the existence of dark matter and dark energy, both of which have not been observed directly. Many modern theories of quantum gravity have recently provided particles which can act as models for these observations, and is one of the most active areas of research in present times. While the CDM-Λ model poses these open problems and more, it is still the most successful model at explaining the observations of our universe that we know.

In the light of huge success of the predictions by this model, we look at extensions of the model which explain some of the problems in it. One such major contender is the theory of inflation. It corrects some of the issues in the model, while also predicting slight deviations from homogeneity of the universe, which is measurable in the Cosmic Microwave Background from the early universe. It is also possible that this inflationary phase might have operated at a Planck scale, in which case: we would be looking at the working of a quantum theory of gravity in the early universe.

The hope is that a theory of quantum gravity will explain, in a consistent manner, all the shortcomings of the standard model of cosmology. Our first insights into such a theory, however, might emerge from looking at inflation.

2.1 Standard model of cosmology and its problems

The initial attempts at understanding the structure of our universe came from the assumptions of homogeneity and isotropy. The entire universe was taken to be a manifold subject to Einstein's equations. This is justified as we are working on huge length scales, and thus energy scales are low; a classical theory of gravity is well justified here. On very large scales, we suppose that the universe is homogeneous and isotropic to a free falling observer. These assumptions lead to the following metric, called the FRW metric:

$$d\tau^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega\right) \tag{1}$$

This has been confirmed by Large Scale Survey (LSS) and Cosmic Microwave Background (CMB) data surveys, which will be described later. We will often use the Hubble parameter $H = \frac{\dot{a}}{a}$, by convention. The dynamics of the universe can be derived by substituting the above metric solution in the Einstein's field equation. That gives us a relation between $a(t)$, k and the matter dominating the universe.

The left handside of the Einstein's equation is characterized by the following quantities derived from (1):

$$\begin{aligned} R_{ij} &= \delta_{ij}\left[2\,\dot{a}^2 + a\,\ddot{a} + 2\frac{k}{a^2}\right] \\ R &= 6\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a} + \frac{k}{a^2}\right] \end{aligned}$$

We consider a perfect fluid as the source term in our analysis. This is a good approximation to the kind of matter present in the universe on the scale we are working at. The assumptions of homogeneity and isotropy in our coordinates gives us the following stress energy term:

$$T_{00} = \rho(t); T_{0i} = 0; T_{ij} = -p(t)\, \delta_{ij}$$

The diffeomorphism invariance of gravity has an associated symmetry: conservation of stress energy tensor, which gives the following identity:

$$T^{0\mu}_{;\mu} = \frac{\partial T^{0\mu}}{\partial x^{\mu}} + \Gamma^{0}_{\mu\nu} T^{\mu\nu} + \Gamma^{\mu}_{\mu\nu} T^{0\nu} = 0 \Rightarrow \frac{d\rho}{dt} + \frac{3\,\dot{a}}{a}(p + \rho) = 0 \tag{2}$$

Now, to derive the actual dynamics, we substitute the quantities in the Einstein's field equations to get (from the G_{00} term):

$$H^2 + \frac{k}{a^2} = \frac{8\pi G\rho}{3} \tag{3}$$

By taking the trace of the field equations, we get:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{4}$$

Equation (3) and (4) can be used to derive the conservation of S.E equation. Let us review the nature of this system. The universe becomes static when $\dot{a} = 0$. This leads to the following condition from (3):

$$\frac{k}{a^2} = \frac{8\pi G\rho_d}{3}$$

We have the energy condition $\rho > 0$ if any matter exists; This implies that the universe can become static only if $k = 1$. Else, the universe will be dynamic. If $k \neq 1$, then the universe always expands or always contracts (by continuity). We know from astronomical observation that the universe is expanding (Fig 2). So, in the case of $k \neq 1$ for our universe, it has always been expanding ($\dot{a} > 0$). If not, the universe might have been contracting before, and now expanding- and contracting in the future. There are indeed cyclic views of the universe.

At any time t_0, given H_0, we can define the following:

$$\rho_o^{\text{crit}} = \frac{3H_0^2}{8\pi G} \tag{5}$$

We then have the following condition for the type of solution, from substitution in (3):

$$\rho = \rho_0^{\text{crit}} \Rightarrow k = 0$$

$$\rho > \rho_0^{\text{crit}} \Rightarrow k = 1, \rho < \rho_0^{\text{crit}} \Rightarrow k = -1$$

We can express the conditions above, clearly as:

$\rho < \rho_0^{\text{crit}} \Rightarrow$ The universe has always been expanding, and will continue to expand. It can be open or flat.

$\rho > \rho_0^{\text{crit}} \Rightarrow$ The universe is closed. It can start contracting again.

In most cases, there are different types of matter present simultaneously. The contribution then, would be as follows:

$$H^2 = \frac{8\pi G}{3}\left(\sum_i \rho_i\right) - \frac{k}{a^2}$$

Our definition of ρ_0^{crit} is still the same.

We wish to write these in dimensionless parameters. If at time t_0 we had the hubble parameter H_0 and critical density ρ_0^{crit}, it can be done as follows:

$$H^2 = H_0^2\left(\sum_i \rho_i/\rho_0^{\text{crit}}\right) - \frac{k}{a^2} \Rightarrow \left(\frac{H}{H_0}\right)^2 = \left(\sum_i \rho_i/\rho_0^{\text{crit}}\right) - \frac{k}{a^2 H_0^2}$$

We can relate all this to energy density at t_0. Define the ratio of energy density to critical energy density $\Omega = \rho_0/\rho_0^{\text{crit}}$, and the curvature term as $\Omega_k = -\frac{k}{a_0^2 H_0^2}$. We also assume the matter under description can be expressed by the equation of state $P = w\rho$ where w is constant.

We write the above equation as:

$$\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_i \left(\frac{a}{a_0}\right)^{-3(w_i+1)} + \Omega_k \left(\frac{a}{a_0}\right)^{-2} \quad (6)$$

By evaluating the above expression at $t = t_0$, we get the consistency condition:

$$\sum_i \Omega_i + \Omega_k = 1$$

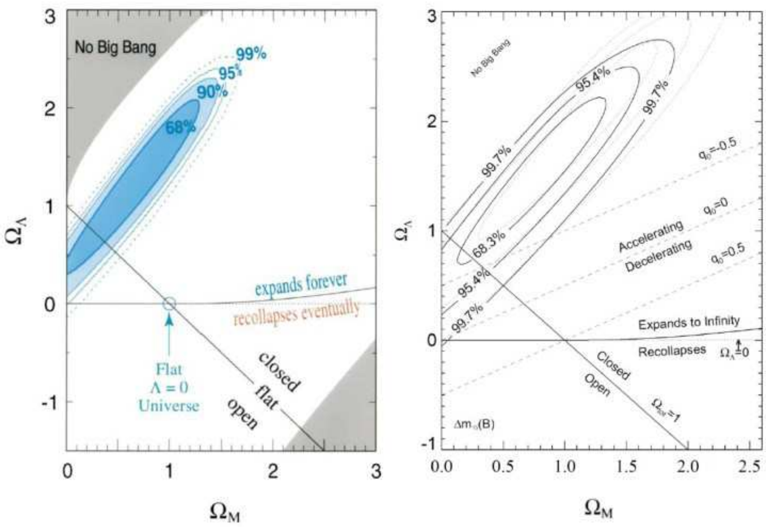


Fig. 1. Fitting of Type IA supernova data (source: Type IA data release)

We know from observations (Fig. 1) that galaxies are moving away from us, i.e $\dot{a}>0$. It also shows us that the present energy densities support the fact that we live in an almost flat universe, which has always been expanding and continue to do so.

For the case of $w>0$ (strong energy condition), we know that $a\sim\tau^{2/(1+3w)}$. Thus, we have a singularity as we approach $\tau\rightarrow 0$. This is the beginning of big-bang.

From Fig (1), we see that the experiments support the evidence of an almost flat universe ($k\approx 0$), considerably accelerating fit from a form of matter corresponding to $P/\rho=-1$.

Given any form of matter, we usually express their equation of state in the form $p=w\rho$. It is quite possible that w changes with time. Now, the conservation equation can be written as:

$$\frac{\dot{\rho}}{\rho}=-\frac{3}{}\frac{\dot{a}}{a}(w+1)$$

Since we have $\rho>0, \dot{a}>0$, one gets the following condition $\dot{\rho}\leqslant 0$ iff $w\geqslant -1$. Under the above conditions, one expects that the energy density of the system to increase as we go back in time. If we were talking about particles, we would expect the temperature of the system to increase. This is the reason why one would expect the early universe to be hot. As one approaches the origin, the energy scales become higher, and we may need to add higher order terms into our effective action. The popular notion is that we might have reached our present state by symmetry breaking as the universe cooled*.

2.1.1 Event horizon, Particle horizon and Horizon problem

The fact that universe seems to be homogeneous at large scales seems to favour the intuition from equilibrium statistics that they must have been in equilibrium at some early stage of the universe. This would seem possible if the observable universe was in causal contact at some point in the early universe. Let us elaborate on the causal structure of our universe.

The null geodesic in the FRW metric is given by:

$$dt^2=a^2(t)\left(\frac{dr^2}{1-kr^2}+r^2d\Omega\right)$$

Since it is isotropic, we can look at the one in which $d\Omega$ is zero in general. This gives:

$$\frac{dr}{dt}=\frac{\sqrt{1-kr^2}}{a(t)}$$

At time t_0, starting at any point $r=0$, the causally accessible region of space-time is given by:

$$\int_0^r \frac{dr}{\sqrt{1-kr^2}} \leqslant \int_{t_0}^t \frac{dt}{a(t)}$$

The maximum distance for a given t in the causally accessible region of space-time in the future ($t > t_0$) by a present observer is called the event horizon. Even if the future t is infinity, there will be a event horizon if the integral converges.

Similiarly, we can look at the causally accessible region of space-time for a point at origin at the beginning of universe. The maximum distance for a given t under this set is called the particle horizon.

The proper distance of the particle horizon is given by:

$$d_{\max}(t) = a(t)\int_0^r \frac{dr}{\sqrt{1-kr^2}} = a(t)\int_0^{t_0} \frac{dt}{a(t)}$$

Presently, the particle is given as:

$$d_{\max}(t_0) = a(t_0)\int_0^{t_0} \frac{dt}{a(t)} = a(t_0)\int_0^{a_0} \frac{da}{Ha^2}$$

By the FRW equation, we can write this as:

$$d_{\max}(t_0) = \frac{a_0}{H_0}\int_0^{a_0} \frac{da}{a^2\sqrt{\sum \Omega_i(a)(a/a_0)^{-3(w_i+1)}}}$$

If we are dealing with matter obeying the strong energy condition, then $(a/a_0)^{-3(w_i+1)}$ decreases with time. This implies that the particle horizon increases with time. But, if this were true since the beginning, then new areas of the universe have been coming in causal contact with one another with time. These areas were not in causal contact with each other before as the particle horizon has been strictly increasing. In fact, one observes the CMB radiation, and sees two points in space which are not in causal contact with each other, yet they are homogeneous. This is called the *horizon problem*.

Homogeneity in itself seems like a fine-tuning apart from the above causality issue. In fact, one knows that in gravity under presence of normal matter- inhomogeneities get amplified. The denser part of the universe become denser and vice versa.

We can always say that all parts of the universe were homogeneous to begin with, but a theory which does not need such "fine-tuning" of parameters would be more attractive.

2.1.2 Planck epoch, GUT epoch

The best experimentally verified theory we have at very small length scales is the Standard model of particle physics. It has been verified to energy scales of 14 TeV at LHC. This, however, is not sufficient to describe the universe which reaches very high energy scales as we go further back in time. It is expected that at the Planck scale, a theory of quantum gravity will be essential to describe the physics at work. It is below the Planck scale that we can describe a theory in which we can treat quantum field theory on a classical gravity background; this is an effective field theory. There is no guarantee however that the quantum field theory below the Planck scale is the standard model we know. It is expected that there is a Grand Unified Theory (GUT) which works at those scales, and at lower energy scales: there is a symmetry breaking which leads to the Standard model we know.

In the early universe, the energy scales are expected to have been at the Planck scale. Many people expect corrections at this scale to our model, because of quantum effects of gravity. One expects these corrections to rectify the singularity at the beginning of the universe. Later in time, after this, when the energy scale of the universe dropped below the Planck scale, we expect the GUT theory, which later gave rise to Standard model.

It is also expected that the dark matter, which has a couple of candidates in some of the GUT, such as axinos from supersymmetry, are expected to have been produced during this time.

This epoch of the universe is not well understood.

2.1.3 Nucleosynthesis and Baryon asymmetry issues

At temperatures around $>10^{11}\,K$, within the range of standard model, the high energy density would have initiated pair production in the vacuum. Initially, the universe is dominated by radiation, as particles are massless and at high energy.

The standard model of physics gives mass to particles through the well-known Higgs mechanism. We know however at higher energy scales, the field settles at the center of the Higgs field giving rise to zero vaccuum expectation value as illustrated in the diagram below. As energy scales lower, a non-zero vaccuum expectation value arises, which causes the spontaneous symmetry breaking of the electro-weak gauge symmetry.

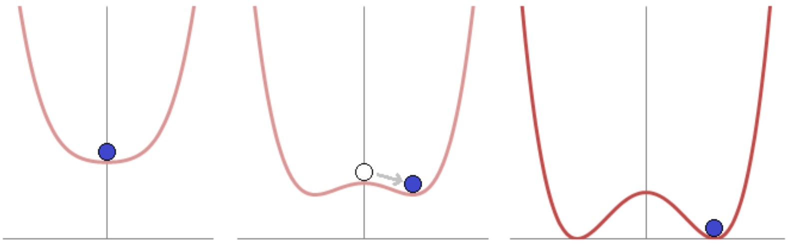


Fig. 2. Electroweak symmetry breaking (source: Wikipedia)

As the universe expanded the electroweak symmetry breaking occured and the Standard model as we know it was formed during this era. However, the energy scales are still too high for hadrons to be formed.

In the very early universe, when the energy density was just about in the regime of standard model, we expect pair production of particles as we know it to have occured. Lighter particles like photons and neutrinos are expected to have formed in larger quantities than heavier particles like leptons and quarks. At this stage, due to the asymptotic freedom of strong interactions, one expects quarks to have been unbounded. As the universe cooled, one expects the quarks to have bounded to form mesons and baryons.

If the particles were produced in the above fashion at the standard model times by particle production alone, one expects the amount of particles observed to be in the same quantity as antimatter. Since, after that point, all processes are covered by the standard model, and standard model respects conservation of baryon/lepton number, we would expect the equality to follow till present date. However, we observe matter around us dominantly, and we have hardly found any trace of antimatter. This is called the baryon asymmetry issue. We expect that this asymmetry might have been carried on from a higher energy scale where a GUT permits a process which violates this conservation.

We approximate matter mostly by non-relativistic matter and radiation at this point. One expects particles in the early universe to have been generated at very high temperatures, which implies radiation mostly. The main interaction of matter from this point onwards can be neatly summarized by the below figure.

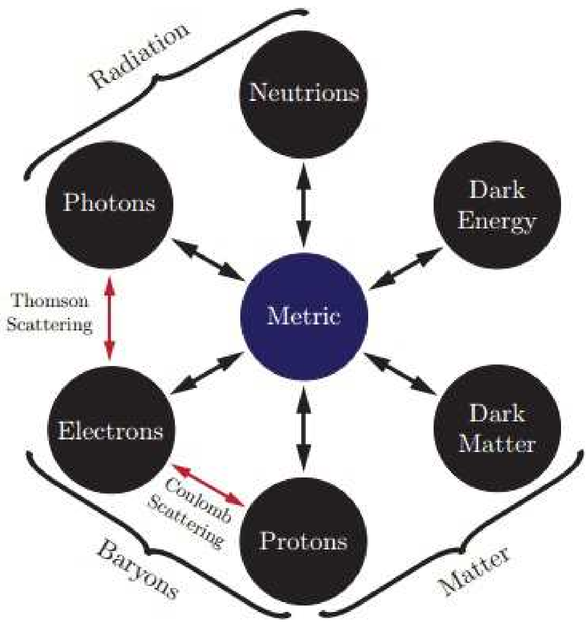


Fig. 3. Interaction of different matter in the universe (source: D. Baumann lecture notes)

2.1.4 Photon decoupling, CMB radiation and inhomogeneities

The rate of interaction between different particles is quite high in the early universe. It is because of that we have matter and radiation in equilibrium with one another. When we talk about radiation, we mainly refer to photon and neutrinos. This is because after the electroweak breaking, all the other particles would have acquired heavy masses. These radiation when not in equilibrium with matter can propagate on the null geodesic undisturbed for large distances, which can be measured in the present time. The process of radiation coming out of equilibrium with matter is called *decoupling*.

The neutrinos are the first to decouple, followed by photons much later. Corresponding to these, we can still measure the radiation as Cosmic Neutrino background (CNB) and the Cosmic Microwave Background (CMB). The former is hard to measure because of the poor detection capabilities with respect to neutrinos. These radiations provide a window into the early universe, and are one of the most valuable cosmological data available to us. The CMB data has been the focus of experiments as COBE, WMAP, and presently Planck.

In the case of photons, they are in equilibrium with electrons via Thomson scattering. At about 10^5 K, the interaction of photons with electrons stopped. Its interaction is generally restricted to free electrons. However, this completely stops when the electrons became bounded in an atom. This phase is called *recombination*.

The radiation from the early universe get doppler shifted because of the expansion of universe. We consider the radiation to be emitted by blackbody, as justified from experimental data summarized in the figure below.

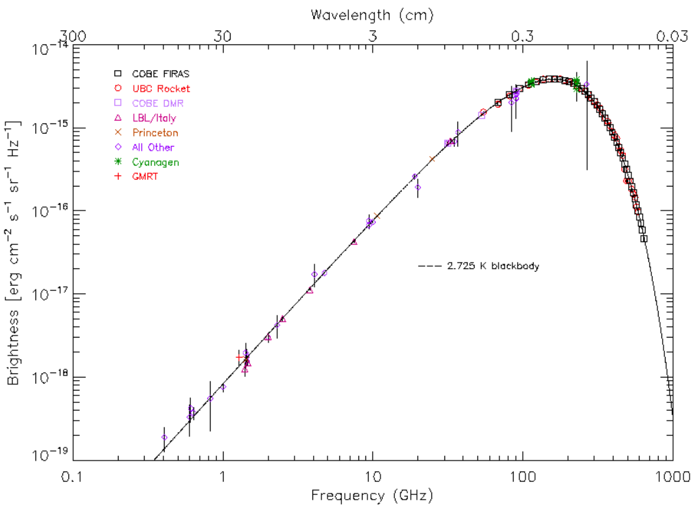


Fig. 4. CMB spectrum with wavelength (source: WMAP data release)

From Wien's displacement law for blackbody radiation, $\lambda T = c$. From doppler shift, we know that $\lambda \sim a$, and hence $aT = c$. We, thus, expect the radiation we observe now to have been redshifted by a huge amount.

The CMB data that we observe is mostly homogeneous. This presents itself as another example of the horizon problem we illustrated before. The angular diameter at time of last scattering is of the order $d_H \sim H_0^{-1}(1+z_L)^{-3/2}$. For a redshift $z_L \sim 1100$, the angular diameter is about 1.6°. However, the entire CMB data is mostly homogeneous. This provides a striking evidence of two parts of the universe which are homogeneous, in spite of being causally disconnected from one another.

While we claimed that the CMB data is mostly homogeneous, it does have slight inhomogeneities. The fluctuations (variance) are of the order 10^{-4} compared to the mean. We are now able to look closely at these inhomogeneities because of the increased accuracy of the Planck mission. There are a couple of factors which contribute to the inhomogenities of the CMB data.

The statistics of inhomogeneities are best described by correlation function. In this case, since we are dealing with photon radiation, we can look at temperature fluctuations in the sky. Let $\mathbf{\Theta}(\hat{n}) = \delta T(\hat{n})/\bar{T}$ be the measured fluctuation at the direction $\hat{n}$ in the sky. The correlation is given by:

$$C(\theta) = \langle \mathbf{\Theta}(\hat{n})\mathbf{\Theta}(\hat{n}') \rangle$$

where $\cos\theta = \hat{n}.\hat{n}'$. The correlation is only taken to depend on the difference in directions.

As with all spherical coordinates, it is convenient to work in spherical harmonic functions ($R = 1$):

$$\mathbf{\Theta}(\hat{n}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Theta_{lm} Y_{lm}$$

The term Θ_{lm} (as is familiar from electrostatics) is called the multipole moment. We define (from statistical isotropy):

$$\langle \Theta_{lm} \Theta^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'}$$

Here, the C_l is the spherical harmonic space equivalent of $C(\theta)$. We can show this as follows: taking the original correlation and substituting the above in it,

$$C(\theta) = \langle \mathbf{\Theta}(\hat{n})\mathbf{\Theta}(\hat{n}') \rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{l'=0}^{\infty} \sum_{m=-l'}^{l'} \langle \Theta_{lm} \Theta^*_{l'm'} Y_{lm} Y^*_{l'm'} \rangle$$

Since the spherical harmonics are orthogonal and taking statistical independence of the two functions,

$$C(\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \langle \Theta_{lm} \Theta^*_{lm} Y_{lm} Y^*_{lm} \rangle = \sum_{l=0}^{\infty} C_l \frac{2l+1}{4\pi} P_l(\cos\theta)$$

We can compute the power spectrum of photon in an inhomogeneous universe, and its effect on the temperature fluctuation observed. We consider slight perturbation of a FRW metric. Considering just the scalar perturbations, we can write the metric as follows:

$$\mathrm{ds}^2 = (1 + 2\psi) dt^2 - a^2(t)(1 - 2\phi)\delta_{ij} dx^i dx^j$$

Before recombination, we expect the photon to be in equilibrium with matter. We make a couple of approximations.

Before the recombination, the photons (γ) and baryons (b) are strongly coupled to each other, by Thomson scattering. So, we can treat the photons and baryons as a single fluid. By the equilibrium of the two (same temperature), we have $v_\gamma = v_b$. This is called the tight-coupling approximation.

Initially the momentum density of baryons is quite lesser than of radiation. This is characterized by the following parameter:

$$R = \frac{(\rho_b + P_b)v_b}{(\rho_\gamma + P_\gamma)v_\gamma} = \frac{3\rho_b}{4\rho_\gamma} \sim 0.6\left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{a}{10^{-3}}\right)$$

As long as this is small, which it is until recombination, we can neglect the effect of baryons. This is called the no-baryon approximation.

We assume the background expansion is matter dominated. This isn't a good approximation as in early times the universe was radiation dominated. We will however need this to simplify our calculations.

By substituting $w_\gamma = 1/3$, we can compute the conservation equation for the photon gas. We will directly cite the expression:

$$(\Theta + \psi)' = -\frac{1}{3}\nabla.v_\gamma$$

$$v_\gamma' = -\nabla(\Theta + \psi)$$

Combining the above equations and expressing in fourier modes,

$$(\Theta + \psi)'' + c_s^2 k^2 (\Theta + \psi) = 0 \,\text{where}\, c_s = 1/3$$

This is a simple harmonic oscillator equation, whose solution can be expressed as:

$$(\Theta + \psi)(\tau) = (\Theta + \psi)(0)\cos(\mathrm{kc}_s\tau) + \frac{(\Theta + \psi)'(0)}{kc_s}\sin(kc_s\tau)$$

After recombination, the photons are free to move independently. The geodesic equation for photon can be written as:

$$\frac{dP^0}{d\tau} = -\Gamma^0_{\mu\nu}P^\mu P^\nu$$

where P is the four momentum. Now the photons are massless, so $(1 + 2\psi)(P^0)^2 = p^2$ where $p^2 = a^2(t)(1 - 2\phi)\delta_{ij}P^iP^j$. To first order, we can express this as:

$$P^0 = p(1 - \psi)$$

If we define orthonormal vectors $\hat{p^i}$, we can easily show upto first order:

$$P^i = (1 + \phi)p\hat{p}^i/a$$

Substituting this in the geodesic equation:

$$\frac{d}{d\tau}[p(1 - \psi)] = \frac{d}{dt}[p(1 - \psi)]p(1 - \psi) = -\Gamma^0_{\mu\nu}P^\mu P^\nu$$

Moving $p(1 - \psi)$ to the other side, upto first order:

$$\frac{d}{dt}[p(1 - \psi)] = -\Gamma^0_{\mu\nu}\frac{P^\mu P^\nu}{p}(1 + \psi)$$

$$-p\frac{d\psi}{dt} + (1 - \psi)\frac{dp}{dt} = -\Gamma^0_{\mu\nu}\frac{P^\mu P^\nu}{p}(1 + \psi)$$

Multiplying by $(1+\psi)$ on both sides, and retaining terms upto first order:

$$\frac{1}{p}\frac{dp}{dt} = \frac{d\psi}{dt} - \Gamma^0_{\mu\nu}\frac{P^\mu P^\nu}{p^2}(1+2\psi)$$

Substituting for the christoffel symbol:

$$\frac{1}{p}\frac{dp}{dt} = \frac{d\psi}{dt} - H + \frac{\partial\phi}{\partial t} - \frac{\partial\psi}{\partial t} - \frac{2\hat{p}^i}{a}\frac{\partial\psi}{\partial x^i}$$

Note the following relation:

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x^i}\frac{dx^i}{dt} = \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x^i}\frac{P^i}{P^0} = \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x^i}\frac{(1+\phi)\hat{p}^i}{a(1-\psi)}$$

By using the above relation, on the equation above:

$$\frac{1}{p}\frac{dp}{dt} + \frac{1}{a}\frac{da}{dt} = -\frac{d\psi}{dt} + \frac{\partial\phi}{\partial t} + \frac{\partial\psi}{\partial t}$$

We finally get the differential equation,

$$\frac{d\ln(ap)}{dt} = -\frac{d\psi}{dt} + \frac{\partial\phi}{\partial t} + \frac{\partial\psi}{\partial t}$$

Recombination usually occurs over a small time period. For simplicity, we assume that it happens instantaneously at t_{rec}, which is well justified as that time is negligible compared to the time over which we are calculating its evolution. We can relate perturbation at recombination to present day as follows:

$$\ln(ap)_{\text{now}} = \ln(ap)_{\text{rec}} + (\psi_{\text{rec}} - \psi_{\text{now}}) + \int_{t_{\text{rec}}}^{t_{\text{now}}} dt\,\frac{\partial(\psi+\phi)}{\partial t}$$

Since the CMB radiation is a blackbody radiation, we have $\lambda T \sim T/p = c \Rightarrow p \sim T$. So, we have $\delta p \sim \delta T = T\Theta(\hat{n}) \Rightarrow p \sim \bar{T}(1+\Theta)$. Now, we have for the above equation:

$$\ln\,[(ap)_{\text{now}}/(ap)_{\text{rec}}] = [a_{\text{now}}\bar{T}_{\text{now}}(1+\Theta_{\text{now}})/a_{\text{rec}}\bar{T}_{\text{rec}}(1+\Theta_{\text{rec}})] = (\psi_{\text{rec}} - \psi_{\text{now}}) + \int_{t_{\text{rec}}}^{t_{\text{now}}} dt\,\frac{\partial(\psi+\phi)}{\partial t}$$

But, we know partition function of photon that $a_{\text{rec}}\bar{T}_{\text{rec}} = a_{\text{now}}\bar{T}_{\text{now}}$. By setting the gauge to get $\psi_{\text{now}} = 0$ and also, by taylor expanding the log, we get:

$$\Theta_{\text{now}} = (\Theta+\psi)_{\text{rec}} + \int_{t_{\text{rec}}}^{t_{\text{now}}} dt\,\frac{\partial(\psi+\phi)}{\partial t}$$

This is the equation widely used in CMB calculations. The $(\Theta+\psi)_{\text{rec}}$ term is called the Sachs-Wolfe term, and the integral the integrated Sachs-Wolfe term.

There are a couple of different reasons for the anisotropy observed in CMB data. The Planck data is one of the latest measurements of CMB data with regards to high precision which is needed for measuring some subtle aspects of the universe.

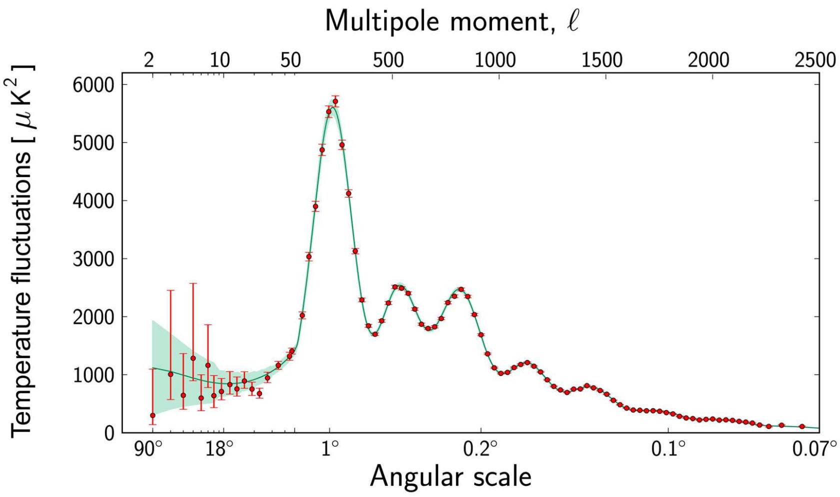


Fig. 5. CMB fluctuation data for different angular scales by Planck data (source: WMAP data release)

2.1.5 Structure formation

The universe at slightly later times after the photon decoupling was dominated by matter. While the ratio of photon number to lepton/baryons is quite high, the energy density of radiation falls rapidly. The matter- mainly dark matter and baryons lead to formation of structures as we know it. The inhomogeneities increase to form galaxies and other large structures as we observe them.

Our present observation suggests a flat universe, but this is a feature connected more to the the early universe than during the later times. Take the curvature term:

$$\mathbf{\Omega}_k = \frac{k}{a^2 H^2}$$

But, if $\ddot{a} < 0$, then $a^2 H^2 = \dot{a}^2$ always decreases. This indicates that the universe shifts away from flatness as it evolves in the presence of normal matter. Since the universe evolves with normal matter for a long time, any small deviations from flatness causes it to shift from flatness even more over time. After the universe was around $10^4\ K$, the universe has expanded as $a(t) \sim t^{2/3} \sim T^{-1}$. So, at around $10^4\,K$, we need around $\Omega_k < 10^{-4}$. Similarly, backwards in time, we observe even higher fine-tuning of Ω_k to match our observations.

This issue of universe to have started very close to flatness initially, is called the *flatness problem*. This is an issue of fine-tuning of the boundary condition of the universe.

2.1.6 Present epoch, and dark energy

The present stage of the universe is one in which the universe has expanded enough to have a low temperature. The effect of matter at this stage should have decreased. We know from astronomical observation, and extrapolation from CMB data that the energy density of matter is given by $\Lambda_m \approx 0.27 \pm 0.03$.

Type Ia supernova (SNIa) have a well known luminosities, and they have enough luminosities to be observed from far enough to determine the redshift over large scales. Two groups, the supernova cosmology project and the High-Z Supernova search team, independently confirmed that the universe is accelerating. This was one of the biggest discoveries in cosmology, and it argued the case for the existence of a form of energy called the “dark energy”.

Given any normal form of matter which satisfies the strong energy condition, we would have from FRW equation:

$$\frac{\ddot{a}}{a} \sim -(\rho + 3p) < 0$$

Thus, ordinary matter cannot account for the observed accleration of the universe. It was for this reason the cosmological constant can be used to describe a new kind of contribution to the source which causes acceleration. Looking at the lagrangian, we have:

$$S = \int d^4x \sqrt{-g}(R - 2\rho_\Lambda)$$

This gives the following field equations:

$$G_{\mu\nu} = \rho_\Lambda g_{\mu\nu} \Rightarrow T_{\mu\nu} \equiv p g_{\mu\nu} + (p + \rho) u_\mu u_\nu = \rho_\Lambda g_{\mu\nu}$$

Taking trace,

$$p_\Lambda = -\rho_\Lambda \Rightarrow w = -1$$

The behaviour of this is quite different from other types of matter. Clearly, this causes acceleration.

Looking at its evolution:

$$\frac{\dot{\rho}}{\rho} = -\frac{3\,\dot{a}}{a}(w + 1) = 0$$

This shows that the energy density never changes. It indicates that its contribution has been the same at all times. It, in fact, causes a de-sitter expansion. The contribution of radiation to energy density currently is negligible as expected. As we know the energy density of non-relativistic matter, by the consistency condition, we can estimate the energy density of dark energy as $\Lambda_D = 0.73 \pm 0.03$. It is clear from this that dark energy has been the major contributor in the late times of the universe, and will continue to be in the future. However it is clear that during earlier times, the radiation and matter components would have had a much higher influence than dark energy as it would have remained constant while the other two would have increased. This justifies most of our calculation till now.

A point to note here is that for dark energy, the particle horizon decreases. So, we have already seen the point where we had the maximum causal contact. Since, the domination of dark energy and the subsequent acceleration of the universe, the causally connected area of the universe is decreasing.

Attempts at a fundamental explanation for dark energy has not been successful till now. One suggestion was that this stems from the vacuum energy of quantum fields. However, the predicted value of it is much higher than what is observed*.

2.2 Motivations for inflation

The concept of inflation was initially proposed by Guth as a mechanism to address the *horizon* and *flatness* problems of the universe. The main idea is that the universe accelerated rapidly in the beginning to account for the horizon and flatness problem. The condition translates to $\ddot{a} > 0$. We can show that in this case, it will solve both the problems. As we showed before, the proper distance of the particle horizon is given by:

$$d_{\max}(t) = a(t_0)\int_{a_i}^{a_o} \frac{da}{aH^2}$$

In an accelerating universe, this decreases similar to dark energy as we explained. As we showed before, this requires matter which violates the strong energy condition. If we were to take a de-sitter expansion of the universe, then we can calculate the particle horizon as follows:

$$d_{\max}(t) = e^{H_0 t_0}\frac{(t_0 - t_i)}{H_0}$$

As we see, for $a_i \to 0$, towards the beginning of big-bang, the time t_i gets pushed to $-\infty$ in this case. So, if the universe started with a de-sitter expansion, then the beginning could be extended to $-\infty$. This would show that the entire universe must have been in causal contact initially, and during inflation: some regions went beyond the horizon. When they re-entered later during the deccelerating phase, they were homogeneous because of the causal contact they had before in the early universe.

During inflation, we can show that the universe tends towards flatness naturally. Take the flatness parameter:

$$\Omega_k = -\frac{k^2}{a^2H^2} = -\frac{k^2}{\dot{a}^2}$$

At the time of inflation $\ddot{a} \gg 0$. This causes $\dot{a}$ to increase, and thus Ω_k to decrease. So, this justifies the universe approaching flatness naturally during inflation.

2.2.1 Crude bounds on amount of inflation

The flatness and homogeneity we observe are known to be valid on the observable part of the universe. The observable part of the universe is however restricted by the causal horizon. This and, also, the knowledge of Ω_k places bounds on the amount of inflation needed to successfully resolve the problems. For convenience, the amount of inflation is defined as e-foldings (N):

$$\Delta N = \int_{t_i}^{t_f} H dt = \int_{a_i}^{a_f} \frac{da}{a} = \ln\frac{a_f}{a_i}$$

This provides a convenient way to express evolution of $a(t_f)$ with respect to some $a(t_i)$, as:

$$a(t_f) = e^{\Delta N} a(t_i)$$

The resolution of the flatness problem is to look at the amount of e-foldings from end of inflation till today. Similarly, for the solution of the horizon problem, one looks at the largest observable area of the universe, and compare the e-foldings. It is known from cosmological observations that $\Delta N > 60$, solves both the problems. In further considerations of theories of inflation, we assume this condition.

2.2.2 *Indirect experimental evidence*

The horizon and flatness problem, while providing a good motivation for inflation, is not an evidence for inflation. It could be that we observe those specific initial conditions for our universe, only because we could have existed in one which had those conditions (anthropic principle). The inflation does however predict other results which quite strongly support its own cause. The most important results come from the power spectrum of CMB at large scales, which are primarily inhomogeneities from the inflationary era.

CMB anisotropies and Planck data

For the different fourier modes of CMB temperature fluctuation, it is related to the curvature perturbation at $t=0$ by the following relation[2]

$$\Theta(k,\hat{n}) = T(k)R_k(0)$$

where $T(k) = T_{\rm sw}(k) + i\cos\theta T_d(k)$. To extract the multipole moments from the above relation, one writes the above relation in a different manner, as follows:

$$\Theta(\hat{n}) = \sum_{lm}\left[4\pi i^l \int \frac{d^3k}{(2\pi)^{3/2}} R_k(0)\Delta_l(k)Y^*_{lm}(k)\right]Y_{lm}(\hat{n})$$

where,

$$\Delta_l(k) \equiv T_{\rm sw}(k)j_l(k\chi_*) + T_d(k)j'_l(k\chi_*)$$

Usually these transfer function are determined numerically. In the case of inflation, since we look at large scales which exit the horizon, one has an analytic expression. At large scales, on can show that:

$$\Delta_l(k) \approx -\frac{1}{5}j_l(k\chi_*)$$

Thus, there is no evolution effects for large scales. The correlation of $R_k(0)$ would directly influence the correlation of the temperature fluctuation that we observe. The fluctuations produced during inflation at large scales which exit the horizon are directly observable through the CMB radiation. This is the crucial data which gives us insight into measurements connected to inflation.

Two-point functions and observables

We primarily look at the power spectrum of R_k at horizon exit. The idea is to look at the bispectrum of the curvature perturbation. This is done by looking at the correlation. The power spectrum is defined by the following notation:

$$\langle R_k R_{k'}\rangle = (2\pi)^3\delta(k+k')P_R(k)$$

We can define a dimensionless power spectrum, as follows:

$$\Delta_s^2(k) = \frac{k^3}{2\pi^2}P_R(k)$$

The variation of the above power spectrum w.r.t the scale k is what will determine if the spectrum is scale invariant or not. Before the model of inflation was proposed, there was a proposed power law for the power spectrum, called the Harrison-Zel'dovich spectrum:

$$\Delta_s^2(k) = k^{n_s-1} \Rightarrow n_s - 1 = \frac{d\ln\Delta_s^2}{d\ln k}$$

The above quantity n_s basically measures the deviation from scale invariance, where $n_s = 1$ if scale invariant. The power law form is assumed, with n_s taken as a contant w.r.t k. In the context of inflation, however, this need not be true. The scalar index can depend on k, usually called as the running of scalar index. In fact, this is true in mostly all the inflationary models.

Planck data 2013 gives a very precise measurement of $n_s = 0.9603 \pm 0.0073$.

E-modes and B-modes of CMB radiation

Apart from scalar perturbations, we also have tensor perturbations, which are characterized in the same way. However, there is a slightly different use of convention.

$$n_t = \frac{d \ln \Delta_D^2}{d \ln k}$$

The important point here is that the tensor perturbations correspond to gravitational waves, in the context of gravity. The measurement of these, are however, an emerging discipline, and not much progress has been made in this regard.

Without dwelling on the specific calculation, it is known that the CMB radiation can be decomposed into curl-free *E mode* and a divergence-free *B mode*. The following facts have been proved:

(a) scalar perturbations create only E-modes and no B-modes.

(b) vector perturbations create mainly B-modes.

(c) tensor perturbations create both E-modes and B-modes.

Since the vector modes decay with expansion of universe, the only modes contributing to B-modes are tensor perturbations. So, observing the B-modes can give us direct contact with primordial gravitational waves.

The last quantity that we are interested in is the tensor-to-scalar ratio. It is the ratio of tensor perturbations to the scalar perturbations. It is defined as:

$$r = \frac{\Delta_D^2}{\Delta_s^2}$$

Lyth bound

The energy scale of inflation is primarily connected to the above quantity. The bound can be computed as follows:

$$r = \frac{8}{m_{\text{pl}}^2} \left(\frac{d\phi}{\text{dN}} \right)^2 \Rightarrow \Delta \phi = m_{\text{pl}} \int \sqrt{\frac{r}{8}} dN$$

In the case of inflation at first order approximation (which will later be formalized as slow-roll):

$$\frac{\Delta \phi}{m_{\text{pl}}} \sim O(1) \sqrt{\frac{r}{0.01}}$$

This gives us an idea of the energy scale of inflation. From the recent input of BICEP2 data, $r \sim 0.2 \gg 0.01$, it shows us that $\Delta \phi \gg m_{\text{pl}}$. This says that the energy scale of the inflation is at a much higher energy scale than previously expected.

In the figure below, the first order results from Planck data are shown. Different popular models are considered usually in the range of 50-60 e-foldings of inflation. While in the below diagram, the prominently favoured model is R^2-inflation (Starobinsky); one can argue that there exists a set of solutions with this feature of late-time solutions in the accepted range for spectral index from Planck dataset.

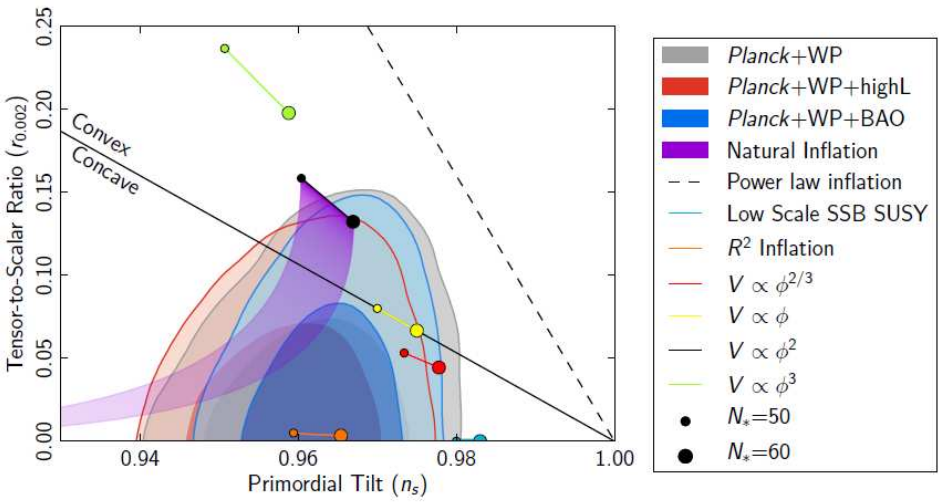


Fig. 6. Constraints from Planck data (source: Planck data release)

Three-point function and non-gaussianity

The first order constraints are not enough to assess the whole variety of models available for inflation. So one of the other signatures we look at is the deviation from gaussianity. This is a higher order constraint. To consider this in more detail, we need the concept of a gaussian random field. Given a function $f(x)$, whose fourier transform has a gaussian distribution associated with it:

$$f(k) = \int d^3x\, f(x) e^{-ik.x} = a_k + i b_k$$

$$P(a_k, b_k) = \frac{1}{\pi\sigma_k^2} e^{-\left(\frac{a_k^2 + b_k^2}{\sigma_k^2}\right)} \Rightarrow P[f(k)] = \frac{1}{\pi\sigma_k^2} e^{-\left(\frac{|f(k)|^2}{\sigma_k^2}\right)}$$

This is called a gaussian random field.

The two point correlation function as we have been calculating is given by (note that mean is zero):

$$\langle f(k) f(k') \rangle = \langle a_k a_{k'} \rangle - \langle b_k b_{k'} \rangle = \sigma^2 \delta(k + k')$$

We can use Isserlis' theorem (wick's theorem in probability) to compute higher order correlation (we need to assume linearity of $f(x)$):

$$\langle f(k) f(k') f(k'') \rangle = 0$$

$$\langle f(k_1) f(k_2) f(k_3) f(k_4) \rangle = \sigma_{k_1}^2 \sigma_{k_2}^2 [\delta(k_1 - k_2)\delta(k_3 - k_4) + \delta(k_1 - k_3)\delta(k_2 - k_4) + \delta(k_1 - k_4)\delta(k_2 - k_3)]$$

Now, if this was not a gaussian random variable, then we would expect the odd correlations not to vanish. This gives us a way to characterize functions using their deviations from *gaussianity*.

We can parametrize this deviation from gaussianity as follows (Komatsu-Spergel Local form):

$$f(x) = f_g(x) + f_{\rm NL}^{\rm local}(f_g(x)^2 - \langle f(x)^2 \rangle)$$

where $f_g(x)$ is a gaussian random variable. One can show that the three-point correlation is non-zero and related to $f_{\rm NL}^{\rm local}$ as follows:

$$\langle f(k_1) f(k_2) f(k_3) \rangle = f_{\rm NL}^{\rm local}[2(2\pi)^3 P(k_1) P(k_2) \delta(k_1 + k_2 + k_3) + ...]$$

where $\int \frac{d^3k}{(2\pi)^3} P(k_1) = \langle f_g(x) f_g(x) \rangle$. Thus, it is a measure of non-gaussianity.

As of 2013 results, no evidence for non-gaussian statistics have been found in the CMB anisotropies.

Galaxy bias

When we look at the power-spectrum of the galaxies, it need not be the same as the part which we cannot observe- dark matter. The dark matter distribution has a different transfer function for the propagation of perturbation with time. So, its distribution will differ from the baryonic matter that we observe. To connect this, a parameter called the *galaxy bias* (b) was introduced as follows:

$$P_{\delta_g} = b^2 P_\delta$$

where P_{δ_g} is the power spectrum of the galaxies observed, P_δ is the underlying dark matter power spectrum which we cannot observe. There is a connection between the dark matter power spectrum and the inflationary power spectrum, through the dark matter transfer function, which also gives us insights into the inflationary power spectrum.

In short, the spectral index, tensor index and tensor-to-scalar ratio form the linear order constraints from Planck data. At higher orders, the deviation from gaussianity in the form of local non-gaussianity places tight bounds on inflationary models. This coupled with a broad set of initial conditions leading to the observed flatness and solving the horizon problem give us a good motivation of the model.

2.3 References

[1] The standard cosmology results have mostly been taken from: S. Weinberg, “Cosmology”; D. Baumann, “Lecture notes in Cambridge”.

[2] E. Komatsu, D.N Spergel; “Acoustic Signatures in the Primary Microwave Background Bispectrum” arXiv: astro-ph/0005036v2 (2000). The original account on non-gaussianity in CMB data. In the context of primordial non-gaussianity, the following reference was relevant: X. Chen, “Primordial Non-Gaussianities from Inflation models”, arXiv: 1002.1416.

3 Mechanism of inflation

3.1 Characteristics of a generic inflationary model

To address the problem of initial conditions previously described (mainly the horizon and flatness problem), we need a model of the early era of the universe which has: (i) flatness as an attractor solution, (ii) parts of the universe exit the horizon, thus losing causal contact; hence, before this era, they were in causal contact. The above conditions are necessary conditions for a successful inflationary model.

There are more features to inflation though. A typical inflationary scenario is characterized by the beginning of inflation, the dynamics of inflation, and inflation exit.

(i) Initial conditions: Inflation begins at high energy scales typically. The initial conditions that inflation start with arise from the features of a theory of quantum gravity. However, in the case of inflation, these initial conditions do not usually influence the features of the inflation itself.

In the case of chaotic inflation, inflation begins with the rolling of a field toward the attractor, with the main area of calculation being the last few e-foldings. The original proposal was that inflation begins by a field tunneling from a false vaccuum, to a true vaccuum followed by inflation. However this was shown to be rather violent leading to large perturbations, and is disfavoured currently. In general, when one refers to inflationary model, they are usually referring to chaotic inflation.

(ii) Dynamics of inflation: The universe goes through a stage of accelerated expansion ($w<1$); this can be modelled by a minimally coupled scalar field- as a simple example.

$$L=\frac{m_{\rm pl}^2}{2}R-\frac{1}{2}(\nabla\phi)^2+V(\phi)$$

A trivial calculation leads us to modelling scalar field behaviour as a perfect fluid[a]. In case of FRW metric, we have

$$T_{00}=\rho_\phi, T_{ij}=a^2(t)\,p_\phi\,\delta_{ij}$$

which gives:

$$\begin{aligned}\rho_\phi &= \frac{1}{2}\dot{\tilde{\phi}}^2+\tilde{V}(\tilde{\phi})\\ p_\phi &= \frac{1}{2}\dot{\tilde{\phi}}^2-\tilde{V}(\tilde{\phi})\end{aligned}$$

Now, the Friedman equation for $k=0$ takes the following form:

$$H^2=\frac{8\pi G}{3}\left(\frac{1}{2}\dot{\tilde{\phi}}^2+\tilde{V}(\tilde{\phi})\right) \tag{7}$$

Along with this we have the other two equation:

$$\dot{H}=-4\pi G\,\dot{\tilde{\phi}}^2$$

$$\ddot{\tilde{\phi}}+3H\dot{\tilde{\phi}}+V_{,\tilde{\phi}}=0$$

These equations are redundant. From any two of these equations, one can derive the third one.

Condition for acceleration implies:

$$\frac{1}{2}\dot{\tilde{\phi}}^2 < \tilde{V}(\tilde{\phi}) \Rightarrow -4\pi G\dot{\tilde{\phi}}^2 < -\frac{8\pi G}{3}\left(\frac{1}{2}\dot{\tilde{\phi}}^2 + \tilde{V}(\tilde{\phi})\right) \Rightarrow |\dot{H}| < H^2$$

From the above condition, we see that any scalar field which starts like this rolls down the potential towards the minima.

The spectrum of perturbation is usually calculated at the late stage of the inflation.

(iii) Inflation exit: When the condition for accelerated expansion gets violated, one expects inflation to end. A suitable mechanism is needed for this to happen in any given inflationary problem, generally called the *"graceful exit"* problem.

3.1.1 Attractor solution and late time behaviour

The observable window to inflation is quite small. The largest observed scale that exit the horizon does not signify the entirety of inflation. It is for this reason that we are only interested in the late time solutions of inflation. However for the inflation to occur, we expect there to be an attractor solution around which inflation occurs. A class of models called the slow roll inflation usually assure the existence of such attractor solutions with inflationary solutions.

We look at the behaviour of the scalar field:

$$\ddot{\phi} = -3H\dot{\phi} - V_{,\phi};\, \ddot{\phi}_{\text{initial}} \approx -V_{,\phi}$$

The above causes the following changes: $\dot{\phi}^2 > 0$, and thus H will decrease. Given that $\dot{\phi}$ remains small, it will keep on increasing until the condition for expansion is not valid anymore. This requires H to be small however, because if it increases drastically, then $\dot{\phi}$ will not increase enough. As it approaches the minima, the inflation will end as the condition for inflation will be violated.

3.2 Reheating

The end of inflation is followed by the standard big-bang scenario, after particle production through a process called *reheating*. There are a couple of scenarios that have been put forth to explain this, but is still a hugely unresolved issue.

The general idea is that it is initiated by a period of non-perturbative particle production called pre-heating followed by a perturbative manner. A popular mechanism is the *parametric resonance*. We will not go into the details, as reheating is a phenomenon relating inflation to the standard big-bang cosmology, but can be worked out separately from inflation; it is not the focus of this thesis.

3.2.1 References

[1] Andrei Linde, "Inflationary cosmology after Planck 2013", arXiv: [hep-th] 1402.0526v1 (2014). This is one of the most up-to-date articles on the topic, by one of the founders of the theory.

4 Prediction of inhomogeneities by inflation

The inhomogenities in the early universe can be accounted by quantum fluctuations. These fluctuations, while in present day are negligible, would have been quite significant at early times due to high energy scale. During inflation, these perturbative scales would have expanded exponentially, exiting the horizon and re-entering later at the time of recombination. During this while, it would have remained frozen outside the horizon.

The observation of these quantized perturbations form one of the signatures of inflation. We treat these quantum fluctuation by quantizing perturbations of the Einstein's field equations. The following section treats this at first order, while usually higher order corrections are treated directly from the Hamiltonian.

4.1 Perturbed field equations on manifolds

The fields we usually work with, in general relativity, are of tensor nature. One of the techniques to work with field equations in general relativity: is to use the method of perturbation. There are subtle issues, however, in this approach.

The tensors do not have any form of order associated with them, and the only way to say that the perturbed solution is a small correction to the original unperturbed case is if we have an order property. This notion of "something small" can be associated with the metric attributed to the manifold. If the original solution corresponds to a background spacetime (manifold) B with a metric g, then the perturbed solution corresponds to a manifold M with a metric g'.

Under the assumption: the two manifolds can be related by a diffeomorphism ψ: $B \rightarrow M$, we can relate the metrics in a perturbative manner as follows:

$$g'(\psi^* X, \psi^* Y) \equiv g(X, Y) + h(X, Y) \text{ with } |h(X, Y)| \ll |g(X, Y)| \, \forall X, Y \in T(M) \tag{8}$$

The above equation can be written in a more compact form, by choosing a basis $\{e^{(\mu)} | \mu = 0, 1, ...3\}$ for the vector space at each point.

$$g(X, Y) = \sum_{\mu,\nu} g\big(e^{(\mu)}, e^{(\nu)}\big) X_\mu X_\nu \equiv g^{\mu\nu} X_\mu X_\nu \tag{9}$$

It is also textbook material to show that the pushforward of a vector is a linear map*, so g' can also be written in the same manner as $g'_{\mu\nu}$. The choice of basis on the background spacetime gives a natural basis on the tangent space of M too. This allows us to express, in terms of that basis, equation (8) as (also from defn. of pullback of 2-forms):

$$[\psi_* g]_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \text{ with } |h_{\mu\nu}| \ll |g_{\mu\nu}| \tag{10}$$

There is still an inconsistency in the above argument. The technique of perturbation works well only under the assumption of the inequality in (8), but the pushforward can give rise to a certain ambiguity from the choice of the pushforward. Given a diffeomorphism ψ, $k\psi$ ($k \in \mathbb{R}$) is also a diffeomorphism. Now, we can compute the perturbation as $h_{\mu\nu} = k[\psi_*(g)]_{\mu\nu} - g_{\mu\nu}$. This causes the perturbation to acquire an arbitrarily large value because of k. There exists a chart on M by definition, and general relativity works irrespective of the charts involved. Now, we can pullback the chart $f: U \subseteq N \to R^d$ to B by $\psi_* f \equiv f \circ \psi$, which will define a natural chart on B, for the given diffeomorphism. For the other map $k\psi$, the natural chart is $k(f \circ \psi)$, which is different. This change of coordinates on B can correspond to cases which result in the inequality in (10) to be violated. This shows us, that it is essential to restrict this freedom before solving small perturbations on manifolds.

A whole class of diffeomorphisms can be considered from a vector field $\xi^\mu(x)$ on B. It is known that vector fields on manifolds generate a one-parameter group of diffeomorphism onto itself. Let's say that it produces $\phi_\varepsilon: M \to M$ group of diffeomorphisms. For any ε, the map $\psi \circ \phi_\varepsilon: M \to N$ represents a diffeomorphism generated over ψ. The natural chart they define on B is $f \circ \psi \circ \phi_\varepsilon: M \to R^d$. This corresponds to a set of different charts on B depending on ε; however, all these choices lead to the same g' on M. The perturbation can now be written for these different redundant choices:

$$\begin{aligned} h_{\mu\nu}^{(\varepsilon)} &= [(\psi \circ \phi_\varepsilon)_* g]_{\mu\nu} - g_{\mu\nu} \\ &= [\phi_{\varepsilon *}(\psi_* g)]_{\mu\nu} - g_{\mu\nu} \end{aligned}$$

From equation (10),

$$\begin{aligned} h_{\mu\nu}^{(\varepsilon)} &= [\phi_{\varepsilon *}(g+h)]_{\mu\nu} - g_{\mu\nu} \\ &= \phi_{\varepsilon *}(g)_{\mu\nu} + \phi_{\varepsilon *}(h)_{\mu\nu} - g_{\mu\nu} \end{aligned}$$

In keeping with the inequality of (10), we keep $\varepsilon \ll 1$, allowing us to express the change as a Lie derivative. Rewriting the above as Lie derivative (from defn.), the set of perturbations which result in the same g' are:

$$h_{\mu\nu}^{(\varepsilon)} = h_{\mu\nu} + 2\varepsilon \nabla_{(\mu}\xi_{\nu)} \tag{11}$$

This is the gauge freedom associated with perturbative gravity theories.

Keeping in mind this gauge freedom, and its importance in dealing with perturbation, the field equations can be derived. Following from eq (10) with natural basis on M,

$$[\psi_* g]_{\mu\nu} \equiv g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \text{ with } |h_{\mu\nu}| \ll |g_{\mu\nu}| \tag{12}$$

The gauge choice will be fixed at the end. Working at the first order of perturbation, it is convenient to relabel the perturbation as $\delta g_{\mu\nu} = h_{\mu\nu}$, so $\delta g^{\mu\nu} = -h^{\mu\nu}$. The field equations dealt with in General Relativity is the Einstein's field equation:

$$R(g_{\sigma\rho})_{\mu\nu} - \frac{1}{2} R(g_{\sigma\rho})\, g_{\mu\nu} = \kappa T_{\mu\nu}(\alpha_1, \ldots, g_{\sigma\rho}) \tag{13}$$

where $\alpha_1, \ldots \alpha_n$ are variables connected to source. The evolution of source is given by the Bianchi identity:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{14}$$

The equation for the perturbations are given by substituting (12), and $\alpha'_1 \to \alpha_1 + \delta\alpha_1 \ldots$,

$$\delta R(g_{\sigma\rho})_{\mu\nu} - \frac{1}{2} \delta R(g_{\sigma\rho})\, g_{\mu\nu} = \kappa \delta T_{\mu\nu}(\alpha_1, \ldots, g_{\sigma\rho}) \tag{15}$$

From this point on, when we write $R, R_{\mu\nu}$; it is implied that they are $R(g_{\sigma\rho}), R(g_{\sigma\rho})_{\mu\nu}$, and also for the related variables.

The change in levi-civita connection and, so, the curvature can be derived by the following straightforward calculation:

$$\begin{aligned}
\delta\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}\delta[g^{\alpha\lambda}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})] \\
&= -\frac{1}{2}[h^{\alpha\lambda}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) - g^{\alpha\lambda}(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu})] \\
&= \frac{1}{2}g^{\alpha\lambda}[-2h_{\gamma\alpha}\Gamma^{\gamma}_{\mu\nu} + \partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}] \\
&= \frac{1}{2}g^{\alpha\lambda}[h_{\alpha\nu;\mu} + h_{\alpha\mu;\nu} - h_{\mu\nu;\alpha}] \\
\delta R_{\mu\nu} &= \delta\big[\partial_\alpha\Gamma^{\alpha}_{\nu\mu} - \partial_\nu\Gamma^{\alpha}_{\alpha\mu} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{\nu\mu} - \Gamma^{\alpha}_{\nu\beta}\Gamma^{\beta}_{\alpha\mu}\big] \\
&= \nabla_\alpha\delta\Gamma^{\alpha}_{\mu\nu} - \nabla_\nu\delta\Gamma^{\alpha}_{\alpha\mu} \\
&= \frac{1}{2}g^{\alpha\lambda}\Big[h_{\lambda\nu;\mu;\alpha} + h_{\lambda\mu;\nu;\alpha} - h_{\mu\nu;\lambda;\alpha} - h_{\alpha\lambda;\mu;\nu}\Big] \qquad (16) \\
\delta R &= \nabla_\alpha\nabla_\beta h^{\alpha\beta} - g^{\alpha\beta}\nabla_\alpha\nabla_\beta h \qquad (17) \\
\text{where}\, h &= h^{\alpha}_{\alpha}
\end{aligned}$$

In the case of a single scalar field source, the perturbation $\phi \to \phi' = \phi + \delta\phi$, gives:

$$\delta T_{\mu\nu} = \partial_\mu\bar{\phi}\delta(\partial_\nu\bar{\phi}\,) + \partial_\nu\bar{\phi}\delta(\partial_\mu\bar{\phi}\,) - h_{\mu\nu}\left(\frac{1}{2}\partial^\rho\bar{\phi}\partial_\rho\bar{\phi}\; + V(\bar{\phi}\,)\right) - g_{\mu\nu}\Big[-\frac{1}{2}h^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\; + g^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\; + V'(\phi)\delta\phi\Big] \qquad (18)$$

Using the perturbed field equations (15), and the above simplification (18) along with the trace of the field equation (13) $[R = 8\pi G T^{\lambda}{}_{\lambda} = \partial^\rho\bar{\phi}\partial_\rho\bar{\phi}\; + 4V(\bar{\phi}\,)]$, we get:

$$\begin{aligned}
\delta R_{\mu\nu} &= \frac{1}{2}h_{\mu\nu}R + \frac{1}{2}g_{\mu\nu}\delta R + \partial_\mu\bar{\phi}\delta(\partial_\nu\bar{\phi}\,) + \partial_\nu\bar{\phi}\delta(\partial_\mu\bar{\phi}\,) - h_{\mu\nu}\left(\frac{1}{2}\partial^\rho\bar{\phi}\partial_\rho\bar{\phi}\; + V(\phi)\right) - \\
&\quad g_{\mu\nu}\Big[-\frac{1}{2}h^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\; + g^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\; + V'(\phi)\delta\phi\Big] \\
&= \frac{1}{2}h_{\mu\nu}\partial^\rho\bar{\phi}\partial_\rho\bar{\phi}\; + 2h_{\mu\nu}V(\bar{\phi}\,) - g_{\mu\nu}\left(\frac{1}{2}h^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\; - \partial_\sigma\bar{\phi}\partial^\sigma\bar{\phi}\; - 2V'(\bar{\phi}\,)\delta\bar{\phi}\;\right) + \\
&\quad \partial_\mu\bar{\phi}\delta(\partial_\nu\bar{\phi}\,) + \partial_\nu\bar{\phi}\delta(\partial_\mu\bar{\phi}\,) - h_{\mu\nu}\left(\frac{1}{2}\partial^\rho\bar{\phi}\partial_\rho\bar{\phi}\; + V(\phi)\right) - g_{\mu\nu}\Big[-\frac{1}{2}h^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\; + \\
&\quad g^{\rho\sigma}\partial_\rho\bar{\phi}\partial_\sigma\bar{\phi}\; + V'(\phi)\delta\phi\Big] \\
&= \partial_\mu\bar{\phi}\delta(\partial_\nu\bar{\phi}\,) + \partial_\nu\bar{\phi}\delta(\partial_\mu\bar{\phi}\,) + h_{\mu\nu}V(\phi) + g_{\mu\nu}V'(\phi)\delta\phi
\end{aligned}$$

Finally, from (16), we get the perturbed field equations as:

$$\frac{1}{2}g^{\alpha\lambda}\Big[h_{\lambda\nu;\mu;\alpha} + h_{\lambda\mu;\nu;\alpha} - h_{\mu\nu;\lambda;\alpha} - h_{\alpha\lambda;\mu;\nu}\Big] = \partial_\mu\bar{\phi}\delta(\partial_\nu\bar{\phi}\,) + \partial_\nu\bar{\phi}\delta(\partial_\mu\bar{\phi}\,) + h_{\mu\nu}V(\phi) + g_{\mu\nu}V'(\phi)\delta\phi \qquad (19)$$

4.1.1 FRW metric as the background spacetime

Our universe is mostly homogeneous and isotropic, thus justifying the FRW metric to a large degree. Recently, however, as we are approaching an era of precision cosmology, the deviation from FRW metric- as inhomogeneity and anisotropy- yield valuable information about some processes (such as inflation, Sunyaev-Zeldovich effect, dipole anisotropy). It is natural to treat these deviations as small perturbations over FRW spacetime.

Quite often, in cosmology, we prefer using different gauge in different parts of the calculations. For instance, in the inflationary era, to remove the effect of curvature, we would like to work in a spatially flat gauge; while in the recombination time, we prefer the newtonian gauge. To connect variables from one to another, it is preferable to talk about gauge-invariant quantities.

In the case of the original manifold being given by the FRW metric, we have:

$$\bar{g}_{00} = -1, \bar{g}_{ij} = a(t)^2 \delta_{ij}, \bar{g}_{i0} = 0$$

So, the Christoffel symbols are as follows, which also gives us the connection:

$$\Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j, \Gamma^0_{ij} = a\dot{a}\delta_{ij}$$

$$\begin{aligned} \nabla_\mu T^\nu &= \partial_\mu T^\nu + \Gamma^\nu_{\mu\alpha} T^\alpha \\ &= \partial_\mu T^\nu + \Gamma^\nu_{\mu 0} T^0 + \Gamma^\nu_{\mu i} T^i \\ \text{This implies: } \nabla_j T^i &= \partial_j T^i + \frac{\dot{a}}{a}\delta^i_j T^0 \\ \nabla_0 T^\mu &= \partial_0 T^\mu \\ \nabla_j T^0 &= \partial_j T^0 + a\dot{a}\delta_{\mathrm{ij}} T^i \end{aligned}$$

Similiarly, we see for $\nabla_\lambda h_{\mu\nu} = \partial_\lambda h_{\mu\nu} - \Gamma^\alpha_{\mu\lambda} h_{\alpha\nu} - \Gamma^\alpha_{\nu\lambda} h_{\alpha\mu}$:

$$\begin{aligned} \nabla_i h_{jk} &= \partial_i h_{\mathrm{jk}} - a\,\dot{a}\,(\delta_{ij} h_{0k} + \delta_{ik} h_{0j}) \\ \nabla_0 h_{jk} &= \partial_0 h_{\mathrm{jk}} - \frac{\dot{a}}{a}(\delta^i_j h_{ik} + \delta^i_k h_{ij}) \\ \nabla_i h_{j0} &= \partial_i h_{j0} - a\,\dot{a}\,\delta_{ij} h_{00} - \frac{2\dot{a}}{a}\delta^k_i h_{jk} \\ \nabla_0 h_{00} &= \partial_0 h_{00} \end{aligned}$$

For simplicity, we write the perturbation of ricci tensor from (16) as follows:

$$\frac{1}{2} g^{\alpha\lambda}\Big[h_{\lambda\nu;\mu;\alpha} + h_{\lambda\mu;\nu;\alpha} - h_{\mu\nu;\lambda;\alpha} - h_{\alpha\lambda;\mu;\nu} \Big] = \frac{1}{2}\Big[g^{\alpha\lambda} h_{\lambda(\mu;\nu);\alpha} - \nabla^\alpha \nabla_\alpha h_{\mu\nu} - h_{;\mu;\nu} \Big]$$

Using the above relations, after a tedious but straightforward calculation, one shows that:

$$\begin{aligned} \delta R_{\mathrm{jk}} &= -\frac{1}{2}\partial_j \partial_k h_{00} - (2\dot{a}^2 + a\dot{a})\delta_{\mathrm{jk}} h_{00} - \frac{1}{2} a\dot{a}\delta_{\mathrm{jk}} \dot{h}_{00} + \frac{1}{2a^2}(\nabla^2 h_{\mathrm{jk}} - \partial_i \partial_j h_{ik} - \\ &\quad \partial_i \partial_k h_{ij} + \partial_j \partial_k h_{ii}) - \frac{1}{2}\ddot{h}_{\mathrm{jk}} + \frac{\dot{a}}{2a}\big(\dot{h}_{\mathrm{jk}} - \delta_{\mathrm{jk}} \dot{h}_{ii}\big) + \bigg(\frac{\dot{a}^2}{a^2} + 3\frac{\ddot{a}}{a}\bigg) h_{jk} + \frac{\dot{a}^2}{a^2}\delta_{jk} h_{ii} + \\ &\quad \frac{\dot{a}}{a}\delta_{jk}\partial_i h_{i0} + \frac{1}{2}\big(\partial_j \dot{h}_{k0} + \partial_k \dot{h}_{j0}\big) + \frac{\dot{a}}{2a}(\partial_j h_{k0} + \partial_k h_{j0}) \\ \delta R_{j0} &= \frac{\dot{a}}{a}\partial_j h_{00} + \frac{1}{2a^2}(\nabla^2 h_{j0} - \partial_j \partial_i h_{i0}) + \bigg(\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a}\bigg) h_{j0} + \frac{1}{2}\frac{\partial}{\partial t}\bigg[\frac{1}{a^2}(\partial_j h_{kk} - \partial_k h_{kj})\bigg] \\ \delta R_{00} &= \frac{1}{2a^2}\nabla^2 h_{00} + \frac{3\dot{a}}{2a}\dot{h}_{00} - \frac{1}{a^2}\partial_j h_{j0} + \frac{1}{2a^2}\bigg[\ddot{h}_{\mathrm{ii}} - \frac{2\dot{a}}{a}\dot{h}_{ii} + 2h_{ii}\bigg(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\bigg)\bigg] + \\ &\quad 3h_{00}\bigg(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\bigg) \end{aligned}$$

In a similiar manner, for the source part:

$$\kappa \delta T_{\mathrm{ij}} = -h_{ij}\bigg(\frac{1}{2}\dot{\phi}^2 + V\bigg) - a^2(t)\delta_{ij}\bigg[\frac{1}{2}h_{00}\dot{\phi}^2 - \dot{\phi}\delta\dot{\phi} + V'(\phi)\delta\phi\bigg]$$

$$\kappa\delta T_{i0} = \dot\phi\delta\phi_{,i} - \frac{1}{2}h_{i0}\dot\phi^2 - V(\phi)h_{i0}$$
$$\kappa\delta T_{00} = \dot\phi\delta\dot\phi - h_{00}V(\phi) + V'(\phi)\delta\phi$$

In the field equations, we are interested in the following term on the RHS:

$$S_{\mu\nu} \equiv \kappa\delta\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\lambda_\lambda\right) = \partial_\mu\bar\phi\delta(\partial_\nu\bar\phi) + \partial_\nu\bar\phi\delta(\partial_\mu\bar\phi) + h_{\mu\nu}V(\phi) + g_{\mu\nu}V'(\phi)\delta\phi$$
$$S_{ij} = h_{ij}V(\phi) + a^2V'(\phi)\delta\phi\delta_{ij}$$
$$S_{i0} = \dot\phi\delta\phi_{,i} + h_{i0}V(\phi)$$
$$S_{00} = 2\dot\phi\delta\dot\phi + h_{00}V(\phi) - V'(\phi)\delta\phi$$

The FRW metric has the property that scalar, vector and tensor modes evolve independently of each other. It is, thus, wise to split the terms above into scalar, vector and tensor modes. We proceed with the following substitution:

$$h_{00} = -S, h_{i0} = h_{0i} = w_i, h_{ij} = \psi\delta_{ij} + s_{ij} \tag{20}$$

where the trace part of h_{ij} is completely represented by ψ, ie. s_{ij} is traceless. s_{ij} is also symmetric as expected from the property of metric.

Now, we use Helmholtz decomposition theorem to split the 3-vector into divergenceless part and other. With this, we would have completely decomposed the perturbation into scalar, vector and tensor modes.

$$w_i = \partial_i w^p + w_i^c; \text{where } \partial_i w_i^c = 0 \tag{21}$$

$$s_{ij} = \partial_i\partial_j s^p + \partial_i s_j^v + \partial_j s_i^v + s_{ij}^c; \text{where } \partial_i s_i^v = 0, \partial_i s_{ij}^c = 0 \tag{22}$$

The metric has been decomposed into S, ψ, w^p, s^p: four scalar modes; w_i^c, s_i^v: two vector modes; s_{ij}^c: one tensor mode.

Let's see the gauge freedom this decomposition shows us. From (11), considering a parameter ξ, we have a gauge freedom as follows:

$$\Delta h_{\mu\nu} = -\partial_\mu\xi_\nu - \partial_\mu\xi_\nu - 2\Gamma^\alpha_{\mu\nu}\xi_\alpha$$

Writing down the components:

$$\Delta h_{ij} = -\partial_i\xi_j - \partial_j\xi_i + 2a\,\dot a\delta_{ij}\xi_0$$
$$\Delta h_{i0} = -\partial_t\xi_i - \partial_i\xi_0 + 2\frac{\dot a}{a}\xi_i$$
$$\Delta h_{00} = -2\partial_t\xi_0$$

We try to compare this to our equations after SVT decomposition. But first, we need to SVT decompose $\xi_i = \partial_i\xi^s + \xi_i^v$.

$$\Delta h_{ij} = -2\partial_i\partial_j\xi^s - \partial_i\xi_j^v - \partial_j\xi_i^v + 2a\,\dot a\delta_{ij}\xi_0$$
$$\Delta h_{i0} = -\partial_t\partial_i\xi^s - \partial_t\xi_i^v - \partial_i\xi_0 + 2\frac{\dot a}{a}\xi_i^v + 2\frac{\dot a}{a}\partial_i\xi^s$$
$$\Delta h_{00} = -2\partial_t\xi_0$$

The above relation can be used to see the gauge freedom of our original parameters, by comparing with (20), (21) and (22):

$$\Delta S = 2\dot\xi_0$$
$$\Delta w^p = -\dot\xi^s - \xi_0 + 2\frac{\dot a}{a}\xi^s$$

$$
\begin{aligned}
\Delta w^c_i &= 2\frac{\dot a}{a}\xi^v_i - \dot\xi^v_i \\
\Delta s^c_{ij} &= 0 \\
\Delta s^v_i &= -\xi^v_i \\
\Delta s^p &= -2\xi^s \\
\Delta\psi &= 2a\,\dot a\xi_0
\end{aligned}
$$

The different modes of perturbation can, again, be derived by substituting (20), (21) and (22) in (15). The following sections describe each of these modes.

Scalar modes

From the first equation,

$$
\begin{aligned}
\psi\delta_{ij}V(\phi)+V(\phi)\partial_j\partial_k s^p + a^2V'(\phi)\delta\phi\delta_{ij} &= \frac{1}{2}\partial_j\partial_k S + (2\dot a^2 + a\dot a)\delta_{\rm jk}S - \frac{1}{2}a\dot a\delta_{\rm jk}\dot S + \frac{1}{2a^2}(\nabla^2\psi\delta_{jk} + 2\partial_k\partial_j\psi) - \frac{1}{2}\ddot\psi\delta_{\rm jk} - \\
&\quad \frac{1}{2}\partial_j\partial_k\ddot s^p + \frac{\dot a}{2a}\big(-2\dot\psi\delta_{\rm jk} + \partial_j\partial_k\dot s^p - \delta_{\rm jk}\nabla^2\dot s^p\big) + \left(\frac{\dot a^2}{a^2}+3\frac{\ddot a}{a}\right)(\psi\delta_{\rm jk}+\partial_j\partial_k s^p) + \\
&\quad \frac{\dot a^2}{a^2}\delta_{jk}(3\psi + \nabla^2 s^p) + \frac{\dot a}{a}\delta_{jk}\nabla^2 w^p + \partial_j\partial_k\dot w^p + \frac{\dot a}{a}(\partial_j\partial_k w^p)
\end{aligned}
$$

The above equation can be decomposed into the pure and derivative terms to give two equations:

$$
\begin{aligned}
2(2\dot a^2+a\dot a)S - a\dot a\dot S + \frac{1}{a^2}\nabla^2\psi - \ddot\psi + \frac{\dot a}{a}\big(-2\dot\psi - \nabla^2\dot s^p\big) + \left(\frac{\dot a^2}{a^2}+3\frac{\ddot a}{a}\right)2\psi + \frac{2\dot a^2}{a^2}(3\psi+\nabla^2 s^p) + \frac{2\dot a}{a}\nabla^2 w^p &= a^2V'(\phi \\
&\quad \psi V(\phi) \\
\partial_j\partial_k\left[\frac{1}{2}S + \frac{1}{a^2}\psi - \frac{1}{2}\ddot s^p + \frac{\dot a}{2a}\dot s^p + \left(\frac{\dot a^2}{a^2}+3\frac{\ddot a}{a}\right)s^p + \dot w^p + \frac{\dot a}{a}w^p - V(\phi)s^p\right] &= 0
\end{aligned}
$$

We look at the next equation:

$$
\frac{\dot a}{a}\partial_j S + \left(\frac{\dot a^2}{a^2}+2\frac{\ddot a}{a}\right)\partial_j w^p + \frac{1}{2}\frac{\partial}{\partial t}\left[\frac{1}{a^2}(3\partial_j\psi - \partial_j\psi)\right] = \dot\phi\delta\phi_{,i} + \partial_i w^p V(\phi)
$$

The last equation gives:

$$
\frac{1}{2a^2}\nabla^2 S + \frac{3\dot a}{2a}\dot S - \frac{1}{a^2}\nabla^2 w^p + \frac{1}{2a^2}\left[3\ddot\psi - \frac{6\dot a}{a}\dot\psi + 6\psi\left(\frac{\dot a^2}{a^2}-\frac{\ddot a}{a}\right)\right] + 3S\left(\frac{\dot a^2}{a^2}+\frac{\ddot a}{a}\right) = 2\dot\phi\delta\dot\phi - SV(\phi) - V'(\phi)\delta\phi
$$

The conservation of energy-momentum tensor gives us:

$$
\partial_j\left(-(\psi\delta_{ij}+\partial_i\partial_j s^p)\left(\frac{1}{2}\dot\phi^2+V\right) - a^2(t)\delta_{ij}\left[-\frac{1}{2}S\dot\phi^2 - \dot\phi\delta\dot\phi + V'(\phi)\delta\phi\right]\right) + \partial_t\left(\dot\phi\delta\phi_{,i} - \frac{1}{2}\partial_i w^p\dot\phi^2 - V(\phi)\partial_i w^p\right) = 0
$$

$$
\partial_t\big(\dot\phi\delta\dot\phi + {\rm SV}(\phi) + V'(\phi)\delta\phi\big) + \partial_i\left(\dot\phi\delta\phi_{,i} - \frac{1}{2}\partial_i w^p\dot\phi^2 - V(\phi)\partial_i w^p\right) = 0
$$

Vector modes

We take the first equation into account:

$$-\frac{1}{2}\partial_j\ddot{s}^v_k-\frac{1}{2}\partial_k\ddot{s}^v_j+\frac{\dot{a}}{2a}(\partial_j\dot{s}^v_k+\partial_k\dot{s}^v_j)+\left(\frac{\dot{a}^2}{a^2}+3\frac{\ddot{a}}{a}\right)(\partial_j s^v_k+\partial_k s^v_j)+\frac{1}{2}(\partial_j\dot{w}^c_k+\partial_k\dot{w}^c_j)+\frac{\dot{a}}{2a}(\partial_j w^c_k+\partial_k w^c_j)$$
$$=(\partial_j s^v_k+\partial_k s^v_j)V(\phi)$$

Taking one of the derivative part of the above equation:

$$\partial_k\left[-\frac{1}{2}\ddot{s}^v_j+\frac{\dot{a}}{2a}\dot{s}^v_j+\left(\frac{\dot{a}^2}{a^2}+3\frac{\ddot{a}}{a}\right)s^v_j+\frac{1}{2}\dot{w}^c_j+\frac{\dot{a}}{2a}w^c_j-s^v_jV\right]=0$$

Considering the conservation of energy-momentum tensor:

$$\partial_j\left(-(\partial_i s^v_j+\partial_j s^v_i)\left(\frac{1}{2}\dot{\phi}^2+V\right)-a^2(t)\delta_{ij}\left[-\dot{\phi}\delta\dot{\phi}+V'(\phi)\delta\phi\right]\right)+\partial_t\left(\dot{\phi}\delta\phi_{,i}-\frac{1}{2}w^c_i\dot{\phi}^2-V(\phi)w^c_i\right)=0$$

$$\partial_t\left(\dot{\phi}\delta\dot{\phi}+V'(\phi)\delta\phi\right)+\partial_i\left(\dot{\phi}\delta\phi_{,i}-\frac{1}{2}w^c_i\dot{\phi}^2-V(\phi)w^c_i\right)=0$$

Tensor modes

We have only one equation with tensor terms- the first equation.

$$(\nabla^2 s^c_{\mathrm{jk}})-a^2\ddot{s}^c_{\mathrm{jk}}+a\,\dot{a}\,(\dot{s}^c_{\mathrm{jk}})+2(\dot{a}^2+3\ddot{a}a)s_{jk}=0$$

Note that the tensor mode is independent of the source; so the power spectrum of tensor perturbations is independent of the model in the scalar field case.

4.1.2 References

[1] The techniques of helmholtz decomposition, and some of the gauges shown here, are taken from, S. Weinberg, "Cosmology" 2008, and lecture notes by D. Baumann; and Eugene A. Lim. The notations followed here are rather different, however; the calculations redone keeping in mind the context of inflation.

[2] J.M Bardeen, "Gauge-invariant cosmological perturbations", Phys. Rev. D: Vol. 22, No. 8 (1980). Arguably the original and still highly relevant article on the topic.

[3] Some of the inital material on perturbation were inspired by the formal treatment of General Relativity, as followed by Sean Carroll, "Spacetime and Geometry", and Wald's treatment of "General Relativity".

4.2 Quantization of gauge-invariant quantities

Now that we have the perturbed equations at hand, we can fix the gauge and proceed with finding the power spectrum. The observables quantized, in general, are gauge invariant quantities.

Scalar perturbation

We choose the following gauge to simplify the calculation of R such that in these coordinate, we have:

$$\delta\phi=s^p=0$$

We see that this fixes all the gauge freedom for the scalar modes. Let $\psi \rightarrow R$; We get the following set of equations:

$$\begin{aligned} H\dot{S} + 2\big(3H^2 + \dot{H}\big)S + a^{-2}\nabla^2 R - \ddot{R} - 6H\dot{R} + 2H\nabla^2 w^p &= 0 \\ \dot{R} &= HS \\ -\frac{1}{2}\frac{\partial}{\partial t}\big(S\dot{H}\big) - 3H\dot{H}S - \dot{H}\nabla^2 w^p + \frac{3}{2}\dot{H}\dot{R} &= 0 \end{aligned}$$

Eliminating w^p from the equations, we get:

$$\ddot{R} + \dot{R}\left(3H - \frac{2\dot{H}}{H} + \frac{\ddot{H}}{\dot{H}}\right) - a^{-2}\nabla^2 R = 0$$

We can reduce this to the form of a oscillatory equation. But, first, we are interested in the fourier modes of these terms. One can show from the translation invariance of the background that each of the fourier modes evolve independent of each other.

So fourier transforming, the equation reduces to:

$$\ddot{R}_k + \dot{R_k}\left(3H - \frac{2\dot{H}}{H} + \frac{\ddot{H}}{\dot{H}}\right) + a^{-2}k^2 R_k = 0$$

We also want this equation in terms of conformal time. It will then reduce as follows:

$$\begin{aligned} \frac{d}{\mathrm{dt}}(a^{-1}R_k') + a^{-1}R_k'\left(3H - \frac{2\dot{H}}{H} + \frac{\ddot{H}}{\dot{H}}\right) + a^{-2}k^2 R_k &= 0 \\ -a^{-2}R_k'\dot{a} + a^{-2}R_k'' + a^{-1}R_k'\left(3H - \frac{2\dot{H}}{H} + \frac{\ddot{H}}{\dot{H}}\right) + a^{-2}k^2 R_k &= 0 \\ R_k'' + R_k'\left(2aH - \frac{2\dot{H}}{H} + \frac{a\ddot{H}}{\dot{H}}\right) + k^2 R &= 0 \end{aligned}$$

Note that for $z = \frac{a\dot{\phi}^2}{H^2}$, $v_k = zR_k$, we get the desired equation:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0$$

Tensor perturbation

There is no gauge-fixing required here, because the tensor modes are gauge invariant.

$$(\nabla^2 s_{\mathrm{jk}}^c) - a^2\ddot{s}_{\mathrm{jk}}^c + a\,\dot{a}\,(\dot{s}_{\mathrm{jk}}^c) + 2(\dot{a}^2 + 3\ddot{a}a)s_{jk} = 0$$

We see that this is not influenced by the source. It is thus clear that the quantum fluctuations from the inflaton field result in scalar perturbations while the gravitational waves contribute to the power spectrum of the tensor modes. We will reduce these to the same form as the scalar one for convenience.

Taking the polarization $e_{11} = -e_{22} = \frac{1}{\sqrt{2}}$; $e_{\mathrm{ii}} = 0$; $k_i e_{\mathrm{ij}} = 0$, we decompose the tensor as:

$$s_{ij}^c = e_{ij}D_k(t)e^{ikx}$$

We can write the equation in terms of these modes as:

$$k^2 D_k - aD_k'' + 2\dot{a}D'_k + 2(\dot{a}^2 + 3\ddot{a}a)D_k = 0$$

Taking $z=a, v=zD$, we get:

$$v_k''+\left(k^2-\frac{z''}{z}\right)v_k=0$$

This equation can be canonically quantized to give us the power spectrum. We are working in a quantization of fields, where the gravitational perturbations are also considered as fields under FRW metric. Now, irrespective of the actual physical observable under discussion, if they obey the same classical equation, after promoting them to operators, they will have the same form.

The power spectrum for any observable of the form: $\hat{v}_k = v_k(t)\hat{a}_k + v_k^*(t)\hat{a}_k^\dagger$ can be computed as follows:

$$\begin{aligned}
\langle 0|\hat{v}_k\hat{v}_{k'}|0\rangle &= \langle 0|\big(v_k(t)\hat{a}_k+v_k^*(t)\hat{a}_k^\dagger\big)\big(v_{k'}(t)\hat{a}_{k'}+v_{k'}^*(t)\hat{a}_{k'}^\dagger\big)|0\rangle\\
&= \langle 0|v_k(t)\hat{a}_k v_{k'}^*(t)\hat{a}_{k'}^\dagger|0\rangle\\
&= \langle 0|v_k(t)v_{k'}^*(t)\big[\hat{a}_k,\hat{a}_{k'}^\dagger\big]|0\rangle\\
&= (2\pi)^3|v_k(t)|^2\delta(k-k')
\end{aligned}$$

In the de-sitter case, taking large scales, we have the following:

$$\frac{z''}{z}=\frac{a''}{a}=\frac{2}{\tau^2}$$

This gives us the following:

$$\omega(\tau)=k\left(1-\frac{1}{k^2\tau^2}\right)$$

Now, we can solve for the solutions:

$$\begin{aligned}
\hat{v}_k(\tau) &= Ae^{i\left(k\tau-\frac{1}{k\tau}\right)}\hat{a}_k+A^*e^{-i\left(k\tau-\frac{1}{k\tau}\right)}\hat{a}_k^\dagger\\
&= Ae^{-i(k\tau)}\left(1-\frac{i}{k\tau}\right)\hat{a}_k+A^*e^{i(k\tau)}\left(1+\frac{i}{k\tau}\right)\hat{a}_k^\dagger
\end{aligned}$$

This gives us the correlation:

$$\langle 0|\hat{v}_k\hat{v}_{k'}|0\rangle=(2\pi)^3\delta(k+k')|v_k(\tau)|^2=(2\pi)^3\delta(k+k')\left(1+\frac{1}{k^2\tau^2}\right)$$

We can calculate the power spectrum of both scalar and tensor perturbations at horizon exit:

$$\Delta_R^2(k)=\frac{H_*^2}{(2\pi)^2}\frac{H_*^2}{\dot{\phi}_*^2}$$

$$\Delta_D^2(k)=\frac{2H_*^2}{\pi^2}$$

4.2.1 References

[1] Birrell and Davies, “Quantum fields in curved space”, ISBN 0-521-27858-9.

4.3 Slow roll single field inflation

For any scalar field in a potential with minima, one can formalize the slow roll as first order corrections to flat solution (de sitter expansion). Exactly at the minima, we have a flat potential, so the evolution is given by:

$$\dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2} = 0 = -4\pi G \dot{\tilde{\phi}}^2$$

$$H^2 = \frac{8\pi G}{3}\tilde{V}\left(\tilde{\phi}\right) = \text{const.}$$

So, in this context, the slow roll scenario exists for all scalar field models around the potential minima. Take the first order correction to de-sitter expansion:

$$H^2 = H_0^2\left(1 - \frac{\varepsilon_h}{3}\right)^{-1}$$

This implies in general that:

$$\varepsilon_h = 3\frac{H^2 - H_0^2}{H^2} = -\frac{\dot{H}}{H^2}; \text{with condition}\, \varepsilon_h \ll 1$$

The second condition we want is for the above condition to hold for a long time. This is achieved by the following condition in the evolution of the field:

$$\left|\ddot{\tilde{\phi}}\right| \ll H\left|\dot{\tilde{\phi}}\right| \Rightarrow 3H\dot{\tilde{\phi}} = -V_{,\tilde{\phi}}$$

We define the condition as follows for that:

$$\eta_h = -\frac{\ddot{\tilde{\phi}}}{H\dot{\tilde{\phi}}}; |\eta_h| \ll 1$$

Now, one can show that $\varepsilon = 3\frac{\dot{\phi}^2/2}{V + \dot{\phi}^2/2}$. This shows that $\varepsilon < 1$ corresponds to condition for inflation. So, all scalar fields correspond to slow-roll attractor solution, as departure from $\varepsilon = 0$, which is de-sitter case corresponds to an inflating universe. Intuitively, around the minima of any potential, at small distances slow-rolling fields will describe inflation.

The number of e-folding in slow-roll can be obtained as:

$$\begin{aligned}\Delta N &= \int_{t_i}^{t_f} H\,\mathrm{dt}\\ &= \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi\\ &= \int_{\phi_i}^{\phi_f} \frac{2}{m_{\text{pl}}}\frac{1}{\sqrt{\varepsilon_h(\phi)}}\end{aligned}$$

4.3.1 Potential slow-roll parameters (PSR)

It is useful to note here itself that there are two forms of parameters used in slow-roll approximation. One is a parameter of potential V and the other of Hubble quantity H. The Hubble parameters are given as follows:

$$\varepsilon_h(\phi) = \frac{1}{4\pi G}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 = -\frac{\dot{H}}{H^2}; \eta_h(\phi) = \frac{1}{4\pi G}\left(\frac{H''(\phi)}{H(\phi)}\right) = \frac{-\ddot{\phi}}{H\dot{\phi}}$$

The following conditions translate to slow-roll conditions.

$$\varepsilon_h, |\eta_h| \ll 1$$

The condition for inflation is $\varepsilon_h < 1$. We can see that these are good set of parameters to describe conditions for slow-roll inflation. However given a potential, it's easier to work with potential slow roll parameters. These parameters, however, also require inflationary attractor condition to be satisfied to describe slow-roll inflation. We can derive the following relations upto first order:

$$\eta_v = \frac{m_{\rm pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2$$

$$\varepsilon_\nu = \frac{m_{\rm pl}^2}{8\pi}\left(\frac{V''(\phi)}{V(\phi)}\right)$$

We will work in this first order approximation of slow roll.*

4.3.2 Power spectrum

The power spectrum for any observable of the form: $\hat{v}_k = v_k(t)\hat{a}_k + v_k^*(t)\hat{a}_k^\dagger$ can be computed as follows:

$$\begin{aligned}
\langle 0|\hat{v}_k\hat{v}_{k'}|0\rangle &= \langle 0|\big(v_k(t)\hat{a}_k + v_k^*(t)\hat{a}_k^\dagger\big)\big(v_{k'}(t)\hat{a}_{k'} + v_{k'}^*(t)\hat{a}_{k'}^\dagger\big)|0\rangle \\
&= \langle 0|v_k(t)\hat{a}_k v_{k'}^*(t)\hat{a}_{k'}^\dagger|0\rangle \\
&= \langle 0|v_k(t)v_{k'}^*(t)\big[\hat{a}_k, \hat{a}_{k'}^\dagger\big]|0\rangle \\
&= (2\pi)^3|v_k(t)|^2\delta(k-k')
\end{aligned}$$

In the case of single-field slow-roll inflation, we have the following Mukhanov equation:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0, z^2 = 2a^2\varepsilon$$

In the case of slow roll, we can reduce the term as follows:

$$\frac{z''}{z} = \frac{\nu^2 - 1/4}{\tau^2} \text{ where } \nu \sim \frac{3}{2} + 3\varepsilon_h - \eta_h$$

So the equation turns out to be of Bessel form whose solution in this case is given by the Hankel function:

$$v_k(\tau) = (k|\tau|)^{1/2}\Big[c_1 H_\nu^{(1)}(k|\tau|) + c_2 H_\nu^{(2)}(k|\tau|)\Big]$$

Now, let's try to put boundary conditions on this. In the limit $\tau \to \infty$, we need it to reach a de-sitter limit. The limit of the Hankel function is given by:

$$H_\nu^{(1,2)}(k|\tau|) \to \sqrt{\frac{2}{\pi k|\tau|}}e^{\pm i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}$$

This implies the following solution:

$$v_k(\tau) = \left(\frac{\pi|\tau|}{4}\right)^{1/2} e^{i(2\nu+1)\frac{\pi}{4}} H_\nu^{(1)}(k|\tau|)$$

The above solution with boundary conditions applied can give us a solution for de-sitter case. For the sake of clarity, we will rederive it in the following manner: assuming large scales first.

In the de-sitter case, taking large scales, we have the following:

$$\frac{z''}{z} = \frac{a''}{a} = \frac{2}{\tau^2}$$

This gives us the following:

$$\omega(\tau) = k\left(1 - \frac{1}{k^2\tau^2}\right)$$

Now, we can solve for the solutions:

$$\begin{aligned} \hat{v}_k(\tau) &= Ae^{i\left(k\tau - \frac{1}{k\tau}\right)}\hat{a}_k + A^* e^{-i\left(k\tau - \frac{1}{k\tau}\right)}\hat{a}_k^\dagger \\ &= Ae^{-i(k\tau)}\left(1 - \frac{i}{k\tau}\right)\hat{a}_k + A^* e^{i(k\tau)}\left(1 + \frac{i}{k\tau}\right)\hat{a}_k^\dagger \end{aligned}$$

This gives us the correlation:

$$\left\langle 0|\hat{v}_k \hat{v}_{k'}|0\right\rangle = (2\pi)^3\delta(k+k')|v_k(\tau)|^2 = (2\pi)^3\delta(k+k')\left(1 + \frac{1}{k^2\tau^2}\right)$$

We can calculate the power spectrum of both scalar and tensor perturbations at horizon exit:

$$\Delta_R^2(k) = \frac{H_*^2}{(2\pi)^2}\frac{H_*^2}{\dot{\phi}_*^2}$$

$$\Delta_D^2(k) = \frac{2H_*^2}{\pi^2}$$

4.3.3 Spectrum in terms of PSR

In the case of slow-roll we can derive the above results in terms of slow roll parameters. Once we are able to do that, we will able to compute these observed quantities directly in terms of the potential SRP which depend on the shape of the potential. We will work with only slow-roll models for the time-being.

The scalar index can be written as follows:

$$\begin{aligned} \frac{d\ln\Delta_s^2}{d\ln k} &= \frac{d\ln\Delta_s^2}{dN} \times \frac{dN}{d\ln k} \\ &= \left(2\frac{d\ln H}{dN} - \frac{d\ln\varepsilon}{dN}\right)\frac{dN}{d\ln k} \\ n_s - 1 &= (2\eta_* - 6\varepsilon_*) \end{aligned}$$

Similiarly, we get:

$$n_t = -2\varepsilon_*$$

We can compute r in the same fashion to get:

$$r = 16\varepsilon_*$$

4.3.4 Consistency of slow-roll single field

From the above equation, we get a consistency condition as follows:

$$r = -8n_t$$

This follows from the fact that slow-roll is completely described by two parameters while we have three observables. This consistency condition may not hold in general, but its confirmation supports the case of slow-roll inflation.

4.4 References

[1] The standard cosmology results have mostly been taken from: S. Weinberg, "Cosmology"; D. Baumann, "Lecture notes in Cambridge".

[2] The overview of results has been extracted primarily from: Planck 2013 results, "I. Overview of products and scientific results"; ArXiv/0674450.

[3] Planck 2013 results, "XXII. Constraints on Inflation"; ArXiv/0674450.

[4] C. Armendariz-Picon, P.B Greene; "Spinors, Inflation and Non-Singular cyclic cosmologies", hep-th/0301129v1 (2003).

[5] D. Baumann, "TASI lectures on Inflation".

[6] S. Weinberg, "Effective Field theory for Inflation"; arXiv/0804.4291v2 (2008).

5 Present models of inflation

There are numerous models which attempt to explain the underlying dynamics of inflation. The Planck data 2013 strongly supports almost scale invariant, mostly gaussian scalar spectrum.

In the presence of a numerous models, one can motivate an *effective-field theory* style terms in the lagrangian. If we were to suppose that a single scalar field ϕ, and curvature terms from general relativity to be the major cause for the inflation; then the terms in the lagrangian with couplings α_i with ($[\alpha_i] \sim M^d$, $d > 0$) are taken. The leading order $d = 2$ terms:

$$\phi^2, R$$

Similarly, $d = 0$ terms:

$$\phi^4, R^2, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}, \phi^2 R, \nabla_\mu\nabla_\nu R^{\mu\nu}, \partial_\mu\partial^\mu R, \phi(\nabla^2\phi), (\nabla\phi)^2$$

The third and fourth terms above can be removed as they can be incorporated into topological invariants. Neglecting all derivative terms, as they won't contribute to the classical field equations, the lagrangian can be written as:

$$L = \frac{m_{\rm pl}^2}{2}R + \frac{m^2}{2}\phi^2 + \lambda\phi^4 + \alpha R^2 + \beta\phi^2 R + k_1\phi(\nabla^2\phi) - \frac{1}{2}(\nabla\phi)^2$$

The sixth term above can be rewritten in terms of the last term upto a derivative term. The general lagrangian can now be written as:

$$L = \frac{m_{\rm pl}^2}{2}R - \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \lambda\phi^4 + \alpha R^2 + \beta\phi^2 R + k_1\phi(\nabla^2\phi) \tag{23}$$

This can be further simplified through some non-trivial steps, from Brans-Dicke theory and *f(R)* gravity.

The fifth term in (23) is the one used in Starobinsky inflation, and it is be shown to be equivalent to a non-minimally coupled scalar field.

5.1 *Aspects of f(R) gravity*

The metric theory of *f(R)* gravity, can be analyzed to express it as a non-minimally coupled scalar field.

$$\mathcal{L} = \sqrt{-g}\, f(R)$$

Varying this w.r.t metric, one gets:

$$\begin{aligned} \delta S &= \delta\int d^4x\sqrt{-g}\, f(R) \\ &= \int d^4x\Big[-\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu} f(R) + \frac{\partial f(R)}{\partial R}\sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu}\Big] \end{aligned}$$

$$\text{Now, } J = \frac{\partial f(R)}{\partial R}\sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu} = \frac{\partial f(R)}{\partial R}\sqrt{-g}\, g^{\mu\nu}(\nabla_\alpha\delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu\delta\Gamma_{\alpha\mu})$$

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}(\nabla_\mu\delta g_{\nu\alpha} + \nabla_\nu\delta g_{\mu\alpha} - \nabla_\alpha\delta g_{\mu\nu})$$
$$\text{So we get, } J = \frac{\partial f(R)}{\partial R}\sqrt{-g}(g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu})$$

$$\frac{\partial f}{\partial R}R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\frac{\partial f}{\partial R} = 0$$

If one were to express this as: $\phi(R) = \frac{\partial f}{\partial R}$, we get:

$$\phi R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\phi = 0$$

We can show that by coordinate redefinition that this reduces to a Brans-Dicke theory, which after a conformal transformation of the metric that it reduces to a a minimally coupled scalar field to Einstein's metric.

Let us start with the original lagrangian, and see that a Legendre transformation reduces it to Brans-Dicke theory. Take $\phi(R) = \frac{\partial f}{\partial R}$. Now this is taken to be convex which is why the above procedure works. We take the *legendre transformation* as:

$$V(\phi) = R\phi - f(R)$$

From the property of legendre transformation, it is clear that the same system as before can be written to give the same extremization as follows (as long as the original function is convex, in our case the second derivative being non-zero):

$$\mathcal{L}(g_{\mu\nu}, R) = \sqrt{-g}f(R) \to \mathcal{L}(g_{\mu\nu}, \phi) = \sqrt{-g}(\phi R - V(\phi))$$

The above lagrangian is of the Brans-Dicke type with $\omega = 0$.

The sixth term in (23), and the term from above which gives the same form, can be further decoupled by a conformal transformation.

5.1.1 Different conformal frames of Brans-Dicke theory

We have shown that most of metric *f(R)* gravity reduces to Brans Dicke theory with $\omega = 0$. This theory has been known to be described in different frames which are related by conformal maps. The version we are working with is called as the Jordan frame. The idea is that there exists a conformal map from this manifold to another called the Einstein frame which looks like Einstein's gravity minimally coupled to a scalar field.

We now redefine the metric in the above Lagrangian as follows:

$$(\psi_*\tilde{g})_{\mu\nu} = e^{-2\sigma}\tilde{g}$$

Substituting the above change in Ricci scalar, as shown in the appendix, we get:

$$R = e^{-2\sigma}\big[\tilde{R} - 6\big(\tilde{\nabla}^2\sigma\big) - 6\big(\tilde{\nabla}\sigma\big)^2\big]$$

$$g = \tilde{g}e^{8\sigma}$$

We apply the above changes into the lagrangian, and now we have a parametrization of the same laws:

$$\begin{aligned}\mathcal{L}(g_{\mu\nu}, \phi) &= \sqrt{-\tilde{g}}e^{4\sigma}\big[\phi e^{-2\sigma}\big\{\tilde{R} - 6\big(\tilde{\nabla}^2\sigma\big) - 6\big(\tilde{\nabla}\sigma\big)^2\big\} - V(\phi)\big] \\ &= \sqrt{-\tilde{g}}\big[\phi e^{2\sigma}\big\{\tilde{R} - 6\big(\tilde{\nabla}^2\sigma\big) - 6\big(\tilde{\nabla}\sigma\big)^2\big\} - e^{4\sigma}V(\phi)\big]\end{aligned}$$

Eliminating the terms in front of R will make a part of the equation look like Einstein's gravity and the other terms like correction to it. So, we will choose the following:

$$e^{-2\sigma} = \phi \Rightarrow \sigma = -\frac{1}{2}\ln\phi$$

So, we get:

$$\begin{aligned}\mathcal{L}(g_{\mu\nu},\phi) &= \sqrt{-\tilde{g}}\left[\tilde{R} + 3\big(\tilde{\nabla}^2 \ln\phi\big) - \frac{3}{2}\big(\tilde{\nabla}\ln\phi\big)^2 - \phi^{-2}V(\phi)\right] \\ &= \sqrt{-\tilde{g}}\left[\tilde{R} - \frac{1}{2}\big(\tilde{\nabla}\sqrt{3}\ln\phi\big)^2 - \phi^{-2}V(\phi)\right]\end{aligned}$$

We can do a convenient redefinition of the scalar field to express this as a minimally coupled scalar field:

$$\tilde{\phi} = \sqrt{3}\ln\phi$$

$$\tilde{V}\big(\tilde{\phi}\big) \equiv \phi^{-2}V(\phi) = e^{-\frac{2\tilde{\phi}}{\sqrt{3}}}V\left(e^{\frac{\tilde{\phi}}{\sqrt{3}}}\right)$$

So, we get the following familiar lagrangian:

$$\mathcal{L}\big(g_{\mu\nu},\tilde{\phi}\big) = \sqrt{-\tilde{g}}\left[\tilde{R} - \frac{1}{2}\big(\tilde{\nabla}\,\tilde{\phi}\big)^2 - \tilde{V}\big(\tilde{\phi}\big)\right]$$

Finally (23), reduces in general to a lagrangian of the form.

$$L = \frac{m_{\rm pl}^2}{2}R - \frac{1}{2}(\nabla\phi)^2 + V(\phi) \tag{24}$$

This is the lagrangian we dealt with in the previous chapter. Now, we include specific terms for $V(\phi)$ from the above discussion.

5.2 $m^2\phi^2$ toy model

This is the simplest term in the lagrangian (23). We will evaluate the inflation with about $N_* \sim 60$. Using the *PSR relation* of slow-roll in this case, we get

$$\varepsilon_v = \eta_v = 2\left(\frac{M_{\rm pl}}{\phi}\right)^2$$

We can calculate the scalar spectrum as follows:

$$\Delta_s^2 = \frac{m^2}{M_{\rm pl}^2}\frac{N_*^2}{3}$$

The scalar index and tensor-to-scalar ratio can computed by the slow-roll results as follows:

$$n_s = 1 + 2\eta_v - 6\varepsilon_v \approx 0.98$$

$$r \approx 0.1$$

This represents inflation at a very low energy scale. This is quite expected from the shape of the potential which is not steep. The BICEP2 dataset release clearly disfavours this result.

5.3 $\lambda\phi^4$ model

This is again one of the other simple terms under consideration. We will proceed with the calculation for a general monomial model, just as a generic treatment. The potential slow roll parameters for the $\lambda_n\phi^n$ theory are given as:

$$\eta_v = \frac{m_{\rm pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = \frac{m_{\rm pl}^2}{16\pi}\left(\frac{n}{\phi}\right)^2$$

$$\varepsilon_\nu = \frac{m_{\rm pl}^2}{8\pi}\left(\frac{V''(\phi)}{V(\phi)}\right) = \frac{m_{\rm pl}^2}{8\pi}\left(\frac{n(n-1)}{\phi^2}\right)$$

The inflation ends when $\varepsilon = 1$, i.e

$$\frac{m_{\rm pl}^2}{8\pi}n(n-1) = \phi_e^2$$

The e-foldings for which inflation needs to take place can be computed as (at a certain slow roll approximation):

$$\Delta N = \int_{\phi_i}^{\phi_f} \frac{2\pi}{m_{\rm pl}}\frac{1}{\sqrt{\varepsilon_h(\phi)}} = \frac{8\pi^2}{m_{\rm pl}^2}\left(\frac{\phi_*^2 - \phi_e^2}{\sqrt{n(n-1)}}\right) \sim 60$$

So, one can compute the field at horizon exit for the scales that come back at CMB as:

$$\phi_*^2 \sim \frac{m_{\rm pl}^2}{8\pi^2}n(n-1) + \frac{15 m_{\rm pl}^2}{2\pi}\sqrt{n(n-1)}$$

Now, we can compute the spectral index at horizon exit:

$$\begin{aligned} n_s - 1 &= (2\eta_* - 6\varepsilon_*) \\ &= \frac{m_{\rm pl}^2}{8\pi}\left(\frac{n}{\phi_*}\right)^2 - \frac{3m_{\rm pl}^2}{4\pi}\left(\frac{n(n-1)}{\phi_*^2}\right) \\ &= \frac{m_{\rm pl}^2}{8\pi}\left(\frac{6-5n^2}{\phi_*^2}\right) \\ &= \frac{1}{8\pi}\left(\frac{6-5n^2}{\frac{1}{8\pi^2}n(n-1) + \frac{15}{2\pi}\sqrt{n(n-1)}}\right) \end{aligned}$$

Similarly, one can compute the tensor-scalar ratio as:

$$\begin{aligned} r &= 16\varepsilon_* \\ &= \frac{2m_{\rm pl}^2}{\pi}\left(\frac{n(n-1)}{\phi_*^2}\right) \\ &= \frac{2}{\pi}\left(\frac{n(n-1)}{\frac{1}{8\pi^2}n(n-1) + \frac{15}{2\pi}\sqrt{n(n-1)}}\right) \end{aligned}$$

The above results can be worked out independently for each case of n, and they turn out to be inconsistent with Planck data.

5.4 Starobinsky model (R^2 inflation)

f(R) gravity can, in general, be approximated by an analytic expansion, with the leading order corrections included to give,

$$f(R) = R + \alpha R^2 \tag{25}$$

As explained before, this term naturally arises in an effective field theory of gravity.

The equation of motion reads as:

$$G_{\mu\nu} + (2\alpha R)R_{\mu\nu} - \frac{1}{2}R(\alpha R)g_{\mu\nu} + 2\alpha(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R = 0$$

Taking the trace of the above equation gives:

$$-R + 6\alpha\Box R = 0$$

We shall check the stability of the solution to (25). Note the difference between an attractor solution and a stable solution; a stable solution is a solution which will remain unchanged under small perturbations, while an attractor solution is a unique late-time solution to which most cases converge. The desired stable solution in inflation is a flat spacetime; so considering perturbations to that in *TT gauge*:

$$\partial_t^2 h - \nabla^2 h - h/6\alpha = 0$$

We need the condition $\alpha > 0$ for our case of inflationary attractor solution.

For now, we will working in the simple case of slow-roll de-sitter case inflation. The inflation is taken to be initiated by a potential driven inflaton field which arises in the Einstein frame. This will continue for a long time due to our slow-roll conditions that we enforce. Following the general procedure,

$$\phi = 1 + 2\alpha R$$

Since the derivative of this is non-zero, our legendre transformation is well justified.

Now, the potential is:

$$V(\phi) = R\phi - f(R) \Rightarrow V(\phi) = -\frac{(1-\phi)^2}{4\alpha}$$

So, the theory in Brans-Dicke form, but in Einstein's frame, is:

$$\tilde{\phi}(R) = \sqrt{3/\kappa}\ln\phi = \sqrt{3/\kappa}\ln(1 + 2\alpha R)$$

$$\tilde{V}(\tilde{\phi}) = \frac{1}{\phi^2}V(\phi) = -\frac{1}{4\alpha\phi^2}(\phi - 1)^2 = -\frac{1}{4\alpha\kappa}\left(e^{-\sqrt{\kappa}\tilde{\phi}/\sqrt{3}} - 1\right)^2$$

There is extra scaling as shown in the beginning of these notes.

We will compute its derivatives as that will be helpful in the next step:

$$\tilde{V}'(\tilde{\phi}) = \frac{\sqrt{\kappa}}{2\sqrt{3}\alpha\kappa}\left(e^{-2\sqrt{\kappa}\tilde{\phi}/\sqrt{3}} - e^{-\sqrt{\kappa}\tilde{\phi}/\sqrt{3}}\right)$$

$$\tilde{V}''(\tilde{\phi}) = -\frac{1}{6\alpha}\left(2e^{-2\sqrt{\kappa}\tilde{\phi}/\sqrt{3}} - e^{-\sqrt{\kappa}\tilde{\phi}/\sqrt{3}}\right)$$

As we showed before the slow roll parameters can be interpreted in terms of the shape of the potential too. We will use that to develop constraints for slow roll with the working lagrangian.

In the context of slow roll inflation, we are interested in the potential slow roll parameters:

$$\varepsilon_\nu = \frac{2}{9}\left(e^{\sqrt{\kappa}\tilde{\phi}/\sqrt{3}} - 1\right)^{-2}$$

$$\eta_\nu = -\frac{2}{9}\left(e^{\sqrt{\kappa}\tilde{\phi}/\sqrt{3}} - 2\right)\left(e^{\sqrt{\kappa}\tilde{\phi}/\sqrt{3}} - 1\right)^{-2}$$

The slow roll condition is as follows:

$$\varepsilon_\nu, |\eta_v| \ll 1$$

In terms of α, the equation can be reduced as follows:

$$(2\alpha R)^{-2} \ll \frac{9}{2}$$

$$|(1 - 2\alpha R)|(2\alpha R)^{-2} \ll \frac{9}{2}$$

Further analysis of the above depends on the following inequality:

$$\alpha < \frac{1}{R} \text{ or } \alpha > \frac{1}{R}$$

We are working in a case where we want the universe to closely approach flat solution, so at large scales we have $R = 12H^2 \approx 0$. Also, the correction to Einstein's gravity should be negligible on high length scales. So we have the following condition (accelerating universes need to have $R > 0$):

$$\alpha R^2 \ll R \Rightarrow \alpha R \ll 1$$

So, we can get the following condition ($0 < \alpha R < 1/2$):

$$\frac{9}{2} \gg (2\alpha R)^{-2} > |1 - 2\alpha R|(2\alpha R)^{-2}$$

The slow-roll condition is expressed as follows. In slow roll, H decreases as explained before.

$$\alpha \ll \frac{1}{24H^2}\left(\sqrt{\frac{2}{9}}\right) < \frac{2\alpha\kappa}{-3\left(e^{-\sqrt{\kappa}\tilde{\phi_i}/\sqrt{3}} - 1\right)^2}\left(\sqrt{\frac{2}{9}}\right)$$

As we had stressed before, for the above analysis to be meaningful, we need it to be attractor solution we are working with. Now, the inflation ends when the following condition is satisfied:

$$p = \sqrt{\frac{9}{8\pi}}\ln\left(1 + \frac{3}{\sqrt{2}}\right) = 0.45003 \tag{26}$$

We will now first do a first order derivation of the results for slow-roll inflation in the above case, imposing constraints on α accordingly. We need to calculate the slow roll parameters at horizon-exit.

$$\Delta N = \ln\left(\frac{a_f}{a_i}\right) \simeq -\sqrt{\frac{4\pi}{m_{\text{pl}}^2}}\int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{\varepsilon_v(\phi)}} d\phi = \sqrt{\frac{48\pi^2}{m_{\text{pl}}^4}}\left(\sqrt{\frac{3}{\kappa}}e^{\tilde{\phi_*}/\sqrt{3}} - \tilde{\phi}_* - \sqrt{\frac{3}{\kappa}}e^{\tilde{\phi_i}/\sqrt{3}} + \tilde{\phi_i}\right) \approx 55 \tag{27}$$

We expect the horizon exit around $\Delta N \sim 55$ before end of inflation. Numerically solving for $\tilde{\phi}_*$ from (27), and from (26); the expected indices are calculated as follows:

$n_s = 1 - \frac{4}{9}\Big(e^{\sqrt{8\pi}\phi_i/3m_{\rm pl}} - 2\Big)\Big(e^{\sqrt{8\pi}\phi_i/3m_{\rm pl}} - 1\Big)^{-2} - \frac{12}{9}\Big(e^{\sqrt{8\pi}\phi_i/3m_{\rm pl}} - 1\Big)^{-2} =$ 0.9650120

$n_t = -\frac{4}{3}\Big(e^{\sqrt{8\pi}\phi_i/3m_{\rm pl}} - 2\Big)^{-2} \approx -0.0004372$

$r = -8n_t = -0.0004372 = 0.0034976$

Referring to Planck-WP data, we see it is within the bounds of the measurement. The energy scale of the inflation, in the light of recent news, seems to be much lesser than what is observed. A deviation from slow-roll behaviour, or contribution of other non-scalar terms may be responsible for this.

5.5 Standard model fields

Inflationary models are expected to take place at very high-energy scales, where one does not expect the standard model to work. Even so, in the event of new discoveries, one expects much higher-order terms to contribute to the inflationary models; however as a simple motivation for a scalar field, one takes the well-known candidate: the Higgs field.

5.5.1 Minimal coupling of Higgs field

The Higgs field can act as an inflaton when it's value is around the Planck scale. This is under the assumption that Standard Model is valid slightly below the Planck scale. With these assumptions, one gets $n_s = 0.980 - 0.983$, and $r = 0.0062$ with 50-60 e-folding[1]. This theory seems very improbable considering the fact that the energy scales at which they claim this model holds is very close to the Planck scale, where there should be other corrections if one expects a theory of quantum gravity to exist.

5.6 Non-minimal coupling

The non-minimal coupling terms, as we showed could be reduced to the form of minimal coupling with exponential potential. The coupling parameter can be motivated in a couple of ways:

(a) Non-minimal coupling of Higgs field

(b) Conformal coupling of an arbitrary scalar field

In general, this is the same as the Starobinsky model as the shape of the potential turn out to be the same. It is thus no surprise that this will also lead to a consistent result with Planck data.

In the case of Higgs field, the coupling would be given by the following lagrangian:

$$S = \int d^4x\,\sqrt{-g}\left(-\frac{M_{\rm pl}^2}{2}R - \xi H^\dagger H R + \mathfrak{L}_{\rm SM}\right)$$

In the unitary gauge $H^T = \big(0, (h+v)/\sqrt{2}\big)$, one can reduce this as[2]:

$$S = \int d^4x\,\sqrt{-g}\left(-\frac{M_{\rm pl}^2}{2}R - \frac{1}{2}\xi h^2 R + \frac{1}{2}(\partial_\mu h)^2 - \lambda\frac{h^4}{4}\right)$$

We can always switch over to a frame in which this is non-minimally coupled. We then get a potential as follows:

$$U(\chi)=\frac{\lambda M_p^4}{4\xi^2}\left(1-e^{-\frac{\sqrt{2}\chi}{\sqrt{3}M_p}}\right)^2 \text{ where } \frac{d\chi}{dh}=\frac{M_p\sqrt{M_p^2+(6\xi+1)\xi h^2}}{M_p^2+\xi h^2}$$

Following the slow-roll assumptions, one can show that[2], $n_s=0.967$ and $r=0.0032$. This are in good agreement with Planck data.

The model is however fundamentally flawed, as at Planck scale, a theory of quantum gravity needs to be taken into account. The standard model, is quite definitely not one.

5.7 References

[1] Y Hamada, H. Kawai, K-Y Oda, "Minimal Higgs inflation", arXiv: hep-ph/1308.6651v3 (2013).

[2] Dmitry Gorbunov, "Higgs Inflation and QFT".

6 Higher order effective lagrangian models

The recent discovery of B-modes of the photon, and also the ever increasing sensitivity of Planck data motivates us to look at better corrections to the inflationary model. In the context of BICEP2 release, we now have a radical new view of inflation. The Lyth bound clearly indicates that inflation happens at a super-planckian scale (at least in the context of single scalar field models with canonical kinetic terms). All these motivate us to look beyond the first order terms, and also well beyond the simple single-scalar field model.

As expected at this scale, motivations from string theory are more dominant. The axion-monodromy inflation, for instance has come in favour because of the new observations. The future of inflationary cosmology will involve calculations more along these lines. From the effective field theory perspective, it can be shown through a tedious simplification[1] (for a scalar field in gravity, of course) that the theory reduces mostly to the form of a k-inflation*. The model is briefly reviewed, and other string models are further introduced.

6.1 k-inflation

Among the higher derivative terms, a term that can be highly motivated is the non-canonical kinetic term. This also arises as one of the terms in the effective field theory of inflation as first order correction terms[2].

The general lagrangian considered here, is as follows:

$$L = -\sqrt{-g}\left\{\frac{m_{\rm pl}^2}{2}R - p(X,\sigma)\right\} \tag{28}$$

where $X = -\frac{1}{2}(\nabla\sigma)^2$.

The field equations can be derived by $\delta S = \delta\int L d^4x = 0$ (variation w.r.t the metric):

$$\frac{m_{\rm pl}^2}{2}G_{\mu\nu} = \frac{\partial p}{\partial X}(\nabla_\mu\sigma\nabla_\nu\sigma) - \frac{1}{2}p g_{\mu\nu}$$

This can be slightly rearranged, and by adding a few terms, made to look like a perfect fluid:

$$G_{\mu\nu} = \frac{1}{m_{\rm pl}^2}\left[\left\{2X\frac{\partial p}{\partial X} + p - p\right\}\frac{(\nabla_\mu\sigma\nabla_\nu\sigma)}{2X} - p g_{\mu\nu}\right]$$

This is equivalent to a perfect fluid with the following characteristics:

$$\varepsilon \equiv \varepsilon(X,\sigma) = 2X\frac{\partial p}{\partial X} - p, p \equiv p(X,\sigma), u_\mu = \frac{\nabla_\mu\sigma}{\sqrt{2X}}$$

From the Friedmann's equation in the first chapter, we can now write it in the context of k-inflation as:

$$H^2 = \frac{1}{3m_{\rm pl}^2}, \dot{H} = -(\varepsilon + p)\frac{1}{2m_{\rm pl}^2}$$

The condition for inflation is given by (also assuming $\varepsilon > 0$):

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{6m_{\rm pl}^2}(\varepsilon + 3p) > 0 \Rightarrow \frac{X}{p}\frac{\partial p}{\partial X} < -1$$

Scalar perturbation

In the longitudonal gauge, one can write the scalar perturbed metric as:

$$ds^2 = (1 + 2\psi)dt^2 - (1 - 2\psi)\, a^2(t)dx^i dx^j$$

The metric side of the perturbations have been computed before, and are the same here. The source terms, however, are slightly different:

$$\delta T_0^0 = \delta\varepsilon = \varepsilon_{,X}\delta X + \varepsilon_{,\sigma}\delta\sigma = \frac{\varepsilon + p}{c_s^2}\left\{\frac{\delta\sigma}{\dot{\sigma}} - \phi\right\} - 3H(\varepsilon + p)\frac{\delta\sigma}{\dot{\sigma}}$$

$$\delta T_i^0 = (\varepsilon + p)\left(\frac{\delta\sigma}{\dot{\sigma}}\right)_{,i}$$

Looking at the perturbative equations we derived in the early chapters, we get:

$$\frac{1}{a^2}\nabla^2\phi - 3H\dot{\phi} - 3H^2\phi = \frac{1}{2m_{\rm pl}^2}\left[\frac{\varepsilon + p}{c_s^2}\left\{\frac{\delta\sigma}{\dot{\sigma}} - \phi\right\} - 3H(\varepsilon + p)\frac{\delta\sigma}{\dot{\sigma}}\right]$$

$$\left(\dot{\phi} + H\phi\right)_{,i} = \frac{1}{2m_{\rm pl}^2}(\varepsilon + p)\left(\frac{\delta\sigma}{\dot{\sigma}}\right)_{,i}$$

One can construct the gauge invariant quantity R from the above equations:

$$v'' - c_s^2\nabla^2 v - \frac{z''}{z}v = 0$$

where $v = zR$. The above equation can be quantized in the exact way as before.

6.1.1 References

[1] J. Garriga, V.F Mukhanov, "Perturbations in k-inflation", arXiv: hep-th/9904176v1 (1999). The calculations are taken from this paper basically.

[2] S. Weinberg, "Effective Field Theory for Inflation", arXiv: 0804.4291v2 [hep-th] (2008).

6.2 String theory inspired models

As higher order effective actions typically comprise of an inflationary theory in the regions of quantum gravity, there are several string theory inspired inflationary models. One of the recently favoured model is axion monodromy inflation, but we will be looking only at an inflationary scenario arising from a Heterotic string action.

6.2.1 Heterotic $E8 \times E8$ effective field action

This is a particular effective lagrangian of general interest in string theory. Without prior motivation, an analysis of it under inflation is carried out here. The Lagrangian density under consideration is a heterotic $E8 \times E8$ string action compactified on T^6, given as[1]:

$$\mathcal{S}[G_{\mu\nu}, S] = \int d^4x\sqrt{-G}S[R + S^{-2}(\nabla S)^2 + \lambda E] \tag{29}$$

where, E is the Gauss-Bonnet(GB) term given by $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2$. By generalized Gauss-Bonnet theorem[3], its integral corresponds to a multiple of Euler characteristic, and hence by itself will not contribute to the equation of motion (EoM). However, it is coupled to a scalar field here: which retains the GB term in the EoM. We will often call the GB term as eulerian density.

To work in the physical frame, we have to decouple the Ricci scalar from the scalar field by a metric redefinition: expressing a part of it in the form of Einstein's gravity. The obvious way is to do a conformal transformation of the metric.

Before we proceed with the redefinition, we split the euler density term in (29), into the conformal and non-conformal part to make the calculations easier[6].

$$\mathcal{S}[G_{\mu\nu}, S] = \int d^4x\sqrt{-G}\left[SR + S^{-1}(\nabla S)^2 + \lambda S\,(C_{\mu\nu\rho\sigma})^2 - 2\lambda S\,(R_{\mu\nu})^2 + \frac{2}{3}\lambda S R^2\right]$$

The relation between the conformal tensor and the euler density used above is easily derived from the definition:

$$C_{\mu\nu\sigma\rho} = R_{\mu\nu\sigma\rho} + (g_{\mu[\sigma}R_{\rho]\nu} - g_{\nu[\sigma}R_{\rho]\mu}) - \frac{1}{3}Rg_{\mu[\sigma}g_{\rho]\nu}$$

Now, we perform the conformal transformation of the metric. We shift to a new manifold N in the same conformal class as the original manifold M, by a given diffeomorphism ψ: $M \rightarrow N$, such that its pullback is given by:

$$(\psi_*\tilde{g})_{\mu\nu} = e^{2\sigma}\tilde{g}_{\mu\nu}$$

with arbitrary freedom on σ. The lagrangian, in the new conformal frame, reduces to:

$$\begin{aligned}\mathcal{S}[G_{\mu\nu}, S] = &\int d^4x\sqrt{-\tilde{G}}e^{4\sigma}\Big[Se^{-2\sigma}\{\tilde{R} - 6(\tilde{\nabla}^2\sigma) - 6(\tilde{\nabla}\sigma)^2\} + S^{-1}e^{-2\sigma}(\tilde{\nabla}S)^2 + \\ &\lambda Se^{-4\sigma}(C_{\mu\nu\rho\sigma})^2 - 2\lambda Se^{-4\sigma}\{(\tilde{R}_{\mu\nu})^2 - 2\tilde{R}(\tilde{\nabla}^2\sigma) + 8(\tilde{\nabla}^2\sigma) + 4\tilde{R}^{\mu\nu}(\tilde{\nabla}_\mu\sigma\tilde{\nabla}_\nu\sigma - \\ &\tilde{\nabla}_\mu\tilde{\nabla}_\nu\sigma) - 4\tilde{R}(\tilde{\nabla}\sigma)^2 + 12(\tilde{\nabla}\sigma)^4 - 8(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\sigma)(\tilde{\nabla}^\mu\sigma)(\tilde{\nabla}^\nu\sigma) + 4(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\sigma)^2 + \\ &20(\tilde{\nabla}^2\sigma)(\tilde{\nabla}\sigma)^2\} + \frac{2}{3}\lambda Se^{-4\sigma}\{\tilde{R} - 6(\tilde{\nabla}\sigma)^2 - 6(\tilde{\nabla}^2\sigma)\}^2\Big]\end{aligned}$$

Note that from this point on, when we say the lagrangian reduces to a quantity, we imply that upto a total derivative term, i.e all total derivative terms that arise are discarded at every step.

The lagrangian can be split into leading order, and the correction term from operators which have coupling constants of negative mass dimensions (in the usual spirit of effective field theories). We eliminate the non-minimal coupling of R at leading order, by setting $\sigma = -\frac{1}{2}\ln(S/M^2)$. The terms in leading order are as follows:

$$\mathcal{L}_1 = \sqrt{-\tilde{G}}\,M^2\{\tilde{R} - 2(\tilde{\nabla}\sigma)^2\}$$

This is a scalar field on an exactly flat potential. Obviously, this cannot lead to inflation; so, the major potential effects arise from the correction terms, and thus mostly be flat.

The correction from first order mass-suppressed operators are, as follows:

$$\mathcal{L}_2 = \lambda M^2 e^{-2\sigma}\sqrt{-\tilde{G}}\Big[(\tilde{C}_{\mu\nu\sigma\rho})^2 - 2(\tilde{R}_{\mu\nu})^2 + \frac{2}{3}\tilde{R}^2 - 4\tilde{R}(\tilde{\nabla}^2\sigma) - 8\tilde{R}^{\mu\nu}(\tilde{\nabla}_\mu\sigma\tilde{\nabla}_\nu\sigma - \tilde{\nabla}_\mu\tilde{\nabla}_\nu\sigma) + 8(\tilde{\nabla}^2\sigma)^2 + 16(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\sigma)(\tilde{\nabla}^\mu\sigma)(\tilde{\nabla}^\nu\sigma) - 16(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\sigma)^2 - 16(\tilde{\nabla}^2\sigma)(\tilde{\nabla}\sigma)^2\Big] \tag{30}$$

From this point, it is understood that the terms we speak of are in the new conformal frame, and we will omit the *tilda* indicating that.

The fifth term in (30) can be simplified by using Bianchi identity, and the seventh and eighth terms are related by the following relation:

$$16\lambda M^2\sqrt{-G}e^{-2\sigma}(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\sigma)(\tilde{\nabla}^\mu\sigma)(\tilde{\nabla}^\nu\sigma) = 16\lambda M^2\sqrt{-G}e^{-2\sigma}\Big[\frac{5}{2}(\nabla_\mu\sigma)^2(\nabla^2\sigma) - (\nabla^2\sigma)^2 - (\nabla_\mu\sigma)^4 + R^{\mu\nu}(\nabla_\mu\sigma)(\nabla_\nu\sigma)\Big] + (b.t. ...)$$

Field equations derived from the first order now reduce the terms into much simpler forms. If we had taken an equation of motion of higher order at this stage, then it would add a $\lambda^n\{n>1\}$ term to the expression. Those terms are of higher order, and are not considered here. The field equations from leading order are:

$$R_{\mu\nu} = 2\nabla_\mu\sigma\nabla_\nu\sigma, (\nabla^2\sigma) = 0$$

The simplified correction term has now reduced to:

$$\mathcal{L}_2 = \lambda M^2\sqrt{-G}e^{-2\sigma}[(C_{\mu\nu\sigma\rho})^2 - \alpha\{(\nabla\sigma)^2\}^2]$$

We will now compute observable values, such as spectral index, non-gaussianity, from the above lagrangian.

The approach now, as usual, is to consider quantized perturbations over FRW metric[2]. The perturbations, in general, can be written as:

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}..., \phi \to \phi' = \phi + \delta\phi^{(1)} + \delta\phi^{(2)}... \tag{31}$$

The first order Hamiltonian gives the background equations. The second order and higher order lead to the perturbations which we will quantize. Considering the background equation, we get the leading order correction to the field equations as:

$$\mathcal{L}_2 = \lambda M^2\sqrt{-G}e^{-2\sigma}[-\alpha\{(\nabla\sigma)^2\}^2]$$

The convenience of the Weyl tensor in cosmology is that it disappears, as the FRW metric is conformally flat[5]. It is now trivial to look at the background field evolution, from the above equation:

$$H^2 + \frac{k}{a^2} = -2(\nabla\sigma)^2 - 3\alpha\lambda e^{-2\sigma}\{(\nabla\sigma)^2\}^2$$

The background condition gives end of inflation when $\dot{\sigma} = 0$.

The quadratic term of the leading lagrangian is well known from the ADM formalism[10]. The correction term can be easily written as:

$$\mathcal{L}_2^{(2)} = \lambda m_{\rm pl}^2\sqrt{-G}e^{-2\sigma}\Big(g^{\alpha\mu}g^{\beta\nu}g^{\gamma\sigma}g^{\delta\rho}C^{(1)}_{\mu\nu\sigma\rho}C^{(1)}_{\alpha\beta\gamma\delta}\Big) - \alpha\lambda m_{\rm pl}^2\big[\sqrt{-G}e^{-2\sigma}\{(\nabla\sigma)^2\}^2\big]^{(1)}$$

It is convenient to decompose our perturbation into scalar and tensor perturbations. Our further analysis will follow the standard calculations of primordial fluctuations.

The CMB anisotropies are conveniently measured in terms of the gauge-invariant quantity. This is the usual way to work with perturbative theories in gravity, as we can use a convenient set of gauges for different apects of cosmological problems[9]. We characterize the scalar perturbations as follows,

$$h_{ij} = \psi\delta_{ij} + \partial_i\partial_j s^p; h_{0i} = \partial_i w^p; h_{00} = -S$$

where the gauge-invariant quantity of interest is R, given by:

$$\mathcal{R} = \frac{\psi}{2} - \frac{\delta\phi^{(1)}}{\dot{\phi}}H$$

For consistency with literature, we have redefined the scalar field as $\phi \equiv \sqrt{2}\sigma$. The above quantity remains constant outside the horizon, for purely adiabatic fluctuations[2]. The derivation of the fluctuation of the above quantity is done through ADM formalism, and is regular textbook derivation. The quadratic lagrangian for the scalar perturbation is,

$$\mathcal{L}^{(2)} = -\frac{m_{\text{pl}}^2\dot{H}}{H^2}a^3\left[\left\{1 + \frac{16}{3}\dot{H}(3\alpha+1)e^{-2\sigma}\lambda\right\}\dot{\mathcal{R}} - \frac{1}{a^2}(\nabla\mathcal{R})^2\right]$$

This rather corresponds to k-inflation, and the scalar perturbation can computed as in the previous section as:

$$n_s - 1 = -3\left(1 + \frac{p}{\varepsilon}\right) - \frac{1}{H}\left(\ln\left(1 + \frac{p}{\varepsilon}\right)\right) - \frac{1}{H}(\ln c_s)$$

Using the result from above, one gets at around $N \sim 55 - 90$:

$$c_s \sim 0.27417 - 0.294345$$

This is consistent with Planck constraints. This leads to a scalar index (which runs) derived from numerical computation as $n_s \sim 0.9212 - 0.9622$.

The non-gaussianity in the form of a non-local form can be computed by direct results from k-inflation as[11]:

$$f_{\text{NL}} = \frac{85}{324}(1 - c_s^{-2})$$

This follows from the above result as $f \sim -3.12746$, which is quite consistent with the Planck data.

The only contribution to the source of tensor modes come from the Weyl tensor. The Weyl term would be equivalent to perturbation over flat space. Note however that this is significantly different from the other inflationary theories till now, as we have a source term for gravitational waves here.

Tensor perturbations can treated by the following metric:

$$h_{ij} = a^2(t)[e^D]_{ij}, D_{ii} = 0, \partial_i D_{ij} = 0$$

At first order, through considerable simplification, one can show that the source terms are:

$$a^3\lambda e^{-2\sigma}\left[\dot{D}_{il}[2H^2 + 2\nabla^2/a^2]\dot{D}_{ik} - 4H\dot{D}_{ik}(\nabla^2/a^2)D_{ik} + 2D_{ik}(\nabla^4/a^4)D_{ik}\right]$$

The above term from Weyl tensor has been worked out in literature, and those results will be used from this point. In context to previous calculation, it reduces the Mukhanov equation with $z = \frac{m_{\mathrm{pl}}}{\sqrt{2}}\left(-\frac{\eta}{\eta_1}\right)^{\alpha}$ where η is the conformal coordinate.

From the above two equations, one can deduce a derivation for the tensor-to scalar-ratio:

$\Delta_t^2(k) \sim k^{3(1+\beta)}$ where $\beta = 2\alpha/3$

As argued for the modified Lyth bound, to account for trans-Planckian fluctuations[12], the tensor-to-scalar ratio is seen to be consistent with BICEP2 results, with $r \sim 0.182$.

This model is seen to be consistent with Planck results.

6.2.2 References

[1] A. Sen, hep-th/0505122v2, “Stretching the Horizon of a Higher Dimensional Small Black Hole” (2005).

[2] For a reference with basic literature on inflation, I used, among many, S. Weinberg, “Cosmology” 2008, and lecture notes by D. Baumann; and Eugene A. Lim.

[3] The primary strategy followed in this article arise mainly from this reference: S. Weinberg, “Effective Field Theory for Inflation”, arXiv: 0804.4291v2 [hep-th] (2008).

[4] Chern, Shiing-Shen (1945), "On the curvatura integra in Riemannian manifold", Annals of Mathematics 46 (4): 674–684. While this contains the original proof of the generalized Gauss-Bonnet theorem, the statement and its summary (which is usually all that is needed) is given in: Yin Li, “The Gauss-Bonnet-Chern Theorem on Riemannian Manifolds”, arXiv 1111.4972v4 (2011).

[5] A conformal transformation of a FRW metric to a Minkowski metric is commonly known, and thus the FRW metric is conformally flat. For an explicit proof, refer to Masao Iihoshi, Sergei V. Ketov, and Atsushi Morishita; “Conformally Flat FRW metrics”, arXiv: hep-th/0702139v3 (2007).

[6] The results of conformal transformation in general relativity, used here, can be easily derived from first principles, but of useful reference was: D.F. Carneiro, E.A. Freiras, B. Gon¸calves, A.G. de Lima, and I.L. Shapiro; “On useful conformal tranformations in General Relativity”.

[7] Planck Collaboration; “Planck 2013 results”, arXiv: 0674450 [astro-ph.CO] (2013).

[8] E. Komatsu, D.N Spergel; “Acoustic Signatures in the Primary Microwave Background Bispectrum” arXiv: astro-ph/0005036v2 (2000). The original account on non-gaussianity in CMB data. In the context of primordial non-gaussianity, the following reference was relevant: X. Chen, “Primordial Non-Gaussianities from Inflation models”, arXiv: 1002.1416.

[9] J.M Bardeen, “Gauge-invariant cosmological perturbations”, Phys. Rev. D: Vol. 22, No. 8 (1980). Arguably the original and still highly relevant article on the topic.

[10] S. Weinberg, “Quantum contributions to cosmological correlations”, hep-th/0506236.

[11] J. Garriga, V.F Mukhanov, “Perturbations in k-inflation”, arXiv: hep-th/9904176v1 (1999).

[12] Gasperini, “String Cosmology”. The tensor perturbations have been primarily been worked out in a manner as in this reference.

7 Conclusion

In the light of the new results of BICEP2 dataset, the scenario of inflation has reached a wider audience. The Planck dataset, with its extremely sensitive instruments, will continue to give tighter constraints on CMB data. This will lead to a even more selective preference to inflationary models.

Many of the existing models, have been at energy scales much smaller than what was anticipated, and this will lead to much higher interests in theories more closely connected to quantum gravity. Higher order corrections in the form of effective field actions from string theory have already been explored. Analyzing the many different mechanisms in string theory, all leading to inflation, and their relation to each other would be one of the areas of interest in the future.

The release of B-mode data of Planck at the end of 2014, and a refined analysis of non-gaussianity will surely be of interest.

8 Appendix

8.1 Appendix A: Scalar field in gravity

We can show that a scalar field can behave as a source with negative pressure-density ratio. We start with the basic lagrangian:

$$\delta S[g_{\mu\nu},\phi]=\int d^4x\,\sqrt{-g}\left[\frac{1}{2}R-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)\right]=0$$

Variation w.r.t metric gives us:

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi+V(\phi)\right)$$

Variation w.r.t the scalar field gives the equation corresponding to the conservation of energy-momentum.

$$\frac{\delta S}{\delta\phi}=\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi)+V_{,\phi}=0$$

Thus, this translates to a stress-energy tensor as follows:

$$\kappa T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi+V(\phi)\right)=A_\mu A_\nu+g_{\mu\nu}B$$

This is similiar to that of a perfect fluid. We redefine ϕ to $\phi\to\sqrt{\kappa}\tilde{\phi}$, $V(\phi)\to\kappa\tilde{V}(\tilde{\phi})$. The second condition is possible because of dimensional analysis. Either the scalar field should be quadratic or there should be other terms which can be redefined to give the above result.

By comparison to perfect fluid, take: $p=-\left(\frac{1}{2}\partial^\sigma\tilde{\phi}\partial_\sigma\tilde{\phi}+\tilde{V}(\tilde{\phi})\right)$. This implies by taking trace of the stress energy tensor:

$$\begin{aligned}-(\rho+p)+4p &= -\partial^\sigma\tilde{\phi}\partial_\sigma\tilde{\phi}-4\tilde{V}(\tilde{\phi})\\ 3p-\rho &= -\partial^\sigma\tilde{\phi}\partial_\sigma\tilde{\phi}-4\tilde{V}(\tilde{\phi})\\ \rho &= -\left(\frac{1}{2}\partial^\sigma\tilde{\phi}\partial_\sigma\tilde{\phi}-\tilde{V}(\tilde{\phi})\right)\\ (\rho+p)^{1/2}u_\mu &= \partial_\mu\tilde{\phi}\\ u_\mu &= \left(-\partial^\sigma\tilde{\phi}\partial_\sigma\tilde{\phi}\right)^{-1/2}\partial_\mu\phi\end{aligned}$$

Now, we see that $w=\frac{p}{\rho}=\frac{\frac{1}{2}\partial^\sigma\tilde{\phi}\partial_\sigma\tilde{\phi}+\tilde{V}(\tilde{\phi})}{\frac{1}{2}\partial^\sigma\tilde{\phi}\partial_\sigma\tilde{\phi}-\tilde{V}(\tilde{\phi})}$. This can be negative, and that explains accelerated expansion.

8.2 Appendix B: Canonical quantization of harmonic oscillator

This is a standard problem in quantum theories, but there is a subtle difference here. This involves a simple harmonic oscillator modes over curved spacetimes, and an issue of non-uniqueness of vaccuum occurs.

The equation we are dealing with is the following:

$$v_k''+\omega^2(\tau)z=0$$

The Lagrangian for this is constructed as follows:

$$L(v_k, v_k', t) = \frac{1}{2}((v_k')^2 - \omega^2(t) v_k^2)$$

Here, $p_k = v_k'$. Hence the poisson brackets are as follows: $\{v_k, p_k\} = 1$. Let's follow the procedures of canonical quantization now:

$$v_k \to \hat{v}_k, p_k \to \hat{p}_k, \{,\} \to -i[,]$$

In our case, $[v_k, p_k] = i$. Now, working with the hamiltonian:

$$\hat{H} = \frac{\omega^2(t)}{2}\left(\frac{\hat{p}_k^2}{\omega^2(t)} + \hat{v}_k^2\right) = \frac{\omega^2(t)}{2}\left(\hat{v}_k - \frac{i\,\hat{p}_k}{\omega(t)}\right)\left(\hat{v}_k + \frac{i\,\hat{p}_k}{\omega(t)}\right) - \frac{i}{2}[\hat{v}_k, \hat{p}_k]$$

We can write this in a concise form following conventions:

$$\hat{H} = \left(\hat{a}_k \hat{a}_k^\dagger + \frac{1}{2}\right)\omega(t)$$

It is easy to derive from the commutation relation we have that:

$$\left[\hat{a}_k, \hat{a}_k^\dagger\right] = 1; \left[\hat{a}_k, \hat{H}\right] = \hat{a}_k\omega; \left[\hat{a}_k, \hat{H}\right] = -\hat{a}_k\omega$$

We need the time evolution of this, which is given by Heisenberg's equation:

$$\hat{a}_k(\tau) = \hat{a}_k e^{i\int \omega(\tau) d\tau}$$

So our final solution would be:

$$\hat{v}_k(\tau) = \mathrm{A}\mathrm{e}^{i\int \omega(\tau) d\tau}\hat{a}_k + A^* e^{-i\int \omega(\tau) d\tau}\hat{a}_k^\dagger$$

8.3 Appendix C: ADM Hamiltonian

An arbitrary globally hyperbolic manifold has an embedding, such that we can decompose it into a set of hypersurfaces of constant t.

On each of these hypersurfaces, we can define a metric h_{ij}:

$$h_{ij} = g_{ij} + n_i n_j$$

The extrinsic curvature is defined as:

$$K_{ij} = h_i^k \nabla_k n_j$$

The intrinsic curvature of the hypersurface with metric h_{ij}, is calculated as the Riemann curvature of the hypersurface as:

$$R_{\mathrm{abc}}^{(3)\ d}\omega_d = (D_a D_b - D_b D_a)\omega_c$$

By a tedious textbook derivation,

$$R_{\mathrm{abc}}^{(3)\ d} = h_a^f h_b^g h_c^k h_j^d R_{fgk}{}^j - K_{\mathrm{ac}} K_b^d + K_{\mathrm{bc}} K_a^d$$

for 3-dim. hypersurface in ADM formalism.

The ADM formalism has the metric decomposed as follows:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Using standard results from ADM formalism (basically the contraction of the intrinsic curvature above), one can show that:

$$R = R^{(3)} + (K_{ij}K^{ij} - K^2), \sqrt{-g} = \sqrt{-h}N$$

So, we can write the lagrangian as:

$$
\begin{aligned}
L &= \sqrt{-g}\Big[\frac{m_p^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big] \\
&= \frac{1}{2}\sqrt{-h}\big[m_p^2 N R^{(3)} + m_p^2 N(K_{ij}K^{ij} - K^2) - 2NV(\phi) + N^{-1}\big(\dot{\phi} - N^i\partial_i\phi\big)^2 - \\
&\quad N h^{ij}\partial_i\phi\partial_j\phi\big]
\end{aligned}
$$

The above looks like it has constraint equations with parameter N, so we get the constraint equation as:

$$m_p^2 R^{(3)} + m_p^2(K_{ij}K^{ij} - K^2) - 2V(\phi) + N^{-2}\big(\dot{\phi} - N^i\partial_i\phi\big)^2 - h^{ij}\partial_i\phi\partial_j\phi = 0$$

and the momentum constraint:

$$\nabla_i[K_j^i - E\delta_j^i] = 0$$

The above two equations give the constraints in ADM formalism for scalar field non-minimally coupled to gravity.