On metric independent perturbation scheme and associated relativistic acoustic geometry for spherical accretion

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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Certificate

This is to certify that this thesis entitled "On metric independent perturbation scheme and associated relativistic acoustic geometry for spherical accretion" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by " B. Ananda Deepika" at "Harish-Chandra Research Institute", under the supervision of "Tapas Kumar Das" during the academic year 2013-2014.

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"To see a world in a grain of sand
and heaven in a wild flower
hold infinity in the palm of your hand
and eternity in an hour" -Blake

Abstract

Stationary solutions of the fluid dynamic equations governing the infall of matter on compact astrophysical objects have widely been studied by the accretion astrophysicists to probe the nature of the emitted spectra through which one can make predictions about the observational evidences of the black holes in our universe. To have a better understanding of the accretion process, one, however, needs to ensure that such stationary states are stable, and very few works, that too on case by case basis, are available in the literature which provide any comprehensive scheme of the stability analysis of the stationary accretion solutions in curved space time. A linear perturbation analysis of the steady state solutions of the Euler and the continuity equations has been developed in this thesis which works for any space time metric, and hence is sufficiently general to incorporate the stability analysis of non-dissipative continuous medium within a metric independent framework. The aim of this thesis is to show that a generalized linear perturbation scheme, independent of the mode of perturbation can be developed for any general relativistic spherically symmetric static space time which not only ensures the stability of the integral stationary accretion solutions but also leads to the emergence of a relativistic acoustic metric representing a curved manifold. The work presented in this thesis is a part of the ongoing project. The main project has been developed to study, along with the linear perturbation scheme in spherically symmetric static space time, the onset and propagation of any generalized non-linear perturbation in any general stationary axisymmetric space time, even for space time endowed with spin and the cosmological constant, and to investigate the corresponding emergent gravity phenomena. The content of this thesis will be submitted as a manuscript, along with two other manuscripts (under preparation) where all the findings of the aforementioned long term project will be reported.

Contents

Part I

Introduction

Chapter 1

Introduction

Almost all the matter in the Universe is in gaseous form, where particles move randomly and undergo collisions. The dynamics might seem quite complicated to study at the first sight. However, the complexity can be avoided by considering the matter as a *fluid*. A fluid is an idealized "continuous" medium determined by certain macroscopic properties such as density, pressure and velocity. Therefore, the study of flow of matter or the fluid flow for various astrophysical systems describes the evolution of such complicated systems. A flow is described by the complete solution(both time and spatial dependence) of velocity and density fields. Since it is difficult to obtain a complete solution analytically, by considering a time slice one can obtain stationary solutions. However it is important to understand the stability of these solutions, whether they remain stable for a timescale longer than observational time periods. We thus adopt a linear perturbation scheme to study the stability of the stationary solutions. As a by-product we obtain a wave equation which gives a acoustic metric defined on a curved manifold. Literatures exist where acoustic metric has been derived for both non-relativistic [1, 2] and relativistic flows [3, 4]. In these works they have obtained the acoustic metric by perturbing velocity potential. Hence we would like to check if the acoustic metric obtained by perturbing the accretion rate, while studying the stability of stationary solutions is identical to the one obtained via velocity potential perturbation scheme. In the following sections, we will briefly review the concepts of accretion, analogue gravity and look at the physically existing analogue gravity models in the universe.

1.1 Astrophysical accretion onto compact objects

One of the vital processes in astrophysical systems is accretion of mass [5, 6]. Accretion is a process through which the surrounding matter is gravitationally captured by the compact objects. Effectively this process is a conversion of gravitational binding energy or rest mass energy into radiation flux and thus important in making the accreting objects potentially very powerful sources of cosmic energy. In comparison to the energy released in the nuclear fusion reactions by conversion of hydrogen into helium, the energy yielded by accretion process is about twenty times higher. Hence this process has become an important tool to study the physics of central compact objects. The power of accretion as an energy generator is illustrated by accretion efficiency η_{acc}. This measures the amount of energy gained from matter with mass m, in units of its rest mass energy, $E = mc^2$ and is considerably high for black holes when compared to any other astrophysical objects. The key mechanism behind the energy generation by X-ray binaries and by the most luminous objects of the Universe like the Quasars and AGN(Active Galactic Nuclei) is believed to be the black hole accretion and hence studying the accretion process has become increasingly important.

Spherical Accretion

To determine the accretion flow onto a compact object(or a black hole in specific) and the pattern of the radiation emitted, one should first describe the flow geometry. The simplest considered flow geometry is spherical flow, i.e.,when the accreting gas has zero angular momentum. If the gas possess intrinsic angular momentum, then the flow would be either two or three dimensional, depending up on the geometry of the disc.

Many astrophysical phenomena could be described by spherical symmetry. An obvious example could be stars, provided we neglect the effects of rotation and magnetic fields. In this case the stellar wind would be a steady spherical outflow. Spherically symmetric assumption often holds good in the cases where presence of an object with small mass can affect a fluid with a large lengthscale and less angular momentum. An example is gas accreting onto a star from interstellar medium. A scenario like accretion of gas by the nucleus of galaxy from the stars rotating around could be well described as spherical symmetric accretion flow because, matter is accreted in almost al directions and hence the net angular momentum is zero. For more details on spherically symmetric flow, one can refer [7]

Study of accretion has its beginnings in the paper presented by Hoyle & Littleton, [8]. In their work, they computed the rate at which a moving star captures gas (pressure-less matter). Later Bondi investigated the spherically symmetric polytropic accretion onto an accretor under Newtonian potential using non-relativistic hydrodynamics [9]. Further progress has been made by Michel, [10]by discussing the general relativistic spherical accretion of a perfect fluid in Schwarzscild metric. Following Michel's work, Begelman presented some aspects of the critical points of the accretion flow [11]. Another important study by Malec showed that in negligence of back reaction, relativistic effects enhance mass accretion [12]. Many interesting problems like entropic-acoustic or other instabilities in spherical accretion [13, 14, 15, 16, 17], stability properties of Bondi solutions [18], high energy cosmic rays production from AGNs [19], high energy emission from relativistic particles in galactic centre [20], study of hadronic model of AGNs [21], [22], explanation of lithium abundances in the late-type, low-mass companions of the soft X-ray transient [23], accretion powered spherical winds ejecting from galactic and extra galactic black hole environments [24] could be addressed by studying the spherically symmetric accretion.

1.2 Analogue models of(and for) gravity

Incompatibility between quantum mechanics and general relativity is encountered not only on the level of everyday life but at the most fundamental level where the building blocks of matter have their existence. Many attempts have been made to reconcile general relativity with quantum mechanics and one such major attempt was made by Stephen W. Hawking in 1974 [25] where he applied quantum mechanics to the horizon of black hole, a vacuum solution of Einstein's field equations in general relativity. A classical black hole can never radiate, but when we introduce quantum effects, the situation changes radically. Hawking used quantum field theoretic calculation on curved spacetime that finite non-zero value exists for physical temperature and entropy of black hole [26]. A linear quantum field, initially in its vacuum state prior to gravitational collapse was considered to propagate against a dynamical background which is a classical spacetime describing the gravitational collapse leading to the formation of Schwarzschild black hole. Near the event horizon of the black hole the vacuum expectation value of the energy momentum tensor of this field is negative. This phenomena leads to the flux of negative energy into the black hole which decreases the mass of the hole. So the quantum state of the outgoing mode of the field would contain particles. The expected number of the outgoing particles corresponds to black body radiation and hence such radiation is thermal in nature with a finite temperature named Hawking temperature given as

$$T_H = \frac{\hbar c^3}{8\pi k_B G M_{BH}} \tag{1.1}$$

where G is the universal gravitational constant, M_{BH} is the mass of the black hole and c, k_B, $\hbar$ are the velocity of light in vacuum, Boltzmann constant, Dirac's constant respectively. For a lucid description of the physical interpretation of Hawking radiation refer to [27] Substituting for the values of the fundamental constants in Eqn.(1.1) we can rewrite T_H as

$$T_H \sim 6.2 \times 10^{-8} \left(\frac{M_\odot}{M_{BH}}\right) \text{DegreeKelvin} \tag{1.2}$$

Note that for one solar mass black hole, the value of Hawking temperature is too small to be experimentally determined. A rough estimate shows that for stellar mass black holes, Hawking temperature would be 10^7 times colder than the cosmic microwave background radiation. The more the mass of the black hole the lesser would be T_H. Hence, only for primordial black holes with small size and mass, T_H would be a measurable quantity, but the time-scale $\mathcal{T}$ over which the mass of the black hole changes significantly due to Hawking radiation is given as [28]

$$\mathcal{T} \sim \left(\frac{M_{BH}}{M_\odot}\right)^3 10^{65}\text{Years} \tag{1.3}$$

By setting $\mathcal{T}$ equal to the present age of the Universe, the lower bound on the mass of the primordial black hole could be obtained. It turns out to be around 10^{15}gm in mass and of the size 10^{-13}cm and the corresponding Hawking temperature is 10^{11}K. This temperature is comparable with the macroscopic fluid temperature of the freely falling matter(spherically symmetric accretion) onto one solar mass isolated Schwarzschild black hole. Present day technology or instrumental techniques are not sufficient to detect the primordial black holes with such small size and mass(if at all they exist). Hence, an observational evidence of Hawking radiation could not be established.

Another difficulty is the infinite redshift of the photon caused by the event horizon. Due to this, the initial configuration of the emergent Hawking quanta are supposed to possess trans-Planckian frequencies and the corresponding wavelengths are less than Planck length. Hence Hawking radiation cannot be dealt with low energy effective theories. Since we do not have a good understanding of physics at such small length scales, some of the fundamental issues like the statistical meaning of black hole entropy or the exact physical origin of the outgoing mode of the quantum field remains unsolved. The difficulties mentioned hereby rule out the possibility of experimental testing of Hawking radiation and this was the main motivation to launch a new theory analogous to Hawking's one which could be experimentally tested. This theory may serve important in the fields of investigation of acoustic super-radiance [29, 30, 31, 32, 33, 34, 35, 36], FRW cosmology [37], quasi-normal modes [38, 39], quantum gravity, inflationary models and sub-Planckian models of string theory [40]. There is now a lot of interests in stimulating black holes by using fluid dynamics/ condensed matter analogues.

In 1981, Unruh pioneered the subject of analogue gravity and since then many analogies have be established between the dynamical features of an inhomogeneous fluid system and the kinematic features of space-time in general relativity. He showed that the acoustic perturbations in a transonic barotropic irrotational fluid satisfies d'Alembert equation of motion for a massless scalar field propagating in curved space-time with a metric referred as acoustic metric which resembles the Schwarzschild metric near the horizon [1]. Propagation of acoustic perturbations through a classical, dissipationless, inhomogeneous, transonic fluid forms an analogue event horizon located at the transonic point. Collection of these transonic points form a sonic surface, which acts like a trapping surface for outgoing phonons. Beyond the transonic point the flow is supersonic and any acoustic perturbation dragged by supersonically moving fluid can never propagate upstream and escape through sonic surface. Thus acoustic horizon is a hyper null surface, generated by acoustic null geodesics, i.e. the phonons. Similar to the event horizon, acoustic horizon also emits radiation with quasi thermal phonon spectra, which is analogous to the actual Hawking radiation and the corresponding temperature is referred as the analogue hawking temperature given as [2]

$$T_{AH} = \frac{\hbar}{4\pi k_B}\left[\frac{1}{c_s^2}\frac{\partial u_\perp^2}{\partial \eta}\right]_{\text{AcousticHorizon}} \tag{1.4}$$

where c_s is the sound speed, $u_\perp$ is the component of the flow velocity normal to acoustic horizon and $\frac{\partial}{\partial \eta}$ is the normal derivative. To summarize, the sonic surface is actually an acoustic horizon analogous to the black hole event horizon in many ways. Thus the concept of acoustic black holes is stemmed from the existence of close analogy between the propagation of sound in a moving fluid and that of light in curved space time.

Physical models constructed using such analogies are called analogue gravity models. There are many devices that reproduce the essential characteristics of

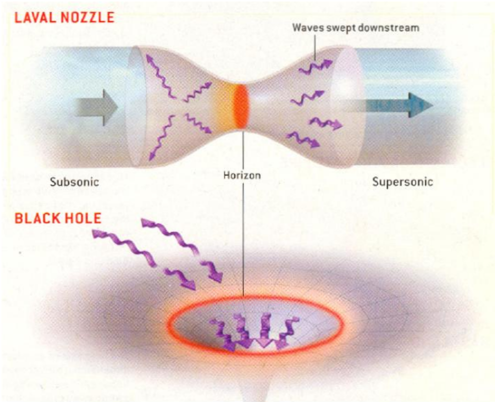

Figure 1.1: *Black hole analogue - Laval nozzle* The incoming fluid is subsonic; the constriction forces it to accelerate to the speed of sound, so that the outgoing fluid is supersonic. Sound waves in the subsonic region can move upstream, whereas waves in the supersonic region cannot.This figure is taken from [42]

a black hole horizon. One such is Laval nozzle. A Laval nozzle is usually found at the end of rockets. One can see from Fig.(1.1) that the incoming fluid is subsonic and the constriction forces the fluid to accelerate. So the subsonic fluid gets accelerated to the sound speed(at the horizon) and keeps accelerating becoming supersonic fluid. Sound waves in the subsonic region can propagate upstream, but in supersonic region they are dragged by supersonically moving fluid. Thus the constriction acts like horizon of a black hole: sound waves can enter the supersonic region through the acoustic horizon but cannot exit. For more insight one can refer [41, 42]

1.2.1 Accreting black hole as classical analogue gravity model

There are analogue models based on classical fluids and quantum fluids, but none of them include gravity. Hence we consider an astrophysical system, accreting fluid onto a compact object. In case of a black hole, the fluid is by default transonic. This is because, far away from the black hole fluid is subsonic and near the horizon due to extreme conditions, fluid has to be supersonic, implying there exists atleast one transonic point. An accreting black hole system [6] as a classical analogue is unique in the sense that only for such a system, both kind of horizons, the gravitational and the acoustic, are simultaneously present in the same system.

As discussed earlier, a general approach to study the acoustic metric is by perturbing the velocity potential, which is an immeasurable physical quantity. Using this approach, properties of acoustic geometries have been investigated for both non- relativistic [1, 2] and for relativistic flows [3, 4]. Another motivation is to attempt this problem from a more astrophysical point of view. Hence we adopted linear perturbation scheme on accretion rate which is a first integral of motion(for non-dissipative flows) and indeed a measurable quantity. There are many astrophysical scenarios where accretion rate could be measured. Recent one is a gas cloud named G2 which is about to collide with the supermassive black hole at our galactic centre. The cloud is as massive as four earths and such an encounter can possibly emit flares. If so, then detection of those flares through telescopes can give an estimation of the variation of accretion rate [43, 44, 45, 46].

In the next chapter, we will briefly discuss how to obtain acoustic metric via velocity potential perturbation for both non-relativistic and relativistic flows by re-deriving the results in [1, 2, 3, 4]. In chapter 4 and 5 we will present our work, i.e. the perturbation analysis on stationary background in both non-relativistic and relativistic regime.

Part II

Acoustic Geometry for Non Gravitating Background Space-time

Chapter 2

Non-relativistic flow in flat space-time

In this chapter we will re-derive the results in [1]. Similar derivation has been presented in Visser's paper [2], but he has further defined the notions of ergo-region, trapped regions, acoustic apparent horizons and acoustic event horizons for supersonic fluid flows. So we start with the fundamental equations of fluid dynamics, [47, 48, 49, 50] which are the equation of Continuity

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \vec{v}) = 0 \tag{2.1}$$

and Euler's equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\,\vec{v} = -\frac{\nabla p}{\rho} - \nabla \Phi \tag{2.2}$$

where ρ is the fluid density, $\vec{v}$ is the velocity of the flow and p is the pressure of the fluid. We assume the flow to be inviscid(zero viscosity), hence the only forces will be due to pressure and possibly any external potential force Φ. We also assume the flow to be irrotational,so

$$\nabla \times \vec{v} = 0 \tag{2.3}$$

so now velocity can be expressed as a gradient of a scalar quantity and this quantity is identified as velocity potential Ψ

$$\vec{v} = -\nabla \Psi \tag{2.4}$$

Along with the governing equations of motion, an equation of state is necessary for studying the fluid flow. We assume the flow to be barotropic (ρ is a function of p only). With this we can define specific enthalpy h as

$$h(p) = \int_0^p \frac{dp^{'}}{\rho(p^{'})} \Rightarrow \nabla h = \frac{1}{\rho} \nabla p \tag{2.5}$$

Using this, we now modify Euler's equation in the form that is easily integrable

$$-\frac{\partial}{\partial t} \nabla \Psi + \frac{1}{2} \nabla (\vec{v}^2) + \nabla h + \nabla \Phi = 0 \tag{2.6}$$

$$\Rightarrow -\partial_t \Psi + h + \frac{1}{2} \left(\nabla \Psi\right)^2 + \Phi = 0 \tag{2.7}$$

Eqn.(2.7) is a version of Bernoulli's equation in the presence of external forces.

Perturbations

It is always interesting to solve the complete equations of motion for the fluid variables(ρ, p, Ψ). But in practice, we separate the exact motion into some average bulk motion(ρ_0, P_0,Ψ_0) plus low amplitude acoustic disturbances($\epsilon\rho^{'}$,$\epsilon p^{'}$ and $\epsilon\Psi^{'}$). Refer to [47, 48, 49, 50] for additional details. So we set

$$\rho = \rho_0 + \epsilon\rho^{'} + \mathcal{O}(\epsilon^2) \tag{2.8}$$

$$p = p_0 + \epsilon p^{'} + \mathcal{O}(\epsilon^2) \tag{2.9}$$

$$\Psi = \Psi_0 + \epsilon\Psi^{'} + \mathcal{O}(\epsilon^2) \tag{2.10}$$

Since the disturbances are low in amplitude, we can ignore the higher order terms and consider only terms linear in perturbed quantities. Substituting for these in continuity equation, Eqn.(2.1) and linearizing gives a pair of equations

$$\partial_t\rho_0 + \nabla.\left(\rho_0 v_0\right) = 0 \tag{2.11}$$

$$\frac{\partial\rho^{'}}{\partial t} + \nabla.\left(\rho_0\vec{v}^{'} + \rho^{'}\vec{v}_0\right) = 0 \tag{2.12}$$

Since specific enthalpy is a function of pressure and density, it is now written as

$$h = h_0 + \frac{\epsilon p^{'}}{\rho_0}\mathcal{O}(\epsilon^2) \tag{2.13}$$

Similarly linearization of Euler equation, Eqn.(2.2) gives

$$-\partial_t\Psi_0 + h_0 + \frac{1}{2}\left(\nabla\Psi_0\right)^2 + \Phi = 0 \tag{2.14}$$

$$-\frac{\partial\Psi^{'}}{\partial t} + \frac{p^{'}}{\rho_0} - \vec{v}_0.\nabla\Psi^{'} = 0 \tag{2.15}$$

The above equation can be rearranged as

$$p^{'} = \rho_0\left(\partial_t\Psi^{'} + \vec{v}_0.\nabla\Psi^{'}\right) \tag{2.16}$$

We now use the barotropic assumption to get the relation

$$\rho^{'} = \frac{\partial\rho}{\partial p}p^{'} = \frac{\partial\rho}{\partial p}\left(\partial_t\Psi^{'} + \vec{v}_0.\nabla\Psi^{'}\right)\rho_0 \tag{2.17}$$

Substituting this in the linearised continuity equation, Eqn.(2.11) gives the wave equation

$$\partial_t\left(\frac{\partial\rho}{\partial p}\rho_0(\partial_t\Psi^{'} + \vec{v}_0\cdot\nabla\Psi^{'})\right) + \nabla\cdot\left(\rho_0\nabla\Psi^{'} - \frac{\partial\rho}{\partial p}\rho_0\vec{v}_0(\partial_t\Psi^{'} + \vec{v}_0\cdot\nabla\Psi^{'})\right) = 0. \tag{2.18}$$

We define local speed of sound

$$c_s^2 \equiv \frac{\partial p}{\partial\rho} \tag{2.19}$$

We introduce a (3+1)-dimensional spacetime coordinates $x^\mu \equiv (t, x, y, z)$. Now we can write the previous equation in a more explicit form as

$$\partial_t \left(\frac{\rho}{c_s^2} \partial_t \Psi^{'} \right) + \partial_t \left(\frac{\rho}{c_s^2} \left\{ v_0^x \partial_x \Psi^{'} + v_0^y \partial_y \Psi^{'} + v_0^z \partial_z \Psi^{'} \right\} \right) + \partial_x \left(\frac{\rho}{c_s^2} v_0^x \partial_t \Psi^{'} \right) + \partial_y \left(\frac{\rho}{c_s^2} v_0^y \partial_t \Psi^{'} \right) +$$
$$\partial_z \left(\frac{\rho}{c_s^2} v_0^z \partial_t \Psi^{'} \right) + \partial_x \left(\frac{\rho}{c_s^2} \left\{ ((v_0^x)^2 - c_s^2) \partial_x \Psi^{'} + v_0^x v_0^y \partial_y \Psi^{'} + v_0^x v_0^z \partial_z \Psi^{'} \right\} \right) +$$
$$\partial_y \left(\frac{\rho}{c_s^2} \left\{ v_0^y v_0^x \partial_x \Psi^{'} + ((v_0^y)^2 - c_s^2) \partial_y \Psi^{'} v_0^y v_0^z \partial_z \Psi^{'} \right\} \right) +$$
$$\partial_z \left(\frac{\rho}{c_s^2} \left\{ v_0^z v_0^x \partial_x \Psi^{'} + v_0^z v_0^y \partial_y \Psi^{'} + ((v_0^z)^2 - c_s^2) \partial_z \Psi^{'} \right\} \right) = 0$$

To get the physical import of the wave equation we construct a matrix and express the wave equation in that matrix. So now we construct a symmetric 4×4 matrix

$$f^{\mu\nu}(t, x, y, z) \equiv \frac{\rho_0}{c_s^2} \begin{bmatrix} 1 & v_0^x & v_0^y & v_0^z \\ v_0^x & ((v_0^x)^2 - c_s^2) & v_0^x v_0^y & v_0^x v_0^z \\ v_0^y & v_0^y v_0^x & ((v_0^y)^2 - c_s^2) & v_0^y v_0^z \\ v_0^z & v_0^z v_0^x & v_0^z v_0^y & ((v_0^z)^2 - c_s^2) \end{bmatrix}$$

$$\equiv \frac{\rho_0}{c_s^2} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v_0^i & \vdots & -(c_s^2 \delta_{ij} - v_0^i v_0^j) \end{bmatrix}$$

such that

$$\partial_\mu \left(f^{\mu\nu} \partial_\nu \Psi^{'} \right) = 0 \tag{2.20}$$

where the greek indices μ, ν run from 0 to 3 and the latin indices i,j from 1 to 3 (i,j = 1, 2, 3 correspond to x, y and z respectively).

In Lorentzian manifold, propagation of photons is well described by the equation of motion for massless scalar field given by d'Alembert in terms of the metric $g_{\mu\nu}$

$$\Delta\psi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \psi \right) \tag{2.21}$$

The wave equation obtained by the linear perturbation analysis could be written in the above form when

$$\sqrt{-g} g^{\mu\nu} = f^{\mu\nu} \tag{2.22}$$

where $f^{\mu\nu}$ corresponds to the elements of wave equation and $g \equiv det(g_{\mu\nu})$. The implication of writing the wave equation in the d'Alembert form is that the propagation of the perturbation in moving fluid is analogous to the propagation of light waves in curved space-time. So we have

$$det\,(f^{\mu\nu}) = (\sqrt{-g}) g^{-1} = g \tag{2.23}$$

From the explicit expression of $f^{\mu\nu}$, we get

$$det(f^{\mu\nu}) = \left(\frac{\rho_0}{c_s^2} \right)^4 \left[(-1) \left(c_s^2 - v_0^2 \right) - (-v_0)^2 \right] [c_s^2][c_s^2] = -\frac{\rho_0^4}{c_s^2} \tag{2.24}$$

Thus we get

$$g = -\frac{\rho_0^4}{c_s^2}; \sqrt{-g} = \frac{\rho_0^2}{c_s} \tag{2.25}$$

Finally by finding $g^{\mu\nu}$ and inverting this matrix gives the acoustic metric as

$$g_{\mu\nu} \equiv \frac{\rho_0}{c_s^2} \begin{bmatrix} -(c_s^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v_0^i & \vdots & \delta_{ij} \end{bmatrix}$$

With this metric one can write the acoustic interval as

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = \frac{\rho_0}{c_s} \left[-c_s dt^2 + (dx^i - v_0^i dt)\delta_{ij}(dx^j - v_0^j dt) \right] \tag{2.26}$$

Assuming that the background flow is spherically symmetric, stationary and convergent flow, we can define a new time coordinate by

$$d\tau = dt + \frac{\vec{v}.d\vec{x}}{c_s^2 - \vec{v}^2} \tag{2.27}$$

Using this, the acoustic line element is written in τ and $\vec{x}$

$$ds^2 = \frac{\rho}{c_s} \left[-(c_s^2 - v^2)d\tau^2 + \left\{ \delta_{ij} + \frac{v^i v^j}{c_s^2 - v^2} \right\} dx^i dx^j \right] \tag{2.28}$$

If we assume that at some r, say $r = R$, the background fluid smoothly exceeds the velocity of the sound, then we can express velocity as

$$v = -c_s + \alpha(r - R) + \mathcal{O}\left((r - R)^2\right) \tag{2.29}$$

Substituting for v in the line element and dropping the angular part gives

$$ds^2 = \rho(R) \left(2\alpha(r - R)d\tau^2 - \frac{dr^2}{2\alpha(r - R)} \right) \tag{2.30}$$

which compare with the Schwarzschild metric

$$ds^2 = (r - 2M)/2M dt^2 - 2M/(r - 2M) dr^2 \tag{2.31}$$

near the horizon of a black hole.

Chapter 3

Background flow in curved space-time

In this section we will show the derivation of acoustic metric by linear perturbation scheme on velocity potential. This has been done for relativistic flows by Billic [3, 4]. In this reference, a general space time metric with Lorentzian signature(+, -, -, -) has been considered and all the components of fluid four-velocity components have been perturbed along with density. So the acoustic metric obtained is a 4×4 matrix, independent of any assumptions of flow geometry, like spherical or axisymmetric. But here we will start with general metric with Lorentzian signature (-, +, +, +). All the flow variables of the fluid carry same denotation as mentioned in appendix A
Equation of continuity for relativistic flows is

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\rho v^\mu\right) = 0 \tag{3.1}$$

The energy-momentum conservation requires

$$\nabla_\nu T^{\mu\nu} = 0 \tag{3.2}$$

This equation when applied for perfect fluids gives

$$(\epsilon + p)v^\nu \nabla_\nu v^\mu + \partial_\mu p + v_\mu v^\nu \partial_\nu p = 0 \tag{3.3}$$

Using the definition of specific enthalpy, $h = \dfrac{\epsilon + p}{\rho}$, we can modify the above equation as

$$v^\nu \nabla_\nu (h v_\mu) - \partial_\mu h = 0 \tag{3.4}$$

Under the assumption of isoentropic flow, we use the Eqn.(A.3) to get the equation of potential flow

$$h v_\mu = -\partial_\mu \phi \tag{3.5}$$

Following the standard procedure, we linearize Eqn.(3.1) and Eqn.(3.5) by introducing

$$\rho \rightarrow \rho + \delta\rho \tag{3.6}$$

$$v^\mu \rightarrow v^\mu + \delta v^\mu \tag{3.7}$$

$$h \rightarrow h + \delta h \tag{3.8}$$

$$\phi \rightarrow \phi + \delta\phi \tag{3.9}$$

For simplicity we will use φ instead of $\delta\phi$. Fluctuations of the metric due to acoustic disturbances will be neglected. From the normalization condition $v^\mu v_\mu = -1$ we get

$$g_{\mu\nu} v^\mu \delta v^\nu = 0 \tag{3.10}$$

Substituting Eqn.(3.9) into Eqn.(3.5) gives

$$\delta h = v^\mu \partial_\mu \left(\delta\varphi\right) \tag{3.11}$$

and

$$h\delta v^\mu = -g^{\mu\nu}\partial_\nu \left(\delta\varphi\right) - v^\mu v^\nu \partial_\nu \left(\delta\varphi\right) \tag{3.12}$$

Similarly linearization of continuity equation gives

$$\partial_\mu \left(\sqrt{-g}\rho\delta v^\mu\right) + \partial_\mu \left(\sqrt{-g}\delta\rho v^\mu\right) = 0 \tag{3.13}$$

From the definition of h and from barotropic relation we get

$$\delta\rho = \frac{v^\nu \partial_\nu \left(\varphi\right)}{c_s^2 h}\rho \tag{3.14}$$

where c_s is the sound speed. Using Eqn.(3.14) and Eqn.(3.12) in Eqn.(3.13) gives

$$\partial_\mu \left\{ \frac{\rho\sqrt{-g}}{h} \left[g^{\mu\nu} + \left(1 - \frac{1}{c_s^2}\right) v^\mu v^\nu \right] \partial_\nu \varphi \right\} = 0 \tag{3.15}$$

So now we can define the symmetric tensor

$$f^{\mu\nu} = \frac{\rho\sqrt{-g}}{h} \left[g^{\mu\nu} + \left(1 - \frac{1}{c_s^2}\right) v^\mu v^\nu \right] \tag{3.16}$$

such that

$$\partial_\mu \left(f^{\mu\nu}\partial_\nu \varphi\right) = 0 \tag{3.17}$$

This wave equation can be put in the form of d'Alembert equation of motion for a massless scalar field propagating in a (3+1) dimensional Lorentzian geometry described by the acoustic metric tensor $G_{\mu\nu}$ such that

$$f^{\mu\nu} = \sqrt{-G}G^{\mu\nu} \tag{3.18}$$

From this we get form of the acoustic metric

$$G_{\mu\nu} = \frac{\rho}{hc_s} \left[g_{\mu\nu} + \left(1 - c_s^2\right) v_\mu v_\nu \right] \tag{3.19}$$

Part III

Spherically Symmetric Black Hole Accretion

Chapter 4

Non-relativistic Bondi Flow

In this chapter we will start with a motivation to study the stability of stationary solutions of Euler and continuity equation describing the Newtonian Bondi flow [9] and later describe how the analogue gravity phenomena emerges as a by-product. We work in non-relativistic regime,i.e. Newtonian limit. Consider inviscid fluid with subsonic velocity at infinity accreting in a spherically symmetric fashion on to a gravitating compact object. We study the accretion from a purely hydrodynamical point of view, without considering any heating or cooling process. Self gravitation of the fluid is neglected. Throughout the work we deal in natural units $G = M = c = 1$

The governing equations

The governing hydrodynamical equations of an inviscid fluid flow are the continuity equation and the Euler equation which are basically the conservation equations of mass and energy respectively. Equation of Continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho v r^2\right) = 0 \tag{4.1}$$

where ρ is fluid density and v is the radial velocity of the flow.
Euler equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\partial \Phi}{\partial r} = 0 \tag{4.2}$$

where p is the pressure and $\Phi(r)$ is an gravitational force potential. One can simply consider Newtonian potential $\Phi = -GM/r$, or any modified Newtonian potential to describe the general relativistic effects [51]. We assume the fluid to be ideal so that for an adiabatic flow we can write the equation of state $p = K\rho^\gamma$, where γ is the polytropic index and K is a constant. For Isothermal case $\gamma = 1$ and it is simply $p = K\rho$. Only in isothermal case we have, $K = \mathcal{R}T/\mu$ where $\mathcal{R}$. For derivation of equation of state refer to appendix A.

4.1 Stationary solutions

As we see from the Euler equation it is difficult to analytically obtain a complete time-dependent solutions. So we consider stationary solutions. In steady state, time dependence vanishes, so the equations of motion now become

$$v\frac{\partial v}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\partial \Phi}{\partial r}=0 \tag{4.3}$$

$$\frac{d}{dr}(\rho v r^2)=0 \tag{4.4}$$

Integrating Eqn.(4.4) over the volume gives

$$4\pi v r^2=\text{constant}=-\dot{M} \tag{4.5}$$

where $\dot{M}$ is recognized as the accretion rate and the negative sign id due to the in fall of the matter.
For integrating the steady state euler equation, Eqn.(4.3), consideration of equation of state is necessary. For adiabatic case it is

$$\frac{v^2}{2}+\frac{c_s^2}{\gamma-1}+\Phi=\text{constant}=\mathcal{E}_a \tag{4.6}$$

and for isothermal case

$$\frac{v^2}{2}+c_s^2\ln\rho+\Phi=\text{constant}=\mathcal{E}_i \tag{4.7}$$

where c_s is the sound speed defined as $c_s^2=\dfrac{\partial p}{\partial \rho}$. So in isothermal case sound speed is constant while it is a function of ρ in adiabatic case. Here the integration constant $\mathcal{E}_a$ can be identified as specific energy (energy per unit mass). $\mathcal{E}_i$, though a constant cannot be specific energy as for the system to be in isothermal state, energy should be dissipated and the integrated equation doesn't account for the dissipation.
For adiabatic case, one can define entropy accretion rate as $\dot{\mathcal{M}}=\dot{M}(\gamma K)^{(1/\gamma-1)}$ [52] such that

$$\dot{\mathcal{M}}=4\pi c_s^{(2/\gamma-1)}v r^2 \tag{4.8}$$

4.2 Critical point analysis

So differentiating Eqn.(4.8) gives the derivative of sound speed

$$\frac{dc_s}{dr}=c_s\left(\frac{1-\gamma}{2}\right)\left[\frac{1}{v}\frac{dv}{dr}+\frac{2}{r}\right] \tag{4.9}$$

Differentiating Eqn.(4.6) and substituting for the derivative of sound speed using Eqn.(4.9) gives

$$\frac{\partial v}{\partial r}=\frac{v(2c_s^2/r-\Phi')}{v^2-c_s^2} \tag{4.10}$$

For isothermal case, since c_s is constant, differentiating Eqn.(4.5) gives

$$\frac{1}{\rho}\frac{d\rho}{dr}=-\left[\frac{2}{r}+\frac{1}{v}\frac{dv}{dr}\right] \tag{4.11}$$

Similarly to obtain velocity derivative we differentiate Eqn.(4.7) and using Eqn.(4.11) gives the same equation as Eqn.(4.10). Though the intermediate equations vary, the final form of the derivative of velocity is same for both adiabatic and isothermal flow. Note from the expression of derivative of velocity, Eqn.(4.10) that the denominator vanishes when $v^2 = c_s^2$. Solution for velocity could be found only by considering a counter integral around it. We would rather like to perform critical point analysis. To obtain critical point, both the numerator and denominator in the right hand side of the Eqn.(4.10) must vanish. This will give the two critical point conditions

$$v_h = c_{s_h} = \sqrt{|\frac{\Phi' r_h}{2}|} \tag{4.12}$$

Subscript h denotes the critical point values. To understand the trajectory of the flow or to plot the phase portraits, we need to know the solution for velocity. Since this is analytically difficult to integrate, one performs integration starting with the critical values of velocity and velocity derivative as initial values and then numerically integrate for four cases, $v > 0$ & $dv/dr > 0$, $v > 0$ & $dv/dr < 0$, $v < 0$ & $dv/dr > 0$, $v < 0$ & $dv/dr < 0$. So we first need to express $v_h,\ c_{s_h},\ \frac{dv}{dr}|_h$ in terms of known values i.e. $r_h,\ \gamma$ so that one can find the numerical values for integrating. In adiabatic case, one can find the location of the horizon by substituting Eqn.(4.12) in Eqn.(4.6) to get the location of the critical point

$$r_h = 4\left(\frac{\gamma - 1}{\gamma + 1}\right)\frac{(\mathcal{E} - \Phi_h)}{\Phi_h'} \tag{4.13}$$

In Isothermal case, one can directly use the Eqn.(4.12) to find the location of horizon in terms of sound speed, which is a constant ans since $c_s = \sqrt{\Theta T}$ we get

$$r_h = \frac{2\Theta T}{\Phi'} \tag{4.14}$$

Now we use the $\mathcal{L}$-Hosipital rule to parametrize the derivative of velocity at critical point and is given as

$$v_h' = \sqrt{\frac{2\Phi_h'}{r_h}}\frac{1}{(1+\gamma)}\left[(1-\gamma) \pm \sqrt{\frac{3-5\gamma}{2} + \frac{\Phi_h'' r_h}{8\Phi_h'}}\right] \tag{4.15}$$

$$v_h' = \sqrt{-\frac{1}{2}\left(\frac{\Phi_h'}{r_h} + \Phi_h''\right)} \tag{4.16}$$

where Eqn.(4.15) corresponds to adiabatic case while Eqn.(4.16) for isothermal case. So with given initial boundary conditions like $\mathcal{E}$ and γ, one can find the location of critical point r_h, then the values of velocity and sound speed at the critical point and finally the velocity derivative value at the critical point. These will serve as the initial values for plotting a phase portrait for further dynamical studies.

4.3 Stability of stationary solutions

We introduce a linear perturbation scheme to study the stability of the stationary solutions obtained in the previous section. We perturb fluid density and the radial velocity around the stationary background flow.

$$\rho(r,t) = \rho_0(r) + \rho^{'}(r,t) \tag{4.17}$$
$$v(r,t) = v_0(r) + v^{'}(r,t) \tag{4.18}$$

We define a new quantity $\Psi = \rho v r^2$ which is similar to accretion rate, differing by a geometrical factor. Substituting for the perturbed quantities gives

$$\Psi_0(r) = \rho_0 v_0 r^2 \tag{4.19}$$
$$\Psi^{'}(r,t) = \rho_0 v^{'} r^2 + \rho^{'} v_0 r^2 \tag{4.20}$$

Similarly substituting for the perturbed quantities in the continuity equation, Eqn.(4.1) and retaining only the linear terms gives

$$\frac{\partial \rho^{'}}{\partial t} + \frac{1}{r^2}\frac{\partial \Psi^{'}}{\partial r} = 0 \tag{4.21}$$

Since sound speed is a function of density, even it gets perturbed as

$$c_s^2(r,t) = c_{s_0}^2(r) + \frac{dc_s^2}{d\rho}|_{t=0}\rho^{'} \tag{4.22}$$

. Now we perturb the Euler equation and up on linearizing it after using the previous equation

$$\frac{\partial v^{'}}{\partial t} + \frac{\partial (v_0 v^{'})}{\partial r} + \frac{\partial}{\partial r}\left(\frac{c_{s_0}^2 \rho^{'}}{\rho_0}\right) = 0 \tag{4.23}$$

Using Eqn.(4.20) consider the time derivative of $\Psi^{'}$ which gives the relation

$$\frac{1}{\Psi}\frac{\partial \Psi^{'}}{\partial t} = \frac{1}{\rho_0}\frac{\partial \rho^{'}}{\partial t} + \frac{1}{v_0}\frac{\partial v^{'}}{\partial t} \tag{4.24}$$

Solving Eqn.(4.21) and Eqn.(4.24) gives

$$\frac{\partial v^{'}}{\partial t} = \frac{v_0}{\Psi}\frac{\partial \Psi^{'}}{\partial t} + \frac{v_0^2}{\Psi_0}\frac{\partial \Psi^{'}}{\partial r} \tag{4.25}$$

Taking time derivative of linearized Euler equation, Eqn.(4.23) and using Eqn.(4.21) and Eqn.(4.25) gives a wave equation

$$\frac{\partial}{\partial t}\left(\frac{v_0}{\Psi_0}\frac{\partial \Psi^{'}}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{v_0^2}{\Psi_0}\frac{\partial \Psi^{'}}{\partial r}\right) + \frac{\partial}{\partial r}\left(\frac{v_0^2}{\Psi_0}\frac{\partial \Psi^{'}}{\partial t}\right) + \frac{\partial}{\partial r}\left(\frac{v_0(v_0^2 - c_{s_0}^2)}{\Psi_0}\frac{\partial \Psi^{'}}{\partial r}\right) = 0 \tag{4.26}$$

Acoustic metric

Let

$$f^{\mu\nu} = \frac{v_0}{\Psi_0}\begin{bmatrix} 1 & v_0 \\ v_0 & v_0^2 - c_{s_0}^2 \end{bmatrix}$$

We see that $f^{\mu\nu}$ satisfies Eqn.(4.26)

$$\partial_\mu\left(f^{\mu\nu}\partial_\nu \Psi^{'}\right) = 0 \tag{4.27}$$

This looks like d'Alembertian equation of motion for massless scalar field

$$\Delta\psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right). \tag{4.28}$$

when $\sqrt{-g}g^{\mu\nu} = f^{\mu\nu}$. Since $\sqrt{-g} = \sqrt{-det(f^{\mu\nu})} = \frac{v_0 c_{s_0}}{\Psi_0}$ one can find $g^{\mu\nu}$

$$g^{\mu\nu} = \frac{1}{c_{s_0}}\begin{bmatrix} 1 & v_0 \\ v_0 & v_0^2 - c_{s_0}^2 \end{bmatrix}$$

Inverse of this metric gives the acoustic metric

$$g_{\mu\nu} = \frac{-1}{c_{s_0}}\begin{bmatrix} v_0^2 - c_{s_0}^2 & -v_0 \\ -v_0 & 1 \end{bmatrix}$$

and finally the line element is given as

$$ds^2 \equiv \frac{1}{c_{s_0}}\left[\left(v_0^2 - c_{s_0}^2\right)dt^2 + dr^2 - 2v_0 dr dt\right] \tag{4.29}$$

The acoustic metric and the corresponding line element are identical to the results obtained in [1, 2] where the perturbation scheme was centered around velocity potential. For proof refer to appendix C.

Solutions of wave equation

Since we restrict our interest to flows which are subsonic at large radii, all the solution except for one solution remain subsonic through out the flow. The one exception is Bondi solution [9]. This solution goes through a sonic point described as a spherical surface where the flow velocity is equal to the sound speed and at the smaller radii the flow becomes supersonic. This solution cannot be smoothly matched on to a stellar surface for obvious reasons that the flow cannot be supersonic at the stellar surface. But one can address this issue by considering discontinuous flows containing a standing shock. Through a shock, which in principle can occur anywhere between the sonic point and the stellar surface, a supersonic flow becomes subsonic and now the flow can be matched on to the hard stellar surface. So we basically have three kinds of solutions: solutions that are subsonic everywhere, smooth Bondi solution, which is an apt solution for black hole accretion and lastly the shocked solution which is subsonic everywhere except between the sonic point and the shock [53]. Now in the rest part we will study the stability of these solutions.
As seen earlier the wave equation obtained is

$$\partial_\mu\left(f^{\mu\nu}\partial_\nu\Psi'\right) = 0. \tag{4.30}$$

Consider a trial acoustic wave solution of the form,

$$\Psi' = g_\omega(r)\exp(-i\omega t) \tag{4.31}$$

where the spatial part of the solution, $g_\omega(r)$ satisfies

$$\left(-\omega^2\right)f^{tt}g_\omega + (-i\omega)\left[f^{tr}\partial_r g_\omega + g_\omega\partial_r f^{rt} + f^{rt}\partial_r g_\omega\right] + \left[g_\omega\partial_r f^{rr} + f^{rr}\partial_r g_\omega\right] = 0. \tag{4.32}$$

This we get by substituting Eqn.(4.31) into Eqn.(4.30), where $f^{tt} = \frac{v_0}{\Psi_0}$,$f^{tr} = f^{rt} = \frac{v_0^2}{\Psi_0}$, $f^{rr} = \frac{v_0}{\Psi_0}\left(v_0^2 - c_{s_0}^2\right)$. Now we shall separately analyse Standing-wave and Travelling-wave solutions.

4.3.1 Standing wave analysis

Standing waves require the perturbation to vanish at the boundaries(two different radii) at all times. The outer boundary could be at very large radius, like radius of the cloud surrounding the compact object from which matter is accreting on to the compact object. The inner boundary can be the surface of the compact object. But if the compact object is a black hole, then the unperturbed flow is described by Bondi solutions and there is no physical mechanism to constrain the flow in supersonic region to have vanishing perturbations. If the compact object is a neutron star, then its surface must be separated from the possible supersonic flow region by a shock and hence it would be a discontinuous flow. But the standing wave analysis rely on the continuity of the solution. So we have to restrict ourselves to completely subsonic flows. So we choose two boundaries at r_1 and r_2 such that the perturbation vanishes at all times, i.e. $g_\omega(r_1) = g_\omega(r_2) = 0$ and we consider the properties of standing wave in the region between these boundaries, i.e. $r_1 \leq r \leq r_2$. Multiply Eqn.(4.33) with g_ω and integrate the above equation over the entire spatial range within which the standing wave is distributed. This gives

$$\omega^2 \int g_\omega^2 f^{tt} dr + i\omega \int \partial_r (g_\omega^2 f^{rt}) dr + \int f^{rr} \left(\partial_r g_\omega\right)^2 dr = 0 \tag{4.33}$$

This looks like $A\omega^2 + Bi\omega + C = 0$, where

$$A = \int g_\omega^2 f^{tt} dr \tag{4.34}$$

$$B = \int [\partial_r (g_\omega^2 f^{rt})] dr \tag{4.35}$$

$$C = \int f^{rr} \left(\partial_r g_\omega\right)^2 dr \tag{4.36}$$

Since $g_\omega(r_1) = g_\omega(r_2) = 0$, after substituting for the explicit forms of f^{tt} and f^{rr} we get

$$\omega^2 = -\frac{\int v_0 \left(v_0^2 - c_{s_0}^2\right) \left(\partial_r g_\omega\right)^2 dr}{\int v_0 g_\omega^2 dr} \tag{4.37}$$

Since we are considering only the solutions which are subsonic everywhere, $v_0^2 < c_{s_0}^2$ always. Hence the value of ω^2 will be positive for any real function $g_\omega(r)$ and both the positive and negative solutions of ω will be real. Therefore, the perturbations of standing wave type are purely oscillatory and can never grow in time implying that the stationary solutions are stable. Remember that by stationary solutions, we mean only those solutions which are entirely subsonic. solutions.

4.3.2 Travelling wave analysis

Here we shall study travelling waves, whose wavelengths are small compared to the radius of the compact object, r_0. Hence for such waves frequency ω is large, so we may try to expand $g_\omega(r)$ in power series. The trial power series solution is

$$g_\omega(r) = \exp\left[\sum_{n=-1}^{\infty} \frac{k_n(r)}{\omega^n}\right] \tag{4.38}$$

Substituting this power solution in Eqn.(4.33) and collecting the coefficients of the same power of ω ($\omega >> 1$) together we get,

$$\text{for } \omega^2 \text{ terms}, -f^{tt} - 2if^{tr}\frac{k_{-1}}{dr} + f^{rr}\left(\frac{k_{-1}}{dr}\right)^2 = 0, \tag{4.39}$$

$$\text{for } \omega^1 \text{ terms}, -2if^{tr}\frac{dk_0}{dr} - i\frac{df^{tr}}{dr} + \frac{df^{rr}}{dr}\frac{dk_{-1}}{dr} + f^{rr}\left[2\frac{dk_{-1}}{dr}\frac{dk_0}{dr} + \frac{d^2k_{-1}}{dr^2}\right] = 0, \tag{4.40}$$

$$\text{for } \omega^0 \text{ terms}, -2if^{tr}\frac{dk_1}{dr} + \frac{df^{rr}}{dr}\frac{dk_0}{dr} + f^{rr}\left[2\frac{dk_{-1}}{dr}\frac{dk_1}{dr} + \left(\frac{dk_0}{dr}\right)^2 + \frac{d^2k_0}{dr^2}\right] = 0. \tag{4.41}$$

From Eqn.(4.39), the leading order coefficients turn out to be

$$k_{-1} = i\int \frac{f^{tr} \pm \sqrt{(f^{tr})^2 - f^{rr}f^{tt}}}{f^{rr}}dr = i\int \frac{dr}{u_0 \mp c_{s0}} \tag{4.42}$$

and putting it into Eqn.(4.40) we get,

$$k_0 = -\frac{1}{2}\ln\left(\sqrt{(f^{tr})^2 - f^{rr}f^{tt}}\right) = -\frac{1}{2}\ln\left(\frac{u_0 c_{s0}}{f_0}\right) \tag{4.43}$$

. For self-consistency, it is necessary to show that the power series given by $g_\omega(r)$ converges rapidly as n increases, i.e.

$$\omega^{-n}|k_n(r)| \gg \omega^{-(n+1)}|k_{n+1}(r)| \tag{4.44}$$

. This requirement can be shown to be very much true, by considering the leading terms in k_n(first three terms). At large length scales, assuming the flow behaves as r^{-2}, and that c_{s_0} approaches its constant ambient value, we get $k_{-1} \sim r$ and $k_0 \sim lnr$ and $k_1 \sim r$. Since $\omega \gg 1$, we see that

$$\omega r \gg lnr \gg \frac{1}{\omega r} \to \omega|k_{-1}(r)| \gg |k_0| \gg \frac{1}{\omega}|k_1(r)| \tag{4.45}$$

Thus the solutions obtained are entirely self-consistent.

Summary

In this chapter we tried to study the stability of the steady state solutions of spherical flow. But the wave equation obtained is similar to the d'Alembert equation of motion for massless scalar fields. So following the standard procedure we obtain the acoustic metric and the corresponding acoustic interval. We also found that the metric is identical to the acoustic metric obtained by velocity potential perturbation scheme. Thereby establishing that the acoustic metric is independent of the perturbation scheme employed for non-relativistic flows. Since the acoustic metric is same in both the cases, the properties of various acoustic geometries and the definitions of notions like ergo-region, acoustic apparent horizon and acoustic event horizon would be same. One can refer to [2] for these details. With these results we were motivated to carry out similar analysis for relativistic flows to check if same results could be established.

Chapter 5

General Relativistic Spherical Accretion

In this chapter we do the similar analysis done in chapter 4, but for fluid flow in curved space-time. General relativistic effects play a crucial role in the regions very close to the central black hole, where most of the gravitational energy is released. Not just black holes, but also to study other astrophysical relevant scenarios like early-bang cosmology, relativistic collisions of elementary particles and ions e.t.c. In such situations, extreme conditions like very high density and temperature are attained due to strong gravity where the velocity of the fluid becomes comparable with the speed of light, c. Even the speed of sound gets close to c. For example, the equation of state for an ideal ultrarelativistic gas yields sound speed as $c_s = c/\sqrt{3}$. Hence in such conditions, one cannot work with Newtonian formalism and relativistic effects have to be considered. We start with a general static space time and later briefly discuss it for Schwarzschild spacetime. Consider a general line element for a static space-time

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \tag{5.1}$$

We assume a perfect fluid which is defined to be a continuous distribution of matter with the energy-momentum tensor of the form

$$T^{\mu\nu} = (\epsilon + p)\, v^\mu v^\nu + pg^{\mu\nu} \tag{5.2}$$

where ρ is the fluid density, v_μ is the fluid-four velocity which obeys the normalization condition, $v^\mu v_\mu = -1$. From this we get

$$v^t = \sqrt{\frac{1 + g_{rr}v^2}{-g_{tt}}} \tag{5.3}$$

As usual we proceed with the conservation equations. Equation of continuity in tensorial notation is given as

$$\nabla_\mu \left(\rho v^\mu\right) = 0 \tag{5.4}$$

which after few manipulations can be written as

$$\frac{1}{\sqrt{-g}}\partial_t \left(\rho v^t \sqrt{-g}\right) + \frac{1}{\sqrt{-g}}\partial_r \left(\rho v \sqrt{-g}\right) = 0 \tag{5.5}$$

where g is the determinant of the background space-time metric, $g_{\mu\nu}$. Our second conservation equation is the conservation of energy-momentum given as

$$\nabla_\mu T^{\mu\nu} = 0 \tag{5.6}$$

For $\mu = t$ we get the energy equation

$$v^t \partial_t v^t + v \partial_r v^t + g^{tt} \left(\partial_r g_{tt}\right) v v^t + \frac{c_s^2}{\rho} \left[\left((v^t)^2 + g^{tt} \right) \partial_t \rho + v v^t \partial_r \rho \right] = 0 \tag{5.7}$$

For $\mu = r$ we get the radial equation(like Euler equation in non-relativistic regime)

$$\sqrt{\frac{1+g_{rr}v^2}{-g_{tt}}}\partial_t v + v\sqrt{\frac{1+g_{rr}v^2}{-g_{tt}}}\frac{c_s^2}{\rho}\partial_t \rho + \partial_r \left(\frac{1+g_{rr}v^2}{2g_{rr}}\right) + \left(\frac{1+g_{rr}v^2}{2g_{rr}}\right)\frac{\partial_r (g_{tt}g_{rr})}{g_{tt}g_{rr}} + \frac{c_s^2}{\rho}\left(\frac{1+g_{rr}v^2}{g_{rr}}\right)\partial_r \rho = 0 \tag{5.8}$$

Refer to appendix D for the derivation of Eqn.(5.7) and Eqn.(5.8)

5.1 Stationary Solutions

We now look at the stationary solutions, where the time derivatives vanish. Starting with the continuity equation, Eqn.(5.5), we get

$$\partial_r \left(\rho v \sqrt{-g}\right) = 0 \tag{5.9}$$

Integrating this gives the stationary solution

$$k \rho v \sqrt{-g} = -\dot{M} = \text{constant} \tag{5.10}$$

where k is a geometrical factor that appears due to integration over the volume. This constant is defined as the accretion rate and the negative sign signifies the infall of the matter. When the time derivatives in the energy equation, Eqn.(5.7) are set to zero, we find

$$h v_t = \mathcal{E} \tag{5.11}$$

where $\mathcal{E}$ is a constant along the geodesic and is identified as the specific energy. h is the specific enthalpy. Similarly making the derivatives vanish in the radial equation, Eqn.(5.8) gives.

$$-\frac{2c_{s0}^2}{\rho_0}\partial_r \rho_0 = \left(\frac{g_{rr}}{1+g_{rr}v_0^2}\right)\partial_r\left(\frac{1+g_{rr}v_0^2}{g_{rr}}\right) + \frac{\partial_r (g_{tt}g_{rr})}{g_{tt}g_{rr}} \tag{5.12}$$

Till now we have worked in the local rest frame, now for convenience we shall work in co-rotating frame of the fluid. In this frame the radial velocity is denoted by u. Necessary transformations are obtained by Lorentz transformations

$$v^t = \frac{1}{\sqrt{-g_{tt}}}\frac{1}{\sqrt{1-u^2}} \tag{5.13}$$

$$v = u\sqrt{\frac{-g_{tt}}{1-u^2}} \tag{5.14}$$

Now the stationary solutions could be expressed in terms of u

$$\dot{M} = k\rho_0 u_0 \sqrt{-g}\sqrt{\frac{-g_{tt}}{1-u_0^2}} \tag{5.15}$$

$$\left(\frac{\gamma-1}{\gamma-1-c_{s0}^2}\right)\sqrt{\frac{-g_{tt}}{1-u_0^2}} = \mathcal{E} \tag{5.16}$$

Take derivatives of both the equations, Eqn.(5.14) and Eqn.(5.16)and solving for $\frac{du}{dr}$ from them gives

$$\frac{du}{dr} = \frac{\frac{2c_{s0}^2}{\sqrt{gg_{tt}}}\partial_r\sqrt{gg_{tt}} - \frac{\left(1-u_0^2\right)\partial_r\left(g_{rr}g_{tt}\right)}{g_{rr}g_{tt}\left(1-u_0^2+g_{rr}g_{tt}u_0^2\right)} + \frac{\partial_r g_{rr}}{g_{rr}}}{\frac{2u_0 g_{tt}g_{rr}}{\left(1-u_0^2+g_{rr}g_{tt}u_0^2\right)} - \frac{2c_{s0}^2}{u_0\left(1-u_0^2\right)}} \tag{5.17}$$

Depending on the metric elements one can find an explicit expression for du/dr and then,the critical point analysis can be carried out similar to what shown in 4.

5.2 Stability of stationary solution

We now perturb the radial velocity, density and accretion rate around the background stationary solutions.

$$v(r,t) = v_0(r) + v^{'}(r,t) \tag{5.18}$$

$$\rho(r,t) = \rho_0(r) + \rho^{'}(r,t) \tag{5.19}$$

We define a new variable $\Psi = \rho v\sqrt{-g}$ which, one can see from Eqn.(5.10) is similar to accretion rate, apart from a geometric constant. From the definition of Ψ we see that

$$\Psi_0(r) = \rho_0 v_0\sqrt{-g} \tag{5.20}$$

$$\Psi^{'}(r,t) = \left(\rho^{'}v_0 + \rho_0 v^{'}\right)\sqrt{-g} \tag{5.21}$$

To express the derivatives of $\rho^{'}$ and $v^{'}$ solely in terms of derivatives of $\Psi^{'}$, we use Eqn.(5.5) and the previous equation. So from the continuity equation we get

$$\partial_t\left[\rho^{'}v_0^t - \frac{g_{rr}\rho_0 v_0 v^{'}}{g_{tt}v_0^t}\right] + \frac{1}{\sqrt{-g}}\partial_r\Psi^{'} = 0 \tag{5.22}$$

where $v_0^t = \sqrt{\frac{1+g_{rr}v_0^2}{-g_{tt}}}$. From Eqn.(5.21) we get

$$v_0\partial_t\rho^{'} + \rho_0\partial_t v^{'} + \frac{1}{\sqrt{-g}}\partial_t\Psi^{'} = 0 \tag{5.23}$$

Solving Eqn.(5.22) and Eqn.(5.23) gives us the following two relations

$$\partial_t v^{'} = \frac{-g_{tt}v_0^t}{\rho_0\sqrt{-g}}\left[v_0^t\partial_t\Psi^{'} + v_0\partial_r\Psi^{'}\right] \tag{5.24}$$

$$\partial_t\rho^{'} = \frac{-1}{\sqrt{-g}}\left[v_0\partial_t\Psi^{'} - g_{tt}v_0^t\partial_r\Psi^{'}\right] \tag{5.25}$$

Substituting for the perturbed quantities in Eqn.(5.8) and using Eqn.(5.12) gives

$$\frac{g_{rr}}{g_{tt}v_0^t}\partial_t v^{'} + \partial_r\left[\frac{g_{rr}v_0v^{'}}{g_{tt}(v_0^t)^2} - \frac{c_{s0}^2\rho^{'}}{\rho}\right] + \frac{g_{rr}v_0c_{s0}^2}{g_{tt}\rho_0v_0^t}\partial_t\rho^{'} = 0 \tag{5.26}$$

Taking the time derivative of the above equation and substituting for the time derivatives of $\rho^{'}$ and $v^{'}$ from Eqn.(5.24) and Eqn.(5.25) we get

$$\partial_t\left(\frac{g_{rr}v_0}{v_0^t}\left[\frac{c_{s_0}^2 + (c_{s_0}^2-1)g_{tt}(v_0^t)^2}{g_{tt}}\right]\partial_t\Psi^{'}\right) + \partial_t\left(\frac{g_{rr}v_0}{v_0^t}\left[v_0v_0^t(c_{s_0}^2-1)\right]\partial_r\Psi^{'}\right) +$$
$$\partial_r\left(\frac{g_{rr}v_0}{v_0^t}\left[v_0v_0^t(c_{s_0}^2-1)\right]\partial_t\Psi^{'}\right) + \partial_r\left(\frac{-v_0}{v_0^t}\left[c_{s_0}^2 + (c_{s_0}^2-1)g_{rr}v_0^2\right]\partial_r\Psi^{'}\right) = 0 \tag{5.27}$$

So now we can define the symmetric tensor

$$f^{\mu\nu} \equiv \frac{g_{rr}v_0}{v_0^t}\begin{bmatrix} \dfrac{c_{s_0}^2 + (c_{s_0}^2-1)g_{tt}(v_0^t)^2}{g_{tt}} & v_0v_0^t(c_{s_0}^2-1) \\ v_0v_0^t(c_{s_0}^2-1) & \dfrac{c_{s_0}^2 + (c_{s_0}^2-1)g_{rr}v_0^2}{g_{rr}} \end{bmatrix}$$

such that

$$\partial_\mu\left(f^{\mu\nu}\partial_\nu\Psi^{'}\right) = 0 \tag{5.28}$$

Acoustic metric

The wave equation looks alike to the equation of motion for massless scalar field given by d'Alembert

$$\Delta\psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right) \tag{5.29}$$

The wave equation obtained by the linear perturbation analysis could be written in the above form when

$$\sqrt{-G}G^{\mu\nu} = f^{\mu\nu} \tag{5.30}$$

where $f^{\mu\nu}$ corresponds to the elements of wave equation and $G^{\mu\nu}$ is the acoustic metric. Once $f^{\mu\nu}$ is known, we can find $\sqrt{-G}$

$$G = det(f^{\mu\nu}) = \frac{g_{rr}c_{s_0}^2v_0^2}{g_{tt}(v_0^t)^2} \Rightarrow \sqrt{-G} = \frac{c_{s_0}v_0}{v_0^t}\sqrt{\frac{-g_{rr}}{g_{tt}}} \tag{5.31}$$

Finally the acoustic metric is given as

$$G_{\mu\nu} \equiv \frac{\sqrt{-g_{tt}g_{rr}}}{c_{s_0}}\begin{bmatrix} \dfrac{c_{s_0}^2 + (c_{s_0}^2-1)g_{rr}v_0^2}{g_{rr}} & v_0v_0^t(1-c_{s_0}^2) \\ v_0v_0^t(1-c_{s_0}^2) & \dfrac{c_{s_0}^2 + (c_{s_0}^2-1)g_{tt}(v_0^t)^2}{g_{tt}} \end{bmatrix}$$

This is same as the acoustic metric obtained through the perturbation of velocity potential approach made in [4]. For proof refer appendix C. This assures that even for relativistic flows, the acoustic metric is independent of the mode of perturbation.

Solution of wave equation

To study the stability of the stationary solutions, we use the trial acoustic wave solution.

$$\Psi^{'} = p(r) \exp(-i\omega t) \tag{5.32}$$

Substituting for the above in Eqn.(5.27), the spatial part p(r) satisfies

$$\omega^2 p(r) f^{tt} + i\omega \left[\partial_r \left(p(r) f^{rt}\right) + f^{tr} \partial_r p(r)\right] - \left[\partial_r \left(f^{rr} \partial_r p(r)\right)\right] = 0 \tag{5.33}$$

5.2.1 Standing wave analysis

Here we perform the standing wave analysis to study the stability of the stationary solutions. As mentioned in section 4.3.1, we restrict ourselves to the solutions which are subsonic everywhere. Multiply with p(r) and integrate the above equation over the entire spatial range within which the standing wave is distributed

$$\omega^2 \int p(r)^2 f^{tt} dr + i\omega \int [\partial_r (p^2 f^{rt})] dr - \int p(r) [\partial_r (f^{rr} \partial_r p(r))] dr = 0 \tag{5.34}$$

This looks like $A\omega^2 + Bi\omega + C = 0$, where

$$A = \int p(r)^2 f^{tt} dr \tag{5.35}$$

$$B = \int [\partial_r (p^2 f^{rt})] dr \tag{5.36}$$

$$C = - \int p(r) [\partial_r (f^{rr} \partial_r p(r))] dr \tag{5.37}$$

Since the amplitude of the wave vanishes at the boundaries, $p(r_1) = p(r_2) = 0$. So we get

$$\omega^2 = - \frac{\int f^{rr} \left(\partial_r p(r)\right)^2}{\int f^{tt} p(r)^2} \tag{5.38}$$

5.2.2 Travelling wave analysis

We use the trial power solution

$$p(r) = \exp\left[\sum_{k=1}^{n} \frac{k_n(r)}{\omega^n}\right] \tag{5.39}$$

Substitute this in Eqn.(B.4) and setting the coefficients of individual powers of ω gives the leading coefficients in the power series. Equating the coefficient of ω^2 term to zero gives

$$k_{-1} = i \int \frac{f^{tr} \pm \sqrt{(f^{tr})^2 - f^{tt} f^{rr}}}{f^{rr}} dr \tag{5.40}$$

Similarly equating the coefficient of ω term to zero gives

$$k_0 = \frac{-1}{2}\ln(\sqrt{(f^{tr})^2 - f^{tt}f^{rr}}) \tag{5.41}$$

In this section 5.1 and 5.2 we provided a general formalism to study the stability of the stationary solutions and to obtain acoustic metric. In the next section, we will work in Schwarzschild geometry. This geometry basically describes the gravitational field around a spherical mass, on the assumption that the electric charge of the mass, angular momentum of the mass, and universal cosmological constant are all zero. Slowly rotating astronomical objects such as many stars, planets and black holes, including Earth and the Sun can be described by Schwarzschild solutions.

5.3 Schwarzschild space-time

The metric of Schwarzschild space time looks like in Eqn.(5.1), with

$$g_{tt} = -g_{rr}^{-1} \equiv f(r) = 1 - \frac{2}{r}, g_{\theta\theta} = \frac{g_{\phi\phi}}{\sin\theta} = r^2 \tag{5.42}$$

From the normalization condition we get

$$v^t = \frac{\sqrt{v^2 + f}}{f} \tag{5.43}$$

Stationary Solutions

Pair of stationary solutions obtained are accretion rate

$$4\pi\rho v r^2 = \dot{M} = \text{constant} \tag{5.44}$$

and the specific energy conserved along the geodesic

$$hv_t = \mathcal{E} \tag{5.45}$$

Following the similar procedure, one can obtain du/dr or just substitute for metric elements in Eqn.(5.17) to get

$$\frac{du}{dr} = \frac{u(1-u^2)\left(3c_{s_0}^2 r - 2c_{s_0}^2 - 1\right)}{(u^2 - c_{s_0}^2)(r-2)} \tag{5.46}$$

The denominator of the above expression vanishes when $u^2 = c_{s_0}^2$. So we carry out the critical point analysis by equating both the numerator and denominator to zero simultaneously. This gives the information about fluid velocity at the critical point. Since the flow velocity equals the sound speed, this critical point can be identified as the sonic point. So for $r < r_h$ we have subsonic point and for $r > r_h$ we have supersonic fluid, with the transition occurring at r_h.

$$u_h = a_h = \sqrt{\frac{1}{3r_h - 2}} \tag{5.47}$$

These results were earlier derived in [54].

Linear perturbation analysis

We define a new variable $\Psi = \rho v r^2$ which, one can see from Eqn.(5.44) is similar to the accretion rate. Now we perturb the radial velocity, density linearly around the stationary solutions and get the wave equation as

$$\partial_t \left(\frac{v_0}{v_0^t f} \left[\frac{v_0^{t^2} f^2 - c_{s_0}^2 v_0^2}{f^2} \right] \partial_t \Psi^{'} \right) + \partial_t \left(\frac{v_0}{v_0^t f} \left[v_0 v_0^t (1 - c_{s_0}^2) \right] \partial_r \Psi^{'} \right) +$$
$$\partial_r \left(\frac{v_0}{v_0^t f} \left[v_0 v_0^t (1 - c_{s_0}^2) \right] \partial_t \Psi^{'} \right) + \partial_r \left(\frac{v_0}{v_0^t f} \left[v_0^2 - c_{s_0}^2 (v_0^2 + f) \right] \partial_r \Psi^{'} \right) = 0 \quad (5.48)$$

where We can identify the symmetric matrix

$$f^{\mu\nu} \equiv \frac{v_0}{v_0^t f} \begin{bmatrix} \frac{v_0^{t^2} f^2 - c_{s_0}^2 v_0^2}{f^2} & v_0 v_0^t (1 - c_{s_0}^2) \\ v_0 v_0^t (1 - c_{s_0}^2) & v_0^2 - c_{s_0}^2 (v_0^2 + f) \end{bmatrix}$$

such that $\partial_\mu \left(f^{\mu\nu} \partial_\nu \Psi^{'} \right) = 0$. As the wave equation looks like the d'Alembert equation, we can finally obtain the acoustic metric

$$G_{\mu\nu} \equiv \frac{-1}{c_{s_0}} \begin{bmatrix} -v_0^2 + c_{s_0}^2 (v_0^2 + f) & v_0 v_0^t (1 - c_{s_0}^2) \\ v_0 v_0^t (1 - c_{s_0}^2) & \frac{-v_0^{t^2} f^2 + c_{s_0}^2 v_0^2}{f^2} \end{bmatrix}$$

Considering the non-relativistic limit, $f \approx 1, v_0^2 \ll 1$ and $c_{s_0}^2 \ll 1$. Now the acoustic metric simplifies to

$$G_{\mu\nu} \equiv \frac{-1}{c_{s_0}} \begin{bmatrix} -v_0^2 + c_{s_0}^2 & v_0 \\ v_0 & -1 \end{bmatrix}$$

This 2x2 matrix is same as the acoustic metric obtained in non-relativistic regime [2].

5.3.1 Stability analysis

To study the stability of the stationary solutions, we use the trial acoustic wave solution.

$$\Psi^{'} = p(r) exp(-i\omega t) \quad (5.49)$$

Substituting for the above in Eqn.(5.48) gives

$$\omega^2 p(r) h^{tt} + i\omega [\partial_r (p(r) h^{rt}) + h^{tr} \partial_r p(r)] - [\partial_r (h^{rr} \partial_r p(r))] = 0 \quad (5.50)$$

Standing wave analysis

After following the same procedure demonstrated for general metric in section 5.2.1 we get

$$\omega^2 = - \frac{\int \frac{v_0}{v_0^t f} \left(v_0^2 \left(1 - c_{s_0}^2 \right) - c_{s_0}^2 f \right) \left(\partial_r p(r) \right)^2 dr}{\int \frac{v_0}{v_0^t f} \left[\frac{v_0^2 \left(1 - c_{s_0}^2 \right) + f}{f^2} \right] p(r)^2 dr} \quad (5.51)$$

Since c_s^2 is always less than 1, denominator is always positive and the overall sign of ω^2 depends on $\left(v_0^2\left(1-c_{s_0}^2\right)-c_{s_0}^2 f\right)$. Recollect the co-moving coordinates in which we worked out the stationary solutions. In this frame, for the subsonic flows $u < c_s$. After the necessary transformation this condition in our local rest frame reads as $v_0^2 < \dfrac{c_{s_0}^2 f}{1-c_{s_0}^2}$. So for subsonic flows, ω^2 is positive and two real solutions exist. Since we restrict ourselves to subsonic flows, there is no problem. So the amplitude of the perturbation never grow in time and the stationary solutions are stable.

Travelling wave analysis

We use the trial power solution

$$p(r) = \exp\left[\sum_{k=1}^{n} \frac{k_n(r)}{\omega^n}\right] \tag{5.52}$$

Following the similar analysis gives

$$k_{-1} = i\int \frac{h^{tr} \pm \sqrt{(h^{tr})^2 - h^{tt}h^{rr}}}{h^{rr}}dr = i\int \frac{v_0\sqrt{v_0^2+f}[(1-c_{s_0}^2) \pm c_{s_0}f]}{v_0^2 - c_{s_0}^2(v_0^2+f)}dr \tag{5.53}$$

Similarly equating the coefficient of ω term to zero gives

$$k_0 = \frac{-1}{2}ln(\sqrt{(h^{tr})^2 - h^{tt}h^{rr}}) = \frac{-1}{2}\ln(\frac{c_{s_0}v_0}{\sqrt{v_0^2+f}}) \tag{5.54}$$

In the asymptotic limit, we get $k_{-1} \sim r, k_0 \sim \ln r$ and $k_1 \sim r^{-1}$. Since $\omega \gg 1$ we can see that $\omega r \gg \ln r \gg \dfrac{\omega}{r}$, the power series converges. Hence the stationary solutions obtained are self-consistent.

Summary

In this chapter, we have dealt with similar formalism presented in chapter 4, but for relativistic flows. The stationary solutions obtained are stable in linear perturbation, and the acoustic metric derived from the wave equation is identical to the result in [4]. Hence we can conclude that though the acoustic metric embeds the properties of the background space, it is independent of the mode of perturbation scheme adopted.

Part IV

Conclusions

Chapter 6

Conclusions and Outlook

In this thesis, we have studied the stability of the stationary solutions of accretion flow on to a gravitating mass. A complete solution of velocity and density fields describes the fluid flow. Since it is analytically difficult to solve the full set of space-time dependent conservation equations, one has to consider stationary solutions. Both the density and velocity fields are connected to the matter accretion rate, first integral obtained from the stationary solution of continuity equation. Therefore, perturbing the accretion rate about a constant stationary value will lead to a wave equation which conveys sufficient information on stability of both the fields. The wave equation obtained is similar to the d'Alembert equation of motion for massless scalar fields. Implies that, the acoustic disturbances(sound) in a moving fluid propagate in a analogous to the way light propagates in curved space-time. Following the standard procedure, we found the acoustic metric. The acoustic metric depends on the fluid's equation of state and involves two tensors: background space-time metric tensor and product of fluid four-velocity. In the initial chapters, 2 and 3, we briefly discussed the velocity potential perturbation scheme, underlying assumptions and re-derived the results for non-relativistic flows [2, 1] and relativistic flows [4]. In the next chapters, 3 and 4, we discussed the stability of the stationary solutions, and then provided a new formalism for obtaining the acoustic metric in which we perturbed an astrophysically measurable quantity, accretion rate. This acoustic metric matches with the results in [2, 4], thereby confirming the physical basis for entire analogue gravity subject. As far as the stability is concerned, the stationary solutions obtained are stable and this is in perfect qualitative conformity with the earlier studies dwelt on the question of stability of the flow solutions.

Motivated by these results in spherical flow, we extended our work to most general axisymmetric space-time endowed with black hole spin as well as cosmological constant and then later to Kerr space-time. For axisymmetric flow, we provided a general form of the acoustic metric, for a general static space time. All our calculations were restricted to the equatorial plane. We found the acoustic metric to be

$$G_{\mu\nu} \equiv \frac{-1}{a_0\sqrt{x}} \begin{bmatrix} -v_0^2 + a_0^2(v_0^2 + f) & v_0 v_0^t(1 - a_0^2) \\ v_0 v_0^t(1 - a_0^2) & \dfrac{-v_0^{t^2} f^2 + a_0^2(v_0^2 + v_0^{\phi^2} r^2 f)}{f^2} \end{bmatrix}$$

where $x = (f + v_0^{\phi^2} r^2 f(1 - a_0^2))$

Note that the acoustic metric is similar to the one for spherical flow, just differing by a conformal factor. Similarly, even for the Kerr-space time, we found the acoustic metric. Since the main line of thesis is Spherical flow, the above mentioned works are not presented in this thesis. Hopefully we will communicate papers based on these works soon.

In future we would like to study non-linear perturbation analysis by including higher order non-linear terms. As sound waves describe the linear fluctuations in background flow, non-linear fluctuations are addressed by blast waves. Since blast waves are encountered in Supernovae explosion, such analysis would provide better understanding about acoustic geometry for Supernovae explosion.

Appendix A

Basic relativstic fluid dynamics

In this section we will discuss the basic assumptions made in chapter 3 and the corresponding relations obtained based on these assumptions, which have been made use while deriving the acoustic metric. Firstly, we assumed the fluid to be ideal. So we consider the energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (p + \epsilon)\, v_\mu v_\nu - p g_{\mu\nu} \tag{A.1}$$

where $g_{\mu\nu}$ is the metric tensor, p, ϵ, v^μ are the pressure, energy density and four-velocity of the fluid. Throughout the thesis Lorentzian signature of the metric considered is (-,+,+,+). In this convention we have,

$$v_\mu v^\mu = g_{\mu\nu} v^\mu v^\nu = -1 \tag{A.2}$$

Secondly, we restrict ourselves to isoentropic flows. A flow is said to be isoentropic when the specific entropy s/ρ is constant, i.e. when

$$\partial_\mu \left(\frac{s}{\rho}\right) = 0 \tag{A.3}$$

where ρ is the fluid density. Thirdly, we assume the flow to be irrotational, i.e. having a vanishing vorticity. For a fluid flow, vorticity $w_{\mu\nu}$ is defined as

$$w_{\mu\nu} = P^\eta_\mu P^\sigma_\nu \nabla_\nu v_\eta \tag{A.4}$$

where

$$P^\mu_\nu = \delta^\mu_\nu - v^\mu v_\nu \tag{A.5}$$

is the projection operator which projects an arbitrary vector in space-time into its component in the subspace orthogonal to v^μ. So for vanishing vorticity, we have

$$w_{\mu\nu} = 0 \tag{A.6}$$

Specific enthalpy h, and specific entropy s/ρ are related by the thermodynamic identity

$$dh = T d\left(\frac{s}{\rho}\right) + \frac{1}{\rho} dp \tag{A.7}$$

where $h = \dfrac{\epsilon + p}{\rho}$.

Appendix B

(1+1)-dimensional acoustic metric in non-relativistic flow.

Here we will show a short derivation of (1+1)-dimensional acoustic metric obtained via linear perturbation of velocity potential. So we start with the (3+1)-dimensional result obtained in chapter 2 and reduce it to (1+1) dimensional metric for the sake of comparison with our results. For a spherical flow, we work in spherical coordinates. So Eqn(2.18) can now be written in terms of t and r coordinates as

$$\partial_t \left(\frac{\rho_0 r^2}{c_{s_0}^2} \left(\partial_t \Psi^{'} + v_0 \partial_r \Psi^{'} \right) \right) + \partial_r \left(\frac{\rho_0 v_0 r^2}{c_{s_0}^2} \partial_t \Psi^{'} + \frac{\rho_0 r^2}{c_{s_0}^2} \left(v_0^2 - c_{s_0}^2 \right) \partial_r \Psi^{'} \right) = 0 \tag{B.1}$$

We can construct a matrix

$$f^{\mu\nu} = -\frac{\rho_0 r 2}{c_{s_0}^2} \begin{bmatrix} 1 & v_0 \\ v_0 & v_0^2 - c_{s_0}^2 \end{bmatrix}$$

such that

$$\partial_\mu \left(f_{\mu\nu} \partial_\nu \Psi^{'} \right) = 0 \tag{B.2}$$

So to find the acoustic metric, we follow the similar procedure starting with the determinant of this matrix

$$det(f^{\mu\nu}) = \frac{\rho_0^2 r^4}{c_{s_0}^4} \left[-c_{s_0}^2 + v_0^2 - v_0^2 = -\frac{\rho_0^2 r^4}{c_{s_0}^2} \right] \tag{B.3}$$

Thus we get

$$g = -\frac{\rho_0^2 r^4}{c_{s_0}^2} ; \sqrt{-g} = \frac{\rho_0 r^2}{c_{s_0}} \tag{B.4}$$

Now we find

$$g^{\mu\nu} = -\frac{1}{c_{s_0}} \begin{bmatrix} -1 & -v_0 \\ -v_0 & -v_0^2 + c_{s_0}^2 \end{bmatrix}$$

and finally the acoustic metric is obtained by inverting the previous matrix

$$g_{\mu\nu} = -\frac{1}{c_{s_0}} \begin{bmatrix} v_0^2 - c_{s_0}^2 & -v_0 \\ -v_0 & 1 \end{bmatrix}$$

With this metric one can write the acoustic interval as

$$ds^2 \equiv \frac{1}{c_{s_0}} \left[\left(v_0^2 - c_{s_0}^2 \right) dt^2 + dr^2 - 2v_0 dr dt \right] \tag{B.5}$$

Appendix C

(1+1)-dimensional acoustic metric in relativistic flow.

In this section, we will present a short derivation of (1+1)-dimensional acoustic metric obtained through the perturbation of velocity potential for relativistic flows. The symmetric tensor for a general metric is derived in chapter 3

$$f^{\mu\nu} = \frac{\rho\sqrt{-g}}{h}\left[g^{\mu\nu} + \left(1 - \frac{1}{c_s^2}\right)v^\mu v^\nu\right] \tag{C.1}$$

In spherical co-ordinates we find it to be

$$f^{\mu\nu} \equiv \frac{\rho\sqrt{-g}}{hc_s^2}\begin{bmatrix} \dfrac{c_s^2 + (c_s^2-1)g_{tt}(v^t)^2}{g_{tt}} & vv^t(c_s^2-1) \\ vv^t(c_s^2-1) & \dfrac{c_s^2+(c_s^2-1)g_{rr}v^2}{g_{rr}} \end{bmatrix}$$

To find the acoustic metric, we follow the standard procedure starting with the determinant of this matrix $f^{\mu\nu}$

$$G = det(f^{\mu\nu}) = \frac{\rho^2(-g)}{g_{tt}g_{rr}c_s^2h^2} \to \sqrt{-G} = \frac{\rho}{c_s h}\sqrt{\frac{-g}{g_{tt}g_{rr}}} \tag{C.2}$$

Now we find

$$G^{\mu\nu} \equiv \frac{\sqrt{g_{tt}g_{rr}}}{c_s}\begin{bmatrix} \dfrac{c_s^2 + (c_s^2-1)g_{tt}(v^t)^2}{g_{tt}} & vv^t(c_s^2-1) \\ vv^t(c_s^2-1) & \dfrac{c_s^2+(c_s^2-1)g_{rr}v^2}{g_{rr}} \end{bmatrix}$$

and finally the acoustic metric is obtained by inverting the previous matrix

$$G_{\mu\nu} \equiv \frac{\sqrt{-g_{tt}g_{rr}}}{c_{s_0}}\begin{bmatrix} \dfrac{c_{s_0}^2 + (c_{s_0}^2-1)g_{rr}v_0^2}{g_{rr}} & v_0v_0^t(1-c_{s_0}^2) \\ v_0v_0^t(1-c_{s_0}^2) & \dfrac{c_{s_0}^2+(c_{s_0}^2-1)g_{tt}(v_0^t)^2}{g_{tt}} \end{bmatrix}$$

and the corresponding acoustic interval is

$$ds^2 \equiv G_{tt}dt^2 + 2G_{tr}dt.dr + G_{rr}dr^2 \tag{C.3}$$

For the Schwarzschild space time, $g_{tt} = -g_{rr}^{-1} \equiv f(r) = 1 - \frac{2}{r}, g_{\theta\theta} = \frac{g_{\phi\phi}}{\sin\theta} = r^2$. Substituting for these in the previous matrix, gives the acoustic metric

$$G_{\mu\nu} \equiv \frac{-1}{c_s} \begin{bmatrix} -v^2 + c_s^2(v^2 + f) & vv^t(1 - c_s^2) \\ \\ vv^t(1 - c_s^2) & \dfrac{-(v^t)^2 f^2 + c_s^2 v^2}{f^2} \end{bmatrix}$$

Appendix D

Derivation of Radial parts of the Euler equation and the Continuity equation

Continuity equation is

$$(\rho v^{\mu})_{;\mu} = 0 \tag{D.1}$$

$$\Rightarrow \frac{1}{\sqrt{-g}}\partial_t\left(\rho v^t\sqrt{-g}\right) + \frac{1}{\sqrt{-g}}\partial_r\left(\rho v\sqrt{-g}\right) = 0 \tag{D.2}$$

Energy-momentum conservation

$$T^{\mu\nu}_{;\nu} = 0 \tag{D.3}$$

where

$$T^{\mu\nu} = (\epsilon + p)\, v^{\mu}v^{\nu} + pg^{\mu\nu} \tag{D.4}$$

Since

$$\nabla_{\nu}v^{\mu} = \partial_{\nu}v^{\mu} + \Gamma^{\mu}_{\nu\lambda}v^{\lambda} \tag{D.5}$$

Now it looks like

$$(\epsilon + p)\left[\frac{v^{\mu}}{\sqrt{-g}}\partial_{\nu}(\sqrt{-g}v^{\nu}) + v^{\nu}\partial_{\nu}v^{\mu} + \Gamma^{\mu}_{\nu\lambda}v^{\nu}v^{\lambda}\right] + v^{\mu}v^{\nu}\partial_{\nu}\left(\epsilon + p\right) + g^{\mu\nu}\partial_{\nu}P = 0 \tag{D.6}$$

Using the continuity equation, Eqn.(D.2), this can be recast as

$$v^{\nu}\partial_{\nu}v^{\mu} + \Gamma^{\mu}_{\nu\lambda}v^{\nu}v^{\lambda} + \frac{a^2}{\rho}\left(v^{\mu}v^{\nu} + g^{\mu\nu}\right)\partial_{\nu}\rho = 0 \tag{D.7}$$

Christoffel symbols can be evaluated using the metric through

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}g^{\mu\eta}\left(\partial_{\nu}g_{\eta\lambda} + \partial_{\lambda}g_{\eta\nu} - \partial_{\eta}g_{\nu\lambda}\right) \tag{D.8}$$

Since v^{θ} and v^{ϕ} are zero, the only surviving connection term when we consider $\mu = t$ is

$$\Gamma^{t}_{tr} = \frac{1}{2}g^{tt}\partial_r g^{tt} \tag{D.9}$$

Using this one can find that for $\mu = t$, the energy conservation equation is

$$v^t\partial_t v^t + v\partial_r v^t + g^{tt}\left(\partial_r g_{tt}\right)vv^t + \frac{a^2}{\rho}\left[\left((v^t)^2 + g^{tt}\right)\partial_t\rho + vv^t\partial_r\rho\right] = 0 \tag{D.10}$$

Similarly when $\mu = r$, the non-vanishing connections are

$$\Gamma^r_{tt} = \frac{-1}{2} g^{rr} \partial_r g_{tt} \tag{D.11}$$

$$\Gamma^r_{rr} = \frac{-1}{2} g^{rr} \partial_r g_{rr} \tag{D.12}$$

Substituting for all these gives the radial equation

$$v^t \partial_t v + v \partial_r v - \frac{(v^t)^2 \partial_r g_{tt}}{2 g_{rr}} + \frac{v^2 \partial_r g_{rr}}{2 g_{rr}} + \frac{a^2}{\rho} v v^t \partial_t \rho + \frac{a^2}{\rho} \left[v^2 + g^{rr} \right] \partial_r \rho = 0 \tag{D.13}$$

Using the normalization condition $v^\mu v_\mu = -1$ one can write

$$\frac{v^2}{2 g_{rr}} \partial_r g_{rr} + \frac{1}{2} \partial_r v^2 = - \left[\frac{(v^t)^2 \partial_r g_{tt}}{2 g_{rr}} + \frac{g_{tt} \partial_r (v^t)^2}{2 g_{rr}} \right] \tag{D.14}$$

Substituting the previous equation in Eqn.(D.10) gives the necessary equation

$$\sqrt{\frac{1 + g_{rr} v^2}{-g_{tt}}} \partial_t v + v \sqrt{\frac{1 + g_{rr} v^2}{-g_{tt}}} \frac{a^2}{\rho} \partial_t \rho + \partial_r \left(\frac{1 + g_{rr} v^2}{2 g_{rr}} \right) + \left(\frac{1 + g_{rr} v^2}{2 g_{rr}} \right) \frac{\partial_r (g_{tt} g_{rr})}{g_{tt} g_{rr}} + \frac{a^2}{\rho} \left(\frac{1 + g_{rr} v^2}{g_{rr}} \right) \partial_r \rho = 0 \tag{D.15}$$

The radial equation is manipulated to the form in Eqn.(D.13) just for convenience purpose.It is easy to perform perturbation scheme on this equation as the coefficient of the term containing spatial derivative of velocity is 1.

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