Noncommutative Quantum Mechanics and Noncommutative Geometry

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled "Noncommutative Quantum Mechanics and Noncommutative Geometry" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by " SHIVRAJ PRAJAPAT" at "S N Bose National Centre For Basic Sciences Kolkata", under the supervision of "Prof Biswajit Chakraborty" during the academic year 2013-2014.

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Abstract

The quantum mechanics in Moyal plane, which has noncommutativity $[\hat{x}, \hat{y}] = i\theta(\theta = \text{constant})$, was formulated using Hilbert-Schmidt operators. This formulation is shown to allow to construct the Schwinger's SU(2) generators in this noncommutative case. Using these generators we revisited the problem of 2D noncommutative harmonic oscillator, which earlier was being studied by star product formulation of quantum mechanics in noncommutative space. The energy spectrum of this 2D noncommutative oscillator is obtained where it comes with the renormalized parameters different from the bare parameters occuring in the Hamiltonian, from which it can be inferred that observable mass and frequency of 2D noncommutative oscillator will be different from the bare mass and frequency. The breaking of SU(2) and time reversible symmetry in also shown.
The noncommutative geometry, which allows to do the geometry not only for the commutative/usual space but also for the noncommutative /pseudo space in the algebric manner, where the topological and geometrical informtion of a space is captured through the spectral triple (Algebra ($\mathcal{A}$), Hilbert space($\mathcal{H}$), Dirac operator($\mathcal{D}$)). We applied the approach of noncommutative geometry to compute this spectral triple in the noncommutative space whose coordinates satisfy SU(2) algebra type noncommutativity- $[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k$. With this noncommutativity, the radius-squared operator, which is Casimir operator, is quantized, so the space can be viewed as concentric spheres, each sphere being named "fuzzy sphere". We calculated the infinitesimal distance function between two "points" (taken to be pure states of $C^* - algebra$) on these fuzzy spheres and studied its closeness with the commutative case, calculating the distance function using the coherent state.

Contents

Chapter 1

Introduction

The work of my MS thesis is related to the field of "Noncommutative quantum mechanics and Noncommutative geometry".

1.1 History/Literature survey

Theory of noncommutativity deals in atleast two arena. One is noncommutativity in the space-time coordinates and other is the noncommutativity which arises by defining mapping from the differentiable manifolds to the matrix spaces, these maps becomes noncommutative due to the matrices being noncommutaive. These kinds of noncommutativity arises in the so-called "Almost commutative spaces", used in the spectral formulation of standard model - a la Connes et.al [17]

The spacetime noncommutativty was first proposed by Heisenberg himself in 1930. His idea behind the proposal was to suggest one possible way to remove the divergences occuring in the quantum field theory. The first paper on the subject appeared in 1947 by Hartland Snyder. By that time the renormalization technique to remove the divergences of quantum field theory took the attention of the scientific community and the field of noncommutativity got subsided. After that it only got the boost in mid 1990's with the paper of Sergio Doplicher, Klaus Fredenhagen and John Roberts in 1994 ([1]). They set out an another motivation for the spacetime noncommutativity from the argument of both Quantum mechanics and Einstein's theory of general relativity. The argument is given as follows: To probe the distance of the order of the Planck length l_p given by-

$$ {}_p = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-33} cm \tag{1.1} $$

the probe wavelength has to be of the order of compton wavelength. Alternatively, the probe mass/energy has to be huge (the probe mass/energy is inversely proportional to the Compton wavelength). But from the consideration of the general relativity , the localization of the huge mass/energy in the scale of Planck length will automatically leads to the formation of Black-hole and event horizon will take place, which in turn will make the measurement impossible. So, it seems natural to propose the certain spacetime noncommutativity which will prohibit to make the simultaneous measurement of all the space-time coordinates.

Further, it got boost with the paper by Nathan Seiberg and Edward Witten in 1999 [2]. In the context of string theory they argued that the low energy limit of the string theory leads to the spatial noncommutativity in the presence of constant B-field (analogous to the constant magnetic field.)

Also, Since the space-time noncommutativity does not allows to make a precise measurement of the space-time coordinates, the notion of the usual/differentiable space is lost. The usual points, paths do not make sense in the noncommutative case. The development of the "Noncommutative Geometry" around 1980's by French mathematician Alain Connes and his co-workers helped to study these type of noncommutative spaces [15]. The development of the noncommutative geometry was based on the famous theorem - "Galfand -Naimark theorem" given by Galfand and Naimark around 1940's , which estabilish a duality between a space and the algebra of the smooth functions defined on it. Connes in the noncommutative geometry developed the so called "Spectral Triplet" , comprising of - Algebra , Hilbert space and the Dirac operator $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, (The algebra and Dirac operator both acts on the Hilbert space through some appropriate representation) where the algebra captures the topological information and the Dirac operator captures the geometrical information of the space. The algebra can then be taken to be noncommutative and which is thought to be the algebra of the functions defined on some "virtual noncommutative spaces." The attention from the noncommutative spaces (which does not even concretely exist) is switched to the noncommutative algebra, which is assumed to capture the information of the noncommutative spaces.

It was argued by Alain Connes in one of his paper " Noncommutative geometry and Physics"([16]) that at the level of the grand unification scale (10^{16} Gev) with is three decimal lower than the Planck scale (10^{19} Gev) , unification of gravity with the standard model of particle physics at the classical level will lead to a manifold which will not be a differentiable manifold, instead it will be pseudo/noncommutative manifold which is described through the noncommutative algebra. Using the approach of Noncommutative geometry the unification of gravity with the standard model of particle physics has been given by Connes and his co-workers, the detail of which can be found in the review article "Particle Physics from almost Commutative

Spacetime" ([17]), by Walter D. Van Suijlecom and Koen Van Den Dungen , where the Higgs field comes out naturally. Infact, the approach of noncommutative geometry has been able to correctly postdicted the Higgs mass around 126 Gev, ([18]) which was observed recently in 2012 at LHC , Cern.

Also, the spatial noncommutativity can be realized in the presence of the strong arbitrary magnetic field , where the field is assumed to be so high that the particle can be assumed to remain in the lowest Landau level (the energy level separation is directly proportional to the magnetic field). The projected coordinates on the lowest Landau level than satisfy the certain spatial noncommutativity. The realization of spatial noncommutativity can also be obtained through the classical approach using the Dirac bracket analysis, where the requirement of the strong arbitrary magnetic field imposes certain constrained equation, which in the Hamiltonian formulation are handled using the Dirac's prescription , where the Poisson bracket are replaced with the Dirac Bracket and the first approximation to quantum theory is obtained through Dirac bracket instead of Poisson bracket ([19]).

The quantum mechanics and quantum field theory in the simplest choice of noncommutativity of type

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \tag{1.2}$$

is being studied using the star product([3]), mainly the Voros star product and the Moyal star product. Recently, it was shown that the different star product leads to the different physical consequences ([5]) so it was necessary to have an operatorial framework/formulation. The operatorial formulation for the 2D plane with the noncommutativity $[\hat{x}, \hat{y}] = i\theta$, which is called Moyal plane, is given through the Hilbert-Schmidt operators in the paper ([7]).

The Hilbert-Schmidt operatorial formulation in the Moyal plane has been studied since its development to check its consistenty with the quantum mechanical interpretation . The various problems in noncommutative quantum mechanics itself like- free particle, harmonic oscillator, Coulomb potential, finite square well and many others has been studied since then using this approach.

1.2 What underlies my work

The noncommutative harmonic oscillator has been studied widely but mostly with the use of star products. We revisited the problem of noncommutative harmonic oscillator in 2D Moyal plane in parallel to the 2D harmonic oscillator in the commutative case , using the approach of Hilbert-Schmidt operatorial formulation with the help of Schwinger's SU(2) generators. It has been shown that the Hilbert-Schmidt operator formulation allows to construct the Schwinger's SU(2) generators in noncommutative Moyal plane and using which it is also shown that the SU(2) symmetry of noncommutative harmonic oscillator is broken to U(1) symmetry along with the breaking of time reversible symmetry. The energy spectrum is also obtained where it comes with the renormalized parameters different from the bare parameters occuring in the Hamiltonian, from which it can be inferred that observable mass and frequency of 2D noncommutative oscillator will be different from the bare mass and frequency. This has been uploaded in the arXiv: 1312.3095 [23].

The above used Schwinger's SU(2) generators is now can be reinterpreated as the position operators in 3D with the SU(2) noncommutative algebra. Such a spatial noncommutativity is realized in the presence of the strong magnetic field produced by the magnetic monopole. This type of spatial noncommutativity is similar to the noncommutativity satisfied by the angular momentum operators. One can define the simultaneous eigenstate $|n, n_3\rangle$ of radius -squared operator $\hat{\vec{x}}^2$, which is Casimir operator, and the $\hat{x}_3$ operator similar to the case of angular momentum. The radius-squared operator takes the value $\lambda^2 n(n+1)$, so that radius is quantized. Each sphere with radius indexed by n is called fuzzy sphere. The applicability of the noncommutative geometry can be tested by applying to these simple noncommutative spaces (fuzzy sphere), so we desired to apply the formulation of noncommutative Geometry to these type of noncommutative spaces to construct the spectral triplet $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and to find the analytic formula for the Connes infinitesimal distance function between two "points"(taken to be pure states of an appropriate C^*-algebra) derived in [21]. Also following the paper [22] we constructed the perelemov coherent state on the sphere and applied the formalism of [21] to obtain the metric of the sphere in the stereographic variables.

Chapter 2

Essential background theory

2.1 Noncommutativity in magnetic field

Noncommutativity can be realized in the strong magnetic field. The noncommutativity in magnetic field arises when one consider the magnetic field to be so high that particle can be expected to remain in the ground state (The seperation between energy levels is directly proportional to the strength of the magnetic field), then the projected coordinates on the ground state satisfies the certain noncommutativity. In the Landau problem , where one consider the magnetic field in the Z-direction and assume the motion of the particle in the x-y plane. The projected coordinates on the lowest Landau level satisfies the Heisenberg algebra type noncommutativity

$$[\hat{x}, \hat{y}] = i\theta \qquad \theta = \frac{-1}{qB} \tag{2.1}$$

The noncommutativity in the magnetic field produced by the magnetic monopole has also been obtained quantum mechanically from the projection of the coordinate variable to the ground state in [24]. The projected coordinates are found to satisfy the Lie-algebra type noncommutativity:

$$[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k \tag{2.2}$$

where λ is inversely related to the strength of the magnetic field.

2.2 Formulation of quantum mechanics in noncommutative Moyal plane

In this section we discuss the formulation of quantum mechanics in noncommutative Moyal plane which has noncommutativity in coordinates $\hat{x}_1$ and $\hat{x}_2$ of the type-

$$[\hat{x}_1, \hat{x}_2] = i\theta\varepsilon_{ij} \tag{2.3}$$

In the noncommutative quantum mechanics there is a spatial noncommutativity which imposes restriction on the simultaneous measurement of the coordinates x_1 and x_2 of the form $\Delta x_1 \Delta x_2 \geq \frac{\theta}{2}$. Due to this uncertainity , noncommutative quantum mechanics does not allow to make a precise measurement of position and the notion of points are lost. The configuration space itself becomes blurred as it happens with the phase space in quantum mechanics since coordinates and momentum does not commute. Here in the nocommutative case the 2D configuration space itself is replaced by the Hilbert space which is named as classical Hilbert space $\mathcal{H}_c$. The physical state of the system under consideration is taken to be the Hilbert Schmidt operators $|\psi)$ (denoted by this symbol to distinguish from the usual $|\psi\rangle$), which are traceclass (well defined trace) and bounded operators (bounded norm) on the $\mathcal{H}_c$. The Hilbert -Schmidt operators also forms a Hilbert space which is named quantum Hilbert space $\mathcal{H}_q$ to distinguish it from the previously defined classical Hilbert space $\mathcal{H}_c$. The element of $\mathcal{H}_q$ which are operators (polynomials in operators $\hat{x}_1$ and $\hat{x}_2$), now serve the purpose of ket in noncommutative case. The wave function in the appropriate representaion is obtained by taking the inner product with the corresponding basis. The observables are then taken to be Hermitian operators on $\mathcal{H}_q$ with some approprite representation.

Because of the Hermiticity of the observables here too there eigenvalues are real and there eigenstates provides the complete basis for $\mathcal{H}_q$. The postulates of quantum mechanics still remains valid here. The position measurement needs special attention since they do not commute so the simultaneous eigenstates of both the position observables can not be constructed and there is always an uncertainity in the simultaneous measurement. The best thing one can do is to take the coherent state since they are the minimum uncertainity state to give the notion of position measurement. But since the coherent states provides the over complete basis for the $\mathcal{H}_q$, one can not construct the projective valued measure (PVM) to assign the probability of particular outcome, instead one uses positive operator valued measure (POVM) to assign the probability of particular outcome.

2.2.1 Formulation

In 2D, the noncommutative coordinate algebra(in the unit $\hbar = 1$) can be written as

$$[\hat{x}_i, \hat{x}_j] = i\theta\epsilon_{ij} \tag{2.4}$$

Defining the creation $b^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 - i\hat{x}_2)$ and annihilation operators $b = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 + i\hat{x}_2)$ such that $[b, b^\dagger] = 1$. The noncommutative two dimensional classical configuration space can be written as:

$$\mathcal{H}_c = \text{span}\{| \ n\rangle\}_{n=0}^{\infty} \tag{2.5}$$

where $|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle$ is the eigenstate of the operator $b^\dagger b : (b^\dagger b|n\rangle = n|n\rangle)$ The corresponding quantum Hilbert space, the elements of which represent the physical states, can then be taken as the set of Hilbert- Schmidt operators which are all bounded trace-class operators over $\mathcal{H}_c$

$$\mathcal{H}_q = \{\psi : \text{tr}_c(\psi^\dagger \psi) < \infty\} \tag{2.6}$$

The elements of $\mathcal{H}_q$ are denoted by a round bracket $|\psi) = span\{\big|m\big\rangle\big\langle n\big|\}$ (most general state) and the inner product between them is defined as

$$(\phi|\psi) = \text{tr}_c(\phi^\dagger \psi) \tag{2.7}$$

where the subscript c refers to tracing over $\mathcal{H}_c$ and $\dagger$ denotes Hermitian conjugation on $\mathcal{H}_c$ while $\ddagger$ will denotes the same on $\mathcal{H}_q$.
If $\hat{X}_i$ and $\hat{P}_i$ are the representations of the operators $\hat{x}_i$ and the conjugate momentum respectively acting on $\mathcal{H}_q$, then a unitary representation is obtained by the following action:

$$\hat{X}_i\psi = \hat{x}_i\psi, \quad \hat{P}_i\psi = \frac{1}{\theta}\epsilon_{ij}[\hat{x}_j, \psi] = \frac{1}{\theta}\epsilon_{ij}\left(\hat{X}_j^L - \hat{X}_j^R\right)\psi, \tag{2.8}$$

where $\hat{X}_i^L$ and $\hat{X}_i^R$ refer to left and right action respectively i.e. $\hat{X}_i^L\psi = \hat{x}_i\psi$ and $\hat{X}_i^R\psi = \psi\hat{x}_i$. Note that, we have taken the action of $\hat{X}_i$ to be left action ($\hat{X}_i \equiv \hat{X}_i^L$) some what arbitrarily and the momentum operator is taken to act adjointly. This ensures that $\hat{X}_i$ and $\hat{P}_i$ satisfy non-commutative Heisenberg algebra:

$$[\hat{X}_i^L, \hat{X}_j^L] \equiv [\hat{X}_i, \hat{X}_j] = i\theta\epsilon_{ij} \ ; \ [\hat{X}_i, \hat{P}_j] = i\delta_{ij} \ ; \ [\hat{P}_i, \hat{P}_j] = 0 \tag{2.9}$$

It can be checked easily that the right action satisfies

$$[\hat{X}_i^R, \hat{X}_j^R] = -i\theta\epsilon_{ij} \ ; \ [\hat{X}_i^L, \hat{X}_j^R] = 0 \tag{2.10}$$

The corresponding right actions of anihilation operator is introduced in an analogous manner:

$$\langle n| = \frac{1}{\sqrt{n!}}\langle 0|b^n \equiv (b_R)^n\left(\frac{1}{\sqrt{n!}}\langle 0|\right) \tag{2.11}$$

so that for any $\psi \in \mathcal{H}_q$,one define $B_L\psi = B\psi$ and $B_R\psi = \psi B$ and similarly for $B_{L/R}^\ddagger$

2.3 Non-commutative Geometry

In this section we briefly discuss the noncommutative geometry developed by the French mathematician Alain Connes which allows us to give the concept of geometry not only for the differentiable manifolds but also for the pseudo spaces (noncommutative spaces).

The field of noncommutative geometry is based on the famous theorem by Galfand and Naimark known as "Galfand- Naimark theorem." The statement of the theorem is stated as: "The category of the locally compact Hausdorff space is anti equivalent to the category of the commutative C^*-algebra."

$$\text{locally compact Hausdorff space} \simeq \text{commutative C*-algebra}^{op} \tag{2.12}$$

The term used in the statement of the theorem will be shortly reviewed in the subsequent sections. The above stated Galfand- Naimark Theorem implies that there is a duality between a locally compact Hausdorff space and a commutative C^*-algebra. In short, all the information about the space is encoded through C^*-algebra. One can now take the algebra to be noncommuative and the corresponding space will then be called the noncommutative space. Alain Connes developed the formulation of noncommutative geometry where he defines the so called "spectral triplet" $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

1. Algebra $(\mathcal{A})$
2. Hilbert Space $(\mathcal{H})$
3. Dirac Operator $(\mathcal{D})$

The algebra $(\mathcal{A})$ captures the topological information of space and the Dirac operator $(\mathcal{D})$ captures the information of geometry in these spaces. The choice of the algebra and Dirac operator constitutes the main input of the model while the choice of the Hilbert space is irrelevant. The algebra element acts on Hilbert space $\mathcal{H}$ through some appropriate representation. The Dirac operator also acts on the Hilbert space. The commutator $[\mathcal{D}, a]$,a $\in \mathcal{A}$ plays the role of the differential $\frac{da}{ds}$, with ds the length element, in the commutative case. The geodesic formula of the Riemannian geometry-

$$d(x,y) = Inf \int_\gamma ds \tag{2.13}$$

is replaced in noncommutative geometry with-

$$\begin{aligned} d(x,y) &= sup_B\{|a(x) - a(y)|\} \\ B &= \{a \in \mathcal{A}, \|[\mathcal{D}, a]\| \leq 1\} \end{aligned} \tag{2.14}$$

2.3.1 Duality between space and algebra

Algebra

An algebra is a set of elements together with the property of vector space as well as the group. Any algebra A will be an algebra over some field C . This means that objects like $\alpha a + \beta b \in A$ with a,$b \in A$ and $\alpha, \beta \in C$. Also, it is closed under product $A \times A \to A$, $A \ni (a, b) \to ab \in A$, which is distributive over addition,

$$a(b + c) = ab + ac, \quad a, b, c \in A \tag{2.15}$$

There are different class of algebra-

1. Banach Algebra -An Algebra with an additional property of definite norm such that-
$$a, b \in A : \|ab\| = \|a\|\|b\| \tag{2.16}$$

2. C*-Algebra -A Banach algebra with an additional property of the notion of involution map $^* : A \to A$ with the property
$$a, b \in A, \quad a^{**} = (a^*)^* = a \tag{2.17}$$
$$(a + b)^* = a^* + b^* \tag{2.18}$$
$$(ab)^* = b^* a^* \tag{2.19}$$

Category

A category C consist of collection of three main elements

1. Class ob(C) - The elements of it are called objects.

2. Class hom(C)- The element of it are called morphisms or arrows or maps. Each arrow f $\in hom(C)$ connects a particular object $a \in ob(C)$ to another obeject $b \in ob(C)$,$f : a \to b$.

3. Composition of arrows- such that for any three objects a,b,c$\in$ ob(C) we have hom(a,b)$\times$ hom(b,c) = hom(a,c). hom(a,b) is the notation for - hom class of all arrows from a to b

Galfand- Naimark Theorem

The celebrated Gelfand-Naimark theorem states that the category of locally compact Hausdorff spaces is anti-equivalent to the category of commutative C*-algebras:

$$\{\text{locally compact Hausdorff space}\} \simeq \{\text{commutative C*-algebra}\}^{op} \tag{2.20}$$

For a locally compact Hausdoff space X, let $C_0(X)$ denote the algebra of complex valued continuous functions on X that "vanish at ∞" . This means for any $\epsilon > 0$ there is a compact subset $K \subset X$ such that $|f(x)| < \epsilon$ for $x \notin K$

$$C_0(X) = \{f : X \to C, f \text{ is continuous and } f(\infty) = 0\}. \tag{2.21}$$

Under pointwise addition and scalar multiplication $C_0(X)$ is obviously an algebra over the field of complex number C, with the sup-norm

$$\|f\| = sup\{|f(x)|; x \in X\} \tag{2.22}$$

Also, let A be a commutative C^* -algebra. Let $\Omega(A)$ = set of characters of A , where a character is simply a nonzero algebra map

$$\Omega : A \to C \tag{2.23}$$

From the theorem it follows that there is a duality between locally compact Hausdorff space X and algebra $C_0(X)$ in the form-

$$X \simeq \Omega(C_0(X)) \tag{2.24}$$

And for given C^*- algebra 'A' the duality is of the form-

$$A \simeq C_0(\Omega(A)) \tag{2.25}$$

As a generalization one can now take the algebra to be noncommutative. A noncommutative C^*-algebra will be now thought of as the algebra of continuous functions on some 'virtual noncommutative space'. The attention will be switched from spaces, which in general do not even exist 'concretely', to algebra of functions.

Alain Connes has developed the new calculus which replaces the usual differential calculus. He defines the so called "spectral triple" ($\mathcal{A}, \mathcal{H}, \mathcal{D}$) mentioned in the introduction part. The algebra $\mathcal{A}$ acts on the Hilbert space $\mathcal{H}$ and captures the topological information discussed in the previous section. The Dirac operator is needed to give a notion of distance.

2.3.2 Commutative space through noncommutative geometry

In the commutative, for a 'n' dimensional Riemannian manifold (X,g), $C_0(X)$ the algebra of complex valued continuous functions on X that "vanish at infinity" provides the algebra $\mathcal{A}$ of the spectral triplet. The Hilbert space is taken to be square integrable sections of the spinor bundle over X , $\mathcal{H} = L^2(X, S)$. The Dirac operator $\mathcal{D}$ is taken to be $\mathcal{D} = i\gamma^\mu \nabla_\mu$, where ∇_μ is the covariant derivative associated with the Levi-Civita (metric and torison free) connection of the metric tensor 'g' of the manifold X. The gamma metric γ^μ obey the relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2.26}$$

The elements of $\mathcal{A}$ acts as multiplicative operator on $\mathcal{H}$.

$$(f.\psi)(x) := f(x)\psi(x), \qquad f \in \mathcal{A}, \quad \psi \in \mathcal{H} \tag{2.27}$$

Let us apply it to the simplest case of one dimensional line. The spectral triplet for the line is -

1. Algebra ($\mathcal{A}$) - continuous functions f(x) defined on the line with f($\pm\infty$)=0
2. Hilbert space ($\mathcal{H}$)- same as algebra (only in this case)
3. Dirac operator ($\mathcal{D}$)- $\quad -i\partial_x$

The point of the space (1D line) is obtained by defining the map (2.24). The dirac delta function provides the mapping:

$$\int dx f(x)\delta(x - x_0) = f(x_0) = \text{C-number} \tag{2.28}$$

In the dirac delta function $\delta(x-x_0)$, x_0 specifies the point of the 1D line. The distance between two point x and y of the line is obtained through (2.14).

$$d(x, y) = Sup\{|f(x) - f(y)|, \|\partial_x(f(x))\| \leq 1\} \tag{2.29}$$

the suprimum will be obtained when the function has the slope of 1 i.e. f(x) = x+c , so that we get

$$d(x, y) = |x - y| \tag{2.30}$$

2.3.3 Noncommutative Case: Moyal plane

As earlier discussed the Moyal plane is defined to be the plane whose coordinate satisfy the commutator algebra-$[x, y] = i\theta$ and we defined the noncommuative configutaion space by-

$$\mathcal{H}_c = Span\{|n\rangle\}_{n=0}^{\infty} \tag{2.31}$$

where $|n\rangle$ is the eigenstate of the number operator $b^\dagger b$, where $b = \frac{1}{\sqrt{2\theta}}(x + iy)$.The number operator is related to the radius-squared operator by-

$$\hat{r}^2 = \hat{x}^2 + \hat{y}^2 = 2\theta(b^\dagger b + 1/2) \tag{2.32}$$

so the state $|n\rangle$ is an eigenstate of the radius-squared operator with eigenvalue-

$$\hat{r}^2|n\rangle = 2\theta(b^\dagger b + 1/2)|n\rangle = 2\theta(n + 1/2) \tag{2.33}$$

starting from the smallest radius $\sqrt{\theta}$, the radius is quantized (2.33). The physical state of the system is taken to be element of the quantum Hilbert space $\mathcal{H}_q$, whose elements are Hilbert-Schmidt operators (trace-class and bounded operators) which acts on $\mathcal{H}_c$, so they are function of coordinate operator $\hat{x}$ and $\hat{y}$. The general form is written as-

$$\mathcal{H}_q = Span\{|n\rangle\langle m|\} \tag{2.34}$$

as introduced earlier the action of coordinate $\hat{X}_i$ and $\hat{P}_i$ on $\mathcal{H}_q$ are given as-

$$\hat{X}_i\psi = \hat{x}_i\psi, \quad \hat{P}_i\psi = \frac{1}{\theta}\epsilon_{ij}[\hat{x}_j, \psi] = \frac{1}{\theta}\epsilon_{ij}\left(\hat{X}_j^L - \hat{X}_j^R\right)\psi, \tag{2.35}$$

The elements of $\mathcal{H}_q$ forms an algebra. This algebra is then can be taken to be the algebra $\mathcal{A}$ in the spectral triplet. It is a generalization from the commutative case where the algebra is taken to be the function defined on the space. The only difference is that now the coordinates becomes operator in the noncommuative case. The Hilbert space $\mathcal{H}$ for the spectral triplet is taken to be two copy of $\mathcal{H}_c$ i.e. $\mathcal{H}_c \bigotimes C^2$ $(|\psi\rangle, |\phi\rangle)$. The action of the algebra on this Hilbert space is $\pi(a)(|\psi\rangle, |\phi\rangle) = (a|\psi\rangle, a|\phi\rangle)$. The Dirac operator $\mathcal{D}$ is taken to be $i\sigma^i P_i$ (i=1,2, σ^i are Pauli matrices) ,where P_i are the momentum operator. Let us write the elements of spectral triplet together for the Moyal plane-

1. Algebra $\mathcal{A} - Span\{|n\rangle\langle m|\}$
2. Hilbert space $\mathcal{H} - \begin{pmatrix} span\{|n\rangle\} \\ span\{|n'\rangle\} \end{pmatrix}$
3. Dirac operator $\mathcal{D} = -i\sigma^i P_i = -i\sqrt{\frac{2}{\theta}}\begin{bmatrix} 0 & b^\dagger \\ b & 0 \end{bmatrix}$

The action of the Dirac operator on the two component coulmn vector is well defined. The action of the algebra on the Hilbert space is also well defined and is the left multiplication. From the Galfand -Naimark theorem we know the duality between the algebra $\mathcal{A}$ and a space represented by it, and is defined through (2.24). The "point" (taken to be the pure state of the C^*-algebra) of the noncommutative space are then obtained by (2.24). In this case the mapping is defined to be-

$$\Omega : |n\rangle\langle m| \to tr_c((|n\rangle\langle m|)^\dagger |n'\rangle\langle m'|) = C \tag{2.36}$$

The element $|n'\rangle\langle m'|$ belongs to the dual of the algebra , which coincidently is same as the algebra itself. The pure state ρ of this dual are the elements of the form - $|n\rangle\langle n|$, which represents the "points" of the noncommutative space. Infact, it is the density matrix corresponding to the state $|n\rangle$ viewed from the perspective of classical Hilbert space $\mathcal{H}_c$ and the state $|n\rangle$ are the

eigenstate of the radius-squared operator with the eigenvalue (2.33). So, the space can be viewed as the concentric circles , each circle representing one "point" of the noncommutative space.
Distance formula in the noncommuative geometry is given by

$$d(\rho, \rho') = Sup_B\{\rho(a) - \rho'(a)\} \tag{2.37}$$

$$B = \{a \in \mathcal{A}, \|[\mathcal{D}, a]\| \leq 1\} \tag{2.38}$$

This formula is further simplified in our problem derived in [21] in the form -

$$d(\rho, \rho') = \frac{tr_c(d\rho)^2}{\|[\mathcal{D}, \pi(d\rho)]\|_{op}} \tag{2.39}$$

where

$$\|A\|_{op} = sup_{\phi \in \mathcal{H}} \frac{\|A\phi\|}{\|\phi\|} \tag{2.40}$$

and ρ is the pure density matrix corresponding to the state $|n\rangle$. The $d\rho$ is the infinitesimal change given as-

$$d\rho = |n+1\rangle\langle n+1| - |n\rangle\langle n| \tag{2.41}$$

The remaining calculation is now straightforward-

$$[\mathcal{D}, \pi(d\rho)] = -i\sqrt{\frac{2}{\theta}} \begin{bmatrix} 0 & [b^\dagger, d\rho] \\ ,[b, d\rho] & 0 \end{bmatrix} \tag{2.42}$$

In order to take the norm we first diagonalize it

$$[\mathcal{D}, \pi(d\rho)]^\dagger [\mathcal{D}, \pi(d\rho)] = \frac{2}{\theta} \begin{bmatrix} [b, d\rho]^\dagger [b, d\rho] & 0 \\ 0 & [b^\dagger, d\rho]^\dagger [b^\dagger, d\rho] \end{bmatrix} \tag{2.43}$$

and one can calculate-

$$[b, d\rho]^\dagger [b, d\rho] = 4(n+1)|n+1\rangle\langle n+1| + n|n\rangle\langle n| \\ +(n+2)|n+2\rangle\langle n+2| \tag{2.44}$$

$$.[b^\dagger, d\rho]^\dagger [b^\dagger, d\rho] = 4(n+1)|n\rangle\langle n| + n|n-1\rangle\langle n-1| + \\ (n+2)|n+1\rangle\langle n+1| \tag{2.45}$$

Both of these operators are diagonal. The operator norm, defined to be the largest eigenvalue, can be read off exactly from both the equation (2.45,2.46)- 4(n+1), which gives the norm-

$$\|[\mathcal{D}, \pi(d\rho)]\| = 2\sqrt{\frac{2(n+1)}{\theta}} \tag{2.46}$$

Using $tr_c(d\rho)^2 = 2$, the distance between the state $|n+1\rangle$ and $|n\rangle$ is obtained from (2.39)

$$d(n+1, n) = \sqrt{\frac{\theta}{2(n+1)}} \tag{2.47}$$

One can now calculate the infinitesimal distance function between the coherent states. The coherent state in the Moyal plane is -

$$|z\rangle = e^{zb^\dagger - \bar{z}b} = e^{\frac{-z\bar{z}}{2}} e^{zb^\dagger} |0\rangle \tag{2.48}$$

The pure state density matrix is $\rho = |z\rangle\langle z|$. The $d\rho$ is then-

$$d\rho = |z + dz\rangle\langle z + dz| - |z\rangle\langle z| \tag{2.49}$$

Using (2.48), we can write -

$$|z + dz\rangle = \big(1 + dzb^\dagger - \frac{zd\bar{z}}{2} - \frac{\bar{z}dz}{2} + O(dz^2, d\bar{z}^2, dzd\bar{z})\big)|z\rangle \tag{2.50}$$

Using (2.50) and retaining the first order term in dz and $d\bar{z}$,the $d\rho$ is written as

$$d\rho = (b^\dagger - \bar{z})|z\rangle\langle z|dz - |z\rangle\langle z|(b - z)d\bar{z} \tag{2.51}$$

defining the operator $\tilde{b} = b - z$ and $\tilde{b}^\dagger = b^\dagger - \bar{z}$, it is easily seen that these operators satisfy-

$$[\tilde{b}, \tilde{b}^\dagger] = [b, b^\dagger] = 1 \tag{2.52}$$

$$(b - z)|z\rangle = \tilde{b}|z\rangle = \tilde{b}|\tilde{0}\rangle = 0 \tag{2.53}$$

Where we have identify $|z\rangle \equiv |\tilde{0}\rangle$. The states with tilde are then obtained by the raising operator $\tilde{b}^\dagger$.

$$|\tilde{n}\rangle = \frac{(\tilde{b}^\dagger)^n}{\sqrt{n!}}|\tilde{0}\rangle \tag{2.54}$$

The $d\rho$ is now written in terms of the states with tilde.

$$d\rho = |\tilde{1}\rangle\langle\tilde{0}|dz - |\tilde{0}\rangle\langle\tilde{1}|d\bar{z} \tag{2.55}$$

Similar to the earlier case of the discrete basis , now we apply the Connes' infinitesimal distance formula (2.39)for the coherent basis, using (2.55). For that, we calculate following quantity-

$$[b, d\rho] = dz|\tilde{0}\rangle\langle\tilde{0}| - \sqrt{2}d\bar{z}|\tilde{0}\rangle\langle\tilde{2}| - dz|\tilde{1}\rangle\langle\tilde{1}| \tag{2.56}$$

$$.[b^\dagger, d\rho] = \sqrt{2}|\tilde{2}\rangle\langle\tilde{0}|dz - |\tilde{1}\rangle\langle\tilde{1}|d\bar{z} + |\tilde{0}\rangle\langle\tilde{0}|d\bar{z} \tag{2.57}$$

Similar to the discrete case , to diagonalize the $[\mathcal{D}, \pi(a)]$, we need to calculate-

$$[b, d\rho]^\dagger[b, d\rho] = d\bar{z}dz|\tilde{0}\rangle\langle\tilde{0}| - \sqrt{2}(dz)^2|\tilde{0}\rangle\langle\tilde{2}| + 2d\bar{z}dz|\tilde{2}\rangle\langle\tilde{2}| + d\bar{z}dz|\tilde{1}\rangle\langle\tilde{1}| \tag{2.58}$$

$$.[b^\dagger, d\rho]^\dagger[b^\dagger, d\rho] = 3d\bar{z}dz|\tilde{0}\rangle\langle\tilde{0}| + d\bar{z}dz|\tilde{1}\rangle\langle\tilde{1}| \tag{2.59}$$

By diagonalising the first one we get-

$$\|[\mathcal{D}, \rho(a)]\|_{op} = \sqrt{\frac{6d\bar{z}dz}{\theta}} \tag{2.60}$$

The distance is then-

$$d(z, z + dz) = \sqrt{\frac{2\theta}{3}}\sqrt{d\bar{z}dz} \tag{2.61}$$

Chapter 3

On the role of Schwinger's SU(2) generators for Simple Harmonic Oscillator in 2D Moyal plane

3.1 Angular momentum

We are now going to study angular momentum using Schwinger's representation. First we will discuss angular momentum operators in commutative 2D case with the help of two decoupled simple harmonic oscillators. Then we will move to non-commutative case where we will see how the roles of angular momentum operators gets interchanged with the operators in commutative case. This is based on [23].

3.1.1 Schwinger's representation of Angular momentum

Schwinger introduced a method to express a general angular momentum in quantum mechanics in terms of the creation ($\hat{a}^\dagger_\alpha$) and annihilation ($\hat{a}_\alpha$) operators of a pair of independent harmonic oscillators satisfying:

$$[\hat{a}_\alpha, \hat{a}_\beta] = 0 = [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] \text{ and } [\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta} \;\; \forall \alpha, \beta = 1, 2 \tag{3.1}$$

With the help of these operators the number operator $\hat{N}$ and angular momentum operator $\hat{\vec{J}}$ is defined as follows:

$$\hat{N} = \hat{a}^\dagger_\alpha \hat{a}_\alpha \tag{3.2}$$

$$\hat{\vec{J}} = \frac{1}{2}\hat{a}^\dagger_\alpha \{\vec{\sigma}\}_{\alpha\beta}\hat{a}_\beta \;;\; [\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k, \tag{3.3}$$

with $\vec{\sigma}$ being our usual Pauli matrices.

3.1.2 Role of Angular momentum operators in commutative 2-D plane

In the commutative case let us choose our basis for a pair of decoupled harmonic oscillators which can be regarded as $|m\rangle \otimes |n\rangle \in \mathcal{H}_c \otimes \mathcal{H}_c$. We define the creation operators $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$ as $(\hat{a}^\dagger \otimes 1)$ and $(1 \otimes \hat{a}^\dagger)$ respectively, and their Hermitian conjugates as annihilation operators. Now using Schwinger's prescription (3.3) we obtain the three angular momentum operators as

$$\hat{J}_1 = \frac{1}{2}\left(\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2\right), \quad \hat{J}_2 = \frac{i}{2}\left(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2\right) \quad \text{and} \quad \hat{J}_3 = \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2\right) \tag{3.4}$$

such that they satisfy the $su(2)$ algebra

$$\left[\hat{J}_i, \hat{J}_j\right] = i\epsilon_{ijk}\hat{J}_k \tag{3.5}$$

in which $\hat{J}_3$ and the Casimir $\hat{\vec{J}}^2$ operators satisfy following eigen-value equations:

$$\hat{J}_3\big(|m\rangle \otimes |n\rangle\big) = j_3\big(|m\rangle \otimes |n\rangle\big), \quad \text{where } j_3 = \frac{1}{2}(m-n) \tag{3.6}$$

$$\hat{\vec{J}}^2\big(|m\rangle \otimes |n\rangle\big) = j(j+1)\big(|m\rangle \otimes |n\rangle\big), \quad \text{where } j = \frac{1}{2}(m+n) \tag{3.7}$$

Let us consider the Hamiltonian of two independent harmonic oscillators with same mass μ and frequency ω as follows:

$$\hat{H} = \frac{1}{2}\mu\omega^2\left(\hat{X}_1^2 + \hat{X}_2^2\right) + \frac{1}{2\mu}\left(\hat{P}_1^2 + \hat{P}_2^2\right) = \frac{\omega}{2}\left(\frac{\hat{\vec{P}}^2}{\mu\omega} + \mu\omega\hat{\vec{X}}^2\right) \tag{3.8}$$

This can be identified with the 2D harmonic oscillator on a commutative plane. We can then write the 4D phase space variables in terms of the respective ladder operators as:

$$\hat{X}_\alpha = \frac{1}{\sqrt{2\mu\omega}}\left(\hat{a}_\alpha + \hat{a}_\alpha^\dagger\right) \tag{3.9}$$

$$\hat{P}_\alpha = i\sqrt{\frac{\mu\omega}{2}}\left(\hat{a}_\alpha^\dagger - \hat{a}_\alpha\right) \quad \forall \ \alpha = 1,2 \tag{3.10}$$

Now we perform following canonical transformation(which would preserve the commutation relation $[\hat{X}_\alpha, \hat{P}_\beta] = i\delta_{\alpha\beta}$):

$$\hat{P}_\alpha \to \hat{p}_\alpha = \frac{\hat{P}_\alpha}{\sqrt{\mu\omega}} \quad \text{and} \quad \hat{X}_\alpha \to \hat{x}_\alpha = \sqrt{\mu\omega}\hat{X}_\alpha$$

With this our Hamiltonian becomes $\hat{H} = \frac{\omega}{2}\left(\hat{x}_1^2 + \hat{x}_2^2 + \hat{p}_1^2 + \hat{p}_2^2\right)$, which clearly enjoys SO(4) symmetry in the 4D phase space.

To understand the action of the Schwinger's angular momentum operators (3.4) let us calculate the following commutation relations:

$$[\hat{J}_3, \hat{x}_\alpha] = \frac{i}{2}\left(\delta_{\alpha 2}\hat{p}_2 - \delta_{\alpha 1}\hat{p}_1\right) \quad \text{and} \quad [\hat{J}_3, \hat{p}_\alpha] = \frac{i}{2}\left(\delta_{\alpha 1}\hat{x}_1 - \delta_{\alpha 2}\hat{x}_2\right) \qquad (3.11)$$

$$[\hat{J}_1, \hat{x}_\alpha] = \frac{-i}{2}\left(\delta_{\alpha 1}\hat{p}_2 + \delta_{\alpha 2}\hat{p}_1\right) \quad \text{and} \quad [\hat{J}_1, \hat{p}_\alpha] = \frac{i}{2}\left(\delta_{\alpha 1}\hat{x}_2 + \delta_{\alpha 2}\hat{x}_1\right) \qquad (3.12)$$

$$[\hat{J}_2, \hat{x}_\alpha] = \frac{i}{2}\epsilon_{\alpha\beta}\hat{x}_\beta \quad \text{and} \quad [\hat{J}_2, \hat{p}_\alpha] = \frac{i}{2}\epsilon_{\alpha\beta}\hat{p}_\alpha; \quad \text{with} \quad \alpha, \beta = 1, 2. \qquad (3.13)$$

From the above relations we can conclude that $\hat{J}_3$ generates simultaneous SO(2) rotation in x_1p_1 and x_2p_2 planes. Like-wise in case of $\hat{J}_1$ the rotation occurs in x_1p_2 and x_2p_1 planes, where as for $\hat{J}_2$ the rotation occurs in x_1x_2 and p_1p_2 planes.
This is however a part of the $SO(4)$ symmetry only, as the $SU(2)$ symmetry generated by this $\hat{\vec{J}}$'s corresponds to one of the $su(2)$'s in the decomposition of so(4) Lie algebra as $su(2) \oplus su(2)$. To get the other $su(2)$, one has to just flip the sign of one of the momenta components, say of p_2.

3.1.3 Schwinger's Angular momentum operators in non-commutative 2-D plane

As we have discussed the quantum Hilbert space comprises of Hilbert-Schmidt operators, and therefore any generic Hilbert-Schmidt operator can be written as:

$$\left|\Psi\right) = \sum_{m,n} C_{mn}|m\rangle\langle n| \in \mathcal{H}_q$$

The $\mathcal{H}_q$ can be identified with $\mathcal{H}_c \otimes \tilde{\mathcal{H}}_c$, where $\tilde{\mathcal{H}}_c$ is the dual of $\mathcal{H}_c$. Since, there is a one-to-one map between the basis $|m\rangle \otimes |n\rangle$ and $|m\rangle \otimes \langle n|$, the Hilbert spaces, span$\{|m\rangle \otimes |n\rangle\}$=$\mathcal{H}_c \otimes \mathcal{H}_c$ and $\mathcal{H}_q$ are isomorphic. In order to obtain the angular momentum operators acting on $\mathcal{H}_q$ i.e the counterpart of the expressions in (3.4), let us replace $\hat{a}_1$ with $\hat{B}_L$ and $\hat{a}_2^\dagger$(and not $\hat{a}_2$) with $\hat{B}_R$ and their respective Hermitian conjugates. Here the operators $\hat{B}_{L/R}$ and $\hat{B}^\ddagger_{L/R}$ are the operators on quantum Hilbert space $\mathcal{H}_q$ such that $\hat{B}_L = \hat{b} \otimes 1$ and $\hat{B}_R = 1 \otimes \hat{b}_R$(2.11). Hence we get:

$$\hat{J}_1 = \frac{1}{2}\left(\hat{B}_R\hat{B}_L + \hat{B}_L^\ddagger\hat{B}_R^\ddagger\right), \quad \hat{J}_2 = \frac{i}{2}\left(\hat{B}_R\hat{B}_L - \hat{B}_L^\ddagger\hat{B}_R^\ddagger\right) \quad \text{and} \quad \hat{J}_3 = \frac{1}{2}\left(\hat{B}_L^\ddagger\hat{B}_L - \hat{B}_R\hat{B}_R^\ddagger\right) \qquad (3.14)$$

Clearly they satisfy the su(2) algebra

$$\left[\hat{J}_i, \hat{J}_j\right] = i\epsilon_{ijk}\hat{J}_k, \qquad (3.15)$$

in which $\hat{J}_3$ satisfies the eigen-value equation:

$$\hat{J}_3|m\rangle\langle n| = j_3|m\rangle\langle n|, \quad \text{where } j_3 = \frac{1}{2}(m-n) \tag{3.16}$$

and the corresponding Casimir operator

$$\hat{J}^2 = \frac{1}{4}\left(\hat{B}_L^\ddagger \hat{B}_L + \hat{B}_R \hat{B}_R^\ddagger\right)\left(\hat{B}_L^\ddagger \hat{B}_L + \hat{B}_R \hat{B}_R^\ddagger + 2\right), \tag{3.17}$$

satisfying following eigen-value equation:

$$\hat{J}^2|m\rangle\langle n| = j(j+1)|m\rangle\langle n|, \quad \text{where } j = \frac{1}{2}(m+n) \tag{3.18}$$

These pair of eigen-values $j_3(3.16)$ *and* $j(j+1)(3.18)$ are exactly identical to the ones in (3.6) and (3.7). This suggests that both the basis states $\big|m\big\rangle \otimes \big|n\big\rangle \in \mathcal{H}_c \otimes \mathcal{H}_c$ and $\big|m,n\big) \equiv \big|m\big\rangle \otimes \big\langle n\big| \in \mathcal{H}_c \otimes \tilde{\mathcal{H}}_c$ can be labeled alternatively as $\big|j,j_3\big) = \big|m,n\big)$ where $j = \frac{1}{2}(m+n)$ and $j_3 = \frac{1}{2}(m-n)$. This is illustrated in following diagram(figure 1). We see from the diagram that all the discrete points with integer-valued coordinates (m,n) with $m,n \geq 0$ in the first quadrant represent a basis element $|m,n) \equiv |m\rangle\langle n|$ of the Hilbert space of states $\mathcal{H}_q$. Hence we find that the Hilbert space gets split up into states of constant j value lying on the straight line running parallel to j_3 axis e.g. $j = \frac{1}{2}$ line, $j = 1$ line and so on, and j_3 takes values within the interval $-j \leq j_3 \leq j$

We can then construct the usual ladder operators $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$, connecting all states belonging to fixed $'j'$ subspace of $(2j+1)$ dimension and the extremal points are annihilated by $\hat{J}_\pm$. This is just what happens in usual quantum mechanics.

We can now express the position and momentum operators in terms of $\hat{B}_L, \hat{B}_L^\ddagger$, $\hat{B}_R$ and $\hat{B}_R^\ddagger$. By making use of equation (2.35), the definition $\hat{B}_{L/R} = \frac{1}{\sqrt{2\theta}}(\hat{X}_1^{L/R} + i\hat{X}_2^{L/R})$ [5] and its Hermitian conjugate, and the adjoint action of momenta(2.35) we get:

$$\hat{X}_1^L = \sqrt{\frac{\theta}{2}}(\hat{B}_L + \hat{B}_L^\ddagger) \tag{3.19}$$

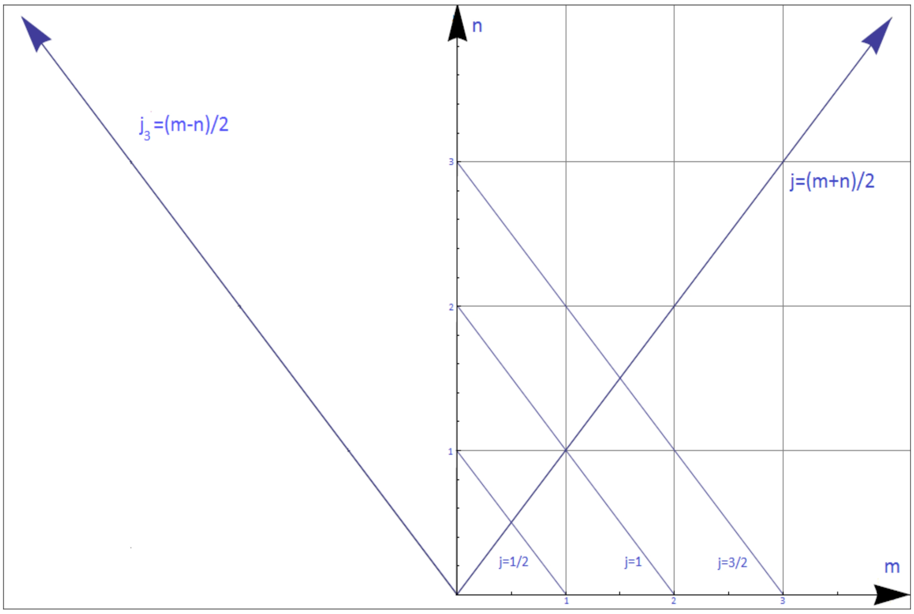
$$\hat{X}_2^L = i\sqrt{\frac{\theta}{2}}(\hat{B}_L^\ddagger - \hat{B}_L) \tag{3.20}$$

$$\hat{P}_1 = \frac{i}{\sqrt{2\theta}}\left(\hat{B}_L^\ddagger - \hat{B}_L - \hat{B}_R^\ddagger + \hat{B}_R\right) \tag{3.21}$$

$$\hat{P}_2 = \frac{1}{\sqrt{2\theta}}\left(\hat{B}_R^\ddagger + \hat{B}_R - \hat{B}_L^\ddagger - \hat{B}_L\right) \tag{3.22}$$

Further, the commuting coordinates [5] introduced as

$$\hat{X}_i^c = \frac{1}{2}\left(\hat{X}_i^L + \hat{X}_i^R\right) = \hat{X}_i + \frac{\theta}{2}\epsilon_{ij}\hat{P}_j \tag{3.23}$$



satisfying $[\hat{X}^c_i, \hat{X}^c_j] = 0$, can be expressed like-wise as:

$$\hat{X}^c_1 = \frac{1}{2}\sqrt{\frac{\theta}{2}}\left(\hat{B}_R + \hat{B}_L + \hat{B}^\ddagger_L + \hat{B}^\ddagger_R\right) \tag{3.24}$$

$$\hat{X}^c_2 = \frac{i}{2}\sqrt{\frac{\theta}{2}}\left(\hat{B}^\ddagger_L - \hat{B}_L + \hat{B}^\ddagger_R - \hat{B}_R\right) \tag{3.25}$$

To see the similarity of the action of Schwinger's angular momentum operators with their commutative counterpart(3.11-3.13), we first perform following canonical transformation, to construct dimensionless phase space variables:

$$\hat{X}^c_i \to \hat{x}^c_i = \frac{\hat{X}^c_i}{\sqrt{\theta}} \quad \text{and} \quad \hat{P}_i \to \hat{p}_i = \sqrt{\theta}\hat{P}_i \qquad \forall\ i = 1, 2 \tag{3.26}$$

Then we calculate following commutation relations:

$$[\hat{x}^c_i, \hat{J}_1] = \frac{i}{2}\left(\delta_{i1}\hat{p}'_1 - \delta_{i2}\hat{p}'_2\right) \quad \text{and} \quad [\hat{p}'_i, \hat{J}_1] = \frac{i}{2}\left(\delta_{i2}\hat{x}_2 - \delta_{i1}\hat{x}_1\right) \tag{3.27}$$

$$[\hat{x}^c_i, \hat{J}_2] = -\frac{i}{2}\left(\delta_{i1}\hat{p}'_2 + \delta_{i2}\hat{p}'_1\right) \quad \text{and} \quad [\hat{p}'_i, \hat{J}_2] = \frac{i}{2}\left(\delta_{i1}\hat{x}_2 + \delta_{i2}\hat{x}_1\right) \tag{3.28}$$

$$[\hat{x}^c_i, \hat{J}_3] = \frac{i}{2}\epsilon_{ij}\hat{x}_j \quad \text{and} \quad [\hat{p}'_i, \hat{J}_3] = \frac{i}{2}\epsilon_{ij}\hat{p}'_j \tag{3.29}$$

Here we have introduced $\hat{p}'_i = \frac{\hat{p}_i}{2}$. So we see that the angular momentum operators $\hat{J}_i$ are responsible for inducing simultaneous SO(2) rotations in two orthogonal planes of our 4-D phase space. In fact for $\hat{J}_1$ the rotation occurs in $X^c_1P'_1$ and $X^c_2P'_2$ planes. In case of $\hat{J}_2$ the rotation occurs in $X^c_1P'_2$ and $X^c_2P'_1$

planes, where as for $\hat{J}_3$ the rotation occurs in $X_1^c X_2^c$ and $P_1' P_2'$ planes. Here we can also appreciate that roles of commutative and non-commutative angular momentum operators are exchanged in following manner (where superscript C refers to commutative case(3.11-3.13) and NC for non-commutative(3.27-3.29) case):

$$\hat{J}_1^{NC} \leftrightarrow \hat{J}_3^C, \quad \hat{J}_2^{NC} \leftrightarrow \hat{J}_1^C \text{ and } \hat{J}_3^{NC} \leftrightarrow \hat{J}_2^C \tag{3.30}$$

Again like the commutative case this is also a part of $SO(4)$ rotation in X_1^c, X_2^c, P_1, P_2 space, as $\hat{J}^{NC}$'s corresponds to only one of the $su(2)$'s of $so(4) = su(2) \oplus su(2)$. In either case, this is just reminiscent of the splitting of the SL(2,C) algebra by taking suitable combinations of spatial rotations with Lorentz boost in $3+1$ dimensional Minkowski space-time, and SL(2,C) happens to be just the double cover of the Lorentz group SO(3,1),whose Euclidean version is the spin group Spin(4), which is the double cover of SO(4). Now we can obtain the rotation matrix for finite rotation $R_i(\lambda_i)$ generated by $\hat{J}_i$ in the $4D$ phase space by using above commutation relations (3.27-3.29) which is implemented by the following unitary transformation

$$\Xi \longrightarrow \Xi' = e^{-i\vec{\lambda}.\vec{J}} \Xi e^{i\vec{\lambda}.\vec{J}} = R(\vec{\lambda})\Xi \tag{3.31}$$

where the 4-component column matrix Ξ comprises of phase space variables and is given by:

$$\Xi = \left(\hat{x_1}^c, \hat{x_2}^c, \frac{\hat{p}_1}{2}, \frac{\hat{p}_2}{2} \right)^T \tag{3.32}$$

The matrix representations of the corresponding generators in the 4-dimensional representation are given by $\hat{J}_i = -i \frac{\partial R_i(\vec{\lambda})}{\partial \lambda_i}\Big|_{\vec{\lambda}=0}$. This yields

$$\hat{J}_1 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \hat{J}_2 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \hat{J}_3 = \frac{i}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.33}$$

The corresponding Casimir operator is then given by:

$$\hat{\vec{J}}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2 = \frac{1}{2}\left(\frac{1}{2} + 1\right) I \tag{3.34}$$

indicating that this corresponds to spin $\frac{1}{2}$ representation of $SU(2) \subset SO(4)$, which looks odd from a naive consideration, where it appears to be $j = \frac{3}{2}$ representation as the dimensionality is 2j+1=4. This is however easily explained by fact that the $\hat{J}'s$ are making simultaneous rotation in two different orthogonal planes through $j = \frac{1}{2}$ representation. However, the various basis states $|j, j_3)$ in fig.1 will transform according to the $j-$th representation of Schwinger's SU(2) generators.

Finally, we would like to point out the important role played by the commuting but unphysical position-like operators $\hat{X}_i^c$(3.23). It is only $\hat{X}_i^c$, rather than the physical non-commuting position operators $\hat{X}_i$, that transform covariantly under SU(2). It is only under SO(2) rotation, generated by $\hat{J}_3$, that both $\hat{X}_i^c$ and $\hat{X}_i$ transform covariantly. As has been shown in [14], that in 3D Moyal space also, it is only $\hat{X}_i^c$ that transform covariantly under SO(3) rotation, although $\hat{X}_i^c$ does not correspond to any physical observable [5]. Further, as we shall see later that this is also true for Bogoliubov transformation, where it undergoes simple scaling transformation and plays a very important role in our analysis, despite being an unphysical observable [5].

3.2 Simple Harmonic Oscillator in commutative plane

In commutative ($\theta = 0$) case the Hamiltonian is given by(3.8):

$$\hat{H}_I = \frac{1}{2}\mu\omega^2\hat{\vec{X}}^2 + \frac{1}{2\mu}\hat{\vec{P}}^2 \tag{3.35}$$

Using the form of position and momentum operators in terms of ladder operators (3.9-3.10), we get following result:

$$\hat{\vec{X}}^2 = \frac{1}{2\mu\omega}\left(\hat{a}_1\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_1^\dagger + \hat{a}_2\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_2^\dagger\right) + \frac{1}{\mu\omega}\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1\right) \tag{3.36}$$

$$\hat{\vec{P}}^2 = -\frac{\mu\omega}{2}\left(\hat{a}_1\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_1^\dagger + \hat{a}_2\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_2^\dagger\right) + \mu\omega\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1\right) \tag{3.37}$$

so that our Hamiltonian (3.35) becomes:

$$\hat{H}_I = \omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right), \tag{3.38}$$

which is clearly SU(2) invariant as it commutes with angular momentum operators $\hat{J}_i$(3.4). Using our most general state $|m\rangle \otimes |n\rangle$, we get following spectrum:

$$\hat{H}_I|m\rangle \otimes |n\rangle = \omega(2j+1)|m\rangle \otimes |n\rangle, \quad \text{where} \quad j = \frac{1}{2}(m+n), \tag{3.39}$$

which was expected as our state $|m\rangle \otimes |n\rangle$ corresponds to pair of simple harmonic oscillator (one ket for each SHO), and the energy of any SHO, say $|m\rangle$ is $\omega\left(m + \frac{1}{2}\right)$. SU(2) invariance of the Hamiltonian is also evident from above as spectrum is given by $E(j, j_3) = \omega(2j+1)$ and is independent of j_3 and (2j+1) being the dimensionality of j-th subspace.

3.3 Simple Harmonic oscillator in non-commutative plane

Here we shall first analyze the unphysical Hamiltonian of simple harmonic oscillator involving the commuting coordinates $\hat{X}_i^c$ [5], which is shown to yield the same spectrum as in the commutative case(3.39). This will then pave the way to analyze the physical oscillator involving non-commuting operators.

3.3.1 Unphysical SHO involving $(\hat{X}_i^c)$

Let us first construct following quadratic form:

$$\hat{H}_1 = \left(\frac{1}{\theta}(\hat{\vec{X}}^c)^2 + \frac{\theta}{4}\hat{\vec{P}}^2\right), \tag{3.40}$$

which can be regarded formally as a Hamiltonian of a harmonic oscillator with particular choices of mass and angular frequency parameters. But since it involves $\hat{X}_i^c$ (3.23), which is just a mathematically constructed commuting observable and is devoid of any physical interpretation [5], this oscillator is really unphysical in nature. Now using canonical transformation (3.26) above Hamiltonian becomes:

$$\hat{H}_1 = \left((\hat{x}_1^c)^2 + (\hat{x}_2^c)^2 + \left(\frac{\hat{p_1}}{2}\right)^2 + \left(\frac{\hat{p_2}}{2}\right)^2\right) = \Xi^T\Xi \;\;\textsf{using}\;\; (3.32), \tag{3.41}$$

which is manifestly invariant under one of the SU(2) symmetry group in the decomposition of SO(4) i.e $SO(4) = SU(2)\otimes SU(2)$ analogous to commuting case.[1] This is evident from commutation relations (3.27-3.29) and unitary transformation (3.31).
Then (using 3.21-3.25), one gets

$$(\hat{\vec{X}}^c)^2 = (\hat{X}_1^c)^2 + (\hat{X}_2^c)^2 = \frac{\theta}{2}\left(\hat{B}_L^\ddagger\hat{B}_L + \hat{B}_R\hat{B}_R^\ddagger + 1 + \hat{B}_L\hat{B}_R^\ddagger + \hat{B}_L^\ddagger\hat{B}_R\right) \tag{3.42}$$

$$\hat{\vec{P}}^2 = (\hat{P}_1)^2 + (\hat{P}_2)^2 = \frac{2}{\theta}\left(\hat{B}_L^\ddagger\hat{B}_L + \hat{B}_R\hat{B}_R^\ddagger + 1 - \hat{B}_L\hat{B}_R^\ddagger - \hat{B}_L^\ddagger\hat{B}_R\right), \tag{3.43}$$

so that our Hamiltonian (3.41) becomes:

$$\hat{H}_1 = \frac{1}{2}\left(\frac{2}{\theta}(\hat{\vec{X}}^c)^2 + \frac{\theta}{2}\hat{\vec{P}}^2\right) = \left(\hat{B}_L^\ddagger\hat{B}_L + \hat{B}_R\hat{B}_R^\ddagger + 1\right) \tag{3.44}$$

Hence for a general basis state $|m\rangle\langle n|$ we get following spectrum:

$$\hat{H}_1|j, j_3) = \hat{H}_1|m\rangle\langle n| = (m+n+1)|m\rangle\langle n| = (2j+1)|j, j_3), \;\;\textsf{using}\;\; (\mathbf{??}) \tag{3.45}$$

and the spectrum can be read-off easily, yielding just $(2j+1)$.
We then consider following Hamiltonian, which is a slight variant for the

[1] Actually, it is invariant under the entire SO(4) group itself; the other SU(2) symmetry is realized by flipping the sign of one of the momenta component say of $\hat{p}_2$, as was observed earlier. In our present analysis, we shall not make use of this second SU(2). We shall therefore continue to refer to the first SU(2) symmetry only.

non-physical planar harmonic oscillator(3.40), by introducing mass (μ) and angular frequency (ω):

$$\hat{H}_2 = \frac{1}{2\mu}\hat{\vec{P}}^2 + \frac{1}{2}\mu\omega^2(\hat{\vec{X}}^c)^2 \tag{3.46}$$

Naively we expect that this Hamiltonian to be equivalent to $\hat{H}_I$(3.35), however this is not the case because there is a major difference in the construction of actually commuting (3.35) and mathematically constructed commuting $\hat{X}_i^c$ coordinates. The annihilation operators a_1 and a_2 occurring in(3.9) act on the left and right slots separately, annihilating the vacuum $|0\rangle \otimes |0\rangle$. In this case we can define creation and annihilation operators in an analogous manner to(3.9 and 3.10) as:

$$\hat{C}_i^\dagger = \frac{1}{\sqrt{2\mu\omega}}\left(\mu\omega\hat{X}_i^c - i\hat{P}_i\right), \quad \hat{C}_i = \frac{1}{\sqrt{2\mu\omega}}\left(\mu\omega\hat{X}_i^c + i\hat{P}_i\right) \quad \forall \;\; i = 1, 2, \tag{3.47}$$

which, however involves $\hat{X}_i^c$ and $\hat{P}_i$ acting simultaneously in the left and right slots. One can see that the relevant ground state here i.e. $|0\rangle \otimes \langle 0|$ is annihilated by C_i only under a special choice of parameters μ and ω, which we call as a *critical point* :

$$\mu_0 = \frac{\omega_0}{2} = \frac{1}{\sqrt{\theta}} \tag{3.48}$$

Also we notice that at this critical point $\hat{H}_2$(3.46) reduces to $\hat{H}_1$(3.40), up to an overall constant: $\hat{H}_2 = \frac{1}{\sqrt{\theta}}\hat{H}_1$ and furthermore, the ladder operators $\hat{C}_i$ and $\hat{C}_i^\dagger$ (3.47) reduce to the following linear combination of operators $\hat{B}_L$ and $\hat{B}_R^\ddagger$:

$$\hat{C}_1 = \frac{1}{\sqrt{2}}\left(\hat{B}_L + \hat{B}_R^\ddagger\right); \quad \hat{C}_2 = \frac{-i}{\sqrt{2}}\left(\hat{B}_L - \hat{B}_R^\ddagger\right) \tag{3.49}$$

It is therefore interesting to see how our Hamiltonian(3.46) behaves under above condition(3.48). We find that:

$$\hat{H}_2 = \omega_0\left(\frac{1}{\theta}(\hat{\vec{X}}^c)^2 + \frac{\theta}{4}\hat{\vec{P}}^2\right) = \omega_0\left(\hat{B}_L^\ddagger\hat{B}_L + \hat{B}_R\hat{B}_R^\ddagger + 1\right), \tag{3.50}$$

so that its action on $|m, n) \equiv |m\rangle\langle n| \equiv |j, j_3)$ gives

$$\hat{H}_2|m\rangle\langle n| = \omega_0(2j+1)|m\rangle\langle n| \tag{3.51}$$

yielding the spectrum as

$$E(j, j_3) = \omega_0(2j+1) \tag{3.52}$$

matching with the spectrum of commutative case(3.39), as was expected. It should be mentioned here that the contrasting situation in the non-commutative context from the commutative case discussed in the previous section becomes clear here. The mass and frequency parameter μ and ω requires to be compatible with each other, as in (3.48) and also with the natural mass scale introduced by $(\frac{1}{\sqrt{\theta}})$.

Now in order to calculate the spectrum of the Hamiltonian $\hat{H}_2$(3.46) for arbitrary value of parameters μ and ω, we write it in following form by using(3.42-3.43):

$$\hat{H}_2 = \alpha\left(\hat{B}_L^\ddagger \hat{B}_L + \hat{B}_R^\ddagger \hat{B}_R\right) + \beta\left(\hat{B}_L^\ddagger \hat{B}_R + \hat{B}_R^\ddagger \hat{B}_L\right) \tag{3.53}$$

where

$$\alpha = \frac{\mu\omega^2\theta}{4} + \frac{1}{\mu\theta} \quad \text{and} \quad \beta = \frac{\mu\omega^2\theta}{4} - \frac{1}{\mu\theta} \tag{3.54}$$

Clearly Hamiltonian contains off-diagonal terms. To diagonalize the Hamiltonian, let us introduce the new set of operators $\hat{B}'_L$ and $\hat{B}'_R$, which are related to un-primed ones via Bogoliubov transformation as:

$$\begin{pmatrix} \hat{B}'_L \\ \hat{B}'_R \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \hat{B}_L \\ \hat{B}_R \end{pmatrix} \tag{3.55}$$

This ensures $[\hat{B}'_L, \hat{B}'^\ddagger_L] = -[\hat{B}'_R, \hat{B}'^\ddagger_R] = 1$ like their un-primed counterparts. With above transformation our Hamiltonian becomes:

$$\begin{aligned}
\hat{H}_2 = \quad & \left[\alpha\left(\cosh^2\phi + \sinh^2\phi\right) - 2\beta\sinh\phi\cosh\phi\right]\left(\hat{B}'^\ddagger_L \hat{B}'_L + \hat{B}'_R \hat{B}'^\ddagger_R\right) \\
+ \quad & \left[\beta\left(\cosh^2\phi + \sinh^2\phi\right) - 2\alpha\sinh\phi\cosh\phi\right]\left(\hat{B}'^\ddagger_L \hat{B}'_R + \hat{B}'^\ddagger_R \hat{B}'_L\right) \\
+ \quad & 2\alpha\sinh^2\phi - 2\beta\sinh\phi\cosh\phi + \alpha
\end{aligned}$$

For diagonalizing the Hamiltonian, let us set the coefficient of the non diagonal term zero i.e.

$$\left[\beta\left(\cosh^2\phi + \sinh^2\phi\right) - 2\alpha\sinh\phi\cosh\phi\right] = 0, \tag{3.56}$$

which gives

$$\coth\phi + \tanh\phi = \frac{2\alpha}{\beta} \tag{3.57}$$

and the Hamiltonian becomes:

$$\hat{H}_2 = 2\beta\sinh\phi\cosh\phi\left[\left(\frac{\alpha}{\beta}\right)^2 - 1\right]\left(\hat{B}'^\ddagger_L \hat{B}'_L + \hat{B}'_R \hat{B}'^\ddagger_R + 1\right) \tag{3.58}$$

We can now determine the spectrum easily by acting above Hamiltonian on an appropriate basis $|m\rangle'\langle n|'$ which is related to older basis $|m\rangle\langle n|$ by a unitary transformation, as we will show later.

Now we proceed to show that the Bogoliubov transformation used above is equivalent to a certain canonical transformation. To achieve this we expand the ladder operators in matrix equation(3.55) in terms of position operators acting from left and right to get the following set of equations:

$$\begin{aligned} \hat{X}_1'^L &= \cosh\phi \hat{X}_1^L + \sinh\phi \hat{X}_1^R \ ; \ \hat{X}_2'^L = \cosh\phi \hat{X}_2^L + \sinh\phi \hat{X}_2^R \\ \hat{X}_1'^R &= \cosh\phi \hat{X}_1^R + \sinh\phi \hat{X}_1^L \ ; \ \hat{X}_2'^R = \cosh\phi \hat{X}_2^R + \sinh\phi \hat{X}_2^L , \end{aligned} \tag{3.59}$$

which when re-expressed in terms of $\hat{X}_i^c$(3.23) and $\hat{P}_i$(2.35), takes the following simple form:

$$\hat{X}_i'^c = e^{\phi} \hat{X}_i^c \ ; \ \ \hat{P}_i' = e^{-\phi} \hat{P}_i \tag{3.60}$$

Clearly, this is a canonical transformation in the $(\hat{X}_i^c, \hat{P}_j)$ space, where $\hat{X}_i^c$'s are scaled and $\hat{P}_i$'s are de-scaled. The scaling factor e^{ϕ} can be determined easily by solving the quadratic equation in $\tanh\phi$(3.57). The two roots are evidently reciprocal to each other and choosing one smaller than unity yields the value

$$\tanh\phi = \frac{1}{\beta}(\alpha - \omega) \tag{3.61}$$

Putting back the value of α and β we finally get:

$$e^{\phi} = \sqrt{\frac{\mu\omega\theta}{2}}, \tag{3.62}$$

which clearly reduces to unity at the critical point(3.48). This has some points of contact with the Eq.(18) in [9]. It can be easily checked that under this scaling transformation, the spatial non-commutative parameter preserves its value: $[\hat{X}_i, \hat{X}_j] = [\hat{X}_i', \hat{X}_j'] = i\theta\epsilon_{ij}$. It is interesting to note here that the canonical relation(3.60) is equivalent to following unitary transformation:

$$\hat{X'}_i^c = e^{\phi} \hat{X}_i^c = e^{-\frac{i}{2}\phi\hat{D}} \hat{X}_i^c e^{\frac{i}{2}\phi\hat{D}}, \tag{3.63}$$

where $\hat{D} = \frac{1}{2}(\hat{X}_i^c \hat{P}_i + \hat{P}_i \hat{X}_i^c) = i(\hat{B}_L^\ddagger \hat{B}_R - \hat{B}_L \hat{B}_R^\ddagger)$ is the dilatation operator. Correspondingly, one finds

$$\hat{B}_L' \pm \hat{B}_R' = e^{-\frac{i}{2}\phi\hat{D}} (\hat{B}_L \pm \hat{B}_R) e^{\frac{i}{2}\phi\hat{D}} = e^{\pm\phi}(\hat{B}_L \pm \hat{B}_R) \tag{3.64}$$

and the basis states in $\mathcal{H}_q$ undergo following transformation:

$$|m\rangle\langle n| \rightarrow |m\rangle'\langle n|' = e^{i\phi\hat{D}} |m\rangle\langle n| = e^{\phi(\hat{B}_R^\ddagger \hat{B}_L - \hat{B}_L^\ddagger \hat{B}_R)} |m\rangle\langle n| \tag{3.65}$$

Hence instead of doing Bogoliubov transformation, we could have equivalently done the canonical transformation(3.60) to re-write the Hamiltonian(3.46) in a manifestly SU(2) invariant form in the primed frame as:

$$\hat{H}_2 = \omega \left(\frac{1}{\theta} (\hat{\vec{X}}'^c)^2 + \frac{\theta}{4} (\hat{\vec{P}}')^2 \right) = \omega \left(B_L'^\ddagger B_L' + B_R' B_R'^\ddagger + 1 \right) \ \ \text{using} \ \ (3.62) \tag{3.66}$$

We can check that the coefficient of Hamiltonian(3.58) is indeed ω, by using the values of $\tanh\phi, \alpha$ and β so that one gets the spectrum of the same form(3.52) with replacement $\omega_0 \to \omega$:

$$E(j, j_3) = \omega(2j+1) \tag{3.67}$$

This demonstrates that $SU(2)$ symmetry is preserved even at a point away from the critical point (3.48) in the $\mu - \omega$ plane. As we shall show in the next sub-section that this $SU(2)$ symmetry will, however, be broken in the physical oscillator, where unphysical commuting " position -like" observable $\hat{X}_i^c$ in (3.46) will be replaced by the physical non-commuting position operator $\hat{X}_i$ in the Hamiltonian below (3.68).

3.3.2 Physical SHO involving non-commuting coordinates

Let us finally consider the Hamiltonian for the physical simple harmonic oscillator involving non-commuting position operators:

$$\hat{H}_3 = \frac{1}{2\mu}\hat{\vec{P}}^2 + \frac{1}{2}\mu\omega^2(\hat{\vec{X}})^2 \tag{3.68}$$

Using(3.23), this can be re-expressed as:

$$\hat{H}_3 = \frac{1}{2\mu}\hat{\vec{P}}^2 + \frac{1}{2}\mu\omega^2\left[(\hat{\vec{X}}^c)^2 + \frac{\theta^2}{4}\hat{\vec{P}}^2 + 2\theta\hat{J}_3\right], \tag{3.69}$$

which can be brought to the same form as $\hat{H}_2$(3.46), up to a Zeeman term as:

$$\hat{H}_3 = \frac{1}{2\mu'}\hat{\vec{P}}^2 + \frac{1}{2}\mu'\omega'^2(\hat{\vec{X}}^c)^2 + \mu\theta\omega^2\hat{J}_3, \tag{3.70}$$

with re-normalized parameters μ' and ω', satisfying $\mu\omega^2 = \mu'\omega'^2$, are given by:

$$\frac{1}{\mu'} = \frac{1}{\mu} + \frac{\mu\omega^2\theta^2}{4} \quad \text{and} \quad \omega'^2 = \omega^2\left(1 + \frac{\mu^2\omega^2\theta^2}{4}\right) \tag{3.71}$$

The presence of Zeeman term however, breaks its SU(2) symmetry to U(1), as the Hamiltonian does not commute with angular momentum operators $\hat{J}_1$ and $\hat{J}_2$ anymore. But the Hamiltonian is still symmetric under rotation generated by $\hat{J}_3$. Now we can again do the same canonical transformation(3.60), with new parameters μ' and ω' satisfying

$$e^{2\phi} = \frac{\mu'\omega'\theta}{2}, \tag{3.72}$$

and keeping in mind that $\hat{J}_3 = \epsilon_{ij}\hat{X}_i^c\hat{P}_j$ remains unaffected under this transformation, to get:

$$\hat{H}_3 = \omega'\left(B_L'^\ddagger B_L' + B_R' B_R'^\ddagger + 1\right) + \frac{\mu'\theta\omega'^2}{2}\left(B_L'^\ddagger B_L' - B_R' B_R'^\ddagger\right) \tag{3.73}$$

The corresponding spectrum can be easily read as

$$E(j, j_3) = \omega'(2j+1) + \theta\mu'\omega'^2 j_3 \tag{3.74}$$

The presence of j_3-dependent term is clearly due to the presence of the $SU(2)$- symmetry breaking Zeeman term in in (3.70). The occurrence of these renormalised parameters ω' and μ' in the spectrum (3.74) indicates that these parameters, rather than their 'bare' counterparts ω and μ occurring in (3.68), are the observable quantities of the theory. Indeed, it can be easily checked, using that for μ' and ω' at the critical point (3.48) i.e. $\mu' = \frac{\omega'}{2} = \frac{1}{\sqrt{\theta}}$, the corresponding bare mass diverges $\mu \to \infty$ and frequency $\omega \to 0$. Nevertheless, in order to establish compatibility with [8], it is desirable to express (3.73) in terms of the bare quantities μ and ω. Using the definition of μ' and ω'(3.71), we can easily write above Hamiltonian (3.73) as:

$$\hat{H}_3 = \frac{\lambda_+}{2\mu}\left(2B_L'^{\ddagger}B_L' + 1\right) + \frac{\lambda_-}{2\mu}\left(2B_R'B_R'^{\ddagger} + 1\right), \tag{3.75}$$

where λ_+ and λ_- are given by:

$$\lambda_{\pm} = \frac{1}{2}\left(\mu\omega\sqrt{4+\mu^2\omega^2\theta^2} \pm \mu^2\omega^2\theta\right) \tag{3.76}$$

This clearly reproduces spectrum of [8] calculated using a different approach. We therefore turn our attention now towards the determination of the ground state. To begin with, let us note that, unlike in the commutative case, where the ground $|0\rangle\otimes|0\rangle$ is annihilated by operators $\hat{a}_i$ with $\hat{a}_1$ and $\hat{a}_2$ annihilating the ground states $|0\rangle$ occurring in the left and right sectors respectively in $|0\rangle\otimes|0\rangle$, in the non-commutative case the corresponding ground state $|0\rangle\langle 0|$ is not annihilated by operators $\hat{C}_i$(3.47) except at the critical point(3.48). Besides, the form of $\hat{C}_i$ (3.49) at the critical point (3.48) clearly indicates that these $\hat{C}_i$'s have simultaneous actions on both the slots. Indeed, the analogous expressions for the commutative case will be combinations $\frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2)$ and $\frac{-i}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2)$ respectively. But away from the critical point (3.48) i.e. for general values of μ and ω there must exist another ground state $|0\rangle'\langle 0|'$, related to the older one as in (3.65), which should be annihilated by a transformed $\hat{C}_i'$.

$$\hat{C}_i'|0\rangle'\langle 0|' = 0 \ \ ; \ \ |0\rangle'\langle 0|' = e^{-\phi(\hat{B}_L^{\ddagger}\hat{B}_R - \hat{B}_L\hat{B}_R^{\ddagger})}|0\rangle\langle 0| \tag{3.77}$$

where $\hat{C}_i'$ is defined analogously as in (3.49)

$$\hat{C}_1' = \frac{1}{\sqrt{2}}\left(\hat{B}_L' + \hat{B}_R'^{\ddagger}\right); \ \ \hat{C}_2' = \frac{-i}{\sqrt{2}}\left(\hat{B}_L' - \hat{B}_R'^{\ddagger}\right) \tag{3.78}$$

in terms of $\hat{B}_L'$ and $\hat{B}_R'$ introduced in (3.55). Equivalently,

$$\hat{B}_L'|0\rangle'\langle 0|' = 0 \text{ and } \hat{B}_R'^{\ddagger}|0\rangle'\langle 0|' = 0 \tag{3.79}$$

which are clearly not independent of each other, as they are related by Hermitian conjugation. One can easily see at this stage that either of these equations are trivially satisfied by making use of (3.64,3.65,3.77). But, before going ahead with the straightforward computation of $|0\rangle'\langle 0|'$ and compare with the normalised version of the ground state obtained in [8], we would like to point out certain exact parallel between the approach followed here with [8]. To that end, note that, using these equations (3.79) and making use of (3.55) we get

$$\hat{b}|0\rangle'\langle 0|' = -\tanh\phi|0\rangle'\langle 0|'\hat{b} \tag{3.80}$$

and its Hermitian conjugate. Then using (3.72) and the counterpart of (3.61) i.e $\tanh\phi = \frac{1}{\beta'}(\alpha' - \omega')$, where α' and β' defined analogously as in (3.54), with the replacements $\mu \to \mu'$ and $\omega \to \omega'$, yielding

$$(1 + \theta\lambda_+)\hat{b}|0\rangle'\langle 0|' = |0\rangle'\langle 0|'\hat{b} \tag{3.81}$$

just from the first equation in (3.79). As expected, the second equation of (3.79) yields the Hermitian conjugate of (3.81) and these pair of equations exactly reproduces the defining equations (3.53 , 3.55) of ref [8]. One can then proceed analogously, as in [8] to get the normalized version of the ground state as

$$\psi_0 \equiv |0\rangle'\langle 0|' = \frac{1}{\cosh\phi}e^{\gamma(\hat{b}^\dagger\hat{b})} \tag{3.82}$$

where

$$\gamma = \log(1 - \theta\lambda_-) = \log(-\tanh\phi) \tag{3.83}$$

so that

$$(\psi_0|\psi_0) = \frac{1}{\cosh^2\phi}Tr_c(e^{2\gamma\hat{b}^\dagger\hat{b}}) = 1. \tag{3.84}$$

Here, we have made use of the identities

$$(1 + \theta\lambda_+)(1 - \theta\lambda_-) = 1 \text{ and } \tanh\phi = \frac{\lambda_+}{\lambda_-} = \frac{1}{1 + \theta\lambda_+} \tag{3.85}$$

Now expanding (3.82), using the complete set of basis $\sum_{m=0}^{\infty}|m\rangle\langle m| = 1\!\!1$, as

$$\psi_0 = \text{sech}\phi\Sigma_{m=0}^{\infty}(-1)^m(\tanh\phi)^m|m\rangle\langle m| = \text{sech}\phi\left(|0\rangle\langle 0| - \tanh\phi|1\rangle\langle 1| + ...\right) \tag{3.86}$$

we can easily see that the Taylor's expansion of $\tanh\phi$, $\text{sech}\phi$ and their product exactly reproduces (3.77)

$$\begin{aligned} |0\rangle'\langle 0|' &= e^{-\phi(\hat{B}_L^\ddagger\hat{B}_R - \hat{B}_L\hat{B}_R^\ddagger)}|0\rangle\langle 0| \\ &= (1 - \frac{\phi^2}{2!} + \frac{5}{4!}\phi^4 - ..)|0\rangle\langle 0| + (\phi - \frac{5\phi^3}{3!} + \frac{5}{4!}\phi^5 - ..)|1\rangle\langle 1| + .. = \psi_0 \end{aligned}$$

Finally, note that the sign alternates from one term to another, because of the presence of the factor $(-1)^m$ in ψ_0 in (3.86) and therefore cannot be interpreted as a mixed density matrix, from the perspective of classical Hilbert space $\mathcal{H}_c$.

3.4 Time reversal

We proceed to check how the Hamiltonian $\hat{H}_3$, defined through (3.73), behaves under time reversal. Using the defination of $\hat{X}_i^c$ defined through (3.23), $\hat{X}_i^L$ can be written as-

$$\hat{X}_i^L = \hat{X}_i^c + \frac{\theta}{2}\epsilon_{ij}\hat{P}_j \tag{3.87}$$

Now using the fact that $\hat{X}_i^c$, which is mathematically commuting coordinate, does not undergo any change: $\hat{X}_i^c \to \hat{X}_i^c$ under time reversal and $\hat{P}_i \to -\hat{P}_i$ under time reversal, the following transformation property of $\hat{X}_i^L$ under time reversal can be obtained

$$\Theta\hat{X}_i^L\Theta^{-1} = \hat{X}_i^L + \theta\epsilon_{ij}\hat{P}_j, \tag{3.88}$$

where Θ is the time reversal operator. Similarly, we obtain the transformation property of $\hat{X}_i^R$ under time reversal to be

$$\Theta\hat{X}_i^R\Theta^{-1} = \hat{X}_i^R - \theta\epsilon_{ij}\hat{P}_j \tag{3.89}$$

Further, we proceed to check the transformation property of $\hat{B}_{L/R}$ (defined earlier)

$$\Theta\hat{B}_{L/R}\Theta^{-1} = \hat{B}^{\ddagger}_{R/L} \tag{3.90}$$

$$\Theta\hat{B}^{\ddagger}_{L/R}\Theta^{-1} = \hat{B}_{R/L} \tag{3.91}$$

So that the Hamiltonian $\hat{H}_3$ (3.68) and angular momentum $\hat{J}_3$, under time reversal transform to

$$\begin{aligned}
\Theta\hat{H}_3\Theta^{-1} &= \omega'(\hat{B}'^{\ddagger}_L\hat{B}'_L + \hat{B}'_R\hat{B}'^{\ddagger}_R + 1) - \frac{\mu'\theta\omega'^2}{2}(\hat{B}'^{\ddagger}_L\hat{B}'_L - \hat{B}'_R\hat{B}'^{\ddagger}_R) \neq \hat{H}_3 \\
\Theta\hat{J}_3\Theta^{-1} &= -(\hat{B}'^{\ddagger}_L\hat{B}'_L - \hat{B}'_R\hat{B}'^{\ddagger}_R) = -\hat{J}_3
\end{aligned} \tag{3.92}$$

This clearly indicates that time reversal symmetry is broken and that the Zeeman term is solely responsible for breaking the both SU(2) and time reversal symmetry.

Chapter 4

Distance function on Fuzzy/noncommutative sphere

Here we would like to discuss the work which we have carried out on the "distance calculation on fuzzy sphere" through the formalism of noncommutative geometry described earlier and the analytic formula for the Connes infinitesimal distance function derived in [21] and also discussed in the distance calculation of Moyal plane (2.39).

The spatial noncommutativity of Lie algebra type, where the coordinates satisfy SU(2) algebra is given by

$$[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k \tag{4.1}$$

and is similar to the noncommutativity satisfied by the angular momentum operators $\hat{J}$. As earlier discussed, this type of spatial noncommutativity is realized in the presence of magnetic field produced by magnetic monopole. One can define the simultaneous eigenstate $|n, n_3\rangle$ of radius-squared operator ($\hat{\vec{x}}^2$), which is Casimir operator, and the $\hat{x}_3$ operator similar to the case of angular momentum where one defines the simultaneous eigenstate $|j, j_3\rangle$ of $\hat{J}^2$ and $\hat{J}_3$. The notion of 3D configuration space is now replaced with the Hilbert space spanned by the ket $|n, n_3\rangle$. The eigenvalues of the radius-squared operator are $\lambda^2 n(n+1)$ and the eigenvalues of $\hat{x}_3$ are λn_3 where $-n \leq n_3 \leq n$. The radius is quantized and each sphere with the fixed radius is called the fuzzy sphere.

In this work we apply the Connes formalism of the noncommutative geometry for the computation of infinitesimal distance between any two generic "points" (taken to be pure states of an appropriate C^* algebra) on these fuzzy sphere. We construct the spectral triplet $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ discussed earlier, for these fuzzy spheres and apply the Connes distance formula for the computation of infintesimal distance. Also, using the construction of perelemov of the coherent states on these spheres [22] and applied the formalism of [21] in these coherent states to obtain the metric in stereographic variables.

In the next section we will discuss the construction of Dirac operator on the spheres S^3 and S^2 and subsequently the construction of spectral triplet and the distance computation will be discussed for the noncommutative case.

4.1 Construction of Dirac operators on S^3 and S^2 and noncommutative geometry

In this section the construction of Dirac operator for commutative 2 and 3-spheres i.e. S^2 and S^3 will be discussed. This construction then will pave the way to generalize it for noncommutative case.

4.1.1 Commutative case

The C_0^2 manifold which is isomorphic to $R^4 - \{0\}$ is defined through the nonzero complex doublets.

$$C_0^2 = \{\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \in C^2 | \chi \neq 0\} \tag{4.2}$$

The CP^1 manifold consist of nonzero complex doublet with the identification

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \sim \begin{pmatrix} \lambda\chi_1 \\ \lambda\chi_2 \end{pmatrix}; \quad C \ni \lambda \neq 0 \tag{4.3}$$

which are complex lines passing through the origin in C^2. The representative points on each line is choosen by first imposing the restriction $\chi^\dagger\chi = r =$ *constant*.

$$\{\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \in C^2 \; |\chi^\dagger\chi = |\chi_1|^2 + |\chi_2|^2 = r\} \tag{4.4}$$

Which clearly is S^3. With $r = 1$ we have $S^3 \sim SU(2)$. We then have still U(1) freedom left in the constraint in (4.4) with respect to the transformation

$$\chi_i \rightarrow e^{i\theta}\chi_i \tag{4.5}$$

If we choose the gauge from this U(1) freedom, we get $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ i.e. choosing the section from the U(1) bundle. Two particular ways of doing this are -

1. Choose a section from the U(1) bundle where χ_1 is real- $\chi_1^* = \chi_1$ in the neighborhood $U_+ : \chi_1 \neq 0$. With this the constraint in (4.4) reduces to -

$$\chi_1^2 \; + \; |\chi_2|^2 = r \tag{4.6}$$

Which clearly is S^2. This can equivalently be obtained by the transformation

$$U_+ : \chi \to \chi' = \begin{pmatrix} \chi_1' \\ \chi_2' \end{pmatrix} = \left(\frac{\chi_1^*}{\chi_1}\right)^{1/2} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \left(\frac{\chi_1^*}{\chi_1}\right)^{1/2} \chi \qquad (4.7)$$

with the identification (4.3). Now choosing $\lambda = 1/\chi_1$ we can write

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \approx \begin{pmatrix} 1 \\ \rho \end{pmatrix} \qquad (4.8)$$

where ρ is a complex number representing the stereographic projection of a point of the sphere from the south pole. Thus U_+ corresponds to the northern hemisphere.

2. Choose a section from the U(1) bundle where χ_2 is real- $\chi_2 = \chi_2^*$ in the neighborhood $U_- : \chi_2 \neq 0$. With this the constraint in (4.4) reduces to -

$$|\chi_1|^2 + \chi_2^2 = r, \qquad (4.9)$$

which again clearly is S^2. Here also similar to the previous case the doublet can be written as

$$\chi = \begin{pmatrix} \eta \\ 1 \end{pmatrix} \qquad (4.10)$$

where $\eta = \chi_1/\chi_2$ is a complex number representing the stereographic projection of a point of the sphere from the north pole. Thus U_- corresponds to the southern hemisphere.

Both the chart U_+ and U_- is interepreted as covering the northern hemisphere and southern hemisphere respectively. The chart U_+ misses only south pole and the chart U_- misses only north pole. In the overlapping region $U_+ \bigcap U_-$, one therefore has $\eta = 1/\rho$. For unit sphere S^2, the transition function relating sections $\psi_\pm$ of the tautological line bundle thus takes the form:

$$\psi_+ = e^{i\varphi}\psi_- \qquad (4.11)$$

The cartesian coordiantes of the S^2 in the respective chart is defined through the Hopf map -

$$x_i' = \chi'^\dagger \sigma_i \chi' \in U_+, i = 1, 2, 3 \qquad (4.12)$$
$$x_i'' = \chi'^\dagger \sigma_i \chi' \in U_-, i = 1, 2, 3 \qquad (4.13)$$

where $\sigma' s$ are the pauli matrices. In the region $U_+ \cap U_-$ the coordinates x_i' and x_i'' coincide i.e. $x_i' = x_i'' = x$, as they are gauge invariant.

The $\chi \in S^3$ can be parametrized in terms of Euler's angle as-

$$\chi_1 = r^{\frac{1}{2}} \cos\frac{\theta}{2} e^{\frac{i}{2}(\varphi+\psi)} \tag{4.14}$$

$$\chi_2 = r^{\frac{1}{2}} \sin\frac{\theta}{2} e^{-\frac{i}{2}(\varphi-\psi)} \tag{4.15}$$

The $\chi', \chi'' \in S^2$ is then obtained as- [1]

$$\chi' = e^{-\frac{i}{2}(\psi+\varphi)}\chi \tag{4.16}$$

$$\chi'' = e^{\frac{i}{2}(\varphi-\psi)}\chi \tag{4.17}$$

In the region $U_+ \cap U_-$ we have the transition map to go from one chart to other.

$$\chi' = e^{i\varphi}\chi'', \tag{4.18}$$

which clearly has the same form as (4.11). Before, we construct the spinor-bundle , let us re-look at (4.8) , where ρ representating a point in the complex plane which coordinatises a point of CP^1. Now if we consider a map $\rho \to \rho^n$ (say), with 'n' being a positive integer, then it is clear that ρ^n winds around the complex plane 'n' times in the counter-clock-wise direction. Like-wise, η-occuring in (4.10) will also wind around 'n' times -but in the clock-wise direction. For complex conjugate variables ρ^* and η^* these are just opposites, as one can easily see. Consequently considering any function

$$f(\rho, \rho^*) = (\rho^*)^{m_1}\rho^{m_2} \tag{4.19}$$

given in terms of the monomial, and defined in the region U_+, the total winding number is $m_2 - m_1$, which is nothing but the algebric sum of the winding numbers. For generic cases this function can not be regarded as a section in the tautological line-bundle, rather on other non-canonical U(1) bundle over S^2, with the asscociated transition function $e^{in\varphi}$

$$\psi_+ = e^{in\varphi}\psi_- \tag{4.20}$$

It is only for n=1, that one can identify the bundle to be S^3, with this background let us now consider first spinor bundle over the C_0^2 manifold whose sections are of the form-

$$\Psi = \begin{pmatrix} \psi_1(\chi_\alpha, \chi_\beta^*) \\ \psi_2(\chi_\alpha, \chi_\beta^*) \end{pmatrix} \qquad \alpha, \beta = 1, 2, \tag{4.21}$$

where ψ_α are polynomials of homogeneous function $\chi_\alpha' s$ and χ_α^*'s.

$$\psi_\alpha = \sum_{m_1,m_2,n_1,n_2} C^\alpha_{m_1,m_2,n_1,n_2}(\chi_1^*)^{m_1}(\chi_2^*)^{m_2}\chi_1^{n_1}\chi_2^{n_2} \quad , \quad m_1, m_2, n_1, n_2 \in Z \tag{4.22}$$

[1]The parametrization of χ (4.15,4.14) when used in the Hopf map (4.12,4.13), gives the familiar coordinates of S^2 , $x_1 = r\sin\theta\cos\varphi$,$x_2 = r\sin\theta\sin\varphi$ and $x_3 = r\cos\theta$)

Note that, we have used here the homogeneous coordinates $\chi_\alpha, \chi^*_\alpha$- rather than the stereographic variables ρ and η, in the argument of both the variables ψ_1 and ψ_2. These can be regarded as a doublet of scalar fields like (4.19), but transforming under SU(2). This is a trivial bundle, defined globally on C^2_0. The section of the spinor bundle over S^3 is defined with the restriction on $\chi' s$ of the form

$$\chi^\dagger \chi = r \tag{4.23}$$

and the sections of the spinor bundle over S^2 is defined using χ' in U_+ and χ'' in U_-(4.16,4.17)

The differential operator

$$J_i = \frac{1}{2}(\chi_\alpha \sigma_i^{\beta\alpha} \partial_{\chi_\beta} - \chi^*_\alpha (\sigma^*)_i^{\beta\alpha} \partial_{\chi^*_\beta}) \tag{4.24}$$

acts on the spinors(4.21) defined on C^2_0. Since these differential operators are independent of r and depends upon θ , φ, and ψ, these operators therefore act on S^3 as well and can be identified with rotation generators for the spinors on S^3 (See appendix A). Similarly the dilatation operator, is given by the differential operator

$$K = \frac{1}{2}(\chi_\alpha \partial_{\chi_\beta} - \chi^*_\alpha \partial_{\chi^*_\beta}) \tag{4.25}$$

Since each term in (4.22) can be factored in to homogeneous holomorphic and anti-holomorphic functions, K yields the net winding number. The operator J_i and K can be expressed in terms of Euler's angle using (4.15,4.14).

$$\begin{aligned}
J_1 &= i \sin\varphi \frac{\partial}{\partial\theta} + i \cos\varphi \cot\theta \frac{\partial}{\partial\varphi} - i \frac{\cos\varphi}{\sin\theta} \frac{\partial}{\partial\psi} \\
J_2 &= -i \cos\varphi \frac{\partial}{\partial\theta} + i \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} - i \frac{\sin\varphi}{\sin\theta} \frac{\partial}{\partial\psi} \\
J_3 &= -i \frac{\partial}{\partial\varphi} \\
K &= i \frac{\partial}{\partial\psi}
\end{aligned} \tag{4.26}$$

The operator J_i satisfy SU(2) algebra -

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{4.27}$$

Now, since the component of the sections of the spinor bundle on unit (r=1) S^3 of the form (4.22), it can be written in terms of the Euler's angles, introduced in (4.15,4.14),

$$\psi_\alpha = \sum_{m_1,m_2,n_1,n_2} C^\alpha_{m_1,m_2,n_1,n_2} (\cos\theta)^{m_1+n_1} (\sin\theta)^{n_2+m_2} \tag{4.28}$$

$$\times e^{-i\frac{\varphi}{2}(m_1-m_2-n_1+n_2)} e^{-i\frac{\psi}{2}(m_1+m_2-n_1-n_2)} \tag{4.29}$$

the sections on $S(C^2)$ and $S(S^3)$ can be further divided in to class of different subbundles , indexed by the eigenvalue k of the dilatation operator K(4.26): $S_k(C^2)$ and $S_k(S^3)$ where

$$k = m_1 + m_2 - n_1 - n_2 \tag{4.30}$$

is the eigenvalue of the operator K (4.26)

$$K\Psi = k\Psi \tag{4.31}$$

The section of the spinor bundle over S^2 as earlier mentioned,[2] has the form

$$\Psi' = \begin{pmatrix} \psi'_1(\chi'_\alpha, \chi'^*_\beta) \\ \psi'_2(\chi'_\alpha, \chi'^*_\beta) \end{pmatrix} \qquad \alpha, \beta = 1, 2 \quad on \quad U_+ \tag{4.32}$$

$$\Psi'' = \begin{pmatrix} \psi''_1(\chi''_\alpha, \chi''^*_\beta) \\ \psi''_2(\chi''_\alpha, \chi''^*_\beta) \end{pmatrix} \qquad \alpha, \beta = 1, 2 \quad on \quad U_- \tag{4.33}$$

In terms of Euler's angle they can be written as-

$$\begin{aligned} \Psi'_\alpha &= \sum a_{n_1,n_2,m_1,m_2} \chi_1'^{*m_1} \chi_2'^{*m_2} \chi_1'^{n_1} \chi_2'^{n_2} \quad \text{in} \quad U_+ & (4.34) \\ &= \sum a_{n_1,n_2,m_1,m_2} r^{\frac{m_1+m_2+n_1+n_2}{2}} (\cos\frac{\theta}{2})^{m_1+n_1} (\sin\frac{\theta}{2})^{m_2+n_2} e^{i\varphi(m_2-n_2)} \\ \Psi''_\alpha &= \sum a_{n_1,n_2,m_1,m_2} \chi_1''^{*m_1} \chi_2''^{*m_2} \chi_1''^{n_1} \chi_2''^{n_2} \quad \text{in} \quad U_- & (4.35) \\ &= \sum a_{n_1,n_2,m_1,m_2} r^{\frac{m_1+m_2+n_1+n_2}{2}} (\cos\frac{\theta}{2})^{m_1+n_1} (\sin\frac{\theta}{2})^{m_2+n_2} e^{i\varphi(n_1-m_1)} \end{aligned}$$

these are independent of ψ, so they live on S^2. Since the coefficient in both the expression are same in the vicinity of equator ($\theta = \pi/2$) so that in $U_+ \bigcap U_-$ we have

$$\Psi' = e^{ik\varphi + i\delta} \Psi'' \tag{4.36}$$

where k is the topological index (Chern class) given in (4.30), and δ may be constant or globally defined function on S^2. Consequently, the two sections Ψ_0 and Ψ_1 on S^2 is called equivalent if

$$\Psi'_0 = \Psi'_1 \quad , \quad \Psi''_0 = e^{i\delta} \Psi''_1 \tag{4.37}$$

[2]The section of the spinor bundle over S^2 can be obtained from (4.15,4.14) by the gauge fixing i.e. by setting $\varphi = \psi$ in U_+ and $\varphi = -\psi$ in U_-. In both the charts the component of the section differs in the exponential factor in the form $e^{i\varphi(m_2-n_2)}$ in U_- and $e^{i\varphi(n_1-m_1)}$ in U_+. If we have the condition k = 0, the components matche in both the chart and Ψ is globally defined on S^2 and we obtain a section on the trivial bundle. On the other hand, for $k \neq 0$, then we have the transition rule to go from one chart to another in the form $e^{i\varphi(m_2-n_2)} = e^{i\varphi(n_1-m_1-k)}$, the k is called the winding number, introduced earlier and is the eigenvalue of the K operator (4.31)

The equivalance class of a given section $\Psi \in S_k(S^2)$ is denoted as $\tilde{\Psi}$. The representative section can be given as

$$\tilde{\Psi}'_\alpha(\chi', \chi'^*) = \sum a_{n_1,n_2,m_1,m_2} \chi_1'^{*m_1} \chi_2'^{*m_2} \chi_1'^{n_1} \chi_2'^{n_2} \quad on \quad U_+ \quad (4.38)$$

$$\tilde{\Psi''}_\alpha(\chi'', \chi''^*) = \sum a_{n_1,n_2,m_1,m_2} \chi_1''^{*m_1} \chi_2''^{*m_2} \chi_1''^{n_1} \chi_2''^{n_2} \quad on \quad U_- \quad (4.39)$$

with $k = m_1 + m_2 - n_1 - n_2$. The coefficients in the both the expressions is same and the transition rule

$$\tilde{\Psi}'(\chi', \chi'^*) = e^{ik\varphi} \tilde{\Psi}''(\chi'', \chi''^*) \quad (4.40)$$

is satisfied.
The free Dirac operator $\tilde{D}'_k : \tilde{S}(S^2) \to \tilde{S}(S^2)$ is defined by

$$\tilde{D}'_k = [i\sigma'^\mu(\partial'_\mu + A'_\mu)] \quad on \quad U_+ \quad (4.41)$$

$$\tilde{D}''_k = [i\sigma''^\mu(\partial''_\mu + A''_\mu)] \quad on \quad U_- \quad (4.42)$$

Where ∂_μ denotes the derivative ∂_θ, ∂_φ in the local coordiantes θ and φ in $U_+ \bigcap U_-$ and $(\partial_\mu + iA_\mu)$ represents covariant derivative. The $\sigma's$ (σ^θ and σ^φ) satisfies the following Clifford algebra

$$\{\sigma^\mu, \sigma^\upsilon\} = 2g^{\mu\nu}, \quad \mu, \nu = 1, 2 \quad (4.43)$$

in $U_+ \bigcap U_-$, where $g^{\mu\upsilon}$ is the inverse of the metric tensor [3] of S^2. and A_μ is the k- monopole field and is given as (which are derived in Appendix B)-

$$A'_\mu = ik\chi'^\dagger \partial'_\mu \chi' \quad on \quad U_+ \quad (4.44)$$

$$A''_\mu = ik\chi''^\dagger \partial''_\mu \chi'' \quad on \quad U_+ \quad (4.45)$$

The field A'_μ and A''_μ in $U_+ \bigcap U_-$ are related by the gauge transformation

$$A'_\mu = A''_\mu - ih\partial_\mu h^{-1} \quad , \; h = e^{ik\varphi} \quad (4.46)$$

The J's in (4.26) are the vector field on S^3 reperesented in the coordinate basis $(\partial_\theta, \partial_\varphi, \partial_\psi)$. These J's provide an orthonormal basis at $T_p(S^3)$ and $g(J_i, J_j) = \delta_{ij}$ and correspond to the rotation generators in three independent directions. Correspondingly, the dual vectors e^i's provide an orthonormal basis in the cotangent space $T^*_p(S^3)$, $< e^i, J_i >= \delta_{ij}$. Both of these

[3]starting from the metric $ds^2 = 4d\chi^\dagger d\chi$ for C_0^2, we can obtain the metric on unit S^3 (r=1) by the parametrization (4.14,4.15) to get $ds^2 = 1/4[d\theta^2 + d\varphi^2 + d\psi^2 + 2\cos\theta d\varphi d\psi]$. Upon gauge-fixing $\psi = \pm\varphi$ one gets the following expressions of metric on S^2 , $ds^2 = (d\frac{\theta}{2})^2 + \sin^2\frac{\theta}{2}d\varphi^2$ for U_- and $ds^2 = (d\frac{\theta}{2})^2 + \cos^2\frac{\theta}{2}d\varphi^2$ for U_+. The expression for U_+ takes the canonical form upon the replacement $\theta \to (\pi - \theta)$, as here the stereographic projection is being made from the south pole, Further since a point on S^2, having the polar angle θ, subtends an angle $\theta/2$ at the south pole, it gives rise to an additional factor of $1/4$ for the metric ds^2 for S^3, apart from explaining the occurence of $\theta/2$ rather than θ in the metric for S^2

can be obtained, also from the Maurer-Cartan left-invariant 1-form (See Appendix D). So we should be able to write them locally in the form-

$$J_i = \frac{\partial}{\partial s_i} \qquad i = 1, 2, 3 \tag{4.47}$$

where $s_i's$ are the 3- affine parameters along the integral curve of J_i. The tri-beins-

$$J_i \xi^\mu = \frac{\partial \xi^\mu}{\partial s_i} = e_i^\mu \qquad \text{where}, \mu = 1, 2, 3 \text{ s.t.} \qquad \xi^1 = \theta \ , \ \xi^2 = \phi \text{ and } \xi^3 = \psi \tag{4.48}$$

by considering ξ^1 and ξ^2 only, the tri-beins connects the coordiantes basis to the orthonormal basis on the cotangent $T_p^*(S^2)$

$$d\xi^\mu = (J_i \xi^\mu) ds^i \tag{4.49}$$

The σ^μ matrices which satisfy the clifford algebra on S^2 are obtained from the usual pauli matrices σ^i from these tri-beins in the following way-

$$\sigma^\mu = (J_i \xi^\mu) \sigma^i; \quad \mu = 1, 2 \tag{4.50}$$

A straightforward computation yields,

$$\sigma^\theta = \begin{bmatrix} 1 & -\cot\theta e^{-i\varphi} \\ -\cot\theta e^{i\varphi} & -1 \end{bmatrix}$$

$$\sigma^\varphi = \begin{bmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{bmatrix}$$

satisfying (4.43) and

$$A'_\theta = 0, \qquad A'_\varphi = \frac{k}{2}(\cos\theta - 1) \tag{4.51}$$

$$A''_\theta = 0, \qquad A''_\varphi = \frac{k}{2}(\cos\theta + 1) \tag{4.52}$$

In S^2 the eigenvalue problem of the Dirac operator has to do in the patches U_+ and U_- differently, so the $\tilde{D}_k$ in $S_k(S^2)$ is switched to the problem in $S_k(S^3)$, which will be defined globally on S^3, with the transformation of the sections and $\tilde{D}_k$

$$\Psi = e^{-\frac{i}{2}k(\varphi+\psi)} \tilde{\Psi}' \quad on \ \ U_+ \tag{4.53}$$

$$\Psi = e^{\frac{i}{2}k(\varphi-\psi)} \tilde{\Psi}' \quad on \ \ U_- \tag{4.54}$$

And

$$D_k = e^{-\frac{i}{2}k(\varphi+\psi)} \tilde{D}_k e^{\frac{i}{2}k(\varphi+\psi)} \quad on \ \ U_+ \tag{4.55}$$

$$D_k = e^{-\frac{i}{2}k(\varphi+\psi)} \tilde{D}_k e^{\frac{i}{2}k(\varphi+\psi)} \quad on \ \ U_- \tag{4.56}$$

A straightforward computation shows these special cases of the Dirac operator on S^3, given as ,

$$D_k = \frac{1}{r} \sigma_j (J_j - \frac{k}{2} \frac{x_j}{r}) \tag{4.57}$$

4.1.2 Noncommutative Case

In this section we will describe the noncommutativity of the type-

$$[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k, \tag{4.58}$$

where λ is the noncommutative parameter having the dimension of length. This can be obtained from the commutative version by making the previously defined $\chi's$ and χ^*'s to be annihilation and creation operator respectively satisfying

$$[\chi_\alpha, \chi_\beta^\dagger] = \frac{\lambda}{2}\delta_{\alpha\beta} \quad , \quad [\chi_\alpha, \chi_\beta] = 0 = [\chi_\alpha^\dagger, \chi_\beta^\dagger] \tag{4.59}$$

The coordiantes obtained through operatorial version of the Hopf map, which is nothing but the schwinger's construction (3.3),

$$\hat{x}_i = \chi^\dagger\sigma_i\chi = \chi_\alpha^\dagger\sigma_{\alpha\beta}^i\chi_\beta \tag{4.60}$$

which clearly satisfy the spatial nonommutativity (4.58). The algebra defined in (4.59) is isomorphic to the two sets of harmonic oscillator so that any generic state is labeled by a pair of integers n_1 and n_2 as $|n_1, n_2\rangle$ (n_1 is a label for first harmonic oscillator while n_2 is for the second one) and can be obtained by applying the creation operators $\chi_1^\dagger$ and $\chi_2^\dagger$ on the "ground state" $|0,0\rangle \equiv |0\rangle$:

$$|n_1, n_2\rangle = \sqrt{\frac{(2/\lambda)^{n_1+n_2}}{n_1!n_2!}}\chi_1^{\dagger n_1}\chi_2^{\dagger n_2}|0\rangle \tag{4.61}$$

and the operator $\chi^\dagger\chi$ takes the value

$$\chi^\dagger\chi|n_1, n_2\rangle = \frac{\lambda}{2}(n_1 + n_2)|n_1, n_2\rangle \tag{4.62}$$

The Casimir operator $\hat{x}_i^2$ and $\hat{x}_3$ takes the value

$$\hat{x}_i^2|n_1, n_2\rangle = \lambda^2 n(n+1)|n_1, n_2\rangle \ , \ n = \frac{n_1 + n_2}{2} \tag{4.63}$$

$$\hat{x}_3|n_1, n_2\rangle = \lambda n_3|n_1, n_2\rangle \quad , \quad n_3 = \frac{n_1 - n_2}{2} \tag{4.64}$$

calling

$$\frac{n_1 + n_2}{2} = n \quad , \quad \frac{n_1 - n_2}{2} = n_3 \ , \ \text{for fixed n} \ , \ -n \leq n_3 \leq n \tag{4.65}$$

the state $|n_1, n_2\rangle$ can be relabeled by $|n, n_3\rangle$ where we have

$$\hat{x}_i^2|n, n_3\rangle = \lambda^2 n(n+1)|n, n_3\rangle \tag{4.66}$$

$$\hat{x}_3|n, n_3\rangle = \lambda n_3|n, n_3\rangle \tag{4.67}$$

Clearly n is an integral multiple of 1/2 . The ladder operators $\hat{x}_+ = \hat{x}_1 + i\hat{x}_2$ and $\hat{x}_- = \hat{x}_1 - i\hat{x}_2$ can be checked to satisfy

$$[\hat{x}_3, \hat{x}_+] = \lambda\hat{x}_+ \quad , \quad [\hat{x}_3, \hat{x}_-] = \lambda\hat{x}_- \quad , \quad [\hat{x}_+, \hat{x}_-] = 2\lambda\hat{x}_3 \tag{4.68}$$

so that,

$$\hat{x}_+|n, n_3\rangle = \sqrt{n(n+1) - n_3(n_3+1)}|n, n_3+1\rangle \tag{4.69}$$
$$\hat{x}_-|n, n_3\rangle = \sqrt{n(n+1) - n_3(n_3-1)}|n, n_3-1\rangle \tag{4.70}$$

The 3D configuration space is replaced by the Hilbert space $\mathcal{F}$ spanned by $|n, n_3\rangle$

$$\mathcal{F} = Span\{|n, n_3\rangle\} \tag{4.71}$$

where radius is quantized (4.66). Each n corresponds to the fixed sphere of radius $\lambda^2 n(n+1)$. For fixed n the Hilbert space is restricted to (2n+1)-dimensional space

$$\mathcal{F}_n = \{|n, n_3\rangle \ \mid \ \text{n is fixed}, -n \leq n_3 \leq n\} \tag{4.72}$$

The Hilbert-Schmidt operators which act on the $\mathcal{F}$ can be written as

$$\Psi = Span\{|n, n_3\rangle\langle n', n_3'|\} \tag{4.73}$$

of course, subject to additional conditions like "trace-class" and boundedness. The previously defined operator K and rotation generator J_i (4.24, 4.25) in the commutative case acts on the section of the spinor bundle. The action of J_i and K on the Ψ_α in the noncommutative case is given by the following adjoint actions:

$$J_i\Psi_\alpha = \frac{1}{\lambda}[x_i, \Psi_\alpha] \tag{4.74}$$
$$K\Psi_\alpha = [\hat{N}, \Psi_\alpha] \quad , \quad \hat{N} = \chi_1^\dagger\chi_1 + \chi_2^\dagger\chi_2 \tag{4.75}$$

These actions of J_i and K in the noncommutative case is obtained by replacing the derivatives in (4.24, 4.25) with the following commutators-

$$\partial_{\chi_\alpha}\Psi = [\chi_\alpha^\dagger, \Psi] \ , \ \partial_{\chi_\alpha^\dagger}\Psi = [\chi_\alpha, \Psi] \tag{4.76}$$

The action of the position X_i on the Ψ_α is taken to be left multiplication in S_k for $k > 0$ and the right multiplication for $k < 0$.

$$X_i\Psi = \hat{x}_i\Psi \ , \ k > 0 \tag{4.77}$$
$$X_i\Psi = \Psi\hat{x}_i \ , \ k < 0 \tag{4.78}$$

Spectral Triplet For The Noncommutative/Fuzzy Sphere

In this section we discuss the construction of the spectral triplet $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for the fuzzy sphere. In the commutative case as earlier discussed , the complex valued functions defined on the manifold serve the purpose of the elements of the algebra $(\mathcal{A})$. The function of χ's , $\chi^{\dagger}s \in S^3$ of the type -

$$\Psi = \sum_{m_1, m_2, n_1, n_2} \chi_1^{*m_1} \chi_2^{*m_2} \chi_1^{n_1} \chi_2^{n_2} \tag{4.79}$$

with $k = m_1 + m_2 - n_1 - n_2 = 0$ are the functions defined on S^2, which is clear from (4.29), since ψ dependence goes away . So in the commutative case the algebra $(\mathcal{A})$ is-

$$\mathcal{A} = \{\Psi = \sum \chi_1^{*m_1} \chi_2^{*m_2} \chi_1^{n_1} \chi_2^{n_2} \quad | \quad k = m_1 + m_2 - n_1 - n_2 = 0\} \tag{4.80}$$

The multiplication of two elements of $\mathcal{A}$ also preserve the condition $k = 0$, so indeed they forms an algebra.

In the noncommutative case also the algebra elements are taken to be the elements as in (4.79) but $\chi' s$ and $\chi^{\dagger}$'s are now operators, which are taken in the normal-ordered form:

$$\mathcal{A} = \{\hat{\Psi} = \hat{\chi}_1^{\dagger m_1} \hat{\chi}_2^{\dagger m_2} \hat{\chi}_1^{n_1} \hat{\chi}_2^{n_2} \quad | \quad k = m_1 + m_2 - n_1 - n_2 = 0\} \tag{4.81}$$

In the noncommutative case consider the sphere indexed by n. The elements of $\mathcal{F}_n$ (4.72) are the states which describe the sphere indexed by n. Consider the operators which acts on $\mathcal{F}_n$ and constructed from the elements of $\mathcal{F}_n$-

$$\hat{\Omega} = |n, n_3\rangle\langle n, n_3'| \tag{4.82}$$

The operator K (4.75) has the eigenvalue k=0 on these $\hat{\Omega}$ (4.82).

$$K|n, n_3\rangle\langle n, n_3'| = [\hat{N}, |n, n_3\rangle\langle n, n_3'|] = 0 \tag{4.83}$$

So these $\hat{\Omega}$ should be analouge of the Ψ (4.80) in the noncommutative case. So in the noncommutative case we take the algebra $\mathcal{A}$ to be-

$$\mathcal{A} = Span\{|n, n_3\rangle\langle n, n_3'| \quad | \quad -n \leq n_3, n_3' \leq n\} \tag{4.84}$$

The Hilbert space $(\mathcal{H})$ on which the algebra and Dirac opertor acts is taken to be section of the spinor bundle over $S_k(S^3)$, with both the component of the section having same k value. The Dirac operator $(\mathcal{D})$ is given in (4.57). Let us write the spectral triplet for this fuzzy sphere.

1. The Algebra $\mathcal{A} = \text{Span}\{|n, n_3\rangle\langle n, n_3'|, -n \leq n_3, n_3' \leq n\}$

2. The Hilbert space $\mathcal{H} = \Psi = \begin{pmatrix} \psi_1(\chi_\alpha, \chi_\beta^{\dagger}) \\ \psi_2(\chi_\alpha, \chi_\beta^{\dagger}) \end{pmatrix}$

3. Dirac operator $\mathcal{D} = D_k = \frac{1}{r}\sigma_j(J_j - \frac{k}{2}\frac{x_j}{r})$

The "points" of the noncommutative space are defined through the mapping-

$$\Omega : \mathcal{A} \to C \tag{4.85}$$

As discussed in the Moyal case the "points" of the noncommutative space are the pure state $|n, n_3\rangle\langle n, n_3|$, which belongs to the dual of the algebra (here algebra and its dual are same). This is the pure density matrix corresponding to the $|n, n_3\rangle$, so these $|n, n_3\rangle$, for fixed n and $-n \leq n_3 \leq n$, represents the "points" of the fuzzy/noncommutative sphere whose radius is indexed by n.

Connes distance calculation on Fuzzy sphere

In this section we discuss the Connes' distance calculation on the fuzzy sphere. The "points" of the fuzzy sphere are described by -

$$|n, n_3\rangle \in \mathcal{H}_c \quad , \quad -n \leq n_3 \leq n \tag{4.86}$$

we would like to calculate the distance between the two nearest points $|n, n_3\rangle$ and $|n, n_3+1\rangle$ on the sphere. The Connes distance formula between the two points ω and ω' of the noncommutative space as mentioned in the Moyal case, following [21], is given by -

$$d(\omega, \omega') = Sup\{|(\omega) - a(\omega')| \quad , \quad \|[\mathcal{D}, a]\|_{op} \leq 1, a \in \mathcal{A}\} = \frac{tr(d\rho^2)}{\|[D, \pi(d\rho)]\|_{op}} \tag{4.87}$$

In this case $\rho = |n, n_3\rangle\langle n, n_3|$ and $\rho' = |n, n_3+1\rangle\langle n, n_3+1|$ and $d\rho$ is-

$$d\rho = |n, n_3+1\rangle\langle n, n_3+1| - |n, n_3\rangle\langle n, n_3| \tag{4.88}$$

First we need to calculate $\|[\mathcal{D}, \pi(d\rho)]\|_{op}$. The $\mathcal{D}$ (4.57) is,-

$$\mathcal{D} = \frac{1}{r}\begin{bmatrix} J_3 - \frac{k}{2}\frac{x_3}{r} & J_1 - iJ_2 - \frac{k}{2}\frac{x_1 - ix_2}{r} \\ J_1 + iJ_2 - \frac{k}{2}\frac{x_1 + ix_2}{r} & -J_3 + \frac{k}{2}\frac{x_3}{r} \end{bmatrix} \tag{4.89}$$

Dirac operator $\mathcal{D}_k$ (4.89) acts on the sections of the spinor subbundle $S_k(S^3)$. The Algebra has k=0, so in the Dirac operator we take k=0. The commutator $[\mathcal{D}, \pi(d\rho)]$ now reduces to

$$[\mathcal{D}, \pi(d\rho)] = \frac{1}{r}\begin{bmatrix} 0 & \frac{1}{\lambda}[(\hat{x}_1 - i\hat{x}_2), d\rho] \\ \frac{1}{\lambda}[(\hat{x}_1 + i\hat{x}_2), d\rho] & 0 \end{bmatrix} \tag{4.90}$$

$$= \frac{1}{r}\begin{bmatrix} 0 & \frac{1}{\lambda}[(\hat{x}_-, d\rho]) \\ \frac{1}{\lambda}[(\hat{x}_+), d\rho] & 0 \end{bmatrix} \tag{4.91}$$

where the commutator $[J_3, (d\rho)]$ vanishes. The action of J_i on the Hilbert space is defined through (4.74). In order to take the norm we write-

$$[\mathcal{D}, \pi(d\rho)]^\dagger[\mathcal{D}, \pi(d\rho)] = \frac{1}{r^2}\begin{bmatrix} \frac{1}{\lambda^2}[(\hat{x}_-), d\rho]^\dagger[(\hat{x}_-), d\rho] & 0 \\ 0 & \frac{1}{\lambda^2}[(\hat{x}_+), d\rho]^\dagger[(\hat{x}_+), d\rho] \end{bmatrix} \tag{4.92}$$

An explicit computation yields

$$[x_+,\pi(d\rho)] = \sqrt{n(n+1)-(n_3+1)(n_3+2)}|n,n_3+2\rangle\langle n,n_3+1| \\ -2\sqrt{n(n+1)-n_3(n_3+1)}|n,n_3+1\rangle\langle n,n_3| \\ +\sqrt{n(n+1)-n_3(n_3-1)}|n,n_3\rangle\langle n,n_3-1| \quad (4.93)$$

$$[x_-,\pi(d\rho)] = \sqrt{n(n+1)-(n_3+1)(n_3+2)}|n,n_3+1\rangle\langle n,n_3+2| \\ +2\sqrt{n(n+1)-n_3(n_3+1)}|n,n_3\rangle\langle n,n_3+1| \\ +\sqrt{n(n+1)-n_3(n_3-1)}|n,n_3-1\rangle\langle n,n_3| \quad (4.94)$$

so that

$$[(\hat{x}_+),d\rho]^\dagger[(\hat{x}_+),d\rho] = [n(n+1)-(n_3+1)(n_3+2)]|n,n_3+1\rangle\langle n,n_3+1| \\ +4[n(n+1)-(n_3+1)n_3]|n,n_3\rangle\langle n,n_3| \\ +[n(n+1)-(n_3-1)n_3]|n,n_3-1\rangle\langle n,n_3-1| \quad (4.95)$$

$$[(\hat{x}_-),d\rho]^\dagger[(\hat{x}_-),d\rho] = [n(n+1)-(n_3+1)(n_3+2)]|n,n_3+2\rangle\langle n,n_3+2| \\ +4[n(n+1)-(n_3+1)n_3]|n,n_3+1\rangle\langle n,n_3+1| \\ +[n(n+1)-(n_3-1)n_3]|n,n_3\rangle\langle n,n_3| \quad (4.96)$$

Since both of these operators are diagonal, the operator norm, defined to be the largest eigenvalue, can be read off exactly from both the equation, yielding - $4[n(n+1)-n_3(n_3+1)]$. This gives the operator norm of the commutator as,

$$||[\mathcal{D},\pi(d\rho)]|| = \frac{2\sqrt{[n(n+1)-n_3(n_3+1)]}}{\lambda\sqrt{n(n+1)}} \quad (4.97)$$

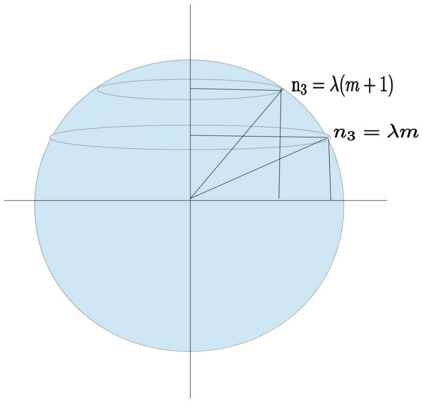
The infinitesimal distance between the state $|n,n_3\rangle$ and $|n,n_3+1\rangle$ is then easily obtained from (2.39), using $tr_c(d\rho)^2=2$,

$$d(n_3+1,n_3) = \frac{\lambda\sqrt{n(n+1)}}{\sqrt{[n(n+1)-n_3(n_3+1)]}} \quad (4.98)$$

On the infinitesimal nature of the distance formula (4.98)

To clarify the adjective "infinitesimal" in the context, let us recall from the theory of angular momentum that the state $|n,n_3\rangle$ can be visualised as the vector $\vec{x}$ preceesing the x_3- axis along a cone, in such a manner that the tip of the vector $\vec{x}$ lies on the circle of latitude on a sphere of radius $\lambda\sqrt{n(n+1)}$, m being an integer $-n \leq m \leq n$ (see fig.). The associated polar angles are therefore quantized as,

$$\theta_m = \sin^{-1}(\frac{m}{\sqrt{n(n+1)}}) \quad (4.99)$$



Now, let us treat 'm' to be a continuous variable for a moment.

$$d\theta_m = \frac{dm}{\sqrt{n(n+1) - m^2}} \tag{4.100}$$

So the distance is obtained by multiplying with quantized radius to get

$$ds = \frac{\lambda\sqrt{n(n+1)}dm}{\sqrt{n(n+1) - m^2}} \tag{4.101}$$

This almost matches with the distance expression (4.98); infact with the replacement $dm \rightarrow \triangle\ m = 1$ and $m^2 \rightarrow n_3(n_3+1)$ one reproduces (4.98) exactly.

Construction of Perelemov Coherent state on S^2 and distance function

We now provide a brief review of the construction of Perelemov's SU(2) coherent, as given in [22]. Let us consider a general Lie group G, whose unitary irreducible representation on some Hilbert space $\mathcal{H}$ is denoted as T(g). Consider a fixed vector in the Hilbert space denoted as $|x_0\rangle$ and consider $|x\rangle$ obtained as, $|x\rangle = T(g)|x_0\rangle$, where g is any element of the group G. The two state $T(g_1)|x_0\rangle$ and $T(g_2)|x_0\rangle$ are called equivalent if they differ by a phase factor-

$$T(g_1)|x\rangle = e^{i\alpha}T(g_2)|x_0\rangle \Rightarrow T(g_2^{-1}g_1)|x_0\rangle = e^{i\alpha}|x_0\rangle \quad , \quad |e^{i\alpha}| = 1 \tag{4.102}$$

Consider the subgroup H of the group G with the property-

$$T(h)|x_0\rangle = e^{i\alpha(h)}|x_0\rangle \tag{4.103}$$

This construction shows that the vectors $|x\rangle_g$ for all group element g, which belongs to the left coset class of group G with respect to the subgroup H , differs only in the phase factor, so they determine the same state. Choosing a representative g(x) in the equivalance class $x \in G/H$,one gets the associated state $|x\rangle$. These states are the Perelemov coherent states on the base manifold G/H. Here, particularly we are interested in the construction of coherent state on $S^2 = SU(2)/U(1)$, with G= SU(2) and H=U(1). We write an element of SU(2) as the matrices-

$$g = \begin{bmatrix} u & v \\ -\bar{v} & \bar{u} \end{bmatrix} \tag{4.104}$$

which can be parametrized in terms of Euler angles exactly like (4.15,4.14). The subgroup H=U(1) is

$$h = \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} \quad , \quad u = e^{\frac{-i}{2}\psi} \tag{4.105}$$

is a stability subgroup. As discussed earlier the S^2 manifold is described by the doublet- $\begin{pmatrix} 1 \\ z \end{pmatrix}$. Upon normalization,one can identify, $u = 1/\sqrt{(1+|z|^2)}$ and $v = z/\sqrt{(1+|z|^2)}$, so that the associated SU(2) group element can be written-

$$g = \frac{1}{\sqrt{1+|z|^2}} \begin{bmatrix} 1 & z \\ -\bar{z} & 1 \end{bmatrix} \tag{4.106}$$

Here $z \in C$ represents the stereographic projected coordiantes of the points of the S^2 manifold from the south pole. The coherent state on the S^2 is then -

$$|z\rangle = T(g(z))|0\rangle \tag{4.107}$$

where $|x_0\rangle \equiv |0\rangle$ represents the north pole, and $T(g(z))$ is given by (4.106). This can be re-casted as-

$$|z\rangle = \frac{1}{\sqrt{1+|z|^2}}|0\rangle + \frac{1}{\sqrt{1+|z|^2}}(z\sigma_+ - \bar{z}\sigma_-)|0\rangle \tag{4.108}$$

where $\sigma_+ = \sigma_1 + i\sigma_2$ and $\sigma_- = \sigma_1 - i\sigma_2$ and σ_1, σ_2 are the two pauli matrices in the defining fundamental j=1/2 representation of the SU(2) Lie group generators. In an arbitrary j-th representtion, we can write the coherent state as-

$$|z\rangle = \frac{1}{\sqrt{1+|z|^2}}|0\rangle + \frac{1}{\sqrt{1+|z|^2}}(zJ_+ - \bar{z}J_-)|0\rangle \quad , \quad J_\pm = J_1 \pm iJ_2 \tag{4.109}$$

with J_i being the corresponding generators. In the noncommutative case the $|0\rangle$ which represents the north pole is identified with the state $|n, n_3 = n\rangle (r = \lambda\sqrt{n(n+1)}, x_3 = \lambda n)$, which represent the north pole closely and J_1, J_2 can be taken to be x_1 and x_2 since they satisfy the SU(2) Lie algebra noncommutativity. So, we can write the coherent state as-

$$|z\rangle = \frac{1}{\sqrt{1+|z|^2}}|n,n\rangle + \frac{1}{\sqrt{1+|z|^2}}(zx_+ - \bar{z}x_-)|n,n\rangle \tag{4.110}$$

For the infinitesimal distance function on S^2 using coherent state , letus write $d\rho$ introduced earlier-

$$d\rho = |z+dz\rangle\langle z+dz| - |z\rangle\langle z| \tag{4.111}$$

This is obtained as (see Appendix D for the derivation)

$$d\rho = \frac{i\sqrt{2n}}{1+|z|^2}(|n,n-1\rangle\langle n,n|d\bar{z} - dz|n,n\rangle\langle n,n-1|) \tag{4.112}$$

Similar to the earlier case we calculate $[\mathcal{D}, \pi(d\rho)]$ -

$$[\mathcal{D}, \pi(d\rho)] = \frac{1}{r}\begin{bmatrix} [J_3, d\rho] & [J_-, d\rho] \\ .[J_+, d\rho] & -[J_3, d\rho] \end{bmatrix} \tag{4.113}$$

At this stage, we observe that any point on S^2 can be regarded as a north-pole, upon suitable SO(3) rotation of the coordinate axis and $J_i's$ can be regarded as the SU(2) generators, associated to these new axes. With this the point coordinatized by z, will remain invariant under the action of the stability subgroup U(1) contained in SU(2), for which the generator is J_3. Consequently, the state $|z\rangle$ will now correpond to $|n, n\rangle$ and J_3 terms, occuring in the diagonal elements of (4.113) can be disregarded. Indeed, the Clifford algebra for 2-d manifold like S^2 should be generated by just 2 Pauli matrices. This indicates that the structure of the Dirac operator on S^2 has a similar form as that of the Moyal plane- $\frac{1}{r}\begin{bmatrix} 0 & J_- \\ J_+ & 0 \end{bmatrix}$, but $J_\pm$ defined locally, by identifying the point in question to the north pole. This allows us to write the commutator $[\mathcal{D}, \pi(d\rho)]$ in the following form:

$$[\mathcal{D}, \pi(d\rho)] = \frac{1}{r}\begin{bmatrix} 0 & [J_-, d\rho] \\ .[J_+, d\rho] & 0 \end{bmatrix} \tag{4.114}$$

or rather in terms of SU(2) Lie algebra-valued position coordinates as,

$$[\mathcal{D}, \pi(d\rho)] = \frac{1}{r}\begin{bmatrix} 0 & [x_-, d\rho] \\ .[x_+, d\rho] & 0 \end{bmatrix} \tag{4.115}$$

Clearly, the rest of the computation is just like the Moyal case. To obtain the operatorial norm, we first write-

$$[\mathcal{D}, \pi(d\rho)]^\dagger [\mathcal{D}, \pi(d\rho)] = \frac{1}{r^2} \begin{bmatrix} [x_+, d\rho]^\dagger [x_+, d\rho] & 0 \\ 0 & [x_-, d\rho]^\dagger [x_-, d\rho] \end{bmatrix} \quad (4.116)$$

and calculate

$$[x_+, d\rho] = \frac{i\sqrt{2n}}{1+|z|^2} (d\bar{z}\sqrt{2n}|n\rangle\langle n| - d\bar{z}\sqrt{2n}|n-1\rangle\langle n-1| + dz\sqrt{2(2j-1)}|j\rangle\langle j-2|) \quad (4.117)$$

$$[x_-, d\rho] = \frac{i\sqrt{2n}}{1+|z|^2} (d\bar{z}\sqrt{2(2j-1)}|j-2\rangle\langle j| - dz\sqrt{2j}|j-1\rangle\langle j-1| + dz\sqrt{2n}|j\rangle\langle j|) \quad (4.118)$$

The maximum eigenvalue λ_{max} is obtained by diagonalisation to yield,

$$\|[\mathcal{D}, \pi(d\rho)]\|_{op} = \sqrt{\frac{4n(3n-1)}{(1+|z|^2)^2} d\bar{z}dz} \quad (4.119)$$

Finally, using

$$tr_c(d\rho)^2 = \frac{4n}{(1+|z|^2)^2} d\bar{z}dz \quad (4.120)$$

and multiplying by radius $\lambda\sqrt{n(n+1)}$ we get using (2.39) the infinitesimal distance on S^2:

$$d(|z\rangle, |z+dz\rangle) = \frac{\lambda\sqrt{n(n+1)}}{1+|z|^2} \sqrt{\frac{4n}{3n-1} d\bar{z}dz} = \lambda\sqrt{\frac{4n^2(n+1)}{3n-1}} \frac{\sqrt{d\bar{z}dz}}{1+|z|^2} \quad (4.121)$$

This is precisely, the form of the metric on S^2 in the stereographic variable. Further the linear scaling with 'n' for large 'n' is manifest in the expression.

Chapter 5

Conclusuion/Result

5.1 Result

A. The noncommutative 2D harmonic oscillator energy spectrum is obtained as-

$$E(j, j_3) = \omega'(2j+1) + \theta\mu'{\omega'}^2 j_3 \quad , \quad J = \frac{m+n}{2} \quad , \quad J_3 = \frac{m-n}{2} \tag{5.1}$$

where ω' and μ' are given as-

$$\frac{1}{\mu'} = \frac{1}{\mu} + \frac{\mu\omega^2\theta^2}{4} \quad \textsf{and} \quad \omega'^2 = \omega^2\left(1 + \frac{\mu^2\omega^2\theta^2}{4}\right) \tag{5.2}$$

The spectrum in the limit $\theta = 0$ reduces to the commutative result $E(j, j_3) = \omega'(2j+1)$. Also, The spectrum of the 2D oscillator comes with the renormalized parameters ω' and μ', so the observable mass and frequency will be these renormalised mass and frequency rather than their bare counterparts. In the noncommutative 2D oscillator the $|0\rangle\langle 0|$ state is the ground state only for the particular value of the parameters named as critical parameters:

$$\mu_0 = \frac{\omega_0}{2} = \frac{1}{\sqrt{\theta}} \tag{5.3}$$

This is unlike to commuative case where $|0\rangle\langle 0| \equiv |0, 0\rangle$ is always the ground state. In the noncommutaive case the hamiltonian of the harmonic oscillator does not respect the time reversible symmetry.

$$\Theta\hat{H}_3\Theta^{-1} = \omega'(\hat{B}_L'^{\ddagger}\hat{B}_L' + \hat{B}_R'\hat{B}_R'^{\ddagger} + 1) - \frac{\mu'\theta\omega'^2}{2}(\hat{B}_L'^{\ddagger}\hat{B}_L' - \hat{B}_R'\hat{B}_R'^{\ddagger}) \neq \hat{H}_3 \tag{5.4}$$

Where Θ is the time reversible operator. Also, the SU(2) symmetry is broken to U(1) symmetry due to the presence of J_3 term. in the hamiltonian.

$$\hat{H}_3 = \frac{1}{2\mu'}\hat{\vec{P}}^2 + \frac{1}{2}\mu'\omega'^2(\hat{\vec{X}}^c)^2 + \mu\theta\omega^2\hat{J}_3 \tag{5.5}$$

B. In the noncommutaive geometry part the space with the noncommutativity $[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k$ can be viewed as concentric sphere, because the Casimir operator is the radius-squared operator which is quantized. Each sphere named as fuzzy sphere. the spectral triplet for these fuzzy sphere , which captures the topological as well as the geometrical property of these noncommutative space is obtained as-

1. The Algebra $\mathcal{A} = \text{Span}\{|n, n_3\rangle\langle n, n_3'|, -n \leq n_3, n_3' \leq n\}$
2. The Hilbert space $\mathcal{H} = \Psi = \begin{pmatrix} \psi_1(\chi_\alpha, \chi_\beta^\dagger) \\ \psi_2(\chi_\alpha, \chi_\beta^\dagger) \end{pmatrix}$
3. Dirac operator $\mathcal{D} = D_k = \frac{1}{r}\sigma_j(J_j - \frac{k}{2}\frac{x_j}{r})$

The state $|n, n_3\rangle$is the simultaneous eigenstate of the radius-squared operator$(\hat{\vec{x}})^2$ and $\hat{x}_3$. The distance between the state $|n, n_3\rangle$and$|n, n_3 + 1\rangle$ which are the closest "point" on the fuzzy sphere is obtained as-

$$d(n_3 + 1, n_3) = \frac{\lambda\sqrt{n(n+1)}}{\sqrt{[n(n+1) - n_3(n_3+1)]}} \tag{5.6}$$

The infinitesimal distance between the coherent state on the fuzzy sphere is obtained as-

$$d(|z\rangle, |z + dz\rangle) = \lambda\sqrt{\frac{4n^2(n+1)}{3n-1}}\frac{\sqrt{d\bar{z}dz}}{1+|z|^2} \tag{5.7}$$

The scaling property for large n is clearly linear in 'n'.

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Appendix A

Appendix

A.1 Appendix A

The χ transform as -

$$\chi \to \chi' = \chi + (i/2)\varepsilon^i \sigma^i_{\alpha\beta}\chi_\beta \tag{A.1}$$

Taking each component Ψ_α of the section of the spinor bundle over S^3 to transform as a scalar,

$$\Psi_\alpha(\chi_\alpha, \chi^*_\alpha) \to \Psi'_\alpha(\chi'_\alpha, \chi'^*_\alpha) = \Psi_\alpha(\chi_\alpha, \chi^*_\alpha) \tag{A.2}$$

we get

$$\Psi'_\alpha(\chi_\alpha, \chi^*_\alpha) + i\varepsilon^i \frac{1}{2}[\chi_\beta(\sigma^i)_{\alpha\beta}\partial_\alpha - \chi^*_\beta(\sigma^i)^*_{\alpha\beta}\partial^*_\alpha]\Psi_\alpha = \Psi_\alpha(\chi_\alpha, \chi^*_\alpha) \tag{A.3}$$

so that

$$\delta\Psi_\alpha = \Psi'_\alpha(\chi'_\alpha, \chi'^*_\alpha) - \Psi_\alpha(\chi_\alpha, \chi^*_\alpha) = -i\varepsilon^i J_i \Psi_\alpha \tag{A.4}$$

where J_i are given as

$$J_i = \frac{1}{2}[\chi_\beta(\sigma^i)_{\alpha\beta}\partial_\alpha - \chi^*_\beta(\sigma^i)^*_{\alpha\beta}\partial^*_\alpha] \tag{A.5}$$

A.2 Appendix B

Since S^3 is the U(1) bundle over $CP^1 \sim S^2$, there exist a natural U(1) connection (gauge field) over S^2. We would like to calculate this connection term A_μ on the S^2 manifold using Atiyah's method.
We shall first discuss the general construction and then the special case of CP^1. Consider a vector bundle E over the base space M consists of a family of vector space E_y parametrised by points $y \in M$. Also, let E be a subbundle of the trivial bundle $M \times R^N$, such that E_y can be embeddedas a vector

subspace in R^N and any section f(y) of E taking its values in E_y can be thought of as a function taking value in E_y. The partial drivative of f, may not take value in R^N . The projection of the ordinary derivatives in to E_y defines the covariant derivative on E,

$$\nabla f = Pdf, \qquad P - \text{projection operator} \tag{A.6}$$

If E is the tangent bundle over M and P be the orthogonal projection, we get Levi-Civita connection of the riemannian geometry. Choosing an orthogonal gauge/local frame for the bundle E gives the linear maps $R^n \to E_y$ which are isomorphisms preserving orthogonality. Composing these isomorphisms with the continuous embedding of E_y in R^N, we can write $U_y : R^n \to R^N$. The $U_y U_y^\dagger = P_y$ is the projection operator and projects orthogonal elements in R^N on to E_y. To calculate the covariant derivative ∇ in the gauge 'U' we put f=Ug, where g is function on M which takes values in R^n.

$$\nabla(Ug) = Pd(Ug) = UU^\dagger d(Ug) = U[dg + i(-iU^\dagger dU)g] \tag{A.7}$$

which shows that the gauge field is

$$A = -iU^\dagger dU \tag{A.8}$$

To determine the U(1) gauge field over CP^1, consider the tautological line bundle over CP^1. with the gauge $z_1 = \bar{z}_1$ the complex doublet $z = \begin{pmatrix} 1/\sqrt{1+\bar{\rho}\rho} \\ \rho/\sqrt{1+\bar{\rho}\rho} \end{pmatrix}$ represent a unit vector in C^2. and multiplication by the nonzero complex numbers λ generates a whole comlex line through this point in CP^1 and defines section/gauge in this line bundle over S^2. The U_y here is given by-

$$U_y = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{1+\bar{\rho}\rho} \\ \rho/\sqrt{1+\bar{\rho}\rho} \end{pmatrix} \tag{A.9}$$

and one gets using (A.8),

$$A = -iZ^\dagger dZ \tag{A.10}$$

A.3 Appendix C

Using the χ parametrization (4.15,4.14) one can obtained the metric over S^3, as induced from the flat metric on C_0^2, given as

$$ds^2 = 4d\chi^\dagger d\chi \tag{A.11}$$

This is obtained as

$$ds^2 = 4d\chi^\dagger d\chi = [d\theta^2 + d\varphi^2 + d\psi^2 + 2\cos\theta d\varphi d\psi] \tag{A.12}$$

Identifying this with the metric tensor through $ds^2 = g_{ij}dx^i dx^j$ with $(x^1 = \theta, x^2 = \varphi, x^3 = \psi)$ one gets,

$$g_{ij} == \begin{bmatrix} g(\partial_\theta, \partial_\theta) & g(\partial_\theta, \partial_\varphi) & g(\partial_\theta, \partial_\psi) \\ g(\partial_\varphi, \partial_\theta) & g(\partial_\varphi, \partial_\varphi) & g(\partial_\varphi, \partial_\psi) \\ g(\partial_\psi, \partial_\theta) & g(\partial_\psi, \partial_\varphi) & g(\partial_\psi, \partial_\psi) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos\theta \\ 0 & \cos\theta & 1 \end{bmatrix} \quad \text{(A.13)}$$

Using this the inner product between J_i (4.24) (4.26) over S^3 can be easily computed to get

$$g(J_i, J_j) = \delta_{ij} \quad \text{(A.14)}$$

A.4 Appendix D

We know that S^2 is ahomogeneous coset space: $S^2 \approx SU(2)/U(1)$. Consider

$$SU(2) \ni g(z) = (1+|z|^2)^{-1/2} \begin{bmatrix} 1 & z \\ -\bar{z} & 1 \end{bmatrix} \quad \text{(A.15)}$$

parametrized using the stereographically projected variables (4.8,4.10) it is then straightforward to calculate the left invariant Maurer-Cartan 1-form $g^{-1}(z)dg(z)$ to get

$$g^{-1}(z)dg(z) = i\frac{A}{2}\sigma_3 + \frac{i}{1+|z|^2}[-i\sigma_+ dz + i\sigma_- d\bar{z}] \quad \text{(A.16)}$$

where

$$A = i(\frac{\bar{z}dz - zd\bar{z}}{1+|z|^2}) \quad \text{(A.17)}$$

is the U(1) connection 1-form. The state $|z + dz\rangle$ is written as-

$$|z + dz\rangle = (1 + g^{-1}dg - i\frac{A}{2}\sigma_3)|z\rangle \quad \text{(A.18)}$$

Note that we have excluded the σ_3 term, as this is asscociated with the stability subgroup U(1). This finally yields

$$|z + dz\rangle = |z\rangle + \frac{i}{1+|z|^2}[-i\sigma_+ dz + i\sigma_- d\bar{z}]|z\rangle \quad \text{(A.19)}$$

The $|z\rangle$ is identified with $|z\rangle \equiv |n, n\rangle$,by rotating the coordinate axes, so that-

$$|z + dz\rangle = |z\rangle + \frac{i\sqrt{2j}d\bar{z}}{1+|z|^2}|n, n-1\rangle \quad \text{(A.20)}$$

and the corresponding $d\rho$ is defined as -

$$d\rho = |z + dz\rangle\langle z + dz| - |z\rangle\langle z| \quad \text{(A.21)}$$

On a straightforward computation, this gives

$$d\rho = \frac{i\sqrt{2j}}{1+|z|^2}(d\bar{z}|n, n-1\rangle\langle n, n| - dz|n, n\rangle\langle n, n-1|) \tag{A.22}$$

Finally, we would like to mention that, using Eulerian angle parametrisation, one can compute the left-invariant 1-form e^i's occuring in the Mauree-Cartan 1-form $g^{-1}dg$ as

$$g^{-1}dg = i/2(e^i\sigma^i) \tag{A.23}$$

to get

$$e^1 = \sin\psi d\theta - \sin\theta\cos\psi d\varphi \tag{A.24}$$

$$e^2 = -\cos\psi d\theta - \sin\theta\sin\psi d\varphi \tag{A.25}$$

$$e^3 = -\cos\theta d\varphi - d\psi \tag{A.26}$$

They provide an orthonormal basis in $T_p^*(S^2)$ and their duals are precisely the SO(3) rotation generators J_i (4.26,4.24) satisfying,

$$< e^i, J_i > = \delta^i_j \tag{A.27}$$

$$g(J_i, J_j) = \delta_{ij} \tag{A.28}$$