Estimation of the Dimension of Cuspidal and Total Cohomology

A thesis
submitted in partial fulfilment of the requirements
for the degree of

Doctor of Philosophy

by

Chaitanya Ambi
ID: 20143346

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE

2019

Dedicated to
My Mother

Certificate

Certified that the work incorporated in the thesis entitled "**Estimation of the Dimension of Cuspidal and Total Cohomology**", submitted by **Chaitanya Ambi** was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: **Prof. A. Raghuram**

(Thesis Supervisor)

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

Date: *Chaitanya Ambi*

Roll Number: 20143346

Abstract.

We consider the Weil restriction of a connected reductive algebraic group over a number field to the rational numbers. For a level structure in the group of its adèlic points, we form an adèlic locally symmetric space. A finite-dimensional, algebraic, irreducible representation of the group of real points of the Weil restriction induces an associated sheaf on this space.

Raghuram and Bhagwat found certain necessary conditions for non-vanishing of the cuspidal part of the respective sheaf cohomology in case of the general linear group under some additional assumptions on the number field and the weight of the representation. Motivated by this, we estimate the growth rate of cuspidal cohomology with varying level structure as well as weight in case of automorphic induction from GL(1) over imaginary quadratic fields to GL(2) over the rationals and also that of symmetric square transfer from GL(2) to GL(3); both over the rationals.

We also present bounds on the dimension of the total sheaf cohomology which apply to an arbitrary connected reductive algebraic group with varying level structure or weight. The bounds thus obtained are consistent with the classical dimension formulae as well as several known results.

Acknowledgements

The author was supported by a Doctoral Fellowship by NBHM throughout his research. He wishes to express special gratitude for the same.
The author is indebted to Prof. A. Raghuram for suggesting this problem. His constant guidance as well as encouragement made this research inspiring and delightful. The author would also like to thank both Prof. C. Bhagwat and Prof. U. K. Anandavardhanan for their suggestions and very helpful conversations. The author is grateful to the faculty, staff and fellow students at IISER Pune for their support and cooperation.
Most importantly, the author is indebted to his mother for her unwavering love and support and wishes to dedicate this work to her.

Chaitanya Ambi.

Statement of Originality

The main results of this thesis which constitute original research are the following four Theorems: 1.4.1, 1.4.3, 1.4.5 and 1.4.9. These theorems have been restated in the subsequent chapters as 7.4.3, 7.3.2, 8.1.2 and 3.3.1 respectively, for convenience of the reader.

All the auxilliary lemmas, including 5.3.2, 5.3.3, 7.4.2, 8.1.1, as well as the remarks and corollaries following the main theorems are original.

Also, each method used and all computations performed represent original work of the author.

Estimation of the Dimension of Cuspidal and Total Cohomology

Chaitanya Ambi

September 10, 2019

Contents

Chapter 1

Introduction.

1.1 Motivation.

The far-sighted conjectures of Langlands on functoriality have given an impetus not only to modern number theory but also to many other areas of mathematics. These conjectures have made allowence for the application of techniques of Representation Theory, Algebraic Geometry as well as Analysis to number theoretic problems. Automorphic forms, together with the analytic theory of L-functions, lie at the heart of these conjectures. These forms are constituents of automorphic resperentations. The notion of an automorphic representation and the associated automorphic form for a general algebraic group over a number field has been studied extensively due to this.

The well-known correspondence of Eichler-Shimura for modular cusp forms permits us to interpret cuspidal automorphic representations cohomologically. More precisely, it relates the cohomology with appropriate coefficients of a certain adèlic locally symmetric space to classical holomorphic modular forms. One can deduce the existence or vanishing of cusp forms with prescribed parameters by studying this cohomology and also prove finiteness or multiplicity one results for the corresponding automorphic representations. Such results have deep number theoretic implications.

It is, then, natural to ask how the dimension of the space of cusp forms grows when we vary the parameters involved therein. More generally, we may ask the same question in case of the group of real points of a connected, reductive linear algebraic group. The corresponding adèlic locally space can be compactified in order that this problem becomes amenable to geometric and analytic methods. Several such compactifications have been constructed and used depending on the purpose, such as the Borel-Serre compactification.

This work was motivated by a result by Raghuram and Bhagwat (see [2]). They found the conditions on the weight of an irreducible representation of the general linear group (over a totally real or a CM field) which ensure the non-vanishing of the cuspidal cohomology for *some* level structure. With the additional hypothesis of the weight being *parallel*, they generalised the result to an arbitrary number field.

Based on this result on existence of cusp forms, it is interesting to find the dimension of the space of automorphic forms for *varying* level structure or as the irreducible representation varies. This is the problem we have attempted to address in the current work. We present estimates on the dimension of the total cohomology complex as either of the representation and the level structure varies.

One can also ask how the cuspidal part of the cohomology grows with the representation or the level structure. Therefore, in a similar vein, we also investigate how *cuspidal* automorphic representations are related to those obtained by Langlands transfer from groups of lower rank under certain L-homomorphisms. This is carried out for automorphic induction from GL_1/E to $GL_2/\mathbb{Q}$, (where $E/\mathbb{Q}$ is an imaginary quadratic field) and also for symmetric square transfer from $GL_2/\mathbb{Q}$ to $GL_3/\mathbb{Q}$. The results obtained are consistent with some of the known bounds and provide a good idea of the exact growth rate of the dimension of the space of those automorphic cusp forms.

1.2 Statement of the problems.

1.2.1 The case of $GL_2/\mathbb{Q}$.

Consider a quadratic extension of the field of rational numbers. We form the group of idèles over this field. The characters of this group can be canonically related to classical Hecke characters (also known as grossencharacters) of the quadratic extension.

In accordance with Langlands functoriality, every algebraic Hecke character yields an automorphic form for $GL_2(\mathbb{A}_{\mathbb{Q}})$ by automorphic induction. Under certain conditions, one obtains a Maass form when the extension is real and a holomorphic modular form for an imaginary quadratic extension.

Furthermore, this modular form is a cusp form whenever the inducing algebraic Hecke character is Galois regular (and an Eisenstein series, otherwise).

Fix a weight and a level structure for $GL_2(\mathbb{A}_{\mathbb{Q}})$. The space of cusp forms for this weight and level is finite dimensional. Each imaginary quadratic extension of $\mathbb{Q}$ may potentially contribute some cusp forms via automorphic induction of its algebraic Hecke characters. We ask the following question about these cusp forms:

Question 1. *How much of the cuspidal cohomology of $GL_2/\mathbb{Q}$ is obtained by automorphic induction?*

We shall find estimates for this total dimension for a fixed weight as the level varies. Also, we shall estimate the growth rate of the total with a fixed level as the weight varies.

1.2.2 The case of $GL_3/\mathbb{Q}$.

Consider a cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$. One may construct an L-homomorphism (termed as the *symmetric square transfer*) from $GL_2(\mathbb{A}_{\mathbb{Q}})$ to $GL_3(\mathbb{A}_{\mathbb{Q}})$ using its Langlands parameters placewise. By Langlands Functoriality, one obtains an automorphic representation of $GL_3(\mathbb{A}_{\mathbb{Q}})$ for each cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$. We ask the following:

Question 2. *How much of the cuspidal cohomology of $GL_3/\mathbb{Q}$ is obtained by symmetric square transfer from $GL_2/\mathbb{Q}$?*

We shall estimate for the above number as either of the weight and the level structure varies.

1.2.3 The case of a connected, reductive linear algebraic group.

Consider a connected, reductive linear algebraic group over a number field. For a finite dimensional irreducible representation of the group of its real points, we may consider vector-valued automorphic forms. We also consider a level structure for the group and form the corresponding adèlic locally symmetric space. The inclusion of cusp forms (with coefficients in the vector space of the irreducible representation), into the space of vector-valued smooth functions on the group of adèlic points of the algebraic group was considered by Borel. He proved that this inclusion produces an injection into the cohomology with the pertinent coefficients. We shall term the latter as cuspidal cohomology.

Raghuram and Bhagwat ([2]) considered the general linear group over a totally real or a CM field. They found that for certain highest weights, the cuspidal cohomology is non-vanishing for some level structure. Motivated by this, we ask the following similar and quantitative question about the total dimension of the respective cohomology complex:

Question 3. *For a connected, reductive linear algebraic group* $\mathbf{G}/F$*, how does the cohomology complex grow with varying level structure as well as the weight of representation?*

We shall find the growth rate of the total dimension as either of the finite dimensional representation and level structure varies.

1.3 An outline of the approach.

We maintain the terminology of the previous section.

1.3.1 The strategy for $GL_2/\mathbb{Q}$.

We use the well-known criterion for cuspidality of automorphic induction mentioned in the previous section, that only the Galois regular Hecke characters of a fixed, compatible infinity-type yield cusp forms on automorphic induction to $GL_2/\mathbb{Q}$. Hence, we need to estimate the former in order to obtain bounds.

To begin, consider a fixed imaginary quadratic extension of the rational numbers. We have a classical relation between an algebraic Hecke character and the induced modular form, which shows that its weight and level are both determined completely by that of the infinity-type of the Hecke character and the norm of the conductor, respectively. We count the number of such characters with an appropriate conductor and sum over all possible imaginary quadratic extensions. More precisely, we obtain bounds on the order of growth of the total number of such characters as the norm of the conductor varies.

Next, we note that the Galois regular Hecke characters are precisely those which do not factor through the idèlic norm map while other Hecke characters are compositions of the idèlic lifts of Dirichlet characters with the said map. The total number of Dirichlet characters of the corresponding conductor is easy to compute. We subtract this from the total number of Hecke characters computed above. Summing over all imaginary quadratic fields and estimating this difference gives us the bounds.

1.3.2 The strategy for $GL_3/\mathbb{Q}$.

We use the criterion for cuspidality established by Gelbart and Jacquet ([18]) for the case of symmetric square transfer. The criterion asserts that a cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ which is not induced from any algebraic Hecke character of a quadratic field yields a cuspidal automorhic representation of $GL_3(\mathbb{A}_\mathbb{Q})$.

Hence, we may draw upon the upper bound obtined in the previous case. More precisely, we estimate the difference between the total number of newforms of a fixed weight as well as level and the total number of algebraic Hecke characters with the relevent conductors computed previously.

We also carry out out similar estimation for constant level structure but varying weight of the resulting respesentation. Note that under certain conditions, the weight of the modular form being transferred completely determines the infinity type of the cusp form for $GL_3(\mathbb{A}_\mathbb{Q})$.

1.3.3 The strategy for a connected, reductive linear algebraic group.

So far, we have used Langlands Functoriality to obtain bounds on the dimension of cuspidal cohomology. It would be neither tractable nor computationally feasible to employ this technique for the transfer by each L-homomorphism from every lower rank group to the group under consideration. Therefore, we consider the total cohomology and adopt a different strategy (at the cost of sharpness of the bounds).

Consider a connected, reductive linear algebraic group over an arbitrary number field. Each topological connected component of the adèlic locally symmetric space corresponding to a fixed level structure of the adèlic points of the group is an orbifold (i.e., quotient of a manifold by a proper group action). If the level structure is deep enough, each of these connected components is a locally symmetric space and corresponding arithmetic groups are all torsion-free. More interestingly, each of these manifolds is an Eilenberg-MacLane space, with the only non-vanishing homotopy group being the fundamental group. Therefore, the cohomology with coefficients in a local system of complex vector spaces coincides with the group cohomology (with the respective coefficients) of the corresponding arithmetic groups.

This is precisely the fact which we shall exploit. By comparison with the level structure for reference and some elementary considerations, we deduce the growth rate of the cohomology with varying volume and also the dimension of the irreducible representation of the coefficient system.

Now, we fix a level structure supporting cohomology (with coefficients in the corresponding vector space) for reference. For each level structure which is deeper than the reference structure but has finite index in it, we consider the pertinent adèlic locally symmetric space. We relate its sheaf cohomology with the group cohomology by means of elementary topology.

Using Shapiro's Lemma, this can be compared with the cohomology of the adèlic locally symmetric space for reference. In doing so, we use the fact that the degree of a finite covering map between the locally symmetric spaces under consideration equals the index of the corresponding arithmetic subgroup. This, in turn, is cast in terms of volume of the total adèlic locally symmetric space for convenience (note that by classical finiteness results, each of these has finite volume).

1.4 Statement of the results.

1.4.1 The bounds in case of $GL_2/\mathbb{Q}$.

Notation 1. *For integers $k, N \geq 1$, let $C_k(N)$ denote the number of cusp eigenforms of normalised Hecke operators for $\Gamma_1(N) \subseteq SL_2(\mathbb{Z})$ having weight k which are obtained by automorphic induction from Hecke characters of every possible imaginary quadratic extension of $\mathbb{Q}$.*

Notation 2. *For an integer $N \geq 1$, define*

$$\hat{N} := \prod_{p|N} p. \tag{1.1}$$

Firstly, we show that $\liminf C_k(N) = 0$ when either of k or N approaches infinity while the other is fixed. This bears out the necessity of passing to deeper level structures in order to find nontrivial lower bounds thereon.

When $k \geq 2$ is fixed, we have the following lower bound:

Theorem 1.4.1. *Every positive integer* n *has a fixed multiple* $N_0 = N_0(n)$ *such that we have*

$$C_k(NN_0) \gg_{k,\epsilon,N_0} N^{1-\epsilon}\hat{N}^{1/2-\epsilon} \quad \forall\epsilon \in (0, 1/2) \tag{1.2}$$

as $N \to \infty$. *Here, the implied constant depends upon the chosen values of* k, N_0 *and* ϵ.

Remark 1.4.2. *Note that when* $k \geq 2$, *each cusp form obtained by automorphic induction from some Hecke character of an imaginary quadratic field corresponds to a* ***cohomological*** *cuspidal automorphic representation of* $GL_2(\mathbb{A}_{\mathbb{Q}})$.

Theorem 1.4.3. *Let* $k, N \geq 1$ *be integers. Let* $C_k(N)$ *be as above. We have*

$$C_k(N) \ll_k N \cdot \hat{N}^{3/2-\epsilon} \quad \forall\epsilon \in (0, 1/2) \tag{1.3}$$

as $N \to \infty$.

Remark 1.4.4. *By Lemma 5.3.2, we have*

$$\dim_{\mathbb{C}} S_k(\Gamma_1(N)) \sim_k N^2.$$

Thus, the above theorem implies that when N *is divisible by large powers of several primes, only a* ***negligible fraction*** *of the holomorphic cusp forms in* $S_k(\Gamma_1(N))$ *is obtained by automorphic induction from Hecke characters.*

This fraction, however, can be rendered ***nonzero*** *by passage to a deeper level structure by virtue of the lower bound obtained earlier.*

1.4.2 The bounds in case of $GL_3/\mathbb{Q}$.

For each finite place p of $\mathbb{Q}$ and integer $n \geq 0$, we define

$$H_p(n) = \{x = (x_{ij})_{3\times 3} \in GL_3(\mathbb{Z}_p) : x_{31}, x_{32} \in p^n\mathbb{Z}_p\}. \tag{1.4}$$

Let $N = \prod_{i=1}^{r} p_i^{n_i}$ be a positive integer. Set

$$H_f(N) := \prod_{i=1}^{r} H_{p_i}(n_i). \tag{1.5}$$

Definition 1. *For each $N \geq 1$, we shall refer to the compact open subgroup $H_f(N) \subset GL_3(\mathbb{A}_f)$ as the* ***level structure*** *corresponding to N.*

We shall denote the set of cuspidal automorphic representations of $GL_2(\mathbb{A}_\mathbb{Q})$ whose non-archimedean part π_f has non-zero K_f-fixed vectors and the infinitesimal character of whose finite part π_∞ equals $\lambda + \rho$, (where ρ denotes half the sum of positive roots of $\mathfrak{gl}_3(\mathbb{C})$) by $\mathcal{A}_{cusp}(GL_2, \lambda, K_f)$.
We define $\mathcal{A}_{cusp}(GL_3, \mu, H_f)$ analogously for $GL_3/\mathbb{Q}$.

Notation 3. *Set $\lambda_k := (k/2 - 1, 1 - k/2)$. Let $\mu_k = (k-2, 0, 2-k)$ be the dominant integral weight for $GL(3, \mathbb{R})$ obtained by symmetric square transfer from λ_k. For $N \geq 1$, consider the level structure $H_f(N)$.*
Define

$$D_k(N) = \{\Pi \in \mathcal{A}_{cusp}(GL_3/\mathbb{Q}, \mu_k, H_f(N)) : \exists \pi \text{ and } M \geq 1 \text{ such that } \\ \pi \in \mathcal{A}_{cusp}(GL_2/\mathbb{Q}, \lambda_k, K_f(M)), \quad \Pi = sym^{\otimes 2}(\pi)\}.$$

Here, $sym^{\otimes 2}$ denotes the symmetric square transfer; see 27 for the definition.

Theorem 1.4.5. *Let $k \geq 2$ be an* ***even*** *integer. Let $p \geq 2$ be a fixed prime. With $D_k(.)$ as above, we have*

$$\# D_k(p^n) \gg_k p^{2n} \quad \text{as } n \to \infty. \tag{1.6}$$

Remark 1.4.6. *Note that we have considered even values of k merely to simplify computations. An analogous result can be proved for odd values of k.*

Remark 1.4.7. *We could also have contribution to cuspidal cohomology for $GL_3/\mathbb{Q}$ from automorphic induction of Hecke characters of* ***cubic extensions*** *of $\mathbb{Q}$. It is well-known that such an extension cannot be totally real and hence cannot be Galois.*

Remark 1.4.8. *When N is not necessarily a prime power, we have the upper bound*

$$\# D_k(N) \ll_k N^2 \quad \text{as } N \to \infty, \tag{1.7}$$

which follows from 3.3.1.

1.4.3 The bounds in case of a connected, reductive linear algebraic group.

Theorem 1.4.9. *Consider a number field F having signature $(r_1, 2r_2)$. Let $\mathbf{G}$ be a connected, reductive linear algebraic group over F and $G := Res_{F/\mathbb{Q}}(\mathbf{G})$*

be the Weil restriction of scalars to $\mathbb{Q}$. *Consider an algebraic, irreducible, finite dimensional complex representation* $(\pi, \mathcal{V})$ *of* $G(\mathbb{R})$. *Select a neat level structure* $K_f^{'}$ *for reference. Let* $S^G_{K_f^{'}}$ *be the corresponding adèlic locally symmetric space such that we have*

$$H^{\bullet}(S^G_{K_f^{'}}, \tilde{\mathcal{V}}) \neq 0,$$

$\tilde{\mathcal{V}}$ *being the associated sheaf. Fix a Haar measure on* $G(\mathbb{A}_{\mathbb{Q}})$. *Then, there exist constants*

$$0 \leq c = c(\mathbf{G}, (r_1, 2r_2), K_f^{'}) \text{ and } C = C(\mathbf{G}, (r_1, 2r_2), K_f^{'})$$

which depend on $\mathbf{G}, K_f^{'}$, *and* $(r_1, 2r_2)$ *such that for every neat level structure* K_f *which satisfies*

$$K_f \trianglelefteq K_f^{'} \text{ and } [K_f^{'} : K_f] < \infty,$$

the following bounds hold:

$$c \leq \frac{\dim_{\mathbb{C}} H^{\bullet}(S^G_{K_f}, \tilde{\mathcal{V}})}{\dim_{\mathbb{C}} \mathcal{V} \cdot vol(S^G_{K_f})} \leq C, \tag{1.8}$$

where

$$H^{\bullet}(S^G_{K_f}, \tilde{\mathcal{V}}) = \bigoplus_{q \in \mathbb{Z}} H^q(S^G_{K_f}, \tilde{\mathcal{V}}).$$

Remark 1.4.10. *Whenever the Borel-Serre compactification* $\bar{S}^G_{K_f^{'}}$ *of* $S^G_{K_f^{'}}$ *has a connected component with non-vanishing Euler characteristic, the constant* c *as above is in fact strictly positive and the lower bound is non-trivial. Using a result by Dodziuk (see [14]), it can be shown that this occurs whenever* $G(\mathbb{R})/K^0_{\infty}$ *is a hyperbolic manifold of even real dimension. This occurs, for instance, in case of elliptic as well as Hilbert modular forms.*

Remark 1.4.11. *The above theorem seems to be known to experts. Yet, it is apparently not available in the literature as stated above to the best of our knowledge. Also, our proof is elementary in that it does not use the Trace Formula or any analytic means (unlike the analogous known results).*

Part I

Estimation of Total Cohomology.

Chapter 2

Algebraic and Topological Preliminaries.

2.1 Notation and Preliminaries.

2.1.1 The number field.

Let F be a finite extension of $\mathbb{Q}$. Let S_∞ denote the set of its archimedean places, which consists of r_1 real places and r_2 pairs of complex conjugate places (denoted by S_r and S_c, respectively). The degree of this field extension is equal to $d_F := r_1 + 2r_2$.

The completion of F at a place v will be denoted by F_v. For $v \notin S_\infty$, we let O_v be the ring of integers of F_v. We shall denote the ring of adeles over F by $\mathbb{A}_F$ and its finite part by $\mathbb{A}_{F,f}$. Similarly, $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_f$ will denote the adeles over $\mathbb{Q}$ and the finite adeles over $\mathbb{Q}$, respectively.

2.1.2 Weil restriction of scalars.

Consider the algebraic closure $\bar{\mathbb{Q}} \subset \mathbb{C}$ of $\mathbb{Q}$. Let Σ be the set of all distinct field homomorphisms $\sigma : F \to \bar{\mathbb{Q}}$. For an affine F-variety V and a ring R, we shall denote its R-points of V by $V(R)$.

Definition 2. *Let V be an affine F-variety. There exists an affine $\mathbb{Q}$-variety denoted by $Res_{F/\mathbb{Q}}V$ and called the **Weil restriction** of scalars which, for any commutative $\mathbb{Q}$-algebra A, has the following property:*

$$Res_{F/\mathbb{Q}}V(A) = V(F \otimes_\mathbb{Q} A)$$

Such a variety is unique up to a canonical isomorphism over $\mathbb{Q}$ and satisfies the following over $\bar{\mathbb{Q}}$:

$$Res_{F/\mathbb{Q}} V \times \bar{\mathbb{Q}} = \prod_{\sigma \in \Sigma} V \times_{F,\sigma} \bar{\mathbb{Q}} \tag{2.1}$$

2.1.3 The algebraic group.

Let $\mathbf{G}$ be a connected, reductive linear algebraic group over F. Denote its centre by $\mathbf{Z}$. Choose a Borel subgroup $\mathbf{B} = \mathbf{TU}$ with its corresponding maximal torus $\mathbf{T}$ and unipotent radical $\mathbf{U}$. Define the following:

$$\begin{aligned} G &:= Res_{F/\mathbb{Q}}(\mathbf{G}), \\ B &:= Res_{F/\mathbb{Q}}(\mathbf{B}), \\ Z &:= Res_{F/\mathbb{Q}}(\mathbf{Z}), \\ T &:= Res_{F/\mathbb{Q}}(\mathbf{T}). \end{aligned}$$

Let C_∞ be the maximal compact subgroup of $G(\mathbb{R})$ corresponding to the Iwasawa decomposition

$$G(\mathbb{R}) = B(\mathbb{R}) C_\infty. \tag{2.2}$$

Set

$$K_\infty = Z(\mathbb{R})^0 C_\infty, \tag{2.3}$$

where $Z(\mathbb{R})^0$ is the topological connected component of identity of $Z(\mathbb{R})$. Similarly, let K_∞^0 be the topological connected component of identity of K_∞.

2.2 The adèlic locally symmetric space.

Let $K_f = \prod_{p<\infty} K_p$ be a compact open subgroup of $G(\mathbb{A}_f) = \mathbf{G}(\mathbb{A}_{F,f})$. Here, K_p is a compact open subgroup of $G(\mathbb{Q}_p)$ such that we have

$$K_p = \prod_{v|p} \mathbf{G}(O_v) \tag{2.4}$$

for all but finitely many non-archimedean places p. We call K_f a **level structure**. Also, let D denote the symmetric space $G(\mathbb{R})/K_\infty^0$.

Now, we embed $G(\mathbb{Q})$ diagonally in $G(\mathbb{A}_\mathbb{Q})$. It embedds as a discrete subgroup. Hence, the action of $G(\mathbb{Q})$ on $G(\mathbb{A}_\mathbb{Q})$ by left multiplication is properly discontinuous. This action descends to the right coset space $D \times G(\mathbb{A}_f)/K_f$ and continues to be properly discontinuous because of the compactness of both K_f and K_∞^0.

Definition 3. *We define the* ***adèlic locally symmetric space*** *with level structure* K_f *as the quotient of* $D \times G(\mathbb{A}_f)/K_f$ *by the left action of* $G(\mathbb{Q})$*:*

$$S^G_{K_f} := G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K^0_\infty K_f.$$

The left action of $G(\mathbb{Q})$ on $G(\mathbb{A}_f)/K_f$ is known to produce a finite number of double cosets. Let

$$G(\mathbb{A}_f) = \amalg_{1\le i\le m} G(\mathbb{Q}) g_i K_f \tag{2.5}$$

be a set of double coset representatives.

Definition 4. K_f *is called* ***neat*** *if the coset stabilisers*

$$\Gamma_i := Stab_{G(\mathbb{Q})}(g_i K_f), \quad (1 \le i \le m) \tag{2.6}$$

are all torsion-free.

We also note that

$$\Gamma_i = G(\mathbb{Q}) \cap g_i K_f g_i^{-1}, \quad (1 \le i \le m). \tag{2.7}$$

When K_f is neat, $S^G_{K_f}$ is the following finite union of manifolds:

$$S^G_{K_f} = \amalg_{1\le i\le m} \Gamma_i \backslash D. \tag{2.8}$$

Each $X_i := \Gamma_i\backslash D$ is a locally symmetric space with the simply connected cover $D = G(\mathbb{R})/K^0_\infty$.

2.3 The associated sheaf.

For the algebraic group $G/\mathbb{Q}$ and an affine $\mathbb{Q}$-vector space $\mathcal{V}$, consider an algebraic representation

$$\pi : G \to Aut(\mathcal{V}). \tag{2.9}$$

For each open subset $U \subset S^G_{K_f}$, consider the functor $\mathcal{F}$ defined as follows:

$$\begin{aligned}\mathcal{F}(U) = \{f : p^{-1}(U) \to \mathcal{V} \otimes_\mathbb{Q} \mathbb{C} : f \text{ locally constant and}\\ f(gu) = \pi(g^{-1})f(u) \quad \forall g \in G(\mathbb{Q}), u \in U\}\end{aligned}$$

This defines a presheaf of complex vector spaces on $S^G_{K_f}$. Here, p denotes the quotient by the action of $G(\mathbb{Q})$:

$$p : D \times G(\mathbb{A}_f)/K_f \twoheadrightarrow S^G_{K_f}. \tag{2.10}$$

Remark 2.3.1. *It is known that* $\mathcal{F}$ *extends to a sheaf* $\tilde{\mathcal{F}}$ *which is nonzero only if the cental character of* π *is an algebraic Hecke character of* F*. We refer the reader to [22] for details.*

2.3.1 Dominant integral weights.

Let $\mathfrak{g}$ (resp., $\mathfrak{b}$) denote the complexified Lie algebra of $G(\mathbb{R})$ (resp., $B(\mathbb{R})$). A dominant integral weight of $\mathfrak{g}$ with respect to $\mathfrak{b}$ is of the form

$$\mu = (\mu^v)_{v \in S_\infty}, \tag{2.11}$$

where we have

$$\mu^v = (\mu^w, \mu^{\bar{w}}) \tag{2.12}$$

whenever $v = \{w, \bar{w}\} \in S_c$ and each μ^v is a dominant integral weight of $\mathbf{G}(F_v)$.

2.3.2 The induced local sytem.

Let μ be a dominant integral weight of $G(\mathbb{R})$ as above. We shall denote by $(\pi_\mu, \mathcal{V}_\mu)$ the finite dimensional complex representation of $G(\mathbb{R})$ with highest weight μ. Note that

$$\pi_\mu = \bigotimes_{v \in S_\infty} \pi_{\mu^v}, \tag{2.13}$$

where $\pi_{\mu^v} = \pi_{\mu^w} \otimes \pi_{\mu^{\bar{w}}}$ whenever $v = \{w, \bar{w}\} \in S_c$.

Definition 5. *By a $\mathbb{C}$-**local system** of rank r on a topological space, we mean a locally constant sheaf whose stalks are all isomorphic to $\mathbb{C}^r$.*

Remark 2.3.2. *Set $\mathcal{V} = \mathcal{V}_\mu$ and consider the sheaf $\tilde{\mathcal{F}}$ (with the notation as above). It is noteworthy that when the subgroups Γ_i are all torsion free (i.e., the level structure is neat), the sheaf $\tilde{\mathcal{F}}$ defined as above is a $\mathbb{C}$-local system. We shall denote this local system by $\tilde{\mathcal{V}}_\mu$.*

2.4 Borel-Serre compactification.

We saw that the adèlic locally symmetric space under consideration is a finite disjoint union of manifolds:

$$S^G_{K_f} = \bigcup_i X_i. \tag{2.14}$$

Although each X_i is known to have finite volume, it is neither compact nor symmetric in general. In order that the problem becomes tractable with geometric methods, we work with the **Borel-Serre compactification** $\bar{X}_i^{BS}$.

This is a compact, connected manifold with corners whose interior is diffeomorphic to X_i (see [4]). It may be realised as the quotient

$$\bar{X}_i^{BS} = \Gamma_i \backslash {}_{\mathbb{Q}}\bar{D}^{BS}. \tag{2.15}$$

Here, ${}_{\mathbb{Q}}\bar{D}^{BS}$ is the partial Borel-Serre compactification as in [4].

Remark 2.4.1. *The associated sheaf is known to extend to $\partial \bar{X}_i$, which we shall denote by the same symbol $\tilde{\mathcal{V}}$. Most importantly, the inclusion $i : X \hookrightarrow \bar{X}$ is known to be a homotopy equivalence and hence, preserves the de Rham cohomology of $X_i \quad \forall 1 \leq i \leq m$.*

Notation 4. *Henceforth, we shall denote ${}_{\mathbb{Q}}\bar{D}^{BS}$ by $\bar{D}$ for brevity. Let D^0 denote the connected space $G(\mathbb{R})^0/K_\infty^0$ (where $G(\mathbb{R})^0$ is the connected component of identity in $G(\mathbb{R})$).*

Using the Borel-Serre partial compactification $\bar{D}$, we observe the following:

Lemma 2.4.1. *For each $1 \leq i \leq m$, the symmetric space $\Gamma_i \backslash D$ is homotopy equivalent to a finite CW-complex. $\Gamma_i \backslash D$ is the union of connected components as follows:*

$$\Gamma_i \backslash D = \amalg_{1 \leq j \leq \# \pi_0(D)} Y_{ij}.$$

Furthermore, if we set $\Gamma_{ij} := \pi_1(Y_{ij}) \quad (1 \leq j \leq \# \pi_0(D))$, then each of the connected components Y_{ij} is an Eilenberg-MacLane space up to homotopy equivalence:

$$Y_{ij} \approx K(\Gamma_{ij}, 1), \qquad \textit{for each } 1 \leq j \leq \# \pi_0(D) \textit{ and } 1 \leq i \leq m.$$

Proof. The compactification

$$\Gamma_i \backslash D \mapsto \Gamma_i \backslash \bar{D} \tag{2.16}$$

is known to be a homotopy equivalence. The first claim follows form [5] as $\Gamma_i \backslash \bar{D}$ admits a finite triangulation.

Recall that $D = G(\mathbb{R})/K_\infty^0$. As K_∞^0 is connected, D has the same number of connected components as $G(\mathbb{R})$. Since $G(\mathbb{R})^0/K_\infty^0$ is known to be a contractible manifold, it is simply connected. Hence, $G(\mathbb{R})^0/K_\infty^0$ is the universal cover of each of the connected components of $\Gamma_i \backslash \bar{D}$ and $n \geq 1$.

Since the higher homotopy groups $\pi_n, n \geq 2$ are unaffected on passing to universal covers, we have the following for each connected component Y_{ij} of $\Gamma_i \backslash D$ and $n \geq 1$:

$$\pi_n(Y_{ij}) = \begin{cases} \Gamma_{ij} \text{ if } n = 1, \\ 0 \text{ for } n \geq 2 \end{cases} \tag{2.17}$$

(0 denotes the trivial group here). Hence the last assertion of the lemma follows. ■

2.5 Cuspidal cohomology.

It is well-known that the inclusion

$$\mathcal{C}^{\infty}_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A})) \xrightarrow{\iota} \mathcal{C}^{\infty}(G(\mathbb{Q})\backslash G(\mathbb{A})) \tag{2.18}$$

of automorphic cusp forms into the space of smooth functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ induces an inclusion of $(\mathfrak{g}, K^0_\infty)$-cohomology (see [3]). On taking K_f-invariants, this yields an injection $i^\bullet_{K_f}$ as follows:

$$H^\bullet_{(\mathfrak{g},K^0_\infty)}(\mathcal{C}^{\infty}_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \otimes \mathcal{V}_\mu) \xrightarrow{i^\bullet_{K_f}} H^\bullet_{(\mathfrak{g},K^0_\infty)}(\mathcal{C}^{\infty}(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \otimes \mathcal{V}_\mu).$$

By virtue of the isomorphism

$$j^\bullet_{K_f} : H^\bullet_{(\mathfrak{g},K^0_\infty)}(\mathcal{C}^{\infty}(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \otimes \mathcal{V}_\mu) \approx H^\bullet(S^G_{K_f}; \tilde{\mathcal{V}}_\mu),$$

we may define cuspidal cohomology as follows.

Definition 6. *Let $\tilde{\mathcal{V}}_\mu$ be the associated sheaf for the representation $(\pi, \mathcal{V}_\mu)$ defined in the previous section. We define its **cuspidal cohomology** with coefficients in $\tilde{\mathcal{V}}_\mu$ as follows:*

$$H^\bullet_{cusp}(S^G_{K_f}; \tilde{\mathcal{V}}_\mu) := im(j^\bullet_{K_f} \circ i^\bullet_{K_f})$$
$$(i.e.,\ H^\bullet_{cusp}(S^G_{K_f}; \tilde{\mathcal{V}}_\mu) \approx H^\bullet_{(\mathfrak{g},K^0_\infty)}(\mathcal{C}^{\infty}_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \otimes \mathcal{V})).$$

Remark 2.5.1. *Note that whenever K_f is neat, $\tilde{\mathcal{V}}_\mu$ is a local system of rank $dim_{\mathbb{C}}\mathcal{V}_\mu$.*

Chapter 3

Bounds for a connected, reductive linear algebraic group.

We mention that a stronger form of our result to follow was obtained by Rohlfs and Speh in [33] by means of the Trace formula. Although our result is a partial rediscovery, its proof is elementary.

3.1 The result by Raghuram and Bhagwat.

As in the previous chapter, let $S^G_{K_f}$ denote an adèlic locally symmetric space for some level structure K_f. Let $\mathcal{V}_{\mu,\mathbb{C}}$ be the algebraic irreducible representation of $G(\mathbb{R})$ having highest weight μ. Let $\tilde{\mathcal{V}}_{\mu,\mathbb{C}}$ denote the associated sheaf on $S^G_{K_f}$.

When F is either totally real or a CM field, Raghuram and Bhagwat (see [2]) have ascertained the conditions on μ which ensure the non-vanishing of the cuspidal cohomology for **some** level structure, i.e.,

$$\varinjlim_{K_f} H^{\bullet}_{cusp}(S^G_{K_f}, \tilde{\mathcal{V}}_{\mu,\mathbb{C}}) \neq 0. \tag{3.1}$$

The aim is to estimate $\dim_{\mathbb{C}} H^{\bullet}_{cusp}(S^G_{K_f}, \tilde{\mathcal{V}}_{\mu,\mathbb{C}})$ **in terms of** $\dim_{\mathbb{C}} \mathcal{V}_{\mu,\mathbb{C}}$ **and the volume of** $S^G_{K_f}$. (By 'volume', we mean the measure induced on $S^G_{K_f}$ by a suitable choice of Haar measure on $G(\mathbb{A}_F)$ here.)

3.2 Some well-known auxilliary results.

Here, we state some auxiliary results for the sake of completeness.

Lemma 3.2.1. *Consider a finite CW-complex X and a finite dimensional complex vector space $\mathcal{V}$. Let $\tilde{\mathcal{V}}$ be a local system of $\mathbb{C}$-vector spaces on X with stalk $\mathcal{V}$. If*

$$X = \amalg_i X_i \tag{3.2}$$

are the path components of X, then we have

$$H^q(X;\tilde{\mathcal{V}}) = \bigoplus_i H^q(X_i;\tilde{\mathcal{V}}|_{X_i}) \quad \forall q \geq 0. \tag{3.3}$$

Proof. The assertion of the lemma holds for **homology** with coefficients in the dual local system $\check{\tilde{\mathcal{V}}} := Hom(\tilde{\mathcal{V}}, \mathbb{C}_X)$ (see [40], Chap. VI, Thm.3.1, p.275). Since $\mathcal{V}$ is finite dimensional, we render it applicable to cohomology by means of [13] (Chap.2, (2.1), p.50):

$$H^q(X;\tilde{\mathcal{V}}) = H_q(X;(\check{\tilde{\mathcal{V}}})\check{}). \tag{3.4}$$

■

Lemma 3.2.2. *Let Γ be a group and $\mathcal{V}$ be a finite dimensional complex Γ-module. Let $\tilde{\mathcal{V}}$ be a local system of $\mathbb{C}$-vector spaces on $K(\Gamma,1)$ with stalk $\mathcal{V}$. Then we have*

$$H^q(K(\Gamma,1);\tilde{\mathcal{V}}) = H^q(\Gamma;\mathcal{V}) \quad \forall q \geq 0, \tag{3.5}$$

where $H^\bullet(\Gamma;\mathcal{V})$ denotes the group cohomology of Γ with coefficients in $\mathcal{V}$:

$$H^\bullet(\Gamma;\mathcal{V}) := Ext^\bullet_\Gamma(\mathbb{C},\mathcal{V}). \tag{3.6}$$

Proof. This is [40] (Chap. VI, Thm.3.5, p.281) used with complex coefficients. ■

Lemma 3.2.3. *Let X be a finite CW-complex with $C^\bullet(X;\mathbb{C})$ as its cochain complex. Consider a local system $\tilde{\mathcal{V}}$ of $\mathbb{C}$-vector spaces having stalk $\mathcal{V}$ on X. Then we have*

$$\dim_\mathbb{C} H^q(X;\tilde{\mathcal{V}}) \leq \dim_\mathbb{C} \mathcal{V} \cdot \dim_\mathbb{C} C^q(X;\mathbb{C}) \quad \forall q \geq 0. \tag{3.7}$$

If we set

$$\chi(X;\tilde{\mathcal{V}}) := \sum_{q\geq 0} (-1)^q \dim_\mathbb{C} H^q(X;\tilde{\mathcal{V}})$$

and $\chi(X)$ denotes the Euler characteristic of X, then we also have

$$\chi(X;\tilde{\mathcal{V}}) = \chi(X) \cdot \dim_\mathbb{C} \mathcal{V}. \tag{3.8}$$

Proof. The first assertion is in [13] (Prop. 2.5.4 (i), p.49.). As for the Euler characteristic, see [13] (Prop. 2.5.4 (i), p.49.). ∎

Lemma 3.2.4 (Shapiro)**.** *If $H \leq G$ and M is an H-module, then we have a canonical isomorphism*

$$H^\bullet(H, M) \approx H^\bullet(G, co\text{-}Ind_H^G M), \tag{3.9}$$

where

$$co\text{-}Ind_H^G M := Hom_{\mathbb{C}H}(\mathbb{C}G, M). \tag{3.10}$$

Proof. See [7], Prop.3.3, p.147. ∎

3.3 An Estimate for the Total Cohomology.

3.3.1 Statement of the Main Result.

We shall present an elementary proof of the following result:

Theorem 3.3.1. *Consider a number field F having signature $(r_1, 2r_2)$. Let $\mathbf{G}$ be a connected, reductive linear algebraic group over F and $G := Res_{F/\mathbb{Q}}(\mathbf{G})$ be the Weil restriction of scalars to $\mathbb{Q}$. Consider an algebraic, irreducible, finite dimensional complex representation $(\pi, \mathcal{V})$ of $G(\mathbb{R})$. Select a neat level structure $K_f^{'}$ for reference. Let $S^G_{K_f^{'}}$ be the corresponding adèlic locally symmetric space for such that we have*

$$H^\bullet(S^G_{K_f^{'}}, \tilde{\mathcal{V}}) \neq 0, \tag{3.11}$$

$\tilde{\mathcal{V}}$ being the associated sheaf. Fix a normalisation of the Haar measure on $G(\mathbb{A}_\mathbb{Q})$ such that $vol(S^G_{K_f}) = 1$. Then, there exist constants

$$0 \leq c = c(\mathbf{G}, (r_1, 2r_2), K_f^{'}) \text{ and } C = C(\mathbf{G}, (r_1, 2r_2), K_f^{'}) \tag{3.12}$$

which depend on $\mathbf{G}, K_f^{'}$ and $(r_1, 2r_2)$ such that for every neat level structure K_f which satisfies

$$K_f \trianglelefteq K_f^{'} \text{ and } [K_f^{'} : K_f] < \infty, \tag{3.13}$$

the following bounds hold:

$$c \leq \frac{\dim_\mathbb{C} H^\bullet(S^G_{K_f}, \tilde{\mathcal{V}})}{\dim_\mathbb{C} \mathcal{V} \cdot vol(S^G_{K_f})} \leq C, \tag{3.14}$$

where

$$H^\bullet(S^G_{K_f}, \tilde{\mathcal{V}}) := \bigoplus_{q \in \mathbb{Z}} H^q(S^G_{K_f}, \tilde{\mathcal{V}}).$$

Remark 3.3.2. *Whenever $S^G_{K'_f}$ has a connected component with non-vanishing Euler characteristic, the constant c as above is in fact strictly positive and the lower bound is non-trivial.*
Using a result by Dodziuk (see [14]), it can be shown that this occurs whenever $G(\mathbb{R})/K^0_\infty$ is a hyperbolic manifold of even real dimension.

3.3.2 Proof.

The proof will be presented in several small parts.

Proof. Fix a neat level structure K'_f with the said property. Let $K_f \trianglelefteq K'_f$ be an arbitrary level structure having $[K'_f : K_f] < \infty$.

The level structure $K_f \trianglelefteq K'_f$.
We begin with the following claim:

$$K_f \trianglelefteq K'_f \text{ and } [K'_f : K_f] < \infty \Rightarrow \exists p : S^G_{K_f} \twoheadrightarrow S^G_{K'_f} \tag{3.15}$$

such that p is a covering map and $S^G_{K_f}$ is a finite normal cover of $S^G_{K'_f}$.
Let

$$G(\mathbb{A}_f) = \amalg_{1\leq i\leq m} G(\mathbb{Q}) g_i K'_f \text{ and}$$
$$K'_f = \amalg_{1\leq j\leq [K'_f:K_f]} h_j K_f, \quad (h_j \in K'_f).$$

Hence,

$$G(\mathbb{A}_f) = \coprod_{\substack{1\leq i\leq m \\ 1\leq j\leq [K'_f:K_f]}} G(\mathbb{Q}) g_i h_j K_f. \tag{3.16}$$

Set

$$\Gamma'_i := Stab_{G(\mathbb{Q})}(g_i K'_f) \quad (1 \leq i \leq m)$$
$$\text{and } \Gamma_{ij} := Stab_{G(\mathbb{Q})}(g_i h_j K_f) \quad (1 \leq j \leq [K'_f : K_f]).$$

We have

$$\Gamma_{ij} = G(\mathbb{Q}) \cap g_i(h_j K_f h_j^{-1}) g_i^{-1} \tag{3.17}$$

because $\Gamma'_i = G(\mathbb{Q}) \cap g_i K_f g_i^{-1}$. Since K_f is normal in K'_f, we have

$$g_i K'_f g_i^{-1} \trianglerighteq g_i K_f g_i^{-1} = g_i(h_j K_f h_j^{-1}) g_i^{-1}. \tag{3.18}$$

Intersecting with $G(\mathbb{Q})$, we infer that

$$\Gamma_{ij} \trianglelefteq \Gamma_i' \quad \forall 1 \leq i \leq m, \text{ and } 1 \leq j \leq [K_f' : K_f]. \tag{3.19}$$

Incidentally, this argument shows that the index $[\Gamma_i' : \Gamma_{ij}]$ is independent of $j \quad (\forall 1 \leq i \leq m)$ and also that K_f is neat.
This proves the claim, as we have

$$S^G_{K_f} = \coprod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq [K_f' : K_f]}} \Gamma_{ij} \backslash D \text{ and} \tag{3.20}$$

$$S^G_{K_f'} = \coprod_{1 \leq i \leq m} \Gamma_i' \backslash D. \tag{3.21}$$

The locally symmetric spaces.

Since Γ_i' is discrete and torsionless for each i, we get

$$\Gamma_i' \cap C^0_\infty = \{1\}, \quad \forall 1 \leq i \leq m.$$

As $K^0_\infty = C^0_\infty Z(\mathbb{R})^0$, we see that

$$\Gamma_i' \cap K^0_\infty \subseteq Z(\mathbb{R})^0. \tag{3.22}$$

Also, each Γ_i' has trivial intersection with the group of deck transformations of the covering map p as above (since the latter is finite and hence torsion). These considertaions apply verbatim to each $\Gamma_{ij}, \quad (1 \leq j \leq [K_f' : K_f])$. Therefore, we may write

$$S^G_{K_f} = \coprod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq [K_f' : K_f]}} \ \coprod_{1 \leq k \leq \#\pi_0(D)} \Gamma_{ij} \backslash D^0_k, \quad \text{and} \tag{3.23}$$

$$S^G_{K_f'} = \coprod_{1 \leq i \leq m} \ \coprod_{1 \leq k \leq \#\pi_0(D)} \Gamma_i' \backslash D^0_k, \tag{3.24}$$

where we have denoted diffeomorphic copies of D^0 by the suffix k merely for the sake of keeping a count.

Computation of cohomology.

Now, we compute $H^\bullet(S^G_{K_f}, \tilde{\mathcal{V}})$. Note that the connected components of $S^G_{K_f}$ are manifolds (even locally symmetric spaces) because K_f is neat. Thus, the connected components coincide with path components and Lemma 3.2.1 applies:

$$H^\bullet(S^G_{K_f}, \tilde{\mathcal{V}}) = \bigoplus_{\substack{1\le i\le m \\ 1\le j\le [K'_f:K_f]}} \bigoplus_{1\le k\le \#\pi_0(D)} H^\bullet(\Gamma_{ij}\backslash D^0_k; (\tilde{\mathcal{V}})|_{\Gamma_{ij}\backslash D^0_k}). \tag{3.25}$$

Since $\pi_1(\Gamma_{ij}\backslash D^0_k) = \Gamma_{ij}$, Lemma 2.4.1 applies and we have

$$\Gamma_{ij}\backslash D^0_k \approx K(\Gamma_{ij}, 1). \tag{3.26}$$

We combine this with Lemma 3.2.2 to obtain the following (note that each Γ_{ij} is a subgroup of $G(\mathbb{Q})$):

$$H^\bullet(S^G_{K_f}, \tilde{\mathcal{V}}) = \bigoplus_{\substack{1\le i\le m \\ 1\le j\le [K'_f:K_f]}} \bigoplus_{1\le k\le \#\pi_0(D)} H^\bullet(\Gamma_{ij}; \mathcal{V}). \tag{3.27}$$

Now, we use Shapiro's lemma to get

$$H^\bullet(\Gamma_{ij}; \mathcal{V}) = H^\bullet(\Gamma'_i; \textit{co-Ind}^{\Gamma'_i}_{\Gamma_{ij}} \mathcal{V}). \tag{3.28}$$

Therefore, we obtain

$$H^\bullet(S^G_{K_f}, \tilde{\mathcal{V}}) = \bigoplus_{\substack{1\le i\le m \\ 1\le j\le [K'_f:K_f]}} \bigoplus_{1\le k\le \#\pi_0(D)} H^\bullet(\Gamma'_i; \textit{co-Ind}^{\Gamma'_i}_{\Gamma_{ij}} \mathcal{V}). \tag{3.29}$$

Since $\dim_{\mathbb{C}} \mathcal{V}$ and $[\Gamma'_i : \Gamma_{ij}]$ are both finite, we can identify the co-induced modules with induced ones throughout:

$$\textit{co-Ind}^{\Gamma'_i}_{\Gamma_{ij}} \mathcal{V} \approx \textit{Ind}^{\Gamma'_i}_{\Gamma_{ij}} \mathcal{V}. \tag{3.30}$$

The total volume.

We may identify the finite index $[\Gamma'_i : \Gamma_{ij}]$ of each subgroup with the degree of the corresponding covering map. Furthermore, this degree is the ratio of the respective volumes. This is because all these locally symmetric spaces have finite volume under the invariant metric induced by the choice of the Haar measure on $G(\mathbb{A}_\mathbb{Q})$:

$$[\Gamma'_i : \Gamma_{ij}] = \frac{vol(\Gamma_{ij}\backslash D^0_k)}{vol(\Gamma'_i\backslash D^0_k)}. \tag{3.31}$$

Upper bound.

We have

$$\dim_{\mathbb{C}} H^\bullet(S^G_{K_f};\tilde{\mathcal{V}}) = \sum_{\substack{1\le i\le m\\ 1\le j\le [K'_f:K_f]}} \left\{ \sum_{1\le k\le \#\pi_0(D)} \dim_{\mathbb{C}} H^\bullet(\Gamma'_i; Ind^{\Gamma'_i}_{\Gamma_{ij}}\mathcal{V})\right\} \tag{3.32}$$

By Lemma 3.2.2 and Lemma 3.2.3, we have

$$\dim_{\mathbb{C}} H^\bullet(\Gamma'_i; Ind^{\Gamma'_i}_{\Gamma_{ij}}\mathcal{V}), \le \dim_{\mathbb{C}} C^\bullet(\Gamma'_i\backslash\bar{D}^0_k;\mathbb{C})\cdot \dim_{\mathbb{C}} Ind^{\Gamma'_i}_{\Gamma_{ij}}\mathcal{V},$$
$$\dim_{\mathbb{C}} C^\bullet(\Gamma'_i\backslash\bar{D}^0_k;\mathbb{C})\cdot \dim_{\mathbb{C}} Ind^{\Gamma'_i}_{\Gamma_{ij}}\mathcal{V} = \dim_{\mathbb{C}} C^\bullet(\Gamma'_i\backslash\bar{D}^0_k;\mathbb{C})\cdot[\Gamma'_i:\Gamma_{ij}]\cdot\dim_{\mathbb{C}}\mathcal{V}. \tag{3.33}$$

Therefore,

$$\begin{aligned}
&\dim_{\mathbb{C}} H^\bullet(S^G_{K_f};\tilde{\mathcal{V}}) \le \\
&\quad \sum_{\substack{1\le i\le m\\ 1\le j\le [K'_f:K_f]}} \left\{ \sum_{1\le k\le \#\pi_0(D)} \dim_{\mathbb{C}} C^\bullet(\Gamma'_i\backslash\bar{D}^0_k;\mathbb{C})\cdot[\Gamma'_i:\Gamma_{ij}]\cdot\dim_{\mathbb{C}}\mathcal{V}\right\}\\
&= \sum_{\substack{1\le i\le m\\ 1\le j\le [K'_f:K_f]}} \left\{ \sum_{1\le k\le \#\pi_0(D)} \dim_{\mathbb{C}} C^\bullet(\Gamma'_i\backslash\bar{D}^0_k;\mathbb{C})\cdot\frac{vol(\Gamma_{ij}\backslash D^0_k)}{vol(\Gamma'_i\backslash D^0_k)}\cdot\dim_{\mathbb{C}}\mathcal{V}\right\}\\
&\qquad\le (\dim_{\mathbb{C}}\mathcal{V})\cdot\left(\frac{\max_{1\le i\le m}\dim_{\mathbb{C}} C^\bullet(\Gamma'_i\backslash\bar{D}^0_k;\mathbb{C})}{\min_{1\le i\le m} vol(\Gamma'_i\backslash D^0_k)}\right)\\
&\qquad\quad\cdot\left\{\sum_{\substack{1\le i\le m\\ 1\le j\le [K'_f:K_f]}}\ \sum_{1\le k\le \#\pi_0(D)} vol(\Gamma_{ij}\backslash D^0_k)\right\}\\
&\qquad\quad = \left(\frac{\max_{1\le i\le m}\dim_{\mathbb{C}} C^\bullet(\Gamma'_i\backslash\bar{D}^0;\mathbb{C})}{\min_{1\le i\le m} vol(\Gamma'_i\backslash D^0)}\right)\cdot\dim_{\mathbb{C}}\mathcal{V}\cdot vol(S^G_{K_f})
\end{aligned}$$

But the fraction above depends solely on $S^G_{K'_f}$, which in turn depends on $\mathbf{G}, (r_1, 2r_2)$ and the reference level structure K'_f for a fixed $\mathcal{V}$. This establishes the upper bound with

$$C = C(\mathbf{G},(r_1,2r_2),K'_f) := \frac{\max_{1\le i\le m}\dim_{\mathbb{C}} C^\bullet(\Gamma'_i\backslash\bar{D}^0_k;\mathbb{C})}{\min_{1\le i\le m} vol(\Gamma'_i\backslash D^0_k)}. \tag{3.34}$$

Lower bound.

By the hypotheses of the theorem, the total cohomology is non-vanishing for $K_f^{'}$ and hence for K_f. The Euler characteristic

$$\chi(\bar{S}^G_{K_f};\tilde{\mathcal{V}}) = \sum_{q\geq 0}(-1)^q \dim_{\mathbb{C}} H^q(S^G_{K_f};\tilde{\mathcal{V}}) \tag{3.35}$$

provides a lower bound on the total dimension as follows:

$$\dim_{\mathbb{C}} H^\bullet(S^G_{K_f};\tilde{\mathcal{V}}) \geq |\chi(\bar{S}^G_{K_f};\tilde{\mathcal{V}})|. \tag{3.36}$$

By Lemma 3.2.3, we have

$$\chi(\bar{S}^G_{K_f};\tilde{\mathcal{V}}) = \dim_{\mathbb{C}} \mathcal{V}_\mu \cdot \chi(\bar{S}^G_{K_f}). \tag{3.37}$$

But for each covering $\Gamma_{ij}\backslash\bar{D}^0_k \twoheadrightarrow \Gamma_i^{'}\backslash\bar{D}^0_k$, we have

$$\chi(\Gamma_{ij}\backslash\bar{D}^0_k) = [\Gamma_i^{'} : \Gamma_{ij}] \cdot \chi(\Gamma_i^{'}\backslash\bar{D}^0_k). \tag{3.38}$$

Hence, we obtain

$$|\chi(\bar{S}^G_{K_f})| = |\sum_{i,j,k} \frac{vol(\Gamma_{ij}\backslash D^0_k)}{vol(\Gamma'_i\backslash D^0_k)} \cdot \chi(\Gamma'_i\backslash D^0_k)| \tag{3.39}$$

$$\geq (\frac{\min_{1\leq i\leq m}|\chi(\Gamma'_i\backslash D^0)|}{\max_{1\leq i\leq m} vol(\Gamma'_i\backslash D^0)}) \cdot vol(S^G_{K_f}). \tag{3.40}$$

Hence, we get

$$\dim_{\mathbb{C}} H^\bullet(S^G_{K_f};\tilde{\mathcal{V}}) \geq |\chi(\bar{S}^G_{K_f^{'}})| \cdot \dim_{\mathbb{C}} \mathcal{V}_\mu \cdot vol(S^G_{K_f}). \tag{3.41}$$

This establishes the lower bound with

$$c = c(\mathbf{G},(r_1,2r_2),K_f^{'}) := \left(\frac{\min_{1\leq i\leq m}|\chi(\Gamma'_i\backslash D^0)|}{\max_{1\leq i\leq m} vol(\Gamma'_i\backslash D^0)}\right) \tag{3.42}$$

whenever $\chi(\bar{S}^G_{K_f}) \neq 0$. ■

As a simple consequence of the above result, we mention the following:

Corollary 3.3.1. *With the same assumptions and notation as in the preceding theorem, we have*

$$\dim_{\mathbb{C}} H^\bullet_{cusp}(S^G_{K_f};\tilde{\mathcal{V}}) \ll vol(S^G_{K_f}) \text{ as } vol(K_f) \to 0 \tag{3.43}$$

for a fixed $\mathcal{V}$. Furthermore, if there exists a level structure $K_f^{'}$ such that $\chi(\bar{S}^G_{K_f^{'}}) \neq 0$, then

$$\dim_{\mathbb{C}} H^{\bullet}(S^G_{K_f};\tilde{\mathcal{V}}) \gg vol(S^G_{K_f}) \text{ as } vol(K_f) \to 0 \tag{3.44}$$

for a fixed $\mathcal{V}$.
We also have

$$\dim_{\mathbb{C}} H^{\bullet}_{cusp}(S^G_{K_f}, \tilde{\mathcal{V}}) \ll \dim_{\mathbb{C}} \mathcal{V} \text{ as } \dim_{\mathbb{C}} \mathcal{V} \to \infty \tag{3.45}$$

for a fixed neat level structure $K_f \trianglelefteq K_f^{'}$ as $\mathcal{V}$ varies.

3.4 Nontriviality of the Lower bound.

For a fixed $\mathcal{V}$, our results imply that $\dim_{\mathbb{C}} H^{\bullet}(S^G_{K_f};\tilde{\mathcal{V}})$ grows almost linearly with volume whenever $S^G_{K_f}$ has nonvanishing Euler characteristic. Here, we show that an instance of this occurs whenever the common real dimension of the constituent locally symmetric spaces is even and D^0 is a symmetric space of the non-compact type.

3.4.1 A result by Dodziuk.

Theorem 3.4.1 (Dodziuk)**.** *For a hyperbolic closed Riemannian manifold M of dimension $2m$ and its universal cover $\tilde{M}$, the following holds true for the L^2- Betti numbers ($q \geq 0$):*

$$b_q^{(2)}(\tilde{M}) = \begin{cases} 0 \text{ if } q \neq m, \\ > 0 \text{ for } q = m \end{cases}$$

The above result is proved in [14]. Since the Euler characteristic of $\tilde{M}$ computed with L^2- Betti numbers (see [29]) is known to coincide with the topological Euler characteristic of M, it follows that

$$(-1)^m \chi(M) > 0. \tag{3.46}$$

Whenever the universal cover D^0 of each connected component of $S^G_{K_f}$ is a complete hyperbolic manifold and D^0 is even dimensional, the result is applicable. (This occurs in case of elliptic as well as Hilbert modular forms where $\dim_{\mathbb{R}} G/K^0_{\infty}$ is even.)

Chapter 4

Comparison with known bounds.

Our bounds serve to explain why most dimension formulae and bounds are of the form 'dimension times volume' up to the first order. We have shown that the dimension of cohomology grows at most linearly with either of volume and dimension when the other is kept fixed.

Remark 4.0.1. *As mentioned in the previous chapter, Rohlfs and Speh (see [33]) obtained a stonger form of our bounds for a general connected reductive algebraic group. This was accomplished by means of the Trace formula. Our method, on the other hand, is topological and does not rely on analytic arguments.*

4.1 Classical Dimension Formulae.

4.1.1 Elliptic Modular Forms.

Consider the space $\mathcal{M}_k(\Gamma_1(N))$ of holomorphic modular forms of weight k for the congruence subgroup $\Gamma_1(N) \subset SL_2(\mathbb{Z})$ consisting of matrices which are unipotent modulo $N \in \mathbb{N}$. We assume that $k \geq 3$ and $N \geq 5$ in order to simplify the dimension formula (as in [36], p.96, Prop. 6.6.6). In this case,

$$\dim_{\mathbb{C}}(\mathcal{M}_k(\Gamma_1(N))) = (1/2)(k-1)(2g(N) + c(N) - 2) + c(N)/2, \qquad (4.1)$$

where $g(N)$ is the genus of the Bailey-Borel compactification and $c(N)$ is the number of cusps added.

We have used the Borel-Serre compactification for estimation instead of this. The Borel-Serre compactification is homeomorphic to the surface obtained on adjoining $c(N)$ circles (instead of adding $c(N)$ cusps) the Bailey-Borel compactification. Being a compact surface with boundary, its Euler characteristic χ_N is readily computable:

$$\chi_N = 2 - 2g(N) - c(N). \tag{4.2}$$

We compare this with our lower bound here (with $\mathbf{G} = GL_2/\mathbb{Q}$ and the level structure corresponding to $\Gamma_1(N)$). We get the following:

$$(k-1)|\chi_N| \leq 2\dim_{\mathbb{C}}(\mathcal{M}_k(\Gamma_1(N))). \tag{4.3}$$

Note that the lower bound falls short of the exact dimension of holomorphic modular forms by $c(N)/2$.

For a discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ for which

$$vol(\Gamma\backslash SL_2(\mathbb{R})/SO(2)) < \infty,$$

We have the following upper bound as in [8] (see Chap.1, Prop.3, p.12)

$$\dim_{\mathbb{C}} \mathcal{M}_k(\Gamma) \leq k \cdot \left(\frac{vol(\Gamma\backslash SL_2(\mathbb{R})/SO(2))}{4\pi}\right) + 1. \tag{4.4}$$

This also resembles our upper bound.

4.1.2 Hilbert Modular Forms.

Let $\mathbf{G} = SL_2/F$ where $F/\mathbb{Q}$ is a totally real extension of degree n. Consider a discrete subgroup $\Gamma \subseteq SL_2(\mathbb{R})^n$ which is not co-compact. Under the assumption of irreducibility of Γ (see [17],Chap.I, p.31 for the definition), it can be shown that the total dimension of the space $[\Gamma; (2r, ..., 2r)]_0$ of Hilbert cusp forms with parallel weight $(2r, ..., 2r)$, $(r > 1)$ equals the following (see [17],Thm.3.5, p.110):

$$\dim_{\mathbb{C}}[\Gamma; (2r, ..., 2r)]_0 = vol(\Gamma\backslash(SL_2(\mathbb{R})/SO(2))^n) \cdot (2r-1)^n$$
$$+\text{lower order terms in } vol \text{ and } r.$$

The above formula, obtained by means of the Selberg trace formula, is consistent with 3.3.1. (We only need to recognise that $(2r-1)^n$ as the rank of the local system involved therein.)

4.1.3 Siegel Modular Forms

Similar considerations apply in case of Siegel modular forms as 3.3.1 is also applicable to the symplectic group. In particular, the explicit dimension formulae for vector valued Siegel modular forms (given in Wakatsuki ([38],Thm.3.2) and Tsushima ([37])) both follow the same pattern (namely, that the total dimension grows almost linearly with weight of the involved representation as well as the volume of the pertinent locally symmetric space).

4.2 Analogy with Lück's Approximation Theorem.

Consider the cohomology with constant coefficients (in $\mathbb{C}$) in the setting of 3.3.1. Due to the topological methods used in the proof, the main result bears a striking resemblance with a result by Lück. We define the notion of residual finiteness of a group first.

Definition 7. *A discrete group* Γ_0 *is said to be residually finite whenever there is a descending sequence of normal subgroups* $\Gamma_i \trianglelefteq \Gamma_0, \quad [\Gamma_0 : \Gamma_i] < \infty$ *with trivial intersection:*

$$\bigcap_{i\geq 1} \Gamma_i = 1. \tag{4.5}$$

Betti numbers with constant coefficients are usually not multiplicative under finite covering maps. However, it was conjectured by Gromov (and proved by Lück) how Betti numbers grow asympotically.

Theorem 4.2.1 (Lück)**.** *Consider a connected CW-complex* X *of finite type with universal cover* $\tilde{X}$*. Assume that* $\Gamma_0 := \pi_1(X)$ *is residually finite so that there is a descending chain of normal subgroups of finite index as follows:*

$$\Gamma_0 \supset \Gamma_1 \supset \dots \Gamma_n \supset \dots \tag{4.6}$$

Let X_n *denote the* $[\Gamma_0 : \Gamma_n]$*-sheeted cover of* X *for each* $n \geq 1$*. Then the following holds true for the Betti numbers* $b_\bullet$*:*

$$\lim_{n\to\infty} \frac{b_q(X_n)}{[\Gamma_0 : \Gamma_n]} = b_q^{(2)}(\tilde{X}) \quad \forall q \geq 0, \tag{4.7}$$

where $b_\bullet^{(2)}$ *denotes the respective* L^2*-Betti number.*

(See [29] for the definition of L^2-cohomology and the proof.)
Consider the setting of Thm. 3.3.1. All the connected components of $S^G_{K'_f} = \amalg \Gamma_{ij} \backslash D^0$ are locally symmetric spaces whenever a level structure K'_f chosen for reference is neat. We may choose a descending chain of its compact open normal subgroups of K'_f having finite index with trivial intersection. (This is possible because $1 \in G(\mathbb{A}_f)$ has a neighbourhood basis of compact open subgroups.) The corresponding descending chain of arithmetic subgroups will also have trivial intersection. For such a K'_f, all the groups are thus residually finite.
Furthermore, existence of the Borel-Serre compactification implies that each $\Gamma_{ij} \backslash \bar{D}^0$ is finite. Therefore, Lück's result is applicable to each $\Gamma_{ij} \backslash D^0$. In particular, the constants c and C in 3.3.1 are related to the L^2-Betti numbers with constant coefficients.

4.2.1 Samet's Upper Bound.

Samet ([34]) has obtained an upper bound on the sum of the Betti numbers (with coefficients in an integral domain of characteristic zero in case of a Hadamard manifold (i.e., a simply connected, complete Riemannian manifold of non-positive sectional curvature bounded below by -1). In particular, his result applies to symmetric spaces G/K for a semisimple Lie group with trivial centre and without compact factors (K being its maximal compact subgroup). For a discrete group Γ of isometries of G/K, he shows that the sum of Betti numbers of $\Gamma \backslash G/K$ is bounded above by a constant times its volume. This is in keeping with our upper bound.

4.3 Comparison with known bounds on Cuspidal Cohomology.

4.3.1 The result by Donnelly

Donnelly ([15], Thm. 1.1) has obtained an upper bound for the cuspidal eigenfunctions of the Laplacian over a locally symmetric space. It is interesting to compare our bounds with this for the case of $GL_2/\mathbb{Q}$. Since cusp forms of weight k are eigenfunctions of the Laplacian corresponding to the eigenvalue $4\pi(k/2)(k/2-1)$, Donnelly's bound is in keeping with 3.3.1 for varying weight as well as volume.

4.3.2 The bounds by Calegari and Emerton.

Calegari and Emerton [10] have considered the case of a semisimple algebraic group over a general number field. They prove the bound

$$dim_{\mathbb{C}} H^q_{cusp}(S^G_{K_f}, \tilde{\mathcal{V}}_\mu) \ll vol(S^G_{K_f})^{1-\frac{1}{\dim G(\mathbb{R})}} \quad \forall q \geq 0 \tag{4.8}$$

under certain assumptions and for a fixed dominant integral weight μ. Our upper bound 3.3.1 (on the *total* dimension of the pertinent cohomology ring) is coarser by an exponent of $\frac{1}{\dim G(\mathbb{R})}$ than this.

4.3.3 The Case when F is imaginary quadratic:

The experimental data of [16] and [1] indicates that for GL_2/F, the rate of growth of the total dimension of the space of cusp forms of a parallel weight (d, d) with a fixed level structure is actually $\sim d$. [30] and [24] have obtained the upper bounds $\ll d^{\frac{5}{3}+\epsilon}$ and $\ll d^{\frac{3}{2}+\epsilon}$, respectively. Our upper bound 3.3.1 gives $\ll d^2$, which is coarser than these.

4.3.4 The Case when F is totally real:

Shimizu ([35]) proved the formula

$$\dim_{\mathbb{C}} H^\bullet_{cusp}(S^G_{K_f}, \tilde{\mathcal{V}}_\mu) \sim \dim_{\mathbb{C}} \mathcal{V}_\mu \tag{4.9}$$

for the total dimension of the space of cusp forms with a fixed level structure for a totally real field. If there exists a level structure K_f' with $\chi(S^G_{K_f'}) \neq 0$, then our bound 3.3.1 for a fixed level structure applies to GL_2/F and is in accordance with Shimizu's result.

4.3.5 Grobner's bound.

The case when $G = Sp(n, 1), \quad n \geq 2$ has been considered by Grobner (see [20], Thm.4.1). We see that Grobner's result is consistent with our estimate 3.3.1, since the formal degree computed therein is proportional to the dimension of the relevent representation.

Part II

Estimation of Cuspidal Cohomology.

Chapter 5

An overview of automorphic forms.

5.1 Generalities on Automorphic Forms.

Let F be a number field and $\mathbb{A}_F$ be the ring of adeles of F. Denote the completion of F at a place v by F_v. If v is non-archimedean, denote the ring of integers in F_v by $\mathcal{O}_v$ and let $\mathfrak{p}_v$ denote the maximal ideal in $\mathcal{O}_v$. Set $F_\infty := \prod_{v \text{ infinite}} F_v$. We shall denote the ring of finite adeles by $\mathbb{A}_{F,f}$.

Let $\mathbf{G}$ be a connected reductive linear algebraic group over F. Set $G_\infty := \mathbf{G}(F_\infty)$. Denote the complexified Lie algebra of G_∞ by $\mathfrak{g}_\infty$. Let $\mathcal{Z}(\mathfrak{g}_\infty)$ be the centre of its universal enveloping algebra. Also, let K_∞ be the maximal compact subgroup of G_∞.

Definition 8. *By a* ***level structure*** *in* $\mathbf{G}(\mathbb{A}_{F,f})$*, we mean a compact open subgroup* K_f *satisfying*

$$K_f \subseteq \mathbf{G}(\prod_{v \text{ finite}} \mathcal{O}_v).$$

Let θ denote the Cartan involution of G with respect to K_∞. For $g \in G$, define its **Hilbert-Schmidt norm** as

$$\|g\| := Tr(Ad(\theta g^{-1})Adg)^{1/2}. \tag{5.1}$$

Definition 9. *A function* $f : G \to \mathbb{C}$ *is said to be* ***of moderate growth*** *(or* ***slowly increasing****) whenever there exists* $n \in \mathbb{N}$ *such that*

$$|f(g)| \ll \|g\|^n \quad \forall g \in G.$$

We shall call a function $f : \mathbf{G}(\mathbb{A}_F) \to \mathbb{C}$ **smooth** if it locally constant at the non-archimedean places and infinitely differentiable at the archimedean places of $\mathbf{G}(\mathbb{A}_F)$.

Definition 10. *A smooth function $f : \mathbf{G}(\mathbb{A}_F) \to \mathbb{C}$ is said to be an* ***automorphic form*** *relative to K_f whenever it satisfies the following conditions:*

1. *f is K_∞-finite on the right (i.e., the span of the right translates of f by elements of K_∞ is finite-dimensional),*
2. *$f(gk) = f(g) \quad \forall k \in K_f$,*
3. *f is $\mathcal{Z}(\mathfrak{g})$-finite,*
4. *f is of moderate growth.*

Remark 5.1.1. *The third condition above is equivalent to the existance of an ideal $J \subseteq \mathcal{Z}(\mathfrak{g})$ having finite codimension. Thus, we may speak of an automorphic form relative to the pair (J, K_f).*
Note that we may further impose the condition $f(zg) = \omega(z)f(g) \quad \forall z \in \mathbf{Z}(\mathbb{A}_F)$, where ω is a quasicharacter of $\mathbf{Z}(F)\backslash\mathbf{Z}(\mathbb{A}_F)$.

Example 1. *A Hecke characters $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ is a basic example of automorphic form for GL_1/F. Also, a holomorphic modular eigenform can be adelised into an automorphic form for $GL_2/\mathbb{Q}$.*

Definition 11. *An automorphic form $f \in \mathcal{A}(J, K_f)$ is called a* ***cusp form*** *if for the unipotent radical $N_\mathbf{P}$ of every proper parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$, we have*

$$\int_{N_\mathbf{P}(F)\backslash N_\mathbf{P}(\mathbb{A}_F)} f(ng)dn = 0 \quad \forall g \in \mathbf{G}(\mathbb{A}_F).$$

5.2 Hecke characters.

5.2.1 Idèlic Hecke characters.

We are interested in algebraic Hecke characters of imaginary quadratic fields. Therefore, we begin with a brief discussion of both classical and idèlic Hecke characters.

Definition 12. *Let E be a number field. An idèlic Hecke character χ of E is a continuous quasicharacter*

$$\chi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times.$$

Remark 5.2.1. *By the term quasicharacter, we mean that χ is not necessarily unitary. However, every idèlic Hecke character χ factorises as*

$$\chi = \chi^0 \otimes \|\|^r,$$

for some $r \in \mathbb{R}$. Here, we have denoted the character determined by the idèlic norm by $\|.\|$. That is,

$$\|x\| := \prod_v |x_v|_v, \tag{5.2}$$

with $|.|_v$ being the normalised valuation at place v.
*Therefore, no loss of information incurs when we restrict to **unitary** Hecke characters. We shall work with unitary Hecke characters from now on.*

The character χ factorises over all the places v of E as

$$\chi = \bigotimes_v \chi_v. \tag{5.3}$$

Definition 13. *The character χ is said to be **unramified** at an non-archimedean place v if $\chi_v|_{\mathcal{O}_v^\times}$ is the trivial character and **ramified** otherwise.*

We have,

$$\mathcal{O}_v^\times \subseteq ker(\chi_v) \quad \forall v \notin \mathcal{F}. \tag{5.4}$$

except for a finite set $\mathcal{F}$ of non-archimedean places. This is because of the inductive limit topology imposed on idèles.

Let $x = (x_v)_v \in \mathbb{A}_E^\times$. At each unramified place $w \notin \mathcal{F}$, we have $t_w \in \mathbb{C}$ such that

$$\chi_w : x_w \mapsto |x_w|_w^{t_w}. \tag{5.5}$$

However, at each place $v \in \mathcal{F}$, there will be a smallest positive integer $f_v \in \mathbb{Z}$ such that $\mathfrak{p}_v^{f_v} \subseteq \ker \chi_v$.

Definition 14. *The ideal*

$$\mathfrak{f} := \prod_{v \in \mathcal{F}} \mathfrak{p}_v^{f_v} \subseteq \mathcal{O}_E$$

*is termed as the **conductor** of χ.*

5.2.2 Classical Hecke characters.

Consider an imaginaryquadratic field E. Let $\mathfrak{m}$ a proper integral ideal of the ring $\mathcal{O}_E$ of its algebraic integers. Let $I_{\mathfrak{m}}$ denote the group of fractional ideals of E coprime to $\mathfrak{m}$. Denote the subgroup of $I_{\mathfrak{m}}$ consisting of the principal fractional ideals coprime to $\mathfrak{m}$ by $P_{\mathfrak{m}}$.

Definition 15. *The quotient group $H_{\mathfrak{m}} := I_{\mathfrak{m}}/P_{\mathfrak{m}}$ is called the **ray class group** modulo $\mathfrak{m}$.*

The group defined above is known to be of finite order, which we shall denote by $h_E(\mathfrak{f})$.

Definition 16. *Let $\mathfrak{m}$ be an integral ideal of $\mathcal{O}_E$. Given a character $\alpha : H_{\mathfrak{m}} \to \mathbb{C}^\times$, let $\mathfrak{f}$ be the smallest divisor $\mathfrak{f}|\mathfrak{m}$ such that there exists $\alpha' : H_{\mathfrak{f}} \to \mathbb{C}^\times$ which restricts to α on $H_{\mathfrak{m}}$. The ideal $\mathfrak{f}$ is termed as the **conductor** of α.*

The conductor of α is thus the largest ideal $\mathfrak{f} \supseteq \mathfrak{m}$ such that α can be regarded as a Hecke character for $H_{\mathfrak{f}}$.

Notation 5. *Let*

$$U_{\mathfrak{m}} := \{u \in \mathcal{O}_E^\times : u \equiv 1 \pmod{\mathfrak{m}}\}.$$

Thus, $U_{\mathfrak{m}}$ is the group of units of E which are invertible modulo $\mathfrak{m}$.

Definition 17. *A classical Hecke character ψ modulo $\mathfrak{m}$ for E is a quasicharacter of $I_{\mathfrak{m}}$ for which there exists a pair (χ_∞, χ) of unitary characters of $\mathbb{C}^\times$ and $(\mathcal{O}_E/\mathfrak{m})^\times$, respectively, which satisfies the following for every $a \in \mathcal{O}_E$ which is invertible modulo $\mathfrak{m}$:*

$$\psi(a\mathcal{O}_E) = \chi_\infty(a)\chi(a) \qquad (*).$$

Remark 5.2.2. *This pair is completely determined once we specify ψ. This is because the elements invertible modulo $\mathfrak{m}$ are dense in $\mathbb{C}^\times$ and thus fix χ_∞ by continuity. The character χ_∞, in turn, fixes χ by the condition $(*)$ specified above.*
However, a pair (χ_∞, χ) satisfying the above condition determines ψ only u pto a character of the class group; thereby corresponding to exactly h_E Hecke characters (we shall denote the class number of E by h_E.)

Definition 18. *When E is an imaginary quadratic field and $\chi_\infty$is unitary, there exists $l \in \mathbb{Z}$ and $s \in \mathbb{R}$ such that*

$$\chi_\infty : z \mapsto \left(\frac{z}{\bar{z}}\right)^{-l} |z|^{is}. \tag{5.6}$$

We shall refer to l as the -type of the Hecke respective character.

With this in mind, we quote the following formula (as in [32], Thm. 3.25(i), p.109):

Theorem 5.2.3. *Let $\mathfrak{f}$ be a non-zero integral ideal of $\mathcal{O}_E$. Let $\psi_E(\mathfrak{f})$ denote the number of residue classes modulo $\mathfrak{f}$ which can be represented by units of $\mathcal{O}_E$. We have*

$$h_E(\mathfrak{f}) = \frac{h_E \Phi_E(\mathfrak{f})}{\psi_E(\mathfrak{f})}, \tag{5.7}$$

where we have set $\Phi_E(\mathcal{O}_E) = 1$ and

$$\Phi_E(\mathfrak{f}) := N_{E/\mathbb{Q}}(\mathfrak{f}) \prod_{\mathfrak{p}|\mathfrak{f},\mathfrak{p}\ prime} (1 - N_{E/\mathbb{Q}}(\mathfrak{p})^{-1}). \tag{5.8}$$

Next, we shall show how to relate an idèlic Hecke character to a classical one. Let a character $\chi_f : I_\mathfrak{m} \to \mathbb{C}^\times$ and another character χ_∞ of the archimedean part of $\mathbb{A}_E^\times$ be given. Assume that these satisfy the following:

$$U_\mathfrak{m} \subseteq \ker(\chi_\infty), \text{ and} \tag{5.9}$$

$$\chi_\infty^{-1}(a) = \chi_f(a\mathcal{O}_E), \quad \forall a \in E^\times. \tag{5.10}$$

Now, $E^\times$ has the weak approximation property among the finitely many places $v \in \mathcal{F}$ of $\mathbb{A}_E^\times$ where $\mathfrak{p}_v|\mathfrak{m}$. Hence, the above conditions determine an idèlic Hecke character $\chi = \chi_\infty \cdot \chi_f$ once we let

$$\chi_w(\varpi_w) = \chi_f(\mathfrak{p}_w) \quad \forall w \notin \mathcal{F}. \tag{5.11}$$

Remark 5.2.4. *If χ_f is primitive of conductor $\mathfrak{f}$, χ will also have conductor $\mathfrak{f}$.*

5.2.3 Algebraic Hecke Characters.

For a number field F, we shall deal with a particular type of Hecke character of $GL_1(\mathbb{A}_F)/\mathbb{F}^\times$. We define this type here.

Let S_r (resp., S_c) denote the union of the real (resp., complex) places of F. Set $S_\infty := S_r \cup S_c$. For $x = (x_v)_v \in \mathbb{A}_F^\times$, define $x_\infty := (x_v)_{v \in S_\infty}$. Also, let $|.|_\infty$ be the restriction of the idèlic norm map to S_∞.

For a complex number $z = x + iy \in \mathbb{C}, \quad x, y \in \mathbb{R}$ define

$$|x + iy|_\mathbb{C} = x^2 + y^2, \tag{5.12}$$

$$|x + iy|_\mathbb{R} = \sqrt{x^2 + y^2}. \tag{5.13}$$

Every character of $\mathbb{C}^\times$ is of the form

$$z \mapsto (\frac{z}{|z|_{\mathbb{R}}})^{-a}|z|_{\mathbb{C}}^{b} \quad \text{where } a \in \frac{1}{2}\mathbb{Z}, b \in \mathbb{C}. \tag{5.14}$$

Hence, for

$$\chi_\infty := \prod_{v \in S_\infty} \chi_v, \tag{5.15}$$

we must have

$$\chi_\infty(x_\infty) = |x_\infty|_\infty^s \cdot \prod_{v \in S_r} \left(sgn(x_v)^{-a_v} \cdot |x_v|_v^{i\alpha_v} \right) \cdot \prod_{w \in S_c} \left((\frac{x_w}{|x_w|_{\mathbb{R}}})^{-a_w} \cdot |x_w|_w^{i\alpha_w} \right), \tag{5.16}$$

for some $s \in \mathbb{R}$. Here, $a_v \in \{0, 1\} \quad \forall v \in S_r$ and $\alpha_u \in \mathbb{R} \quad \forall u \in S_\infty$.

Not every idèlic Hecke character $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ is the restriction of a homomorphism of algebraic groups $(Res_{F/\mathbb{Q}}\mathbb{G}_m)(\mathbb{A}_\mathbb{Q}) \to \mathbb{G}_m(\mathbb{C})$. The characters which satisfy this condition will play an important role in automorphic induction.

Definition 19. *Let $F/\mathbb{Q}$ be an imaginary quadratic extension. A unitary idèlic Hecke character $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ is said to be* ***algebraic*** *with parallel weight whenever there exists an integer $a \in \mathbb{Z}$ such that*

$$\chi_\infty(z) = \left(\frac{z}{|z|_{\mathbb{R}}} \right)^{-a}. \tag{5.17}$$

5.3 Modular forms.

Classical modular forms are certain automorphic forms for $GL_2/\mathbb{Q}$. Although modular forms can be defined over an arbitrary number field, we shall restrict to the relevent case of $\mathbb{Q}$ in this section.

Modular forms are intimately connected with the representations of $GL_2(\mathbb{Q}_p)$ for each place $p \leq \infty$ of $\mathbb{Q}$. In order to state the main results of this chapter, we shall review these representations briefly.

5.3.1 Representations of $GL_2(\mathbb{R})$.

Firstly, note that $GL_2(\mathbb{R})$ is generated by $\mathbb{R}_+^\times$ and its subgroup $SL_2^\pm(\mathbb{R})$ consisting of matrices having determinant ± 1. The latter contains $SL_2(\mathbb{R})$ as a subgroup of index two. Every irreducible representation of $SL_2(\mathbb{R})$ can be induced to $SL_2^{\pm}(\mathbb{R})$ to obtain at most two irreducible representations. We may then construct representations of $GL_2(\mathbb{R})$ by tensoring these with the characters of $\mathbb{R}_+^\times$. Therefore, it suffices to focus on the representations of $SL_2(\mathbb{R})$.

5.3.2 The notion of Admissibility.

Following [19], we shall call a unitary irreducible representation of $SL_2(\mathbb{R})$ **admissible** if its restriction to $SO(2)$ contains each irreducible representation of the latter group at most finitely many times. This means that the $SO(2)$-finite vectors in the underlying vector space span a dense subspace and leads to the notion of a $(\mathfrak{g}, K)$ module in general. We shall concentrate on such representations from now on.

Definition 20. *For the finite places p, **admissibility** of an irreducible unitary representation of $GL_2(\mathbb{Q}_p)$ means that the following conditions are satisfied:*

- *Every vector in the underlying vector space is fixed by some compact open subgroup of $K_p \subseteq GL_2(\mathbb{Z}_p)$,*
- *The space of vectors fixed by each such K_p is finite dimensional.*

5.3.3 Hecke Algebra.

For $p \leq \infty$, it is possible to construct an algebra $\mathcal{H}_p$, the simple modules of which are the admissible irreducible unitary representations of $GL_2(\mathbb{Q}_p)$. This algebra is called the **Hecke algebra** for the group over the respective local field (see [19] for a detailed description).

At archimedean places, this algebra serves as a substitute for the universal enveloping algebra of the underlying (complexified) Lie algebra. Its product structure is given by convolution of certain distributions. For example, the Hecke algebra of $SL_2^{\pm}(\mathbb{R})$ is given by

$$\mathcal{H}(SL_2^{\pm}(\mathbb{R})) = \mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) \oplus \delta * \mathcal{U}(\mathfrak{sl}_2(\mathbb{C})), \tag{5.18}$$

δ being the Dirac distribution supported at $diag(-1, 1) \in GL_2(\mathbb{R})$.

For non-archimedean $\mathbb{Q}_p$, the corresponding Hecke algebra is the convolution algebra of locally constant and compactly supported functions on $GL_2(\mathbb{Q}_p)$. These conditions are imposed due to the totally disconnected nature of $GL_2(\mathbb{Q}_p)$.

5.3.4 Irreducible Admissible Representations.

Consider the Borel subgroup $B \subseteq GL_2(\mathbb{R})$ consisting of the upper triangular matrices. Each pair of quasicharacters of $\mathbb{R}^\times$ may be regarded as a representation of B. In order to construct representations of $GL_2(\mathbb{R})$, we induce each such pair to it. In fact, every irreducible unitary representation of $GL_2(\mathbb{R})$ occurs as a subquotient of the induced module for some such pair of characters. These representations are further classified into the principal, discrete, continuous and complementary series according to the ratio of and the parameters defining the involved characters. By contrast, $GL_2(\mathbb{C})$ has no discrete series.

Some of the admissible irreducible representations of $GL_2(\mathbb{Q}_p)$ (for finite p) are obtained in a similar manner- the principal series and special representations are examples of this. There also exist (pre-unitary) continuous and complementary series when the involved pair of characters is not unitary. However, there exist **supercuspidal** representations in addition to these. Being supercuspidal is also equivalent to having each matrix coefficient compactly supported modulo the centre of the respective group. This means that the matrix coefficient vanishes outside the product of the centre with a compact subset of $GL_2(\mathbb{Q}_p)$.

5.3.5 A brief overview of modular forms.

Let $\mathcal{H}$ denote the complex upper half plane:

$$\mathcal{H} := \{z \in \mathbb{C} : \Im(z) > 0\}. \tag{5.19}$$

The group $SL_2(\mathbb{R})$ acts on $\mathcal{H}$ by fractional linear transformations as follows:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z \mapsto \gamma \cdot z := \frac{az+b}{cz+d}. \tag{5.20}$$

Consider a discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ such that we have

$$[SL_2(\mathbb{Z}) : (\Gamma \cap SL_2(\mathbb{Z}))] < \infty \text{ and } [\Gamma : (\Gamma \cap SL_2(\mathbb{Z}))] < \infty. \tag{5.21}$$

(We may take Γ to be a subgroup of finite index in $SL_2(\mathbb{Z})$).

Definition 21. *Let Γ be as above and $k \in \mathbb{Z}$. A **modular form** of weight k is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ which satisfies the following:*

$$f(\gamma \cdot z) = det(\gamma)^{1-k}(cz+d)^k f(z) \quad \forall \gamma \in \Gamma, z \in \mathcal{H}.$$

We shall denote the space of all modular forms for Γ having weight k by $M_k(\Gamma)$.

In particular, a modular form yields an automorphic form for $GL_2(\mathbb{A}_\mathbb{Q})$ on adelisation; so the definitions of the first section of this chapter apply. We refer the reader to [19] (Chap. 1-5) for details and also for the definitions of Petersson inner product as well as newforms.

Furthermore, we note that the unipotent radical of the upper triangular Borel subgroup of $GL_2(\mathbb{A}_\mathbb{Q})$ is isomorphic to the additive group $\mathbb{A}_\mathbb{Q}$. Hence, we may formulate the condition of being a (holomorphic) cusp form in terms of vanishing of the constant term in its the Fourier expansion. We shall denote the space of all cusp forms for Γ having weight k by $S_k(\Gamma)$.

5.3.6 Counting the dimension of the space of cusp forms.

For $N = \prod_{i=1}^{r} p_i^{n_i} \in \mathbb{N}$, set

$$K_f(N) = \prod_p \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) : c \in p_i^{n_i}\mathbb{Z}_p \}. \tag{5.22}$$

If normalise the Haar measure of $GL_2(\mathbb{A}_f)$ so that $vol(GL_2(\mathbb{Z}_p)) = 1 \quad \forall p < \infty$, then we have

$$\begin{aligned} vol(K_f(N))^{-1} &= \prod_{i=1}^{r} [GL_2(\mathbb{Z}_{p_i}) : \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_{p_i}) : ord_{p_i} c \geq n_i \}], \\ &= \psi(N), \end{aligned} \tag{5.23}$$

where

$$\psi(N) := N \prod_{p|N} (1 + p^{-1}) \quad (= [SL_2(\mathbb{Z}) : \Gamma_0(N)]). \tag{5.24}$$

Define the convolution of the Euler totient function φ with itself as follows:

$$(\varphi * \varphi)(N) = \sum_{d|N} \varphi(d)\varphi(N/d). \tag{5.25}$$

We shall denote the number of prime divisors of an integer $N \geq 2$ by $\sigma_0(N)$ and define $\sigma_0(1) = 1$. For $N \geq 5$ and $k \geq 3$, the dimension formula given in [Stein, Prop.6.6, p.96] simplifies to the following:

$$\dim_\mathbb{C} S_k(\Gamma_1(N)) = (\frac{(k-1)\varphi(N)\psi(N)}{24}) - \frac{(\varphi * \varphi)(N)}{4}. \tag{5.26}$$

In order to estimate the total number of cusp forms with character corresponding to a particular weight and level, we shall need the following lemma.

Lemma 5.3.1.

$$(\varphi * \varphi)(N) \leq \sigma_0(N)\varphi(N) \quad \forall N \geq 1. \tag{5.27}$$

Proof. The inequality holds trivially for $N = 1$; so we may assume that $N \geq 2$. We call an arithmetic function f multiplicative if

$$f(mn) = f(m)f(n) \text{ whenever } \gcd(m, n) = 1. \tag{5.28}$$

Since φ is multiplicative, so is $\varphi * \varphi$. Hence, both the sides of the claimed inequality are multiplicative. Therefore, it suffices to prove the inequality when $N = p^n, \quad (n \geq 1)$ for some prime p.

For such $N = p^n$, we have

$$\begin{aligned}(\varphi * \varphi)(p^n) &= \sum_{i=0}^{n} \varphi(p^i)\varphi(p^{n-i}),\\ &= p^n\left\{\sum_{i=1}^{n-1}(1-1/p)^2 + 2(1-1/p)\right\},\\ &= \varphi(p^n)\left\{(n-1)(1-1/p) + 2\right\},\\ &\leq \varphi(p^n)\{(n-1)+2\} = \sigma_0(p^n)\varphi(p^n).\end{aligned}$$

■

Lemma 5.3.2. *We have*

$$\frac{\dim_{\mathbb{C}} S_k(\Gamma_1(N))}{N^2} = (\frac{k-1}{4\pi^2}) + o(1) \text{ as } N \to \infty.$$

Proof. The standard formulae for $\varphi(N)$ and $\psi(N)$ yield the following:

$$\varphi(N)\psi(N) = N^2 \prod_{p|N}(1-1/p^2) \geq N^2 \prod_{p \text{ prime}}(1-1/p^2) = (\frac{6}{\pi^2})N^2.$$

Hence, we have

$$\begin{aligned}\dim_{\mathbb{C}} S_k(\Gamma_1(N))/N^2 &= (\frac{(k-1)\varphi(N)\psi(N)}{24N^2}) - \frac{(\varphi * \varphi)(N)}{4N^2},\\ &\geq (\frac{k-1}{4\pi^2}) - \frac{(\varphi * \varphi)(N)}{4N^2}.\end{aligned}$$

But the standard asymptotic formula for the divisor function σ_0 (see [23], Thm. 315, p. 343), when combined with the trivial estimate $\varphi(N) \le N$, shows that

$$\sigma_0(N)\varphi(N) = O(N^{1+\epsilon}) \quad \forall \epsilon > 0. \tag{5.29}$$

In view of the preceeding lemma, this implies

$$\frac{(\varphi * \varphi)(N)}{4N^2} = o(1) \text{ as } N \to \infty, \tag{5.30}$$

and thus the assertion of the lemma follows. ■

We shall also require estimates on the dimension of the space of newforms.

Lemma 5.3.3. *Let $n \ge 1$ be an integer and $p \ge 2$ be a prime. We have*

$$\frac{\dim_{\mathbb{C}} S_k^{new}(\Gamma_1(p^n))}{p^{2n}} = (\frac{k-1}{4\pi^2})(1 - \frac{1}{p^2})^2 + o(1) \tag{5.31}$$

as $n \to \infty$.

Proof. For each $M \in \mathbb{N}$, define

$$\bar{\mu}(M) := \begin{cases} 0 \text{ if } p^3 | M \text{ for some prime } p, \\ \prod_{p||M}(-2) \text{ otherwise,} \end{cases} \tag{5.32}$$

where empty product is understood to be 1.
By [36] (Prop. 6.6, p.96), we have

$$\dim_{\mathbb{C}} S_k^{new}(\Gamma_1(M)) = \sum_{d|M} \bar{\mu}(M/d) \dim_{\mathbb{C}} S_k(\Gamma_1(d)). \tag{5.33}$$

For $M = p^n$, the above sum contains exactly three terms. Adding these with aid of the previous Lemma, we obtain the desired expression. ■

Chapter 6

Use of Langlands Functoriality.

6.1 An overview of Langlands Functoriality.

Let F be a number field. Consider a linear algebraic group $\mathbf{G}/F$. Assume that $\mathbf{G}$ is connected, reductive and split over F. This means that it has a maximal torus isomorphic to $\mathbb{G}_m^r$ over F for some integer $r \geq 1$.

6.1.1 Characters of T.

Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ over F, i.e., a maximal connected solvable subgroup. Select a maximal split torus $\mathbf{T}$ in $\mathbf{B}$ so that $\mathbf{B} = \mathbf{TU}$ for the unipotent radical $\mathbf{U}$ of $\mathbf{G}$. Denote the character group of $\mathbf{T}$ by $X^*(\mathbf{T})$ and the set of its cocharacters by $X_*(\mathbf{T})$ (both over F).

We observe that $\mathbf{U}$ determines a set Δ of simple roots as well as a set $\check{\Delta}$ of simple coroots. Thus, for each $\alpha \in \Delta$, we have $\check{\alpha} \in \check{\Delta}$ normalised so that

$$\alpha(\check{\alpha}(t)) = t^2. \tag{6.1}$$

6.1.2 L-groups.

Definition 22. *By the* ***root datum*** *for* $\mathbf{G}$*, we mean the quadruple* $\Psi(\mathbf{G}) := (X^*(\mathbf{T}), \Delta, X_*(\mathbf{T}), \check{\Delta})$.

By Chevalley's theorem, there exists a connected reductive group $\check{G}$ over $\mathbb{C}$ such that

$$\Psi(\check{G}) = (X_*(\mathbf{T}), \check{\Delta}, X^*(\mathbf{T}), \Delta). \tag{6.2}$$

Note that we may define an action of $Gal(\bar{F}/F)$ on $\check{G}$ by dualising that on Δ to $\check{\Delta}$. This can be done by virtue of the split exact sequence

$$1 \to Int(\mathbf{G}) \to Aut(\mathbf{G}) \to Aut(\Psi(\mathbf{G})) \to 1 \tag{6.3}$$

.

Definition 23. *The L-**group** of $\mathbf{G}$ is defined as*

$$^{L}G = \check{G} \rtimes Gal(\bar{F}/F).$$

Note that $\check{G} = {}^{L}G^{0}$, where ${}^{L}G^{0}$ denotes the connected component of the identity in the respective group.

6.1.3 The Principle of Functoriality.

Now, let $\mathbf{H}$ be another connected reductive algebraic group over F.

Definition 24. *An L-**homomorphism** $\phi : {}^{L}H \to {}^{L}G$ is a continuous homomorphism over $Gal(\bar{F}/F)$ whose restriction to ${}^{L}H^{0}$ is holomorphic.*

When $\mathbf{G}$ is quasi-split over F, **Langlands' principle of functoriality** predicts a transfer of every automorphic representation of $\mathbf{H}$ to that of $\mathbf{G}$ associated with each such L-homomorphism ϕ. Automorphic induction and symmetric square transfer are special instances of this.

6.2 Automorphic Induction.

We shall restrict the discussion to the relevent case of $GL(2)$ over a local field.

6.2.1 Weil Representation.

We shall outline the construction of Weil representation over a local field F here. This representation provides concrete examples of supercuspidal representations and will be required for our considerations on automorphic induction.

Observe that the construction of principal series is carried out by induction of a pair of quasicharacters of the diagonal torus which are regarded as being defined for the Borel subgroup of upper triangular matrices. This construction neglects the unipotent radical of the Borel subgroup (recall that the Borel subgroup is its semidirect product with the diagonal torus). Hence, there could exist irreducible representations of $GL_2(F)$ which are not accounted for even by the subquotients of the principal and the special series.

For instance, choose an additive character ψ of F and regard it as a character of the unipotent radical of the standard Borel subgroup of upper triangular matrices. Induce it to the whole group. The vector space of the induced representation is related to the corresponding Whittaker model. These models enjoy the property of uniqueness for non-archimedean F, which is crucial in proving the multiplicity one theorem for cuspidal automorphic representations of $GL_2/\mathbb{Q}$.

Further, recall the Iwasawa decomposition of $GL_2(F)$: it is the product of its centre, the diagonal torus, the unipotent radical and the corresponding maximal compact subgroup. The maximal compact subgroup determines a symmetric bilinear form, which is a quadratic form q over F.

But now q determines a quadratic extension E/F such that the norm one subgroup of $E^\times$ coincides with the maximal compact subgroup above. Define an action π on the Schwartz-Bruhat class functions on $GL_2(F)$ as follows:

$$\pi(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})f(x) = \psi(uq(x))f(x), \quad (x \in GL_2(F), u \in F). \tag{6.4}$$

This action, when modified suitably with the nontrivial character of $E^\times/F^\times$, is independent of ψ over $GL_2(F)$. The corresponding representation is related to the sought **Weil representation**. For non-archimedean F, this is **supercuspidal** whenever a certain character of $E^\times$ is ramified (and one of the principal or special series otherwise).

At archimedean places, however, the Weil construction produces principal or discrete series representations. The non-existence of supercuspidals at the archimedean places may be attributed to the fact that we have neither small subgroups nor 'ramified' characters at these places.

6.2.2 Automorphic Induction from $GL(1)/E$ to $GL(2)/\mathbb{Q}$.

Definition 25. *Let E/F be a finite extension of an algebraic number fields. The* ***idèlic norm map***

$$\mathcal{N}_{E/F} : \mathbb{A}_E^\times \to \mathbb{A}_F^\times \tag{6.5}$$

is the continuous homomorphism determined placewise as follows:

$$a \mapsto \left((\prod_{v|w} N_{E_v/F_w}(a_v))_w \right), \quad (a = (a_v) \in \mathbb{A}_E^\times). \tag{6.6}$$

6.2.3 A criterion for cuspidality.

Consider a place v of $\mathbb{Q}$. We set π_v to be the principal series representation induced by $(\chi_w, \chi_{w'})$ if v splits in E with $w, w'|v$ and the Weil representation corresponding to χ_u otherwise, with u lying over v. Define

$$\pi(\chi) := \bigotimes_v \pi_v. \tag{6.7}$$

Denote the central character of $\pi(\chi)$ by ω.

Now, Thm. 7.11 on p.148 in [19] states that $\pi(\chi)$ is cuspidal whenever

$$\chi \neq \delta \circ \mathcal{N} \tag{6.8}$$

for any Hecke character $\delta : \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times \mapsto \mathbb{C}^\times$ (i.e., a Dirichlet character with some conductor). Here, $\mathcal{N}_{E/\mathbb{Q}} : \mathbb{A}_E^\times \to \mathbb{A}_\mathbb{Q}^\times$ is the idèlic norm map. This condition is equivalent to

$$\chi \neq \chi^\sigma, \tag{6.9}$$

where σ is the nontrivial element of the Galois group of $E/\mathbb{Q}$. As E is an imaginary quadratic extension, σ is the restriction of the complex conjugation on $\mathbb{C}$.

In view of Remark 7.7 on p.142 in [19], the infinity component $\pi(\chi)_\infty$ must be a discrete series representation. More precisely, if

$$\chi_\infty : z \mapsto \left(\frac{z}{\bar{z}}\right)^{k-1} \quad \text{for some } k \in \mathbb{Z},$$

then there exist $l \in \mathbb{C}$ such that $\pi(\chi)_\infty$ is isomorphic to the irreducible quotient of the representation induced by the pair of characters

$$(t \to |t|_\mathbb{R}^l t^{k-1}, \quad t \to |t|_\mathbb{R}^l sgn(t)). \tag{6.10}$$

6.3 Automorphic representations of $GL_3(\mathbb{Q})$.

6.3.1 The level structure.

For each finite place p of $\mathbb{Q}$ and integer $n \geq 0$, we define

$$H_p(n) = \{x = (x_{ij})_{3\times 3} \in GL_3(\mathbb{Z}_p) : x_{31}, x_{32} \in p^n\mathbb{Z}_p\}. \tag{6.11}$$

Let $N = \prod_{i=1}^{r} p_i^{n_i}$ be a positive integer. Set

$$H_f(N) = \prod_{i=1}^{r} H_{p_i}(n_i). \tag{6.12}$$

For each $N \geq 1$, we shall refer to the compact open subgroup $H_f(N) \subset GL_3(\mathbb{A}_f)$ as the **level structure** corresponding to N.

6.3.2 The associated idèlised Dirichlet character.

Let $(\Pi, W) = (\bigotimes_{p\leq\infty} \Pi_p, \bigotimes_{p\leq\infty} W_p)$ be a cuspidal automorphic representation of $GL_3(\mathbb{A}_\mathbb{Q})$. Its central character Ω_Π factorises over the places of $\mathbb{Q}$ as

$$\Omega_\Pi = \bigotimes_{p\leq\infty} \Omega_p. \tag{6.13}$$

For each p, Ω_p determines a character Ω'_p of $GL_3(\mathbb{Q}_p)$ as follows:

$$\Omega_p(x) = \Omega'_p(x_{33}) \quad (x = (x_{ij}) \in H_p(n)), \tag{6.14}$$

where $p^n \| cond(\Omega_p)$.

6.3.3 The conductor of Π.

Notation 6. *For an integer $M \geq 0$ and each finite place $p < \infty$ of $\mathbb{Q}$, define*

$$W_p^{H_p(M),\Omega'_p} := \{w \in W_p : \Pi_p(x)w = \Omega'_p(x_{33})w \quad \forall x = (x_{ij})_{3\times3} \in H_p(M)\}.$$

It is well-known that there exists an integer $M_0 \geq 0$ such that $W_p^{H_p(M),\Omega'_p} \neq 0$ for each $M \geq M_0$ (see [27]).

Definition 26. *The smallest integer $c(\Pi_p) \geq 0$ for which $W_p^{H_p(c(\Pi_p)),\Omega'_p} \neq 0$ is termed as the **conductor** of Π_p. We shall also refer to*

$$N_\Pi := \prod_{p<\infty} p^{c(\Pi_p)}$$

*as the **conductor** of Π.*

6.3.4 The conductor of π.

Now, we define a level structure of $GL_2(\mathbb{A}_\mathbb{Q})$ as follows:

Notation 7. *For an integer $n \geq 0$ and each prime $p < \infty$, define*

$$K_p(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) : ord_p(c) \geq n\}.$$

Also, if $N = \prod_{i=1}^{r} p_i^{n_i}$ is the prime factorisation of an integer $N \geq 1$, then set

$$K_f(N) := \prod_{i=1}^{r} K_p(n_i) \cdot \prod_{q \nmid N,\ q\ prime} GL_2(\mathbb{Z}_q) \tag{6.15}$$

Remark 6.3.1. *$K_f(N)$ is precisely the level structure corresponding to $\Gamma_0(N)$ defined in the previous chapter.*
Further, if $\omega_\pi := \otimes_{p \leq \infty} \omega_p$ denote the central character of π, there exists a unique primitive Dirichlet character ϵ whose idèlisation (denoted by $\tilde{\epsilon}$) equals ω_π.

Note that for each prime $p < \infty$ and $n \geq 0$, the Dirichlet character ω_p defines a character of $K_p(n)$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \omega_p(d), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p(n). \tag{6.16}$$

We shall denote this character by $\omega'_p : K_p(n) \to \mathbb{C}^\times$.

Next, define

$$V_p^{K_p(n),\omega'_p} := \{v \in V_p : \pi_p(k)v = \omega'_p(k)v \quad \forall k \in K_p(n)\}. \tag{6.17}$$

We quote the following with the notation as above (see [11]):

Theorem 6.3.2 (Casselman)**.** *There exists a minimal integer $c(\pi_p) \geq 0$ (termed as the* ***conductor*** *of π_p) such that*

$$\dim V_p^{K_p(c(\pi_p)),\omega'_p} = 1. \tag{6.18}$$

Furthermore, we have

$$p^{c(\pi_p)} \geq cond(\omega_p), \tag{6.19}$$

where $cond(\omega_p)$ is the conductor of the Dirichlet character ω_p in the usual sense.

Remark 6.3.3. *For example, we have $c(\pi_p) \geq 2$ whenever π_p is supercuspidal (see [26] for details).*

6.4 Langlands Transfer in case of the Symmetric Square.

Let $\pi = \otimes_{p\leq\infty}\pi_p$ be a cuspidal automorphic representation of $GL_2(A_{\mathbb{Q}})$, each π_p being an irreducible admissible representation of $GL_2(\mathbb{Q}_p)$. Thus, π is an irreducible subspace of

$$\mathcal{L}^2_{cusp}(GL_2(\mathbb{Q})\backslash GL_2(A_{\mathbb{Q}}), \omega_\pi) \tag{6.20}$$

(the space of square-integrable cusp forms with central character ω_π).

Next, let $\mathcal{W}_p$ denote the Weil group at $p = \infty$ and the Weil-Deligne group for finite p. (See [39] and [12], resp., for the definitions of these groups.) By the local Langlands correspondence for $GL_2(\mathbb{Q}_p)$, there exists a semisimple representation

$$\sigma(\pi_p) : \mathcal{W}_p \to GL_2(\mathbb{C}) \tag{6.21}$$

associated with each π_p.

If π_p is unramified and its Hecke matrix is $diag(a_p, b_p)$, we form the 3×3 matrix $diag(a_p^2, a_pb_p, b_p^2)$.

When π_p is ramified at p, we consider the map $sym^{\otimes 2} : GL_2(\mathbb{C} \to GL_3(\mathbb{C}$. The composite map

$$sym^{\otimes 2} \circ \sigma(\pi_p) : \mathcal{W}_p \to GL_3(\mathbb{C}) \tag{6.22}$$

defines a representation of $GL_3(\mathbb{C}$. Now, using the local Langlands correspondence for $GL_3(\mathbb{Q}_p)$, we obtain a representation of $GL_3(\mathbb{C}$ which we denote by $sym^{\otimes 2}(\pi_p)$. (Its Hecke matrix is given by $diag(a_p^2, a_pb_p, b_p^2)$ for unramified pi_p.)

Definition 27. *The restricted tensor product representation*

$$sym^{\otimes 2}(\pi) := \otimes'_{p\leq\infty} sym^{\otimes 2}(\pi_p)$$

is said to be obtained by ***symmetric square transfer*** *of* π.

Remark 6.4.1. *By the Langlands principle of functoriality,* $\Pi := sym^2(\pi)$ *ought to be an automorphic representation of* $GL_3(\mathbb{A}_{\mathbb{Q}})$*. Gelbart and Jacquet (see [18]) showed that it is indeed isomorphic to an irreducible subquotient of*

$$\mathcal{L}_2^{cusp}(GL_3(\mathbb{Q})\backslash GL_3(A_{\mathbb{Q}}), \omega_\pi^6),$$

provided that π *is not obtained by automorphic induction of a Hecke character.*

6.4.1 Cohomological representations.

Consider the finite dimensional irreducible representation of $GL_3(\mathbb{R})$ corresponding to a dominant integral weight μ. We shall denote the underlying $\mathbb{C}$-vector space by $\mathcal{V}_\mu$. Let $\Pi = \Pi_\infty \otimes \Pi_f$ be a cuspidal automorphic representation of $GL_3(\mathbb{A}_\mathbb{Q})$, where Π_∞ and Π_f are its archimedean and non-archimedean parts, respectively. For each compact open subgroup $K_f \subseteq \prod_{p<\infty} GL_3(\mathbb{Z}_p)$, we consider $\Pi_f^{K_f}$, the subspace of K_f-fixed vectors of Π_f.

Definition 28. Π *is said to be* ***cohomological*** *with coefficients in* $\mathcal{V}_\mu$ *for the level structure* K_f *if the following holds for the relative Lie algebra cohomology:*

$$H^\bullet(\mathfrak{gl}_3(\mathbb{C}), \mathbb{R}_+^\times \cdot SO(3,\mathbb{R}); \Pi_\infty \otimes \mathcal{V}_\mu) \otimes \Pi_f^{K_f} \neq 0. \tag{6.23}$$

We denote this by

$$\Pi \in Coh(GL_3, \mu, K_f),$$

where $Coh(GL_3, \mu, K_f)$ *is the set of all cohomological cuspidal automorphic representations so defined.*

Chapter 7

Bounds for Automorphic Induction.

We shall consider the case of automorphic induction from GL_1 over a quadratic field to $GL_2/\mathbb{Q}$ in this chapter.

7.1 Notation and Preliminaries.

Notation 8. *For an integer $N \geq 1$, let $S_k(\Gamma_1(N))$ denote the space of cusp forms for the **congruence subgroup** $\Gamma_1(N)$, where we define*

$$\Gamma_1(N) := \{X \in SL_2(\mathbb{Z}) : X \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}\}.$$

For the congruence subgroup

$$\Gamma_0(N) := \{X \in SL_2(\mathbb{Z}) : X \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N}\},$$

we define $S_k(\Gamma_0(N))$ analogously.

Remark 7.1.1. *Note that*

$$S_k(\Gamma_1(N)) = \bigoplus_{\epsilon:(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times} S_k(\Gamma_0(N), \epsilon),$$

where ϵ ranges over all Dirichlet characters modulo N and $S_k(\Gamma_0(N), \epsilon)$ denotes the subspace of holomorphic cusp forms having character ϵ.

The aim is to investigate how much of cuspidal cohomology is obtained by automorphic induction from the Hecke characters of various imaginary quadratic fields. Quantitatively, we shall find bounds on the number of eigenforms of normalised Hecke operators in $S_k(\Gamma_1(N))$ which are obtained in this manner.

Remark 7.1.2. *Note that the irreducible cuspidal automorphic representation corresponding to such an eigenform is* ***cohomological****.*

Notation 9. *For integers $k, N \geq 1$, let $C_k(N)$ denote the number of cusp eigenforms of normalised Hecke operators for $\Gamma_1(N) \subseteq SL_2(\mathbb{Z})$ having weight k which are obtained by automorphic induction from Hecke characters of imaginary quadratic extensions of $\mathbb{Q}$.*

Let $E = \mathbb{Q}(\sqrt{-d})$ be a quadratic extension where $d > 0$ is an arbitrary squarefree integer. We note that the discriminant of $E/\mathbb{Q}$ equals

$$D_{E/\mathbb{Q}} := \begin{cases} -d \text{ if } d \equiv 3 (\text{mod } 4), \\ -4d \text{ otherwise;} \end{cases} \tag{7.1}$$

The prime 2 ramifies only in the latter case. Also, let $\mathbb{A}_E^\times$ denote the idèles over E.

Notation 10. *For functions f, g on $\mathbb{N}$ taking non-negative real values, we shall write $f \ll g$ if there exists a constant C such that $f(n) \leq C \cdot g(n)$ for all sufficiently large values of n.*
We shall also write $f \sim g$ when $f \ll g$ and $g \ll f$.

The following result connects the conductor and ∞-type of a Hecke character to the level and weight of the modular form obtained by automorphic induction.

Theorem 7.1.3 (Hecke)**.** *Let $k \geq 2$ be an integer. Let $E = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic extension where $d > 0$ is an arbitrary squarefree integer. Assume that χ is a Hecke character of E with conductor $\mathfrak{f} \subseteq \mathcal{O}_E$ and ∞-type $(k-1)$ which yields a cusp form*

$$f_\chi \in S_l(\Gamma_0(N), \epsilon)$$

on automorphic induction (for a certain Dirichlet character ϵ). Then

$$l = k \text{ and } N_{E/\mathbb{Q}}(\mathfrak{f}) \cdot |D_{E/\mathbb{Q}}| | N.$$

7.2 The number of unitary Hecke characters.

In this section, we recall some facts about Hecke characters. We use these facts to estimate the total number of unitary Hecke characters of a given ∞-type for an ideal $\mathfrak{f} \subseteq \mathcal{O}_E$. This will be required to obtain bounds on $C_k(N)$.

Notation 11. *For an ideal $\mathfrak{f} \subseteq \mathcal{O}_E$, let $\psi_E(\mathfrak{f})$ denote the number of residue classes modulo $\mathfrak{f}$ representable by the units of E, i.e.,*

$$\psi_E(\mathfrak{f}) := \#\{u + \mathfrak{f} : u \in \mathcal{O}_E^\times\}.$$

Note that since $E = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field, $1 \in \mathcal{O}_E^\times$ is the only unit both of whose embeddings in $\mathbb{C}$ are positive. (1 is clearly invertible modulo every ideal $\mathfrak{f} \subseteq \mathcal{O}_E$.) Therefore, the order $h_E(\mathfrak{f})$ of the ray class group modulo $\mathfrak{f}$ coincides with that of the **narrow ray class group** modulo the same. The former is given by the following formula in [32] (Thm. 3.25(i), p.109), where h_E is the class number of E.

$$h_E(\mathfrak{f}) = h_E \Phi_E(\mathfrak{f})/\psi_E(\mathfrak{f}) \tag{7.2}$$

As $1 \in \mathcal{O}_E^\times$ no matter what $\mathfrak{f}$ is, we have $\psi_E(\mathfrak{f}) \geq 1$. Hence,

$$h_E(\mathfrak{f}) \leq h_E \Phi_E(\mathfrak{f}), \tag{7.3}$$

with Φ_E as in 5.2.3.

Remark 7.2.1. *Once the ∞-type of a unitary Hecke character modulo $\mathfrak{f}$ is fixed, it corresponds to a character of the narrow ray class group modulo the same ideal (see [32], Prop. 7.7.7, p.330). Conversely, every character of the narrow ray class group modulo $\mathfrak{f}$ along with a consistent choice of the ∞-type defines a Hecke character.*

7.3 Upper Bound.

For $M \in \mathbb{N}$, let $\varphi_0(M)$ denote the number of primitive Dirichlet characters of conductor M. Using the Möbius inversion formula for the equation

$$\varphi(N) = \sum_{d|N} \varphi_0(d) \tag{7.4}$$

(see [25], Chap.3, p.46), we get

$$\varphi_0(M) = M \prod_{p||M} (1 - \frac{2}{p}) \prod_{p^2|M} (1 - \frac{1}{p})^2, \tag{7.5}$$

whenever $M \geq 2$.

We shall need the following result in the sequel:

Lemma 7.3.1. *Let $n \geq 1$ be an integer and $r \in (0,1)$. We have*

$$\prod_{p|n} \frac{1}{1-p^{-r}} \ll_\epsilon n^{1-r-\epsilon} \tag{7.6}$$

for each $0 < \epsilon < \min\{r, 1-r\}$ as n approaches infinity.

Proof. See [21], Eq.(29). ■

Keeping the criterion 6.8 in mind, we describe the strategy in short.

Remark 7.3.1. *An upper bound on the total number of normalised Hecke eigenforms for $\Gamma_1(N)$ of weight $k \geq 2$ obtained by automorphic induction is clearly provided by the* ***total*** *number of pertinent Hecke characters (whether those factor through the norm or not).*

Notation 12. *For an integer $N \geq 1$, define*

$$\hat{N} := \prod_{p|N} p.$$

Theorem 7.3.2. *Let $k, N \geq 1$ be integers and $C_k(N)$ be as in 9. We have*

$$C_k(N) \ll_{k,\epsilon} N \cdot \hat{N}^{3/2-\epsilon} \quad \forall \epsilon \in (0, 1/2) \tag{7.7}$$

as $N \to \infty$.

Remark 7.3.3. *By the dimension formula for cusp forms, we have*

$$\dim_{\mathbb{C}} S_k(\Gamma_1(N)) \sim_k N^2.$$

The upper bound shows that $C_k(N)$ is only a negligible fraction when N is divisible by large powers of primes.

Proof. By Hecke's theorem, the ∞-type of each Hecke character contributing to $C_k(N)$ equals $(k-1)$.

For each ideal $\mathfrak{d} \subseteq \mathcal{O}_E$, let $h^0_E(\mathfrak{d})$ denote the number of unitary Hecke characters having conductor **exactly** $\mathfrak{d}$. The number of characters of the narrow ray class group modulo $\mathfrak{d} \subseteq \mathcal{O}_E$ equals $h_E(\mathfrak{d})$. Each such character will have a conductor $\mathfrak{f}|\mathfrak{d}$. Thus, with Remark 7.2.1 of the previous section in mind, we also see the following:

$$\sum_{\mathfrak{f}|\mathfrak{d}} h_E^0(\mathfrak{f}) \leq h_E(\mathfrak{d}). \tag{7.8}$$

We need to estimate the sum

$$\sum_{\substack{d|N \\ d \text{ squarefree}}} \left\{ \sum_{\substack{E=\mathbb{Q}(\sqrt{-d}) \\ N_{E/\mathbb{Q}}(\mathfrak{f})|\frac{N}{|D_{E/\mathbb{Q}}|}}} h_E^0(\mathfrak{f}) \right\} \tag{7.9}$$

over various conductors $\mathfrak{f}$ in order to find an upper bound on $C_k(N)$.

So, let $M = M(d)$ be the smallest positive integer for which $\frac{MN}{|D_{E/\mathbb{Q}}|}$ is a perfect square. If we replace the condition $N_{E/\mathbb{Q}}(\mathfrak{f})|\frac{N}{|D_{E/\mathbb{Q}}|}$ above by the **weaker** $N_{E/\mathbb{Q}}(\mathfrak{f})|\frac{MN}{|D_{E/\mathbb{Q}}|}$, we sum over a **larger** number of conductors. This increases the inner summand for each $E = \mathbb{Q}(\sqrt{-d})$ as follows:

$$\left\{ \sum_{N_{E/\mathbb{Q}}(\mathfrak{f})|\frac{N}{|D_{E/\mathbb{Q}}|}} h_E^0(\mathfrak{f}) \right\} \leq \left\{ \sum_{N_{E/\mathbb{Q}}(\mathfrak{f})|\frac{MN}{|D_{E/\mathbb{Q}}|}} h_E^0(\mathfrak{f}) \right\} \tag{7.10}$$

Set $\frac{MN}{|D_{E/\mathbb{Q}}|} = \bar{M}^2$. We observe the following for every integer $l \geq 1$ and each prime ideal $\mathfrak{p}|\mathfrak{f}$, the following holds:

$$p^l | N_{E/\mathbb{Q}}(\mathfrak{p}^l) \Leftrightarrow \mathfrak{p}^l | p^l \mathcal{O}_E. \tag{7.11}$$

In particular, if $p^{2l} \| N_{E/\mathbb{Q}}(\bar{M}\mathcal{O}_E) = \bar{M}^2$ and $\mathfrak{p}|p\mathcal{O}_E$, then $\mathfrak{p}^l|\bar{M}\mathcal{O}_E$. Therefore, if we substitute $N_{E/\mathbb{Q}}(\mathfrak{f})|\frac{MN}{|D_{E/\mathbb{Q}}|}$ by the **still weaker** condition $\mathfrak{f}|\bar{M}\mathcal{O}_E$, we get

$$\left\{ \sum_{N_{E/\mathbb{Q}}(\mathfrak{f})|\frac{N}{|D_{E/\mathbb{Q}}|}} h_E^0(\mathfrak{f}) \right\} \leq \sum_{\mathfrak{f}|\bar{M}\mathcal{O}_E} h_E^0(\mathfrak{f}) = h_E(\bar{M}\mathcal{O}_E). \tag{7.12}$$

Summing over various E, we obtain the following upper bound:

$$C_k(N) \leq \sum_{\substack{d|N \\ d \text{ squarefree}}} h_E(\bar{M}\mathcal{O}_E) \tag{7.13}$$

Using the bound $h_E(\mathfrak{f}) \leq h_E \Phi_E(\mathfrak{f})$, we get

$$C_k(N) \leq \sum_{\substack{d|N \\ d \text{ squarefree}}} h_E \Phi_E(\bar{M}\mathcal{O}_E). \tag{7.14}$$

We have

$$\Phi_E(\bar{M}\mathcal{O}_E) = \frac{MN}{|D_{E/\mathbb{Q}}|} \prod_{\mathfrak{p}|\bar{M}\mathcal{O}_E} (1 - N_{E/\mathbb{Q}}(\mathfrak{p})^{-1}) \tag{7.15}$$

where $\mathfrak{p}$ ranges over prime ideals. Since $M|\hat{N}$, it follows that

$$C_k(N) \leq N\hat{N} \cdot \sum_{\substack{d|N \\ d \text{ squarefree}}} \frac{h_E}{|D_{E/\mathbb{Q}}|}. \tag{7.16}$$

As d is squarefree for each summand, we have

$$d|N \Leftrightarrow d|\hat{N}. \tag{7.17}$$

Thus,

$$C_k(N) \leq N\hat{N} \cdot \sum_{\substack{d|\hat{N} \\ d \text{ squarefree}}} \frac{h_E}{|D_{E/\mathbb{Q}}|} \tag{7.18}$$

Now, we shall use the following asymptotic formula for class number (see [32], Cor.1 to Thm.8.14, p.434):

$$\log h_{\mathbb{Q}(\sqrt{-D})} = (1/2 + o(1)) \log |D| \text{ as } |D| \to \infty. \tag{7.19}$$

This implies

$$\frac{h_E}{|D_{E/\mathbb{Q}}|} \ll_\epsilon |D_{E/\mathbb{Q}}|^{-(1/2)+\epsilon}. \tag{7.20}$$

for each $\epsilon \in (0, 1/2)$. Now,

$$\sum_{\substack{d|\hat{N} \\ d \text{ squarefree}}} |D_{E/\mathbb{Q}}|^{-(1/2)+\epsilon} \leq \sum_{\substack{d|\hat{N} \\ d \text{ squarefree}}} d^{-(1/2)+\epsilon}, \tag{7.21}$$

$$= \prod_{p|\hat{N}} \big(1 + \frac{1}{p^{1/2-\epsilon}}\big) \tag{7.22}$$

This product is less than

$$\prod_{p|\hat{N}} \frac{1}{1 - (1/p)^{1/2+\epsilon}} \ll_\epsilon \hat{N}^{1/2-\epsilon}, \quad \epsilon \in (0, 1/2). \tag{7.23}$$

Here, we have used Lemma 7.3.1. We conclude the proof as this yields the desired upper bound:

$$C_k(N) \ll_\epsilon (N\hat{N}) \cdot \hat{N}^{1/2-\epsilon} = N\hat{N}^{3/2-\epsilon} \tag{7.24}$$

■

7.4 Lower Bound.

We begin with a lemma in order to emphasise the necessity of passing to deeper level structures in order to get reasonable lower bounds.

Lemma 7.4.1. *Let* $\{p_n\}_{n=1}^{\infty}$ *be an increasing sequence of primes each of which satisfies* $p_n \equiv 1 \pmod 4$. *Then*

$$C_k(p_n) = 0 \quad \forall n \geq 1 \text{ and } k \geq 2. \tag{7.25}$$

Proof. Let $E = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic extension with $d > 0$ squarefree. Consider an idelic Hecke character $\chi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ of E. Assume that χ yields a modular cusp form f_χ (with some Dirichlet character ϵ) by automorphic induction. We have already observed that the ∞-type of χ must be

$$\chi_\infty : z \mapsto (z/|z|_{\mathbb{R}})^{k-1}. \tag{7.26}$$

Let α be the classical Hecke character determined by χ and $\mathfrak{f}$ be its conductor. Define

$$E_{\mathfrak{f}} := \left\{\frac{a}{b} \in E^\times : a, b \in \mathcal{O}_E, a \equiv b \pmod{\mathfrak{f}} \text{ and } (ab\mathcal{O}_E, \mathfrak{f}) = 1\right\}. \tag{7.27}$$

Since χ is trivial on $E^\times$, we have

$$\alpha(c\mathcal{O}_E) = \chi_\infty^{-1}(c) \quad \forall c \in E^\times. \tag{7.28}$$

Therefore,

$$\ker \chi_\infty \supseteq E_{\mathfrak{f}} \cap \mathcal{O}_E^\times. \tag{7.29}$$

(This holds because each unit u of E which is invertible modulo $\mathfrak{f}$ must satisfy $\alpha(u\mathcal{O}_E) = 1$).

Fix a value of n. Consider an arbitrary Hecke character χ of E having conductor $\mathfrak{f} \subseteq \mathcal{O}_E$ and ∞-type $(k-1)$. If χ were to yield a cusp form $f_\chi \in S_k(\Gamma_1(p_n))$ on automorphic induction, then we would have

$$N_{E/\mathbb{Q}}(\mathfrak{f}) \cdot |D_{E/\mathbb{Q}}| | p_n. \tag{7.30}$$

This is impossible as $|D_{E/\mathbb{Q}}| = -4d$ is even and p_n is odd (note that the only permissible values of d are 1 and p_n).
Therefore, no imaginary quadratic extension can contribute to cuspidal cohomology even as $n \to \infty$ and regardless of k. ∎

Remark 7.4.1. *As a consequence, we see that* $\liminf C_k(N) = 0$ *when either of* k *or* N *approaches infinity while the other is fixed. Hence, we must allow replacement of* N *by its sufficiently large multiples in order to get reasonable bounds. Such a replacement amounts to choosing a deeper level structure.*

With the above remark in mind, we give effect to the following strategy:

Remark 7.4.2. *We need to find a lower bound on the number of normalised Hecke eigenforms corresponding to a level structure for* $GL_2(\mathbb{A}_\mathbb{Q})$ *and a fixed weight. This can be accomplished by subtracting an upper bound on the number of Hecke characters factor through the idèlic norm from a lower bound on the total number of pertinent Hecke characters.*

Firstly, we find an upper bound on the number of idèlic Hecke characters of which factoring through the idèlic norm map

$$\mathcal{N}_{E/\mathbb{Q}} : \mathbb{A}_E^\times \to \mathbb{A}_Q^\times. \tag{7.31}$$

These are precisely the characters of the type $\tilde{\delta} \circ \mathcal{N}_{E/\mathbb{Q}}$ for the idèlisation $\tilde{\delta}$ of some Dirichlet character δ.

To this end, we define the following:

Notation 13. *For* $N \in \mathbb{N}$, *let*

$$\Delta_k(N) := \#\Big\{\chi : \mathbb{A}_E^\times/E^\times \to S^1 : \chi = \tilde{\delta} \circ \mathcal{N}_{E/\mathbb{Q}} \text{ for some Dirichlet character with } cond(\delta)|N \text{ and some } E = \mathbb{Q}(\sqrt{-d}), d > 0\Big\}.$$

Here, $\tilde{()}$ *denotes idèlisation of the respective Dirichlet character and* $\mathcal{N}_{E/\mathbb{Q}}$ *is the idèlic norm map.*

Lemma 7.4.2. *For each* $E = \mathbb{Q}(\sqrt{-d})$, *write* $\frac{N}{|D_{E/\mathbb{Q}}|} = M^2 \cdot \bar{M}$, *where* $M = M(E)$ *depends on* E *and* $\bar{M}$ *is squarefree. Then*

$$\Delta_k(N) \leq \sum_{\substack{d|N \\ d \text{ squarefree}}} h_E \varphi(M). \tag{7.32}$$

Proof. Consider a Dirichlet character δ having conductor $f \geq 2$. Let $f = \prod_{i=1}^r p_i^{f_i}$ be its factorisation into distinct primes. For an imaginary quadratic field E, the idèlic character $\tilde{\delta} \circ \mathcal{N}_{E/\mathbb{Q}}$ is a Hecke character for E

only if its ∞-type satisfies satisfies the constraint 7.29. The total number of such characters, therefore, **exceeds** that of the Hecke characters for $\mathbb{A}_E^\times/E^\times$. We proceed to estimate the former in order to find an upper bound on $\Delta_k(N)$.

Firstly, we investigate the relation between the conductor of δ and that of the finite part of $\tilde{\delta}\circ\mathcal{N}_{E/\mathbb{Q}}$. For each $1\le i\le r$, let

$$f_i' := \begin{cases} f_i & \text{if } p_i|D_{E/\mathbb{Q}}, \\ 2f_i & \text{otherwise} \end{cases} \tag{7.33}$$

If χ is the character of the narrow ray class group associated with $\tilde{\delta}\circ\mathcal{N}_{E/\mathbb{Q}}$ (see the Remarks 2.2.1 and 2.2.3), then we get

$$cond(\chi) = f'\mathcal{O}_E, \tag{7.34}$$

where $f' = \prod_{i=1}^r p_i^{f_i'}$. (Here, we have considered whether or not a prime p_i ramifies in E).

Hence, if $N_{E/\mathbb{Q}}(cond(\chi))|\frac{N}{|D_{E/\mathbb{Q}}|}$, then clearly $f^2|\frac{N}{|D_{E/\mathbb{Q}}|}$.
But

$$f^2|\frac{N}{|D_{E/\mathbb{Q}}|} \Leftrightarrow f|M. \tag{7.35}$$

(Note that $M = M(E)$ is dependent on E).

As there are $\varphi_0(f)$ such characters for each $E=\mathbb{Q}(\sqrt{-d})$, we get

$$\Delta_k(N) \le \sum_{\substack{d|N \\ d \text{ squarefree}}} \Big(\sum_{\substack{E=\mathbb{Q}(\sqrt{-d}) \\ f^2|\frac{N}{|D_{E/\mathbb{Q}}|}}} h_E\varphi_0(f)\Big). \tag{7.36}$$

Substituting the condition on the inner sum by the equivalent $f|M$, we get

$$\Delta_k(N) \le \sum_{\substack{d|N \\ d \text{ squarefree}}} h_E\Big(\sum_{\substack{E=\mathbb{Q}(\sqrt{-d}) \\ f|M}} \varphi_0(f)\Big) \tag{7.37}$$

$$= \sum_{\substack{d|N \\ d \text{ squarefree}}} h_E\varphi(M). \tag{7.38}$$

This is precisely the upper bound claimed. ■

Theorem 7.4.3. *Let $k \geq 1$ be an integer. Define $C_k(.)$ as in 9. Every positive integer n has a fixed multiple $N_0 = N_0(n)$ such that we have*

$$C_k(NN_0) \gg_{k,\epsilon,N_0} N^{1-\epsilon}\hat{N}^{1/2-\epsilon} \quad \forall \epsilon \in (0, 1/2) \tag{7.39}$$

as $N \to \infty$. Here, the implied constant depends upon the chosen values of k, N_0 and ϵ.

Proof. Given an integer $n \geq 1$, select a fixed multiple $N_0 = N_0(n)$ of n such that

$$4\hat{N}_0 | N_0 \text{ and } 6\varphi(\check{N}_0) < \check{N}_0. \tag{7.40}$$

This is possible as we have

$$\liminf_{r\to\infty} \varphi(r)/r = 0. \tag{7.41}$$

For $N \in \mathbb{N}$, consider a positive squarefree divisor d of NN_0. Set $E = \mathbb{Q}(\sqrt{-d})$. Further, assume that $d \neq 1, 3$ so that $\mathcal{O}_E^\times = \{1, -1\}$. This assumption ensures that for an ideal $\mathfrak{f} \subseteq \mathcal{O}_E$, the following relation is satisfied whenever $\mathfrak{f} \nmid 2\mathcal{O}_E$:

$$(-1) \notin E_\mathfrak{f} \cap \mathcal{O}_E^\times, \tag{7.42}$$

with $E_\mathfrak{f}$ as in 7.29. The constraint 7.29 is satisfied trivially under these assumptions. Hence, whenever $d \neq 1, 3$ and $\mathfrak{f} \nmid 2\mathcal{O}_E$, every character of the respective narrow ray class group is a Hecke character for E.

As we are omitting certain ideals by the above conditions, we clearly have

$$C_k(NN_0) + \Delta_k(NN_0) \geq \sum_{\substack{d|N \\ d \text{ squarefree} \\ d\neq 1,3}} \Big(\sum_{\substack{E=\mathbb{Q}(\sqrt{-d}) \\ N_{E/\mathbb{Q}}(\mathfrak{f}) | \frac{NN_0}{|D_{E/\mathbb{Q}}|}}} h_E^0(\mathfrak{f}) \Big) \tag{7.43}$$

with $h_E^0(\mathfrak{f})$ as in the proof of the upper bound.

Write $\frac{NN_0}{|D_{E/\mathbb{Q}}|} = M^2\bar{M}$, where $\bar{M} = \bar{M}(E)$ is squarefree. If we substitute the condition on the inner sum by the **stronger** $\mathfrak{f}|M\mathcal{O}_E$ (whereby counting over **fewer** conductors), we get the following lower bound:

$$C_k(NN_0) + \Delta_k(NN_0) \geq \sum_{\substack{d|N \\ d \text{ squarefree} \\ d\neq 1,3}} \Big(\sum_{\substack{E=\mathbb{Q}(\sqrt{-d}) \\ N_{E/\mathbb{Q}}(\mathfrak{f}) | M\mathcal{O}_E}} h_E^0(\mathfrak{f}) \Big), \tag{7.44}$$

$$\geq \sum_{\substack{d|N \\ d \text{ squarefree} \\ d\neq 1,3}} \Big(\sum_{\substack{E=\mathbb{Q}(\sqrt{-d}) \\ N_{E/\mathbb{Q}}(\mathfrak{f}) | M\mathcal{O}_E \\ \mathfrak{f}\nmid 2\mathcal{O}_E}} h_E^0(\mathfrak{f}) \Big). \tag{7.45}$$

Here, we have further omitted each ideal dividing $2\mathcal{O}_E$ from the sum. Since there are at most two such ideals, we may neglect those without affecting the order of the lower bound. Doing so, we obtain

$$\sum_{\substack{E=\mathbb{Q}(\sqrt{-d}) \\ N_{E/\mathbb{Q}}(\mathfrak{f})|M\mathcal{O}_E \\ \mathfrak{f}\nmid \ 2\mathcal{O}_E}} h_E^0(\mathfrak{f}) \approx \sum_{\substack{E=\mathbb{Q}(\sqrt{-d}) \\ N_{E/\mathbb{Q}}(\mathfrak{f})|M\mathcal{O}_E}} h_E(\mathfrak{f}) \geq h_E\Phi_E(M\mathcal{O}_E)/6, \tag{7.46}$$

(because $\psi_E(\mathfrak{f}) \leq 6$ trivially). Subtraction of the upper bound on $\Delta_k(N)$(as in Lemma 7.4.2) from this yields the following:

$$C_k(NN_0) \gg_{k,N_0} \sum_{\substack{d|N \\ d \text{ squarefree} \\ d\neq 1,3}} h_E\{\Phi_E(M\mathcal{O}_E) - 6\varphi(M)\} \tag{7.47}$$

$$= \sum_{\substack{d|N \\ d \text{ squarefree} \\ d\neq 1,3}} h_E M\big\{M \prod_{\mathfrak{p}|M\mathcal{O}_E} (1 - N_{E/\mathbb{Q}}(\mathfrak{p})^{-1}) - 6\varphi(M)/M\big\} \tag{7.48}$$

Here, we have noted that $N_{E/\mathbb{Q}}(M\mathcal{O}_E) = M^2$.

But for each prime ideal $\mathfrak{p}|M\mathcal{O}_E$, we have $N_{E/\mathbb{Q}}(\mathfrak{p}) = p$ or p^2 for a prime integer $p|M$. Therefore, we get

$$C_k(NN_0) \gg_{k,N_0} \sum_{\substack{d|N \\ d \text{ squarefree} \\ d\neq 1,3}} h_E\varphi(M)(M-6). \tag{7.49}$$

As $N \to \infty, \quad M \to \infty$. The condition 7.40 on N_0 then ensures that $M \gg 6$. Also, note that the formula (see [23], Chap. 18, Thm. 328, p.352)

$$\liminf_{r\to\infty} \frac{\varphi(r)\log\log r}{r} = e^{-\gamma} \tag{7.50}$$

(where $\gamma = 0.57721\ldots$ denotes the Euler's constant) implies that

$$\varphi(M) \gg_\epsilon M^{1-2\epsilon} \quad \forall \epsilon \in (0, 1/2). \tag{7.51}$$

Therefore,

$$C_k(NN_0) \gg_{k,N_0,\epsilon} \sum_{\substack{d|N \\ d \text{ squarefree} \\ d\neq 1,3}} h_E M^{2-2\epsilon} \approx (NN_0)^{1-\epsilon} \cdot \sum_{\substack{d|N \\ d \text{ squarefree} \\ d\neq 1,3}} h_E/|D_{E/\mathbb{Q}}|. \tag{7.52}$$

We may estimate the latter sum exactly as in the proof of Thm. 7.3.2. If we use Lemma 7.3.1 along with the Siegel bound

$$h_E \gg_\epsilon |D_{E/\mathbb{Q}}|^{1/2-\epsilon}, \tag{7.53}$$

the expression simplifies further to

$$C_k(NN_0) \gg_{k,N_0,\epsilon} N^{1-\epsilon}\hat{N}^{1/2-\epsilon}. \tag{7.54}$$

This is the lower bound claimed. ■

Remark 7.4.4. *We have kept the weight of the modular form fixed so far. We shall consider a fixed congruence subgroup as weight varies for the sake of completeness below.*

Let Γ be a fixed congruence subgroup of $SL(2,\mathbb{Z})$. There exists a fixed integer $N = N(\Gamma)$ such that

$$C_k(N) \ll_N k \text{ as } k \to \infty \tag{7.55}$$

and for which both the following indices are finite:

$$[\Gamma : \Gamma \cap \Gamma_1(N)], \quad [\Gamma_1(N) : \Gamma \cap \Gamma_1(N)] < \infty \tag{7.56}$$

To deduce this upper bound, we simply recall the dimension formula

$$\dim_{\mathbb{C}} S_k(SL(2,\mathbb{Z})) \sim k \tag{7.57}$$

as $k \to \infty$. As Γ has finite index in $SL(2,\mathbb{Z})$ too, we have

$$C_k(N) \ll_N k \text{ as } k \to \infty \tag{7.58}$$

trivially.

Chapter 8

Bounds for Symmetric Square Transfer.

Our aim is to find the growth rate of the size of the set of automorphic representations of $GL_3(\mathbb{A}_{\mathbb{Q}})$ which are obtained by symmetric square transfer from modular forms.

8.1 Bounds for $GL(3)/\mathbb{Q}$.

We maintain the same notation as in 2 to express our result.

By [18], a cuspidal automorphic representation π of $GL_2(\mathbb{A}_{\mathbb{Q}})$ yields a cuspidal automorphic form for $GL_3(\mathbb{A}_{\mathbb{Q}})$ on symmetric square transfer precisely when it is **not** induced from any Hecke character of a quadratic extension of $\mathbb{Q}$. We restate this criterion as a theorem here:

Theorem 8.1.1 (Gelbart-Jacquet)**.** *Let $\pi = \bigotimes_{p\leq\infty} \pi_p$ be an irreducible cuspidal automorphic representation of $GL_2/\mathbb{Q}$. Then $sym^{\otimes 2}\pi$ is a cuspidal automorphic representation of $GL_3/\mathbb{Q}$ precisely when*

$$\pi \neq \pi_\chi \tag{8.1}$$

for any Hecke character of a quadratic extension of $\mathbb{Q}$.

We use the above criterion to deduce the following:

Lemma 8.1.1. *Let $\pi = \bigotimes_{p\leq\infty} \pi_p$ be an irreducible cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$. Denote its conductor by N_π. Assume that the infinitesimal character of π_∞ corresponds to the weight*

$$\lambda_k := (k/2 - 1, 1 - k/2),$$

*where $k \geq 2$ is an **even** integer. Assume that*

$$\pi \in Coh(GL_2/\mathbb{Q}, \lambda_k, K_f(n)) \quad \forall n \geq 0.$$

Let $\Pi := sym^{\otimes 2}\pi$ be the representation of $GL_3(\mathbb{A}_\mathbb{Q})$ obtained by symmetric square transfer. Denote its conductor by N_Π and the highest weight corresponding to Π_∞ by μ_k. Then, we have

$$\Pi \in Coh(GL_3/\mathbb{Q}, \mu_k, H_f(N_\Pi)), \tag{8.2}$$
$$\mu_k = (k-2, 0, 2-k), \tag{8.3}$$
$$N_\Pi \cdot \hat{N}_\Pi | N_\pi^2 \text{ and} \tag{8.4}$$
$$N_\pi | N_\Pi \cdot \hat{N}_\Pi^2. \tag{8.5}$$

Proof. The first assertion is proved in [28]. It follows from the result in [6] (Chap. II, Prop 6.12 (i),p.46) which states that μ_k must be equal to its twist by the Cartan involution for π_∞ to be cohomological. Thus, it has the following form:

$$\mu_k = (a, b, c) \quad a \geq b \geq c \quad a, b, c \in \mathbb{Z} \tag{8.6}$$

with $a + c = b = 0$. If the newform associated with π has even weight $k \geq 2$, we have $\lambda_k = (\frac{k}{2} - 1, 1 - \frac{k}{2})$. On symmetric square transfer, this becomes $\mu_k = (k-2, 0, 2-k)$.

We use a formula by Bushnell, Henniart and Kutzko (see [9],Thm. 6.5(i)) in order to compute the conductor of the symmetric square transfer. With the notation as in the previous section, we have the following for each prime $p|N_\Pi$:

$$c(\Pi_p) = 2c(\pi_p) - c(\omega_p) - 2/e_p, \tag{8.7}$$

where $1 \leq e_p \leq 2$ is a certain ramification index and $p^{c(\omega_p)} = cond(\omega_p)$.

Now, the result by Casselman quoted in the previous section implies that $0 \leq c(\omega_p) \leq c(\pi_p)$. Also, we clearly have $1 \leq 2/e_p \leq 2$. Combining these, we obtain the following:

$$c(\Pi_p) + 2 \geq c(\pi_p) \text{ and } c(\Pi_p) + 1 \leq 2c(\pi_p) \tag{8.8}$$

The claim follows now from the definition of the conductor when we multiply over all primes $p|N_\Pi$. ■

For the dominant integral weight $\mu_k = (k-2, 0, 2-k)$ as in Lemma 8.1.1, we define the following.

Notation 14. *For $N \geq 1$, consider the level structure $H_f(N)$. Define*

$$D_k(N) := \{\Pi \in \mathcal{A}_{cusp}(GL_3/\mathbb{Q}, \mu_k, H_f(N)) : \exists \pi \text{ and } M \geq 1 \text{ such that } \pi \in \mathcal{A}_{cusp}(GL_2/\mathbb{Q}, \lambda_k, K_f(M)), \quad \Pi = sym^{\otimes 2}(\pi)\},$$

which is the set of cuspidal automorphic representations of $GL_3(\mathbb{A}_{\mathbb{Q}})$ which are obtained by symmetric square transfer from holomorphic cusp forms.

We shall consider the case when k as above is even for the sake of simplicity. We restate our result with the notation as above.

Theorem 8.1.2. *Let $k \geq 2$ be an **even** integer. Let $p \geq 2$ be a prime. We have*

$$\#D_k(p^n) \gg_k p^{2n} \quad \text{as } n \to \infty, \tag{8.9}$$

with the implied constant depending only on k.

Remark 8.1.3. *By Lemma 5.3.2, we also have the obvious upper bound*

$$\#D_k(p^n) \ll_k p^{2n} \quad \text{as } n \to \infty.$$

*Thus, the above theorem implies that the holomorphic cusp forms in $S_k(\Gamma_1(p^n))$ which yield a cuspidal automorphic representation of $GL_3(\mathbb{A}_{\mathbb{Q}})$ on symmetric square transfer constitute **a substantial fraction** of the total.*

Proof. Let $\Pi := \otimes_{q \leq \infty} \Pi_q \in D_k(p^n)$. Clearly, Π_q is unramified except when $q = p$. If $\Pi = sym^{\otimes 2}\pi$, it follows that π_q is also unramified unless $q = p$. Therefore, it suffices to consider the conductor of π_p; call it m.

Next, we use Lemma 8.1.1 to estimate m as follows:

$$m - 2 \leq n \leq 2m - 1 \Rightarrow (n+1)/2 \leq m \leq n + 2. \tag{8.10}$$

Further, it is well-known that π_p corresponds to a newform (with character) in $S_k^{new}(\Gamma_1(p^m))$. The above inequality, thus, shows that a newform in $S_k^{new}(\Gamma_1(p^i))$ can contribute to $D_k(p^n)$ only if $(n+1)/2 \leq i \leq n + 2$.

Therefore, with $C_k(.)$ as in 9, we have

$$\#D_k(p^n) + C_k(p^{n+2}) \geq \sum_{(n+1)/2 \leq i \leq n+2} \dim_{\mathbb{C}} S_k^{new}(\Gamma_1(p^i)). \tag{8.11}$$

But, by Thm. 7.3.2 we have the upper bound

$$C_k(p^{n+2}) \ll_{k,\epsilon} p^{n+2} p^{3/2-\epsilon} \quad \forall \epsilon \in (0, 1/2). \tag{8.12}$$

In view of Lemma 5.3.3, we have

$$\frac{\dim_{\mathbb{C}} S_k^{new}(\Gamma_1(p^i))}{p^{2i}} = (\frac{k-1}{4\pi^2})(1-\frac{1}{p^2})^2 + o(1) \tag{8.13}$$

as $i \to \infty$.

As $p \geq 2$, we trivially have $(1 - 1/p^2) \geq 3/4$. Hence,

$$\dim_{\mathbb{C}} S_k^{new}(\Gamma_1(p^i)) \gg_k p^{2i}. \tag{8.14}$$

The sum in Eq.8.11 above is, therefore, of the order

$$\sum_{(n+1)/2 \leq i \leq n+2} p^{2i} \gg p^{2n+4} \tag{8.15}$$

as $n \to \infty$.

Now, a comparison of the above equation with the upper bound on $C_k(p^{n+2})$ shows that $C_k(p^{n+2})$ is negligible as compared to p^{2n+4} as $n \to \infty$. In effect, we may subtract the **upper bound** on $C_k(p^{n+2})$ from a **lower bound** on the sum in Eq.8.11. This produces the claimed lower bound on $\#D_k(p^n)$ for each fixed prime p follows:

$$\#D_k(p^n) \gg_k p^{2n+4} \tag{8.16}$$

as $n \to \infty$. ■

8.2 Comparison with Marshall's bound.

By Remark 8.1.3 and Thm. 8.1.2, we may also infer the following for sufficiently large fixed n:

$$\#D_k(p^n) \gg_{p,n} k \text{ as } k \to \infty. \tag{8.17}$$

This is consistent with a prediction in [31]; as we shall discuss in this section.

Marshall ([31]) has considered the case of $GL_3/\mathbb{Q}$ with a fixed level structure H_f but varying weight $\mu = (k, 0, -k)$. He proves the bounds

$$k \ll_{\epsilon, H_f} \#Coh(GL_3, \mu, H_f) \ll_{\epsilon, H_f} k^{3-\frac{4}{27}+\epsilon}. \tag{8.18}$$

(Note that $\dim \mathcal{V}_\mu \sim k^3$ by the Weyl dimension formula.) He also predicts that the exact growth rate is $\#Coh(GL_3, \mu, H_f) \sim k$ as $k \to \infty$. As $D_k(N) \subseteq Coh(GL_3/\mathbb{Q}, \mu, H_f(N))$, the consequence our result Eq. 8.17 provides further evidence in favour of this prediction.

Bibliography

[1] Ash, A., Grayson, D., and Green, P. Computations of cuspidal cohomology of congruence subgroups of $\mathrm{SL}(3, \mathbf{Z})$. *J. Number Theory 19*, 3 (1984), 412–436.

[2] Bhagwat, C., and Raghuram, A. Endoscopy and the cohomology of $GL(n)$. *Bull. Iranian Math. Soc. 43*, 4 (2017), 317–335.

[3] Borel, A. Regularization theorems in Lie algebra cohomology. Applications. *Duke Math. J. 50*, 3 (1983), 605–623.

[4] Borel, A., and Ji, L. Compactifications of symmetric and locally symmetric spaces. 69–137.

[5] Borel, A., and Serre, J.-P. Théorèmes de finitude en cohomologie galoisienne. *Comment. Math. Helv. 39* (1964), 111–164.

[6] Borel, A., and Wallach, N. R. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, vol. 94 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980.

[7] Brown, K. S. *Cohomology of groups*, vol. 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[8] Bruinier, J. H., van der Geer, G., Harder, G., and Zagier, D. *The 1-2-3 of modular forms*. Universitext. Springer-Verlag, Berlin, 2008. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.

[9] Bushnell, C. J., Henniart, G. M., and Kutzko, P. C. Local Rankin-Selberg convolutions for GL_n: explicit conductor formula. *J. Amer. Math. Soc. 11*, 3 (1998), 703–730.

[10] Calegari, F., and Emerton, M. Bounds for multiplicities of unitary representations of cohomological type in spaces of cusp forms. *Ann. of Math. (2) 170*, 3 (2009), 1437–1446.

[11] Casselman, W. On some results of Atkin and Lehner. *Math. Ann. 201* (1973), 301–314.

[12] Deligne, P. Les constantes des équations fonctionnelles des fonctions *L*. 501–597. Lecture Notes in Math., Vol. 349.

[13] Dimca, A. *Sheaves in topology.* Universitext. Springer-Verlag, Berlin, 2004.

[14] Dodziuk, J. L^2 harmonic forms on rotationally symmetric Riemannian manifolds. *Proc. Amer. Math. Soc. 77*, 3 (1979), 395–400.

[15] Donnelly, H. On the cuspidal spectrum for finite volume symmetric spaces. *J. Differential Geom. 17*, 2 (1982), 239–253.

[16] Finis, T., Grunewald, F., and Tirao, P. The cohomology of lattices in $\mathrm{SL}(2,\mathbb{C})$. *Experiment. Math. 19*, 1 (2010), 29–63.

[17] Freitag, E. *Hilbert modular forms.* Springer-Verlag, Berlin, 1990.

[18] Gelbart, S., and Jacquet, H. A relation between automorphic representations of GL(2) and GL(3). *Ann. Sci. École Norm. Sup. (4) 11*, 4 (1978), 471–542.

[19] Gelbart, S. S. *Automorphic forms on adèle groups.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1975. Annals of Mathematics Studies, No. 83.

[20] Grobner, H. An explicit growth condition for middle-degree cuspidal cohomology of arithmetically defined quaternionic hyperbolic n-manifolds. *Monatsh. Math. 159*, 4 (2010), 335–340.

[21] Gronwall, T. H. Some asymptotic expressions in the theory of numbers. *Trans. Amer. Math. Soc. 14*, 1 (1913), 113–122.

[22] Harder, G., and Raghuram, A. *Eisenstein cohomology and the special values of Rankin-Selberg L-functions*, vol. 203 of *Annals of Mathematical Studies.* Princeton University Press, 2020.

[23] Hardy, G. H., and Wright, E. M. *An introduction to the theory of numbers.* Oxford, at the Clarendon Press, 1954. 6th ed.

[24] Hu, Y. Multiplicities of cohomological automorphic forms. *arXiv preprint arXiv:1801.10074* (2018).

[25] Iwaniec, H., and Kowalski, E. *Analytic number theory*, vol. 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[26] Jacquet, H., and Langlands, R. P. *Automorphic forms on* GL(2). Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970.

[27] Jacquet, H., Piatetski-Shapiro, I. I., and Shalika, J. Conducteur des représentations du groupe linéaire. *Math. Ann. 256*, 2 (1981), 199–214.

[28] Labesse, J.-P., and Schwermer, J. On liftings and cusp cohomology of arithmetic groups. *Invent. Math. 83*, 2 (1986), 383–401.

[29] Lück, W. Approximating L^2-invariants by their classical counterparts. *EMS Surv. Math. Sci. 3*, 2 (2016), 269–344.

[30] Marshall, S. Bounds for the multiplicities of cohomological automorphic forms on GL_2. *Ann. of Math. (2) 175*, 3 (2012), 1629–1651.

[31] Marshall, S. Bounds for the number of cohomological automorphic representations of gl3/q in the weight aspect. *arXiv preprint arXiv:1812.03134v1* (2018).

[32] Narkiewicz, W. a. a. *Elementary and analytic theory of algebraic numbers*, third ed. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004.

[33] Rohlfs, J., and Speh, B. On limit multiplicities of representations with cohomology in the cuspidal spectrum. *Duke Math. J. 55*, 1 (1987), 199–211.

[34] Samet, I. Betti numbers of finite volume orbifolds. *Geom. Topol. 17*, 2 (2013), 1113–1147.

[35] Shimizu, H. On discontinuous groups operating on the product of the upper half planes. *Ann. of Math. (2) 77* (1963), 33–71.

[36] Stein, W. *Modular forms, a computational approach*, vol. 79 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. With an appendix by Paul E. Gunnells.

[37] TSUSHIMA, R. A formula for the dimension of spaces of Siegel cusp forms of degree three. *Proc. Japan Acad. Ser. A Math. Sci. 55*, 9 (1979), 359–363.

[38] WAKATSUKI, S. The dimensions of spaces of Siegel cusp forms of general degree. *Adv. Math. 340* (2018), 1012–1066.

[39] WEIL, A. Sur la théorie du corps de classes. *J. Math. Soc. Japan 3* (1951), 1–35.

[40] WHITEHEAD, G. W. *Elements of homotopy theory*, vol. 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978.