# Arbitrary spin theories in light-cone gauge

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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This thesis is based on the following papers by the author and his collaborators:

- Y. S. Akshay and S. Ananth, "Fermi-Bose couplings in light-cone field theories", *currently under review*
- Y. S. Akshay and S. Ananth, "Factorization of cubic vertices involving three different higher spin fields", *Nucl. Phys.* **B887**, 168 (2014) [arXiv:1404.2448]
- Y. S. Akshay, S. Ananth and M. Mali, "Light cone gravity in $AdS_4$", *Nucl. Phys.* **B884**, 66 (2014) [arXiv:1401.5933]
- Y. S. Akshay and S. Ananth, "Cubic interaction vertices in higher spin theories", *J. Phys.* **A47**, 045401 (2014) [arXiv:1304.8082]

# Certificate

This is to certify that this thesis entitled "Arbitrary spin theories in light-cone gauge" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by Akshay Y. S. at the Indian Institute of Science Education and Research Pune, under the supervision of Dr. Sudarshan Ananth during the academic year 2014-2015.

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# Declaration

I hereby declare that the matter embodied in the report entitled "ARBITRARY SPIN THEORIES IN LIGHT-CONE GAUGE" are the results of the investigations carried out by me at the Department of Physics, Indian Institute of Science Education and Research Pune, under the supervision of Dr. Sudarshan Ananth and the same has not been submitted elsewhere for any other degree.

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Abstract

We outline an approach which is purely based on symmetry considerations, for the construction of interaction vertices involving arbitrary spin fields. We review existing literature in which this approach is used to construct interactions between bosonic fields of arbitrary spin. We extend this formalism to include interaction vertices involving arbitrary spin fermionic fields. We use these vertices to derive three point scattering amplitudes and find remarkable factorization properties. Further, we find that the form of these amplitudes in momentum space is identical to the bosonic case.

# Contents

# Chapter 1

# Introduction

The Standard Model of particle physics has been phenomenally successful in describing physics upto energy scales of a few TeV. It includes spin 1 gauge bosons($\gamma, W^{\pm}, Z, g$) which mediate the electromagnetic, weak and strong forces. It also includes the spin 0 Higgs which generates masses for $W^{\pm}$ and $Z$ bosons. The theoretical framework underlying the Standard Model is quantum field theory. The fourth fundamental force of Nature, Gravity, is described by Einstein's general theory of relativity. While the field theories underlying the Standard Model(QED, QCD and the Electroweak theory) are renormalizable, general relativity is a non-renormalizable theory and can be used only as an effective field theory. However, general relativity has passed stringent experimental tests for nearly a hundred years now. It is thus likely that a consistent quantum theory of gravity will be a modification of general relativity rather than a complete replacement. In particular, it seems plausible that in a consistent quantum theory, gravity would be mediated by a spin 2 particle, the Graviton. Apart from the gauge bosons, the Standard Model also leptons($e^{-}, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau$), antileptons, quarks($u, c, t, d, s, b$) and antiquarks. These particles possess spin $\frac{1}{2}$.

The fact that particles of spin 0, $\frac{1}{2}$, 1 and possible 2 exist in nature while particles with higher spin seem to be absent is very puzzling. The most direct method to understand the reason behind this is to construct a consistent quantum field theory of higher spin(spin>2) particles. Several attempts have been made in this direction. In the section below, we give a brief overview of these attempts and highlight some of the issues of current interest in higher spin theories.

## 1.1 The history

The earliest attempts to describe higher spin theories were made by Pauli, Dirac, Fierz, Bargmann and Wigner [1].They derived relativistic wave equations describing particles of arbitrary spin and initiated the construction of Lagrangians for the free theories. This was completed for the massive case by Haagen and Singh [2]. Fronsdal and Fang examined these Lagrangians in the massless case and discovered remarkable simplifications [3]. However, interacting higher spin theories in flat spacetime were difficult to construct. A slew of no-go theorems exist which prohibit the existence of such theories [4]. For an accessible overview of several of these theorems, see [5]. However, most of these results are valid only for theories with manifest locality and manifest Lorentz invariance. An apparent exception to these theorems is the construction of cubic vertices in flat spacetime in [6]. However, this theory is not complete without the addition of quartic and higher order interactions. In this thesis, we outline the procedure followed in [6] and develop it further. We begin by describing the symmetries which these theories must possess.

## 1.2 Poincaré invariance

All theories of Physics must be consistent with the principle of special relativity which states that the laws of physics must be the same in all inertial frames. This implies that the equations describing them must retain their form under Poincaré transformations which include rotations, translations and boosts. These operations form a group and its Lie algebra is known as the Poincaré algebra. The generators of the Poincaré group are the Lorentz generators $J_{\mu\nu}$ and translations generators $P_\mu$. They satisfy the commutation relations given below.

$$\begin{aligned} i\,[J_{\mu\nu}, J_{\rho\sigma}] &= \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\sigma\mu}J_{\rho\nu} + \eta_{\nu\sigma}J_{\rho\mu} \\ i\,[P_\mu, J_{\rho\sigma}] &= \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho \\ i\,[P_\mu, P_\nu] &= 0 \end{aligned} \tag{1.1}$$

## 1.3 Gauge invariance

Some theories have an additional symmetry called gauge invariance. We demonstrate this for the case of the free electromagnetic fields whose Action takes the following form.

$$S = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} \tag{1.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A_\mu$ is the photon field. The corresponding Euler-Lagrange equations are

$$\partial_\mu F^{\mu\nu} = 0 \tag{1.3}$$

We note that both the action (1.2) and the equations of motion (1.3) are invariant under a transformation of the form

$$A_\mu \to A_\mu + \partial_\mu \Lambda(x). \tag{1.4}$$

This redundancy in the theory is known as gauge invariance. Choosing $\Lambda(x)$ to be a particular function is called gauge fixing. This eliminates spurious degrees of freedom from the theory. This is illustrated in (2.1).

# Chapter 2

# Free field theories in light-cone gauge

To understand theories of interacting arbitrary spin fields, we must first construct the corresponding free theories. In this thesis, our focus will be on constructing theories of massless fields of arbitrary spin in Minkowski space-time. The free fields are described by gauge theories. They involve many unphysical degrees of freedom which are ultimately eliminated either by gauge fixing or the introduction of auxiliary fields. In the following sections we describe the light-cone gauge in which all the unphysical degrees of freedom are eliminated systematically by gauge choice and imposing the constraint equations. All massless fields in four dimensions have two degrees of freedom. This makes the light-cone gauge especially well suited to massless theories as it deals solely with the physical degrees of freedom.

We define light-cone co-ordinates in Minkowski space-time with signature $(-,+,+,+)$ by

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}}\ , \qquad x^- = \frac{x^0 - x^3}{\sqrt{2}}\ , \qquad x = \frac{x^1 + ix^2}{\sqrt{2}}\ , \qquad \bar{x} = \frac{x^1 - ix^2}{\sqrt{2}}.$$

The corresponding derivatives are

$$\partial_+ = \frac{\partial_0 + \partial_3}{\sqrt{2}}\ , \qquad \partial_- = \frac{\partial_0 - \partial_3}{\sqrt{2}}\ , \qquad \partial = \frac{\partial_1 + i\partial_2}{\sqrt{2}}\ , \qquad \bar{\partial} = \frac{\partial_1 - i\partial_2}{\sqrt{2}}.$$

The co-ordinate $x^+$ is chosen to be time. The corresponding derivative, $\partial_+$ is the light-cone time derivative. Equations involving $\partial_+$ are the dynamical equations. They relate the configuration of the system at one point in light-cone time to another. Equations free of $\partial_+$ are kinematical equations. They serve to constrain the configuration of the system at each instant in time. We will not illustrate this difference for the case of electromagnetism.

## 2.1 Electromagnetism in light-cone gauge

We now return to electrodynamics and illustrate the method of light-cone reduction. Analogous to the definition of light-cone co-ordinates, we can define light-cone components of any vector or tensor. The photon field $A^\mu$, written in terms of its light cone components is $A^\mu = (A^+, A^-, A, \bar{A})$. Note that $A_+ = -A^-$ and $A_- = -A^+$. The light-cone gauge is the choice $A^+ = 0$. Using this in the $\nu = +$ component of (1.3), we get

$$\partial_-^2 A^- + \partial_- \partial_i A^i = 0 \tag{2.1}$$

We can define an operator $\frac{1}{\partial_-}$ which is the Green function of the differential operator $\partial_-$. This will involve a careful choice of boundary conditions, which are explained in [7]. With these conditions, we can write

$$A^- = -\frac{\partial_i}{\partial_-} A_i \tag{2.2}$$

$A^\mu$ originally had four components. Imposing the gauge condition and a constraint equation, we are able to reduce it to two independent components. In the A, we show that these correspond to the two helicity eigenstates of the photon. Plugging all this into (1.2) and performing a few partial integrations, the Action can be cast in the form

$$S = \int d^4x \; \bar{A} \Box A \tag{2.3}$$

## 2.2 Dirac fields in light-cone gauge

We now demonstrate a similar reduction procedure for massless Dirac fields. We begin by defining the following matrices.

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad C = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}. \tag{2.4}$$

Here C is the charge conjugation matrix. The Action describing a free Dirac field is

$$S = \int d^4x \, i \, \bar{\Psi} \gamma^\mu \partial_\mu \Psi \tag{2.5}$$

At the heart of the light-cone formalism is the separation of dynamical and kinematical degrees of freedom. To this end, we define the following projection operators.

$$P_+ \equiv \frac{1}{2}\gamma_+\gamma_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_- \equiv \frac{1}{2}\gamma_-\gamma_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.6}$$

where

$$\gamma_+ = \frac{1}{\sqrt{2}}(\gamma_0 + \gamma_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1+\sigma^3 \\ -1-\sigma^3 & 0 \end{pmatrix}$$
$$\gamma_- = \frac{1}{\sqrt{2}}(\gamma_0 - \gamma_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1-\sigma^3 \\ -1+\sigma^3 & 0 \end{pmatrix}. \tag{2.7}$$

That these are projection operators is easy to see as they satisfy

$$P_+^2 = P_+ \qquad P_-^2 = P_- \qquad P_+P_- = 0 \qquad P_+ + P_- = 1.$$

We can now begin the procedure of light-cone reduction. Let

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{2.8}$$

be a Dirac spinor with Grassmann valued components. Imposing the Majorana condition $\Psi = C\bar{\Psi}^T$ yields

$$\psi_1 = \bar{\psi}_4 \qquad ; \qquad \psi_2 = -\bar{\psi}_3\ . \tag{2.9}$$

Define

$$\Psi_+ = P_+\Psi \quad ; \quad \Psi_- = P_-\Psi\ . \tag{2.10}$$

Acting the projection operators on the Dirac equation yields the following two equations

$$\partial_-\Psi_- = \frac{1}{2}\gamma_-\gamma_i\partial_i\Psi_+\ , \qquad\qquad i = 1,2 \tag{2.11}$$

$$\partial_+\Psi_+ = \frac{1}{2}\gamma_+\gamma_i\partial_i\Psi_-\ , \qquad\qquad i = 1,2 \tag{2.12}$$

This first of these is kinematical and can be solved to yield

$$\psi_1 = \frac{\bar{\partial}}{\partial_-}\psi_2\ . \tag{2.13}$$

The entire spinor can now be written in terms of $\psi_3 \equiv \psi$ as

$$\Psi = \begin{pmatrix} -\frac{\bar{\partial}}{\partial_-}\bar{\psi} \\ -\bar{\psi} \\ \psi \\ -\frac{\partial}{\partial_-}\psi \end{pmatrix}\ . \tag{2.14}$$

This completes the reduction procedure. Using (2.14) in (2.5) and keeping track of Grassmann signs, we arrive at the form

$$S = \int d^4x \;\; i\bar{\psi}\,\frac{\Box}{\partial_-}\,\psi. \tag{2.15}$$

## 2.3 Arbitrary spin fields in light-cone gauge

In (2.1) and (2.2), we have derived the Action for spin 1 and spin $\frac{1}{2}$ fields respectively. One of the great advantages of the light-cone gauge is that free bosonic and fermionic fields of higher helicity can be described by the same Action. This is consistent with the fact that all massless fields in four dimension have two physical degrees of freedom. However, the transformation properties of fields of different helicity will indeed be different. This can be seen from the action of the Lorentz generators on the fields, which are shown in (3.1).

This simplicity comes at the cost of manifest covariance. The light-cone reduction yields a Lagrangian that suffers from a loss of manifest covariance. The covariance of the free theory is ensured by the fact that it is the gauge fixed version of a covariant theory. In order to construct interacting theories in light-cone gauge, we need to find another method of checking covariance. To this end, we represent all the Poincaré generators on the fields and explicitly check for the closure of the algebra. The Hamiltonian is one of the generators of the algebra. We can add interaction terms to this to generate interacting theories of arbitrary spin. However, maintaining covariance of the theory demands that the algebra still closes. This imposes several constraints on the form of the interactions. In the following chapters, we use this scheme to derive cubic interaction vertices for theories involving arbitrary spin bosons and fermions.

# Chapter 3

# Construction of bosonic interactions

*The materiel in this chapter is largely based on the second reference in [9]. This represents original work carried out by the author in collaboration with Dr. Ananth*

We are now in a position to systematically construct interaction vertices for theories involving arbitrary spin fields. In this chapter, we focus on the interactions involving only bosonic fields. The physical degrees of freedom of any bosonic field are the positive and negative helicity states. We will denote these by $\phi$ and $\bar{\phi}$ respectively. The value of the helicity is determined by the action of the rotation generators on them. The Action describing a massless, spin-$\lambda$ bosonic field is

$$S = \int d^4x \ \frac{1}{2}\,\bar{\phi}\,\Box\,\phi \tag{3.1}$$

We can construct the Poincaré generators of the free theory following the procedure in (A). The action of a generator $\mathcal{O}$ on a field is

$$\delta_{\mathcal{O}}\phi = \{\phi, \mathcal{O}\}$$

We now list the actions of all Poincaré generators on a massless field of helicity $\lambda$.

$$\begin{aligned} &\delta_{p^-}\phi = -i\frac{\partial\bar{\partial}}{\partial^+}\phi = -\delta_{p_+}\phi \qquad && \delta_{p^+}\phi = -i\partial^+\phi = -\delta_{p_-}\phi \\ &\delta_{p}\phi = -i\partial\phi && \delta_{\bar{p}}\phi = -i\bar{\partial}\phi \end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\delta_j\phi = i(x\bar\partial - \bar x\partial - \lambda)\phi && \delta_{j^{+-}}\phi = (x^+\frac{\partial\bar\partial}{\partial^+} - x^-\partial^+)\phi \\
&\delta_{j^+}\phi = (x^+\partial - x\partial^+)\phi && \delta_{\bar j^+}\phi = (x^+\bar\partial - \bar x\partial^+)\phi \\
&\delta_{j^-}\phi = (x^-\partial - x\frac{\partial\bar\partial}{\partial^+} + \lambda\frac{\partial}{\partial^+})\phi && \delta_{\bar j^-}\phi = (x^-\partial - x\frac{\partial\bar\partial}{\partial^+} - \lambda\frac{\partial}{\partial^+})\phi
\end{aligned} \tag{3.3}$$

The Hamiltonian for the free field theory is (A)

$$H \equiv P_+ = \int d^3x\,\mathcal{H} = \int d^3x\,\partial_-\bar\phi\,\frac{\partial\bar\partial}{\partial_+}\,\phi\ , \tag{3.4}$$

This is rewritten as

$$H \equiv \int d^3x\,\mathcal{H} = \int d^3x\,\partial_-\bar\phi\,\delta_{\mathcal{H}}\phi\ , \tag{3.5}$$

in terms of the time translation operator

$$\delta_{\mathcal{H}}\phi \equiv \partial_+\phi = \{\phi, \mathcal{H}\}\ , \tag{3.6}$$

where $\{,\}$ denotes the Poisson bracket.

The action of all the free Poincaré generators on the fields are linear. Non linear actions of these generators represent interacting theories. In particular, it can be seen from (3.5) that a quadratic term in $\delta_{\mathcal{H}}\phi$ represents a cubic interaction in the Lagrangian. We can thus systematically add non linear terms and generate all possible interactions. Other dynamical generators also pick up corrections which can be written in terms of $\delta_{\mathcal{H}}\phi$. We organize the interaction terms order by order in a coupling constant $\alpha$. The forms of the corrections to the Lorentz generators are

$$\begin{aligned}
&\delta_{j^{+-}}\phi = \delta^0_{j^{+-}}\phi - ix^+\delta^\alpha_{\mathcal{H}}\phi + O(\alpha^2) \\
&\delta_{j^-}\phi = \delta^0_{j^-}\phi + ix\delta^\alpha_{\mathcal{H}}\phi + \delta^\alpha_s\phi + O(\alpha^2) \\
&\delta_{\bar j^-}\phi = \delta^0_{\bar j^-}\phi + i\bar x\delta^\alpha_{\mathcal{H}}\phi + \delta^\alpha_{\bar s}\phi + O(\alpha^2).
\end{aligned} \tag{3.7}$$

Here, $\delta^\alpha_s$ and $\delta^\alpha_{\bar s}$ represent corrections to the spin part of the generators. We assume these to be of the form $\bar\phi\phi$ as this form agrees with the known spin 1 corrections. At cubic order, these do not mix with any of the other terms and are therefore not relevant to the calculations in this paper.

## 3.1 Deriving cubic interaction vertices

We focus on the following three structures for cubic interaction vertices at order $\alpha$.

$$\delta_{\mathcal{H}}^{\alpha}\phi_1 \sim \phi_2\phi_3 \ ; \quad \delta_{\mathcal{H}}^{\alpha}\phi_2 \sim \phi_1\phi_3 \ ; \quad \delta_{\mathcal{H}}^{\alpha}\phi_3 \sim \phi_1\phi_2 \ , \tag{3.8}$$

where the fields $\phi_1$, $\phi_2$ and $\phi_3$ have integer spins $\lambda_1$, $\lambda_2$ and $\lambda_3$ respectively. The first of these structures, at the level of the action would correspond to terms of the form

$$S \sim \int d^4x \ \bar{\phi}_1\, \phi_2\, \phi_3 \ + c.c. \, . \tag{3.9}$$

We enhance this basic form with derivatives to arrive at the Ansatz

$$\delta_{\mathcal{H}}^{\alpha}\phi_1 = \alpha\, A\, \partial^{+\mu} \left[\bar{\partial}^a \partial^{+\rho}\phi_2 \bar{\partial}^b \partial^{+\sigma}\phi_3\right] \ , \tag{3.10}$$

where $\mu, \rho, \sigma, a, b$ are integers and $A$ is a numerical factor that could depend on the variables and spins. Note that terms of the form $\bar{\phi}_2\bar{\phi}_3$ in $\delta_H^{\alpha}\phi_1$ are independent of terms of the form $\phi_2\phi_3$ and hence will not 'talk' to one another. The commutators

$$\begin{aligned} &[\delta_j, \delta_{\mathcal{H}}^{\alpha}]\,\phi_1 = 0 \ , \\ &[\delta_{j^{+-}}, \delta_{\mathcal{H}}^{\alpha}]\,\phi_1 = -\delta_{\mathcal{H}}\phi_1 \ , \end{aligned} \tag{3.11}$$

impose the following conditions on our Ansatz

$$\begin{aligned} a + b \ &= \lambda_2 + \lambda_3 - \lambda_1 \\ \mu + \rho + \sigma \ &= -1 \ . \end{aligned} \tag{3.12}$$

Since $a, b > 0$, the first of these conditions[1] implies that the vertex cannot exist unless $\lambda_2 + \lambda_3 > \lambda_1$. Now, let $\lambda \equiv \lambda_2 + \lambda_3 - \lambda_1$ so the first equation of (4.6) reads $a + b = \lambda$. There are precisely $(\lambda + 1)$ possible values for a pair $(a, b)$. We now rewrite our Ansatz in (4.5) as a sum of these $(\lambda + 1)$ terms

$$\delta_H^{\alpha}\phi_1 = \alpha \sum_{n=0}^{\lambda} A_n\, \partial^{+\mu_n} \left[\bar{\partial}^n \partial^{+\rho_n}\phi_2\, \bar{\partial}^{(\lambda-n)}\partial^{+\sigma_n}\phi_3\right] \ + c.c. \, . \tag{3.13}$$

The next set of commutators are

$$\left[\delta_{\bar{j}^-}, \delta_H\right]^{\alpha}\phi_1 = 0 \qquad \left[\delta_{j^+}, \delta_H\right]^{\alpha}\phi_1 = 0 \, , \tag{3.14}$$

[1] Note that this reduces to the condition in [6] if we set $\lambda_1 = \lambda_2 = \lambda_3$.

and yield the following conditions

$$\sum_{n=0}^{\lambda} A_n \Big\{ (\mu_n+1-\lambda_1)\partial^{+(\mu_n-1)}\bar\partial(\bar\partial^n\partial^{+\rho_n}\phi_2\bar\partial^{(\lambda-n)}\partial^{+\sigma_n}\phi_3)$$
$$+(\rho_n+\lambda_2)\partial^{+\mu_n}(\bar\partial^{(n+1)}\partial^{+(\rho_n-1)}\phi_2\bar\partial^{(\lambda-n)}\partial^{+\sigma_n}\phi_3) \tag{3.15}$$
$$+(\sigma_n+\lambda_3)\partial^{+\mu_n}(\bar\partial^n\partial^{+\rho_n}\phi_2\bar\partial^{(\lambda-n+1)}\partial^{+(\sigma_n-1)}\phi_3)\Big\} = 0 \ ,$$

$$\sum_{n=0}^{\lambda} A_n \Big\{ n\,\partial^{+\mu_n}(\bar\partial^{(n-1)}\partial^{+(\rho_n+1)}\phi_2\bar\partial^{(\lambda-n)}\partial^{+\sigma_n}\phi_3)$$
$$+(\lambda-n)\partial^{+\mu_n}(\bar\partial^n\partial^{+\rho_n}\phi_2\bar\partial^{(\lambda-n-1)}\partial^{+(\sigma_n+1)}\phi_3)\Big\} = 0 \ . \tag{3.16}$$

These conditions are satisfied if the coefficients obey the following recursion relations.

$$A_{n+1} = -\frac{(\lambda-n)}{(n+1)}A_n \ = \ (-1)^{(n+1)}\binom{\lambda}{n+1}A_0 \ ,$$
$$\rho_{n+1}=\rho_n-1 \quad ; \quad \sigma_{n+1}=\sigma_n+1 \quad ; \quad \mu_{n+1}=\mu_n \ , \tag{3.17}$$

with the last condition showing that $\mu_n$ is independent of $n$. The following "boundary" conditions are also necessary.

$$\rho_{_{n=\lambda}} = -\lambda_2 \qquad \sigma_{_{n=0}} = -\lambda_3. \tag{3.18}$$

The solution of the recursion relations for $\rho$, $\sigma$ and $\mu$ subject to (3.18) is

$$\rho_n = \lambda-\lambda_2-n \ ; \qquad \sigma_n = n-\lambda_3 \ ; \qquad \mu_n = \lambda_1 - 1 \ . \tag{3.19}$$

Thus (3.13) reads

$$\delta_H^\alpha\phi_1 = \alpha\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\,\partial^{+(\lambda_1-1)}\left[\bar\partial^n\partial^{+(\lambda-\lambda_2-n)}\phi_2\,\bar\partial^{(\lambda-n)}\partial^{+(n-\lambda_3)}\phi_3\right] .. \tag{3.20}$$

Since

$$H = \int d^3x\ \partial_-\bar\phi_1\,\delta_{\mathcal{H}}\phi_1, \tag{3.21}$$

the interaction Hamiltonian is

$$H^\alpha = \alpha\int d^3x\ \sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\,\bar\phi_1\,\partial^{+\lambda_1}\left[\bar\partial^n\partial^{+(\lambda-\lambda_2-n)}\phi_2\,\bar\partial^{(\lambda-n)}\partial^{+(n-\lambda_3)}\phi_3\right] + c.c \tag{3.22}$$

Notice that if we set $\lambda_1 = \lambda_2 = \lambda_3 = \lambda'$ in (3.22)with $\lambda'$ odd, $H^\alpha$ vanishes. Hence, a non-vanishing self-interaction Hamiltonian, for odd integer spins exists, if and only if we introduce a gauge group. However, a consistent non-trivial vertex, coupling three fields of different spins, exists irrespective of whether the spins are even or odd.

We note that if the action obtained from the above Hamiltonian is to describe a theory involving fields of odd integer spins with cubic self interaction terms, the existence of a gauge group is forced upon the theory. Interestingly, the three fields could, in principle, carry different gauge groups.

## 3.2 Factorization properties and perturbative relations

We now rewrite the above results in the language of spinor helicity [8] where a four-vector is expressed as a bispinor using $p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}}$, with $\det(p_{a\dot{a}})$ yielding $-p^\mu p_\mu$. The spinor products are

$$< kl >= \sqrt{2}\,\frac{(kl_- - lk_-)}{\sqrt{k_- l_-}} \qquad [kl] = \sqrt{2}\,\frac{(\bar{k}l_- - \bar{l}k_-)}{\sqrt{k_- l_-}} \tag{3.23}$$

Equation (3.22) involves the sum of two kinds of terms: $\bar{\phi}\phi\phi$ and $\phi\bar{\phi}\bar{\phi}$. In Fourier space, the coefficient of the second kind of term $\phi_1(p)\bar{\phi}_2(k)\bar{\phi}_3(l)\,\delta^4(p+k+l)$ up to a sign reads

$$\frac{p_-^{\lambda_1}}{k_-^{\lambda_2} l_-^{\lambda_3}}(lk_- - l_- k)^{\lambda_2+\lambda_3-\lambda_1}, \tag{3.24}$$

which may be rewritten as

$$\frac{1}{\sqrt{2^\lambda}} \;\; < pk >^{(-\lambda_1+\lambda_2-\lambda_3)} \;\; < kl >^{(\lambda_1+\lambda_2+\lambda_3)} \;\; < lp >^{(-\lambda_1-\lambda_2+\lambda_3)} \;\; . \tag{3.25}$$

It is clear that (**??**) exhibits the factorization property mentioned in the introduction. It is clear from the expression that given vertices for spins $(\lambda_1, \lambda_2, \lambda_3)$ and $(\lambda_1', \lambda_2', \lambda_3')$, their product yields the vertex for $(\lambda_1 + \lambda_1', \lambda_2 + \lambda_2', \lambda_3 + \lambda_3')$. As a corollary, note that the coefficient for the coupling of three fields $(n\lambda_1, n\lambda_2, n\lambda_3)$ is the $n$-th power of the coefficient for the coupling $(\lambda_1, \lambda_2, \lambda_3)$.

# Chapter 4

# Construction of fermionic interactions

*The materiel in this chapter is largely based on the third reference in [9]. This represents original work carried out by the author in collaboration with Dr. Ananth*

We can extend the method of the previous chapter to construct interaction vertices involving fermions. Recall from (1) that the light-cone Action for a fermionic field of half-integer spin $\lambda$ is

$$S = \int d^4x \;\, i\bar{\psi}\,\frac{\Box}{\partial_-}\,\psi. \tag{4.1}$$

From (A), the Poincaré generators represented on this field are

$$\begin{aligned} &\delta_{p^-}\psi = -i\frac{\partial\bar{\partial}}{\partial^+}\psi = -\delta_{p_+}\psi \qquad && \delta_{p^+}\psi = -i\partial^+\psi = -\delta_{p_-}\psi \\ &\delta_{p}\psi = -i\partial\psi && \delta_{\bar{p}}\psi = -i\bar{\partial}\psi \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\delta_j\psi = i(x\bar{\partial} - \bar{x}\partial + \lambda)\psi \qquad && \delta_{j^{+-}}\psi = (x^+\frac{\partial\bar{\partial}}{\partial^+} - x^-\partial^+ - \frac{1}{2})\psi \\ &\delta_{j^+}\psi = (x^+\partial - x\partial^+)\psi && \delta_{\bar{j}^+}\psi = (x^+\bar{\partial} - \bar{x}\partial^+)\psi \\ &\delta_{j^-}\psi = (x^-\partial - x\frac{\partial\bar{\partial}}{\partial^+} + (\lambda + \frac{1}{2})\frac{\partial}{\partial^+})\psi && \delta_{\bar{j}^-}\psi = (x^-\partial - x\frac{\partial\bar{\partial}}{\partial^+} - (\lambda + \frac{1}{2})\frac{\partial}{\partial^+})\psi. \end{aligned} \tag{4.3}$$

Note that $\bar{\psi}$ carries helicity of $\lambda$ which is a positive half integer, in contrast with the bosonic case where $\bar{\phi}$ carried negative helicity. Another difference, from the bosonic case, is the presence of a third piece in $\delta_{j^{+-}}$.

## 4.1 Deriving cubic interaction vertices

To begin the construction of interaction vertices, we start with the Hamiltonian for a free theory involving a massless boson of spin $\lambda$ and a massless fermion of spin $\lambda'$.

$$H = \int d^3x \; \left( -\bar{\phi}\, \partial\bar{\partial}\, \phi \; + i\bar{\psi}\, \frac{\partial\bar{\partial}}{\partial_-}\, \psi \right) = \int d^3x \; \left( \partial_- \bar{\phi}\, \delta_{\mathcal{H}} \phi + i\bar{\psi}\, \delta_{\mathcal{H}}\, \psi \right) . \quad (4.4)$$

As we did in the bosonic case, we add non-linear terms order by order in a coupling constant $\alpha$ to the Hamiltonian. The dynamical generators pick up corrections which have that same form as in 3.7.

We make a preliminary Ansatz for $\delta^{\alpha}_{\mathcal{H}}\phi_1$ of the skeletal form $\bar{\phi}_1\psi_2\psi_3$ where the fields $\phi_1$, $\psi_2$ and $\psi_3$ have spins $\lambda_1$, $\lambda_2$ and $\lambda_3$ respectively. $\lambda_1$ is an integer while $\lambda_2$ and $\lambda_3$ are half-integers. From (**??**), we can see the corrections to all other dynamical generators can be expressed in terms of $\delta^{\alpha}_{\mathcal{H}}\phi$. The precise form of the Ansatz is

$$\delta^{\alpha}_{\mathcal{H}}\phi_1 = \alpha\, A\, \partial^{+\mu} \left[ \partial^a \partial^{+\sigma} \psi_2 \partial^b \partial^{+\rho} \psi_3 \right] \; , \quad (4.5)$$

where $\mu, \rho, \sigma, a, b$ are integers and $A$ is a numerical factor that could depend on the variables and spins.

To ensure that the theory resulting from such a vertex is Lorentz covariant, we demand the closure of the Poincaré algebra at this order in $\alpha$. We begin by demanding closure with the simplest kinematical generators.

$$[\delta_j, \delta^{\alpha}_{\mathcal{H}}]\, \phi_1 = 0 \qquad\qquad [\delta_{j^{+-}}, \delta_{\mathcal{H}}]^{\alpha}\, \phi_1 = -\delta^{\alpha}_{\mathcal{H}}\phi_1 \; ,$$

which impose the following constraints on $a, b, \mu, \rho$ and $\sigma$.

$$\begin{aligned} a + b \; &= \lambda_1 + \lambda_2 + \lambda_3 \equiv \lambda \\ \mu + \rho + \sigma \; &= -2 \; . \end{aligned} \quad (4.6)$$

We then demand,

$$\left[\delta^{-}_{j}, \delta_H\right]^{\alpha} \phi_1 = 0 \qquad \left[\delta_{\bar{j}^{+}}, \delta_H\right]^{\alpha} \phi_1 = 0 \, , \quad (4.7)$$

which yield the following constraints,

$$
\begin{aligned}
\sum_{n=0}^{\lambda} \ A_n \Big\{ & -(\mu_n + 1 + \lambda_1)\partial^{+(\mu_n-1)}\partial(\partial^n\partial^{+\rho_n}\psi_2\partial^{(\lambda-n)}\partial^{+\sigma_n}\psi_3) \\
& -(\sigma_n + \lambda_2 + \frac{1}{2})\partial^{+\mu_n}(\partial^{(n+1)}\partial^{+(\sigma_n-1)}\psi_2\partial^{(\lambda-n)}\partial^{+\rho_n}\psi_3) \qquad (4.8) \\
& -(\rho_n + \lambda_3 + \frac{1}{2})\partial^{+\mu_n}(\partial^n\partial^{+\sigma_n}\psi_2\partial^{(\lambda-n+1)}\partial^{+(\rho_n-1)}\psi_3)\Big\} \ = 0 \ , \\
\sum_{n=0}^{\lambda} \ A_n\Big\{ & \ n\,\partial^{+\mu_n}(\partial^{(n-1)}\partial^{+(\sigma_n+1)}\psi_2\partial^{(\lambda-n)}\partial^{+\rho_n}\psi_3) \\
& +(\lambda-n)\partial^{+\mu_n}(\partial^n\partial^{+\sigma_n}\psi_2\bar{\partial}^{(\lambda-n-1)}\partial^{+(\rho_n+1)}\psi_3)\Big\} = 0 \ . \qquad (4.9)
\end{aligned}
$$

The solution to these constraints is given below.

$$
A_n = (-1)^{(n)}\binom{\lambda}{n}; \qquad \sigma_n = -\lambda-\lambda_2+n-\frac{1}{2}\,; \qquad \rho_n = n-(\lambda_3+\frac{1}{2})\,; \qquad \mu_n = -(\lambda_1+1)\,.
$$

Thus (4.5) reads

$$
\delta_H^{\alpha}\phi_1 = \alpha\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\frac{1}{\partial^{+(\lambda_1+1)}}\left[\partial^n\frac{\partial^{+(\lambda-n)}}{\partial^{+(\lambda_2+\frac{1}{2})}}\psi_2\,\partial^{(\lambda-n)}\frac{\partial^{+n}}{\partial^{+(\lambda_3+\frac{1}{2})}}\psi_3\right]. \quad (4.10)
$$

Using this, from (4.4) we obtain the Hamiltonian and thus the Action

$$
\begin{aligned}
S = \int d^4x \ & \Big[\frac{1}{2}\bar{\phi}_1\Box\phi_2 + i\bar{\psi}_2\,\frac{\Box}{\partial_-}\,\psi_2 + i\bar{\psi}_3\,\frac{\Box}{\partial_-}\,\psi_3 \qquad (4.11) \\
& + \alpha\bar{\phi}_1\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\frac{1}{\partial^{+\lambda_1}}\left[\partial^n\frac{\partial^{+(\lambda-n)}}{\partial^{+(\lambda_2+\frac{1}{2})}}\psi_2\,\partial^{(\lambda-n)}\frac{\partial^{+n}}{\partial^{+(\lambda_3+\frac{1}{2})}}\psi_3\right]\Big]
\end{aligned}
$$

This Action may be supplemented with a variety of other interactions. In particular, the algebra tells us that if $\phi_1$ has odd helicity and is to have cubic self-interactions, then it must carry an internal symmetry group with an antisymmetric structure constant [9]. We can thus have two kinds of vertices having the schematic forms $t_a\bar{\phi}_1^a\psi_2^b\psi_{3b}$ and $\bar{\phi}_1^a\psi_2^b\psi_3^c f_{abc}$. The first of these has the bosonic field in the Adjoint representation of the gauge group($t_a$ is its generator) with the fermionic fields in any other representation. The second vertex has all three fields in the same representation and linked by an antisymmetric structure constant.

For example, consider the case where $(\lambda_1, \lambda_2, \lambda_3) = (1, \frac{1}{2}, -\frac{1}{2})$. With an SU(3) internal symmetry, the first kind of vertex is the coupling of gluons to quarks in QCD with the second variety corresponding to the cubic coupling in $\mathcal{N} = 4$ SYM. If we compare this second type of vertex, we find an exact match with [10] (line 5, equation 3.13).

## 4.2 Relation to scattering amplitudes

We now evaluate the three point amplitude resulting from the vertex in (4.11) using the spinor helicity notation introduced in (3.2). The cubic vertex from (4.11) in momentum space is

$$\alpha \int \frac{d^4p}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} (2\pi)^4 \delta^4(p+k+l) \frac{(k_{\_}l - l_{\_}k)^{\lambda}}{p_{\_}^{\lambda_1} k_{\_}^{\lambda_2+\frac{1}{2}} l_{\_}^{\lambda_3+\frac{1}{2}}} \tilde{\bar{\phi}}_1(p) \tilde{\psi}_2(k) \tilde{\psi}_3(l) \ , \tag{4.12}$$

To obtain the amplitude, we simply put all three fields on mass shell. Along with the constraint ($p^2 = 0$) on the momenta, this yields a numerical factor for bosonic fields and a factor proportional to the square root of momentum for fermionic fields (from Appendix A). The final amplitude has the following form

$$< pk >^{(-\lambda_1+\lambda_2-\lambda_3)} \ < kl >^{(\lambda_1+\lambda_2+\lambda_3)} \ < lp >^{(-\lambda_1-\lambda_2+\lambda_3)} \ . \tag{4.13}$$

# Chapter 5

# Discussion and future directions

The tree level, three point amplitude obtained in (4.13) has exactly the same form as the amplitude for scattering of three bosons derived in (3.25). This is a simple and unifying form for three point amplitudes. This represents an off-shell generalization of the KLT relations. These results are consistent with the general result for three point amplitudes derived in [11] using S-matrix arguments, in [12] using little group scaling and [13, 14] using a Fock-space approach.

The next logical step in this procedure would be to derive quartic interaction vertices for higher spin theories. However, this is algebraically involved. For spin 1 quartic self-interactions, the closure of some commutators impose the Jacobi identity on the structure constants. It is gratifying to see that the internal symmetry group emerges from purely algebraic considerations.

Another application of this method would be to derive interaction vertices in non-flat backgrounds. The corresponding isometry algebra must be used to generate constraints on the form of the interactions. In (B), we have derived cubic interaction vertices for a spin 2 field on an $AdS_4$ background from first principles. This derivation is based on [15]. This will serve as a check when we obtain the results using our approach. If successful, this might provide a Lagrangian origin to Vasiliev's equations of motion [**?**]. It would also be interesting to investigate whether KLT-like relations extend to curved backgrounds.

A third way in which our method might prove useful is to shed light on the elusive $\mathcal{N} = (2, 0)$ theory in six dimensions [17]. The application of these techniques to superconformal theories is not new and it has already proven successful in deriving $\mathcal{N} = 4$ super Yang-Mills [18]. In the particular case of the $\mathcal{N} = (2,0)$ theory, the minimal assumptions in our method might prove useful in discovering any new degrees of freedom or algebraic structure present in the theory.

# Appendix A

# Light-cone formalism

In this appendix, we illustrate the method to obtain light-cone realizations of the Poincaré generators on fields. We consider the case of a free scalar(complex) theory. The Action is,

$$S = \int d^4x \ \mathcal{L} = \frac{1}{2}\int d^4x \ \partial_\mu\bar{\phi}\partial^\mu\phi. \tag{A.1}$$

This Action is invariant under a all transformations of the Poincaré group $ISO(3,1)$. Each symmetry will have a corresponding conserved current. These currents can be used to construct charges which act as generators of the symmetry. We use this procedure to construct all the generators of the Poincaré algebra. For now, we note that after integrating by parts, (A.1) can be rewritten as

$$S = \int d^4x \ (\partial_-\bar{\phi}\partial_+\phi + \bar{\phi}\partial\bar{\partial}\phi). \tag{A.2}$$

The canonical momentum corresponding to $\phi$ is

$$\pi \equiv \frac{\partial\mathcal{L}}{\partial(\partial_+\phi)} = \partial_-\bar{\phi} \tag{A.3}$$

The Noether cuurent associated with translations is the energy momentum tensor.

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \tag{A.4}$$

The conserved charges are

$$P^\mu = \int_\Sigma d^3x \, T^{+\mu} = \int_\Sigma d^3x \, (\partial_-\bar{\phi}\partial^\mu\phi - \eta^{+\mu}\mathcal{L}) \tag{A.5}$$

Here $\Sigma$ is a hypersurface of constant $x^+$ and $d^3x = dxd\bar{x}dx^-$. The action of these generators on the fields is

$$\delta_{\mathcal{O}}\phi \equiv \{\phi, \mathcal{O}\} \tag{A.6}$$

Acting in this way on the fields, these generate spacetime translations. In particular, the light-cone Hamiltonian, $P_+ = -P^-$ generates time translations.

$$\begin{aligned} P_+ &= -\int_\Sigma d^3x\,(\partial_-\bar{\phi}\partial^-\phi + \mathcal{L}) = -\int_\Sigma d^3x\,(\partial_-\bar{\phi}\partial^-\phi + \partial_-\bar{\phi}\partial_+\phi + \bar{\phi}\partial\bar{\partial}\phi) \\ &= \int_\Sigma d^3x\,\partial_-\bar{\phi}\frac{\partial\bar{\partial}}{\partial_-}\phi \end{aligned} \tag{A.7}$$

The remaining generators are the conserved charges derived from the Noether current corresponding to the Lorentz transformations. The currents are

$$\mathcal{M}^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}. \tag{A.8}$$

The corresponding charges are

$$\begin{aligned} J^{\mu\nu} &= \int_\Sigma d^3x\mathcal{M}^{+\mu\nu} = \int_\Sigma d^3x\,(x^\mu\frac{\partial\mathcal{L}}{\partial(\partial_+\phi)}\partial^\nu\phi - x^\nu\frac{\partial\mathcal{L}}{\partial(\partial_+\phi)}\partial^\nu\phi) \\ &= \int_\Sigma d^3x\,\partial_-\bar{\phi}(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi \end{aligned} \tag{A.9}$$

So far, we have only considered spinless fields. In the case of fields with spin, the Lorentz generators receive an additional spin dependant contribution. We illustrate this for a massless spin $1(A^\mu)$ field below. To find the contribution arising from the spin of the field, we find the change in the field at the same value of the co-ordinate under a Lorentz transformation $x^\mu \to \Lambda^\mu_\nu x^\nu$.

$$\delta A^\mu \equiv A'^\mu(x) - A^\mu(x) = \omega^\mu{}_\nu A^\nu + \partial^\mu\xi \tag{A.10}$$

Here $\xi$ is a gauge transformation which we use to preserve the gauge choice by requiring $\delta_s A^+ = 0$. This gives,

$$\xi = -\frac{1}{\partial_-}\omega_{-i}A^i \tag{A.11}$$

From this, we have

$$\delta_s A^\mu = \omega^\mu{}_\nu A^\nu + \partial^\mu\xi \tag{A.12}$$

We now consider the transformation of the combination $\frac{1}{\sqrt{2}}(A^1 + iA^2)$.

$$\delta_s(\frac{1}{\sqrt{2}}(A^1 + iA^2)) = \frac{1}{\sqrt{2}}(\omega^1_\nu + i\omega^2_\nu)A^\nu + \partial\xi \tag{A.13}$$

Define

$$\omega \equiv \omega_{12} \qquad \omega_- \equiv \frac{1}{\sqrt{2}}(\omega_{-1} + i\omega_{-2}) \tag{A.14}$$

Then

$$\delta_s A = (-i\omega - i\omega_- \frac{\bar\partial}{\partial_-} + i\bar\omega_- \frac{\partial}{\partial_-})A \tag{A.15}$$

It is easy to see that A is indeed a helicity eigenstate and has helicity 1. It is straightforward to obtain the action of the generator for a field of helicity $\lambda$. The resulting expressions are displayed in (3.3) for the bosonic case and (4.3) for the fermionic case.

# Appendix B

# Light-cone gravity in $AdS_4$

In this Appendix, we will describe how pure gravity is formulated in light-cone gauge in an $AdS_4$ background. This is achieved by making suitable gauge choices and using the constraint relations to eliminate the unphysical degrees of freedom. This will allow us to describe the Action of light-cone gravity on $AdS_4$ in a closed form purely using the physical degrees of freedom. We also perform a perturbative expansion of this gauge-fixed Action to first order in the gravitational coupling constant and comment on the interaction vertex.

## B.1 Preliminaries

The Einstein-Hilbert action reads

$$S_{EH} = \int d^4x\, L \;=\; \frac{1}{2\,\kappa^2}\,\int d^4x\,\sqrt{-g}\,(\,\mathcal{R}\,-\,2\,\Lambda\,)\;, \tag{B.1}$$

where $g = \det g_{\mu\nu}$, $\mathcal{R}$ is the curvature scalar, $\Lambda$ is the cosmological constant of $AdS_4$and $\kappa^2 \;=\; 8\pi G_N$ is the coupling constant in terms of the Newton constant. This is the Action on manifold $M$ with boundary $\partial M$. The Action thus has the form

$$S_{EH} = \int_M d^4x\, L_M \,+ \int_{\partial M} d^3x\, L_{_{\partial M}}. \tag{B.2}$$

Three levels at which this theory can be studied, roughly in order of precision, are: on-shell, off-shell and off-shell+boundary terms. Our analysis is closely tied to the equations of motion, does not require boundary terms and hence falls into the second of these three categories. We note that one may add boundary terms [22] to (B.1) that cancel surface terms resulting from partial integrations. In an arbitrary gauge, such a cancelation between boundary terms is not always guaranteed however, the difference is always a local function of the boundary fields. It is the first term in (B.2) that we derive in the following pages.

The light-cone gauge approach to formulating pure gravity in flat spacetime backgrounds has been studied in [23, 24, 25]. Here we formulate pure gravity in $AdS_4$ characterized by a cosmological constant $\Lambda$. As one would expect, this involves considerable deviations from the flat background approach of [23] and we comment on these changes as and when they occur.

## B.2 $AdS_4$

Consider a five-dimensional flat spacetime with metric $\eta_{MN} \equiv (-1,1,1,1,-1)$ and co-ordinates $\xi^M$, $M = 0\dots4$. On this manifold, $AdS_4$ is defined as the four-dimensional hypersurface

$$-(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 - (\xi^4)^2 = R^2 \ , \tag{B.3}$$

with radius $R$. We now introduce local (Poincaré) co-ordinates $x^\mu \equiv (x^0, x^1, z, x^3)$ on $AdS_4$

$$\xi^0 = \frac{R}{z}x^0 \qquad \xi^1 = \frac{R}{z}x^1 \qquad \xi^3 = \frac{R}{z}x^3 \ , \tag{B.4}$$

$$\xi^2 = \frac{1}{2z}\left[R^2 - \{-(x^0)^2 + (x^1)^2 + (x^3)^2 - z^2\}\right] \ , \tag{B.5}$$

$$\xi^4 = \frac{1}{2z}\left[R^2 + \{-(x^0)^2 + (x^1)^2 + (x^3)^2 - z^2\}\right] \ , \tag{B.6}$$

which satisfy $(B.3)$. $z$ plays the role of a radial coordinate and divides the spacetime into two regions. We work here in the 'patch' $z > 0$ with $z = 0$ being part of the $AdS$ boundary. The induced metric on this space is

$$g^{(0)}_{\mu\nu} = \partial_\mu \xi^M \partial_\nu \xi^N \eta_{MN} = \frac{R^2}{z^2}\eta_{\mu\nu} \ , \tag{B.7}$$

where $\eta_{\mu\nu}$ is the usual Minkowski metric. We now switch to light cone coordinates $x^\mu \equiv (x^+, x^-, x^1, z)$ where

$$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}} \ . \tag{B.8}$$

The cosmological constant for $AdS_4$ is

$$\Lambda = -\frac{3}{R^2} \ . \tag{B.9}$$

## B.3 Light-cone formulation

Our aim is to study fluctuation on the $AdS_4$background. The dynamical variable is the metric $g_{\mu\nu}$, which in the absence of all perturbations must reduce to $g^{(0)}_{\mu\nu}$. We work in light-cone gauge by making the following three gauge choices [23]

$$g_{--} \;=\; g_{-i} \;=\; 0 \quad , \; i=1,z \; . \tag{B.10}$$

Note that these choices are consistent with $g^{(0)}_{\mu\nu}$ since, in light-cone coordinates, $\eta_{--}=\eta_{-i}=0$. A fourth gauge choice will be made shortly.
The metric is parametrized as follows

$$\begin{aligned} g_{+-} \;&=\; -\,e^{\phi} \;, \\ g_{i\,j} \;&=\; e^{\psi}\,\gamma_{ij} \;. \end{aligned} \tag{B.11}$$

The fields $\phi$, $\psi$ are real while $\gamma_{ij}$ is a $2\,\times\,2$ real, symmetric matrix.

The Euler-Lagrange equations corresponding to the Einstein-Hilbert Action read

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\,\mathcal{R} \;=\; -\Lambda\, g_{\mu\nu} \;. \tag{B.12}$$

The light-cone formulation of the theory relies on the fact that a subset of these Euler-Lagrange equations (those not involving time derivatives) are constraint equations. The first relevant constraint relation is $\mathcal{R}_{--}=0$ which reads

$$2\,\partial_-\,\phi\,\partial_-\,\psi \;-\; 2\,{\partial_-}^2\,\psi \;-\; (\partial_-\,\psi)^2 \;+\; \frac{1}{2}\,\partial_-\,\gamma^{kl}\,\partial_-\,\gamma_{kl} \;=\; 0 \;. \tag{B.13}$$

A simple solution to this constraint relation may be obtained by making a fourth gauge choice

$$\phi \;=\; \frac{1}{2}\,\psi \;. \tag{B.14}$$

This reduces equation (B.13) to a quadrature and yields

$$\psi = \frac{1}{4}\frac{1}{{\partial_-}^2}(\partial_-\gamma^{ij}\partial_-\gamma_{ij}) + 2\,ln\,\frac{R^2}{z^2} \;, \tag{B.15}$$

The second term, in $\psi$, is essential to ensure that $g_{ij}$ and $g_{+-}$ reduce correctly to $g^{(0)}_{ij}$ and $g^{(0)}_{+-}$ respectively.

In a flat background [23, 24] the solution to $\psi$ is simply $\psi_{\text{flat}} = \frac{1}{4}\frac{1}{\partial_-{}^2}(\partial_-\gamma^{ij}\partial_-\gamma_{ij})$ and the second term in (B.15) is absent.

We now compute the determinant of $\gamma_{ij}$ from the second relation in (B.11) which implies that

$$\det g^{(0)}_{ij} = \left(\frac{R^2}{z^2}\right)^4 \det \gamma^{(0)}_{ij} \ , \tag{B.16}$$

with the $\{\ \}^{(0)}$ superscripts implying that all fluctuations are switched off. In this limit, the metric is simply $\frac{R^2}{z^2}$ times the Minkowski metric so the L.H.S of (B.16) is $\left(\frac{R^2}{z^2}\right)^2$ thus implying that

$$\det \gamma^{(0)}_{ij} = \left(\frac{z^2}{R^2}\right)^2 \ . \tag{B.17}$$

Note that in contrast to our result above, on a flat background, $\gamma_{ij}$ is unimodular [23, 24]. We choose the determinant of $\gamma_{ij}$ (which includes fluctuations) to be the same as in (B.17). This will ensure that the fluctuation-field, to be introduced shortly, is traceless and makes our calculations easier.

The second constraint relation is $\mathcal{R}_{-i} = 0$ which yields

$$\begin{aligned} g^{-i} \ = \ -\mathrm{e}^{-\phi}\,\frac{1}{\partial_-}\Bigg[\ \gamma^{ij}\,\mathrm{e}^{\phi\,-\,2\,\psi}\,\frac{1}{\partial_-}\,\Big\{\mathrm{e}^{\psi}\,\Big(\frac{1}{2}\,\partial_-\,\gamma^{kl}\,\partial_j\,\gamma_{kl}\ -\ \partial_-\,\partial_j\,\phi \\ -\ \partial_-\,\partial_j\,\psi\ +\ \partial_j\phi\,\partial_-\,\psi\Big)\ +\ \partial_l\,\Big(\mathrm{e}^{\psi}\,\gamma^{kl}\,\partial_-\,\gamma_{jk}\Big)\Big\}\Bigg] \ . \end{aligned} \tag{B.18}$$

### B.3.1 Light-cone Action

The light-cone Action for gravity is

$$S = \int d^3x \int dz\,\mathcal{L} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\left(2g^{+-}\mathcal{R}_{+-} + g^{ij}\mathcal{R}_{ij} - 2\Lambda\right) \ . \tag{B.19}$$

We now compute each term in the above expression, using the results listed thus far. We derive the following closed form expression for the Action in $AdS_4$purely in terms of the physical degrees of freedom.

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int d^3x \int dz \Bigg\{ \frac{z^2}{R^2}e^{\psi}\left(2\partial_+\partial_-\phi + \partial_+\partial_-\psi - \frac{1}{2}\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right) \\ -\frac{z^2}{R^2}e^{\phi}\gamma^{ij}\left(\partial_i\partial_j\phi + \frac{1}{2}\partial_i\phi\partial_j\phi - \partial_i\phi\partial_j\psi - \frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right) \\ -\frac{z^2}{2R^2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}A_i\frac{1}{\partial_-}A_j + \frac{2}{R^2}e^{\phi}\gamma^{zz} - 2\,\frac{z^2}{R^2}\,e^{\psi}\,e^{\phi}\,\Lambda \Bigg\} \ , \end{aligned} \tag{B.20}$$

with

$$A_i = \mathrm{e}^{\psi}\left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk} - \partial_-\partial_i\phi - \partial_i\partial_-\psi + \partial_i\phi\partial_-\psi\right)$$
$$+\partial_k\left(\mathrm{e}^{\psi}\gamma^{jk}\partial_-\gamma_{ij}\right) .$$

In obtaining this result, we have dropped several boundary terms following the arguments outlined in Section B.1.

*Deviations from flat spacetime results*

The three main differences between our result (B.20) and the flat background Action in [23, 24] are the overall factor of $\frac{z^2}{R^2}$ in front of each line, the penultimate term proportional to $\gamma^{zz}$ and the last term, prorptional to the cosmological constant.

## B.4 Perturbative expansion

In this section we obtain a perturbative expression, to cubic order in the fields, for the Action in (B.20). We do this by making the following choice

$$\begin{aligned}\gamma_{ij} &= \frac{z^2}{R^2}\left(\mathrm{e}^{H}\right)_{ij} ,\\ H &= \begin{pmatrix} h_{11} & h_{1z} \\ h_{1z} & -h_{zz}\end{pmatrix} ,\end{aligned} \tag{B.21}$$

with $h_{zz} = -h_{11}$ as explained below equation (B.17). In terms of these fields, equation (B.15) reads

$$\psi = -\frac{1}{4}\frac{1}{\partial_-{}^2}\left[\partial_- h_{ij}\partial_- h_{ij}\right] + 2\,ln\,\frac{R^2}{z^2} + O(h^4) . \tag{B.22}$$

In order to obtain a perturbative expansion of (B.20) we simply use the results (B.21) and (B.22).

We now redefine

$$h \to \frac{1}{\sqrt{2}\kappa}\,h . \tag{B.23}$$

In terms of these fields, the Action at $O(h^2)$ is

$$S_2 = \int d^3x \int dz\,\mathcal{L}_2 , \tag{B.24}$$

with

$$\mathcal{L}_2 = +\frac{R^2}{2z^2}\partial_+ h_{ij}\partial_- h_{ij} - \frac{R^2}{4z^2}\partial_i h_{kl}\partial_i h_{kl} - 2\frac{R^2}{z^3}h_{ik}\partial_k h_{iz} + \frac{R^2}{z^4}h_{zk}h_{kz} \ .$$

In the above we have made use of both (B.9) and the fact that $h_{kk} = 0$. Notice from (B.20) that the cosmological constant only appears in interaction vertices involving an even number of fields.

At $O(h^3)$, the Action reads

$$S_3 = \int d^3x \int dz \ \frac{1}{\sqrt{2}} \ \mathcal{L}_3 \ , \tag{B.25}$$

where

$$\begin{aligned}
\mathcal{L}_3 = \kappa \Big\{ &- 2\frac{R^2}{z^3}h_{iz}\frac{1}{\partial_-}(\partial_- h_{lm}\partial_i h_{lm}) - 12\frac{R^2}{z^3}h_{iz}\partial_i\phi^{(0)} - 16\frac{R^2}{z^4}h_{zz}\phi^{(0)} + 16\frac{R^2}{z^4}h_{iz}\frac{1}{\partial_-}(h_{lz}\partial_- h_{il}) \\
&-4\frac{R^2}{z^3}h_{iz}\frac{1}{\partial_-}(\partial_k h_{lk}\partial_- h_{il}) - 4\frac{R^2}{z^3}h_{iz}\frac{1}{\partial_-}(h_{lk}\partial_-\partial_k h_{il}) + \frac{1}{2}\frac{R^2}{z^2}\partial_k h_{ik}\frac{1}{\partial_-}(\partial_- h_{lm}\partial_i h_{lm}) \\
&+3\frac{R^2}{z^2}\partial_k h_{ik}\partial_i\phi^{(0)} + 4\frac{R^2}{z^3}\partial_k h_{zk}\phi^{(0)} - 4\frac{R^2}{z^3}\partial_k h_{ik}\frac{1}{\partial_-}(h_{lz}\partial_- h_{il}) + \frac{R^2}{z^2}\partial_k h_{ik}\frac{1}{\partial_-}(\partial_l h_{ml}\partial_- h_{im}) \\
&+\frac{R^2}{z^2}\partial_k h_{ik}\frac{1}{\partial_-}(h_{lm}\partial_-\partial_m h_{il}) - \frac{1}{2}\frac{R^2}{z^2}\partial_k h_{ik}\partial_l h_{ij}h_{jl} + 2\frac{R^2}{z^3}h_{iz}\partial_k h_{ij}h_{jk} \\
&+\frac{R^2}{z^2}h_{ij}\partial_i\partial_j\phi^{(0)} + 2\frac{R^2}{z^4}h_{zz}\phi^{(0)} + \frac{R^2}{3z^4}h_{zz}^3 + 4\frac{R^2}{z^3}\partial_i\phi^{(0)}h_{iz} \\
&+\frac{R^2}{4z^2}h_{ij}\partial_i h_{kl}\partial_j h_{kl} - \frac{R^2}{6z^3}\partial_z h_{kk}^3 - \frac{R^2}{2z^2}h_{ij}\partial_i h_{kl}\partial_k h_{jl} \Big\} \ .
\end{aligned}$$

* * *

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