Some aspects of entanglement entropy

A thesis submitted towards partial fulfilment of
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by

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under the guidance of

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CERTIFICATE

This is to certify that this thesis entitled "SOME ASPECTS OF ENTANGLEMENT ENTROPY" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research (IISER) Pune represents original research carried out by "SAGAR FAKIRCHAND LOKHANDE" at the "INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH (IISER) PUNE", under the supervision of "PROFESSOR SUNIL MUKHI" during the academic year 2014-2015.

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DECLARATION

I hereby declare that the matter embodied in the report entitled "SOME ASPECTS OF ENTANGLEMENT ENTROPY" are the results of the investigations carried out by me at the "DEPARTMENT OF PHYSICS, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH (IISER), PUNE", under the supervision of "PROFESSOR SUNIL MUKHI" and the same has not been submitted elsewhere for any other degree.

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Dedication

To **Advaita Vedanta**

and all the faithful Seers of It

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Abstract

Entanglement entropy has emerged as an important quantity in quantum field theory. In this report, we present some aspects of entanglement entropy in the context of string theory. We assume the AdS/CFT Correspondence. String theory has provided a microscopic description of black hole entropy in the case of certain black holes. We examine the relation between the black entropy and entanglement entropy. This relation is described in detail for a special case and comments are made about approaches in finding such a relation in general case. In the second and the main part of this report, we present some first and original attempts at studying the modular invariance of Rényi and entanglement entropies. We clarify subtle issues in the application of Replica Trick to theories at finite temperature. We propose a modular invariant expression for entanglement entropy for free fermion conformal field theories in 2 dimensions, with a finite size entangling interval and finite temperature. We study single Dirac fermion, two correlated Dirac fermions and multiple correlated Dirac fermions in detail. Comments are also made about the T-duality invariance and modular invariance of the Rényi entropy for a single free boson CFT. Our proposal is shown to obey several tests and is consistent with known answers in limiting situations.

Contents

Chapter 1

Introduction

1.1 Entanglement Entropy

Classical dynamics of physical systems differs markedly from their quantum dynamics. The classical framework does not predict quantization of energy levels in atoms, interference between particles, etc. which are regularly seen in experiments and which are an integral part of the quantum framework. But, many consider the distinguishing characteristic of the quantum nature of physical systems to be 'quantum entanglement' [1].

Quantum entanglement is the strong non-local correlation that exists between parts of a composite quantum system. Measurements of observables performed on the parts are found to be strongly correlated, even though before the measurement, the parts may be separated by arbitrarily large distances. This comes as a simple consequence of the fact that even if the whole system can be represented as a ray in some Hilbert space, the parts may not have such a description.

Then, if quantum entanglement is the distinguishing characteristic of quantum systems, how does one quantify it? Quantifying this property of quantum systems will eventually help us say which of two given systems is more "quantum". The answer to the above question was given by John Von Neumann, who generalized the expression for Shannon's Entropy in information theory to Entanglement Entropy (EE) in quantum theory. Entanglement entropy, which is also called Von Neumann entropy, is defined as follows:

Consider a quantum system in a pure state $|0\rangle$. The density matrix in this state is

$$\rho = |0\rangle\langle 0| \tag{1.1}$$

Let A and B be two parts of the quantum system such that the total Hilbert space of the quantum system is a direct product of the Hilbert spaces for each of the parts A and B $\mathcal{H} = \mathcal{H}_A \times \mathcal{H}_B$. Then the reduced density matrix for the part A is defined as follows by tracing over the degrees of freedom belonging to part B:

$$\rho_A = Tr_B \rho \tag{1.2}$$

The entanglement entropy of the quantum system with respect to the bipartition A, B is then defined as the Von Neumann entropy of the reduced density matrix of A. This is given as:

$$S_A = -Tr_A \left(\rho_A \log \rho_A\right) \tag{1.3}$$

If the quantum system was initially in a pure state $|0\rangle$, it turns out that $S_A = S_B$, where S_B is defined similar to S_A except that the partial trace is now over the degrees of freedom in A.

Entanglement entropy defined as above is not a good measure of quantum entanglement between the partitions A and B if the quantum system is initially in a mixed state. There are other quantitative measures of quntum entanglement [2] which one can use in such a case.

1.2 Rényi Entropy

Entanglement entropy can be generalized so as to be applicable to a wider class of systems. Rényi entropy is one such generalization. It is defined as [3] :

$$S_n(\rho_A) = \frac{1}{1-n} \log \left(Tr_A \left({\rho_A}^n\right)\right) \tag{1.4}$$

where n>0 and $n \neq 1$.

In the limit $n \to 1$, the Rényi entropy reduces to the entanglement entropy. One of the most important motivation to study Rényi entropy is the fact that it is difficult to calculate $\log \rho_A$. The Rényi entropy on the other hand has ${\rho_A}^n$ which is not as difficult. Furthermore, the Rényi entropies for different values of n give us a more refined information about the reduced density matrix ρ_A - the knowledge of ${S_A}^n$ for all n is equivalent to knowing the full eigenvalue distribution of ρ_A.

1.3 Entanglement in quantum field theory

Consider a lattice quantum theory in one space and one time dimension, initially on the infinite line. Let the lattice spacing be ϵ and the lattice sites be labelled by a discrete variable x. Let time t be continuous. Let $\{\hat{\phi}(x)\}$ denote a complete set of commuting observables in the lattice theory under consideration. $\epsilon \to 0$ will be the limit in which we get the quantum field theory.

Consider the lattice theory at zero temperature. The theory is then in a pure state. The density matrix of the lattice quantum theory in this state is given by:

$$\rho\Big(\{\phi_1(x_1)\}|\{\phi_2(x_2)\}\Big) = \frac{1}{Z}\Big\langle\{\phi_1(x_1)\}|e^{-\beta\hat{H}}|\{\phi_2(x_2)\}\Big\rangle \tag{1.5}$$

where $Z = Tre^{-it\hat{H}}$ is the partition function of the lattice quantum theory.

This may be expressed in the standard way as a path integral:

$$\rho = \frac{1}{Z}\int[d\phi(x,t)]\prod_x \delta(\phi(x,0)-\phi_2(x_2))\prod_x \delta(\phi(x,0)-\phi_1(x_1))\,e^{-S} \tag{1.6}$$

where $S = \int_0^\infty L dt$, with L being the Lagrangian. We note an important point here - the partition function comes with a boundary condition (say in the presence of a branch cut).

To define the entanglement entropy of the lattice quantum theory, we consider a bipartition of the theory into two parts A and B. Let the subsytem A be the set of all the points x in the interval (u_1, v_1) . Then an expression for the reduced density matrix ρ_A may be found from equation (1.6) by sewing together only those points x which are not in A. This will leave open cut for the interval (u_1, v_1) along the line $t = 0$.

Now we compute the Rényi entropy using the replica trick [4]. We make n copies of the above geometrical structure, and compute $Tr{\rho_A}^n$ by sewing the copies together cyclically along the cuts so that $\phi_1(x)_{(k)} = \phi_2(x)_{(k+1)}$ for all x in A. Let $Z_n(A)$ denote the path integral on this n-sheeted Riemann surface. Whether we keep the same or different boundary conditions over the n replicas is an important issue we will address as we go along. Then

$$Tr\rho_A^n = \frac{Z_n(A)}{Z^n} \tag{1.7}$$

The Rényi entropy is then given by:

$$S_n = \frac{1}{1-n}\log\left(Tr\rho_A^n\right) = \frac{1}{1-n}\log\left(\frac{Z_n(A)}{Z_1^n}\right) \tag{1.8}$$

Using the above equation for Rényi entropy, we can obtain the entanglement entropy of the lattice quantum theory by taking the limit $n \to 1$.

$$S_A = \lim_{n\to 1} \frac{1}{1-n} \log \frac{Z_n(A)}{Z^n} = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n} \tag{1.9}$$

Since $Tr\rho_A^n = \sum_\lambda \lambda^n$ where λ are the eigenvalues of ρ_A (which lie in the interval $[0,1)$) and since $Tr\rho_A = 1$, the left hand side of the above equation is absolutely convergent and therefore analytic for all Re$n > 1$. Then, we can also take the derivative wrt to n and then take the limit. Thus the procedure to get the entanglement entropy from the Rényi entropy is well-defined.

Recall the lattice spacing ϵ of the lattice quantum theory. This acts as the UV cutoff and the limit $\epsilon \to 0$ gives us the entanglement entropy of the quantum field theory.

1.4 Some results about entanglement entropy

In this section, we review two main results about entanglement entropy. We work in 1+1 dimensions because in this case the special nature of the 1+1 dimensional conformal group has facilitated the calculations of the entanglement entropy. The first case refers to a finite-sized entangling interval in a finite-sized system at zero temperature. The second case refers to a finite-sized entangling interval in an infinitely long system with finite temperature β^{-1}.

1. Consider a 1+1 dimensional quantum field theory at critical point (thus it is a conformal field theory). Let the whole system have a finite length L and the subsystem A be a single interval of length l in the system. Suppose we impose periodic boundary conditions on the whole system. Then the entanglement entropy of the system is given by [4]:

$$S_A = \frac{c}{3} \log \left(\left(\frac{L}{\pi\epsilon}\right) \sin\left(\frac{\pi l}{L}\right)\right) + c_1' \tag{1.10}$$

2. Consider a 1+1 dimensional quantum field theory. Let $L \to \infty$. Let the system be in a thermal state (mixed state) at a finite temperature $\frac{1}{\beta}$. Let the subsytem A be again an interval of length l. Then the entanglement entropy of the system is given by [4]:

$$S_A = \frac{c}{3} \log \left(\left(\frac{\beta}{\pi \epsilon} \right) \sinh \left(\frac{\pi l}{\beta} \right) \right) + c_1' \tag{1.11}$$

As observed in [4], the constant c_1' is the same in both cases.

In second part of this report (chapters [3], [4], [5]), we will present some original attempts in trying to derive the expressions for entanglement entropy in 1+1 dimensional free fermion conformal field theories, with a finite length interval as the entangling interval in a finite-sized system at a finite temperature.

1.5 Black holes in string theory

Black holes have an entropy associated to them. In the early 1970's, a remarkable similarity was found between laws of thermodynamics and laws of black hole mechanics. This led Bekenstein to conjecture that black holes have an entropy [5]:

$$S_{BH} = \frac{1}{4} \frac{A}{l_P^2} \tag{1.12}$$

where $l_P = \frac{c^3}{\hbar G}$ is the Planck length.

This area law dependence of the black hole entropy is puzzling and has no explanation in general relativity. Also, since the gravitational field inside a black hole is very high, we expect general relativity to break down. Also, entropy is inherently a quantum notion in the sense that even the thermodynamic entropy is defined as a logarithm of the total number of quantum microstates available to the particular thermodynamic macrostate. It is not clear why the microstates of the black hole should localize near the horizon so as to give an area law dependence to the entropy. These facts motivate us to expect that a full explanation of some properties of black holes can only be found in a quantum theory of gravity.

In the late 1970's and 1980's, string theory emerged as one of the most prominent candidate for a quantum theory of gravity. Since then, the study of black holes in string theory has been focussed towards providing an account for the area law and other properties of the black hole entropy. Black holes in string theory is a vast topic and we give a very brief presentation here just so as to set the context for chapter [2] where a special class of black holes is studied following [16].

Black holes in string theory come as solutions to string equations of motion. Since there are gauge fields that form part of these solutions and the black holes carry charge under these gauge fields, these are charged black holes. If a black hole carries two charges, say electric (Q) and magnetic (P), then it is called a dyonic black hole (or dyon) and its mass M, as is shown in [6], satisfies the relation $M \geq a\sqrt{Q^2 + P^2}$ where a is a constant that comes with the solution (technically, the vacuum expectation value of a scalar field of the solution). The lower bound on the mass is called Bogomol'nyi-Prasad-Sommerfield (BPS) bound [7].

Some black hole solutions in string theory satisfy the BPS bound. In [8], it was shown that these BPS states describe extremal black holes. Extremal black holes are special class of black holes with zero Hawking temperature. These will be important for us as we discuss the entropy of these black holes in the next chapter.

In 1996, Strominger and Vafa [9] showed that one can excatly reproduce the area law of the black hole entropy from string theory. Specifically, we start with Type II string theory on $K3 \times S^1$. Black holes in these theories carry two charges. Thus they are dyons. We then look for those solutions in string theory which satisfy the BPS bound. One then counts the exact degeneracy of these states. The entropy found from this is seen to exactly match the Bekenstein-Hawking entropy of black holes as obtained from the low-energy effective action of the same string theory.

1.6 The AdS/CFT Correspondence

Equation (1.12) tells us that entropy in a spherical region of spacetime scales as the square of the radius and not cube. This observation first gave rise to the Holographic Principle - the statement that all the information inside a volume can be represented as a 'hologram' so that the boundary gives all the information contained in its bulk.

The AdS/CFT Correspondence [10], [11], [12] is a concrete realization of the Holographic Principle. It provides a relation between a string theory, which is a theory of quantum gravity and superconformal field theory without gravity. Moreover, these theories live in different spacetime dimensions. As originally presented, the AdS/CFT Correspondence states that - Type IIB string theory on $AdS_5 \times S^5$ (with appropriate boundary conditions) is dual to conformally invariant $\mathcal{N} = 4$ supersymmetric Yang-Mills theory defined on the 4-dimensional boundary of the AdS_5.

In terms of relating the parameters on both sides, the duality takes the form:

$$g_S = g_{YM}^2, a_0 = \frac{\theta}{2\pi}, Q = N \tag{1.13}$$

where g_S is the string coupling constant, g_{YM} is the Yang-Mills coupling constant of the guage theory, a_0 is the constant value of a real scalar field in the string theory, θ is the instanton angle in the Yang-Mills theory, Q is the quantized flux of the 5-form field strength in the string theory and N is the N of $SU(N)$ - the internal symmetry group of the Yang-Mills theory. The supersymmetry group of $AdS_5 \times S^5$ is the same as that of the $\mathcal{N} = 4$ SYM theory in 3+1 dimensions and the operators in the SYM theory act as sources for the fields in the AdS_5 supergravity theory.

The AdS/CFT Correspondence is remarkable in that it is a kind of strong-weak duality. When the effective coupling $g_{YM}^2 N$ becomes large, the perturbative expansions in the SYM theory cannot be trusted but as $g_S Q$ becomes large in this limit, the perturbative calculation done in supergravity on $AdS_5 \times S^5$ can be trusted. The flux Q measures the size of geometry in Planck units and hence quantum effects in $AdS_5 \times S^5$ correspond to $\frac{1}{N}$ effects in the gauge theory. Thus the AdS/CFT Correspondence is an important calculational tool, apart from being an indicator towards more fundamental properties of quantum gravity.

More general statements of the AdS/CFT Correspondence also exist, which relate string theory on some product space containing an AdS factor to conformal field theory on the boundary of the AdS.

Chapter 2

QEF and entanglement entropy

It can be shown that entanglement entropy satisfies an area law [13], [14]:

$$S_A = \text{constant}\,\frac{\text{Area}(\partial A)}{\epsilon^2} + \text{subleading terms} \tag{2.1}$$

where ϵ is the UV-cutoff and ∂A is the boundary of the entangling surface A

If we compare this to equation (1.12), we see a similarity between leading terms in entanglement entropy and black hole entropy.

Is this similarity a co-incidence? Black hole entropy, as we argued in Chapter [1], can shed light on the nature of quantum gravity. We argue for the importance of entanglement entropy in the present context as follows:

Consider a theory of supergravity as a low-energy limit of a string theory defined in the bulk AdS spacetime. As per the AdS/CFT Correspondence, this is equivalent to a strongly coupled quantum field theory with supersymmetry and conformal invariance and defined on the boundary of the AdS spacetime. Now, the distinguishing feature of a quantum field theory is quantum entanglement which is quantified by the entanglement entropy. Because the quantum theory of gravity defined in the bulk is equivalent to the boundary quantum field theory, the "quantumness" in the bulk theory should somehow be related to the "quantumness" in the boundary theory, which is measured by the entanglement entropy. Black holes provide a natural partition of the Hilbert space of the universe into two and hence one can measure entanglement entropy with such a partition. This, and the area laws of the two entropy, motivates a relation between the black hole entropy in the bulk gravity theory and the entanglement entropy in the boundary field theory.

Having motivated why should black hole entropy and entanglement entropy be related, in the context of the AdS/CFT Correspondence, we will review the work [16] in this chapter where an exact equality was shown between quantum entropy of black holes in the bulk AdS spacetime (defined in the sections that follow) and entanglement entropy in the boundary conformal field theory.

Such a relation is not known for general AdS_{d+1}/CFT_d setups. We describe in the following sections a special case of the problem - that of the AdS_2/CFT_1. Specifically, following [16], we review the state operator correspondence in CFT_1 , entanglement entropy and the quantum entropy of extremal black holes in AdS_2.

2.1 State Operator Correspondence

We know about radial quantization as a way to quantize a Conformal Field Theory (CFT). Here, time runs along the radial direction. This makes it possible to define a one-one map between local operators in the CFT on S^d and states in the same CFT on $S^{d-1} \times \mathbb{R}$. This is the State Operator Correspodence in a general CFT [15]

If ξ labels a coordinate on the cylinder $S^{d-1} \times \mathbb{R}$ and z on S^d, then the map

$$z = e^{\frac{2\pi\xi}{L}} \tag{2.2}$$

takes us from $S^{d-1} \times \mathbb{R}$ to S^d.

If $\phi(z, \bar{z})$ is a field operator in a CFT on S^d, it corresponds to the following state in the CFT on $S^{d-1} \times \mathbb{R}$

$$|\phi\rangle_{in} = \lim_{z,\bar{z}\to 0} \phi(z, \bar{z})|0\rangle \tag{2.3}$$

Now consider the case of CFT_1 - 0+1 dimensional CFT.

If we start with the CFT on S^1, the state operator correspondence becomes a map from local operators on S^1 to states defined in the same CFT on $S^0 \times \mathbb{R}$.

This can be thought of as a map from operators on the Hilbert space $\mathcal{H}_c$ of CFT1 to states in the product Hilbert space $\mathcal{H}_1 \times \mathcal{H}_2$. The Hilbert spaces $\mathcal{H}_c$, $\mathcal{H}_1$, $\mathcal{H}_2$ are all isomorphic to each other.

The mapping from S^1 to $S^0 \times \mathbb{R}$ is -

$$\sigma + i\tau = 2\ \tan^{-1} \tanh\left(\frac{i\theta}{2}\right) \tag{2.4}$$

where $\theta \in [0, 2\pi]$; $\sigma = -\pi, 0$ and $-\infty < \tau < \infty$.

Because CFT_1 is defined only at one spatial point, all operators in CFT_1 are local. If $\hat{M}$ is an operator at $\theta = -\frac{\pi}{2}$ on S^1, it gets mapped to the state -

$$|M\rangle\rangle = M_{ab}\,|a\rangle_1 \times |b\rangle_2 \tag{2.5}$$

where $M_{ab} = \langle a|\hat{M}|b\rangle$, the states $|a\rangle$ and $|b\rangle$ here are defined on $\mathcal{H}_c$ and summation is implied.

Because $\hat{M}$ is localized at $\theta = -\frac{\pi}{2}$, the state $|M\rangle\rangle$ is situated at $\sigma = -\pi$, $\tau = -\infty$.

For the above state, reduced density matrix of the CFT (obtained after tracing over the states in the second copy of $\mathcal{H}$) is -

$$\rho_M = (MM^\dagger)_{ac}|a\rangle\langle c| \tag{2.6}$$

where the states $|a\rangle$ and $|c\rangle$ are now defined in $\mathcal{H}_1$

As a special case which we will refer to later on, consider the identity operator on $\mathcal{H}_c$. This gets mapped to the state $|I\rangle\rangle = |a\rangle_1 \times |a\rangle_2$ on $S^0 \times \mathbb{R}$. The reduced density matrix corresponding to the state $|I\rangle\rangle$ is $\rho_I = |a\rangle\langle a|$. Thus the identity operator gets mapped to a maximally entangled state.

The CFT defined on S^1 forms the boundary of Euclidean local AdS_2 and the one defined on $S^0 \times \mathbb{R}$ forms the boundary of Lorentzian local AdS_2 spacetime. In the following section we prove this statement by presenting AdS_2 metric in both Euclidean and Lorentzian signature.

2.2 Different versions of AdS_2 spacetime

The objects of study in this chapter are a special class of black holes called extremal black holes. Extremal black holes are those charged black holes whose inner and outer horizons coincide. Equivalently, these are defined as black holes with Hawking temperature $T_H = 0$. Since their Hawking temperature is zero, they do not radiate and hence are stable. They are described by those solutions of string theory which satisfy the BPS bound.

Furthermore, we are going to take near-horizon limit of these extremal black holes. Near-horizon geometry, also sometimes called as the geometry near the 'stretched horizon' is thought to encode all the information about microstates of the black hole [17]

We start with a Reissner-Nordstrom black hole solution in 3+1 dimensions. We will take the near-horizon geometry of this 3+1 dimensional black hole and arrive at AdS_2 spacetime. The metric part of the Reissner-Nordstrom solution is:

$$ds^2 = -\left(1 - \frac{a}{\rho}\right)\left(1 - \frac{b}{\rho}\right)d\tau^2 + \frac{d\rho^2}{\left(1 - \frac{a}{\rho}\right)\left(1 - \frac{b}{\rho}\right)} + \rho^2\left(d\theta^2 + sin^2\theta\, d\phi^2\right) \tag{2.7}$$

where the length scales a and b refer to the inner and the outer horizons.

Now define new coordinates (t, r) as [16]:-

$$r = \frac{\rho - \frac{a+b}{2}}{\epsilon} \,,\, t = \frac{\epsilon\tau}{a^2} \tag{2.8}$$

where $\epsilon = \frac{a-b}{2}$.

Now we want to take the near-horizon limit of the metric (2.7) in its extremal form. It is not obvious which limit - the near-horizon limit or the extremal limit - is to be taken first in metric (2.7). If we first take the extremal limit ($a = b$) and then take the near-horizon limit, we will end up with metric:

$$ds^2 = c\left[-r^2 dt^2 + \frac{dr^2}{r^2} + r^2\left(d\theta^2 + sin^2\theta\, d\phi^2\right)\right] \tag{2.9}$$

where c is a constant.

But this is not the correct near-horizon metric because this way zero-energy excitations at asymptotic infinities ($r \to \pm\infty$) may be finite-energy excitations close to the horizon ($\rho \to a$). This is not allowed because AdS_2 cannot support finite-energy excitations [18]. This order of taking limits does not respect the instability of AdS_2 to finite energy excitations.

Instead, as shown in [16], the correct way to take near-horizon limit of this black hole is to take $\epsilon \to 0$ defined through equation (2.8) while keeping (t, r) fixed. This leads to the following near-horizon metric-

$$ds^2 = c\left[-(r^2 - 1)dt^2 + \frac{dr^2}{(r^2 - 1)} + r^2\left(d\theta^2 + sin^2\theta\, d\phi^2\right)\right] \tag{2.10}$$

In this case, the constant $c = a^2$.

From now onwards, we will focus only on the AdS_2 part of this near-horizon metric. Thus the near-horizon geometry of an extremal black hole contains a local AdS_2 factor with metric-

$$ds^2 = a^2\left[-(r^2 - 1)\, dt^2 + \frac{dr^2}{(r^2 - 1)}\right] \tag{2.11}$$

upto a reparametrization, (t, r) can be considered to be as the time and the radial coordinate of the full black hole solution.

Black hole horizons are objects naturally defined in a global spacetime. So, we have to find the global extension of the above local AdS_2 spacetime to study the presence of horizons. Local AdS_2 spacetime can be extended to global AdS_2

spacetime, keeping the signature of the metric Lorentzian. If (T, σ) are global coordinates, then the map which achieves this is-

$$T \pm \sigma = 2 \tan^{-1} \tanh \frac{t \pm \eta}{2} \tag{2.12}$$

where $\eta = \frac{1}{2}\, log \frac{(r-1)}{(r+1)}$; $-\pi < \sigma < 0$ and $-\infty < T < \infty$.
The Lorentzian global AdS_2 metric we thus get is-

$$ds^2 = \frac{a^2}{\sin^2 \sigma} \left[-dT^2 + d\sigma^2 \right] \tag{2.13}$$

Geometrically, this represents an infinite strip, where the coordinate T runs along the boundaries of the strip.

In our presentation of the AdS/CFT equivalence relating to near-horizon geometry of a black hole, we would be interested in defining partition function on AdS_2 space. For this purpose, we need to know Euclidean versions of the local and global AdS_2 metrics, which we present below.

In equation (2.11), if we put $t = -i\theta$, then we get the Euclidean local AdS_2 spacetime -

$$ds^2 = a^2 \left[(r^2 - 1) d\theta^2 + \frac{dr^2}{(r^2 - 1)} \right] \tag{2.14}$$

If we define a new coordnate $\alpha = \sqrt{\frac{r-1}{r+1}}$, then the metric above takes the form-

$$ds^2 = \frac{4a^2}{(1 - \alpha^2)^2} \left[d\alpha^2 + \alpha^2 d\theta^2 \right] \tag{2.15}$$

Since $\alpha = 0$ has no conical singularity, θ is periodic with period 2π. Thus it is clear that Euclidean local AdS_2 space is a disk.

We can Euclideanize the global AdS_2 space as well. In equation (2.13) put $T = -i\,\tau$. Then

$$ds^2 = \frac{a^2}{\sin^2 \sigma} \left[d\tau^2 + d\sigma^2 \right] \tag{2.16}$$

In the state operator correspondence in a CFT, we map a time coordinate running along a real line to a circle. As the CFT we will consider will form the boundary of the AdS spacetime, the state operator correspondence will map the strip-like AdS_2 spacetime to a disk-like spacetime, or the Lorentzian to the Euclidean AdS_2. This is the motivation behind presenting both Lorenzian and Euclidean AdS_2 metrics.

2.3 Quantum entropy of extremal black holes

When we study the entropy of black holes in string theory, the Bekenstein-Hawking entropy equation (1.12) turns out to be the leading contribution to the black hole entropy. The Strominger-Vafa result [9] is proved in the case of large charges of the dyonic black holes under study. The subleading corrections are either due to the classical "stringy" nature of the theory (α' corrections) or due to the quantum nature of the strings (string loop corrections) (or both). There have been several attempts to study these corrections to the Bekenstein-Hawking entropy formula. Please see the review [19] for details about these.

These attempts have culminated in the so-called Quantum Entropy Function (QEF). It gives us a way to compute the fully quantum corrected entropy of extremal black holes [20].

The partition function is (weighted) sum of degeneracies and hence is useful in computing the statistical entropy. The near-horizon geometry of an extremal black hole contains an AdS_2 factor and a compact manifold (denoted by $\mathcal{K}$). The partition function Z_{AdS_2} in this near-hrizon geometry is defined as a path integral over all the string fields in $AdS_2 \times \mathcal{K}$ weighted by e^{-S_E}, where S_E is the Euclidean action evaluated in the AdS_2. In $AdS_2 \times \mathcal{K}$ backround, using symmetry consideration, the metric has a general form and the action becomes-

$$S_E \approx r_0(2\pi v\mathcal{L} + K) + 2\pi v\mathcal{L} \tag{2.17}$$

where r_0 is an upper bound on the radius of Euclidean local AdS_2, placed so as to regularize the infinite volume of AdS_2, v is a constant and $\mathcal{L}$ is the lagrangian density integrated over AdS_2 coordinates. The term involving r_0K comes from the boundary contribution to S_E with K being a constant.

We observe that the term linear in r_0 is ambiguous since it can be changed by changing the boundary terms. However the finite part in Z_{AdS_2} is independent of bounary terms and is unambiguous. We write -

$$Z^{finite}_{AdS_2} = e^{-2\pi v\mathcal{L}} \tag{2.18}$$

Now we adopt a boundary condition where we fix the asymptotic electric field configuration. In this case the path intergral defining Z_{AdS_2} needs to weighted also by a factor $e^{-\left[iq_i \oint d\theta A^{(i)}_\theta\right]}$. The partition function then becomes -

$$Z_{AdS_2} = e^{r_0(2\pi v\mathcal{L} + K - 2\pi\vec{e}\cdot\vec{q}) + 2\pi(\vec{e}\cdot\vec{q} - v\mathcal{L})} \tag{2.19}$$

The finite part of the partition function now becomes-

$$Z_{AdS_2}^{finite} = e^{2\pi(\vec{e}\cdot\vec{q} - v\mathcal{L})} \tag{2.20}$$

We define the quantum entropy function QEF as -

$$\text{QEF} \equiv Z_{AdS_2}^{finite} \tag{2.21}$$

By the AdS/CFT Correspondence, the partition function of string theory on $AdS_2 \times \mathcal{K}$ is equal to that of the boundary CFT_1.

$$Z_{AdS_2} = Z_{CFT_1} \tag{2.22}$$

If H is the Hamiltonian of the CFT_1,

$$Z_{CFT1} = Tr\left(e^{-2\pi r_0 H}\right) = e^{-2\pi r_0 E_0} \sum_{\vec{q}} d\left(\vec{q}\right) e^{-2\pi \vec{e}\cdot\vec{q}} \tag{2.23}$$

where E_0 is the energy of the ground state of the CFT_1. As the spectrum of extremal black holes have a gap separating the BPS ground states from the non-BPS excited states, the extremal BPS black hole is described by a CFT having a similar gap and furthermore, a finite number $d(\vec{q})$ of energy states corresponding to the charges $\vec{q}$.

Therefore,

$$Z_{AdS_2} = e^{-2\pi r_0 E_0} \sum_{\vec{q}} d\left(\vec{q}\right) e^{-2\pi \vec{e}\cdot\vec{q}} \tag{2.24}$$

Extracting the finite part of this full partition function and using equation (2.21),

$$\text{QEF} = \sum_{\vec{q}} d\left(\vec{q}\right) e^{-2\pi \vec{e}\cdot\vec{q}} \tag{2.25}$$

The quantity $\sum_{\vec{q}} d\left(\vec{q}\right) e^{-2\pi \vec{e}\cdot\vec{q}}$ is just equal to the total number of states the CFT_1 has. By statistical theory, logarithm of this gives us the entropy of the system. As this CFT_1 describes the extremal black hole, making use of equation (2.25), we get the result that the full quantum corrected entropy of the extremal black hole is

$$S_{BH} = \log\left(\text{QEF}\right) \tag{2.26}$$

2.4 Calculating observables in CFT_1

The CFT_1 has a finite number of degenerate ground states, say N. Therefore, the only observables in CFT_1 are $N \times N$ matrices. We take linearly independent unitary matrices as a basis and then the CFT_1 has exact $U(N)$ symmetry because the N degenerate states get rotated into themselves under the action of the operators.

By AdS/CFT Correspondence, this symmetry must also be present in the string theory on $AdS_2 \times \mathcal{K}$. Hence, if W is an $N \times N$ matrix acting on CFT_1, there must also be a corresponding transformation $\tilde{W}$ acting on the fields in string theory.

Then $Tr(W)$, an observable in CFT_1 is equivalent to the partition function of string theory on $AdS_2 \times \mathcal{K}$ with a $\tilde{W}$ twist [16].

2.5 States in string theory on $AdS_2 \times K$

Let $|W\rangle\rangle$ be a state in CFT_1 on $S^0 \times \mathcal{R}$. To $|W\rangle\rangle$ and $|V\rangle\rangle$, we want to associate wave-functionals in string theory f_W and f_V such that

$$\text{Inner Product}\,(f_w; f_V) = Tr\left(W^{-1}V\right) = Z_{AdS_2}^{twisted}\left(W^{-1}V\right) \tag{2.27}$$

For this, we define [16] $f_W \equiv$ string theory path integral over the half disk with a cut corresponding to the transformation $\tilde{W}$ that reaches the bounday of the half-disk.

2.6 Entanglement Entropy and QEF

We have shown that (equation (2.26)) - $S_{BH} = \log(\text{QEF})$.

Now consider the string theory on $AdS_2 \times K$ in the Hartle-Hawking vacuum state f_{HH} [21]. Hartle-Hawking vacuum state is a state constructed by path integrating Euclidean action over the half disk (without any cut) and then time evolving it from there with Lorentzian signature. Here we are only interested in its definition at $t = 0$ and not its evolution.

As is defined in absence of cuts, the Hartle-Hawking state is a wave-functional associated to the identity operator acting on CFT_1. In section 1, we presented the state in the CFT on $S^0 \times \mathcal{R}$ corresponding to the identity operator -

$$\hat{I} \;\rightarrow\; |I\rangle\rangle = |a\rangle_1 \times |a\rangle_2 \tag{2.28}$$

Thus string theory on $AdS_2 \times \mathcal{K}$ in the Hartle-Hawking vacuum state f_{HH} correspondes to the CFT_1 in a maximally entangled state -

$$f_{HH} \;\rightarrow\; |I\rangle\rangle \tag{2.29}$$

The entanglement entropy between the two copies 1 and 2 of the CFT_1s is given by $logN$. N is equal to the right-hand side of equation (7). Hence the fully quantum corrected entropy of the extremal black hole in the Hartle-Hawking of string theory on $AdS_2 \times \mathcal{K}$ to be equal to the entanglement entropy between the two copies 1 and 2 of CFT_1 s which reside in the maximally entangled state $|I\rangle\rangle$ -

$$\text{QEF} = \text{Entanglement Entropy} \tag{2.30}$$

The equality between the Quantum Entropy Function of the black hole and the entanglement entropy between the boundary CFTs can also be proved by working fully in the bulk.

Using the replica trick in CFT,

$$S_{ent} = -\lim_{n\to 1} \frac{d}{dn} \frac{Tr\,(\rho^n)}{(Tr\rho)^n} \tag{2.31}$$

Generally, $Tr\,(\rho^n)$ is obtained from the bulk by calculating the partition function over an n-fold cover of AdS space. But in the case of AdS_2, $Tr\,(\rho^n)$ can also be obtained from the partition function of string theory over all spaces each of which is asymptotically AdS_2 and contains a boundary circle of length $n \times 2\pi r_0$. Let us denote this partition function by $\hat{Z}_{AdS_2}(n)$.

It can be shown that $\hat{Z}_{AdS_2}(n) = N\, e^{-n2\pi r_0 E_0}$.

Then $Tr\,(\rho^n) = N\, e^{-n2\pi r_0 E_0}$ and $Tr(\rho) = N\, e^{-2\pi r_0 E_0}$ and

$$S_{ent} = -\lim_{n\to 1} \frac{d}{dn} \frac{N\, e^{-n2\pi r_0 E_0}}{N^n\, e^{-n2\pi r_0 E_0}} = -\lim_{n\to 1} \frac{d}{dn} N^{1-n} = \log N \tag{2.32}$$

Therefore,

$$\text{QEF} = \text{Entanglement Entropy} \tag{2.33}$$

2.7 Discussion

In this chapter, which forms the first part of this report, we have reviewed the work of [16]. We have provided motivation for the existence of a relation between the quantum entropy of black holes in AdS_{d+1} and the entanglement entropy of the CFT_d that forms the boundary of this spacetime. We have discussed why such a relation, in a precise form, is significant. There have been many attempts at finding such a relation.

The Quantum Entropy Function of extremal black holes describes the fully quantum corrected entropy of these black holes. In the special case of AdS_2/CFT_1 , we have reviewd in detail the solution provided in [16] - that the quantum entropy function of the extremal black holes in AdS_2 and the entanglement entropy of the two CFT_1s which form the boundary of the AdS_2 are exactly equal.

Finding such a precise relation in the case of general AdS_{d+1}/CFT_d has not been successful as of now.

Chapter 3

Modular invariance and quantum entropies: 1

In this and the next two chapters, we present some original attempts at studying modular properties of Rényi and entanglement entropies. The modular covariant/invariant expressions we derive pass many tests. Besides being interesting in its own right, the study of modular properties of these quantum entropies reveal some subtle issues related with applying the Replica Trick to CFTs at finite temperature.

One of our primary motivations to study the modular properties (covariance or invariance) of Rényi and entanglement entropies is the disparity in literature between the symmetry properties of these quantities. For example, it can be seen from the expression of [32] that Rényi entropy is T-duality invariant and shifts under modular transformations by a factor dependent on the modular parameter and the central charge of the CFT. However, the expression given in [25] is in a particular spin structure and hence not well-defined under modular transformations. We mention here the specifics that the conformal field theory (CFT) considered in [32] is a theory of free bosons on a torus whereas [25] work with a CFT of free fermions at finite temperature and in a specific spin structure (the Neveu-Schwarz sector or $\nu = 3$, please see Section [3.2] for more details). But the two CFTs are equivalent by Fermi-Bose duality at boson compactification radius $R = 1$. We shall discuss all these ideas in detail as we go along.

3.1 Modular invariance in CFTs

If we have a CFT defined on a finite spacetime (perhaps in order to take care of

IR divergences), then we need to provide boundary conditions for the fields in the CFT. Choosing the boundary conditions to be periodic (or anti-periodic) naturally leads to the spacetime being a torus. With this motivation, we will study CFTs defined on a torus.

A torus may be defined by specifying two linearly independent lattice vectors on the complex plane and identfying all the points that differ by an integer combination of these vectors. If complex numbers ω_1 and ω_2 represent the lattice generators then their ratio $\tau = \frac{\omega_1}{\omega_2}$ is called the 'modular parameter' of the torus and is the only complex parameter that is required to define the torus. The partition function of the CFT and the correlation functions then depend on the modular parameter τ.

If we choose one of the latice generators to be along the real axis of the complex plane, it immediately forces the other generator to lie in the upper (or lower) half-space. Let us choose the generator to lie in the upper half-space. This means $\tau > 0$.

A transformation of the type

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1 \tag{3.1}$$

where $a, b, c, d \in \mathbb{Z}$, is called a modular transformation. The integers a, b, c, d arranged as a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

form the group $PSL_2(\mathbb{Z})$. The group is sometimes called the 'modular group'. If we take τ to be the length of one of the lattice generators of the torus, the length of the other by default becomes 1.

Consider the modular transformation $\tau \to -\frac{1}{\tau}$. This transformation exchanges the two lattice generators into each other. We observe that such a transformation is essentially a composition of a scaling and a rotation both of which are conformal maps. As the CFT should be invariant under any conformal transformation, the CFT should also be invariant under the transformation $\tau \to -\frac{1}{\tau}$.

Now consider the transformation $\tau \to \tau + 1$. This is a discrete translation and the CFT is invariant with respect to this.

It can be shown that the two transformations $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau + 1$ generate the whole of the modular group [22]. Hence the CFT should be invariant under the whole of the modular group. This is termed as 'modular invariance' of the CFT. This invariance places constraints on the operator content of the theory. Furthermore, the modular transformations are now a symmetry of the theory and hence quantities of interest like the partition function should be modular invariant.

3.2 Free fermions at finite temperature

Let us denote the holomorphic component of a fermion field by $D(z)$ and the anti-holomorphic component by $\bar{D}(\bar{z})$. The free theory in 1+1 dimensions is described by the action:

$$S = \frac{1}{2\pi} \int d^2z \left(\bar{D}\partial\bar{D} + D\bar{\partial}D\right) \tag{3.2}$$

Let $(z, \bar{z})$ take values on a torus. We then need to impose boundary conditions on the fermion fields in space and time both of which are finite. We take space to be along the real axis. Then the general transformation of the fermion field, under a translation by the lattice generators of the torus, becomes:

$$D(z+1) = e^{2\pi i a}\, D(z)\ , \quad D(z+\tau) = e^{2\pi i b} D(z) \tag{3.3}$$

where $a, b \in \mathbb{Z}$.

The anti-holomorphic component satisfies the same transformation rule. Now, under the above two translations of z the action must be periodic (i.e. remain unchanged) because the field theory is defined on a torus. The fact that the action is quadratic in each of the fermion field component implies the following possibilities for the boundary conditions [22]:

$$\begin{aligned}
(a,b) &= \big(0,0\big) && \big(R,R\big)\\
(a,b) &= \big(0,\tfrac{1}{2}\big) && \big(R,NS\big)\\
(a,b) &= \big(\tfrac{1}{2},0\big) && \big(NS,R\big)\\
(a,b) &= \big(\tfrac{1}{2},\tfrac{1}{2}\big) && \big(NS,NS\big)
\end{aligned}$$

where R stands for Ramond or the periodic boundary condition and NS stands for Neveu-Schwarz or the anti-periodic boundary condition. A set of numbers (a, b) which represents one boundary condition on the fermion fields in conformal field theories is called a 'spin structure' of the fermion. For later reference we will denote these spin structures as follows:

$$\begin{array}{lll} \nu = 1 & (a,b) = \big(0,0\big) & \big(R,R\big) \\ \nu = 2 & (a,b) = \big(0,\frac{1}{2}\big) & \big(R,NS\big) \\ \nu = 4 & (a,b) = \big(\frac{1}{2},0\big) & \big(NS,R\big) \\ \nu = 3 & (a,b) = \big(\frac{1}{2},\frac{1}{2}\big) & \big(NS,NS\big) \end{array}$$

3.3 Rényi entropy in a modular invariant CFT

Consider a free fermion CFT in 1+1 dimensions. Let the CFT be defined on a torus. In section [1.3], we have discussed the replica trick used to calculate the Rényi entropy and subsequently the entanglement entropy. In this section, we apply the replica trick to a free fermion CFT on a torus, building up on the work in [25].

In the replica trick as applied to the CFT on torus, we extend the original torus to an n-fold cover with branch cuts along spatial intervals from 0 to l, where $[0,l]$ is the entangling interval. This n-fold space can be equivalently thought of as a single torus, with the same modular parameter as the original one, with the insertion of twist fields at the end points of the entangling interval [23].

Let k be the label of the k^{th} replicated torus. Let k range from $-\frac{n-1}{2}$ to $\frac{n-1}{2}$ in integral steps. Then the twist field $\sigma_k(z,\bar{z})$ is defined by its action on the free fermion field:

$$\begin{aligned} \sigma_k(z,\bar{z})\, D(z') \quad &\sim \quad (z-z')^{\frac{k}{n}} \\ \sigma_k(z,\bar{z})\, \bar{D}(\bar{z'}) \quad &\sim \quad (z-z')^{-\frac{k}{n}} \end{aligned} \tag{3.4}$$

One can show that [24],

$$Tr\rho_A^n = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \langle \sigma_k(l,l)\sigma_{-k}(0,0)\rangle \tag{3.5}$$

It is convenient to think of the product of un-normalized correlators of the twist fields as defining the replica partition function.

$$Z_n = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} Z_1\langle \sigma_k(l,l)\sigma_{-k}(0,0)\rangle = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \langle\langle \sigma_k(l,l)\sigma_{-k}(0,0)\rangle\rangle \tag{3.6}$$

This definition is consistent with equation (1.7) where the partition function on the n-sheeted Riemann surface $Z_n(A)$ appears.

Let us consider the free CFT with one Dirac fermion. This has central charge $c = 1$ and consists of two Majorana fermions with correlated spin structures. Denote the Dirac fermion by $(D(z), \bar{D}(\bar{z}))$. By definition, these have Hermitian conjugates $\left(D^\dagger(z), \bar{D}^\dagger(\bar{z})\right)$. The local operators in this theory of dimension $\left(\frac{1}{2}, \frac{1}{2}\right)$ are -

$$D(z)\bar{D}(\bar{z}) \ , \ D^\dagger(z)\bar{D}(\bar{z}) \ , \ D(z)\bar{D}^\dagger(\bar{z}) \text{ and } D^\dagger(z)\bar{D}^\dagger(\bar{z}) \ .$$

The modular invariant partition function in this theory can be found by calculating the partition function in each spin structure and then summing the expressions over all the spin structures. The result is [22]:

$$Z_1 = \frac{1}{2}\sum_{\nu=2}^{4}\left|\frac{\theta_\nu(0|\tau)}{\eta(\tau)}\right|^2 \tag{3.7}$$

and for future reference, we note that:

$$Z_1[m] = \frac{1}{2}\sum_{\nu=2}^{4}\left|\frac{\theta_\nu(0|\tau)}{\eta(\tau)}\right|^{2m} \tag{3.8}$$

where $\nu = 1, 2, 3, 4$ represent the four spin structures described in the previous section and $\theta_\nu(z|\tau)$ are the Jacobi theta functions. Our definitions of the Jacobi theta functions and some useful identities are given in Appendix A.

The following result was obtained for the replica partition function of this theory in [25]

$$Z_n^\nu = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} Z_1^\nu \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{4\Delta_k} \left|\frac{\theta_\nu(\frac{kl}{nL}|\tau)}{\theta_\nu(0|\tau)}\right|^2 \tag{3.9}$$

where $\Delta_k = \frac{k^2}{2n^2}$, $\nu = 2, 3, 4$ is a fixed spin structure, $\tau = i\frac{\beta}{L}$ and $Z_1^\nu(0|\tau)$ is ordinary partition function in the ν^{th} spin structure . The result would formally be the same in the case of $\nu = 1$ but the normalized 2-point function of the twist fields in this case turns out to be divergent because of the identical vanishing of $\theta_1(0|\tau)$. The implicit assumption in [25] is that the replica partition function and hence the Rényi entropy should be computed spin structure by spin structure. The entanglement entropy following from this way of computing the Rényi entropy satisfies the relation proposed in [25], based on holography, relating its small and large interval limits related to the thermal entropy of the same system. However, applying equation (1.7) in this case, we see that the Rényi entropy cannot be defined for the spin structure $\nu = 1$. Also, the given expression for replica partition function is ill-defined under modular transformations.

On the other hand, the un-normalized 2-point function, which is what defines the replica partition function Z_n, is non-singular. As emphasized in [26], entanglement entropy should be a feature of a definite quantum field theory and independent of the presentation of that theory. In a CFT of fermions, quantities which are invariant under different presentations (see bosonization in the next section) are the modular invariant partition function and the correlation functions. Hence, we propose to compute Rényi entropy by first obtaining Z_n transforming nicely under modular transformations and then dividing by Z_1^n where Z_1 is the modular-invariant ordinary partition function.

In the next section we explicitly carry out the computation following our proposal and obtain a modular invariant expression for the entanglement entropy.

3.4 Central charge $c = 1$ theories

Consider a free fermion CFT with only one Dirac field - $(D(z), \bar{D}(\bar{z}))$. This CFT has central charge $c = 1$. Let the entangling interval A be an interval of length l. Following [25], we will first bosonize the theory and then find the twist field in terms of the boson. Then we will calculate the correlation function of the twist field and find the Rényi entropy $S_n(A)$. Taking the limit $n \to 1$ will give us the entanglement entropy.

Bosonization [27] is the equivalence between a theory of free (or interacting) fermions and a theory of free bosons with the same central charge. Bosonization usually works in 1+1 dimensions. A CFT of one Dirac fermion is equivalent to a CFT of a compact free boson on a circle, with the circle of radius $R = 1$ (in our conventions, $\alpha' = 2$. So T-duality acts by $R \to \frac{2}{R}$ and $R = \sqrt{2}$ is the self-dual radius).

$$D(z) = e^{i\phi(z)} \qquad \bar{D}(\bar{z}) = e^{i\bar{\phi}(\bar{z})}$$

where holomorphic component of the boson $\phi(z)$ has the following propagator

$$\langle \phi(z)\phi(0) \rangle = -\log z \tag{3.10}$$

At a general radius R, the free compact boson has vertex operators labelled by integers (e, m) :

$$\mathcal{O}_{e,m} = \mathcal{V}_{e,m}(z)\, \bar{\mathcal{V}}_{e,m}(\bar{z}) \tag{3.11}$$

where

$$\begin{aligned}\mathcal{V}_{e,m}(z) &=: e^{i\alpha_{e,m}\phi(z)}: \;=\; :e^{i\left(\frac{e}{R}+\frac{mR}{2}\right)\phi(z)}: \\ \bar{\mathcal{V}}_{e,m}(\bar{z}) &=: e^{i\bar{\alpha}_{e,m}\bar{\phi}(\bar{z})}: \;=\; :e^{i\left(\frac{e}{R}-\frac{mR}{2}\right)\bar{\phi}(\bar{z})}:\end{aligned} \tag{3.12}$$

of conformal dimension :

$$\left(\Delta_{e,m},\bar{\Delta}_{e,m}\right)=\left(\frac{1}{2}\left(\frac{e}{R}+\frac{mR}{2}\right)^2,\frac{1}{2}\left(\frac{e}{R}-\frac{mR}{2}\right)^2\right) \tag{3.13}$$

The operator product expansion between the holomorphic part of the vertex operators is given by :

$$\mathcal{V}_{e,m}(z)\mathcal{V}_{e',m'}(0)\sim z^{\left(\frac{e}{R}+\frac{mR}{2}\right)\left(\frac{e'}{R}+\frac{m'R}{2}\right)}\,\mathcal{V}_{e+e',m+m'}(z) \tag{3.14}$$

When the boson is compactified at radius $R=1$, the fermion field has conformal dimensions $\left(\Delta,\bar{\Delta}\right)=\left(\frac{1}{2},\frac{1}{2}\right)$ and is given, in the bosonic presentation, by $\mathcal{O}_{1,0}$.

The twist field in this single Dirac fermion theory is given, in the bosonic language, by:

$$\sigma_k=\mathcal{O}_{0,\frac{2k}{n}} \tag{3.15}$$

for $k=-\frac{n-1}{2},-\frac{n-1}{2}+1,\cdots,\frac{n-1}{2}$. These operators have dimensions $\left(\Delta,\bar{\Delta}\right)=\left(\frac{k^2}{2n^2},\frac{k^2}{2n^2}\right)$. Since the winding number m for these operators is not an integer, they are not included in the set of local operators of the theory. This is what one expects for twist operators. The twist operators have the following OPEs:

$$\sigma_k(z,\bar{z})\,\mathcal{V}_{1,0}(z')\sim(z-z')^{\frac{k}{n}},\;\;\sigma_k(z,\bar{z})\,\bar{\mathcal{V}}_{1,0}(z')\sim(\bar{z}-\bar{z}')^{-\frac{k}{n}} \tag{3.16}$$

Having made n-copies of the original torus, now we need to only compute $\langle\langle\,\sigma_k(z,\bar{z})\,\sigma_{-k}(0,0)\,\rangle\rangle$ to get the replica partition function. Using the general result from [28] that:

$$\begin{aligned}\langle\langle\,\mathcal{O}_{e,m}(z,\bar{z})\,\mathcal{O}_{-e,-m}(0,0)\,\rangle\rangle &= \left|\frac{\theta_1'(0|\tau)}{\theta_1(z|\tau)}\right|^{4\Delta_{e,m}}\frac{1}{|\eta(\tau)|^2} \\ &\times\sum_{e',m'}q^{2\Delta_{e',m'}}\,\bar{q}^{2\bar{\Delta}_{e',m'}}\,e^{4\pi i\left(\alpha_{e',m'}\,\alpha_{e,m}z-\bar{\alpha}_{e',m'}\bar{\alpha}_{e,m}\bar{z}\right)}\end{aligned} \tag{3.17}$$

With $(e,m)=(0,\frac{2k}{n})$, $z=\frac{l}{L}$ and $R=1$, this reduces to :

$$
\begin{aligned}
\langle\langle\, \sigma_k\left(\frac{l}{L},\frac{l}{L}\right)\, \sigma_{-k}(0,0)\,\rangle\rangle \;&=\; \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{2k^2}{n^2}} \frac{1}{|\eta(\tau)|^2}\,\sum_{e,m} q^{2\Delta_{e,m}}\bar{q}^{2\bar{\Delta}_{e,m}}\, e^{4\pi i\frac{kl}{nL}e} \\
&=\; \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{2k^2}{n^2}} \times \frac{1}{2}\sum_{\nu=1}^{4}\frac{\left|\theta_\nu\left(\frac{kl}{nL}|\tau\right)\right|^2}{|\eta(\tau)|^2}
\end{aligned}
\tag{3.18}
$$

At this point, we need to decide how to take the product over replicas. We can take the product over all k in the above expression. We would then have the spin structures summed over before the replication is carried out. Any replica in the n-sheeted Riemann surface is then uncorrelated with other replicas. We denote the replica partition function thus obtained by 'uncorrelated replica partition function' Z_n^u -

$$
Z_n^u(L,\beta;l) = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{1}{6}\left(n-\frac{1}{n}\right)} \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{2}\sum_{\nu=1}^{4}\frac{\left|\theta_\nu\left(\frac{kl}{nL}|\tau\right)\right|^2}{|\eta(\tau)|^2} \tag{3.19}
$$

where we have used the result

$$
\sum_{k=-\frac{(n-1)}{2}}^{\frac{(n-1)}{2}} \frac{k^2}{n^2} = \frac{1}{12}\left(n-\frac{1}{n}\right)
$$

We note that the above expression transforms nicely under modular transformations (in fact, it is modular-covariant, as will be shown below). However, there is another way to obtain a modular-covariant replica partition function. We first isolate the contribution to equation (3.19) from a given spin structure $\nu = 1,2,3,4$ of the fermion. This simply corresponds to picking out the respective function θ_ν. Then, we replicate this contribution n times by taking the product over k. Finally, we sum over the spin structures. Here, the replicas in the n-sheeted Riemann surface are perfectly correlated in the sense that they are defined in the same spin structure and then the spin structures are summed over. We denote the replica partition function thus obtained by 'correlated replica partition function' Z_n^c -

$$
Z_n^c(L,\beta;l) = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{1}{6}\left(n-\frac{1}{n}\right)} \frac{1}{2}\sum_{\nu=1}^{4}\prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{\left|\theta_\nu\left(\frac{kl}{nL}|\tau\right)\right|^2}{|\eta(\tau)|^2} \tag{3.20}
$$

The ordinary ($n = 1$) modular-invariant partition function is a unique expression, as is verified by noticing that $Z_n^u = Z_n^c$ at $n = 1$. Now, corresponding to the two replica partition functions we can define two Rényi entropies:

$$S_n^{u,c} = \lim_{n\to 1} \frac{1}{1-n} \log \frac{Z_n^{u,c}}{(Z_1)^n}$$

Which one of these two Rényi entropies (if any) is the correct Rényi entropy of the modular-invariant free Dirac fermion CFT? We discuss the answer to this question, which is quite subtle, in the next section. We establish here a few properties of the two replica partition functions.

Firstly, we note that the two replica partition functions and the corresponding Rényi entropies can be compared to the non-modular-invariant replica partition function of [25]:

$$Z_n^{(\nu)} = \left| \frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)} \right|^{\frac{1}{6}\left(n-\frac{1}{n}\right)} \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{\left|\theta_\nu\left(\frac{kl}{nL}|\tau\right)\right|^2}{|\eta(\tau)|^2} \tag{3.21}$$

and the corresponding Rényi entropy :

$$S_A^n = \frac{1}{1-n} \log \frac{Z_n^{(\nu)}}{(Z_1^{(\nu)})^n} \tag{3.22}$$

However, as pointed out earlier, this can be done only for the spin structures $\nu = 2, 3, 4$. Also, we observe that two Rényi entroies we have presented are not linear combinations of the above equation for Rényi entropy.

Secondly, the two types of replica partition functions have different behavior in the limit $l \to 0$:

$$\begin{aligned} Z_n^u(L,\beta; l\to 0) \;&\sim\; \left(\frac{l}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \left(\frac{1}{2}\sum_{\nu=1}^{4}\left|\frac{\theta_\nu(0|\tau)}{\eta(\tau)}\right|^2\right)^n \\ Z_n^c(L,\beta; l\to 0) \;&\sim\; \left(\frac{l}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \frac{1}{2}\sum_{\nu=1}^{4}\left|\frac{\theta_\nu(0|\tau)}{\eta(\tau)}\right|^{2n} \end{aligned} \tag{3.23}$$

where one can identify the second factor in the above equations as the ordinary partition functions of n Dirac fermions with uncorrelated and correlated spin structures respectively.

Taking the Rényi entropies in this limit, we see that:

$$S_n^u = \frac{n+1}{6n} \log\frac{l}{L} + 0 \tag{3.24}$$

$$\begin{aligned} S_n^c \;&=\; \frac{n+1}{6n} \log\frac{l}{L} + \frac{1}{1-n}\log\left\{\frac{1}{2}\sum_{\nu=1}^{4}\left|\frac{\theta_\nu(0|\tau)}{\eta(\tau)}\right|^{2n}\right\} \\ &-\; \frac{n}{1-n}\log\left\{\frac{1}{2}\sum_{\nu=1}^{4}\left|\frac{\theta_\nu(0|\tau)}{\eta(\tau)}\right|^{2}\right\} \end{aligned}$$

The entanglement entropy is Rényi entropy in the limit $n \to 1$. Thus the entanglement entropy in the limit $l \to 0$ becomes

$$S = \frac{c}{3} \log \frac{l}{L} \tag{3.25}$$

for only the uncorrelated and not the correlated Rényi entropy. Here, the central charge $c = 1$ in our case and we have assumed commutability of the two limits $n \to 1$ and $l \to 0$. Thus the uncorrelated Rényi entropy has the expected expression for entanglement entropy in this limit [29]. What is the meaning of the correlated Rényi entropy not having the expected behavior in $l \to 0$ limit? Does this mean that the correlated replica partition function, inspite of being modular-invariant is wrong? We will answer these questions in the next section.

Now, we will explicitly check the modular invariance of our expressions for the two types of replica partition functions and their Rényi entropies. A generator of the modular group - the transformation $\tau \to -\frac{1}{\tau}$, as described in section [3.1], exchanges the two cycles of the torus (the two sides of the parallelogram which represents the torus). We have a branch cut along the horizontal (real) axis, which under this transformation, becomes a branch cut along the vertical axis. Thus this transformtation acts as $\beta \leftrightarrow L$ and $l \to il$. This permits us to use the well-known transformation of the Jacobi theta functions:

$$\theta_{\alpha\beta}\left(\frac{z}{\tau}\Big| -\frac{1}{\tau}\right) = (-i)^{\alpha\beta} \, (-i\tau)^{\frac{1}{2}} \; e^{\frac{i\pi z^2}{\tau}} \; \theta_{\beta\alpha}(z|\tau) \tag{3.26}$$

along with the usual identifications: $\theta_{11} \to -\theta_1, \theta_{10} \to \theta_2, \theta_{00} \to \theta_3$ and $\theta_{01} \to \theta_4$ and also the standard modular transformation of the Dedekind eta function:

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} \; \eta(\tau) \tag{3.27}$$

Applying these to the second terms of equations (3.19) and (3.20), we find that we get a multiplicative factor:

$$e^{\frac{i\pi}{6\tau}\left(n-\frac{1}{n}\right)\left(\frac{l}{L}\right)^2}$$

since the summation over ν plays no part in this calculation.

From the θ_1 in the denominator of the first terms of both these equations, we get a corresponding factor:

$$e^{-\frac{i\pi}{6\tau}\left(n-\frac{1}{n}\right)\left(\frac{l}{L}\right)^2}$$

which cancels the previous factor. In the end we remain with the multiplicative factor

$$|\tau|^{\frac{1}{6}\left(n-\frac{1}{n}\right)}$$

Thus under the modular transformation $\tau \to -\frac{1}{\tau}$, both replica partition functions transform as:

$$Z_n^{u,c}(\beta, L; il) = \left(\frac{\beta}{L}\right)^{\frac{1}{6}\left(n-\frac{1}{n}\right)} Z_n^{u,c}(L, \beta; l) \tag{3.28}$$

Perhaps surprisingly, the replica partition function for the torus theory is not modular invariant due to the multiplicative pre-factor. This factor vanishes at $n = 1$. The power of the prefactor is $\frac{c}{6}\left(n - \frac{1}{n}\right)$, an expression that appears frequently in the context of the Rényi entropy (in the above case, $c = 1$).

Now under this same modular transformation, the Rényi entropy shifts as:

$$S_n^{u,c}(\beta, L) = -\frac{n+1}{6n} \log \frac{\beta}{L} + S_n(L, \beta) \tag{3.29}$$

and hence even the entanglement entropy would shift by an additive term.

To have the quantum entropies transforming invariantly under modular transformations, we can modify the prescription for computing the replica partition functions by multiplying them by an external (l-independent) factor that renders them modular invariant:

$$\tilde{Z}_n^{\ u,c} = \left(\frac{\beta}{L}\right)^{\frac{1}{12}\left(n-\frac{1}{n}\right)} Z_n^{u,c} \tag{3.30}$$

The modular-invariant Rényi entropies are then given by:

$$\tilde{S}_n^{\ u,c} = \frac{1}{1-n} \log \frac{\tilde{Z}_n^{\ u,c}}{(Z_1)^n} \tag{3.31}$$

which lead to modular invariant entanglement entropies.

We have thus obtained modular invariant expression for the entanglement entropy of free Dirac fermions at finite temperature with a finite size entangling interval.

3.5 Thermal entropy relation

We see in [25] a relation between small and large interval limit of entanglement entropy and thermal entropy, which we call the thermal entropy relation:

$$\lim_{l\to 0}\Big(S(L-l)-S(l)\Big) = S_{thermal} \tag{3.32}$$

where

$$S_{thermal} = \beta^2 \frac{\partial}{\partial \beta}\left(-\frac{1}{\beta}\log Z_1\right) \tag{3.33}$$

This relation will be useful in determining which of the two Rényi entropies $S_n^{u,c}$ is the correct one. The overall factor used to make the replica partition functions modular-invariant is independent of l and hence drops out of the thermal entropy relation. Now consider the small l behavior of the two types of replica partition functions. As seen in equation (3.23) $Z_n^u \to (Z_1)^n$ but $Z_n^c \nrightarrow (Z_1)^n$ (upto an overall factor of l). It is a physical requirement (see [29]) that for small intervals the replica partition function should indeed tend to $(Z_1)^n$ times the given power of l. Thus, the correct Rényi entropy for small interval should be S_n^u and not S_n^c.

Next, consider the large interval limit $l \to L$. In this case, it has been predicted on general grounds that the replica partition function should tend to $Z_1(n\tau)$ [34], [35], apart from the same power of l as $l \to 0$. Observe that:

$$Z_n^c(l\to L) = \frac{1}{2}\left(\frac{l}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \sum_{\nu=1}^{4} \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left|\frac{\theta_\nu(\frac{k}{n}|\tau)}{\eta(\tau)}\right|^2 \tag{3.34}$$

We now use an identity involving θ functions shifted by a fraction:

$$\prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left|\theta_\nu\left(\frac{k}{n}-z\middle|\tau\right)\right| = \left(\prod_{p=1}^{\infty}\left|\frac{(1-q^{2p})^n}{1-q^{2pn}}\right|\right)|\theta_\nu(nz|n\tau)| \tag{3.35}$$

where $\nu = 2,3,4$. This can be easily derived using the product representation of θ functions. Using this,

$$Z_n^c(l\to L) = \frac{1}{2}\left(\frac{l}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \sum_{\nu=1}^{4}\left|\frac{\theta_\nu(0|n\tau)}{\eta(n\tau)}\right|^2 = \left(\frac{l}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} Z_1(n\tau) \tag{3.36}$$

Thus for large interval Z_n^c is the correct Rényi entropy and not Z_n^u, as $Z_n^u \nrightarrow Z_1(n\tau)$. Hence we are led to propose that the correct modular-invariant partition function for the free Dirac fermion CFT is Z_n^u at small l and is Z_n^c at large l. We then have S_n^u as the correct Rényi entropy at small l and S_n^c at large l. Also,

$$\begin{aligned}\lim_{l\to 0}\Big(S(L-l)-S(l)\Big) &= \lim_{n\to 1}\frac{1}{1-n}\log\left(\frac{Z_1(n\tau)}{(Z_1(\tau))^n}\right)\\ &= \log\left\{Z_1(\frac{\beta}{L})\right\}-\frac{\beta}{L}\frac{Z_1'(\frac{\beta}{L})}{Z_1(\frac{\beta}{L})}\\ &= S_{thermal}\end{aligned} \tag{3.37}$$

The key conclusion of this section is that neither S_n^u nor S_n^c is the correct Rényi entropy for finite l.

3.6 Central charge $c = 2$ theories

Consider a free conformal field theory of two Dirac fermions with correlated spin structures. This theory has the ordinary partition function:

$$Z_1[4]=\frac{1}{2}\sum_{\nu=2}^{4}\left|\frac{\theta_\nu(0|\tau)}{\eta(\tau)}\right|^4 \tag{3.38}$$

where we are using the notation:

$$Z_1[m]=\frac{1}{2}\sum_{\nu=2}^{4}\left|\frac{\theta_\nu(0|\tau)}{\eta(\tau)}\right|^m \tag{3.39}$$

The fundamental fields of this theory are two Dirac fermion fields, which we call $D_1(z)$ and $D_2(z)$. By definition, these have Hermitian conjugates $D_i^\dagger(z)$ that are also holomorphic. Combining with their anti-holomorphic components counterparts, the physical operators in the theory of conformal dimensions $\left(\frac{1}{2},\frac{1}{2}\right)$ are:

$$D_i\bar{D}_j,\, D_i^\dagger\bar{D}_j,\, D_i\bar{D}_j^\dagger,\, D_i^\dagger\bar{D}_j^\dagger$$

where i,j run independently over 1, 2. We thus have 16 local operators of dimensions $\left(\frac{1}{2},\frac{1}{2}\right)$. Had the spin structures not been correlated, the allowed operators would only be those with $i=j$.

This $c=2$ theory can be bosonized into two compact bosons ϕ_1,ϕ_2 that are orthogonal to each other and are compactified at the self-dual radius, which in our conventions is $R=\sqrt{2}$. Thus their periodicity is:

$$\phi_i \to \phi_i + 2\sqrt{2}\pi \tag{3.40}$$

If $\phi_i(z)$ are the two holomorphic components of the bosons and $\bar{\phi}_i(\bar{z})$ the anti-holomorphic components, the Dirac fields are given by:

$$D_1(z) = e^{\frac{i}{\sqrt{2}}\phi_1(z)}\, e^{\frac{i}{\sqrt{2}}\phi_2(z)}, \quad D_2(z) = e^{\frac{i}{\sqrt{2}}\phi_1(z)}\, e^{-\frac{i}{\sqrt{2}}\phi_2(z)} \tag{3.41}$$

The Hermitian conjugates have the same expressions but with $i \to -i$, while the anti-holomorphic conjugates have $\phi_i \to \bar{\phi}_i$ without any change of sign on i.

The allowed general vertex operator in the theory is:

$$\mathcal{O}(z,\bar{z}) = e^{\frac{i}{\sqrt{2}}(e_j+m_j)\phi_j(z)}\, e^{\frac{i}{\sqrt{2}}(e_k-m_k)\bar{\phi}_k} \tag{3.42}$$

where the indices j,k are independently summed over 1, 2.

The 16 local operators obtained after combining the holomorphic and anti-holomorphic components of the Dirac fermions correspond to all pairs of orthogonal vectors of $(\text{length})^2 = 2$ in the unit 2 dimensional square lattice:

$$\begin{aligned}[(e_1,e_2);(m_1,m_2)] \;=\; & \pm[(1,1);(0,0)], \pm[(1,-1);(0,0)], \pm[(0,0);(1,1)], \pm[(0,0);(1,-1)] \\ & \pm[(1,0);(0,1)], \pm[(1,0);(0,-1)], \pm[(0,1);(1,0)], \pm[(0,1);(-1,0)]\end{aligned}$$

In order to find the replica partition function, we need to find the twist fields which satisfy equations

$$\sigma_k(z,\bar{z})D_i(z') \sim (z-z')^{\frac{k}{n}}\ , \qquad \sigma_k(z,\bar{z})\bar{D}_i(\bar{z'}) \sim (\bar{z}-\bar{z'})^{-\frac{k}{n}}$$

From these equations, we see that the monodromy induced by the twist field is:

$$\begin{aligned} e^{\frac{i}{\sqrt{2}}\phi_1(z)}\, e^{\frac{i}{\sqrt{2}}\phi_2(z)} &\to e^{\frac{2\pi i k}{n}} e^{\frac{i}{\sqrt{2}}\phi_1(z)}\, e^{\frac{i}{\sqrt{2}}\phi_2(z)} \\ e^{\frac{i}{\sqrt{2}}\phi_1(z)}\, e^{-\frac{i}{\sqrt{2}}\phi_2(z)} &\to e^{\frac{2\pi i k}{n}} e^{\frac{i}{\sqrt{2}}\phi_1(z)}\, e^{-\frac{i}{\sqrt{2}}\phi_2(z)} \end{aligned}$$

From the above equations, we see that the twist field must leave the boson ϕ_2 inert (otherwise it cannot introduce the same monodromy on both $D_1(z)$ and $D_2(z)$). Also the twist field must shift the boson ϕ_1 by a fraction $\frac{k}{n}$ of its period:

$$\phi_1 \to \phi_1 + 2\sqrt{2}\pi\frac{k}{n} \tag{3.43}$$

Then, we see that the twist field should be:

$$\sigma_k(z,\bar{z}) = e^{\sqrt{2}i\frac{k}{n}\phi_1(z)}\, e^{-\sqrt{2}i\frac{k}{n}\bar{\phi}_1(\bar{z})} \tag{3.44}$$

If we observe closely, we will see that the twist field here is the same as that in the $c = 1$ theory, except that it is evaluated at radius $R = \sqrt{2}$. Due to this change in the radius, the conformal dimension is $\Delta_k = \frac{k^2}{n^2}$, which is twice that in the case of the single Dirac fermion. This ensures that $\sum_k \Delta_k = \frac{c}{24}\left(n - \frac{1}{n}\right)$ with $c = 2$, as required.

To calculate the two-point correlator of the twist field, we use the equation (3.17) and get:

$$\langle\langle \sigma_k(z,\bar{z})\sigma_{-k}(0,0)\rangle\rangle = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{4k^2}{n^2}} \frac{1}{|\eta(\tau)|^4} \sum_{e_i,m_i} q^{2\left(\Delta_{e_1,m_1}+\Delta_{e_2,m_2}\right)} \bar{q}^{2\left(\bar{\Delta}_{e_1,m_1}+\bar{\Delta}_{e_2,m_2}\right)} e^{4\pi i \frac{kl}{nL} e_1}$$

where $\Delta_{e_i,m_i} = \frac{1}{2}\left(\frac{e_i+m_i}{\sqrt{2}}\right)^2$, $\bar{\Delta}_{e_i,m_i} = \frac{1}{2}\left(\frac{e_i-m_i}{\sqrt{2}}\right)^2$ and $\eta(\tau)$ is raised to 4^{th} power because there are two bosons whose spectra are to be summed over. Also, consider only the sum in the above expression, which becomes:

$$F(k) \equiv \sum_{e_i,m_i} q^{\frac{1}{2}\left((e_1+m_1)^2+(e_2+m_2)^2\right)} \bar{q}^{\frac{1}{2}\left((e_1-m_1)^2+(e_2-m_2)^2\right)} e^{4\pi i \frac{kl}{nL} e_1}$$

Define $e_1 = \frac{n_1+n_2+n_3+n_4}{2}$, $e_2 = \frac{n_1-n_2-n_3+n_4}{2}$, $m_1 = \frac{n_1-n_2+n_3-n_4}{2}$ and $m_2 = \frac{n_1+n_2-n_3-n_4}{2}$. Since all e_i, m_i are integers, all n_i are either integers or half-integers and $\sum_i n_i$ should be even. Then $F(k)$ becomes:

$$\begin{aligned} F(k) = & \sum_{n_i\in\mathbb{Z}} \left[\frac{1+(-1)^{\sum_i n_i}}{2}\right] q^{(n_1^2+n_3^2)} \bar{q}^{(n_2^2+n_4^2)} e^{2\pi i \frac{kl}{nL}\sum_i n_i} \\ & + \sum_{n_i\in\mathbb{Z}+\frac{1}{2}} \left[\frac{1+(-1)^{\sum_i n_i}}{2}\right] q^{(n_1^2+n_3^2)} \bar{q}^{(n_2^2+n_4^2)} e^{2\pi i \frac{kl}{nL}\sum_i n_i} \end{aligned}$$

Using the definitions of theta functions given in Appendix A, we see that the above sums become (in that order):

$$F(k) = \frac{1}{2}\left[|\theta_3(z|\tau)|^4 + |\theta_4(z|\tau)|^4 + |\theta_1(z|\tau)|^4 + |\theta_2(z|\tau)|^4\right]$$

where $z = \frac{kl}{nL}$.

Hence, the two-point correlation function of the twist fields is:

$$\langle\langle\, \sigma_k(z,\bar{z})\, \sigma_{-k}(0,0)\,\rangle\rangle = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{4k^2}{n^2}} \times \frac{1}{2}\sum_{\nu=1}^{4} \frac{\left|\theta_\nu\left(\frac{kl}{nL}|\tau\right)\right|^4}{|\eta(\tau)|^4} \tag{3.45}$$

The uncorrelated replica partition function, is given by product over all k of the above expression:

$$Z_n^u(L,\beta;l) = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{c}{6}\left(n-\frac{1}{n}\right)} \prod_{k=-\frac{(n-1)}{2}}^{\frac{(n-1)}{2}} \frac{1}{2}\sum_{\nu=1}^{4} \frac{\left|\theta_\nu\left(\frac{kl}{nL}|\tau\right)\right|^4}{|\eta(\tau)|^4} \tag{3.46}$$

with central charge $c = 2$.

The correlated replica partition function, on the other hand, is given by the same product as above but the sum over ν taken later:

$$Z_n^c(L,\beta;l) = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{c}{6}\left(n-\frac{1}{n}\right)} \frac{1}{2}\sum_{\nu=1}^{4} \prod_{k=-\frac{(n-1)}{2}}^{\frac{(n-1)}{2}} \frac{\left|\theta_\nu\left(\frac{kl}{nL}|\tau\right)\right|^4}{|\eta(\tau)|^4} \tag{3.47}$$

The two Rényi entropies are easily obtained from these and so are the entanglement entropies in the respective limits $l \to 0$ and $l \to L$.

As before, we can modify both the replica partition functions by multiplying with the pre-factor $\left(\frac{\beta}{L}\right)^{\frac{c}{6}\left(n-\frac{1}{n}\right)}$ (with $c = 2$), to make them modular invariant. The entropies thus obtained are modular invariant too.

Here, we would like to mention that although this result looks like a straightforward generalization of that obtained for a single Dirac fermion, it does not actually follow directly from it. The reason is that one needs a bosonic dual theory in order to write the twist field. For two Dirac fermions with correlated spin structures, the bosonic dual is not simply two copies of that for the single Dirac fermion. Rather the bosonic dual is two copies of bosons, compactified at $R = \sqrt{2}$, a different CFT with a completely different spectrum of operators.

In the next chapter, we will present a conjecture for the replica partition function for any number of free Dirac fermions at finite temperature in 1+1 dimensions.

Chapter 4

Modular invariance and quantum entropies: 2

4.1 A conjecture for any number of fermions

Consider a 1+1 dimensional CFT of m free Majorana fermions (or $\frac{m}{2}$ Dirac fermions, where m is even) at finite temperature β^{-1} and in a system of finite spatial size L. Let an interval $[0, l]$ be the entangling interval for the study of entanglement entropy. This theory has a central charge $c = \frac{m}{2}$.

Seeing the success of our procedure, we propose that the correlated replica partition function for this theory is:

$$Z_n^c[m] = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{m}{12}\left(n-\frac{1}{n}\right)} \frac{1}{2}\sum_{\nu=1}^{4}\prod_{k=-\frac{(n-1)}{2}}^{\frac{(n-1)}{2}} \frac{\left|\theta_\nu\left(\frac{kl}{nL}|\tau\right)\right|^m}{|\eta(\tau)|^m} \tag{4.1}$$

and the uncorrelated replica partition function for this theory is:

$$Z_n^u[m] = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{m}{12}\left(n-\frac{1}{n}\right)} \prod_{k=-\frac{(n-1)}{2}}^{\frac{(n-1)}{2}} \frac{1}{2}\sum_{\nu=1}^{4} \frac{\left|\theta_\nu\left(\frac{kl}{nL}|\tau\right)\right|^m}{|\eta(\tau)|^m} \tag{4.2}$$

We have verified the correctness of the above proposal for $m = 2, 4$.

We emphasize that our expressions for the replica partition functions are conjectural in nature and does not have direct derivations without constructing the twist fields in terms of bosons.

Furthermore, we also propose the usual modification of the replica partition functions to render them modular invariant. In the above case the modification term will be: $\left(\frac{\beta}{L}\right)^{\frac{m}{24}\left(n-\frac{1}{n}\right)}$.

4.2 Multiple correlated fermions and lattice bosons

For $d \geq 3$, the theory of $m = 2d$ free Majorana fermions (equivalently, d Dirac fermions) with correlated spin structures (on the original copy of the torus) is dual to a specific compactification of d free bosons on a target-space torus:

$$T^c = \mathbb{R}^d / \Gamma_d$$

where Γ_d is the root lattice of $Spin(2d)$ at self-dual radius $R = \sqrt{2}$. This root lattice can be achieved by starting with a rectangular torus and choosing a suitable constant metric and B-field. The ordinary partition function is well-known and has been directly computed within both the fermion as well as boson representation [30]. It is given by $Z_1[m]$ in terms of the definitions in equation (3.39).

We first identify the bosonic representation of dimensions $\left(\frac{1}{2}, \frac{1}{2}\right)$ operators in the theory. These operators are:

$$D_p \bar{D}_q \ , \ D_p^\dagger \bar{D}_q \ , \ D_p \bar{D}_q^\dagger \ , \ D_p^\dagger \bar{D}_q^\dagger$$

where $p, q = 1, 2, \cdots, d$. In the free boson theory, let Λ_R be the root lattice and Λ_W be the dual weight lattice. Then the vertex operators are:

$$\mathcal{O}_{w^i, \bar{w}^i} = e^{i w^i \phi_i} \, e^{i \bar{w}^i \bar{\phi}_i} \tag{4.3}$$

where $w^i, \bar{w}^i \in \Lambda_W$ and $w^i - \bar{w}^i \in \Lambda_R$. Elements of the weight lattice can be parametrized as:

$$w^i = \frac{1}{\sqrt{2}} g^{ij} v_j, \quad \bar{w}^i = \frac{1}{\sqrt{2}} g^{ij} \bar{v}_j \tag{4.4}$$

where $v_i, \bar{v}_j$ are integer column vectors and g^{ij} is the inverse of g_{ij} which itself is half of the Cartan matrix of $Spin(2d)$. For the difference of w^i and $\bar{w}^i$ to lie in the root lattice Λ_R, we must require that $\frac{1}{\sqrt{2}}(v_i - \bar{v}_i) = \sqrt{2} n_i$ where n_i are integers.

The two-point function of the bosons is:

$$\langle \phi_i(z, \bar{z}) \phi_j(z', \bar{z}') \rangle = -g_{ij} \log |z - z'|^2 \tag{4.5}$$

Hence, the conformal dimension of the above operators is:

$$\left(\Delta_{w^i}, \bar{\Delta}_{\bar{w}^i}\right) = \frac{1}{2} \left(g_{ij} w^i w^j, g_{ij} \bar{w}^i \bar{w}^j\right) \tag{4.6}$$

Thus to reconstruct the fermion operators, we must look for pairs of points of unit length in the weight lattice that differ by an element of the root lattice. If $\vec{\alpha}_i$ are the d simple roots of $Spin(2d)$ and $\vec{\lambda}^i$ are the fundamental weights then:

$$\vec{\alpha}_i \cdot \vec{\lambda}^j = \delta_i^j, \quad g_{ij} = \frac{1}{2}\vec{\alpha}_i \cdot \vec{\alpha}_j, \quad g^{ij} = 2\vec{\lambda}^i \cdot \vec{\lambda}^j \tag{4.7}$$

One also has the dual relation:

$$\left(\vec{\lambda}^i\right)_p (\vec{\alpha}_i)_q = \delta_{pq} \tag{4.8}$$

which will be important in what follows. Here, p, q label the individual components of each vector, and the sum is over the vectors (not components).

For the conformal dimensions, we have:

$$\Delta_{w^i} = \frac{1}{2} g_{ij} w^i w^j = \frac{1}{4} g^{ij} v_i v_j \tag{4.9}$$

Therefore to find operators of dimension $\left(\frac{1}{2}, \frac{1}{2}\right)$, we must look for sets of integers v_i for which:

$$g^{ij} v_i v_j = 2 \tag{4.10}$$

From the above equations, it follows that:

$$g^{ij} (\vec{\alpha_i})_p (\vec{\alpha_j})_q = 2\delta_{pq} \tag{4.11}$$

hence the possible v_i are given by:

$$v_i^{(p)} = (\vec{\alpha_i})_p, \quad p = 1, 2, \cdots, d \tag{4.12}$$

For the anti-holomorphic part we start by picking it independently from the same set: $\bar{v}_i^{(q)} = (\vec{\alpha_i})_q$. However, we need to ensure that $\frac{1}{\sqrt{2}} g^{ij} \left(v_j^{(p)} - \bar{v}_j{}^{(q)}\right)$ is $\sqrt{2}$ times an integer. This is gauranteed by the fact that

$$\frac{1}{\sqrt{2}} g^{ij} \left(v_j^{(p)} - \bar{v}_j{}^{(q)}\right) = \frac{1}{\sqrt{2}} g^{ij} \left[(\vec{\alpha_j})_p - (\vec{\alpha_j})_q\right] = \sqrt{2}\left[\left(\vec{\lambda}^j\right)_p - \left(\vec{\lambda}^j\right)_q\right] \tag{4.13}$$

which is indeed in the root lattice. We conclude that the fermion operators are given in bosonic language by :

$$D_p(z)\bar{D}_q(\bar{z}) \to \mathcal{O}_{p,q}(z, \bar{z}) = e^{iw^{(p)i}\phi_i(z)}\, e^{i\bar{w}^{(q)j}\bar{\phi}_j(\bar{z})} \tag{4.14}$$

where $w^{(p)i} = \sqrt{2}(\vec{\lambda}^i)_p$.

We can now look for the twist field, an operator σ_k inducing the monodromy:

$$\sigma_k : D_p(z) \to e^{\frac{2\pi i k}{n}} D_p(z) \tag{4.15}$$

corresponding to a shift:

$$w^{(p)i}\phi_i(z) \to w^{(p)i}\phi_i(z) + \frac{2\pi k}{n} \tag{4.16}$$

This in turn will be induced by a shift $\phi_i \to \phi_i + 2\pi\zeta_i^{(k)}$ where $\zeta_i^{(k)}$ is a constant vector satisfying:

$$w^{(p)i}\zeta_i^{(k)} = \frac{k}{n} \tag{4.17}$$

for all p. Recalling that the last weight of $Spin(2d)$ is $\lambda^{(d)} = \left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right)$, we easily find that the shift is given by:

$$\zeta_i^{(k)} = \frac{\sqrt{2}k}{n}(0, 0, \cdots, 1) \tag{4.18}$$

Thus the twist field only shifts the last boson ϕ_d. Indeed, the twist field has the explicit form:

$$\sigma_k(z, \bar{z}) = \mathcal{O}_{\zeta^{(k)i}, -\zeta^{(k)i}} = e^{i\zeta^{(k)i}\phi_i(z)}\, e^{-i\zeta^{(k)i}\bar{\phi}_i(\bar{z})} \tag{4.19}$$

where one must keep in mind that $\zeta^{(k)i} = g^{ij}\zeta_j^{(k)}$.

To check that this expression for the twist field is correct, we compute its OPE with $D_p(z)$ to find:

$$\sigma_k(z, \bar{z})e^{iw^{(p)i}\phi_i(z')} \sim (z - z')^{\zeta^{(k)i}g_{ij}w^{(p)j}} = (z - z')^{w^{(p)i}\zeta_i^{(k)}} = (z - z')^{\frac{k}{n}} \tag{4.20}$$

The crucial test of our twist field is whether or not it has the desired conformal dimensions. From the expression for the twist field, its (chiral) conformal dimension is:

$$\Delta_k = \frac{1}{2}g_{ij}\zeta^{(k)i}\zeta^{(k)j} = \frac{1}{2}g^{ij}\zeta_i^{(k)}\zeta_j^{(k)} = \frac{k^2}{n^2}g^{dd} \tag{4.21}$$

Using $g^{dd} = 2\lambda^{(d)} \cdot \lambda^{(d)} = \frac{d}{2}$, we get

$$\Delta_k = \frac{dk^2}{2n^2} \tag{4.22}$$

from which it follows that:

$$\sum_{k=-\frac{(n-1)}{2}}^{\frac{(n-1)}{2}} \Delta_k = \frac{d}{24}\left(n - \frac{1}{n}\right) \tag{4.23}$$

as expected.

Recall that the ordinary partition function of these theories is [?]:

$$
\begin{aligned}
Z_1[2d] &= \frac{1}{|\eta(\tau)|^{2d}} \sum_{w,\bar{w}\in\Lambda_W;w-\bar{w}\in\Lambda_R} q^{w^2}\,\bar{q}^{\bar{w}^2} \\
&= \frac{1}{2}\frac{1}{|\eta(\tau)|^{2d}} \sum_{\nu=2}^{4} |\theta_\nu(0|\tau)|^{2d}
\end{aligned}
\tag{4.24}
$$

where, as usual, $w^2 = g_{ij}w^iw^j$ and similarly for $\bar{w}^2$. The equivalence between the first and the second lines of the above equation is the statement of the Bose-Fermi equivalence between the lattice bosons and the multiple fermions with correlated spin structures. To demonstrate this, one starts with the observation that for $Spin(2d)$, $\Lambda_W = \Lambda_R \oplus \Lambda_V \oplus \Lambda_S \oplus \Lambda_C$ where $\Lambda_{V,S,C}$ are the lattices obtained by shifting Λ_R by a fundamental weight in the vector, spinor and the conjugate spinor representations respectively.

Now we split a generic vector $w \in \Lambda_W$ into two classes: those that lie in $\Lambda_R \cup \Lambda_V$ and those in $\Lambda_S \cup \Lambda_C$. In the former set we can write $w^i = \sqrt{2}\sum_p n_p(\vec{\lambda^i})_p$ for arbitrary integers n_p, while in the latter set, we can write $w^i = \sqrt{2}\sum_p \left(n_p + \frac{1}{2}\right)(\vec{\lambda^i})_p$, where n_p are again arbitrary integers. We similarly write $\bar{w}^i = \sqrt{2}\sum_p m_p(\vec{\lambda^i})_p$ and $\bar{w}^i = \sqrt{2}\sum_p \left(m_p + \frac{1}{2}\right)(\vec{\lambda^i})_p$ in the two respective sets. Finally, the restriction $w - \bar{w} \in \Lambda_R$ is implemented by insering a projection operator that causes both w and $\bar{w}$ to lie in Λ_R or in Λ_V (in the first set) or both to lie in Λ_S or in Λ_C (in the second set). In this way, we end up with four Jacobi theta functions $\theta_\nu(0|\tau), \nu = 1, 2, 3, 4$ of which $\theta_1(0|\tau)$ identically vanishes.

Next we turn towards calculating the two-point correlator of the twist fields. For the un-normalized two-point function of twist fields, we find:

$$
\langle\langle\, \sigma_k(z,\bar{z})\,\sigma_{-k}(0,0)\,\rangle\rangle = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{2dk^2}{n^2}} \frac{1}{|\eta(\tau)|^{2d}} \sum_{\substack{w,\bar{w}\in\Lambda_W,\\ w-\bar{w}\in\Lambda_R}} q^{w^2}\bar{q}^{\bar{w}^2}\; e^{2\pi i\frac{l}{L}g_{ij}(w^i+\bar{w}^i)\zeta^{(k)j}}
\tag{4.25}
$$

We now make the following redefinition of w^i and $\bar{w}^i$:

$$
\begin{aligned}
g_{ij}(w^i+\bar{w}^i)\zeta^{(k)j} &= (w^i+\bar{w}^i)\zeta_i^{(k)} = \frac{\sqrt{2}k}{n}(w^d+\bar{w}^d) \qquad (4.26)\\
&= \frac{k}{n}\sum_{p=1}^{d}(n_p+m_p), \quad w,\bar{w}\in\Lambda_R\cup\Lambda_V\\
&= \frac{k}{n}\sum_{p=1}^{d}(n_p+m_p+1), \quad w,\bar{w}\in\Lambda_S\cup\Lambda_C
\end{aligned}
$$

Then the exponents of q , $\bar{q}$ and e become squares of integers and half-integers respectively. Using the definitions of theta functions:

$$
\langle\langle\, \sigma_k(z,\bar{z})\, \sigma_{-k}(0,0)\, \rangle\rangle = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{2dk^2}{n^2}} \frac{1}{2}\sum_{\nu=1}^{4}\left|\frac{\theta_\nu\left(\frac{kl}{nL}|\tau\right)}{\eta(\tau)}\right|^{2d} \qquad (4.27)
$$

Taking the product over k, we get the uncorrelated replica partition function:

$$
Z_n^u = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{d}{24}\left(n-\frac{1}{n}\right)} \prod_{k=-\frac{(n-1)}{2}}^{\frac{(n-1)}{2}} \frac{1}{2}\sum_{\nu=1}^{4}\left|\frac{\theta_\nu\left(\frac{kl}{nL}|\tau\right)}{\eta(\tau)}\right|^{2d} \qquad (4.28)
$$

If we product over replicas first and sum over spin structures later, we get the correlated replica partition function:

$$
Z_n^c = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{d}{24}\left(n-\frac{1}{n}\right)} \frac{1}{2}\sum_{\nu=1}^{4}\prod_{k=-\frac{(n-1)}{2}}^{\frac{(n-1)}{2}} \left|\frac{\theta_\nu\left(\frac{kl}{nL}|\tau\right)}{\eta(\tau)}\right|^{2d} \qquad (4.29)
$$

Then, the Rényi and entanglement entropies of this theory are easily obtained from the above expressions. We can, as usual, modify the above replica partition functions by multiplying by a prefactor $\left(\frac{\beta}{L}\right)^{\frac{m}{24}\left(n-\frac{1}{n}\right)}$. The Rényi and entanglement entropies thus obtained will be modular invariant.

4.3 Discussion

In this chapter we presented proposals for modular invariant replica partition functions in the limits $l \to 0$ and $l \to L$, of any number of correlated Dirac fermions. Using these, modular invariant Rényi and entanglement entropies are easily calculated.

The thermal entropy relation, which has independent evidence from holography, where quite often the thermal entropy $S_{\text{thermal}} = S_{BH}$ is the entropy of the black hole in the AdS bulk, proved very important in deciding which Rény entropy is correct in different limits. We have verified that our proposal as to which replica partition function is to be used in the two limits $l \to 0$ and $l \to L$ satisfies the thermal entropy relation.

The conjecture we have made fulfills several criteria:

1. entanglement entropy converges to $\frac{c}{3}\log\frac{l}{L}$ in the $l \to 0$ limit, in line with [29];

2. the replica partition function (without the repair term) is modular covariant as in [32] (please see the next chapter for details);

3. the entanglement entropy in limits when temperature vanishes ($q \to 0$) and when the spatial size of the system diverges ($L \to \infty$) reduces to known expressions described in Section [1.4].

4. the thermal entropy relation holds true.

Now, it is interesting to find the modular invariant replica partition function at finite l. For a generic positive integer n, one can write a variety of partition functions. Consider the set $S = \{k_1, k_2, \cdots, k_n\}$ where $k_i = -\frac{n-1}{2} + i - 1$. Suppose that for a fixed value k_i and a fixed spin structure ν, we have a replica partition function $Z_n^{(\nu)}(k_i)$. Partition the set S into subsets $S_1, S_2, \cdots, S_r$ having m_j elements of S in the j^{th} set and denote this partition by Y_i where the Y is a reminder for the often used combinatoric device Young diagram . Now we can define the replica partition function for the Young diagram Y_i:

$$Z_n^{Y_i} = \prod_j \left(\sum_{\nu=1}^{4} \prod_{k \in S_j} Z_n^{(\nu)}(k) \right) \tag{4.30}$$

Here, the fermion spin structures are correlated within each subset S_j but uncorrelated across different subsets. One can easily verify that the replica partition function for an arbitrary Young diagram is modular-covariant, with a pre-factor that can be eliminated as usual. We have seen two particulars cases of Young diagrams - one where each subset contains a single element (this gives us the uncorrelated replica partition function) and the other where all the elements are in the single subset (this gives us the correlated replica partition function). Then, we conjecture that the full replica partition function is a linear combination (with l-dependent coefficients) over all Young diagrams:

$$Z_n = \sum_{\text{all partitions}} a_{Y_i}\left(\frac{l}{L}\right) Z_n^{Y_i} \tag{4.31}$$

Each Young diagram corresponds to one or more number of Young tableaux, depending on different permutations of the coefficients k in equation (4.30). The sum over these tableaux is implied. The physical intuition behind the above expression is that as the branch cut of length l expands from 0 to L, the degree of correlation among replicas steadily increases. The coefficient a must be such that only the terms corresponding to Z_n^u, Z_n^c contribute at $l = 0, L$ respectively.

We observe that as an application of the above ideas, one can use entanglement in more general CFTs at finite size and temperature. In the next chapter we present the expression for the Rényi entropy of a free boson on a torus, with compactification radius R. The CFT of the free boson is equivalent to that of a single Dirac fermion on a torus if the boson is compactified at radius $R = 1$. We check the modular covariance of the replica partition function of the free boson at radius R. This provides a piece of independent evidence for the modular covariance of the replica partition function of the Dirac fermion proposed by us (when we specialize to $R = 1$). Moreover, the replica partition function for the free boson at radius R is manifestly T-duality invariant.

Chapter 5

Compact Bosons

5.1 Replica partition function for compact bosons

We recall that the ordinary partition function for a single boson at compactification radius R is:

$$
\begin{aligned}
Z_1(R) &= \frac{1}{|\eta(\tau)|^2} \sum_{e,m} q^{\left(\frac{e}{R}+\frac{mR}{2}\right)^2} \bar{q}^{\left(\frac{e}{R}-\frac{mR}{2}\right)^2} \\
&= \frac{1}{|\eta(\tau)|^2} \sum_{e,m} q^{\left(\frac{2e^2}{R^2}+\frac{m^2R^2}{2}\right)}
\end{aligned}
\tag{5.1}
$$

where we have used the fact that $q = e^{-\pi\tau_2}$.

To calculate the replica partition function, we have to compute the two-point correlators of the twist fields $\mathcal{T}_k$ (with $k = 0, 1, \cdots, n-1$) which are defined by:

$$
\mathcal{T}_k(z,\bar{z})\phi(w) \sim (z-w)^{\frac{k}{n}}, \quad \mathcal{T}_k(z,\bar{z})\bar{\phi}(\bar{w}) \sim (\bar{z}-\bar{w})^{-\frac{k}{n}} \tag{5.2}
$$

and one has:

$$
Z_n = \prod_{k=0}^{n-1} \langle\langle\, \mathcal{T}_k(z,\bar{z})\mathcal{T}_{-k}(0,0)\,\rangle\rangle \tag{5.3}
$$

An important part of the computation of this quantity was carrried out in [31] for a pair of free bosons compactified on a square torus of size R. The result is a product of a quantum and a classical part. It is rather implicit, involving integrals of products of fractional powers of theta functions which appear in the construction of the cut differentials of the torus. Unfortunately the classical part, which carries all the R dependence is not invariant under the $R \to \frac{\alpha'}{R}$ and so it does not satisfy T-duality. Indeed the classical part of the result of [31] has been corrected in [32]. Accordingly we will analyze the result in the latter reference and compare it with our proposal for modular invariance.

In our notation, $\alpha' = 2$. The replica partition function for a free boson compactified on a radius R, as presented in [32], is:

$$Z_n(R) = Z_n^{(1)} Z_n^{(2)} Z_n^{(3)}(R) Z_n^{(3)}\left(\frac{2}{R}\right) \tag{5.4}$$

where

$$\begin{aligned} Z_n^{(1)} &= \frac{1}{|\eta(\tau)|^{2n}} \prod_{k=0}^{n-1} \frac{1}{\left|W_1^1(k,n;\frac{l}{L}|\tau)\right|} \\ Z_n^{(2)} &= \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{l}{L}|\tau)}\right|^{\frac{1}{6}\left(n-\frac{1}{n}\right)} \\ Z_n^{(3)}(R) &= \sum_{m_j} \exp\left(-\frac{\pi R^2}{2n} \sum_{k=0}^{n-1} \left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right| \sum_{j,j'=0}^{n-1} \left[\cos 2\pi (j-j')\frac{k}{n}\right] m_j m_{j'}\right) \end{aligned} \tag{5.5}$$

We note that here the superscripts do not refer to spin structures and only the last two factors, which form the classical part of the full replica partition function, depend on the compactification radius R. T-duality invariance is already manifest at this stage. Also, W_1^1 and W_2^2 are:

$$\begin{aligned} W_1^1(k,n;\frac{l}{L}|\tau) &= \int_0^1 dz \theta_1(z|\tau)^{-\left(1-\frac{k}{n}\right)} \theta_1(z-\frac{l}{L}|\tau)^{-\frac{k}{n}} \theta_1(z-\frac{kl}{nL}|\tau) \\ W_2^2(k,n;\frac{l}{L}|\tau) &= \int_0^1 d\bar{z}\bar{\theta}_1(\bar{z}|\tau)^{-\frac{k}{n}} \bar{\theta}_1(\bar{z}-\frac{l}{L}|\tau)^{-\left(1-\frac{k}{n}\right)} \bar{\theta}_1(\bar{z}-(1-\frac{k}{n})\frac{l}{L}|\tau) \end{aligned} \tag{5.6}$$

The replica partition function given above converges to the ordinary partition function in the $n \to 1$ limit.

We now will investigate the modular transformation of the above replica partition function. We note that:

$$W_1^1(k,n;\frac{l}{L}|\tau) = \frac{1}{\tau} e^{-\frac{i\pi z^2}{\tau}\frac{k}{n}\left(1-\frac{k}{n}\right)} W_2^2(k,n;\frac{l}{L}|\tau) \tag{5.7}$$

which is the equation (B.41) of [31]. So, under a moodular transformation:

$$\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right| \to \left|\frac{W_1^1(k,n)}{W_2^2(k,n)}\right| \tag{5.8}$$

Following this with a multi-variable Poisson resummation as in [32], we get:

$$\begin{aligned} Z_n^{(3)}\left(R;\frac{z}{\tau}|-\frac{1}{\tau}\right) &= \frac{2^{\frac{n}{2}}}{R^n}\left(\prod_{k=0}^{n-1}\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right|^{\frac{1}{2}}\right) Z_n^{(3)}\left(\frac{2}{R};z|\tau\right) \\ Z_n^{(3)}\left(\frac{2}{R};\frac{z}{\tau}|-\frac{1}{\tau}\right) &= \frac{R^n}{2^{\frac{n}{2}}}\left(\prod_{k=0}^{n-1}\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right|^{\frac{1}{2}}\right) Z_n^{(3)}\left(R;z|\tau\right) \end{aligned} \tag{5.9}$$

Thus the product transforms as:

$$Z_n^{(3)}(R)Z_n^{(3)}(\frac{2}{R}) \to \left(\prod_{k=0}^{n-1}\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right|\right) Z_n^{(3)}(R)Z_n^{(3)}(\frac{2}{R}) \tag{5.10}$$

Putting everything together, we find that:

$$Z_n\left(R;\frac{z}{\tau}|-\frac{1}{\tau}\right) = |\tau|^{\frac{1}{6}\left(n-\frac{1}{n}\right)} Z_n(R;z|\tau) \tag{5.11}$$

which will become modular invariant upon multiplying by the half of the prefactor in the above equation.

5.2 Conclusions and outlook

We have found modular invariant expressions for Rényi and entanglement entropies for 1+1 dimensional free fermion CFTs. Since a CFT is well-defined only if it is modular invariant, our results provide a missing link between numerous results on entropies (which are non-modular invariant) and formal CFTs. There are several research problems which can be pursued from here:

1. What is the replica partition function at finite l ?
   To answer this question we need to find the coefficients $a\left(\frac{l}{L}\right)$. One way to go about finding these coefficients is to use combinatorial techniques. These line of search presents itself naturally as the partitions of the n replicas (with n a positive integer) are easily representable in terms of Young diagrams.

2. Does entanglement entropy repect Fermi-Bose duality?
   To find the replica partition function and the Rényi entropy of the free fermion CFTs, we had to use bosonization. Thus it is a hybrid calculation, where both the fermionic and the bosonic descriptions in the Fermi-Bose duality are used. The expression for the replica partition function for a free boson (equation (5.4)) is calculated purely in the bosonic picture [32]. We have checked that both the replica partition function for the free boson and that for the single Dirac fermion are equal and modular covariant (without the 'repair' term) with the same pre-factor. However, the expression in equation (5.4) is given in terms of integrals of theta functions whereas the single Dirac replica partition function equation (4.31) is given explicitly in terms of theta functions. It would be interesting to compare the former equation at $R = 1$ and the latter (when the coefficients a are determined) to see if they agree. If they do, it would be a strong evidence that Rényi and entanglement entropies respect Fermi-Bose duality.

3. What is the replica partition function and hence the entropies for minimal models?
   Minimal models are well-known CFTs with a finite number of operators. It would be interesting to find the entanglement entropy for minimal models as these are considered dual to higher spin AdS_3.

4. What does modular invariant entanglement entropy mean in the context of holography?
   In holography, using the famous Ryu-Takayanagi prescription [33], the entanglement entropy of the CFT is given by area of a minimal surface drooping in the AdS bulk. It would be interesting to see if the expression derived from holography is modular invariant or not and why.

5. How can one apply the results derived in this paper to the condensed matter problems being studied in holography?
   Recently, many condensed matter problems like strange metals, non-relativistic fluids etc. are being studied using holography as a tool. In the conventional application of CFT to statistical mechanics, modular invariance is an essential property. We would like to see if we can learn more about strange metals or higher spin fields or other interesting topics by taking modular invariant expressions for entropies.

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Appendix A

Definitions and conventions

A.1 Conventions

Our 1+1 dimensional theories are defined on a torus with a modular parameter τ and a spatial circle of perimeter L. The entangling interval has length l. The number R describes the radius of the scalar field compactification.

In our conventions, T-duality is implemented by $R \to \frac{2}{R}$ and hence the self-dual radius is $R = \sqrt{2}$. The number of replicas is denoted by n and we calculate n^{th} Rényi entropy. The finite temperature is β^{-1}.

A.2 Theta functions

We define the nome q as

$$q = e^{\pi i \tau}$$

where $|\tau| > 0$
Also, let

$$y = e^{2\pi i z}$$

As infinite series, the definitions of the Jacobi Theta functions are as follows :

$$\theta_1(z|\tau) = -\sum_{-\infty}^{\infty} (-1)^{(n+\frac{1}{2})} q^{\left(n+\frac{1}{2}\right)^2} y^{\left(n+\frac{1}{2}\right)} \tag{A.1}$$

$$\theta_2(z|\tau) = \sum_{-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} y^{\left(n+\frac{1}{2}\right)} \tag{A.2}$$

$$\theta_3(z|\tau) = \sum_{-\infty}^{\infty} q^{n^2} y^n \tag{A.3}$$

$$\theta_4(z|\tau) = \sum_{-\infty}^{\infty} (-1)^n q^{n^2} y^n \tag{A.4}$$

The Dedekind Eta function is defined as:

$$\eta(\tau) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n}) \tag{A.5}$$

We note the following identity which has been used in different sections of the report:

$$\theta_1'(0|\tau) = 2\eta^3(\tau) \tag{A.6}$$