Pricing in a semi-Markov modulated jump diffusion model

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by

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This is to certify that this dissertation entitled Pricing in a semi-Markov modulated jump diffusion model towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents the research carried out by Akash Krishna at Indian Institute of Science Education and Research under the supervision of Dr. Anindya Goswami, Assistant Professor, Department of Mathematics, during the academic year 2014-2015.

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Dedicated to Renu Joy

Declaration

I hereby declare that the matter embodied in the report entitled Pricing in a semi-Markov modulated jump diffusion model are the results of the investigations carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Anindya Goswami and the same has not been submitted elsewhere for any other degree.

Akash Krishna

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Abstract

In this thesis, we introduce a new market model for the stock price dynamics. It is a regime switching market where the parameters volatility and drift follows a semi-Markov process. In addition to that along with the diffusion process, we incorporate a term which give us the discontinuity in the market. We call this market model a semi-Markov modulated jump diffusion model. Apart from defining a market model by stochastic differential equation (SDE), we find the solution of this SDE. Then we derive the infinitesimal generator associated with this model so that some further investigations can be carried out. Finally we have shown that this model is arbitrage free.

Contents

Chapter 1

Introduction

Finding the value of option has always been a major concern in Mathematical Finance. In 1965, a famous economist named Samuelson found a model for the stock price dynamics called geometric Brownian motion model [33]. Eight years later, in 1973, Black, Scholes and Merton [34] used this model to find a formula for the price of European options. In their model, now known as B-S-M model, it is assumed that the basic market parameters such as volatility, drift, bank interest rate are constant during the entire period of the option. That is clearly not the case in the real market. For rectifying this assumptions people proposed and tested many different models.

Firstly, many studies introduce the regime-switching model supported by a finite state Markov chain to study the changing parameters depending on the state of the economy. For instance, [3, 4, 5] considered the regime switching models of the financial market. Secondly, some rare events may result in the rapid variations in asset prices and many papers resort to jump-diffusion models to discuss the effects. [11] and [31] studied different kinds of jump diffusion models. Unfortunately, an important property of regime switching model or jump diffusion model or their combination is the incompleteness of the financial market and[16, 17] had showed that there would be infinitely many equivalent measures in that kind of market. Ever since then, lots of researches have studied different methods to choose the pricing measures on basis of different objectives.

The purpose of this thesis is to introduce a new market model for stock price dynamics. Before we introduce the model, we will visit chronologically different models which are important and which have a significant improvements from the previous ones. For simplicity, we would often mention just special cases and we would describe the models by stochastic differential equations (SDE).

As we have seen earlier, it is all started with the famous B-S-M model, where the stock price follows a geometric Brownian motion as given below.

$$dS_t = S_t(\mu dt + \sigma dW_t),$$

where $S_0 > 0$, μ denotes the drift(expected return) and σ denotes the volatility of the asset(can be thought of as the standard deviation) and W_t is a Brownian motion.

Some years later, people came up with the regime switching model with the improved version of B-S-M, where μ and σ follow either a Markov process or a semi-Markov process . Markov modulated GBM is given below.

$$dS_t = S_t(\mu(X_t)dt + \sigma(X_t)dW_t),$$

where X_t is a Markov process. The above mentioned model appears in [1, 3, 4, 5, 6, 7, 8, 9].

Then [10] considers the semi-Markov modulated GBM

$$dS_t = S_t(\mu(X_t)dt + \sigma(X_t)dW_t),$$

where X_t is a semi-Markov process.

Some times some rare events may result in the rapid variations in asset prices. So a new model called jump diffusion has proposed to incorporate discontinuity in stock price dynamics.

$$dS_t = S_{t-}(\mu dt + \sigma dW_t + \int \eta(z)N(dt, dz)),$$

where $\eta : \mathbb{R} \to \mathbb{R}$ is continuous, bounded above and $\eta(z) > -1$, and $N(dt, dz)$ is a Poisson random measure with intensity measure $\nu(z)dt$, where ν is a finite Borel measure. [11] and [31] studied different kinds of jump diffusion models with some relaxed assumptions.

Another model which is an improved version of both regime switching and the jump diffusion is the jump diffusion model with regime switching.

$$dS_t = S_{t-}(\mu(X_{t-})dt + \sigma(X_{t-})dW_t + \int \eta(z)N(dt, dz))$$

In [12] and [13] the above kind of model appears.

The model which I am studying is the semi-Markov modulated jump diffusion model.

After specifying a market model, there are some important issues to be investigated. These are no arbitrage (NA), Completeness of the market, derivation of the price equation in the form of a partial differential equation, computation of the solution of the price equation, hedging computation etc. Addressing the above issues for the model in which I am working is not straight forward. And these are not fully answered in this thesis. I have found the strong solution of the stochastic differential equation of this model. And I derived the infinitesimal generator associated with this market model so that some of the above mentioned investigations can be carried out.

The rest of this thesis is arranged in the following manner. In Chapter 2, we discuss briefly about B-S-M model [28], Black-Scholes partial differential equation and its solution. Then we discuss about the arbitrage opportunities and completeness of the market [20]. Finally, a brief description of locally risk minimizing hedging in a general incomplete market is discussed [32]. In Chapter 3, we present the statements of the Itô's formula [22] for a right continuous with left limit [RCLL] path without proof. Apart from this, we describe the new model by stochastic differential equations and derive the strong solution we found for this SDE. We have also come up with an infinitesimal generator associated with this market model, which has applications in studying the model further. And in the final section we have shown that this model is arbitrage free. In Chapter 4, we make some concluding remarks.

Chapter 2

Theory of Option Pricing

2.1 B-S-M Theory

The B-S-M model is a mathematical model of a financial market containing only two assets, of which one is risky and another is riskless. The risky asset is modeled as geometric Brownian motion. And the riskless asset called bond B_t is given by $B_t = B_0 e^{rt}$, where r is risk-free rate of interest associated with the currency in which the asset is quoted. There are no dividends and no transaction costs on this assets. Delta hedging is done continuously and arbitrage opportunities are not allowed in this model. From this model, the Black-Scholes formula which gives the price of European options can be deduced. We would first recall this model briefly.

Let $\phi(S, t)$ be the value of a European call option at time t provided the stock price $S_t = S$. It depends on the following variables and parameters namely, S, t, σ, μ, K, T and r. S, and t are variables for stock price and current time respectively, where as σ and μ are parameters associated with the stock price dynamics called as volatility and drift, respectively. Finally, K and T are parameters associated with the details of the particular contract known as strike price and expiry time and r is as defined above.

Let Π denote the value of a portfolio of one long option position and a short position in some quantity Δ of the underlying:

$$\Pi = \phi(S_t, t) - \Delta S_t. \tag{2.1}$$

In the B-S-M model, the stock price dynamics follows a geometric Brownian motion

$$dS_t = S_t(\mu dt + \sigma dW_t),$$

$S_0 \geq 0$. We can see that change in value of the portfolio from time t to $t + dt$ is given by

$$d\Pi_t = d\phi_t - \Delta dS_t.$$

From Itô's formula we have

$$d\phi(S_t, t) = \frac{\partial\phi(S_t,t)}{\partial t}dt + \frac{\partial\phi(S_t,t)}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2\phi(S_t,t)}{\partial S^2}dt.$$

Therefore the portfolio changes by

$$d\Pi_t = \frac{\partial\phi(S_t,t)}{\partial t}dt + \frac{\partial\phi(S_t,t)}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2\phi(S_t,t)}{\partial S^2}dt - \Delta dS_t. \tag{2.2}$$

In the RHS of (2.2), the terms with the dt are the deterministic terms and those with dS are the random. The random terms indicate the possible risk in our portfolio. The random terms in (2.2) are $(\frac{\partial\phi(S_t,t)}{\partial S} - \Delta)dS_t$. If we choose

$$\Delta = \frac{\partial\phi(S_t,t)}{\partial S}, \tag{2.3}$$

then the randomness is reduces to zero. The phenomenon of reduction in randomness is called hedging and exploiting the correlation between instruments to perfectly eliminate risk is called delta hedging.

Once we choose the value of Δ such that randomness reduces to zero, value of the portfolio changes by the amount as given below:

$$d\Pi_t = (\frac{\partial\phi(S_t,t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2\phi(S_t,t)}{\partial S^2})dt. \tag{2.4}$$

Note that (2.4) does not have any terms which contain dS_t which implies that the change in the value of portfolio is completely riskless. This means that $d\Pi_t$ must be equal to the growth in the amount that is deposited in a risk-free interest bearing account. Thus,

$$d\Pi_t = r\Pi dt. \tag{2.5}$$

The above mentioned is an example of no arbitrage principle.
Substituting (2.1), (2.3) and (2.4) into (2.5) we find that

$$(\frac{\partial\phi(S_t,t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2\phi(S_t,t)}{\partial S^2})dt = r(\phi(S_t,t) - S_t\frac{\partial\phi(S_t,t)}{\partial S})dt.$$

On dividing by dt and rearranging we get

$$\frac{\partial\phi(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\phi(S,t)}{\partial S^2} + rS\frac{\partial\phi(S,t)}{\partial S} - r\phi(S,t) = 0.$$

This is known as Black-Scholes equation. This equation along with some appropriate initial and boundary conditions admit a unique classical solution.

2.2 NA and Completeness

We model the stock price dynamics as some random process. But we need to check whether the market model which we consider is really stable. And the necessary condition for a stable market is no arbitrage.

Consider a market consisting of a bond, whose price at time t is S_t^0, and k stocks, whose prices are S_t^i, $1 \leq i \leq k$ which are assumed to be RCLL process. We will consider a finite time horizon T, thus $t \in [0,T]$.
Let $\tilde{S}_t^i = S_t^i(S_t^0)^{-1}$, $S_0^i = 1$ - discounted price of the i^{th} stock.
We denote $S := (S^0, ..., S^k)$ and $\{\mathcal{F}_t^s\}_{t\geq 0}$ the filtration generated by S satisfying usual hypothesis. Furthermore S is assumed to be a semimartingale w.r.t. $\mathcal{F}_t^s$.

We recall some of the important notions from [20] for our subsequent discussions in the following definitions.

Definition 2.2.1. $\theta = (\pi^0, \pi^1, ..., \pi^k)$ *is said to be a trading strategy if (writing* $\mathcal{F}_t$ *for* $\mathcal{F}_t^s$ *)*
(a) each π_t^i *is* $(\mathcal{F}_t)$*- predictable,*
(b) The stochastic integral $\int_0^T \pi_t^i d\tilde{S}_t^i$ *exists for* $i = 0, ..., k$
π_t^i *- no. or the amount of the* i^{th} *stock held by the investor at time t (*$i = 0$ *corresponds to the bond)*
$\theta = (\pi^0, \pi^1, ..., \pi^k)$ *represents the holding of the investor at time t and is also known as the investor's portfolio.*

Definition 2.2.2. *For a given portfolio* $\theta = (\pi^0, \pi^1, ..., \pi^k)$*, its value or wealth process is defined as* $V_t(\theta) := \sum_{i=0}^{k} \pi_t^i S_t^i$*,* $(t > 0)$*.*

Definition 2.2.3. *The accumulated gains or losses up to (and including) the instant* t *are called the gains process and is given by* $G_t(\theta) := \sum_{i=0}^{k} \int_0^t \pi_u^i dS_u^i$

The discounted value process $\tilde{V}_t(\theta)$ and the discounted gains process $\tilde{G}_t(\theta)$ are respectively given by $\tilde{V}_t(\theta) := \sum_{i=0}^{k} \pi_t^i \tilde{S}_t^i = \pi_t^0 + \sum_{i=1}^{k} \pi_t^i \tilde{S}_t^i$, $\tilde{G}_t(\theta) := \sum_{i=1}^{k} \int_0^t \pi_u^i d\tilde{S}_u^i$

Definition 2.2.4. $\theta = (\pi^0, \pi^1, ..., \pi^k)$ *is said to be self-financing strategy if there is no investment or consumption at any time* $t > 0$*. That is* $\theta = (\pi^0, \pi^1, ..., \pi^k)$ *is a self-financing strategy if* $\tilde{V}_t(\theta) = \tilde{V}_0(\theta) + \tilde{G}_t(\theta)$ *a.s* $0 \le t \le T$*.*

Definition 2.2.5. *A self-financing strategy* $\theta = (x, \pi^1, ..., \pi^k)$ *is admissible (or tame) if for some* $m < \infty$*,* $P\{\tilde{V}_t(\theta) \ge -m \forall t\} = 1$

Definition 2.2.6. *An admissible strategy* $\theta = (x, \pi^1, ..., \pi^k)$ *is said to be an arbitrage opportunity if* $x = 0$*,*

$$\tilde{V}_T(\theta) \ge 0 \tag{2.6}$$

P*- a.s. and*

$$P[\tilde{V}_T(\theta) > 0] > 0. \tag{2.7}$$

Definition 2.2.7. $\tilde{S} = (\tilde{S}^1, ..., \tilde{S}^k)$ *has the no arbitrage property (NA) if* $\nexists$ *an admissible strategy* $\theta = (0, \pi)$ *s.t* (2.6) *and* (2.7) *hold.*

Definition 2.2.8. *A probability measure* Q *is said to be equivalent martingale measure (EMM) if* $Q \equiv P$ *and the discounted stock prices* $\{\tilde{S}_t^i\}$ *are martingales with respect to* Q*. Such a probability measure is also referred to as a risk neutral measure.*

Definition 2.2.9. *A contingent claim is an* $\mathcal{F}_t^s$*- measurable random variable* Z_T *satisfying* $Z_T \ge 0$ *a.s,* $E^Q(Z_T) < \infty$*.*

Definition 2.2.10. *A contingent claim is said to be attainable if there exists a strategy* $\theta = (x, \pi^1, ..., \pi^k)$ *such that* $\tilde{V}_t(\pi)$ *is a* $Q-$ *martingale and*

$$\tilde{V}_T(\theta) = Z_T \tag{2.8}$$

a.s.

Definition 2.2.11. *The strategy $\theta = (x, \pi^1, ..., \pi^k)$ satisfying (2.8) is said to be a hedging strategy for the contingent claim Z_T.*

Definition 2.2.12. *A market consisting of $k-$ stocks $(S^1, S^2, ..., S^k)$ and a bond (S_t^0) is said to be complete if every contingent claim is attainable.*

Let $\mu(P) = \{Q : Q \equiv P \text{ and } \tilde{S}_t^i \text{ a } Q\text{- local martingale}, 1 \leq i \leq k\}$ be the class of equivalent (local) martingale measures (EMM).

Theorem 2.2.1. *Let $Q \in \mu(P)$. For an admissible strategy $\theta = (x, \pi^1, ..., \pi^k)$, the discounted gains process is a $Q-$ local martingale and a $Q-$ super martingale. Thus $\mu(P) \neq \phi \Longrightarrow NA$.*

Proof. Let us note that under Q, $\tilde{S}^i$ is a local martingale and hence the discounted gains process, call it U is a stochastic integral with respect to $\tilde{S}$, is also a local martingale.
If $\{\tau_n\}$ is an increasing sequence of stopping times such that $P(\tau_n = T) \to 1$ and $U_t^n = U_{t\wedge\tau_n}$ is a $Q-$ martingale. Then for $s \leq t$, $E^Q(U_t^n|\mathcal{F}_s^s) = U_s^n$. In view of the admissibility of π, $U_t^n \geq -m$ for some m and hence, by Fatou's lemma for conditional expectation, we get
$E^Q(U_t|\mathcal{F}_s^s) = E^Q(\liminf U_t^n|\mathcal{F}_s^s) \leq \liminf E^Q(U_t^n|\mathcal{F}_s^s) = \liminf U_s^n = U_s$
This proves that U_t is a $Q-$ supermartingale. In particular, $E^Q(U_t) \leq E^Q(U_0) = 0$. Thus if $P(U_T \geq 0) = 1$, then $Q(U_T \geq 0) = 1$. Hence $E^Q(U_t) \leq 0 \Rightarrow Q(U_t = 0) = 1$. Therefore $P(U_t = 0) = 1$. Thus NA holds. □

Remark 2.2.1. *One can show that Black-Scholes market is complete and arbitrage free.*

We have seen that existence of an EMM implies NA. Following theorem from [29] known as Girsanov theorem will give us the existence of an EMM.

Theorem 2.2.2 (Girsanov theorem for Itô processes)**.** *Let $X(t)$ be an n-dimensional Itô process of the form*

$$dX(t) = \alpha(t, \omega)dt + \sigma(t, \omega)dB(t); 0 \leq t \leq T$$

where $\alpha(t) = \alpha(t, \omega) \in \mathbb{R}^n$, $\sigma(t) = \sigma(t, \omega) \in \mathbb{R}^{n \times m}$ *and* $B(t) \in \mathbb{R}^m$. *Assume that there exists a process* $\theta(t) \in \mathbb{R}^m$ *such that*

$$\sigma(t)\theta(t) = \alpha(t)$$

for a.a. $(t, \omega) \in [0, T] \times \Omega$ *and such that the process* $Z(t)$ *defined for*$0 \leq t \leq T$ *by*

$$Z(t) := \exp\{-\int_0^t \theta(s)dB(s) - \tfrac{1}{2}\int_0^t \theta^2(s)ds\}$$

exists. Define a measure Q *on* $\mathcal{F}_t$ *by* $dQ(\omega) = Z(T)dP(\omega)$. *Assume that* $E_P[Z(T)] = 1$. *Then* Q *is a probability measure on* $\mathcal{F}_t$, Q *is equivalent to* P *and* $X(t)$ *is a local martingale with respect to* Q.

For a more general model further extension of this theorem is required.

2.3 Pricing in a Fair Market

In B-S-M market the price function $\phi(S, t)$ of a European Call option is the solution of the following partial differential equation (PDE) known as Black-Scholes PDE

$$\tfrac{\partial \phi}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2 \phi}{\partial S^2} + rS\tfrac{\partial \phi}{\partial S} - r\phi = 0$$

subjected to the boundary condition: $\phi(S, T) = (S - K)^+$, and $\phi(0, t) = 0 \ \forall t$.

Solution of this PDE is unique and is given by

$$\phi(S, t) = S\Phi(g(S, T - t)) - Ke^{-r(T-t)}\Phi(h(S, T - t)).$$

Where g and h are defined as follows.

$$g(S, t) := \frac{\log \frac{S}{K} + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}},$$
$$h(S, t) := g(S, t) - \sigma\sqrt{t}.$$

And Φ is the distribution function of the standard normal distribution.

But when we consider a general market model, it is unlikely that it would become a complete market. In a complete market, we can hedge the contingent claim perfectly with a self-financing strategy as we did in B-S-M market. But in an incomplete market, it is not possible with a self-financing strategy. So pricing problem becomes difficult in an incomplete market. But there are different approach to solve this problem. One of the approach to do this is the locally risk minimization method. In this method, we allow additional cash to flow through out the period of the option. And we replicate the claim at the maturity time by this particular strategy in which one minimizes a certain measure of the accumulated cash flow known as quadratic residual risk (QRR) under a certain set of constraints. This minimizing strategy is known as the optimal hedging. It is shown in [18] that the existence of an optimal hedging is equivalent to that of Föllmer Schweizer decomposition of the relevant discounted claim. Following [18] we present a brief description of locally risk minimizing hedging in a general incomplete market.

Consider a market consist of two assets, a stock $\{S_t^1\}_{t\geq 0}$ and a bond $\{S_t^0\}_{t\geq 0}$. Let $\theta = (\pi^0, \pi^1)$ is an admissible strategy as defined in Definition 2.2.5. One can write the value of the portfolio at time t as

$$V_t = \pi_t^0 S_t^0 + \pi_t^1 S_t^1 \tag{2.9}$$

from Definition 2.2.2. Let C_t be the accumulated additional cash flow due to a strategy θ at time t. Then V_t can also be written as sum of two quantities, one is the return of the investment at an earlier instant $t - \Delta$ and the other one is the instantaneous cash flow (ΔC_t) as shown below.

$$V_t = \pi_{t-\Delta}^0 S_t^0 + \pi_{t-\Delta}^1 S_t^1 + \Delta C_t \tag{2.10}$$

From (2.9) and (2.10), one can write

$$V_t - V_{t-\Delta} = \pi_{t-\Delta}^0 (S_t^0 - S_{t-\Delta}^0) + \pi_{t-\Delta}^1 (S_t^1 - S_{t-\Delta}^1) + \Delta C_t$$

or equivalently the SDE

$$dV_t = \pi_t^0 dS_t^0 + \pi_t^1 dS_t^1 + dC_t.$$

For a self-financing strategy, $dC_t = 0$. So in a complete market we can replicate the claim without adding any external cash. That is not the case in an incomplete market. It is shown in [18] that if the market is arbitrage free, the existence of an optimal strategy for hedging an $\mathcal{F}_T$ measurable claim H, is equivalent to the existence of Föllmer Schweizer decomposition of discounted claim $H^* := B_T^{-1} H$ in the form

$$H^* = H_0 + \int_0^T \pi_t^{H^*} dS_t^* + L_T^{H*}$$

where $H_0 \in L^2(\Omega, \mathcal{F}_0, P)$, $L^{H^*} = \{L_t^{H^*}\}_{0 \leq t \leq T}$ is a square integrable martingale starting with zero and orthogonal to the martingale part of S_t^1. Further $\pi_t^{H^*}$ appeared in the decomposition, constitutes the optimal strategy. Indeed the optimal strategy $\theta = (\pi_t^0, \pi_t^1)$ is given by

$$\begin{aligned}
\pi_t^1 &:= \pi_t^{H^*}, \\
V_t^* &:= H_0 + \int_0^t \pi_u^1 dS_u^* + L_t^{H^*}, \\
\pi_t^1 &:= V_t^* - \pi_t^1 S_t^*,
\end{aligned}$$

and $S_t^0 V_t^*$ represents the pseudo locally risk minimizing pricing at time t of the claim H. Hence the Föllmer Schweizer decomposition is the key thing to verify to settle the pricing and hedging problems in any given market.

Chapter 3

A New Market Model

Finding the value of option has always been a major concern in Mathematical Finance. In 1973 Black and Scholes proposed a model called Black-Scholes model for the option pricing problem. But they assumed that the basic market parameters such as volatility and drift are constant during the entire period of the option. That is clearly not the case in the real market. For rectifying this assumption people proposed and tested many different models. Few of those are stochastic volatility models, jump-diffusion models, and Levy processes, regime-switching models. The market of these models are incomplete. For past few years there has been a considerable amount of attention paid to the regime-switching models. The important aspect of regime-switching models is that in this model we consider volatility and drift to follow either a Markov process or a semi-Markov process whose states represent states of business cycles. One can refer to [6, 12, 30]. In this thesis, we model the stock price by the semi-Markov modulated jump-diffusion model.

In this chapter we first provide the sketch of proof of Itô's formula for continuous path processes. And then we state Itô's formula for RCLL path without a proof. We apply that to obtain infinitesimal generator of a certain Markov process which arises from our stock price dynamics model. To this end we recall few definitions and notations which would be used through out this thesis.

3.1 Itô's Formula

Let x be a real valued function on $[0, \infty)$ which is right continuous and has left limits (RCLL). We use the following notation: $x_t := x(t)$, $\Delta x_t := x_t - x_{t-}$, $\Delta x_t^2 := (\Delta x_t)^2$.

We start with a fixed sequence $(\tau_n)_{n=1,2,...}$ of finite partitions $\tau_n = \{0 = t_0 < t_1 < ... < t_{i_n} < \infty\}$ of $[0, \infty)$ with $t_{i_n} \to_n \infty$ and $|\tau_n| = \sup_{t_i \in \tau_n} |t_{i+1} - t_i| \to_n 0$.

Definition: A measure on real line is called discrete measure if its support is at most a countable set. Equivalently, a measure ε is called discrete measure if it is of the form $\varepsilon = \sum_i a_i \varepsilon_{t_i}$, where $a_i \geq 0$, (t_i) is a sequence of real numbers and $\varepsilon_{t_i}(A) = 1$ if $t_i \in A$, otherwise 0 for a measurable set A.

Definition: We say that x is of *quadratic variation along* (τ_n) if the discrete measures

$$\xi_n = \sum_{t_i \in \tau_n} (x_{t_{i+1}} - x_{t_i})^2 \varepsilon_{t_i} \tag{3.1}$$

converge weakly to a Radon measure ξ on $[0, \infty]$. The distribution function of ξ is denoted by $[x, x]$ and given by $[x, x]_t := \xi(0, t)$ and satisfies

$$[x, x]_t = [x, x]_t^c + \sum_{u \leq t} \Delta x_u^2, \tag{3.2}$$

where $[x, x]_t^c$ is the the distribution function of the absolutely continuous part of ξ and $\sum_{u \leq t} \Delta x_u^2$ is the distribution function corresponding to discrete part of ξ.

Let $X : [0, \infty] \to \mathbb{R}$ be a real valued continuous function and $F \in C^2(\mathbb{R})$. Then Taylor's theorem states

$$\Delta F(X_t) = F(X_{t+\Delta t}) - F(X_t) = F'(X_t)\Delta X_t + \frac{1}{2} F''(X_{t'})(\Delta X_t)^2,$$

with $\Delta X_t = X_{t+\Delta t} - X_t$ and some $t' \in [t, t + \Delta t]$.

If X_t is of bounded variation (B.V), then taking the limit for $\Delta t \to 0$ gives $dF(X_t) = F'(X_t)dX_t$. Since the second term in the Taylor series disappears as $[X]_t = 0$ (Quadratic variation of X_t). If X_t is of unbounded variation, we get $dF(X_t) = F'(X_t)dX_t + \frac{1}{2}F''(X_t)(dX_t)^2$, since $[X_t] \neq 0$. Or, we can write this as,

$$F(X_t) = F(X_0) + \int_0^t F'(X_u)dX_u + \frac{1}{2}\int_0^t F''(X_u)(dX_u)^2$$

The second integral in the above equation is well defined for finite quadratic variation of X_t. However, the task of giving a precise meaning of the first integral where both the argument of the integrand and the integrator are of unbounded variation on any arbitrary small time interval remained unsolved for a long time. This task was first solved by Itô. Hence called as Itô calculus.

So for functions of bounded variation, we can apply classical calculus. But in finance we need functions of unbounded variation as integrator. So we need Itô calculus. We encounter portfolio which should represent in terms of some integral where allocation appears in the integrand and the asset prices should be the integrator. In particular, we need the following theorem from [22] which is renowned as Itô's formula for RCLL path.

Theorem 3.1.1. *Let x be of quadratic variation along (τ_n) and F a function of class C^2 on $\mathbb{R}$. Then the Itô formula*

$$\begin{aligned} F(x_t) &= F(x_0) + \int_0^t F'(x_{u-})dx_u + \frac{1}{2}\int_{(0,t]} F''(x_{u-})d[x,x]_u \\ &\quad + \sum_{u\le t}[F(x_u) - F(x_{u-}) - F'(x_{u-})\Delta x_u - \frac{1}{2}F''(x_{u-})\Delta x_u^2], \end{aligned} \tag{3.3}$$

holds with

$$\int_0^t F^{'}(x_{u-})dx_u = \lim_n \sum_{\tau_n \ni t_i \le t} F^{'}(x_{t_i})(x_{t_{i+1}} - x_{t_i}), \tag{3.4}$$

and the series in (3.4) *is absolutely convergent.*

3.2 Model Description

Our market consist of two types of securities. A risky asset whose price can go up or down and a riskless security called bond, where one always gets back the investment, plus interest. We are going to consider options on this risky asset called the stock.

Let $X := \{X_t\}_{0\le t\le T}$ be a semi-Markov process on the state space $\chi = \{1,2,3,...,k\}$, with conditional distribution of holding time $P(T_{n+1} - T_n \le y | X_{T_n} = i) = F(y|i)$ and the transition probabilities $P(X_{T_n} = j | X_{T_{n-1}} = i) = p_{ij}$. Define $\lambda_{ij}(y) := \frac{f(y|i)}{1-F(y|i)}p_{ij}$ where f is the derivative of F, provided F is differentiable and less than 1. Let

$$\begin{aligned} h(i,y,z) :&= \sum_{j\ne i}(j-i)1_{\Lambda_{ij}(y)}(z) \\ g(i,y,z) :&= y\sum_{j\ne i} 1_{\Lambda_{ij}(y)}(z) \end{aligned}$$

It is shown in [32] that one can write the above semi-Markov process in an integral form as given below.

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} h(X_{u-}, Y_{u-}, z)\wp(du, dz) \tag{3.5}$$

$$Y_t = t - \int_0^t \int_{\mathbb{R}} g(X_{u-}, Y_{u-}, z)\wp(du, dz), \tag{3.6}$$

where $\wp$ is Poisson random measure with intensity measure $dudz$ and Y_t is the holding time. And let $S = (S_t)_{0 \le t \le T}$ be a risky asset which follows a semi-Markov modulated jump diffusion model as given below.

$$dS_t = S_{t-}(\mu_{t-}dt + \sigma_{t-}dW_t + \int_{-\infty}^{\infty} f(z_1)N(dz_1, dt)), \tag{3.7}$$

where $S_0 > 0$, $\mu_{t-} := \mu(X_{t-})$ denotes the drift(expected return) and $\sigma_{t-} := \sigma(X_{t-})$ denotes the volatility of the asset(can be thought of as the standard deviation) that follows a semi-Markov process, $f : \mathbb{R} \to \mathbb{R}$ is continuous, bounded above and $f(z_1) > -1$, W_t is a Brownian motion and $N(dz_1, dt)$ is a Poisson random measure with intensity measure $\nu(z_1)dt$, where ν is a finite Borel measure. Also we assume W, X and $N(dt, dz_1)$ are independent.

It turns out that the SDE (3.7) has a strong solution.

Theorem 3.2.1. *The SDE* (3.7) *has a strong solution which is given by*

$$S_t = S_0 \exp[\int_0^t (\mu_{u-} - \frac{1}{2}\sigma_{u-}^2)du + \int_0^t \sigma_{u-}dW_u + \int_0^t \int_{\mathbb{R}} \ln(1 + f(z_1))N(dz_1, dt)] \tag{3.8}$$

Proof. We can see that jumps of this process are coming from last term on the RHS of (3.7). So we can write

$$\begin{aligned} \Delta S_t &= S_t - S_{t-} = S_{t-} \int_{\mathbb{R}} f(z_1)N(dz_1, dt) \qquad (3.9) \\ S_t &= S_{t-} + S_{t-} \int_{\mathbb{R}} f(z_1)N(dz_1, dt) = S_{t-}(1 + \int_{\mathbb{R}} f(z_1)N(dz_1, dt)) = S_{t-}(1 + f(z_2)) \\ &= \int_{\mathbb{R}} S_{t-}(1 + f(z_1))N(dz_1, dt). \end{aligned}$$

And we observe that

$$dS_u^c = S_{u-}(\mu_{u-}du + \sigma_{u-}dW_u) \tag{3.10}$$

because only first two terms on RHS of (3.7) contributes to the continuous part. And

$$d[S]_u^c = S_{u-}^2 \sigma_{u-}^2 du \tag{3.11}$$

because μ_u is of finite variation and $[W]_u = u$ (Levy's theorem).

Let $\tau = \min(t|S_t \leq 0)$ is a stopping time and let $Z_t = \ln S_t$. Applying Itô's formula on $\ln S_t$ for $0 \leq t < \tau$ and using (3.10) and (3.11), we get

$$\begin{aligned} dZ_t &= \frac{dS_t^c}{S_{t-}} - \frac{1}{2}\frac{d[S^c]_t}{S_{t-}^2} + \ln S_t - \ln S_{t-} \\ &= \mu_{t-}dt + \sigma_{t-}dW_t - \frac{1}{2}\sigma_{t-}^2 dt + \ln S_t - \ln S_{t-} \\ &= (\mu_{t-} - \frac{1}{2}\sigma_{t-}^2)dt + \sigma_{t-}dW_t + \ln S_t - \ln S_{t-}. \end{aligned}$$

Where the last term of the RHS can be written as

$$\begin{aligned} \ln S_t - \ln S_{t-} &= \ln(\tfrac{S_t}{S_{t-}}) = \ln(\tfrac{S_{t-}+S_t-S_{t-}}{S_{t-}}) = \ln(1 + \tfrac{\Delta S_t}{S_{t-}}) \\ &= \ln(1 + \int_{\mathbb{R}} f(z_1)N(dz_1, dt)) = \int_{\mathbb{R}} \ln(1 + f(z_1))N(dz_1, dt). \end{aligned}$$

After substituting back this into the equation, we get

$$dZ_t = (\mu_{t-} - \frac{1}{2}\sigma_{t-}^2)dt + \sigma_{t-}dW_t + \int_{\mathbb{R}} \ln(1 + f(z_1))N(dz_1, dt).$$

Integrate it 0 to t, where $0 \leq t < \tau$ to get

$$Z_t - Z_0 = \ln(\tfrac{S_t}{S_0}) = \int_0^t (\mu_{u-} - \frac{1}{2}\sigma_{u-}^2)du + \int_0^t \sigma_{u-}dW_u + \int_0^t \int_{\mathbb{R}} \ln(1 + f(z_1))N(dz_1, dt).$$

So solution of the SDE (3.7) has the above form for $0 \leq t < \tau$.

Choose $\omega \in \Omega$ such that $\tau(\omega)$ is finite else we are done as $\tau = \infty$ P a.s. and $S_t > 0$ a.s. Now let $t \to \tau(\omega)$ and see by (3.8) $S_{\tau(\omega)-} > 0$. Hence non-positivity may occur only by jump. And (3.9) makes clear that non-positivity of S_t does not happen with our assumption $f(z_1) > -1$. Hence $\tau = \infty$ P a.s. and $S_t > 0$ P a.s. $\forall t \in (0, \infty)$.

Hence the proof. □

Let Y_t be the holding time for the semi-Markov process X_t. Then we know that (S_t, X_t, Y_t) is a Markov process. We also know that if A is the infinitesimal generator of (S_t, X_t, Y_t) then for any $\phi \in C_c^\infty$, $\phi(S_t, X_t, Y_t) - \phi(S_0, X_0, Y_0) - \int_0^t A\phi(S_{u-}, X_{u-}, Y_{u-})du$ is martingale w.r.t.$\mathcal{F}_t$. We would use this result for the following derivation. Now we are going to find the generator of this Markov process. Applying Itô's formula on $\phi(S_t, X_t, Y_t)$, we get

$$\begin{aligned}\phi(S_t, X_t, Y_t) &= \phi(S_0, X_0, Y_0) + \int_0^t \tfrac{\partial}{\partial S}\phi(S_{u-}, X_{u-}, Y_{u-})dS_u^c \\ &\quad + \frac{1}{2}\int_0^t \tfrac{\partial^2}{\partial S^2}\phi(S_{u-}, X_{u-}, Y_{u-})d[S]_u^c + \int_0^t \tfrac{\partial}{\partial y}\phi(S_{u-}, X_{u-}, Y_{u-})dY_u \\ &\quad + \sum_{u\le t}[\phi(S_u, X_u, Y_u) - \phi(S_{u-}, X_{u-}, Y_{u-})]. \end{aligned} \tag{3.12}$$

Substitute the expression of dS_u^c and $d[S]_u^c$ in (3.12), RHS becomes

$$\begin{aligned}&\phi(S_0, X_0, Y_0) + \int_0^t \tfrac{\partial}{\partial S}\phi(S_{u-}, X_{u-}, Y_{u-})[S_{u-}(\mu_u du + \sigma_u dW_u)] \\ &+ \frac{1}{2}\int_0^t \tfrac{\partial^2}{\partial S^2}\phi(S_{u-}, X_{u-}, Y_{u-})S_{u-}^2\sigma_{u-}^2 du + \int_0^t \tfrac{\partial}{\partial y}\phi(S_{u-}, X_{u-}, Y_{u-})du \\ &+ \sum_{u\le t}[\phi(S_u, X_u, Y_u) - \phi(S_{u-}, X_{u-}, Y_{u-})].\end{aligned}$$

Taking all du terms together, we can rewrite this as

$$\begin{aligned}&\phi(S_0, X_0, Y_0) + (\int_0^t \tfrac{\partial}{\partial S}\phi(S_{u-}, X_{u-}, Y_{u-})S_{u-}\mu_u \\ &+ \frac{1}{2}\int_0^t \tfrac{\partial^2}{\partial S^2}\phi(S_{u-}, X_{u-}, Y_{u-})S_{u-}^2\sigma_{u-}^2 + \int_0^t \tfrac{\partial}{\partial y}\phi(S_{u-}, X_{u-}, Y_{u-}))du \\ &+ \int_0^t \tfrac{\partial}{\partial S}\phi(S_{u-}, X_{u-}, Y_{u-})S_{u-}\sigma_u dW_u + \sum_{u\le t}[\phi(S_u, X_u, Y_u) - \phi(S_{u-}, X_{u-}, Y_{u-})].\end{aligned}$$

Define $dM_t^1 := \int_0^t \frac{\partial}{\partial S}\phi(S_{u-}, X_{u-}, Y_{u-})S_{u-}\sigma_u dW_u$ which is a martingale process. Thus our equation becomes

$$\begin{aligned}\phi(S_t, X_t, Y_t) &= \phi(S_0, X_0, Y_0) + \int_0^t (S_{u-}\mu_{u-}\tfrac{\partial}{\partial S} + \frac{1}{2}S_{u-}^2\sigma_{u-}^2\tfrac{\partial^2}{\partial S^2} + \tfrac{\partial}{\partial y})\phi du + dM_t^1 \\ &\quad + \sum_{u\le t}[\phi(S_u, X_u, Y_u) - \phi(S_{u-}, X_{u-}, Y_{u-})]. \qquad (3.13)\end{aligned}$$

To compute the last term on the RHS, we observe

$$\begin{aligned}&\phi(S_u, X_u, Y_u) - \phi(S_{u-}, X_{u-}, Y_{u-}) \\ &= (\phi(S_u, X_u, Y_u) - \phi(S_{u-}, X_u, Y_u)) + \phi(S_{u-}, X_u, Y_u) - \phi(S_{u-}, X_{u-}, Y_{u-}) \qquad (3.14) \\ &= \phi(S_{u-}(1 + \int f(z_1)N(du, dz_1)), X_u, Y_u) - \phi(S_{u-}, X_u, Y_u) \\ &\quad +\phi(S_{u-}, X_{u-} + \int h(X_{u-}, Y_u, z)\wp(du, dz), Y_{u-} - \int g(X_{u-}, Y_{u-}, z)\wp(du, dz)) \\ &\quad -\phi(S_u, X_{u-}, Y_{u-}).\end{aligned}$$

As I shown before, we can take the integration outside. Then we get

$$\begin{aligned}= &\int (\phi(S_{u-}(1 + f(z_1)), X_u, Y_u) - \phi(S_{u-}, X_u, Y_u))N(du, dz_1) \\ &+ \int (\phi(S_{u-}, X_{u-} + h(X_{u-}, Y_u, z), Y_{u-} - g(X_{u-}, Y_{u-}, z)) - \phi(S_u, X_{u-}, Y_{u-}))\wp(du, dz).\end{aligned}$$

$$\begin{aligned}= &\int (\phi(S_{u-}(1 + f(z_1)), X_u, Y_u) - \phi(S_{u-}, X_u, Y_u))(n(du, dz_1) + \nu(dz_1)du) \\ &+ \int (\phi(S_{u-}, X_{u-} + h(X_{u-}, Y_u, z), Y_{u-} - g(X_{u-}, Y_{u-}, z)) \\ &-\phi(S_u, X_{u-}, Y_{u-}))(p(du, dz) + dudz),\end{aligned}$$

where n and p are compensated measures corresponding to N and $\wp$. Now define $dM_t^2 := \int(\phi(S_{u-}(1+f(z_1)), X_u, Y_u) - \phi(S_{u-}, X_u, Y_u))n(du, dz_1)$ and $dM_t^3 := \int(\phi(S_{u-}, X_{u-} + h(X_{u-}, Y_u, z), Y_{u-} - g(X_{u-}, Y_{u-}, z)) - \phi(S_u, X_{u-}, Y_{u-}))p(du, dz)$ which are martingale processes. Thus equation (3.14) becomes

$$\begin{aligned}
&\phi(S_u, X_u, Y_u) - \phi(S_{u-}, X_{u-}, Y_{u-}) \\
&= dM_t^2 + dM_t^3 + \int (\phi(S_{u-}(1+f(z_1)), X_u, Y_u) - \phi(S_{u-}, X_u, Y_u))\nu(dz_1)du \qquad (3.15) \\
&\quad + \int (\phi(S_{u-}, X_{u-} + h(X_{u-}, Y_u, z), Y_{u-} - g(X_{u-}, Y_{u-}, z)) - \phi(S_u, X_{u-}, Y_{u-}))dudz.
\end{aligned}$$

Using the definition of h and g, we get

$$\begin{aligned}
i + h(i, y, z) &= \sum_j (j-i)1_{\Lambda_{ij}(y)}(z) + i(\sum_j 1_{\Lambda_{ij}(y)} + 1_{\bigcup_j \Lambda_{ij}(y)^c}) \\
&= \sum_j j1_{\Lambda_{ij}(y)}(z) + i1_{\bigcup_j \Lambda_{ij}(y)^c}(z) \\
X_{u-} + h(X_{u-}, Y_{u-}, z) &= \sum_j j1_{\Lambda_{X_{u-}j}(y)}(z) + (X_{u-})1_{\bigcup_j \Lambda_{X_{u-}j}(y)^c}(z) \\
y - g(i, y, z) &= y1_{\bigcup_j \Lambda_{X_{u-}j}(y)^c}(z) = y1_{k_i}(z).
\end{aligned}$$

And we substitute above expressions into (3.15), we get

$$\begin{aligned}
&\phi(S_u, X_u, Y_u) - \phi(S_{u-}, X_{u-}, Y_{u-}) \\
&= dM_t^2 + dM_t^3 + \int (\phi(S_{u-}(1+f(z_1)), X_u, Y_u) - \phi(S_{u-}, X_u, Y_u))\nu(dz_1)du \\
&\quad + \int [\phi(S_{u-}, \sum_j j1_{\Lambda_{X_{u-}j}(y)}(z) + X_{u-}1_k(z), y1_{kX_{u-}}(z)) - \phi(S_u, X_{u-}, Y_{u-})]dudz \\
&= dM_t^2 + dM_t^3 + \int [\phi(S_{u-}(1+f(z_1)), X_u, Y_u) - \phi(S_{u-}, X_u, Y_u)]\nu(dz_1)du \\
&\quad + \sum_{X_{u-}\neq j} [\phi(S_{u-}, j, 0) - \phi(S_u, X_{u-}, Y_{u-})]\lambda_{X_{u-},j}(Y_{u-})du.
\end{aligned}$$

Now replace (3.13) by above expression, we get

$$\begin{aligned}
\phi(S_t, X_t, Y_t) &= \phi(S_0, X_0, Y_0) + \int_0^t (S_{u-}\mu_{u-}\tfrac{\partial}{\partial S} + \frac{1}{2}S_{u-}^2\sigma_{u-}^2\tfrac{\partial^2}{\partial S^2} + \tfrac{\partial}{\partial y})\phi du \\
&\quad + \int_0^t \sum_{X_{u-}\neq j} [\phi(S_{u-}, j, 0) - \phi(S_u, X_{u-}, Y_{u-})]\lambda_{X_{u-},j}(Y_{u-})du \\
&\quad + \int_0^t \int_{\mathbb{R}} (\phi(S_{u-}(1+f(z_1)), X_u, Y_u) - \phi(S_{u-}, X_u, Y_u))\nu(dz_1)du + dM_t,
\end{aligned}$$

where $dM_t = dM_t^1 + dM_t^2 + dM_t^3$.

Let

$$\begin{aligned} D\phi(S,i,y) &= (S\mu(i)\tfrac{\partial}{\partial S} + \frac{1}{2}S^2\sigma^2(i)\tfrac{\partial^2}{\partial S^2} + \tfrac{\partial}{\partial y})\phi(S,i,y) \\ L\phi(S,i,y) &= \sum_{i\neq j}[\phi(S,j,0) - \phi(S,i,y)]\lambda_{i,j}(y) \\ I\phi(S,i,y) &= \int_{\mathbb{R}}(\phi(S(1+f(z_1)),i,y) - \phi(S,i,y))\nu(dz_1) \end{aligned}$$

Then we can write

$$\phi(S_t,X_t,Y_t) \quad = \phi(S_0,X_0,Y_0) + \textstyle\int_0^t A\phi(S_{u-},X_{u-},Y_{u-})du + dM_t$$

where $A = D + L + I$ is the generator.

3.3 NA

Now we will prove in this section that our model is arbitrage free. For proving no arbitrage we need to find an EMM for this market model. From the following lemma we can construct such a martingale measure. Before stating the lemma, we will recall the definition of Borel previsible process.

Let P denote the previsible $\sigma-$ algebra on $\Omega \times \mathbb{R}^+$ associated with the filtration $\{\mathcal{F}_t\}$ and let $\tilde{P} = P \times B$ where B is the Borel $\sigma-$ algebra on $\mathbb{R}$. A function $H(\omega,t,x)$ which is $\tilde{P}-$ measurable will be called Borel previsible. Thus, suppressing the explicit dependence on ω, a Borel previsible function or process $H(t,x)$ is one such that the process $t \to H(t,x)$ is previsible for fixed x and the function $x \to H(t,x)$ is Borel-measurable for fixed t.

Lemma 3.3.1. *Let $Z = \{Z_t; t \in [0,T]\}$ be a Radon-Nikodym process which is defined as follows*

$$Z_t = \exp\{\int_0^t \phi_u dW_u - \tfrac{1}{2}\int_0^t \phi_u^2 du + \int_0^t\int_{\mathbb{R}} \ln H(z,u)N(dz,du) - \int_0^t\int_{\mathbb{R}}[H(z,u)-1]\nu(dz)du\},$$

where $\phi = \{\phi_t; t \in [0,T]\}$ and $H = \{H(.,t); t \in [0,T]\}$ are previsible and Borel previsible processes such that $\mathbb{E}[\int_0^t \phi_u^2 du] < \infty$ and $H > 0$, respectively. Then Z is a positive local martingale under $\mathbb{P}$ with $Z_0 = 1$.

Proof. Z is always positive and $Z_0 = 1$.

$$\begin{aligned}
\Delta Z_t &= Z_t - Z_{t-} \\
&= \exp\{\int_0^t \phi_u dW_u - \tfrac{1}{2}\int_0^t \phi_u^2 du - \int_0^t \int_{\mathbb{R}} [H(z,u)-1]\nu(dz)du\} \\
&\quad \exp\{\int_0^t \int_{\mathbb{R}} \ln H(z,u) N(dz,du)\} \\
&\quad - \exp\{\int_0^t \phi_u dW_u - \tfrac{1}{2}\int_0^t \phi_u^2 du - \int_0^t \int_{\mathbb{R}} [H(z,u)-1]\nu(dz)du\} \\
&\quad \exp\{\int_{[0,t)} \int_{\mathbb{R}} \ln H(z,u) N(dz,du)\} \\
&= Z_{t-}[(\int_0^t \int_{\mathbb{R}} H(z,u)N(dz,du) - 1] \\
&= Z_{t-}[H(\Delta Z_t, t) - 1]
\end{aligned}$$

Apply Itô formula on $Z_t = \exp\{Y_t\}$.

Where

$$Y_t := \int_0^t \phi_u dW_u - \tfrac{1}{2}\int_0^t \phi_u^2 du + \int_0^t \int_{\mathbb{R}} \ln H(z,u)N(dz,du) - \int_0^t \int_{\mathbb{R}} [H(z,u)-1]\nu(dz)du.$$

We get

$$\begin{aligned}
Z_t - Z_0 &= \int_0^t Z_{u-} dY_u + \tfrac{1}{2}\int_0^t Z_{u-} d[Y]_u^c + \sum_{0<u\le t} [Z_u - Z_{u-} - Z_{u-} \ln H(\Delta Z_u, u)] \\
Z_t - 1 &= \int_0^t Z_{u-}[\phi_u dW_u - \tfrac{1}{2}\phi_u^2 du + \int_{\mathbb{R}} \ln H(z,u)N(dz,du) - \int_{\mathbb{R}} [H(z,u)-1]\nu(dz)du] \\
&\quad + \tfrac{1}{2}\int_0^t Z_{u-}\phi_u^2 du + \int_0^t \int_{\mathbb{R}} Z_{u-}[H(z,u) - 1 - \ln H(z,u)]N(dz,du) \\
&= \int_0^t Z_{u-}\phi_u dW_u + \int_0^t \int_{\mathbb{R}} Z_{u-}[H(z,u)-1]\tilde{N}(dz,du) \qquad (3.16)
\end{aligned}$$

The last formula states that $Z = \{Z_t; t \in [0,T]\}$ is a local $\mathbb{P}-$martingale. □The next theorem will give us the existence of an EMM.

Theorem 3.3.1. *Let Q be defined on $\mathcal{F}_t$ by $\frac{dQ}{dP} = Z_T$. Then Q is a martingale measure.*

We need the following lemma from [26] to prove this theorem.

Lemma 3.3.2. *Define a new measure Q as above. Then the process $\tilde{W}_t = W_t - \int_0^t \phi_u du$ is a Winer process under Q and*

$$\int_0^t \int_{\mathbb{R}} [H(z,u)-1](N(dz,du) - H(z,u)\nu(dz)du) := \int_0^t \int_{\mathbb{R}} [H(z,u)-1]\tilde{M}(dz,du)$$

is a $Q-$martingale with respect to its natural filtration which implies that the compensator measure of $N(dz,dt)$ is given by $\tilde{\nu}(dz,dt) = H(z,t)\nu(dz)dt$.

We prove the Theorem 3.3.1 below.

Proof. Recall that our stock price satisfying the following SDE.

$$dS_t = S_{t-}(\mu_{t-}dt + \sigma_{t-}dW_t + \int_{\mathbb{R}} f(z)N(dz,dt))$$

Solution of this SDE is

$$S_t = S_0 \exp\{\int_0^t [\mu(X_u) - \tfrac{1}{2}\sigma^2(X_u)]du + \int_0^t \sigma(X_u)dW_u + \int_0^t \int_{\mathbb{R}} \ln(1 + f(z)N(dz,du)\}$$

Discounted stock price is given by

$$\tilde{S}_t = \tfrac{S_t}{B_t} = \tfrac{S_t}{\exp\{\int_0^t r(X_u)du\}} = \exp\{-\int_0^t r(X_u)du\}S_t$$

Apply Itô's formula on $\tilde{S}_t$. We get

$$\begin{aligned}
d\tilde{S}_t &= \exp\{-\int_0^t r(X_u)du\}dS_t - S_t \exp\{-\int_0^t r(X_u)du\}r(X_t)dt \\
&= \exp\{-\int_0^t r(X_u)du\}[dS_t - S_{t-}r(X_t)dt] \\
&= \exp\{-\int_0^t r(X_u)du\}[S_{t-}(u_t dt + \sigma_t dW_t + \int_{\mathbb{R}} f(z)N(dz,dt)) - S_{t-}r(X_t)dt] \\
&= [\mu(X_t) - r(X_t)]\tilde{S}_{t-}dt + \sigma(X_t)\tilde{S}_{t-}dW_t + \int_{\mathbb{R}} \tilde{S}_{t-}f(z)N(dz,dt)
\end{aligned}$$

Replace dW_t and $N(dz, dt)$ as below.

$$\begin{aligned} d\tilde{S}_t &= [\mu(X_t) - r(X_t)]\tilde{S}_{t-}dt + \sigma(X_t)\tilde{S}_{t-}[d\tilde{W}_t + \phi(X_t)dt] + \int_{\mathbb{R}} \tilde{S}_{t-}f(z)[\tilde{M}(dz, dt) + H(z,t)\nu(dz)dt] \\ &= [\mu(X_t) - r(X_t) + \sigma(X_t)\phi(X_t) + \int_{\mathbb{R}} f(z)H(z,t)\nu(dz)]\tilde{S}_{t-}dt + \tilde{S}_{t-}\sigma(X_t)d\tilde{W}_t \\ &\quad + \int_{\mathbb{R}} \tilde{S}_{t-}f(z)\tilde{M}(dz, dt) \end{aligned} \tag{3.17}$$

It is well known that when the price of the underlying asset is governed by the classical Black-Scholes model, the unique equivalent martingale measure Q is given by

$$\tfrac{dQ}{dP}|_{\mathcal{F}_t} = \Lambda_T,$$

where Λ satisfies

$$\Lambda_t = \Lambda_0 + \int_0^t \Psi_u \Lambda_u dW_u$$

and the process Ψ is chosen so as to ensure the discounted process $\tilde{S}$ a $Q-$ martingale. In our model (3.7), a natural analogue of this would be to use the equivalent martingale measure Q and the Radon-Nikodym process Z will satisfies the following equation

$$dZ_t = \Psi_t Z_{t-}(\sigma(X_t)dW_t + \int_{\mathbb{R}} f(z)N(dz, dt)). \tag{3.18}$$

Compare (3.18) and (3.16) we get

$$\Psi_t \sigma(X_t) = \phi_t$$

and

$$\Psi_t f(z) = H(z,t) - 1$$

Discounted stock price becomes martingale under Q only when the drift term becomes zero. So we have

$$\mu(X_t) - r(X_t) + \phi(X_t)\sigma(X_t) + \int_{\mathbb{R}} f(z)H(z,t)\nu(dz) = 0$$

From this, we get

$$\phi(X_t)\sigma(X_t) = r(X_t) - \mu(X_t) - \int_{\mathbb{R}} f(z)H(z,t)\nu(dz)$$

Replace ϕ_t with $\Psi_t\sigma(X_t)$ and $H(z,t)$ with $1 + \Psi_t f(z)$ we get

$$\begin{aligned}\Psi_t\sigma^2(X_t) &= r(X_t) - \mu(X_t) - \int_{\mathbb{R}} f(z)(1 + \Psi_t f(z))\nu(dz) \\ &= r(X_t) - \mu(X_t) - \int_{\mathbb{R}} f(z)\nu(dz) - \int_{\mathbb{R}} f^2(z)\Psi_t\nu(dz)\end{aligned}$$

Take Ψ_t terms together

$$\Psi_t[\sigma^2(X_t) + \int_{\mathbb{R}} f^2(z)\nu(dz)] = r(X_t) - \mu(X_t) - \int_{\mathbb{R}} f(z)\nu(dz)$$

$$\Rightarrow \Psi_t = \frac{r(X_t)-\mu(X_t)-\int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_t)+\int_{\mathbb{R}} f^2(z)\nu(dz)}$$

And

$$H(z,t) = \frac{r(X_t)-\mu(X_t)-\int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_t)+\int_{\mathbb{R}} f^2(z)\nu(dz)} f(z) + 1$$

$$\phi_t = \frac{r(X_t)-\mu(X_t)-\int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_t)+\int_{\mathbb{R}} f^2(z)\nu(dz)}\sigma(X_t)$$

We need the following condition

$$\frac{r(X_t)-\mu(X_t)-\int_{\mathbb{R}} f(z)\nu(dz)}{\sigma^2(X_t)+\int_{\mathbb{R}} f^2(z)\nu(dz)} f(z) > -1$$

to ensure that $H(z,t) > 0$. Now substitute the expression of H and ϕ into (3.17). $[\mu(X_t) - r(X_t) + \sigma(X_t)\phi(X_t) + \int_{\mathbb{R}} f(z)H(z,t)\nu(dz)]$ becomes zero. Therefore,

$$\begin{aligned}\tilde{S}_t - \tilde{S}_0 &= \int_0^t \tilde{S_{u-}}\sigma(X_u)dW_u + \int_0^t \tilde{S}_{u-}[\mu(X_u) - r(X_u)]du + \int_0^t\int_{\mathbb{R}} \tilde{S}_{u-}f(z)N(dz,du) \\ &= \int_0^t \tilde{S}_{u-}\sigma(X_u)d\tilde{W}_u + \int_0^t\int_{\mathbb{R}} S_{u-}f(z)\tilde{M}(dz,du)\end{aligned}$$

becomes a $\mathcal{Q}-$martingale by the Lemma 3.3.2. Hence the proof. □

Therefore we have shown the existence of an Equivalent Martingale Measure for our market model. By Theorem 2.2.1 we get that the market model which we considered is arbitrage free.

Chapter 4

Conclusion

The model presented in Chapter 3 is a generalization of jump-diffusion model available in the literature [12, 13, 11]. We have shown that this model we proposed is arbitrage free. Our next aim is to address the pricing problem in this market model. It is unlikely that the market would be complete. So the risk-minimizing pricing approach could be adapted in this regard. Currently I am doing research in the above direction.

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