Kähler Geometry with a view towards the Calabi Conjecture

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Certificate

This is to certify that this dissertation entitled *Kähler Geometry with a view towards the Calabi Conjecture* towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Vishnu N at Indian Institute of Science Education and Research under the supervision of Vamsi Pritham Pingali (IISc), Assistant Professor, Department of Mathematics, Indian Institute of Science, Banglore during the academic year 2019-2020.

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This thesis is dedicated to my Mother.

Declaration

I hereby declare that the matter embodied in the report entitled *Kähler Geometry with a view towards the Calabi Conjecture* , is the result of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Vamsi Pritham Pingali (IISc) and the same has not been submitted elsewhere for any other degree.

Vishnu N

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Abstract

Complex manifolds provide a fertile ground for studying Riemannian geometry as well as algebraic geometry. Many complex manifolds admit Kähler metrics. Kähler metrics are Riemannian metrics which tie in well with the complex structure and have a compatible symplectic structure. In the 1930s, E. Calabi conjectured the existence of Kähler metrics with good curvature properties on some compact complex manifolds. This conjecture was resolved by Aubin and Yau in the 70s. In parallel, Yau also proved the existence of Kähler metrics that are Einstein ($Ric(\omega) = \lambda\omega$) in many cases ($c_1(M) > 0$ and $c_1(M) = 0$). In the case of Fano manifolds ($c_1(M) > 0$), the existence of Kähler-Einstein metrics is not always true and is a much harder question. It was only recently completed thanks to the works of Chen, Donaldson, Sun, and Tian (among others). The primary aim of the present thesis is to study Yau's proof of the Calabi conjecture (Chapter 4), as a part of which we study the basics of complex and Kähler geometry (Chapter 2) and the theory of the Monge-Ampère equation (Chapter 3). We will also look into a couple of applications of the Calabi conjecture, and discuss about Kähler-Einstein metrics (Chapter 5). The necessary preliminaries are presented in Chapter 1.

Contents

Introduction

Kähler metrics are Riemannian metrics on complex manifolds that are compatible with the complex structure as well as with the symplectic structure. From an analytic point of view, the existence of local holomorphic normal coordinates on Kähler manifolds is important. This concept was first introduced by Erich Kähler. Kähler manifolds are in the domain of intersection of various research topics like algebraic geometry, differential geometry, complex analysis, geometric analysis and symplectic geometry.

The Calabi Conjecture is about the existence of "nice" Riemaniann metrics on Kähler manifolds. It was first stated by Aubin in the 1950s and Yau proved the conjecture in 1978. Yau won the fields medal partly for the proof [3] and its applications in algebraic geometry [4]. A problem closely related to the Calabi conjecture is the existence of Kähler-Einstein metrics on Kähler manifolds. The existence of Kähler Einstein metrics for the cases $c_1(M) < 0$ and $c_1(M) = 0$ is very similar to the Calabi Conjecture and was studied alongside it. When $c_1(M) > 0$, Kähler-Einstein metrics do not always exist. There are certain algebro-geometric obstructions to their existence. The case $c_1(M) > 0$ was fully solved in 2015 by Chen-Donaldson-Song and Tian (among others). Other problems like the existence of constant scalar metrics are related to the existence of Kähler-Einstein metrics. The Calabi-Yau theorem which is stated below and is discussed in Chapter 4,

Theorem 4.0.1. (Calabi-Yau theorem) *Let (M, ω_0) be a compact Kähler manifold such that $[\omega_0] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$. Given any $\Omega \in 2\pi c_1(M)$, there exists a unique Kähler metric $\omega \in [\omega_0]$ such that $Ric(\omega) = \Omega$.*

We will define and discuss most of the terminologies in the above theorem over the first few chapters. But notice that the very nature of the theorem relates a topological invariant $(c_1(M))$ with a geometric concept (Ricci curvature).

Overview of chapters

The thesis mainly deals with the Yau's proof of the Calabi conjecture, also known as the Calabi-Yau theorem. In Chapter 1, we introduce vector bundles and Riemannain metrics. Here the intention is to review some of the definitions as well as set the tone for the notation throughout thesis. We will also have a glance into connections on vector bundles. Chapter 2 begins by studying complex manifolds and later shifts the focus onto Kähler manifolds. We will also look into a special connection called Chern connection on Kähler manifolds. The primary example through out the chapter is mainly based on projective spaces. The chapter sets up the geometric framework required for understanding the Calabi-Yau theorem.
We build the analytic framework, from analysis and PDE that will be required in the proof of the Calabi-Yau theorem in Chapter 3. We will briefly touch up on Green's function on a manifold. Here, we will also look at some properties of Elliptic PDE. Chapter 4 is completely dedicated to the proof of the Calabi-Yau theorem. We will see how the geometric problem is converted into a nonlinear Partial Differential Equation (PDE). We will prove uniqueness and existence of solutions to the PDE. This chapter is the focal point of the thesis. Finally, in Chapter 5 we look at a couple of applications of the Calabi-Yau theorem. It also discusses the existence of Kähler-Einstein metrics on Kähler manifolds.

Chapter 1

Preliminaries

Einstein summation convention

Throughout the thesis we will use the Einstein summation convention. As per the convention, when an index occurs twice or more in a single term, then we will sum over that index. The convention is used for notational easiness. As a part of the convention we will denote coordinates of vectors with upper indices. For example, is v is a vector in a vector space V with a basis $\{e_i\}$, it will be denoted as,

$$v = v^i e_i.$$

Here v^i does not mean we are taking powers, but rather $\{v^i\}$ is a collection of scalars.

1.1 Vector Bundles

A smooth vector bundle on a manifold M, is a smoothly varying vector spaces at each point on M. Formally, it is defined as,

Definition 1.1.1. (Smooth Vector Bundle) *A* smooth Vector bundle *of dimension r over a manifold M, is a triplet is (V, M, π), where*

- *V and M are smooth manifolds, $\pi : V \to M$ is a smooth map,*

- *for $p \in M$, $\pi^{-1}(p)$ known as the fiber at p, is a r-dimensional vector space over $\mathbb{R}$, and vector space structure varies smoothly, that is the scalar multiplication is a smooth map from $\mathbb{R} \times V \to V$. Similarly we will have a smoothness criterion for scalar addition,*

that satisfies,

- *(local triviality) For any point $p \in M$, there exist an open neighborhood $U \subset M$ and a diffeomorphism*
$$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r,$$
where for each $x \in U$, ϕ maps $\pi^{-1}(x)$ to $\{x\} \times \mathbb{R}^r$ as a vector space isomorphism.

The simplest example of a vector bundle is the trivial bundle $M \times \mathbb{R}^r$. The local triviality condition says that, locally in a neighborhood, a vector bundle is trivial. Equivalently, vector bundles can be defined as a collection of open sets $\{U_\alpha \subset M : \alpha \in \Lambda\}$ that cover M, and smooth functions (called transition functions) $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$, where $\alpha, \beta \in \Lambda$, satisfying

- $g_{\alpha\alpha} = Id$
- $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and,
- $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = Id.$

Given a collection of functions $g_{\alpha\beta}$ that satisfies the above three conditions, then the manifold

$$V := \frac{\sqcup U_\alpha \times \mathbb{R}^r}{(p, v_\alpha) \sim (p, g_{\alpha\beta} v_\beta)},$$

is a vector bundle over M, with $g_{\alpha\beta}$ as transition functions. A vector bundle is a smoothly varying choice of basis, and $g_{\alpha\beta}$ tells us how the basis transform on the intersection. Some fundamental examples of vector bundles we will be interested in are, Tangent bundle, Cotangent bundle, Normal bundle and bundle of differential forms. Studying vector bundles helps to learn and define properties common to all these bundles.
Given vector bundles V and W over M, we can derive new bundles like:

- dual bundle, V^*; the fibers of V^* are dual spaces to the fibers of V. The transition functions are inverse transpose, $g^*_{ij} = (g^\intercal)^{-1}$. The cotangent bundle is the dual to the Tangent Bundle.
- direct sum, $V \oplus W$; the fibers of $V \oplus W$ are direct sum of the fibers of V and W. The

transition functions of $V \oplus W$, is the direct sum of the transition functions of V and W, that is

$$g_{\alpha\beta}^{V \oplus W} = \begin{bmatrix} [g_\alpha^V] & 0 \\ 0 & [g_\alpha^W] \end{bmatrix}$$

- tensor product, $V \otimes W$; a fiber of $V \otimes W$ is obtained by taking fiberwise tensor product. The transition functions $g_{\alpha\beta}^{V\otimes W}$ is the Kronecker product of $g_{\alpha\beta}^V$ and $g_{\alpha\beta}^W$.

A *section* s of a vector bundle V over M is a smooth map $s : M \to V$ such that, $s(p) \in \pi^{-1}(p)$, that is for each point in the manifold it maps to a vector on its fiber in a smooth way. The simplest example is the zero section s_0, where $s_0(p) = 0$ for all p. The zero section gives an embedding of M into V. In fact, every smooth real valued function on a manifold, is a section of the trivial bundle. We denote $\Gamma(V)$, the set of all sections of a vector bundle V over M. If $\{e\}_i$ is a local basis of a vector bundle V, then locally a section s can be written as

$$s(p) = s^i(p)e_i(p),$$

where s^i is a local real valued function. The basis elements e_i, are examples what are known as "local" sections. Sections of the tangent bundle are called vector fields. In local coordinates a vector field is denoted as,

$$X = a^i \frac{\partial}{\partial x^i}.$$

Vector bundles over complex manifolds (see Chapter 2), with the vector spaces being over $\mathbb{C}$, are known as complex vector bundles. More details on vector bundles are available in [12], [5].

1.2 Riemannian manifolds

To do geometry on manifolds, we need the concept of length and angle. To do so we will need a smoothly varying inner product on the tangent space. A metric on a vector bundle V, is a section of $V^* \otimes V^*$, such that on each fiber it is symmetric and positive definite.
A metric on the tangent bundle TM, is known as *Riemannian metric*. In local coordinates,

a Riemannian metric g is represented as,

$$g = g_{ij} dx^i \otimes dx^j,$$

where $g_{ij} = g_{ji}$ and (g_{ij}) is positive definite.
Once we have an inner product on the tangent space, we can calculate length of curves. If $\gamma : [0, 1] \to M$ is a smooth path, then its length is given by

$$L(\gamma) = \int_0^1 \sqrt{g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}.$$

We can find the minimizer of the length functional, known as geodesics (these are equivalent to straight lines in $\mathbb{R}^n$) by using the energy functional $E(\gamma) = \int_0^1 g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)$. Geodesics are solution to

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij} \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0,$$

where

$$\Gamma^k_{ij} = g^{rl} \frac{1}{2} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right). \tag{1.1}$$

Theorem 1.2.1. (normal coordinates) *Given a Riemannian manifold (M, g), for a point $p \in M$, there exists a neighborhood U along with a coordinate system $\{x_i\}$, such that*

$$g_{ij}(x) = \delta_{ij}(x) + O(|x^2|).$$

The theorem says that we can choose coordinates such that they are euclidean upto first order. Interestingly the second order term cannot be removed and it is directly related to the curvature of the manifold. There are a couple of simple ways to prove this, one is using the exponential map and the fact that the exponential map is a local diffeomorphism around 0. The other way is to make a suitable change of coordinates and eliminate the first order terms.

1.3 Connections

A central idea in calculus is the notion of directional derivatives. In $\mathbb{R}^n$, given two vector fields X and Y, we can look at the directional derivative of Y along X, which is another vector field. Further note that, to define the notion of parallel vectors, we need the concept of directional derivatives on manifolds.

Definition 1.3.1. Let $V \to M$ be a smooth r-vector bundle. Let X be a vector field over M, then a *connection* ∇ on V is a map $\nabla_X : \Gamma(TM) \times \Gamma(V) \to \Gamma(V)$ such that,

- (*Tensoriality in* X) if X_1 and X_2 are vector fields over M, and f_1, f_2 are functions on M, then
$$\nabla_{f_1 X_1 + f_2 X_2} = f_1 \nabla_{X_1} + f_2 \nabla_{X_2},$$
- (*Linearity in* $\Gamma(V)$) if X is a vector field, s_1, s_2 be sections of V, and c_1, c_2 are constants, then
$$\nabla_X (c_1 s_1 + c_2 s_2) = c_1 \nabla_X s_1 + c_2 \nabla_X s_2,$$
- (*Leibniz rule*) if X is a vector field, s is a section of V, f be a function on M, then
$$\nabla_X (fs) = X(f)s + f \nabla_X s.$$

A connection ∇, is a function from $\Gamma(TM) \times \Gamma(V) \to \Gamma(V)$, and is tensorial in $\Gamma(TM)$, hence ∇ can be seen as a function $\nabla : \Gamma(V) \to \Gamma(V \otimes T^*M)$. Let s be a section of a vector bundle V over M. In local coordinates, if $s = s^i e_i$, where e_i is a local basis. Then,

$$\begin{aligned} \nabla s = \nabla(s^i e_i) &= ds^i \otimes e_i + s_i \nabla e_i \\ &= (ds^i + A^i{}_j s^j) \otimes e_i, \end{aligned}$$

where $A^i{}_j$ is a matrix of one forms, which determines the connection. Given connections on vector bundles V and W over M, we can define natural connections on their dual, direct sum, tensor product. (see [12])
We will now see a special connection on the tangent bundle of a manifold.

Definition 1.3.2. The *Levi-Civita connection* over a Riemannian manifold (M, g) is the unique connection on the tangent bundle that satisfies,

- (*metric compatability*) $d(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla Y)$,
- (*torsion free*) $\nabla_X Y - \nabla_Y X = [X, Y]$,

where X and Y are vector fields and d is the covarient derivative.

The first condition is what we expect when we differentiate across an inner product. It is equivalent to saying that the inner product is preserved by parallel transport. If x^i are local coordinates, then .

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i},$$

where Γ^i_{jk} known as Christoffel symbols are given by equation (1.1).
Curvature is the connection composed with itself, very closed to the idea of attributing curvature of a curve to its acceleration. Further details can be found in [7]. In this chapter we have only covered an overview of vector bundles and Riemannian geometry. For more details we refer to [5], see also [8], [7] and [12].

Chapter 2

Complex and Kähler Manifolds

The aim of the present chapter is to introduce the concepts of Complex and Kähler Manifolds. Complex manifolds arise naturally in the study of Riemannian geometry, complex analysis and algebraic geometry. The central object of study in the thesis is about Kähler manifolds. The material presented in this chapter is based on the book [1], see also [12] and [9].

2.1 Complex Manifolds

A *complex manifold* of dimension n is a smooth manifold M with an open cover $\{U_\alpha \subset M : \alpha \in \Lambda\}$ and homeomorphisms $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ such that

$$\Phi_{\alpha\beta} \coloneqq \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a holomorphic diffeomorphism for all $\alpha, \beta \in \Lambda$. The collection $\{(U_\alpha, \phi_\alpha) : \alpha \in \Lambda\}$ is called an *atlas* on the complex manifold. Here Λ is an index set.
A smooth manifold along with a complex atlas is called a complex structure. A complex manifold of dimension n by definition is a real smooth manifold of dimension $2n$, with the additional structure on a complex manifold being that the transition functions are holomorphic. The transitions functions being holomorphic helps one define the concept of a holomorphic function on complex manifolds.
A function on a complex manifold M, $f : M \to \mathbb{C}^n$ is *holomorphic* if for every chart $\{U_\alpha, \phi_\alpha\}$,

the map $f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbb{C}^n$ is holomorphic as a function on $\mathbb{C}^n$. This is well defined by the very fact that transition functions are holomorphic. This idea helps us to look at complex manifolds as locally complex spaces $\mathbb{C}^n$, and hence we can push forward many results in several complex variables like Lioville's theorem, maximum modulus principle, local power series etc to complex manifolds. Let us look at some examples of complex manifolds. The simplest example of complex manifolds are open subsets of $\mathbb{C}^n$ with the atlas being the identity map.

Example 1. (*Riemann Sphere*) Consider $S^2 \subset \mathbb{R}^3$, and identify the xy-plane with $\mathbb{C}$. Let $U_1 = S^2 \setminus \{(0,0,1)\}$ and $\phi_1 : U_1 \to \mathbb{C}$ be the stereo-graphic projection from the north pole. Let $U_2 = S^2 \setminus \{(0,0,-1)\}$ and $\phi_2 : U_2 \to \mathbb{C}$ be the complex conjugate of the stereo-graphic projection from the south pole. Then $\{(U_1,\phi_1),(U_2,\phi_2)\}$ is an atlas. More precisely,

$$\phi_1(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right), \quad \phi_2(x,y,z) = \left(\frac{x}{z+1}, \frac{-y}{z+1}\right).$$

Let $Z = X + \sqrt{-1}Y$, then the transition function Φ_{12} is,

$$\begin{aligned}
\phi_1 \circ \phi_2^{-1}(X,Y) &= \phi_1\left(\frac{2X}{X^2+Y^2+1}, \frac{-2Y}{X^2+Y^2+1}, \frac{1-X^2-Y^2}{X^2+Y^2+1}\right) \\
&= \left(\frac{X}{X^2+Y^2}, \frac{-Y}{X^2+Y^2}\right) = \frac{\overline{z}}{|z|^2} = \frac{1}{z},
\end{aligned}$$

which is biholomorphic on $\mathbb{C} \setminus \{0\}$, where $z = X + \sqrt{-1}Y$.

Example 2. (*Zero sets of holomorphic mappings*) Consider the following subset of $\mathbb{C}^2$,

$$S = \left\{(z^1,z^2) \in \mathbb{C}^2 \ : \ (z^1)^2 + (z^2)^2 - 1 = 0\right\},$$

which is a submanifold with transition functions that are holomorphic and hence it is a complex manifold in its own right. Despite the reminiscence of this equation to a circle in the real plane, the set S is not compact, for any large z^1, we will still have a z^2 such that $(z^1)^2 + (z^2)^2 = 1$. In fact the zero set of any holomorphic function is always non-compact. To see this, let $f : \mathbb{C}^n \to \mathbb{C}^n$ be holomorphic, and $D = \{z \in \mathbb{C}^n : \ f(z) = 0\}$ be a submanifold. For if D is compact, then the maximum modulus principle on the inclusion function will imply that the inclusion function is constant. By the same reasoning one can see that, unlike the theory of smooth manifolds we do not have a Whiteny/Nash Embedding type theorem for complex manifolds, i.e not every complex manifold can be embedded in $\mathbb{C}^n$ as a submanifold.

The next example will be an important class of compact manifolds which will also be the foundation for many more examples of compact complex manifolds.

Example 3. (*Complex projective spaces and Projective manifolds*) An important and larger class of compact complex manifolds are *Complex projective spaces* $\mathbb{CP}^n$. The n^{th} complex projective space consists of lines in $\mathbb{C}^{n+1}$, or can also be seen as an equivalence relation on $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$, where $\overrightarrow{X} \sim \lambda\overrightarrow{X}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Here the topology on $\mathbb{CP}^n$ is the quotient topology on this equivalence relation. We will denote the equivalence class $[\overrightarrow{X}]$ as $[X^0 : \cdots : X^n]$, where $\overrightarrow{X} = (X^0, \ldots, X^n)$. To see why $\mathbb{CP}^n$ is a complex manifold, we define the following atlas: Let $U_i = \{[X^0 : \cdots : X^n] \mid X^i \neq 0\}$ and

$$\phi_i : U_i \to \mathbb{C}^n, \text{ as } [X^0 : \cdots : X^n] \mapsto \left(\frac{X^0}{X^i}, \ldots, \widehat{\frac{X^i}{X^i}}, \ldots, \frac{X^n}{X^i}\right),$$

where $\widehat{\frac{X^i}{X^i}}$ denotes that the term is excluded. It is an easy exercise to see that the transition functions are holomorphic diffeomorphisms. Topologically, $\mathbb{CP}^n \cong S^{2n+1}/S^n$ and hence $\mathbb{CP}^n$ is compact.

In projective spaces, homogeneous polynomials play an important role. Homogeneous polynomials are not well defined functions on projective spaces, but in the next section we will realize them as sections of a complex line bundle. Momentarily to emphasis its importance, note that given a homogeneous polynomial f on $\mathbb{C}^{n+1}$, if $z \in \mathbb{C}^{n+1} \setminus \{0\}$ such that $f(z) = 0$, then f is zero on the equivalence class $[z]$. Hence it makes sense to talk of the zero set of f in $\mathbb{CP}^n$. If z is a regular value of f, then the zero set will be a complex submanifold which is compact. The intersection of zero sets in $\mathbb{CP}^n$ of one or more homogeneous polynomial are called *Projective manifolds.* Projective manifolds are of high research interest and these are important examples of compact complex manifolds.

2.2 Holomorphic Vector bundles

Holomorphic vector bundles are complex vector bundles with holomorphic transition functions. Some of the principal examples are the holomorphic tangent and cotangent bundles. In the following subsection we will look at the holomorphic line bundles on complex projective spaces.

2.2.1 Line Bundles on the Complex Projective Space

The definition of a complex projective space, where every point in $\mathbb{CP}^n$ is an equivalence class which contains a line in $\mathbb{C}^{n+1}$. This itself hints at a natural line bundle on $\mathbb{CP}^n$, where for each point we look at the line passing through that point. More precisely, we consider the manifold,

$$L = \{(X, [l]) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n;\ X = \lambda l, \lambda \in \mathbb{C}\},$$

and the projection map $\pi : L \to \mathbb{CP}^n$, where $(X, [l]) \mapsto [l]$. The fiber at a point in $\mathbb{CP}^n$ is the line passing through it. This line bundle is called the *tautological line bundle* and is denoted by $\mathcal{O}(-1)$. For the trivializing open cover U_i described in Example 3 in the previous section, define $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$

$$\phi_i(\lambda \overrightarrow{Z}, [\overrightarrow{Z}]) = ([\overrightarrow{Z}], \lambda Z^i)$$

Then, the transition functions $g_{ij}^{(-1)} = \phi_i \circ \phi_j^{-1}$ are given by

$$g_{ij}^{(-1)}([\overrightarrow{Z}]) = \frac{Z^i}{Z^j}. \tag{2.1}$$

Let s be a global section of $\mathcal{O}(-1)$, and let $\pi_1 : \mathcal{O}(-1) \to \mathbb{C}^{n+1}$ be the projection map into the first variable. Then $\pi_1 \circ s$ is a holomorphic function from a compact complex manifold to $\mathbb{C}^{n+1}$ and hence it is a constant. Any non zero holomorphic function on $\mathbb{C}^{n+1}$ will have a non trivial kernel and hence s is a zero section.

The dual bundle of $\mathcal{O}(-1)$ is denoted as $\mathcal{O}(1)$ and it is called the *hyperplane line bundle*. The transition functions of $\mathcal{O}(1)$ will be $g_{ij}^{(1)}([\overrightarrow{Z}]) = \frac{Z^j}{Z^i}$. Another way to define $\mathcal{O}(1)$ is to construct it using these as transition functions,

$$\mathcal{O}(1) = \frac{\sqcup U_i \times \mathbb{C}}{(p, v_i) \sim (p, g_{ij}^{(1)} v_j)}.$$

Let f be a homogeneous 1-degree polynomial in $\mathbb{C}^{n+1}$. On U_i we can see this as the local section corresponding to $(Z^i)^{-1} f$. Then the transition functions help us to patch up to get a

well defined global section of $\mathcal{O}(1)$. In fact, we can prove that any section of $\mathcal{O}(1)$ corresponds to a homogeneous degree 1 polynomials on $\mathbb{C}^{n+1}$.
One of the operation on line bundles that produce new line bundles are by taking tensor products. We can define new line bundles $\mathcal{O}(m)$ for integers m, which are obtained by taking tensor product of $\mathcal{O}(1)$ or $\mathcal{O}(-1)$, and it is defined as follows,

$$\mathcal{O}(m) \coloneqq \begin{cases} \mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1), & m \text{ times if } , > 0 \\ \mathcal{O}(-1) \otimes \cdots \otimes \mathcal{O}(-1), & -m \text{ times if } m < 0. \end{cases}$$

Similar to equation 2.1, the transition function of $\mathcal{O}(m)$ is given by

$$g_{ij}^{(m)}([\overrightarrow{Z}]) = \left(\frac{Z_j}{Z_i}\right)^m. \tag{2.2}$$

For $m > 0$, sections of $\mathcal{O}(m)$ can be thought as homogeneous polynomials of degree m, where restricted to U_i, a degree m homogeneous polynomial f can be thought of as a $(Z^i)^{-m} f$. In fact every line bundle on $\mathbb{CP}^n$ is isomorphic to $\mathcal{O}(m)$ for some integer m.

2.3 Almost Complex Structure

An *almost complex structure* on a smooth manifold is a bundle endomorphism, $J : TM \to TM$ such that $J^2 = -Id$. The intuition is, J acts like multiplication by $\sqrt{-1}$ on tangent vectors. Now, $J^2 = -Id$ implies that M must be an even dimensional smooth manifold, for if it is odd, then it must have a real eigenvalue. More importantly, we show that every complex manifold has a canonical almost complex structure on it.
Given a complex manifold M, let $z^1, \ldots, z^n$ be local holomorphic coordinates around a point, write $z^i = x^i + \sqrt{-1}y^i$. Then,

$$\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right\}$$

is a local basis for TM. Define

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \ J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}. \tag{2.3}$$

This endomorphism is well defined and independent of the choice of holomorphic coordinates. Let $\{\widetilde{z}^i\}$ be another choice of local holomorphic coordinates, and let $\widetilde{z}^i = \widetilde{x}^i + \sqrt{-1}\widetilde{y}^i$. By change of variable formula we have,

$$\frac{\partial}{\partial \widetilde{x}^i} = \frac{\partial x^j}{\partial \widetilde{x}^i}\frac{\partial}{\partial x^j} + \frac{\partial y^j}{\partial \widetilde{x}^i}\frac{\partial}{\partial y^j}, \quad \frac{\partial}{\partial \widetilde{y}^i} = \frac{\partial x^j}{\partial \widetilde{y}^i}\frac{\partial}{\partial x^j} + \frac{\partial y^j}{\partial \widetilde{y}^i}\frac{\partial}{\partial y^j}.$$

Then,

$$J\left(\frac{\partial}{\partial \widetilde{x}^i}\right) = \frac{\partial x^j}{\partial \widetilde{x}^i}\frac{\partial}{\partial y^j} - \frac{\partial y^j}{\partial \widetilde{x}^i}\frac{\partial}{\partial x^j} = \frac{\partial y^j}{\partial \widetilde{y}^i}\frac{\partial}{\partial y^j} + \frac{\partial x^j}{\partial \widetilde{y}^i}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial \widetilde{y}^i},$$

where the last but one equality is by the Cauchy-Riemann equations. Similarly, we obtain that $J\left(\frac{\partial}{\partial \widetilde{y}^i}\right) = -\frac{\partial}{\partial \widetilde{x}^i}$. Hence, given a complex structure on a manifold we have a canonical almost complex structure on it. In general, the converse is not true, that is every almost complex structure does not arise from a complex structure. An example arises in S^6, which has an almost complex structure, which is not arising from a complex structure.
Till now we were dealing with the real tangent bundle TM, which is a C^∞ vector bundle on the manifold. On complex manifolds it is more convenient to work with the complexified tangent bundle,

$$T^{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}.$$

We will denote $v\otimes\sqrt{-1}$ as $\sqrt{-1}v$. The endomorphism J can be extended to an endomorphism on the complex vector bundle $T^{\mathbb{C}}M$. Define

$$\frac{\partial}{\partial z^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right), \quad \frac{\partial}{\partial \overline{z}^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right).$$

Then,

$$\left\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \overline{z}^1}, \dots, \frac{\partial}{\partial \overline{z}^n}\right\} \tag{2.4}$$

is a local basis for $T^{\mathbb{C}}M$. Observe that,

$$J\left(\frac{\partial}{\partial z^i}\right) = \sqrt{-1}\frac{\partial}{\partial z^i}, \ J\left(\frac{\partial}{\partial \overline{z}^i}\right) = -\sqrt{-1}\frac{\partial}{\partial \overline{z}^i}. \tag{2.5}$$

We can decompose

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M, \tag{2.6}$$

where $T^{1,0}M$ and $T^{0,1}M$ are the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces, respectively. By equation (2.5), $\{\frac{\partial}{\partial z^i}\}_{i=0}^n$ and $\{\frac{\partial}{\partial \overline{z}^i}\}_{i=0}^n$ are basis for $T^{1,0}M$ and $T^{0,1}M$ respectively. Observe that $T^{1,0}M$ will be a holomorphic vector bundle, and so will rightly be called the *holomorphic tangent bundle* and $T^{1,0}M$ will be called the *anti holomorphic tangent bundle*. The *cotangent bundle* $\Omega^1_{\mathbb{C}}(M)$ is the dual of the complex tangent bundle $T^{\mathbb{C}}M$. A basis for $\Omega^1_{\mathbb{C}}(M)$ locally is given by

$$\{dz^1, \ldots, dz^n,\ d\overline{z}^1, \ldots d\overline{z}^n\},$$

where $z^i = x^i + \sqrt{-1}y^i$ and $\overline{z}^i = x^i - \sqrt{-1}y^i$. This basis is dual to (2.4). We can write $\Omega^1_{\mathbb{C}}(M) = \Omega^{1,0}M \oplus \Omega^{0,1}M$, where $\Omega^{1,0}M$ and $\Omega^{0,1}M$ are spanned by $\{dz^i\}_{i=1}^n$ and $\{d\overline{z}^i\}_{i=1}^n$. Similarly we can decompose higher order forms as

$$\Omega^r_{\mathbb{C}} = \bigoplus_{p+q=r} \Omega^{p,q}M,$$

where $\Omega^{p,q}M$ in local coordinates is spanned by,

$$dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}$$

for $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_p$.

2.4 Hermitian Metrics

The first chapter dealt with Riemannian metric, which is a smoothly varying inner product on the tangent space of a manifold. Here we will be interested in Riemannian metrics on complex manifolds that is compatible with the complex structure in a nice way. Our naive intuition of an almost complex structure is that it acts like multiplication by $\sqrt{-1}$. So informally, one would expect $g(JX, JY) = \sqrt{-1}\overline{\sqrt{-1}}g(X,Y) = g(X,Y)$. In general, for a Riemannian metric on a complex manifold, this need not be true.

Definition 2.4.1. *A Riemannian metric g on a complex manifold M is called* Hermitian *if $g(JX, JY) = g(X,Y)$.*

This essentially says that Hermitian metrics are Riemannian metrics such that J is orthogonal preserving. Splitting the complexified tangent space as $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, let the matrix corresponding to g in local coordinates be

$$[g] = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.7}$$

Then,

$$\begin{aligned} A_{ij} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) &= g\left(J\left(\frac{\partial}{\partial z^i}\right), J\left(\frac{\partial}{\partial z^j}\right)\right) \\ &= g\left(\sqrt{-1}\frac{\partial}{\partial z^i}, \sqrt{-1}\frac{\partial}{\partial z^j}\right) \\ &= -g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = 0. \end{aligned}$$

Similarly, D is also 0. Now consider,

$$B_{ij} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j}\right) = \overline{g\left(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial z^j}\right)} = \overline{C_{ij}}.$$

Since g is symmetric, B as a matrix will be Hermitian and hence locally,

$$(g) = \begin{pmatrix} 0 & h \\ \overline{h} & 0 \end{pmatrix}$$

where h is a Hermitian matrix, and in local coordinates Hermitian metrics are determined by the entries,

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j}\right), \ \ \overline{g_{i\bar{j}}} = g_{j\bar{i}}.$$

Given a Hermitian metric g consider a 2-tensor ω defined as

$$\omega(X, Y) \ \ := \ \ g(J, Y).$$

Observe that $\omega(X, Y) = g(JX, Y) = g(J^2X, JY) = -g(X, JY) = -\omega(Y, X)$, implies that ω is a $1,1$ form, also called the metric form. In local coordinates, ω can be written as

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\overline{z}^j. \tag{2.8}$$

The term *Hermitian metric* will be interchangeably used to describe g, ω or h. All three encapsulate the same information, but can be useful in different scenarios.

On a real manifold of dimension m, that the volume form is given by

$$vol(M) = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^m.$$

On a complex manifold M of dimension n, equipped with a Riemannian Hermitian metric ω, the volume form is given by

$$vol(M) = \frac{\omega^n}{n!}. \tag{2.9}$$

2.5 Kähler Metrics

Definition 2.5.1. *A Hermitian metric g, with a corresponding $1,1$ form ω is called* Kähler *if $d\omega = 0$.*

In local coordinates if,

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j,$$

then

$$d\omega = \sqrt{-1}\frac{\partial g_{i\bar{j}}}{\partial z^k}dz^k \wedge dz^i \wedge d\bar{z}^j + \sqrt{-1}\frac{\partial g_{i\bar{j}}}{\partial \bar{z}^k}d\bar{z}^k \wedge dz^i \wedge d\bar{z}^j.$$

Hence ω is Kähler if and only if

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{k\bar{j}}}{\partial z^i}. \tag{2.10}$$

A complex manifold M equipped with a Kähler metric ω is called a *Kähler manifold.* We denote a Kähler manifold by (M, ω). The above definition essentially describes a Kähler metric as a complex manifold with a symplectic structure. This definition is useful when we are given a Hermitian metric and want to check if it is Kähler. From an analytic point, of view we will be interested in another equivalent formulation. Recall that, on real manifolds we had the concept of normal coordinates. It is natural to ask if, for a Hermitian metric on a complex manifold do we have holomorphic normal coordinates. In general the answer is NO!!

Theorem 2.5.1. (Local holomorphic normal coordinates on Kähler manifolds) *Let g be a Hermitian metric on a complex manifold. Then g is Kähler if and only if for any point*

$p \in M$, there exist a holomorphic normal coordinates $z^1, \ldots, z^n$, that is

$$g_{i\bar{j}}(p) = \delta_{i\bar{j}} \ ; \ \frac{\partial g_{i\bar{j}}}{\partial z^k}(p) = \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^k}(p) = 0.$$

Proof. The idea of the proof is very similar to one of the proofs of normal coordinates on Riemannian manifolds, but during the computation we will need the Kähler criterion.
Let p be a point on a Kähler manifold (M, ω). We can always choose a local holomorphic coordinates, $w^1, \ldots, w^n$ around p such that the metric $\widetilde{g}_{i\bar{j}}(p) = \delta_{i\bar{j}}(p)$. In local coordinates with respect to w^i, let the metric be,

$$\widetilde{g}_{i\bar{j}} = \delta_{i\bar{j}} + a_{i\bar{j}k} w^k + a_{i\bar{j}\bar{k}} \overline{w}^k + O(|w|^2),$$

where $O(|w|^2)$ denotes the terms of order greater than or equal to 2.
By the inverse function theorem, in a small neighborhood, we can define a new holomorphic coordinates $z^1, \ldots, z^n$ satisfying

$$w^i = z^i - \frac{1}{2} b_{ijk} z^j z^k$$

for some constants b_{ijk}, such that $b_{ijk} = b_{ikj}$. Now,

$$\frac{\partial w^i}{\partial z^j} = \delta^i{}_j - b_{ijk} z^k.$$

Let $g_{i\bar{j}}$ be the metric in local coordinates with respect to z^i. Then,

$$g_{i\bar{j}} = \delta_{i\bar{j}} + a_{i\bar{j}k} z^k + a_{i\bar{j}\bar{k}} \overline{z}^k - b_{jki} z^i - \overline{b_{ikj}} \overline{z}^k + O(|z|^2).$$

To obtain normal coordinates we will need that $b_{jki} = a_{i\bar{j}k}$ and $b_{ikj} = \overline{a_{i\bar{j}\bar{k}}} = a_{\bar{i}jk}$. By equation (2.10), the Kähler criterion is equivalent to $a_{i\bar{j}k} = a_{k\bar{j}i}$, hence by defining $b_{jki} \coloneqq a_{i\bar{j}k}$, we get a local holomorphic normal coordinates at p.
To prove the converse, if in some local coordinates $\omega = \sum_i dz^i \wedge d\overline{z}^i + O(|z|^2)$, then $d\omega = 0$. □

The most trivial example of a Kähler manifold is $\mathbb{C}^n$, where the tangent space at a point can be canonically identified with $\mathbb{C}^n$ and the inner product on the tangent space is the normal inner product on $\mathbb{C}^n$. Let us explore another important example.

Example 4. (*Fubini-study metric*) We have seen that projective spaces $\mathbb{CP}^n$ are one of the important examples of complex manifolds. Additionally, $\mathbb{CP}^n$ has a natural Kähler metric ω_{FS}. Recall the quotient map $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$. Let $s : U \subset \mathbb{CP}^n \to \mathbb{C}^{n+1}$ be a local section, that is $\pi \circ s = id$. Then, define

$$\omega_{FS} = \sqrt{-1}\partial\overline{\partial}\log||s||^2. \tag{2.11}$$

To see this is well defined, if $s' : V \to \mathbb{C}^{n+1}\backslash\{0\}$ is another section, then on $U \cap V$ we have, $s' = fs$ for a holomorphic function f, and

$$\sqrt{-1}\partial\overline{\partial}\log||s'||^2 = \sqrt{-1}\partial\overline{\partial}\log||s||^2 + \sqrt{-1}\partial\overline{\partial}\log f + \sqrt{-1}\partial\overline{\partial}\log\overline{f} = \sqrt{-1}\partial\overline{\partial}\log||s||^2.$$

Now consider a section of $\mathcal{O}(m)$, which is a m degree polynomial on $\mathbb{C}^{n+1}$. Let D be the zero of this section. Assume zero is a regular value of the polynomial, then D is a submanifold of $\mathbb{CP}^n$ and the fubini-study metric restricted to D will be a Kähler metric. This gives us a simple recipe for getting a large number of examples of Kähler metrics.

Given a Kähler manifold (M, ω), we know that ω is a closed real 2-form and hence it will define a cohomology class in $H^2(M, \mathbb{R})$. The following lemma gives a parametrisation of a cohomology class in compact Kähler manifolds. This lemma will play a key role in chapter 4, in converting our geometric problem into a partial differential equation.

Lemma 2.5.2. *($\partial\overline{\partial}$-Lemma) Let M be a compact Kähler manifold. If α and η are two real $(1,1)$ forms in the same cohomology class, then there is a global real valued function $f : M \to \mathbb{R}$ such that,*

$$\eta = \alpha + \sqrt{-1}\partial\overline{\partial}f.$$

The proof uses the Hodge decomposition theorem for Compact Kähler manifolds. It is important to note that this lemma is true for Kähler manifolds and not in general for Hermitian metric, and will be key in the proof of Calabi Conjecture.

2.6 Levi-Civita and Chern Connections

In the previous chapter, we saw connections on vector bundles as differentiating a section along a vector field. More importantly we saw a natural connection on the tangent bundle of a

Riemannian manifold, the Levi-Civita connection. On a Kähler manifold, we can differentiate vector fields and even tensors by using the Levi-Civita connection. Since the connection is compatible with the Riemannian metric we have, $\nabla g = 0$. For a Kähler metric, in holomorphic coordinates the almost complex structure is constant, implies $\nabla J = 0$ and hence $\nabla \omega = 0$.
For notational convenience from now onwards we will denote

$$\partial_i := \frac{\partial}{\partial z^i},\ \partial_{\bar{i}} := \frac{\partial}{\partial \overline{z}^i},\ \ \nabla_i := \nabla_{\frac{\partial}{\partial z^i}},\ \nabla_{\bar{i}} := \nabla_{\frac{\partial}{\partial \bar{z}^i}}.$$

Taking analogy from Riemannian geometry, we can define Christoffel symbols for the Levi-Civita connection on a complex manifold as

$$\nabla_j \partial_k = \Gamma^i{}_{jk} \partial_i + \Gamma^{\bar{i}}{}_{jk} \partial_{\bar{i}}.$$

In a similar manner, we can also define other Christoffel symbols. For any vector field X, we have

$$\begin{aligned} (\nabla J)X &= \nabla(J(X)) - J(\nabla(X)), \\ \nabla(J(X)) &= J(\nabla(X)). \end{aligned}$$

Now,

$$J(\nabla_j \partial_k) = \nabla_j J(\partial_k) = \sqrt{-1} \nabla_j \partial_k$$

and hence $\nabla_i \partial_k \in T^{1,0}(M)$. Similarly, $\nabla_{\bar{i}} \partial_k \in T^{1,0}(M)$, and $\nabla_i \partial_{\overline{k}} \in T^{0,1}(M)$ for all i and k. Since the connection is torsion free, we have,

$$\nabla_{\bar{i}} \partial_k = \nabla_k \partial_{\bar{i}} = 0.$$

Hence the Christoffel symbol is completely determined by $\Gamma^i{}_{jk}$, and

$$\Gamma^i{}_{jk} = \overline{\Gamma^{\bar{i}}{}_{\bar{j}\bar{k}}}\ , \tag{2.12}$$

and every other Christoffel symbol is zero. We can calculate the Christoffel symbol in terms of the metric,

$$\Gamma^i{}_{jk} = g^{i\bar{l}} \partial_j g_{k\bar{l}}. \tag{2.13}$$

This can also be seen from the formula for Christoffel symbols in Riemannian geometry, and substituting zero for the appropriate terms.

2.6.1 Chern connection

We have seen when a Riemannian metric on a complex tangent bundle is Hermitian. In general a Hermitian metric on a complex vector bundle is a section $\mathfrak{h}$ of the bundle $V^* \otimes \overline{V^*}$ such that,

1. (*Positive definite*) $\mathfrak{h}(v \otimes \overline{v}) > 0$ for all $v \in V$, and
2. (*Hermitian Symmetry*) $\mathfrak{h}(v \otimes \overline{w}) = \overline{\mathfrak{h}(w \otimes \overline{v})}$ for all $v, w \in V$.

For a Riemannian Hermitian metric on the tangent bundle, the Hermitian metric corresponds to it's restriction the holomorphic tangent bundle $T^{1,0}M$. The metric $\mathfrak{h}$ in local coordinates corresponds to the matrix B in equation (2.7).

For a Riemannian metric, the Levi-Civita connection was the most natural connection that respected the metric structure. On a holomorphic vector bundle with a Hermitian metric, we need to know what is the most natural connection. In naive terms, the connection has to respect both the structure, namely the Hermitian metric and the holomorphic structure of the vector bundle.

Definition 2.6.1. (Chern Connections) *Let (V, M) be a holomorphic vector bundle equipped with a Hermitian metric $\mathfrak{h}$. The* Chern connection *is the unique connection satisfying*

1. $\partial_k(\langle s_1, s_2\rangle) = \langle \nabla_k s_1, s_2\rangle + \langle s_1, \nabla_{\overline{k}} s_2\rangle$, *for two local sections s_1 and s_2 and*
2. $\nabla_{\overline{i}} s = 0$, *for a holomorphic section s on V.*

where $< .,. >$ is the Hermitian inner product $\mathfrak{h}$, that is $< s_1, s_2 > \mathfrak{h}(s_1, s_2)$ for sections s_1 and s_2.

Locally, the second criterion is the Cauchy Riemann equations. In general, the chern connection does not commute and we can again define the curvature of the chern connection in a similar manner.

For a complex manifold M, with a Hermitian metric on the holomorphic tangent bundle

$T^{1,0}M$, we have two notions to differentiate vector fields, the Levi-Civita connection and the Chern connection. Due to the following theorem, we will not have this ambiguity on Kähler manifolds.

Theorem 2.6.1. *A Hermitian metric on a complex manifold M is Kähler if and only if the Levi-Civita and Chern connections coincide on $T^{1,0}M$.*

Curvature is the failure of the connection to commute, hence

$$\begin{aligned} R_{i\ k\bar{l}}^{\ j}\partial_j &= (\nabla_k\nabla_{\bar{l}} - \nabla_{\bar{l}}\nabla_k)\partial_i \\ &= -\nabla_{\bar{l}}\Gamma^j{}_{ik}\partial_j \\ &= -\partial_{\bar{l}}(\Gamma^j{}_{ik})\partial_j - \Gamma^j{}_{ik}\nabla_{\bar{l}}\partial_j = -\partial_{\bar{l}}(\Gamma^j{}_{ik})\partial_j. \end{aligned}$$

Thus,

$$R_{i\ k\bar{l}}^{\ j} = -\partial_{\bar{l}}\Gamma^j{}_{ik},$$

or

$$R_{i\bar{j}k\bar{l}} = g_{m\bar{j}}R_{i\ k\bar{l}}^{\ m} = g_{m\bar{j}}\partial_{\bar{l}}\Gamma^m{}_{ki}. \tag{2.14}$$

Substituting for $\Gamma^m{}_{ki}$ in terms of the metric, we get

$$R_{i\bar{j}k\bar{l}} = -\partial_k\partial_{\bar{l}}g_{i\bar{j}} + g^{p\bar{q}}(\partial_k g_{i\bar{q}})(\partial_{\bar{l}}g_{p\bar{j}}). \tag{2.15}$$

This expression is similar to that in Riemannian geometry, but actually simpler. In normal coordinates, the expression for curvatures is of the form $\partial\overline{\partial}g$. Hence, similar to Geodesic normal coordinates, the coefficient of the second order terms in holomorphic normal coordinates is the curvature tensor.

The *Ricci curvature* denoted by $Ric(g)$ is the trace of the $R_{i\bar{j}k\bar{l}}$ with two variables fixed. That is

$$Ric(g) := \sqrt{-1}R_{i\bar{j}}dz^i \wedge d\bar{z}^j, \tag{2.16}$$

where,

$$R_{i\bar{j}} = g^{k\bar{l}}R_{i\bar{j}k\bar{l}}.$$

Theorem 2.6.2. *Given a Kähler Manifold (M, g), the Ricci curvature is given by,*

$$Ric(g) = -\sqrt{-1}\partial\overline{\partial}\log\det(g).$$

It is a closed $(1,1)$ form and the cohomology class $[Ric(g)] \in H^2(M,\mathbb{R})$ is independent of the choice of the Kähler metric.

Proof. In local coordinates,

$$\begin{aligned} -\partial_{\bar{j}}\partial_i \log\det(g) &= -\partial_{\bar{j}}(g^{k\bar{j}}\partial_i g_{k\bar{l}}) = -\partial_{\bar{j}}(\Gamma^k{}_{ik}) \\ &= R_k{}^k{}_{i\bar{j}} = R_{i\bar{j}}. \end{aligned}$$

The Ricci curvature $Ric(g)$ is closed, since $d(Ric(g)) = d(-\sqrt{-1}\partial\overline{\partial}\log\det(g)) = 0$. Let g, h be two Riemannian Hermitian metrics and ω_g, ω_h be the respective metric forms. Then

$$\begin{aligned} Ric(g) - Ric(h) &= \sqrt{-1}\partial\overline{\partial}\log\left(\frac{\det(h)}{\det(g)}\right) \\ &= \sqrt{-1}\partial\overline{\partial}\log\left(\frac{\omega_h^n}{\omega_g^n}\right). \end{aligned}$$

Since $\dfrac{\omega_h^n}{\omega_g^n}$ is a globally defined function,

$$[Ric(g)] = [Ric(h)].$$

□

The *first Chern class* is defined as

$$c_1(M) \coloneqq \frac{1}{2\pi}[Ric(g)] \in H^2(M,\mathbb{R}). \tag{2.17}$$

Here the 2π is a normalization factor. The first Chern class $c_1(M)$ is one cohomology class in a larger family of cohomology classes, known as *Chern classes*. Given a Kähler metric ω, we can define the matrix valued curvature 2-form F of type $(1,1)$ as

$$F_i{}^j = g^{j\bar{p}}R_{i\bar{p}k\bar{l}}dz^k \wedge d\bar{z}^l.$$

Consider the polynomial,

$$\det\left(I + t\frac{\sqrt{-1}}{2\pi}F\right) = 1 + tc_1(g) + t^2c_2(g) + \dots$$

where $c_i(g)$ is a closed real (i,i) form. The class $[c_i(g)]$ will be independent of the metric g, and hence we denote $c_i(M) := [c_i(g)]$, also known as the i^{th} chern class of the manifold. In our case $c_1(M) := \frac{1}{2\pi}[Ric(g)]$, which intuitively makes sense as $c_1(M)$ is essentially taking the trace of the curvature F.

Definition 2.6.2. *A $(1,1)$ form α is said to be positive (respectively negative) if in local coordinates, the matrix is positive definite (respectively negative definite), that is if*

$$\alpha = \sqrt{-1}\alpha_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

then α is positive if the matrix $(\alpha_{i\bar{j}})$ is positive definite.
A cohomology class $\Omega \in H^2(M, \mathbb{R})$ is positive (resp negative) if there exists a $\alpha \in \Omega$ such that α is positive (resp negative).
A cohomology class is said to be equal to zero if it is cohomologous to zero.

An important result related to Ricci curvature and the first chern class on Kähler manifolds is the Calabi-Yau theorem. We will discuss more about the theorem, its proof and some applications in chapters 4 and 5.

Chapter 3

Analytic Framework

In this chapter, we discuss various concepts and results from analysis and PDE, that we require over the course in the proof of the Calabi-Yau theorem. The Laplace's and Poisson's equations are among the most basic partial differential equations (PDE), with the equations surfacing in Ohm's law, Brownian motion, Edge detection to name a few. Given a function, $f : \Omega \to \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^n$,

$$\begin{aligned} \text{Laplace's Equation} &: \quad \Delta u = 0 \\ \text{Poisson's Equation} &: \quad \Delta u = f. \end{aligned}$$

Solutions to the Laplace's equation are called *harmonic functions.* An important rational behind the study of Laplace equations is that it is the simplest example of a much larger class of differential operators, called Elliptic PDE. The techniques used to study the Laplacian are heavily used in the study of general elliptic operators.
Harmonic functions can be also be defined via the mean value property due to the following theorem.

Theorem 3.0.1. *(Mean Value Theorem) Let U be a open set in $\mathbb{R}^n$. A smooth function $u : U \to \mathbb{R}$ is harmonic, if and only if for every $x \in U$ and $B_r(x) \subset U$ then,*

$$u(x) = \frac{1}{Vol(\partial B_r(x))} \int_{\partial B_r(x)} u(y)dy.$$

Here $B_r(x)$ is the open ball with center x and radius r.

Harmonic functions are real analytic functions and thereby posses a lot of the properties of real analytic functions (and more strikingly complex functions), like maximum modulus principle, Liouville's theorem and so on. A very standard reference for more properties of the Laplace operator on $\mathbb{R}^n$ can be found in [10].

3.1 Laplacian on manifolds

On a manifold, we cannot define the Laplacian as just $\sum_i \frac{\partial^2}{\partial x^i \partial x^i}$, the problem is that it depends on the choice of coordinates x_i and the expression changes when we change coordinates. Another way to see the Laplacian, is the divergence of the gradient of a function. So, let (M, g) be a Riemannian manifold with a Levi-Civita connection. If $u : M \to \mathbb{R}$ be a function on M. The gradient of u is the vector field dual to the form du. Locally,

$$\nabla u = \frac{\partial u}{\partial x^j} g^{ij} \frac{\partial}{\partial x^i}. \tag{3.1}$$

The divergence is the dot product of the Levi-Civita connection with a vector field. For a vector field X, the divergence is $div(X) = \nabla.X = \frac{\partial X^i}{\partial x^i} + \Gamma^i_{ik} X^k$. The Laplacian of M can be defined as

$$\Delta u \coloneqq div(\nabla u) = \sum_i \nabla.(\nabla u) = \frac{\partial}{\partial x^i}\left(g^{ij} \frac{\partial u}{\partial x^j} \right) + \Gamma^i_{ik} \frac{\partial u}{\partial x^j} g^{jk}. \tag{3.2}$$

In normal coordinates, it looks like the Laplacian on $\mathbb{R}^n$. There are few different ways to define the Laplace operator on riemanian manifolds. But we will stick to the above definition. It is important to note that even for the Laplacian operator on Riemannian manifolds we have results like the Stokes theorem, by parts formula.

On Kähler manifold we will use only one half of the Laplacian. If (M, ω) is a Kähler manifold then,

$$\Delta u = g^{i\bar{j}} \partial_i \partial_{\bar{j}} u. \tag{3.3}$$

Suppose we are looking for solutions to the equation $\Delta u = f$ on a compact complex manifold. By applying integration by parts, we have that, $\int \Delta u dV = 0$. Hence, a necessary condition for $\Delta u = f$ to have a solution, is that $\int f dV = 0$. On Kähler manifolds, in fact this is also sufficient.

Theorem 3.1.1. *Let (M, ω) be a Kähler manifold, f is a smooth function on M, and*

$$\int f\omega^n = 0,$$

then there exist a smooth function u such that $\Delta u = f$. More generally if $f \in C^0(M)$ then there exist a $W^{2,p}(M)$ solution for all p in the weak sense and eventually, we get u is a $C^{1,\alpha}(M)$ solution. Here $W^{2,p}(M)$ are the standard Sobolev spaces and $C^{1,\alpha}(M)$ are the Hölder spaces.

We will not go into the details of the proof.

3.2 Green's function for the Laplacian operator

The fundamental solution of the Laplacian is defined as

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi}\ln|x| & ;\ n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & ;\ n > 2, \end{cases}$$

where $x \neq 0$, and $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. In modern terminology Φ satisfies $\Delta\Phi(x) = \delta_0(x)$ in the sense of distributions. Then,

$$u(x) = \int_M \Delta\Phi(x-y)u(y)dy = \int_M \Phi(x-y)\Delta u(y)dy.$$

We can use this to find solutions to the Poisson's equations in $\mathbb{R}^n$, that is a solution to $\Delta u = f$ in $\mathbb{R}^n$ is $u(x) = \int_M \Phi(x-y)f(y)dy$. The Green's function is the extension of the above idea to the boundary value problem. Let Ω be a bounded domain in $\mathbb{R}$ with C^1 boundary and u is a solution to

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

For a fixed x, let ϕ^x be a solution of

$$\begin{aligned} -\Delta\phi^x &= 0, \text{ in } \Omega \\ \phi^x &= \Phi(y-x), \text{ on } \partial\Omega. \end{aligned}$$

Define the Green's function as

$$G(x, y) := \Phi(y - x) - \phi^x(y)$$

where $x \neq y$. Then,

$$u(x) = \int_\Omega f(y)G(x, y)dy - \int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x, y)dy.$$

As a distribution G solves the equation

$$\begin{aligned} -\Delta G_x(y) &= \delta_x(y) \text{ in } \Omega \\ G_x(y) &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $G_x(y) = G(x, y)$ and δ_x is the Dirac delta with singularity at x.

3.2.1 Green's function on Compact manifolds

The aim of this section is to define a Green's function and Green's function type formula for compact Riemannian manifolds. The first question to ask whether we can again find a function G, such that $\Delta G_x = \delta_x$. In Theorem 3.1.1 we saw that we can hope to have a solution to $\Delta u = f$ only if $\int f = 0$, but here $\int \delta_x \neq 0$. So we will define a Green's function on a manifold satisfying

$$\Delta G_x = \delta_x - \frac{1}{V}, \tag{3.4}$$

where V is the volume of the manifold. Note that if G exists, it is smooth when $x \neq y$. Once we have Green's function that satisfies equation (3.4). Then, for any function ϕ,

$$\begin{aligned} \phi(x) &= \int_M \phi(y)\Delta G_x(y)dy + \frac{1}{V}\int_M \phi(y)dy \\ &= \int_M \Delta\phi(y)G_x(y)dy + \frac{1}{V}\int_M \phi(y)dy. \end{aligned}$$

In deriving the existence Green's function we will need the following lemma

Lemma 3.2.1. *Let f, g be functions defined on $\Omega \times \Omega \setminus \{(x, x) | x \in \Omega\}$, where Ω is a bounded* *domain in* $\mathbb{R}^n$*, and*

$$\begin{aligned} |f(x,y)| &\leq \frac{C}{d(x,y)^p}, \\ |g(x,y)| &\leq \frac{C}{d(x,y)^q}, \end{aligned}$$

where $p, q \in (0, n)$ *and* C *is a constant, then*

$$h(x,y) := \int_M f(x,z)g(z,y)dz$$

is continuous for $x \neq y$ *and*

$$\begin{aligned} &|h(x,y)| < C \text{ if } p+q<n \\ &|h(x,y)| < C(1+|\ln d(x,y)|) \text{ if } p+q=n \\ &|h(x,y)| < Cd(x,y)^{n-p-q} \text{ if } p+q>n. \end{aligned}$$

Further in the first case where $p+q<n$*,* h *is continuous everywhere.*

For a fixed x, in a small neighborhood, we expect the Green's function to behave like the fundamental solution. Consider,

$$\widetilde{G}(x,y) = \begin{cases} -\dfrac{ln(d(x,y))}{2\pi} f(d(x,y)); & n=2 \\ \dfrac{(n-2)\omega(n)}{d(x,y)^{n-2}} f(d(x,y)); & n>2 \end{cases}$$

where f is a real valued function on $\mathbb{R}$, such that it is 1 in a neighborhood of 0, and zero outside the injectivity radius. It is easy to see that,

$$|\Delta \widetilde{G}_x(y)| \leq \frac{C}{d(x,y)^{n-2}}.$$

In a neighborhood, $\widetilde{G}$ behave like the fundamental solution. Looking at Δ as distributional derivative, we know that

$$\int_M \widetilde{G}_x(y)\Delta\phi(y)dy = \int_M \Delta\widetilde{G}_x(y)\phi(y)dy,$$

and

$$\Delta \int_M \widetilde{G}_x(y)\phi(y)dy = \int_M \Delta\widetilde{G}_x(y)\phi(y)dy. \tag{3.5}$$

The naive idea to get a Green's function is to write

$$G(x, y) = \widetilde{G}(x, y) + \psi(x, y),$$

this implies

$$\Delta_y \psi(x, y) = \delta_x(y) - \Delta\widetilde{G}_x(y) - \frac{1}{V}.$$

Here we would want $\delta_x(y) - \Delta\widetilde{G}_x(y)$ to be C^0 or better so that by Theorem 3.1.1, we will have a solution ψ. Even if it was L^2 we would have a result similar to Theorem 3.1.1. But we will not be able to exactly do this, we will rather through an iterative process solve such an equation.

Theorem 3.2.2. *There exists a Green's function G satisfying equation (3.4), such that for any smooth function ϕ*

$$\phi(x) = \frac{1}{V}\int_M \phi(y)dy + \int_M G(x, y)\Delta\phi(y)dy.$$

Proof. Let us define,

$$\psi_1(x, y) = \Delta\widetilde{G}_x(y) - \delta_x(y),$$

and

$$\psi_{i+1}(x, y) = \int_M \psi_i(x, z)\psi_1(z, y)dz,$$

Since

$$|\psi_i(x, y)| \leq \begin{cases} \dfrac{C}{d(x, y)^{n-2i}} & ; \quad i < n/2 \\ C & ; \quad i > n/2 \end{cases}$$

Fix a k such that $\frac{n}{2} < k \leq n$, Consider

$$G(x, y) = \widetilde{G}(x, y) + \sum_{i=0}^{k-1} \int_M \psi_i(x, z)\widetilde{G}(z, y)dz + \psi(x, y). \tag{3.6}$$

Infact, we will show that there exists a ψ such that equation (3.4) is true. Taking Laplacian on both sides,

$$\Delta G_x(y) = \Delta \widetilde{G}_x(y) + \sum_{i=0}^{k-1} \Delta_y \int_M \psi_i(x,z)\widetilde{G}(z,y)dz + \Delta_y \psi(x,y). \tag{3.7}$$

By equation (3.5),

$$\Delta_y \int_M \psi_i(x,z)\widetilde{G}(z,y)dz = \psi_i + \int_M \psi_i(x,z)\psi_1(z,y)dz = \psi_i - \psi_{i+1}$$

Hence, equation (3.7), reduces to

$$\delta_x(y) - \frac{1}{V} = \delta_x(y) - \psi_1 + \sum_{i=0}^{k-1}(\psi_i(x,y) - \psi_{i+1}(x,y)) + \Delta_y \psi(x,y).$$

So we need to find ψ such that,

$$\Delta\psi = \psi_k(x,y) - \frac{1}{V}.$$

Since we started with $k > \frac{n}{2}$, the function $\psi_k(x,y) - \frac{1}{V}$ is C^0 and hence by Theorem 3.1.1, we have a $C^{1,\alpha}$ solution ψ. The function $\psi_k(x,y)$ is smooth when $x \neq y$, hence ψ is smooth when $x \neq y$, and

$$\phi(x) = \frac{1}{V}\int_M \phi(y)dy + \int_M G(x,y)\Delta\phi(y)dy,$$

for any smooth function ϕ. □

The function ψ_1 would not even be L^2, so we iteratively repeated the process to ensure that we have a sufficiently regular function. Since G is defined upto a constant, we can ensure that $0 < G(x,y)$ or even ensure that $\int_M G(x,y) = 0$.

3.3 Elliptic Differential Operators

We will look at a more general class of differential operators called elliptic partial differential operators. For a domain $\Omega \subset \mathbb{R}^n$, a linear partial differential operator

$$Lu = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha u$$

is *elliptic* if $\sum_{|\alpha|=k} a_\alpha v^\alpha$ is invertible for $v \neq 0$. We will be interested in second order linear elliptic PDE. A second order linear PDE,

$$L = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_i b_i \frac{\partial}{\partial x^i} + c$$

is elliptic if and only if the matrix $[a_{ij}]$ is positive definite. Elliptic operators tend to behave like finite dimensional operators. We will need the following estimates on elliptic operators.

Theorem 3.3.1. *(Schauder estimates) Let (M, g) be a compact Riemannian manifold. Let L be a uniformly elliptic operator, and let $\alpha \in (0, 1)$, k be a positive integer. Then, there is a constant C such that,*

$$||u||_{C^{k+2,\alpha}(M)} \leq C(||L(u)||_{C^{k,\alpha}(M)} + ||u||_{L^1(M)}),$$

where C depends on the manifold , k, α, the ellipticity constants and the $C^{k,\alpha}$ norms of the coefficients of L. Additionally, if the coefficients of L are in $C^{k,\alpha}(M)$ and if $u \in C^2$ and $L(u) = f \in C^{k,\alpha}$ then u is in $C^{k+2,\alpha}(M)$.

This is a consequence of the local Schauder estimates on domains in $\mathbb{R}^n$. One of the consequences of the Schauder estimates is, that the solution space of Elliptic PDE is finite dimensional. We will use the Schauder estimate to prove higher regularity of solutions. For example, if we have a C^2 solution for an elliptic PDE with smooth coefficients, then by Schauder estimates we will in fact have smooth solutions. Another important result we will require is given below,

Theorem 3.3.2. *Let L be an elliptic second order operator with smooth coefficients on a compact Riemannian manifold M. For $\alpha \in (0, 1)$ and positive integer k, if $\rho \perp \ker L^*$ with respect to the L^2-norm. Then, there exists a unique u such that*

$$Lu = \rho; \quad u \perp \ker L.$$

Equivalently

$$L : (\ker L)^\perp \cap C^{k+2,\alpha} \to (\ker L^*)^\perp \cap C^{k,\alpha}$$

is an isomorphism.

We will also need the following theorem, on Hölder spaces on compact manifolds.

Theorem 3.3.3. *(Compact Embedding) Given a Riemannian manifold (M, g), then the normed linear space $C^{k,\alpha}(M)$ is compactly contained in $C^{l,\beta}(M)$ for $l + \beta < k + \alpha$, that is if $u_k : M \to \mathbb{R}$ is a bounded sequence of functions in $C^{k,\alpha}(M)$, then it has a convergent subsequence in $C^{l,\beta}$ for $l + \beta < k + \alpha$.*

The proof is a siple argument using the Arzela-Ascoli theorem. Apart from the above result we will need the Poincare inequality and Sobolev inequality, both can be found in Evans [10].

Chapter 4

The Calabi Yau Theorem

This is the main chapter of the thesis, and here we discuss Yau's celebrated proof of the Calabi-Yau theorem [3]. It was first stated as a conjecture by Eugenio Calabi. Yau received the fields medal partly for the proof of the theorem, and proving some results in algebraic geometry using the Calabi-Yau theorem. The techniques used in the proof goes beyond just this result, it has laid a foundation to solve other problems even in today's research. In fact a very closely related problem, the existence of Kähler-Einstein metrics, where the case $c_1(M) > 0$ was only resolved in this decade. A key aspect of the proof of the Calabi-Yau theorem will involve solving complex Monge-Ampere type equations. Large parts of the proof in this chapter are based of the references [2] and [1]. In Theorem 2.6.2, we had seen that for a Kähler metric g, the cohomology class $[Ric(g)]$ is independent of the metric g and this cohomology class is 2π times the first chern class, $c_1(M)$. In this chapter we look at a converse problem to this theorem on Kähler manifolds. A cohomology class in $H^2(M, \mathbb{R})$ contains real valued closed 2 forms on the manifold M, and a cohomology class in $H^{1,1}(M, \mathbb{C})$ contains closed 1,1 form on the manifold M.

Theorem 4.0.1. (Calabi-Yau theorem) *Let (M, ω_0) be a compact Kähler manifold such that $[\omega_0] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$. Given any $\Omega \in 2\pi c_1(M)$, there exists a unique Kähler metric $\omega \in [\omega_0]$ such that $Ric(\omega) = \Omega$.*

This is not just a converse to Theorem 2.6.2, but much stronger. It not only states that every element of $[Ric(g)]$ is Ricci of some Kähler metric, but in fact says that every element of $[Ric(g)]$, is the Ricci of a Kähler metric in any Kähler class. In the next chapter, we will discuss some applications of the theorem. For the rest of this chapter, our primary focus is the proof of the Calabi-Yau theorem.

4.1 Geometric problem to a PDE

The first step in the proof is to reformulate the given geometric problem as a PDE. The $\partial\overline{\partial}-$lemma a the key tool in accomplishing this. In local coordinates, let

$$\omega_0 = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

By theorem 2.6.2, we have that the Ricci curvature of ω_0 is given by,

$$Ric(\omega_0) = -\sqrt{-1}\partial\overline{\partial} \log \det g.$$

Since Ω and $Ric(\omega_0)$ are real $(1,1)$ forms in the same cohomology class, by the $\partial\overline{\partial}$-lemma there exists a smooth function f on M such that,

$$\Omega_0 = Ric(\omega_0) - \sqrt{-1}\partial\overline{\partial} f \tag{4.1}$$

where f is unique upto a constant. Suppose there is a $\omega \in [\omega_0]$ such that $Ric(\omega_0) = \Omega$, then by the $\partial\overline{\partial}$-lemma,

$$\omega = \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi \tag{4.2}$$

where ϕ is a smooth function M. Since $Ric(\omega) = \Omega = Ric(\omega_0) - \sqrt{-1}\partial\overline{\partial} f$, we have

$$-\sqrt{-1}\partial\overline{\partial} \log \det(g + \partial\overline{\partial}\phi) = -\sqrt{-1}\partial\overline{\partial} \log \det(g) - \sqrt{-1}\partial\overline{\partial} f$$

which is a non linear PDE of order 4. We now derive a second order non-linear PDE, integrating both sides,

$$\log \det(g + \partial\overline{\partial}\phi) = \log \det(g) + f, \tag{4.3}$$

where constant due to integration has been absorbed into f. The above equation is equivalent to

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\phi)^n = e^f \omega_0^n.$$

If $\omega = \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi$ is Kähler, then $\int \frac{\omega_0^n}{n!} = Vol(M) = \int \frac{\omega^n}{n!}$, and since f is defined upto a constant, we can choose f such that

$$\int (e^f - 1)\omega_0^n = 0.$$

The Calabi-Yau theorem is equivalent to the existence of $\phi : M \to \mathbb{R}$ such that

$$\left.\begin{array}{l} (\omega_0 + \sqrt{-1}\partial\overline{\partial}\phi)^n = e^f \omega_0^n \\ \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi > 0 \end{array}\right\}. \tag{4.4}$$

Note that f is a given information of the problem and, is determined by Ω, and the normalization criterion $\int (e^f - 1)\omega_0^n = 0$. This equation is a second order non linear PDE, also known as the complex Monge-Ampere equation. In general, Monge-Ampere equations are second order nonlinear PDE whose leading term is the determinant of the hessian of the unknown function.

We begin by proving the uniqueness of ω_0. Let $\omega_0, \omega_1 \in [\alpha]$ be Kähler metrics such that,

$$Ric(\omega_0) = Ric(\omega_1) = \Omega.$$

Then by the $\partial\overline{\partial}$-lemma,

$$\omega_0 = \omega_1 + \sqrt{-1}\partial\overline{\partial}\phi.$$

Since $Ric(\omega_0) = Ric(\omega_1)$, we get $\det(\omega_0) = \det(\omega_1)$. Thus, we have

$$\begin{aligned} 0 &= \omega_1^n - \omega_0^n \\ &= (\omega_0 + \sqrt{-1}\partial\overline{\partial}\phi) - \omega_0^n \\ &= \sqrt{-1}\partial\overline{\partial}\phi \wedge (\omega_0^{n-1} + \omega_0^{n-2} \wedge \omega_1 + \cdots + \omega_1^{n-1}) \end{aligned}$$

Multiplying both sides by $-\phi$ and integrating, we get

$$\begin{aligned} 0 &= -\sqrt{-1} \int \phi \partial\overline{\partial}\phi \wedge (\omega_0^{n-1} + \cdots + \omega_1^{n-1}) \\ &= \sqrt{-1} \int \partial\phi \wedge \overline{\partial}\phi \wedge (\omega_0^{n-1} + \cdots + \omega_1^{n-1}) \end{aligned}$$

Since ω_0 and ω_1 are positive definite, that is $\omega_0, \omega_1 > 0$, we arrive at

$$0 = \partial\phi \wedge \overline{\partial}\phi \wedge (\omega_0^{n-1} + \cdots + \omega_1^{n-1}) \geq \partial\phi \wedge \overline{\partial}\phi \wedge \omega_0^{n-1}.$$

To calculate $\partial\phi \wedge \overline{\partial}\phi \wedge \omega_0^{n-1}$, consider normal coordinates on ω_0, then

$$\begin{aligned}
\partial\phi \wedge \overline{\partial}\phi \wedge \omega_0^{n-1} &= \partial_i\phi\partial_{\bar{i}}\phi dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\
&= |\partial\phi|^2\omega_0^n = \frac{1}{2}|\nabla\phi|^2\omega_0^n.
\end{aligned}$$

Hence we have,

$$\int |\nabla\phi|^2\omega_0^n \leq 0.$$

This implies that, $\nabla\phi = 0$ and thus $\omega_0 = \omega_1$.
We just proved that if a solution exists, it is unique. We are now left with the herculean task of showing that there exists a smooth solution to the equation (4.4).
The strategy is to have a family of equations which also depend on a new parameter s, where $s = 0$ will correspond to a much simpler equation and at $s = 1$ we get the equation (4.4). This method is known as the method of continuity. For $0 \leq s \leq 1$, define

$$f_s \coloneqq sf + c(s) = 0,$$

where $c(s)$ is a function such that

$$\int_M (e^{f_s} - 1)\omega^n.$$

Consider the following equation depending on the parameter s,

$$\left.\begin{aligned}
&(\omega_0 + \sqrt{-1}\partial\overline{\partial}\phi)^n = e^{f_s}\omega_0^n \\
&\omega_0 + \sqrt{-1}\partial\overline{\partial}\phi > 0
\end{aligned}\right\}. \qquad (*_s)$$

Note that $f_1 = f$, and hence for $s = 1$ the equation reduces to the complex Monge-Ampere which we are trying to solve. We will show that the set of s for which $(*_s)$ has a solution is non empty, open, and closed. The idea for each step is as follows:

1. (Non-empty) For $s = 0$, $f_0 = 0$ and hence $\phi = 0$ is a solution to $(*_0)$.
2. (Open) Using the Implicit function theorem for Banach manifolds, we show that if there exists a solution to $(*_s)$, then for t close to s, $(*_t)$ has a solution.
3. (Closed) The aim will be to prove apriori estimates on the solution, and then use Theorem 3.3.3 to prove that for a convergent sequence s_i such that $(*_{s_i})$ has a solution, then the limit of s_i also has a solution. This step is the hardest, and is one of the main contributions by Yau.

4.2 Openness

Assume that for $s \in [0,1]$, $(*_s)$ has a solution. Let ϕ_s be a solution of $(*_s)$, that is

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\phi_s)^n = e^{f_s}\omega_0.$$

We need show that for t close to s, we have a solution for $(*_t)$. We need to find a solution ϕ_t such that,

$$\begin{aligned} \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi_t &= e^{f_t}\omega_0, \\ \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi_t &> 0. \end{aligned}$$

Defining, $\omega_s := \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi_s$, the above equation is equivalent to solving for ϕ_t in,

$$(\omega_s + \sqrt{-1}\partial\overline{\partial}(\phi_t - \phi_s))^n = e^{f_t - f_s}\omega_s^n.$$

Setting, $\psi = \phi_t - \phi_s$, we need to find a ψ such that for t close to s,

$$(\omega_s + \sqrt{-1}\partial\overline{\partial}\psi)^n = e^{f_t - f_s}\omega_s^n. \tag{4.5}$$

We have essentially changed the problem of proving openness around the origin. From the previous section we know that ω_s is unique, and hence ϕ_s is unique upto a constant. So, without loss of generality we can can assume that $\displaystyle\int_M \phi_s\omega_0^n = 0$. Thereby we will restrict ourselves to finding solutions in the space,

$$C_0^{k,\alpha}(M) := \left\{\phi \in C^{k,\alpha} | \int_M \phi\omega_0^n = 0\right\}. \tag{4.6}$$

Define the following function between Banach spaces:

$$G : C_0^{2,\alpha}(M) \times [0,1] \to C^{0,\alpha}(M) \text{ by } (\phi, t) \mapsto \log\frac{(\omega_s + \sqrt{-1}\partial\overline{\partial}\phi)^n}{\omega_s^n} - f_t + f_s.$$

Since ϕ_s is a solution to $(*_s)$, consequently $G(0,s) = 0$. We will use the implicit function theorem for Banach spaces to show that for each t in a small neighborhood of s there is a ψ such that $G(\psi, t) = 0$. To apply the implicit function theorem, we need to show that the Gateaux derivative with respect to the ψ variable, at $\psi \equiv 0$, is invertible.

If we define $\Phi : C^{2,\alpha}_0(M) \to C^{0,\alpha}(M)$, as

$$\Phi(\psi) := \log \frac{(\omega_s + \sqrt{-1}\partial\overline{\partial}\psi)}{\omega_s^n}$$

Then the derivative of Φ at $\psi \equiv 0$ is

$$\begin{aligned} D\Phi_{\psi\equiv 0} : C^{2,\alpha}_0(M) &\to V \subset C^{0,\alpha}(M) \\ D\Phi_{\psi\equiv 0}(u) &= \frac{\frac{d}{dt}|_{t=0}(\omega_s + \sqrt{-1}\partial\overline{\partial}tu)^n}{\omega_s^n} \end{aligned}$$

We can differentiate forms which depend on a variable, just like we do with normal functions. If $\alpha(x), \beta(x)$ are forms which depend on a variable x, then

$$\frac{d}{dx}(\alpha(x) \wedge \beta(x)) = \left(\frac{d}{dx}\alpha(x)\right) \wedge \beta(x) + \alpha(x) \wedge \left(\frac{d}{dx}\beta(x)\right)$$

and hence,

$$D\Phi_{\psi\equiv 0}(u) = \frac{n\sqrt{-1}\partial\overline{\partial}u \wedge \omega_s^{n-1}}{\omega_s^n}.$$

Since ω_s is Kähler, we can choose normal coordinates with respect to ω_s. Then, it is easy to see that

$$D\Phi_{\psi\equiv 0}(u) = \Delta_{\omega_s}(u), \tag{4.7}$$

where Δ_{ω_s} is the Laplace operator with respect to the ω_s metric. The Laplacian is a self adjoint elliptic second order operator, and observe that if $u \in Ker\Delta = Ker\Delta^*$, then

$$0 = 2n\int_M u\Delta u\omega_s^n = \int_M ||\nabla u||^2\omega_s^n.$$

Hence u is a constant, but since $\int u = 0$ we get $u = 0$. Hence, $(Ker\Delta)^\perp = C^{2,\alpha}(M)$. By theorem 3.3.2, $D\Phi_{\psi\equiv 0}$ is an isomorphism. Thus in a neighborhood of s, we have a $C^{2,\alpha}$ function ϕ_t such that

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\phi_t)^n = e^{f_t}\omega_0. \tag{4.8}$$

Observe that the coefficients of equation (4.8) are C^∞ and hence by Schauder estimates, we have that $\phi_t \in C^{k,\alpha}(M)$ for all k which implies ϕ_t is smooth.
Since ω_s is positive definite, if required we can choose a smaller neighborhood of s and ensure that ω_t is also positive definite. Hence, we have shown that the set of s for which $(*_s)$ has a

solution is open.

4.3 Closedness

To show that the set of parameters s is closed, consider a sequence $s_i \in [0,1]$ such that $s_i \to s$. We need to show that if for each s_i, equation $(*_{s_i})$ has a solution, then $(*_s)$ also has a solution. We will find apriori estimates on $||\phi||_{C^{2,\alpha}}$, then by Theorem 3.3.3, we will have a convergent subsequence of ϕ_i in $C^2(M)$. In order to find $C^{2,\alpha}$ estimates we will first find apriori uniform C_0 and $\partial\overline{\partial}$-estimates on the solution.

4.3.1 C_0 estimate

The aim is to find a uniform estimate on the C_0 norm of the solution. Yau in his original paper witfully used a technique called Moser's iteration method as a key step to find the C_0 estimate. Apart from that we will be using the Green's formula for compact manifolds, Sobolev inequality and Poincare inequality.
Since ϕ is unique upto a constant, we can assume that $\sup \phi = -1$ (This will help us in the calculation ahead). Our aim is to prove that $\sup |\phi| < C$ where C does not depend on the parameter s. It is easy to see that if we have a C_0 bound on ϕ such that $\sup \phi = -1$, then we also have a uniform C_0 bound on ϕ with zero integral.
In the previous chapter, we saw a Green's formula for compact manifolds:

$$\phi(x) = \frac{1}{V}\int_M \phi(y)\omega_0^n - \frac{1}{V}\int_M \Delta\phi(y)G(x,y)\omega_0^n, \tag{4.9}$$

where $V = \int_M \omega_0^n$. Since G is unique upto a constant, we can also ensure that,

$$0 \le G(x,y) \le \frac{C}{d(x,y)^{2n-2}},$$

for some constant C. Since M is compact, assume that ϕ achieves its supremum at x_o, then

$$-1 = \phi(x_0) = -\frac{1}{V}\int_M |\phi(y)|\omega_0^n - \frac{1}{V}\int_M \Delta\phi(y)G(x_0,y)\omega_0^n. \tag{4.10}$$

Since $\omega_0 + \sqrt{-1}\partial\overline{\partial}\phi$ is a metric, it is positive definite, hence it has a positive trace. Now ω_ϕ in normal coordinates with respect to ω_0 has diagonal elements $1 + \partial_i\partial_{\bar{i}}$, and hence trace of ω_ϕ is $n + \Delta\phi > 0$. Substituting in equation (4.10), we get

$$-1 + \frac{1}{V}\int_M |\phi(y)|\omega_0^n = -\frac{1}{V}\int_M \Delta\phi(y)G(x_0, y)\omega_0^n \leq \frac{n}{V}\int_M G(x_0, y)\omega_0^n.$$

By using the bound on Green's function and M is compact, we get

$$||\phi||_{L^1} = \frac{1}{V}\int_M |\phi(y)|\omega_0^n \leq C. \tag{4.11}$$

We have obtained a L^1 bound on the solution. Now we get a bound on L^2 norm of ϕ in terms of the L^1 norm.

$$\begin{aligned} ||\phi||_{L^1} = \frac{1}{V}\int_M |\phi(y)|\omega_0^n &\geq \frac{C}{V}\int_M (1 - e^{f_s})\phi\omega_0^n \\ &\geq \frac{C}{V}\int_M (\omega_0^n - (\omega_0 + \sqrt{-1}\partial\overline{\partial}\phi)^n)\phi \\ &\geq \frac{C}{V}\int_M \phi\partial\overline{\partial}\phi \wedge \omega_0^{n-1}. \end{aligned}$$

By the calculation in uniqueness section and by the Poincare inequality we get,

$$\begin{aligned} ||\phi||_{L^1} &\geq \frac{C}{nV}\int_M |\nabla\phi|^2\omega_0^n \\ &\geq \frac{C}{nV}\left(\int_M |\phi|^2\omega_0^n - \left(\int_M \phi\omega_0^n\right)^2\right). \end{aligned}$$

Hence,

$$||\phi||_{L^2}^2 \leq \frac{nV}{C}||\phi||_{L^1} + ||\phi||_{L^1}^2 \leq C. \tag{4.12}$$

The last step to prove the the C^0 estimate, involves showing that the C_0 norm can be bounded by some constant multiple of the L^2 norm. Since $\phi \leq -1$, define $\widetilde{\phi} := -\phi \geq 1$ and $\omega_\phi = \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi = \omega_0 - \sqrt{-1}\partial\overline{\partial}\widetilde{\phi}$. Note that working with $\widetilde{\phi}$ will be easier than ϕ. From the Monge-Ampere equation,

$$\begin{aligned} (e^{f_s} - 1)\omega_0^n &= (\omega_0 - \sqrt{-1}\partial\overline{\partial}\widetilde{\phi})^n - (\omega_0)^n \\ &= -\sqrt{-1}\partial\overline{\partial}\widetilde{\phi} \wedge (\omega_\phi^{n-1} + \cdots + \omega_0^{n-1}). \end{aligned}$$

Let $p \geq 1$ be a real number, multiply both sides of the above equation by $\widetilde{\phi}^p$ and integrate, to get

$$\int_M \widetilde{\phi}^p(e^{f_s}-1)\omega_0^n = -\sqrt{-1}\int_M \widetilde{\phi}^p\partial\overline{\partial}\widetilde{\phi}\wedge(\omega_\phi^{n-1}+\cdots+\omega_0^{n-1}). \tag{4.13}$$

Computing the right hand side, we have

$$\begin{aligned}
-\frac{1}{V}\int_M \widetilde{\phi}^p\partial\overline{\partial}\widetilde{\phi}\wedge(\omega_\phi^{n-1}+\cdots+\omega_0^{n-1}) &\geq \frac{1}{V}\int_M \partial\widetilde{\phi}^p\wedge\overline{\partial}\widetilde{\phi}\wedge\omega_0^{n-1}\\
&= \frac{p}{V}\int_M \widetilde{\phi}^{p-1}\partial\widetilde{\phi}\wedge\overline{\partial}\widetilde{\phi}\wedge\omega_0^{n-1}\\
&= \frac{p}{V}\int_M (\widetilde{\phi}^{\frac{p-1}{2}}\partial\widetilde{\phi})\wedge(\widetilde{\phi}^{\frac{p-1}{2}}\overline{\partial}\widetilde{\phi})\wedge\omega_0^{n-1}\\
&= \frac{4p}{V(p+1)^2}\int_M \partial\widetilde{\phi}^{\frac{p+1}{2}}\overline{\partial}\wedge\widetilde{\phi}^{\frac{p+1}{2}}\wedge\omega_0^{n-1}\\
&= \frac{4p}{V(p+1)^2}\int_M |\nabla\widetilde{\phi}^{\frac{p+1}{2}}|^2\omega_0^n
\end{aligned}$$

Hence by the Poincare inequality, (Here n is the complex dimension, hence the real dim will 2n), the left hand side of the above expression is greater than or equal to

$$\frac{4p}{(p+1)^2}\left(C_1\left(\frac{1}{V}\int_M |\widetilde{\phi}|^{\frac{(p+1)n}{n-1}}\omega_0^n\right)^{\frac{n-1}{n}} - \frac{C_2}{V}\int_M |\widetilde{\phi}|^{p+1}\omega_0^n\right). \tag{4.14}$$

On the other hand, computing the left hand side of equation (4.13),

$$\int_M \widetilde{\phi}^p(e^{f_s}-1)\omega_0^n \leq \frac{C}{V}\int_M \widetilde{\phi}^p\omega_0^n \leq \frac{C}{V}\int_M \widetilde{\phi}^{p+1}\omega_0^n. \tag{4.15}$$

Combining the equations (4.14) and (4.15) we obtain,

$$\frac{4p}{(p+1)^2}\left(C_1\left(\frac{1}{V}\int_M |\widetilde{\phi}|^{\frac{(p+1)n}{n-1}}\omega_0^n\right)^{\frac{n-1}{n}} - \frac{C_2}{V}\int_M |\widetilde{\phi}|^{p+1}\omega_0^n\right) \leq \frac{C}{V}\int_M \widetilde{\phi}^{p+1}\omega_0^n.$$

This implies that

$$\left(\frac{1}{V}\int_M |\widetilde{\phi}|^{\frac{(p+1)n}{n-1}}\omega_0^n\right)^{\frac{n-1}{n}} \leq \frac{1}{V}\left(C(p+1)\left(\frac{p+1}{p}\right)\right)\int_M \widetilde{\phi}^{p+1}\omega_0^n.$$

Since $p \geq 1$, we see that $\frac{p+1}{p} \leq 2$ and thus $\frac{p+1}{p}$ can be absorbed in the constant. Further, assuming $p \geq 2$, we may replace $p+1$ by p in the above inequality to get,

$$\left(\frac{1}{V}\int_M |\widetilde{\phi}|^{\frac{pn}{n-1}}\omega_0^n\right)^{\frac{n-1}{np}} \leq \left(\frac{Cp}{V}\int_M \widetilde{\phi}^p\omega_0^n\right)^{\frac{1}{p}}.$$

This implies,

$$||\widetilde{\phi}||_{L^{\frac{pn}{n-1}}} \leq C||\widetilde{\phi}||_{L^p}, \tag{4.16}$$

where $p \geq 2$. Thus we have got an L^q bound of ϕ in terms of an L^p bound, where $p < q$. We now apply Moser's iteration wherein the above inequality is employed iteratively to obtain a C_0 bound.

Define the following sequence,

$$p_0 = 2; \;\; p_{i+1} = \frac{n}{n-1}p_i \geq 2.$$

Applying inequality (4.16) i times, we get ,

$$||\phi||_{L^{p_i}} \leq \prod_{j=0}^{i-1}(Cp_j)^{\frac{1}{p_j}}||\phi||_{L^2}. \tag{4.17}$$

Claim: $\sup|\phi| = \lim_{p\to\infty}||\phi||_{L^p}$.

Proof. Since M is compact,

$$||\phi||_{L^p}^p = \frac{1}{V}\int_M |\phi|^p\omega_0^n \leq (\sup|\phi|)^p$$

To prove the claim we need to show that $\sup|\phi| \leq \lim_{p\to\infty}||\phi||_{L^p}$.
Let $0 < \epsilon < 1$, define $M_\epsilon := \{x \in M : |\phi(x)| > \sup|\phi| - \epsilon\}$,

$$\begin{aligned} ||\phi||_{L^p} = \left(\frac{1}{V}\int_M |\phi|^p\right)^{\frac{1}{p}} &\geq \left(\frac{1}{V}\int_{M_\epsilon}(\sup|\phi|-\epsilon)^p\right)^{\frac{1}{p}} \\ &\geq (\sup|\phi|-\epsilon)Vol(M_\epsilon)^{\frac{1}{p}} \end{aligned}$$

where $Vol(M_\epsilon)$ is the volume or measure of M_ϵ. Taking limit as $p \to \infty$ on both sides, we get

$$\lim_{p\to\infty}||\phi||_{L^p} \geq \sup|\phi| - \epsilon,$$

for all ϵ. Hence $\sup|\phi| = \lim\limits_{p\to\infty} ||\phi||_{L^p}$. $\square$

Using the above claim and taking limit as i tends to infinity on both sides of equation (4.17), we get

$$\sup|\phi| = \lim_{i\to\infty} ||\phi||_{L^{p_i}} \leq \prod_{j=0}^{\infty}(Cp_j)^{\frac{1}{p_j}}||\phi||_{L^2}.$$

We need to calculate

$$\prod_{j=0}^{\infty}(cp_j)^{\frac{1}{p_j}} = e^{\sum_{j=0}^{\infty}\left(\frac{1}{p_j}\ln C+\frac{1}{p_j}\ln p_j\right)},$$

and show it is finite. Now, $p_j = 2(\frac{n}{n-1})^j$, thus

$$\sum_{j=0}^{\infty}\left(\frac{1}{p_j}\ln C + \frac{1}{p_j}\ln p_j\right) = \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{n-1}{n}\right)^j \ln C + \frac{1}{2}\sum_{j=0}^{\infty} j\left(\frac{n-1}{n}\right)^j \ln 2.$$

The first term is a Geometric progression with a ratio less than 1 and hence finite. The second term is of the form $\sum ja^j$, $|a|<1$, which is a Arithmetico-Geometric progression. It can be calculated by differentiating both sides of $\sum a^n = \frac{1}{1-a}$ and multiplying by a to obtain $\sum na^n = \frac{a}{(1-a)^2}$ for $|a|<1$. Hence,

$$\prod_{j=0}^{\infty}(cp_j)^{\frac{1}{p_j}} < \infty,$$

and finally we have the estimate

$$\sup|\phi| < C||\phi||_{L^2} < C.$$

4.3.2 $\partial\overline{\partial}$-estimates

In this section we derive a $\partial_i\partial_{\bar{j}}$ apriori estimates on the solution, which is a more difficult calculation. Observe that ω_ϕ is a positive definite metric, and hence its eigenvalues are positive. Thus,

$$||g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\phi||_g \leq \text{tr }\omega_\phi = n + \Delta\phi,$$

where tr ω_ϕ is the trace of w_ϕ with respect to the metric g. Hence,

$$||\partial\overline{\partial}\phi||_{C_0} \leq \max\{\text{tr }\omega_\phi, n\}.$$

To get a C_0 bound on $\partial\overline{\partial}\phi$, we need to get a bound on tr $\omega_\phi = n + \Delta\phi$.
Let Δ' denote the Laplacian taken with respect to the ω_ϕ metric. A very naive idea to get C_0 bounds is to get an inequality of the form

$$\Delta'(\text{tr }\omega_\phi) \geq g(\text{tr }\omega_\phi) - C,$$

for some nice enough function g. Then at the point of maximum, Δ' is negative and use this to get a bound on tr ω_ϕ. In this proof, rather than applying this idea on tr ω_ϕ we will work with a different function. For ease of notation, let $g' \coloneqq g + \partial\overline{\partial}\phi$. Firstly, let us calculate $\Delta'(\text{tr }\omega_\phi)$. In local coordinates.,

$$\begin{aligned}
\Delta'(\text{tr }\omega_\phi) &= \Delta'(n + \Delta\phi) \\
&= (g')^{i\bar{j}}\partial_i\partial_{\bar{j}}(g^{k\bar{l}}\partial_k\partial_{\bar{l}}\phi) \\
&= (g')^{i\bar{j}}g^{k\bar{l}}\partial_i\partial_{\bar{j}}\partial_k\partial_{\bar{l}}\phi + (g')^{i\bar{j}}(\partial_i\partial_{\bar{j}}g^{k\bar{l}})(\partial_k\partial_{\bar{l}}\phi) + \text{first order terms containing } \partial g.
\end{aligned}$$

Since g is Kähler, considering normal coordinates with respect to g, and substituting

$$\partial_i\partial_{\bar{j}}g^{k\bar{l}} = -\partial_i\partial_{\bar{j}}g_{k\bar{l}} = R_{i\bar{j}k\bar{l}},$$

we get

$$\Delta'(\text{tr }\omega_\phi) = (g')^{i\bar{j}}\partial_i\partial_{\bar{j}}\partial_k\partial_{\bar{k}}\phi + (g')^{i\bar{j}}R_{i\bar{j}k\bar{l}}\partial_k\partial_{\bar{l}}\phi. \tag{4.18}$$

The Laplacian of the trace has a fourth order term. The idea is to differentiate the Monge-Ampere equation twice to get a similar fourth order term. The complex Monge-Ampere equation in local coordinates is,

$$\log\det(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\phi) = f_s + \log\det g_{i\bar{j}}.$$

Differentiating with respect to ∂_k,

$$\partial_k f_s = (g')^{i\bar{j}}(\partial_k g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\partial_k\phi) - g^{i\bar{j}}\partial_k g_{i\bar{j}},$$

and differentiating again with respect to $\partial_{\bar{l}}$, we arrive at

$$\begin{aligned}\partial_k\partial_{\bar{l}} f_s &= (g')^{i\bar{j}}(\partial_k\partial_{\bar{l}} g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\partial_k\partial_{\bar{l}}\phi) - (g')^{t\bar{j}}(g')^{i\bar{r}}(\partial_{\bar{l}} g_{t\bar{r}} + \partial_t\partial_{\bar{r}}\partial_{\bar{l}}\phi)(\partial_k g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\partial_k\phi) \\ &\quad - g^{i\bar{j}}\partial_k\partial_{\bar{l}} g_{i\bar{j}} + g^{t\bar{j}} g^{i\bar{s}}\partial_{\bar{l}} g_{t\bar{s}}\partial_k g_{i\bar{j}}. \end{aligned} \tag{4.19}$$

Since we are working with normal coordinates, the above equation simplifies to,

$$\Delta f_s = (g')^{i\bar{j}}\partial_i\partial_{\bar{j}}\partial_k\partial_{\bar{k}}\phi + (g')^{i\bar{j}} R_{i\bar{j}k\bar{k}} - (g')^{t\bar{j}}(g')^{i\bar{r}}(\partial_t\partial_{\bar{r}}\partial_{\bar{k}}\phi)(\partial_i\partial_{\bar{j}}\partial_k\phi) - R_{i\bar{i}k\bar{k}}.$$

Substituting the fourth order term in equation (4.18), we get

$$\begin{aligned}\Delta'(\mathrm{tr}\ \omega_\phi) &= (g')^{t\bar{j}}(g')^{i\bar{r}}(\partial_t\partial_{\bar{r}}\partial_{\bar{k}}\phi)(\partial_i\partial_{\bar{j}}\partial_k\phi) - (g')^{i\bar{j}} R_{i\bar{j}k\bar{k}} \\ &\quad + R_{i\bar{i}k\bar{k}} + (g')^{i\bar{j}} R_{i\bar{j}k\bar{l}}\partial_k\partial_{\bar{l}}\phi + \Delta f_s. \end{aligned} \tag{4.20}$$

We can choose coordinates around a point, such that g at the point is identity matrix and $(\partial_i\partial_{\bar{j}}\phi)$ is a diagonal matrix. This is possible, since a symmetric matrix (in fact for any self adjoint matrix) can be diagonalised by an orthogonal matrix. Hence, we can start with normal coordinates on g, and then diagonalise $\partial\overline{\partial}\phi$ using orthogonal matrices.
In this new coordinates system, note that

$$g'_{i\bar{j}} = \delta_{i\bar{j}}(1 + \partial_i\partial_{\bar{i}}\phi), \text{ and } (g')^{i\bar{j}} = \frac{\delta_{i\bar{j}}}{1 + \partial_i\partial_{\bar{i}}\phi}.$$

Thus the equation (4.20) reads as

$$\begin{aligned}\Delta'(\mathrm{tr}\ \omega_\phi) &= \frac{1}{(1 + \partial_i\partial_{\bar{i}}\phi)(1 + \partial_j\partial_{\bar{j}}\phi)}(\partial_{\bar{i}}\partial_j\partial_{\bar{k}}\phi)(\partial_i\partial_{\bar{j}}\partial_k\phi) + \frac{R_{i\bar{i}k\bar{k}}}{1 + \partial_i\partial_{\bar{i}}\phi} \\ &\quad - R_{i\bar{i}k\bar{k}} + \frac{R_{i\bar{i}k\bar{k}}}{1 + \partial_k\partial_{\bar{k}}\phi}\partial_i\partial_{\bar{i}}\phi + \Delta f_s. \end{aligned} \tag{4.21}$$

By symmetry of the curvature tensor, we know that $R_{i\bar{i}k\bar{k}} = R_{k\bar{k}i\bar{i}}$, and hence the above equation reads,

$$\begin{aligned}\Delta'(\mathrm{tr}\ \omega_\phi) &= \frac{1}{(1 + \partial_i\partial_{\bar{i}}\phi)(1 + \partial_j\partial_{\bar{j}}\phi)}(\partial_{\bar{i}}\partial_j\partial_{\bar{k}}\phi)(\partial_i\partial_{\bar{j}}\partial_k\phi) \\ &\quad + R_{i\bar{i}k\bar{k}}\left(\frac{1 + \partial_i\partial_{\bar{i}}\phi}{1 + \partial_k\partial_{\bar{k}}\phi} - 1\right) + \Delta f_s. \end{aligned} \tag{4.22}$$

The estimate we have got on the Laplacian of the trace still has a third order term. We will work with a slightly different function and use the above estimate to remove the third order term. Define a function,

$$\psi := e^{-\lambda\phi}\mathrm{tr}\ \omega_\phi.$$

Then,

$$\Delta'\psi = e^{-\lambda\phi}\Delta'\mathrm{tr}\ \omega_\phi + g_{i\bar{i}}\partial_i\partial_{\bar{i}}(e^{-\lambda\phi})\mathrm{tr}\ \omega_\phi + g'_{i\bar{i}}\partial_i(e^{-\lambda\phi})\partial_{\bar{i}}\mathrm{tr}\ \omega_\phi + g'_{i\bar{i}}\partial_{\bar{i}}(e^{-\lambda\phi})\partial_i\mathrm{tr}\ \omega_\phi. \quad (4.23)$$

Simplifying, we get

$$\begin{aligned} \Delta'\psi &= e^{-\lambda\phi}\Delta'\mathrm{tr}\ \omega_\phi + g_{i\bar{i}}\partial_i\partial_{\bar{i}}(e^{-\lambda\phi})\mathrm{tr}\ \omega_\phi - \lambda(g')_{i\bar{i}}\partial_i\phi e^{-\lambda\phi}\partial_{\bar{i}}(\Delta\phi) \\ &\quad -\lambda(g')_{i\bar{i}}\partial_{\bar{i}}\phi e^{-\lambda\phi}\partial_i(\Delta\phi). \end{aligned} \quad (4.24)$$

Applying the Cauchy-Schwarz inequality on the second last term, we get

$$\begin{aligned} \lambda\partial_i\phi e^{-\lambda\phi}\partial_{\bar{i}}(\Delta\phi) &= (\lambda\partial_i\phi e^{\frac{-\lambda}{2}\phi}(\mathrm{tr}\ \omega_\phi)^{\frac{1}{2}})(e^{\frac{-\lambda}{2}\phi}\partial_{\bar{i}}(\Delta\phi)(\mathrm{tr}\ \omega_\phi)^{\frac{-1}{2}}) \\ &\leq \frac{1}{2}(\lambda^2\partial_i\phi\partial_{\bar{i}}\phi e^{-\lambda\phi}(\mathrm{tr}\ \omega_\phi) + e^{-\lambda\phi}\partial_i(\Delta\phi)\partial_{\bar{i}}(\Delta\phi)(\mathrm{tr}\ \omega_\phi)^{-1}). \end{aligned}$$

The last term of equation (4.24) will also give the same inequality. Hence, we have

$$\begin{aligned} \Delta'\psi &\geq e^{-\lambda\phi}\Delta'\mathrm{tr}\ \omega_\phi + g_{i\bar{i}}\partial_i\partial_{\bar{i}}(e^{-\lambda\phi})\mathrm{tr}\ \omega_\phi \\ &\quad -(g')_{i\bar{i}}\lambda^2\partial_i\phi\partial_{\bar{i}}\phi e^{-\lambda\phi}(\mathrm{tr}\ \omega_\phi) - (g')_{i\bar{i}}e^{-\lambda\phi}\partial_i(\Delta\phi)\partial_{\bar{i}}(\Delta\phi)(\mathrm{tr}\ \omega_\phi)^{-1} \end{aligned}$$

Simplifying,

$$g_{i\bar{i}}\partial_i\partial_{\bar{i}}(e^{-\lambda\phi})\mathrm{tr}\ \omega_\phi = -\lambda\Delta\phi e^{-\lambda\phi}\mathrm{tr}\ \omega_\phi + \lambda^2(g')^{i\bar{i}}\partial_i\phi\partial_{\bar{i}}\phi e^{-\lambda\phi}\mathrm{tr}\ \omega_\phi.$$

Combining the above analysis, we obtain

$$\Delta'\psi \geq e^{-\lambda\phi}\Delta'\mathrm{tr}\ \omega_\phi + \lambda^2(g')^{i\bar{i}}\partial_i\phi\partial_{\bar{i}}\phi e^{-\lambda\phi}\mathrm{tr}\ \omega_\phi - (g')_{i\bar{i}}e^{-\lambda\phi}\partial_i(\Delta\phi)\partial_{\bar{i}}(\Delta\phi)(\mathrm{tr}\ \omega_\phi)^{-1}. \quad (4.25)$$

For the first term of the above equation, we already had an estimate which involves a third order derivative term. Now considering the third term,

$$(g')_{i\bar{i}}\partial_i(\Delta\phi)\partial_{\bar{i}}(\Delta\phi) = \frac{\partial_i(\Delta\phi)\partial_{\bar{i}}(\Delta\phi)}{1+\partial_i\partial_{\bar{i}}\phi} = \frac{|\sum_k\partial_k\partial_{\bar{k}}\partial_i\phi|^2}{1+\partial_i\partial_{\bar{i}}\phi}.$$

Splitting, $|\sum_k \partial_k\partial_{\overline{k}}\partial_i\phi|^2 = \left|\sum_k \frac{\partial_k\partial_{\overline{k}}\partial_i\phi}{(1+\partial_k\partial_{\overline{k}}\phi)^{\frac{1}{2}}}(1+\partial_k\partial_{\overline{k}}\phi)^{\frac{1}{2}}\right|^2$, and applying Cauchy-Schwarz inequality,

$$\begin{aligned}(g')_{i\bar{i}}\partial_i(\Delta\phi)\partial_{\bar{i}}(\Delta\phi) &\le \frac{1}{(1+\partial_i\partial_{\bar{i}}\phi)(1+\partial_k\partial_{\overline{k}}\phi)}(\partial_k\partial_{\overline{k}}\partial_{\bar{i}}\phi)(\partial_k\partial_{\overline{k}}\partial_i\phi)(n+\Delta\phi)\\ &\le \frac{1}{(1+\partial_i\partial_{\bar{i}}\phi)(1+\partial_j\partial_{\bar{j}}\phi)}(\partial_{\bar{i}}\partial_j\partial_{\overline{k}}\phi)(\partial_i\partial_{\bar{j}}\partial_k\phi)(n+\Delta\phi).\end{aligned}$$

Observe that, the above inequality has the same third order term as $\Delta'\mathrm{tr}\ \omega_\phi$. The whole trick was to apply the Cauchy-Schwarz inequality, to ensure that we get the same coefficients. Substituting in equation (4.25), we see that

$$\Delta'\psi \ge e^{\lambda}\left(\Delta f_s + C(n+\Delta\phi)\frac{1}{1+\partial_i\partial_{\bar{i}}\phi} - C_2\right) - \lambda\Delta\phi e^{-\lambda\phi}\mathrm{tr}\ \omega_\phi.$$

Observe that,

$$\Delta'\phi = \frac{\partial_i\partial_{\bar{i}}\phi}{1+\partial_i\partial_{\bar{i}}} = n - \frac{1}{1+\partial_i\partial_{\bar{i}}\phi},$$

and choose λ such that $\lambda + C \ge 1$, then

$$\Delta'\psi \ge -c_1e^{-\lambda\phi} - c_2\psi + e^{-\lambda\phi}\frac{1}{1+\partial_i\partial_{\bar{i}}\phi}\mathrm{tr}\ \omega_\phi. \tag{4.26}$$

Claim:

$$\sum_i \frac{1}{1+\partial_i\partial_{\bar{i}}\phi} = \frac{\sum_i(1+\partial_i\partial_{\bar{i}}\phi)}{\prod_i(1+\partial_i\partial_{\bar{i}}\phi)} = e^{-\frac{f_s}{n-1}}\ (\mathrm{tr}\ \omega_\phi).$$

Proof. Note that $1+\partial_i\partial_{\bar{i}}\phi$ are eigenvalues of a positive definite matrix, and hence are positive. Define,

$$a_i := \frac{1}{1+\partial_i\partial_{\bar{i}}\phi} \ge 0.$$

We need to prove that

$$\left(\sum_i a_i\right)^{n-1} \ge \sum_k\prod_{i\neq k} a_i.$$

Now,

$$\left(\sum_i a_i\right)^{n-1} = \left((n-1)\sum_k\sum_{i\neq k}\frac{a_i}{n-1}\right)^{n-1}.$$

Hence by the AM-GM inequality, we get

$$\left(\sum_i a_i\right)^{n-1} \geq \sum_k \left(\sum_{i\neq k} \frac{a_i}{n-1}\right)^{n-1} \geq \sum_k \left(\prod_{i\neq k} \frac{a_i}{n-1}\right)^{n-1}.$$

$\square$

Substituting in equation (4.26),

$$\begin{aligned} \Delta'\psi &\geq -C_1 e^{-\lambda\phi} - C_2\psi + e^{\frac{f_s}{n-1}} e^{-\lambda\phi} (\text{tr } \omega_\phi)^{\frac{n}{n-1}} \\ &= -C_1 e^{-\lambda\phi} - C_2\psi + C_3\psi^{\frac{n}{n-1}}, \end{aligned}$$

for some constants C_1, C_2 and C_3. Assume that ψ achieves a maximum at x_0, then

$$0 \geq \Delta\psi(x_0) \geq -C_1 - C_2\psi(x_0) + C_3\psi(x_0)^{\frac{n}{n-1}}.$$

Hence for all $x \in M$, we have

$$\psi(x)^{\frac{n}{n-1}} \leq \psi(x_0)^{\frac{n}{n-1}} \leq C(1 + \psi(x_0)) < C,$$

and this implies

$$\text{tr } \omega_\phi \leq C.$$

4.3.3 $C^{2,\alpha}$ estimates

In the previous section, we proved a uniform $\partial\overline{\partial}$ estimate on the solution. Yau in his original proof of the Calabi Conjecture proved the C^3 estimates by a similar method as in the previous section. This calculation involves working with a fifth order differential equation, which is simplified using more tedious calculations. A sufficient, $C^{2,\alpha}$ estimate can be obtained by Evan-Krylov method (see [14] and [11]). The idea there is to adopt the Evans Krylov method for complex spaces. Here we apply another trick using the Christoffel symbol, to obtain a $C^{2,\alpha}$ estimates. This method can be found in Gábor's book [1].
For a Kähler metric the Christoffel symbols are given by,

$$\Gamma^i_{jk} = g^{i\bar{l}} \partial_j g_{k\bar{l}}.$$

Let ${\Gamma^i_{jk}}'$ be the Christoffel symbol corresponding to g' metric. Indeed Γ is not a tensor by itself, but $\Gamma' - \Gamma$ is a tensor. Define,

$$S^i_{jk} \coloneqq {\Gamma^i_{jk}}' - \Gamma^i_{jk}$$

Now, instead of working with third order derivatives, we will get an estimate on the Laplacian of S tensor. Consider the norm of S with respect to $g' = g + \sqrt{-1}\partial\overline{\partial}$ metric,

$$|S|^2 = g'^{j\overline{q}} g'^{k\overline{r}} g'_{i\overline{p}} S^i_{jk} \overline{S^p_{qr}}$$

Taking Laplacian with respect to g' on both sides, we get

$$\begin{aligned}
\Delta'|S|^2 &= g'^{l\overline{m}} \partial_l \partial_{\overline{m}} (g'^{j\overline{q}} g'^{k\overline{r}} g'_{i\overline{p}} S^i_{jk} \overline{S^p_{qr}}) \\
&= g'^{l\overline{m}} g'^{j\overline{q}} g'^{k\overline{r}} g'_{i\overline{p}} \partial_l \partial_{\overline{m}} (S^i_{jk} \overline{S^p_{qr}}) + g'^{l\overline{m}} g'^{k\overline{r}} g'_{i\overline{p}} S^i_{jk} \overline{S^p_{qr}} \partial_l \partial_{\overline{m}} (g'^{j\overline{q}}) \\
&\quad + g'^{l\overline{m}} g'^{q\overline{s}} g'_{i\overline{p}} S^i_{jk} \overline{S^p_{qr}} \partial_l \partial_{\overline{m}} (g'^{k\overline{r}}) + g'^{l\overline{m}} g'^{q\overline{s}} g'^{k\overline{r}} S^i_{jk} \overline{S^p_{qr}} \partial_l \partial_{\overline{m}} (g'_{i\overline{p}}) + \text{ terms containing } \partial g'
\end{aligned}$$

Considering normal coordinates with respect to g', we obtain

$$\Delta|S|^2 = \partial_l \partial_{\bar{l}} (S^i_{jk} \overline{S^i_{jk}}) + S^i_{jk} \overline{S^i_{qk}} R_{l\bar{l}j\bar{q}} + S^i_{jk} \overline{S^i_{jr}} R_{l\bar{l}k\bar{r}} - S^i_{jk} \overline{S^p_{jk}} R_{l\bar{l}i\bar{p}}.$$

Now,

$$\partial_l \partial_{\bar{l}} (S^i_{jk} \overline{S^i_{jk}}) = \partial_l \partial_{\bar{l}} (S^i_{jk}) \overline{S^i_{jk}} + \partial_l \partial_{\bar{l}} (\overline{S^i_{jk}}) S^i_{jk} + \partial_l S^i_{jk} \partial_{\bar{l}} \overline{S^i_{jk}} + \partial_l \overline{S^i_{jk}} \partial_{\bar{l}} S^i_{jk},$$

and note that

$$\partial_l S^i_{jk} \partial_{\bar{l}} \overline{S^i_{jk}} = |\partial_l S^i_{jk}|^2 \geq 0, \;\; \partial_l \overline{S^i_{jk}} \partial_{\bar{l}} S^i_{jk} = |\partial_l \overline{S^i_{jk}}|^2 \geq 0.$$

Hence,

$$\partial_l \partial_{\bar{l}} (S^i_{jk} \overline{S^i_{jk}}) \geq \partial_l \partial_{\bar{l}} (S^i_{jk}) \overline{S^i_{jk}} + \partial_l \partial_{\bar{l}} (\overline{S^i_{jk}}) S^i_{jk}.$$

We will now bound $\partial_l \partial_{\bar{l}} (S^i_{jk})$, note that

$$\partial_l \partial_{\bar{l}} (S^i_{jk}) = \partial_l \partial_{\bar{l}} ({\Gamma^i_{jk}}' - \Gamma^i_{jk}) = \partial_l R^i_{j\ k\bar{l}} - \partial_l R^i_{j\ k\bar{l}},$$

where R' is the curvature with respect to g' metric. By the Bianchi identity,

$$\partial_l R'^i_{j\ k\bar{l}} = \nabla'_l R'^i_{j\ k\bar{l}} = \nabla'_j R'^i_k$$

and

$$\partial_l R^i_{j\ k\bar{l}} = \nabla_l R^i_{j\ k\bar{l}} + \Gamma R^i_{j\ k\bar{l}}.$$

The last term is bounded by bounded by $C|S|$. Hence,

$$\partial_l \partial_{\bar{l}} (S^i_{jk}) < -C|S| - C'. \tag{4.27}$$

Thus,

$$\Delta |S|^2 \geq -C|S|^2 - C'|S|. \tag{4.28}$$

Using the $\partial\overline{\partial}$-estimate, equation (4.21) simplifies to,

$$\Delta \mathrm{tr}\, \omega_\phi \geq -C_1 + \epsilon |S|^2. \tag{4.29}$$

We can multiply equation (4.29) by a large enough constant A, and adding equation (4.28), to get

$$\Delta(|S|^2 + A\mathrm{tr}\, \omega_\phi) \geq |S|^2 - C.$$

Similar to the $\partial\overline{\partial}$-estimate, we will apply the maximum principle. Let $|S|^2 + A\mathrm{tr}\, \omega_\phi$ attain its maximum at $x_0 \in M$, then

$$|S|^2(x_0) \leq C.$$

Hence,

$$|S|^2(x) \leq |S|^2(x) + A\mathrm{tr}\, \omega_\phi(x) \leq |S|^2(x_0) + A\mathrm{tr}\, \omega_\phi(x_0) \leq C'.$$

We have got a bound on the $|S|$. Hence we have mixed third derivative bounds of ϕ. In particular, we get C^α bounds on $\partial\overline{\partial}\phi$. Now we can get uniform $C^{2,\alpha}$ estimates of ϕ. We can use Schauder estimates to get a smooth solution.

Chapter 5

Kähler-Einstein metrics

5.1 Applications of the Calabi-Yau theorem

A metric ω is said to *Ricci flat* if $Ric(\omega) = 0$. By considering $\Omega = 0$ in the Calabi-Yau theorem, we get that every Kähler class has a Ricci flat metric. Flat manifolds are locally euclidean in terms of distance and angles. In fact, prior to the Calabi-Yau theorem, there were no such result related to Ricci flatness on Riemannian manifolds. Ricci flat metrics have application in physics, mainly related to Einstein's equations. In the next section, we will discuss about Kähler-Einstein metrics, where Ricci flat metrics will be one of the cases. Before discussing the next application, observe that the Calabi-Yau theorem relates $c_1(M)$, a topological invariant and the Ricci curvature. Hence, having a constrain or condition on $c_1(M)$, will have implications on the geometry of the manifold. The following application is such an example.

Theorem 5.1.1. *Let M be a compact Kähler manifold such that $c_1(M) > 0$, then M is simply connected.*

Proof. The proof will use some results and ideas from outside this thesis. But the main take away here is to demonstrate how the Calabi-Yau theorem is being used.
The first part of the proof is to show that, if M is a compact Kähler manifold with positive first chern class, then the fundamental group of M is finite. Since $c_1(M) > 0$ there exists $\Omega \in c_1(M)$ such that $\Omega > 0$. Hence, by the Calabi-Yau theorem, there exists a Kähler metric

ω such that $Ric(\omega) = \Omega > 0$. Thus we have got a positively Ricci curved metric. Then by Myer's theorem (see [8]), we get that the fundamental group is finite. Note, this is the only part of the proof, that will use the Calabi-Yau theorem.
The *Dobeault cohomology* groups are defined as follows,

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\ker(\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{im(\partial : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

These are essentially the cohomology groups corresponding to the $\overline{\partial}$ operator. In similar fashion to Euler characteristic on real manifolds, we can define the *holomorphic Euler characteristic* as,

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H^{i,0}_{\overline{\partial}}(M).$$

For any compact connected Kähler manifold with $c_1(M) > 0$, by the Kodiara vanishing theorem (see [9]), we get

$$\dim H^{i,0}_{\overline{\partial}}(M) = 0 \quad \text{for } i > 0.$$

Hence the holomorphic Euler characteristic of M, is $\chi(M) = 1$. The next step is to use all these observations for a finite covering of M. Let $\widetilde{M}$ be a k-fold covering of M. The main reason we are interested in finite coverings of M is to find subgroups of $\pi_1(M)$, and hence deduce more about the fundamental group of M itself.
Since $\widetilde{M}$ is a finite cover, $\widetilde{M}$ is a compact Kähler manifold with $c_1(M) > 0$. Again we have $\chi(\widetilde{M}) = 1$. Similar to the real case for a k-fold covering $\widetilde{M}$ over M, we have

$$\chi(\widetilde{M}) = k\,\chi(M).$$

The fact that the holomorphic Euler characteristic is multiplicative for covers is more difficult and follows from the Hirzebruch-Riemann-Roch theorem. Since $\chi(\widetilde{M}) = 1 = \chi(M)$, we get that $\widetilde{M}$ is a one fold covering. Therefore, there are no non-trivial finite coverings of M. Hence $\pi_1(M)$ has no finite index subgroups. But since $\pi_1(M)$ is a finite group, it implies that M is simply connected □

The only role the Calabi-Yau theorem played is to get a Ricci lower bound, from which we could prove that the fundamental group is finite. Without the fundamental group being finite, we would have only been able to conclude that there are no finite index subgroups.
There are other applications like showing that there is a unique complex structure which

is homotopic to $\mathbb{CP}^2$. Yau along side with the proof of the conjecture, also proved few applications of the theorem in Algebraic geometry [4]. The usefulness of the result goes beyond just these applications, but it set the foundation of this subject as well as a learning tool in solving many other related problems.

5.2 Kähler-Einstein metrics

Definition 5.2.1. *A Kähler metric is called* Kähler-Einstein *if*

$$Ric(\omega) = \lambda\omega,$$

for some $\lambda \in \mathbb{R}$.

By re-scaling the metric, we get three cases, namely

$$Ric(\omega) = \omega, \;\; Ric(\omega) = 0, \;\; Ric(\omega) = -\omega.$$

Einstein metrics appear in the Einstein equation in physics. Kähler-Einstein metrics are also related to constant scalar curvature. The goal here is to study the existence of Kähler-Einstein metrics on Kähler manifolds. Since, $2\pi c_1(M) = [Ric(g)]$, we have the hope of finding a Kähler-Einstein metrics only if $c_1(M)$ is positive, negative or zero. For $c_1(M) = 0$, we already discussed about Ricci flat metrics in the previous section, which was a corollary of the Calabi-Yau theorem. In this case we will have a Kähler-Einstein metric in every Kähler class. In the cases $Ric(\omega) = \omega$ and $Ric(\omega) = -\omega$, we can hope to find Kähelr-Einstein metrics only if $\omega \in 2\pi c_1(M)$ and $\omega \in -2\pi c_1(M)$ respectively.

5.2.1 $C_1(M) < 0$

If $c_1(M) < 0$, then we need to find a Kähler metric such that

$$Ric(\omega) = -\omega,$$

where $\omega \in -2\pi c_1(M)$.

Theorem 5.2.1. *Let M be a Kähler manifold with $c_1(M) < 0$, then there exists a Kähler metric $\omega \in -2\pi c_1(M)$, such that $Ric(\omega) = -\omega$.*

The proof is similar to the Calabi Yau theorem, but some steps are actually simpler. Firstly let us find the corresponding PDE to be solved.
If ω_0 is a Kähler metric in $-2\pi c_1(M)$, we need to find a $\omega \in -2\pi c_1(M)$ such that $Ric(\omega) = -\omega$, then by the $\partial\overline{\partial}$-lemma,

$$Ric(\omega_0) = -\omega_0 + \sqrt{-1}\partial\overline{\partial}f,$$

where f is a smooth function, and

$$Ric(\omega) = Ric(\omega_0) - \sqrt{-1}\partial\overline{\partial}\frac{\omega^n}{\omega_0^m}.$$

Let ϕ be a function such that

$$\omega = \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi.$$

Then, by a simplification similar to the Calabi-Yau theorem, Theorem 5.2.1 is equivalent to solving

$$(\omega + \sqrt{-1}\partial\overline{\partial}\phi)^n = e^{f+\phi}\omega_0^n$$

for ϕ, while ensuring that ω_ϕ is positive definite.
The main difference of the above equation compared to equation (4.4) is the additional factor of ϕ. The additional ϕ factor will be beneficial. The C^0 estimate, in this case can be easily derived using the maximum principle. To derive the C^0 bound, let ϕ achieve its maximum at x_0. Then in local coordinates $\partial\overline{\partial}\phi$ is negative semi definite at x_0. This implies

$$\det(g + \partial\overline{\partial}\phi)(x_0) \leq \det(g)(x_0).$$

Hence,

$$e^{(f+\phi)(x_0)} = \frac{\det(g + \partial\overline{\partial}\phi)(x_0)}{\det(g)(x_0)} < 1,$$

this implies

$$f(x_0) + \phi(x_0) < 0.$$

Since ϕ achieves its maxima at x_0, we have

$$\sup|\phi| \leq \sup|f|.$$

Thus by just using the maximum principle we were able to get a C^0 estimate. The other steps will be similar to the proof of the Calabi-Yau theorem. For the openness argument the derivative will be $\Delta - Id$ rather than Δ. In the $\partial\overline{\partial}$-estimate, we have to proceed in the exactly same way, and the factor of ϕ will get absorbed into the constants.

5.2.2 $c_1(M) > 0$

When $c_1(M) > 0$, the corresponding PDE will be

$$(\omega + \sqrt{-1}\partial\overline{\partial}\phi)^n = e^{f-\phi}\omega_0^n.$$

On the surface, the only difference in comparison to the $c_1(M) < 0$ is the negative sign. Firstly proving openness is not easy, we will have to work with a different family of equations,

$$(\omega + \sqrt{-1}\partial\overline{\partial}\phi)^n = e^{f-t\phi}\omega_0^n$$

where $0 \leq t \leq 1$. This is also equivalent to

$$Ric(\omega_\phi) = t\omega_\phi + (1-t)\omega_0.$$

The main problem in the $c_1(M) > 0$ will be showing the C^0 estimate. Due to the negative sign we cannot apply the maximum principle like in the $c_1(M) < 0$ case. Unfortunately, not all Kähler manifolds with $c_1(M) > 0$ have a Kähler-Einstein metric. One invariant that will help us find obstruction is the Futaki invariant (see [2], [1]). Using these we can get examples of Kähler manifold with $c_1(M) > 0$, that do not admit any Kähler-Einstein metric. One such example is blowup of a point in $\mathbb{CP}^2$ (see [2] for more details). Trying to solve the problem will have some algebro-geometric constraints, and was only resolved recently thanks to the works of Chen, Donaldson, Sun, and Tian (among others).

Conclusion

The thesis describes my one year master's project pursued at IISc Bangalore. The central theme of thie project has been on the Yau's proof of the Calabi Conjecture. The aim in this thesis was to have an understanding of the proof from the perspective of a master's student. In the journey of understanding the proof, I got the opportunity to read more on Riemannian geometry, PDE, algebraic topology and geometric analysis.

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