Rigid Analysis

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by

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Certificate

This is to certify that this dissertation entitled Rigid Analysis towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Nikhil Gupta at Indian Institute of Science Education and Research under the supervision of Prof. Manish Mishra, Assistant Professor, Department of Mathematics , during the academic year 2019-2020.

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This thesis is dedicated to my mother.

Declaration

I hereby declare that the matter embodied in the report entitled Rigid Analysis are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Prof. Manish Mishra and the same has not been submitted elsewhere for any other degree.

Nikhil Gupta

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Abstract

Rigid Analysis is the p-adic analogue of the classical complex geometry. After Hensel discovered the p-adic numbers in 1893, attempts were made to formulate a theory of analytic functions over $\mathbb{Q}_p$. Initially, the question of interest had been to find out if there existed an analog of the theory of classical functions over the field of complex numbers. But then as Algebraic Geometry developed and was applied to number theory, there was a need for a good theory of analytic functions. Modern non-Archimedean geometry was born in 1961 when J. Tate, motivated by the question of characterising elliptic curves with bad reduction, gave a seminar at Harvard with the title "Rigid Analytic Spaces". The theory was subsequently further developed by Kiehl, Remmert, Grauert, Gerritzen, among others. It was apparent from the beginning that rigid geometry was much closer to algebraic geometry than to complex analysis. This algebro-geometric view was worked upon by Raynaud. In this thesis, we give an exposition to Rigid Geometry (in the first five chapters), and then introduce the theory of Formal Geometry.

In the last chapter, we introduce the Ramification Theory of Local Fields. In particular, we introduce the so-called APF extensions and give a characterization of the strictly APF extensions.

Contents

Introduction

Consider a field K. A map $|\cdot| : K \to \mathbb{R}_{\geq 0}$ is said to be a non-Archimedean absolute value if for every $x, y \in K$, these conditions hold:
(a) $|x| = 0 \iff x = 0$,
(b) $|xy| = |x||y|$,
(c) $|x + y| \leq \max\{|x|, |y|\}$.

It is this third condition, also called the ultrametric property, which is a special case of the triangle inequality and that gives rise to the interesting properties of this space that necessitates the need for an alternate version of algebraic geometry. In particular, we see that the topology is totally disconnected which further implies the non-existence of a line integral in this setting. This further means that we can no longer provide a link between holomorphic and analytic functions. We proceed via analyticity and define Tate algebras in the first chapter. Affinoid algebras and Affinoid functions are introduced in the subsequent chapters and we give a proof of the Tate's Acyclicity theorem. We then define the GAGA-functor and go on to prove the Proper Mapping theorem. In the rest of the thesis, Raynaud's view of Formal Geometry is introduced.

Chapter 1

Tate Algebras

1.1 The topology

Definition 1.1.1. *Consider a field K. A map $|\cdot| : K \to \mathbb{R}_{\geq 0}$ is said to be a non-Archimedean absolute value if for every $x, y \in K$, we have:*
(a) $|x| = 0 \iff x = 0$,
(b) $|xy| = |x||y|$,
(c) $|x + y| \leq max\{|x|, |y|\}$.

We say that this absolute value is trivial if $|x| = 1$ for $x \neq 0$. Absolute values $|\cdot|_1$ and $|\cdot|_2$ on K are called equivalent, if we have some $r > 0$ s.t. $|\cdot|_2 = |\cdot|_1^r$.

An absolute value $|\cdot|$ defines a distance function on K. For $x, y \in K$, we put $d(x, y) = |x - y|$. This makes K a metric space, and the completion of K with respect to this metric is denoted by $\hat{K}$. Equivalent absolute values define the same topology on K, and thus give rise to the same completion.

Proposition 1.1.1. *Consider x and y in K s.t. $|x| \neq |y|$. Then, we have $|x + y| = max\{|x|, |y|\}$.*

Proof. WLOG let $|y| < |x|$. So, $|x + y| < |x|$ gives

$$|x| = |(x + y) - y| \leq \max\{|x + y|, |y|\} < |x|,$$

which is a contradiction. □

This means that every triangle in K has to be isosceles. Also, it can be seen that we can take any point inside a disc in K to be its center. So, when two disks intersect, they are concentric.

Now, consider a disc without periphery, around $a \in K$ with radius $r \in \mathbb{R}_{\geq 0}$:

$$D^-(a, r) = \{x \in K : d(x, a) < r\}.$$

This is both open and closed in K. It is said to be the "open" disk of radius r centered at a. In the same way, let's look at the disk again but with the periphery:

$$D^{+}(a,r) = \{x \in K : d(x,a) \leq r\},$$

This is also open and closed in K and is said to be the "closed" disk of radius r centered at a. Also, we have a periphery:

$$\partial D(a,r) = \{x \in K : d(x,a) = r\},$$

This is closed, and open, due to 1.1.1. $\partial D(0,1)$ is known as the *unit tire* in K.

Proposition 1.1.2. *The topology of K is totally disconnected.*

Proof. We need to prove that if a subset of K has two or more points, it can't be connected. Take such a subset in K and consider two different points in it. Then take a small ball around the first point and intersect it with the chosen set. Then this intersection and its complement in the set are both open and closed in the chosen set. Consequently, none of such chosen sets can be connected w.r.t the topology induced from K on the set. □

1.2 Restricted Power Series

As usual, let K be a complete non-Archimedean absolute value that is not trivial and consider the algebraic closure $\overline{K}$. A standard result on field extensions says that the absolute value of K is uniquely extended to $\overline{K}$ and that the absolute value is complete on every finite subextension of $\overline{K}/K$. If $n \geq 1$ where $n \in \mathbb{Z}$, let

$$\mathbb{B}^n(\overline{K}) = \{(x_1, \ldots, x_n) \in \overline{K}^n : |x_i| \leq 1\}$$

be the unit ball in $\overline{K}^n$.

Lemma 1.2.1. *A formal power series*

$$f = \sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu = \sum_{\nu \in \mathbb{N}^n} c_{\nu_1 \ldots \nu_n} \zeta_1^{\nu_1} \ldots \zeta_n^{\nu_n} \in K[[\zeta_1, \ldots, \zeta_n]]$$

converges on $\mathbb{B}^n(\overline{K})$ iff $lim_{|\nu| \to \infty} |c_\nu| = 0$.

Proof. f converging at $(1, \ldots, 1) \in \mathbb{B}^n(\overline{K})$ means that the series $\sum_\nu c_\nu$ is convergent due the non-Archimedean property, which gives us the required result. Conversely, let x be a point in the ball, then we have K', a complete subextension of $\overline{K}$ over K, s.t. the coordinates of x are in K'. So, if $|c_\nu| \longrightarrow 0$, it implies $|c_\nu||x^\nu| \longrightarrow 0$, and that $f(x)$ converges in K'. □

Definition 1.2.1. *The K-algebra $T_n = K\langle \zeta_1, \ldots, \zeta_n \rangle$ of every formal power series*

$$\sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu \in K[[\zeta_1, \ldots, \zeta_n]], \;\; c_\nu \in K, \;\; lim_{|\nu| \to \infty} |c_\nu| = 0,$$

is known as the Tate algebra of restricted, or strictly convergent power series. Also, define $T_0 = K$.

It can be easily checked that T_n is a K-algebra. Now, let's define the Gauss norm on T_n as:

$$|f| = \max |c_\nu| \quad \text{where} \quad f = \sum_\nu c_\nu \zeta^\nu .$$

It can be seen that this definition implies that this norm is in fact a K-algebra norm. In particular, by the multiplicative rule of a norm, we see that T_n is an integral domain.

Proposition 1.2.2. *T_n is complete w.r.t. the Gauss norm , i.e. T_n is a Banach K-algebra.*

Proof. Let us have $\sum_{i=0}^{\infty} f_i$ where $f_i = \sum_v c_{iv}\zeta^v \in T_n$ s.t. $f_i \longrightarrow 0$. Since $|c_{iv}| \leq |f_i|$, we get $|c_{iv}| \longrightarrow 0$ for each v, thus, the summations $c_v = \sum_{i=0}^{\infty} c_{iv}$ exist. Now, it can be seen that $f = \sum_v c_v \zeta^v$ is strictly convergent and $f = \sum_{i=0}^{\infty} f_i$. □

Corollary 1.2.3. *A series $f \in T_n$ where $|f| = 1$ is a unit iff its reduction (modulo the unique maximal ideal) $\widetilde{f} \in K[\zeta_1, \dots, \zeta_n]$ is a unit, i.e. iff $\widetilde{f} \in K^*$. More generally, an arbitrary series $f \in T_n$ is a unit iff $|f - f(0)| < |f(0)|$, or iff the absolute value of the constant term of f is strictly larger than that of the rest of the coefficients of f.*

Proof. WLOG, let $f \in T_n$ s.t. $|f| = 1$. If f is a unit in T_n, it is so in $R\langle\zeta_1, \dots, \zeta_n\rangle$ as well. But then $\widetilde{f}$ is a unit in $k[\zeta_1, \dots, \zeta_n]$ which means it is a unit in k^*. On the converse, if $\widetilde{f} \in k^*$, we have $|f(0)| = 1$. WLOG $f(0) = 1$. Then $f = 1 - g$ for some g s.t. $|g| < 1$. So, $\sum_{i=0}^{\infty} g^i$ is an inverse of f. □

Proposition 1.2.4. *(Maximum Principle). For any $f \in T_n$, for every $x \in \mathbb{B}^n(\overline{K})$, we have $|f(x)| \leq |f|$. Furthermore, we have some $x \in \mathbb{N}^n(\overline{K})$ s.t. the equality actually holds.*

Proof. The former claim directly follows from the ultrametric property and the convergence of the power series to 0. Now, let $|f| = 1$. Also, let $\pi : R\langle\zeta_1, \dots, \zeta_n\rangle \longrightarrow k[\zeta_1, \dots, \zeta_n]$. Then, $\widetilde{f} = \pi(f)$ is non-trivial and there exists $\tilde{x} \in \overline{k}^n$ s.t. $\widetilde{f}(\tilde{x}) \neq 0$. Letting $\overline{R}$ the valuation ring of $\overline{K}$, choose a lift of $\tilde{x}$ in $\mathbb{B}^n(\overline{K})$, say x. Now, we have a commutative diagram:

$$\begin{tikzcd}
R\langle\zeta_1, \dots, \zeta_n\rangle \arrow[r] \arrow[d] & k[\zeta_1, \dots, \zeta_n] \arrow[d] \\
\overline{R} \arrow[r] & \overline{k}
\end{tikzcd}$$

Here, the morphism on the left is the evaluation map at x and the other one, at $\tilde{x}$. Now $f(x) \in \overline{R} \mapsto \widetilde{f}(\tilde{x}) \in \overline{k}$ and $\widetilde{f}(\tilde{x}) \in \overline{k}$ is not trivial, so $|f(x)| = |f| = 1$, and we are done. □

Definition 1.2.2. *An element $g = \sum_{\nu=0}^{\infty} g_\nu \zeta_n^\nu \in T_n$ where $g_\nu \in T_{n-1}$ is called ζ_n-distinguished of some order $s \in \mathbb{N}$ if:*

(a) g_s is a unit in T_{n-1}.

(b) $|g_s| = |g|$ and $|g_s| > |g|$ for all $\nu > s$.

Particularly, consider $g = \sum_{\nu=0}^{\infty} g_\nu \zeta_n^\nu$ s.t. $|g| = 1$. Then g is ζ_n-distinguished of order s iff the quotient $\tilde{g}$ looks like

$$\widetilde{g} = \widetilde{g}_s \zeta_n^s + \widetilde{g}_{s-1} \zeta_n^{s-1} + \ldots + \widetilde{g}_0 \zeta_n^0$$

with a unit $\widetilde{g}_s \in K^*$. So, $g \in T_n$ is ζ_n-distinguished of order 0 iff it's a unit. Also, if $n = 1$, every non-trivial $g \in T_1$ is ζ_1-distinguished of some order $s \in \mathbb{N}$.

Lemma 1.2.5. *For non-trivial $f_1, \ldots, f_r \in T_n$, we have a continuous automorphism of T_n where*

$$\zeta_i \mapsto \begin{cases} \zeta_i + \zeta_n^{\alpha_i} & \text{when } i < n \\ \zeta_n & \text{when } i = n \end{cases}$$

and $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are natural numbers and are suitable exponents s.t. $\sigma(f_1), \ldots, \sigma(f_r)$ are ζ_n-distinguished. Also, $|\sigma(f)| = |f|$ for every $f \in T_n$.

Proof. Define σ^{-1} as

$$T_n \to T_n, \quad \zeta_i \mapsto \begin{cases} \zeta_i - \zeta_n^{\alpha_i} & \text{where } i < n \\ \zeta_n & \text{where } i = n \end{cases}$$

Observe that $|\sigma(f)| = |f|$ since $|\sigma(f)| \leq |f|$ for $f \in T_n$ and similarly for σ^{-1}.
Now, let us consider $r = 1$ and for $f \in T_n$ and letting WLOG $|f| = 1$, look at the image $\widetilde{f}$ of f. Let N be minimal, or, $\widetilde{c}_v \neq 0$ for every $v \in N$. Choosing $r \geq \max\{v_i\}$ which are components of some $v \in N$, consider σ s.t. $\alpha_1 = r^{n-1}, \ldots, \alpha_{n-1} = r$. Then,

$$\begin{aligned} \widetilde{\sigma}(\widetilde{f}) &= \sum_{v \in N} \widetilde{c}_v (\zeta_1 + \zeta_n^{\alpha_1})^{v_1} \ldots (\zeta_{n-1} + \zeta_n^{\alpha_{n-1}})^{v_{n-1}} \zeta_n^{v_n} \\ &= \sum_{v \in N} \widetilde{c}_v \zeta_n^{\alpha_1 v_1 + \ldots + \alpha_{n-1} v_{n-1} + v_n} + \widetilde{g}, \end{aligned}$$

s.t. $\widetilde{g} \in k[\zeta_1, \ldots, \zeta_n]$. Then it can be seen that $\widetilde{c_{\overline{v}}} \neq 0$ and $\sigma(f)$ is ζ_n-distinguished of order s.
Now, this same method can be applied for cases where $g > 1$ by taking r big enough. □

We look at distinguished elements in T_n since there is the so-called Weierstrass division by them, and is the analog of Euclid's division in polynomial rings.

Theorem 1.2.6. *(Weierstrass Division). Consider a ζ_n-distinguished series $g \in T_n$ of order s. Then, for every $f \in T_n$, there exists unique $q, r \in T_n$ s.t. $\deg r < s$ and $f = qg + r$. Also, $|f| = max(|q||g|, |r|)$.*
Remark. This theorem is important as it gives us a version of the usual polynomial divison in the Tate algebra for certain kinds of power series. This incentive of a divison makes these restricted power series useful to work with.

Proof. WLOG assume $|g| = 1$. Consider $f = qg + r$ as desired. Then, $|f| \leq \max(|q||g|, |r|)$. If $|f| <$ RHS, let $\max(|q||g|, |r|) = 1$. Then, $\widetilde{q}\widetilde{g} + \widetilde{r} = 0$ with $\widetilde{q} \neq 0$ and $\widetilde{r} \neq 0$. But this contradicts Euclid's division in $k[\zeta_1, \ldots, \zeta_{n-1}][\zeta_n]$. Hence we are done.
Let's verify the existence now. For this, show that for every $f \in T_n$, we have $q, f_1 \in T_n$ and $r \in T_{n-1}[\zeta_n]$ s.t. $\deg r < s$ and

$$f = qg + f_1 + r$$

where $|q| \leq |f|$, $|r| \leq |f|$ and $|f_1| \leq \varepsilon |f|$. Then our result follows since we form equations

$$\begin{gathered} f_i = q_i g + r_i + f_{i+1}, \\ |q_i|, |r_i| \leq \varepsilon^i |f|, \quad |f_{i+1}| \leq \varepsilon^{i+1} |f|. \end{gathered}$$

and hence,

$$f = \Big(\sum_{i=0}^{\infty} q_i\Big)g + \Big(\sum_{i=0}^{\infty} r_i\Big)$$

which is what we wanted. □

Corollary 1.2.7. *(Weierstrass Preparation Theorem). Consider a ζ_n-distinguished series $g \in T_n$ of order s. We then have unique monic $\omega \in T_{n-1}[\zeta_n]$ having degree s s.t. $g = e\omega$ for a unit $e \in T_n$. Also, $|\omega| = 1$. Thus ω is ζ_n-distinguished and has order s.*

Proof. By the previous theorem, observe that

$$\zeta_n^s = qg + r$$

where $q \in T_n$ and $r \in T_{n-1}[\zeta_n]$ with $\deg r < s$. s.t. $|r| \leq 1$. Let $\omega = \zeta_n^s - r$. So, $\omega = qg$ where ω is ζ_n-distinguished of order s and has norm one. To show that the given decomposition of g exists, we prove that q is a unit in T_n. By 1.2.3, on observing the reductions we see that it is actually the case.
For the uniqueness, take $g = e\omega$ and let $r = \zeta_n^s - \omega$. Then

$$\zeta_n^s = e^{-1}g + r,$$

which gives us that e^{-1} and r are unique by the previous result. □

Corollary 1.2.8. *The Tate algebra $T_1 = K\langle\zeta_1\rangle$ is a Euclidean domain.*

Proof. Each non-zero $g \in T_1$ is ζ_1-distinguished of some order $s \in \mathbb{N}$. So, $T_1\ \{0\} \longrightarrow \mathbb{N}$ is a Euclidean function. Here, the map associates to g its order s. □

Definition 1.2.3. *Monic $\omega \in T_{n-1}[\zeta_n]$ s.t. $|\omega| = 1$, as seen in 1.2.7, are known as the Weierstrass polynomials in ζ_n.*

This means that every ζ_n-distinguished element $f \in T_n$ corresponds to a Weierstrass polynomial. Also, if f is a power series that is not trivial in T_n, notice using 1.2.5 that $\zeta_1, \ldots, \zeta_n \in T_n$ can be chosen s.t. f is ζ_n-distinguished of some order s.

Corollary 1.2.9. *(Noether Normalization). Consider an ideal $\mathfrak{a} \subsetneq T_n$, we then have a K-algebra injective homomorphism $T_d \to T_n$ for some $d \in \mathbb{N}$ s.t. $T_d \to T_n \to T_n/\mathfrak{a}$ is a finite injective homomorphism. Here, d is seen to be the Krull dimension of $T_n/\mathfrak{a}$.*

Proof. Let $\mathfrak{a}$ be non-trivial. So , there exists a non-zero f in $\mathfrak{a}$. Using some automorphism on T_n, g can be taken to be ζ_n-distinguished of order $s \geq 0$. Using 1.2.6, any $f \in T_n$ equals $r \in T_{n-1}[\zeta_n]$ s.t. $\deg r < s$ modulo g, or, the natural map $T_{n-1} \longrightarrow T_n \longrightarrow T_n/(g)$ is finite. We can also use the uniqueness of Weierstraß divison to claim that $T_n/(g)$ is free T_{n-1} module that is generated by the equivalence classes of $\zeta_n^0, \ldots, \zeta_n^{s-1}$.
Take $T_{n-1} \longrightarrow T_n/(g) \longrightarrow T_n/\mathfrak{a}$ with kernel as $\mathfrak{a}_1$. If $\mathfrak{a}_1 = 0$, the proof is over. Otherwise, repeat the same procedure for $\mathfrak{a}_1$ and T_{n-1}. Since we get a finite morphism on composing finite morphisms, we arrive at such a map $T_d \longrightarrow T_n/\mathfrak{a}$ in finitely many steps.
Now, the final statement can be readily seen. □

Corollary 1.2.10. *Consider maximal $\mathfrak{m} \subset T_n$. Then $T_n/\mathfrak{m}$ is finite as a K-vector space.*

Proof. By the last result, we have a finite injective homomorphism $T_d \hookrightarrow T_n/\mathfrak{m}$ where d is a natural number. Since $T_n/\mathfrak{m}$ is a field, so is T_d, which implies $d = 0$, or, $T_d = K$. $\square$

Corollary 1.2.11. *The morphism*

$$\mathbb{B}^n(\overline{K}) \to \operatorname{Max} T_n, \quad x \mapsto \mathfrak{m}_x = \{f \in T_n : f(x) = 0\},$$

from unit ball in $\overline{K}^n$ to the set of all maximal ideals in T_n is a surjection.

Let's now look at a few properties of the Tate algebra T_n.

Proposition 1.2.12. *T_n is Noetherian.*

Proof. We prove that each ideal $\mathfrak{a} \subset T_n$ is finitely generated. We proceed inductively. For the inductive step, we assume a non-trivial ideal inside T_n and use Weierstrass division to decrease the index n. $\square$

Proposition 1.2.13. *T_n is a UFD. In particular, it is integrally closed in its field of fractions.*

Proof. We will prove this inductively. Let T_{n-1} be a UFD. By the Gauß lemma, $T_{n-1}[\zeta_n]$ is a UFD as well. Let $f \neq 0$ in T_n that is not a unit. By the previous results, we can take f to be ζ_n-distinguished and hence a Weierstraß polynomial. Considering $f = \omega_1 \dots \omega_r$ for $\omega_i \in T_{n-1}[\zeta_n]$, ω_i are Weierstraß polynomials as well. Now, we show that ω_i which are primes in $T_{n-1}[\zeta_n]$ are primes in T_n. It suffices to show that $T_{n-1}/(\omega) \longrightarrow T_n/(\omega)$ is an isomorphism. But that is easily seen as they are free T_{n-1}-modules generated by equivalence classes of $\zeta_n^0, \dots, \zeta_n^{s-1}$. $\square$

Proposition 1.2.14. *T_n is Jacobson.*

Proof. Claim: For $\mathfrak{a} \subset T_n$, $\operatorname{rad} \mathfrak{a}$ is the intersection of maximal ideals that contain $\mathfrak{a}$. We know that $\operatorname{rad}(\mathfrak{a})$ is the intersection of all prime ideals of T_n that contains $\mathfrak{a}$. So,

$$\operatorname{rad}(\mathfrak{a}) \subseteq \bigcap_{\mathfrak{m} \in \operatorname{Max}(T_n) \mathfrak{a} \subseteq \mathfrak{m}} \mathfrak{m}.$$

We need to prove that every prime ideal $\mathfrak{p}$ of T_n is an intersection of maximal ideals.
Take $\mathfrak{p} = 0$. If $f \in \cap_{\mathfrak{m} \in \operatorname{Max}(T_n)}$, $f(x) = 0$ for every $x \in \mathbb{B}^n(\overline{K})$. By 1.2.4, we get $|f| = 0$ which implies $f = 0$. In the general case, using 1.2.9, we have an injective homomorphism $T_d \longrightarrow B = T_n/\mathfrak{p}$ s.t. B is finite over T_d. Claim: $\cap_{\mathfrak{m} \in \operatorname{Max}(B)} \mathfrak{m} = 0$. This can be seen by considering some f in the intersection and using the fact that it is integral over T_d. Finally, we get a contradiction to the $\mathfrak{p} = 0$ case by using 1.2.11. $\square$

1.3 Ideals in Tate Algebras

Take some ideal $\mathfrak{a} = (a_1, \dots, a_r)$ s.t. $|a_i| = 1$. We are interested in knowing if every $f \in \mathfrak{a}$ can be represented as $f = \sum_{i=1}^{r} f_i a_i$ where $f_i \in T_n$ s.t. $|f_i| \leq |f|$. If it were actually the case, we see that $\mathfrak{a}$ is complete under the norm on T_n which means $\mathfrak{a}$ is closed in T_n.

Definition 1.3.1. *Consider a ring R. A ring norm on R is a map $|\cdot| : R \to \mathbb{R}_{\geq 0}$ s.t. for $x, y \in R$*
(a) $|x| = 0 \iff x = 0$,
(b) $|xy| \leq |x||y|$,
(c) $|x + y| \leq max\{|x|, |y|\}$,
(d) $|1| \leq 1$.
This is said to be a multiplicative norm if we replace (b) with:
$(b)'$ $|xy| = |x||y|$.

Definition 1.3.2. *Consider a ring R along with a multiplicative ring norm $|\cdot|$ s.t. $|x| \leq 1$ for each $x \in R$.*
(a) R is said to be a B-ring if $\{x \in R : |x| = 1\} \subset R^*$.
(b) R is said to be bald if $sup\{|x| : x \in R \, where \, |x| < 1\} < 1$.

Lemma 1.3.1. *Let $S \subseteq R$ be some bald subring of the valuation ring of field K. Let $a \in R$ s.t. $|a| = 1$. Then, $S[a] \subseteq R$ is a bald subring.*

Proof. We can take S to be a B-ring by localizing S w.r.t. $\{x \in S \,|\, |x| = 1\} \subseteq S$. Let $\mathfrak{m} \subseteq S$ be the maximal ideal and let $T = S/\mathfrak{m}$. Also let $\mathcal{M} \subseteq R$ be the maximal ideal of R and $k = R/\mathcal{M}$.
Then, $\overline{a} \in k$ is either algebraic or transcedental over T. In either of the cases, it can be seen that the lemma is true. ☐

Proposition 1.3.2. *Consider a field K with a valuation on it and call its valuation ring R. Then the smallest subring $R' \subset R$ that contains some sequence $a_0, a_1, \ldots, \in R$ converging to zero is bald.*

Proof. The smallest subring T of R is either $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$ where p is a prime. T is seen to be bald in both of the cases. As $\lim_{n\to\infty}|a_n| = 0$, there are only finitely many a_i with $|a_i| = 1$. Substituting T by $T[a_i||a_i| = 1]$, by 1.3.1, it is still bald. So, assume there exists $\varepsilon < 1$ s.t. $|a_i| \leq \varepsilon$ for all $i \in \mathbb{N}$.
Now let $x \in T[a_i||a_i| = 1]$, then $x = \sum_{\nu\in\mathbb{N}^l} c_v a_1^{v_1} \ldots a_l^{v_l}$. Then, $|x| \leq \max\{|c_v| \cdot \varepsilon^{|v|}\}$. But then, $|c_v| \cdot \varepsilon^{|v|} < 1$ unless $v = 0$, in which case, $|x| = |c_v| < 1$ since $|x| < 1$. ☐

If we have a bald subring $R' \subset R$, we can localize R' with all the elements of norm 1 to get a B-ring R'' in R containing R' and is bald. Also, if R is complete, we can pass on to the completion of R''. Since on completing a B-ring, we get a B-ring again, the smallest complete B-ring in R that contains some bald subring of R is again bald.

Definition 1.3.3. *Consider a vector space V over a field K. A norm on V is a map $|\cdot| : V \to \mathbb{R}_{\geq 0}$, s.t.*
(a) $|a| = 0$ *iff* $a = 0$,
(b) $|a + b| \leq \max\{|a|, |b|\}$,
(c) $|ta| = |t||a|$ *for* $t \in K$ *and* $a, b \in V$.

Definition 1.3.4. *Consider a complete normed K-vector space V. A system $(x_\nu)_{\nu\in N}$ of vectors in V, s.t. N is at most countable, is said to be a (topological) orthonormal basis of V if:*
(a) $|x_\nu| = 1$ for every $\nu \in N$.
(b) Every $x \in V$ may be expressed as a convergent series $x = \sum\limits_{\nu\in N} c_\nu x_\nu$ with coefficients in K.
(c) For each equation $x = \sum\limits_{\nu\in N} c_\nu x_\nu$ as in (b), $|x| = max_{\nu\in N} |c_\nu|$. Particularly, c_ν in (b) are unique.

As an example, $\zeta^\nu \in T_n$ form an orthonormal basis when $T_n = K\langle\zeta\rangle$ is the normed vector space over field

K. For a normed vector space V over K, denote:

$$V^o = \{x \in V : |x| \leq 1\},$$

as its "unit ball" and

$$\tilde{V} = V^o / \{x \in V : |x| \leq 1\},$$

as its reduction.

Theorem 1.3.3. *Consider a field K that has a complete valuation and a complete normed vector space V over the field K with an orthonormal basis $(x_\nu)_{\nu \in N}$. Let R be the valuation ring of K, and let's have a system*

$$y_\mu = \sum_{\nu \in N} c_{\nu\mu} s_\nu \in V^o, \quad \mu \in M,$$

s.t. the smallest subring of R that contains all $c_{\nu\mu}$'s is bald. Then, if the residue classes $\tilde{y}_\mu \in \tilde{V}$ forms a K-basis of $\widetilde{V}$, y_μ's form an orthonormal basis of V.

Corollary 1.3.4. *Consider some ideal $\mathfrak{a} \subset T_n$. We then have generators $a_1, \ldots, a_r$ of $\mathfrak{a}$ s.t.:*
(a) $|a_i| = 1$ for every i.
(b) If $f \in \mathfrak{a}$, we have $f_1, \ldots, f_r \in T_n$ s.t.

$$f = \sum_{i=1}^{r} f_i a_i, \quad |f_i| \leq |f|.$$

Corollary 1.3.5. *Every ideal $\mathfrak{a}$ in T_n is complete, which implies it is closed in T_n.*

Chapter 2

Affinoid Algebras and Spaces

2.1 Affinoid Algebras

If $\mathfrak{a} \subset T_n$ is an ideal, it's zero set defined as

$$V(\mathfrak{a}) = \{x \in \mathbb{B}^n(\overline{K}) : f(x) = 0 \ \forall f \in \mathfrak{a}\}$$

can be looked as restriction of elements of $\mathbb{B}^n(\overline{K})$ to $V(\mathfrak{a})$. Thus, $A = T_n/\mathfrak{a}$ can be interpreted as an algebra of "functions" on $V(\mathfrak{a})$.

Definition 2.1.1. *A K-algebra A is said to be an affinoid K-algebra if we have a K-algebra surjective homomorphism $\alpha : T_n \to A$ where $n \in \mathbb{N}$.*

Proposition 2.1.1. *Consider an affinoid K-algebra A. Then, A is Noetherian, Jacobson, and it satisfies the Noether normalization (or, there exists a finite injective homomorphism $T_d \hookrightarrow A$ where $d \in \mathbb{N}$.)*

Proposition 2.1.2. *Consider an affinoid K-algebra A with an ideal $\mathfrak{q}$ in A s.t. its nilradical is maximal in A. Then $A/\mathfrak{q}$ has a finite dimension as a K-vector space.*

Proof. Take $\mathfrak{m} = \operatorname{rad} \mathfrak{q}$. By the previous result, we have a finite injective morphism $T_d \hookrightarrow A/\mathfrak{q}$ where $d \in \mathbb{N}$. Since $\mathfrak{q} \subset \mathfrak{m}$, we get $T_d \hookrightarrow A/\mathfrak{m}$ and as $A/\mathfrak{m}$ is a field, d is zero. □

We can easily describe a norm on affinoid K-algebras. For a surjective morphism $\alpha : T_n \to A$, the Gauß norm $|\cdot|$ of T_n induces a residue norm $|\cdot|_\alpha$ on A as:

$$|\alpha(f)|_\alpha = \inf_{a \in \ker \alpha} |f - a|.$$

Intuitively, for any $f \in A$, this norm takes the infimum over all the pre-images of f in T_n.

Proposition 2.1.3. *For an ideal $\mathfrak{a} \subset T_n$, let $A = T_n/\mathfrak{a}$ be an affinoid K-algebra with projection $\alpha : T_n \to A$. Then for $|\cdot|_\alpha : A \to \mathbb{R}_{\geq 0}$, we have:*

(i) $|\cdot|_\alpha$ *is a K-algebra norm. Also,* $\alpha : T_n \to A$ *is open and continuous.*
(ii) A *is complete under* $|\cdot|_\alpha$*.*
(iii) If $\overline{f} \in A$*, we have a pre-image* f *in* T_n *s.t.* $|f| = |\overline{f}|_\alpha$*. So, for each* $\overline{f} \in A$*, we have* $c \in K$ *s.t.* $|\overline{f}|_\alpha = |c|$*.*

Viewing the elements f of A as functions taking values in $\overline{K}$ on $V(\mathfrak{a})$, the zero set , consider the supremum $|f|_{\text{sup}}$ of every value taken by f. This value is finite by 1.2.4. To make this independent of the specific representation of A as $T_n/\mathfrak{a}$, we set for $f \in A$, called the supremum norm, by

$$|f|_{\text{sup}} = \sup_{x \in \operatorname{Max} A} |f(x)|$$

where $\operatorname{Max} A$ is the spectrum of maximal ideals of A. Also, if $x \in \operatorname{Max} A$, $f(x)$ is the equivalence class of f in A/x. But A/x is a finite extension of K because of 1.2.4, so $|f(x)|$ is well-defined. Note that this supremum norm is in fact only a semi norm, since it doesn't satisfy $|f|_{\text{sup}} = 0 \Rightarrow f = 0$.

Proposition 2.1.4. *Consider an affinoid K-algebra morphism* $\varphi : B \to A$*. Then, for every* $b \in B$*,* $|\varphi(b)|_{sup} \le |b|_{sup}$*.*
Proof. Since $A/\mathfrak{m}$ is finite over K, we write $\mathfrak{r} = \varphi^{-1}(\mathfrak{m})$ to obtain finite maps $K \hookrightarrow B/\mathfrak{r} \hookrightarrow A/\mathfrak{m}$ and hence $\mathfrak{r} \subset B$ is maximal. Since for $b \in B$, $|b(\mathfrak{r})| = |\varphi(b)(\mathfrak{m})|$, this completes the argument. □

Proposition 2.1.5. *The supremum norm* $|\cdot|_{sup}$ *on* T_n *coincides with the Gauß norm* $|\cdot|$*.*
Proof. Before we start with the proof, note that this means that on T_n, $|\cdot|_{\text{sup}}$ is really a norm. Now, by 1.2.4

$$|f| = \sup\{|f(x)| \,:\, x \in \mathbb{B}^n(\overline{K})\}$$

for all $f \in T_n$. We associate to $x \in \mathbb{B}^n(\overline{K})$, the maximal ideal of T_n given by $\mathfrak{m}_x = \{h \in T_n \,:\, h(x) = 0\}$. Evaluation at x gives $T_n/\mathfrak{m}_x \hookrightarrow \overline{K}$. Also, $f(\mathfrak{m}_x) = f(x)$ which implies $|f(\mathfrak{m}_x)| = |f(x)|$. As $\mathbb{B}^n(\overline{K}) \longrightarrow \operatorname{Max} T_n$ s.t. $x \mapsto \mathfrak{m}_x$ is a surjection by 1.2.11, we have the proof. □

Proposition 2.1.6. *Consider an affinoid K-algbera A. Then, for* $f \in A$*, TFAE:*
(i) $|f|_{sup} = 0$.
(ii) f *is nilpotent.*

We now wish to give some relation on the supremum norm and the residue norm on affinoid K-algebras.

Lemma 2.1.7. *For any polynomial*

$$p(\zeta) = \zeta^r + c_1\zeta^{r-1} + \ldots + c_r = \prod_{j=1}^{r}(\zeta - \alpha_j)$$

in $K[\zeta]$ *with zeroes* $\alpha_1, \ldots, \alpha_r \in \overline{K}$*, we have*

$$\max_{j=1,\ldots,r} |\alpha_j| = \max_{i=1,\ldots,r} |c_i|^{\frac{1}{i}}.$$

Proof. Up to sign, since c_i are the ith elementary symmetric function of zeroes $\alpha_1, \ldots, \alpha_r$, we have

$$|c_i|^{\frac{1}{i}} \le \max_{j=1,\ldots,r} |\alpha_j|$$

for $i = 1, \ldots, r$. If $|\alpha_j|$ is maximal only for $j = 1, \ldots, s$, then $|c_s| = |\alpha_1| \ldots |\alpha_s|$. This implies

$$|c_s|^{\frac{1}{s}} = \max_{j=1,\ldots,r} |\alpha_j|.$$

and hence we are done. $\square$

Consider a monic polynomial $p = \zeta^r + c_1\zeta^{r-1} + \ldots + c_r$ where c_i belong to a normed (or semi-normed) ring A, define

$$\sigma(p) = \max_{j=1,\ldots,r} |c_i|^{\frac{1}{i}}$$

as the spectral value of p. Thus the last lemma implies that the spectral value of p is the maximal value of its zeroes.

Lemma 2.1.8. *Let $T_d \hookrightarrow A$ be a finite monomorphism where A is a K-algebra. Assume some $f \in A$ and let A, as a T_d-module, be torsion-free.*

(i) We have a unique monic polynomial $p_f = \zeta^r + a_1\zeta^{r-1} + \ldots + a_r \in T_d[\zeta]$ of minimal degree s.t. $p_f(f) = 0$. In other words, the kernel of the T_d-homomorphism

$$T_d[\zeta] \longrightarrow A, \quad \zeta \longmapsto f$$

is generated by p_f.
(ii) Taking $x \in Max\, T_d$, assume that $y_1, \ldots, y_s \in Max\, A$ are the maximal ideals that restrict to x on T_d. Then

$$\max_{j=1,\ldots,s} |f(y_j)| = \max_{i=1,\ldots,r} |a_i(x)|^{\frac{1}{i}}.$$

(iii) The sup-norm of f is obtained by

$$|f|_{sup} = \max_{i=1,\ldots,r} |a_i|_{sup}^{\frac{1}{i}}.$$

Proof. Since A is finite over T_d (as ideals in T_d are finitely generated), for $y \in \operatorname{Max} A$, A/y is a finite K-vector space. Hence the norms are well-defined.
Let $F = Q(T_d)$ be the field of fractions of T_d. Also, let $F(A) = A \otimes_{T_d} F$ be the F-algebra obtained from A. As A has no torsion over T_d, we have this commiutative diagram of inclusions:

$$\begin{tikzcd} T_d \arrow[r] \arrow[d] & A \arrow[d] \\ F \arrow[r] & F(A) \end{tikzcd}$$

Take the kernel of the F-homomorphism $F[\zeta] \to F(A)$, s.t. $\zeta \mapsto f$, which is generated by a unique monic $p_f \in F[\zeta]$. Claim: $p_f \in T_d[\zeta]$. Now this can be proved using Gauß Lemma and the fact that A is integral over T_d. This finishes (i).

Now, for (ii), we know from the properties of integral ring extensions that the restrictions of maximal ideals yields surjective maps

$$\operatorname{Max} A \longrightarrow \operatorname{Max} T_d[f] \longrightarrow \operatorname{Max} T_d.$$

Hence, since we are looking at maximal ideals in A that restrict to x, we can look at A instead of $T_d[f]$. So let $A = T_d[f]$. Let $L = T_d/x$. Also, let $\overline{f}$ be f's image in $A/(x)$ and $\overline{p_f}$ that of p_f in $L[\zeta]$, then, we get a finite map $L \to A/(x)$. Here, $A/(x) = L[\zeta]/(\overline{p_f})$ as is evident by the First Isomorphism Theorem. Take $\alpha_1, \ldots, \alpha_r$ to be the zeroes of $\overline{p_f}$ in an algebraic closure of L. Then consider these canonical maps

$$A/(x) = L[\overline{f}] \longrightarrow L[\alpha_i], \quad \overline{f} \mapsto \alpha_i.$$

Their kernels are the maximal ideals in $A/(x)$ and hence they coincide with the maximal ideals of A lying over x.
By 2.1.7, we have

$$\max_{j=1,\ldots,s} |f(y_j)| = \max_{i=1,\ldots,r} |\alpha_i| = \max_{i=1,\ldots,r} |a_i|^{\frac{1}{i}}$$

and this finishes (ii).
As for (iii), it can be seen directly using (ii). $\square$

We now generalize (iii) in the next lemma.

Lemma 2.1.9. *Consider a finite affinoid K-algebra homomorphism $\varphi : B \longrightarrow A$. Then, for every $f \in A$, we have an integral equation*

$$f^r + b_1 f^{r-1} + \ldots + b_r = 0$$

where $b_j \in B$ s.t. $|f|_{sup} = max_{i=1,\ldots,r}\ |b_i|_{sup}^{\frac{1}{i}}$.

Theorem 2.1.10. *(Maximum Principle) For every affinoid K-algebra A and $f \in A$, we have $x \in \operatorname{Max} A$ s.t. $|f(x)| = |f|_{sup}$.*

Proof. Since A is Noetherian, it consists only of a finitely many minimal prime ideals, denoted by $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, and we look at $\operatorname{Max} A$ as the union of $\max A/\mathfrak{p}_j, j = 1, \ldots, s$. Let f_j be the equivalence class of f in $A/\mathfrak{p}_j$. Then we have an index j s.t. $|f|_{\text{sup}} = |f_j|_{\text{sup}}$. So, A can be replaced with $A/\mathfrak{p}_j$ and consider A to be an integral domain. By Noether Normalization, we get a finite monomorphism $T_d \hookrightarrow A$ from which, the result can now be easily derived using 1.2.4 and 2.1.8. $\square$

Lemma 2.1.11. *Consider an affinoid K-algebra A and take $f_1, \ldots, f_n \in A$.*
(i) Let there be a K-map $\varphi : K\langle\zeta_1, \ldots, \zeta_n\rangle \longrightarrow A$ s.t. $\varphi(\zeta_i) = f_i$ for $i = 1, \ldots, n$. Then $|f_i|_{sup} \leq 1$ for every i.
(ii) Conversely, if $|f_i|_{sup} \leq 1$ for every i, we have a unique K-map $\varphi : K\langle\zeta_1, \ldots, \varphi_n\rangle \longrightarrow A$ s.t. $\varphi(\zeta_i) = f_i$ for every i. Also, φ is continuous w.r.t. the Gauß norm on T_n and any residue norm on A.

Proof. By 2.1.5, since $|\zeta_i|_{\text{sup}} = |\zeta_i| = 1$, we have (i). For (ii), fix a residue norm $|\cdot|_\alpha$ on A and define φ as

$$\varphi\,(\sum_{v\in\mathbb{N}^n} c_v \zeta_1^{v_1} \ldots \zeta_n^{v_n}) = \sum_{v\in\mathbb{N}^n} c_v f_1^{v_1} \ldots f_n^{v_n}$$

It can now be seen that $|f_i|_{\text{sup}} \leq 1$ means that the f_i are power bounded w.r.t. the residue norms on A, i.e. the sequence $|f^n|_\alpha, n \in \mathbb{N}$ is bounded. (To do this, first prove that there exists an integral equation

$f^r + a_1 f^{r-1} + \ldots + a_r = 0$ where $a_i \in A$ s.t. $|a_i|_\alpha \leq 1$.) This means that φ is well-defined and is unique as a continuous map for which $\zeta_i \mapsto f_i$. So we just need to prove now that φ is the only K-map $K\langle\zeta_1, \ldots, \zeta_n\rangle \longrightarrow A$ s.t. $\zeta_i \mapsto f_i$.
Let's start with A being finite dimensional as a K-vector space. Then, every K-map $\varphi' : K\langle\zeta_1, \ldots, \zeta_n\rangle \longrightarrow A$ is continuous. Now, any norm on A induces the product topology since any $A \xrightarrow{\sim} K^d$ is a homeomorphism. Now viewing $T_n/\ker\varphi'$ as an affinoid K-algebra with the natural residue norm, it's clear that $T_n/\ker\varphi' \hookrightarrow A$ is continuous since for a finite dimensional vector space V, the linear maps $V \longrightarrow K$ are continuous if V has been given the product topology. So, φ' is continuous.
Now, consider K-maps $\varphi, \varphi' : T_n \longrightarrow A$, s.t. $\zeta_i \mapsto f_i$. Then taking $r > 0$, $A/\mathfrak{m}^r$ is of finite vector space dimension over K for a maximal ideal $\mathfrak{m} \subset A$. Hence, the two induced maps $T_n \to A/\mathfrak{m}^r$ are continuous which means that they coincide. So, all we need to prove is that the map from A to $A/\mathfrak{m}^r$ is trivial. This can be achieved by using 6.1.2 on all localizations $A_\mathfrak{m}$ where $\mathfrak{m} \in \operatorname{Max} A$. □

Proposition 2.1.12. *Any affinoid K-algebra morphism $B \longrightarrow A$ is continuous w.r.t. a residue norm on A and B. Particularly, residue norms on an affinoid K-algebra are equivalent.*

Proof. Take a surjection $T_n \longrightarrow B$ which results in $T_n \longrightarrow B \longrightarrow A$. We know from the previous result that this composition is continuous w.r.t. any residue norm on A, which implies the map from B to A is continuous as well. □

2.2 Affinoid Spaces

Let A be an affinoid algebra over K. The affinoid space associated to A is given by $\operatorname{Sp}(A) = \operatorname{Max}(A)$. For $\mathfrak{a} \subset A$, an ideal in A, its zero set

$$V(\mathfrak{a}) = \{x \in \operatorname{Sp} A \,:\, \mathfrak{a} \subset \mathfrak{m}_x\} = \{x \in \operatorname{Sp} A \,:\, f(x) = 0 \text{ for every } f \in \mathfrak{a}\}$$

is called a Zariski closed subset of A.

With this definition of Zariski closed subset, we state some results similar to as that in Algebraic Geometry.

Lemma 2.2.1. *Consider an affinoid K-algebra A with ideals $\mathfrak{a}$ and $\mathfrak{b}$. Also, assume a family $(\mathfrak{a}_i)_{i\in I}$ of ideals in A. Then:*
(i) $\mathfrak{a} \subset \mathfrak{b} \implies V(\mathfrak{a}) \supset V(\mathfrak{b})$.
(ii) $V(\sum_{i\in I} \mathfrak{a}_i) = \bigcap_{i\in I} V(\mathfrak{a}_i)$.
(iii) $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \bigcup V(\mathfrak{b})$.

Proposition 2.2.2. *Consider an affinoid K-algebra A. Then the sets given by*

$$D_f = \{x \in Sp\, A \,:\, f(x) \neq 0\}, \;\; f \in A$$

form a basis of the Zariski topology of $Sp\, A$.

For $Y \subset \operatorname{Sp} A$, associate to it, the ideal given by

$$\operatorname{id}(Y) = \{f \in A \,:\, f(y) = 0, \text{ for every } y \in Y\} = \bigcap_{y \in Y} \mathfrak{m}_y.$$

Theorem 2.2.3. *(Hilbert's Nullstellensatz) Consider an affinoid K-algebra A and an ideal $\mathfrak{a}$ in A. Then,* $\operatorname{id}(V(\mathfrak{a})) = \operatorname{rad} \mathfrak{a}$.

Proof. By definition,

$$\operatorname{id}(V(\mathfrak{a})) = \operatorname{id}(\{x \in \operatorname{Sp} A \,:\, \mathfrak{a} \subset \mathfrak{m}_x\}) = \bigcap_{\mathfrak{a} \subset \mathfrak{m}_x} \mathfrak{m}_x.$$

Here, the R.H.S. is the nilradical of $\mathfrak{a}$, as affinoid K-algebras are Jacobson. ☐

Corollary 2.2.4. *For an affinoid K-algebra A, assume a set of functions f_i for $i \in I$. Then TFAE:*
(i) f_i's do not have any common zeroes on $\operatorname{Sp} A$.
(ii) f_i's generate the unit ideal in A.

Any affinoid K-algebra map $\sigma : B \longrightarrow A$ induces an associated morphism

$${}^a\sigma \,:\, \operatorname{Sp} A \longrightarrow \operatorname{Sp} B, \quad \mathfrak{m} \mapsto \sigma^{-1}(\mathfrak{m})$$

Here, $\sigma^{-1}(\mathfrak{m}) \subset B$ is a maximal ideal as we have a chain of injective maps

$$K \hookrightarrow B/\sigma^{-1}(\mathfrak{m}) \hookrightarrow A/\mathfrak{m}$$

and as $A/\mathfrak{m}$ is a finite field over K. ${}^a\sigma \,:\, \operatorname{Sp} A \longrightarrow \operatorname{Sp} B$ along with σ is said to be a map of affinoid K-spaces. We write $\varphi : \operatorname{Sp} A \longrightarrow \operatorname{Sp} B$ for an affinoid K-space morphism and we represent the map between the affinoid K-algebras by $\varphi^* : B \longrightarrow A$.

Now, φ^* can also be seen as the pulling back of maps from $\operatorname{Sp} B$ to $\operatorname{Sp} A$ since for $x \in \operatorname{Sp} A$, we have

$$\begin{tikzcd} B \arrow[r, "\varphi^*"] \arrow[d] & A \arrow[d] \\ B/\mathfrak{m}_{\varphi(x)} \arrow[r] & A/\mathfrak{m}_x \end{tikzcd}$$

which gives $\varphi^*(g)(x) = g(\varphi(x))$ for every $g \in B$.

2.3 Affinoid Subdomains

The Zariski topology is quite coarse. One issue with it is that it is exactly the same as in Algebraic Geometry and does not use the non-Archimedean topology of the base field. Another issue is that the open subset $D_f \subset \operatorname{Sp}(A)$ corresponds to the A-algebra $A[\frac{1}{f}]$, which is not complete.

To remedy these issues, we want a finer topology which is directly induced from the topology of K. For some $n \in \mathbb{N}$, see $\operatorname{Sp} A$ as a Zariski closed subspace of $\operatorname{Sp} T_n$. Now, when K is algebraically closed, $\operatorname{Sp} T_n$ can be identified with $\mathbb{B}^n(K)$. So the topology of the affine n-space K^n induces a topology on $\operatorname{Sp} A$, called the canonical topology of $\operatorname{Sp} A$.

Let A be an affinoid K-algebra and $X = \operatorname{Sp}(A)$. If $f \in A$ and $\varepsilon \in \mathbb{R}_{\geq 0}$, we put $X(f, \varepsilon) = \{x \in X : |f(x)| \leq \varepsilon\}$.

Definition 2.3.1. *The natural topology on Sp (A) is the topology generated by every set of type $X(f, \varepsilon)$, i.e. $U \subseteq Sp\, A$ is open for the canonical topology iff it is a union of sets $X(f_1, \varepsilon_1) \cap \cdots \cap X(f_r, \varepsilon_r)$.*

By convention, $X(f) = X(f, 1)$ and $X(f_1, \cdots, f_r) = X(f_1) \cap \cdots \cap X(f_r)$ where $f, f_1, \ldots, f_r \in A$.

Proposition 2.3.1. *The natural topology is generated by sets $X(f)$ for $f \in A$, i.e. each open subset of Sp (A) is a finite union of subsets $X(f_1, \cdots, f_r)$.*

Proof. We know that $|f(x)| \in |\overline{K}^*|$ for each $f \in A$ and $x \in \operatorname{Sp}(A)$. For all $f \in A$, and $\varepsilon \in \mathbb{R}_{>0}$,

$$X(f, \varepsilon) = \bigcup_{\varepsilon' \leq \varepsilon : \varepsilon' \in |\overline{K}^*|} X(f, \varepsilon')$$

As $\varepsilon' \in |\overline{K}^*|$, there exists $c \in K^*$ and $s > 0$ in $\mathbb{Z}$ s.t. $\varepsilon'^s = |c|$. Hence, we have $X(f, \varepsilon') = X(f^s, \varepsilon'^s) = X(c^{-1} f^s)$. $\square$

We now see the next result that helps us derive the openness of various kinds of sets.

Lemma 2.3.2. *Let $f \in A$ and $x \in Sp\, A$ s.t. $\varepsilon = |f(x)| > 0$. Then we have $g \in A$ s.t. $g(x) = 0$ and $|f(y)| = \varepsilon$ for every $y \in X(g)$. Particularly, $X(g)$ is an open nbhd of x in $\{y \in X : |f(y)| = \varepsilon\}$.*

Proof. Let $\overline{f}$ be the equivalence class of f in $A/\mathfrak{m}_x$ and take $P(\zeta) = \zeta^n + c_1 \zeta^{n-1} + \ldots + c_n \in K[\zeta]$ to be the minimal polynomial of $\overline{f}$. Also, set $P(\zeta) = \prod_{i=1}^{n} (\zeta - \alpha_i)$, $\alpha_i \in \overline{K}$ as the factorization into roots. Then, fixing $A/\mathfrak{m}_x \hookrightarrow \overline{K}$, $\varepsilon = |f(x)| = |\overline{f}| = |\alpha_i|$ for every i since $P(\overline{f}) = 0$ and since valuation is unique on $\overline{K}$.

Now, let $g = P(f) \in A$. Then $g(x) = 0$ and the next equation can now be easily seen (hint: contrapostive)

$$y \in X \text{ where } |g(y)| < \varepsilon^n \implies |f(y)| = \varepsilon$$

which gives a contradiction. $\square$

Corollary 2.3.3. *If $f \in A$ and $\varepsilon \in \mathbb{R}_{>0}$, the given sets are open w.r.t. natural topology:* $\{x \in Sp\, A : f(x) \neq 0\}, \{x \in Sp\, A : |f(x)| \leq \varepsilon\}, \{x \in Sp\, A : |f(x)| = \varepsilon\}, \{x \in Sp\, A : |f(x)| \geq \varepsilon\}$.

Proposition 2.3.4. *For φ and φ^*, and $f_1, \ldots, f_r \in A$:*

$$\varphi^{-1}((Sp\, A)(f_1, \ldots, f_r)) = (Sp\, B)(\varphi^*(f_1), \ldots, \varphi^*(f_r)).$$

Particularly, φ is continuous w.r.t. the natural topology.

Proof. For every $y \in \operatorname{Sp} B$, we have

$$\begin{tikzcd} A \arrow[r, "\varphi^*"] \arrow[d] & B \arrow[d] \\ A/\mathfrak{m}_{\varphi(y)} \arrow[r] & B/\mathfrak{m}_y \end{tikzcd}$$

where the bottom row is a monomorphism. Hence, for every $f \in A$, $|f(\varphi(y))| = |\varphi^*(f)(y)|$. This implies

$$\varphi^{-1}((\operatorname{Sp} A)(f)) = (\operatorname{Sp} B)(\varphi^*(f))$$

and by taking intersections, we are done. ☐

Definition 2.3.2. *Consider an affinoid K-space $X = Sp\,A$.*
(i) A subset of X of type

$$X(f_1, \ldots, f_r) = \{x \in X \,:\, |f_i(x)| \leq 1\}$$

where $f_1, \ldots, f_r \in A$ is said to be a Weierstraß domain in X.
(ii) A subset of X of type

$$X(f_1, \ldots, f_r, g_1^{-1}, \ldots, g_s^{-1}) = \{x \in X \,:\, |f_i(x)| \leq 1, |g_j(x)| \geq 1|\}$$

where $f_1, \ldots, f_r, g_1, \ldots, g_s \in A$ is said to be a Laurent domain in X.
(iii) A subset of X of type

$$X(\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0}) = \{x \in X \,:\, |f_i(x)| \leq |f_0(x)|\}$$

where $f_0, \ldots, f_r \in A$ without common zeroes is said to be a rational domain in X.

Lemma 2.3.5. *The domains in the last definition are open in $Sp\,A$ w.r.t. the natural topology. Also, the set of Weierstraß domains is a basis for this topology.*

Definition 2.3.3. *Consider an affinoid K-space $X = Sp\,A$. Then $U \subset X$ is said to be an affinoid subdomain of X if we have a map of affinoid K-spaces $i : X' \longrightarrow X$ s.t. $i(X') \subset U$ and the given universal property is satisfied:*
Any map of affinoid K-spaces $\varphi : Y \longrightarrow X$ for which $\varphi(Y) \subset U$ admits a unique factorization through $i : X' \longrightarrow X$ via a map of affinoid K-spaces $\varphi' : Y \longrightarrow X$.

Lemma 2.3.6. *With the previous definition, let $X = Sp\,A$, $X' = Sp\,A'$ and $i^* : A \longrightarrow A'$ be the K-morphism associated to i. Then:*
(i) i is an injection and $i(X') = U$. Hence, a bijective map $X' \xrightarrow{\sim} U$ is induced.
(ii) For every $x \in X'$ and $n \in \mathbb{N}$, i^* induces an isomorphism of affinoid K-algebras
$A/\mathfrak{m}_{i(x)}^n \xrightarrow{\sim} A'/\mathfrak{m}_x^n$.
(iii) $\mathfrak{m}_x = \mathfrak{m}_{i(x)}A'$ when $x \in X'$, .

Proof. For $y \in U$, there is a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "i^*"] \arrow[d, "\pi"] & A' \arrow[d, "\pi'"] \\ A/\mathfrak{m}_y^n \arrow[r, "\sigma"] & A'/\mathfrak{m}_y^n A' \end{tikzcd}$$

So $\mathrm{Sp}\,(A/\mathfrak{m}_y^n)$ is a singleton space which is mapped to $y \in U$. Using the universal property of affinoid subdomains, we have a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "i^*"] \arrow[d, "\pi"] & A' \arrow[d, "\pi'"] \arrow[dl, "\alpha"'] \\ A/\mathfrak{m}_y^n \arrow[r, "\sigma"] & A'/\mathfrak{m}_y^n A' \end{tikzcd}$$

We claim that the lower triangle is commutative. Firstly, the morphism of affinoid K-spaces corresponding to $\sigma \circ \pi$ is mapped to $y \in U$. Since $\sigma \circ \pi$ factors through i^* via both π^* and $\sigma \circ \alpha$, by the uniqueness, we have $\pi' = \sigma \circ \alpha$.

Since π' is a surjection, the same is true for σ. Also, the surjectivity of π implies that of α. $\ker \pi' = \mathfrak{m}_y^n A' \subset \ker \alpha$, thus, *sigma* is injective, which means it's bijective. For $n = 1$, $\mathfrak{m}_y A'$ is maximal in A'. So, the fiber of i over y is not trivial and is singleton with $x \in X'$ for $\mathfrak{m}_x = \mathfrak{m}_y A'$. Hence, we have shown (i) and (iii). For (ii), note that $\mathfrak{m}_x = \mathfrak{m}_y A' = \mathfrak{m}_{i(x)} A'$. □

When working with affinoid subdomains, we use the previous lemma to identify $U \subset X$ to X'. Thus there is an affinoid K-space structure on every affinoid subdomain U in X which is unique up to natural isomorphism. Consider the affinoid subdomain $X' \hookrightarrow X$. It is said to be open in X if it is open w.r.t. the natural topology.

Proposition 2.3.7. *The Weierstraß, Laurent, and Rational domains for any affinoid K-space $X = Sp\,A$ are open affinoid subdomains. We call them as special affinoid subdomains.*

Proof. We have already seen the openness before. We will only prove the proposition for Weierstraß domains here. The rest of the two domains can be done similarly. Let's look at Weierstraß domain $X(f) \subset X$. Let $A\langle \zeta_1, \dots, \zeta_r \rangle$ be the affinoid K-algebra of restricted power series over A. Here, we have the residue norm on A. Consider an affinoid K-algebra

$$A\langle f \rangle = A\langle f_1, \dots, f_r \rangle = A\langle \zeta_1, \dots, \zeta_r \rangle / (\zeta_i - f_i \,:\, i = 1, \dots, r).$$

There is a canonical affinoid K-algebra morphism $i^* : A \longrightarrow A\langle f \rangle$ with map $i : \mathrm{Sp}\,A\langle f \rangle \longrightarrow X$. Claim: i is mapped to $X(f)$ and every other morphism of affinoid K-spaces $\varphi : Y \longrightarrow X$ with $\mathrm{img}\,\varphi \longrightarrow X(f)$ admits a unique factorization through i.

Take an affinoid K-space morphism $\varphi : Y \longrightarrow X$ and the associated affinoid K-algebra morphism $\varphi^* : A \longrightarrow B$. For every $y \in Y$, observe that

$$|\varphi^*(f_i)(y)| = |f_i(\varphi(y))|, \quad i = 1, \dots, r,$$

since $A/\mathfrak{m}_{\varphi(y)} \hookrightarrow B/\mathfrak{m}_y$ between finite extensions of K. So, $\varphi(Y) \subset X(f)$ is equivalent to $|\varphi^*(f_i)|_{\sup} \leq 1$ for every i. As $i^*(f_i)$ is the equivalence class of ζ_i in $A\langle f\rangle$, $|f_i|_{\sup} \leq 1$ since $|f|_{\sup} \leq |f|_\alpha$ for every f. This proves that i has image in $X(f)$. The rest of the claim can now be proved by extending φ^* from A to $A\langle\zeta\rangle$ s.t. $\zeta_i \mapsto \varphi^*(f_i)$ for every i.
Remark: For rational domain, consider

$$A\langle \frac{f}{f_0}\rangle = A\langle \frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0}\rangle = A\langle\zeta_1, \ldots, \zeta_r\rangle/(f_i - f_0\zeta_i \, : \, i = 1, \ldots, r)$$

$\square$

Proposition 2.3.8. (Transitivity of Affinoid Subdomains) *For affinoid K-space X, take an affinoid subdomain V in X, and an affinoid subdomain U in V. Then U is an affinoid subdomain in X.*

Proposition 2.3.9. *Take an affinoid K-space morphism $\varphi : Y \longrightarrow X$ and also an affinoid subdomain $X' \hookrightarrow X$. Then $Y' = \varphi^{-1}(X')$ is an affinoid subdomain of Y, and we have a unique affinoid K-space map $\varphi' : Y' \hookrightarrow X'$ s.t. the given diagram commutes*

$$\begin{tikzcd}
Y' \arrow[r, hook, "\varphi'"] \arrow[d] & X' \arrow[d] \\
Y \arrow[r, "\varphi"] & X
\end{tikzcd}$$

If X' is Weierstraß, Laurent, or rational in X, the same holds for $Y' \subset Y$.

Proposition 2.3.10. *Consider an affinoid K-space X and affinoid subdomains U, V in X. Then $U \cap V$ is an affinoid subdomain of X. If U and V are any of three domains, the same will hold for $U \cap V$ as well.*

Corollary 2.3.11. *Consider an affinoid K-space $X = Sp\, A$. Every Weierstrasß domain in X is Laurent, and every Laurent domain in X is rational.*

Proposition 2.3.12. *Consider an affinoid K-space $X = Sp\, A$ and a rational subdomain $U \subset X$. Then we have a Laurent domain $U' \subset X$ s.t. $U \subset U'$ is a Weierstraß domain.*
Proof. Take $U = \mathrm{Sp}\, A' = X(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0})$ where f_i's do not have common zeroes on X. Since $|f_i(x)| \leq |f_0(x)|$ for every i, observe that $f_0(x) \neq 0$ for every $x \in U$. Hence, $f_0|_U$ is a unit in A'. Applying 2.1.10 on $(f_0|_U)^{-1}$, we have $c \in K*$ s.t. $|cf_0(x)| \geq 1$ for every $x \in U$. Putting $U' = X((cf_0)^{-1})$, observe $U \subset U'$ and that

$$U = U'(f_1|_{U'} \cdot (f_0|_{U'})^{-1}, \ldots, f_r|_{U'} \cdot (f_0|_{U'})^{-1}).$$

Here $f_0|_{U'}$ is a unit on U'. Hence, U' as constructed above, works. $\square$

Proposition 2.3.13. (Transitivity of Special Affinoid Subdomains). *Consider an affinoid K-space X, Weierstraß (respectively rational) domain $V \subset X$, and a Weierstraß (respectively rational) domain $U \subset V$. Then U is a Weierstraß (respectively rational) domain in X. By the last propositiion, the same does not hold for Laurent domains.*
Proof. Put $X = \mathrm{Sp}\, A$. Let's prove for Weierstraß domains first. Let $V = X(f)$ and $U = V(g)$. Since A's image is dense in $A\langle f\rangle$ and since we can subtract a tuple of supremum norm ≤ 1 from g without altering

$U = V(g)$, we can let g to be the restriction of a tuple of functions in A. In that case, $U = X(f, g)$ and the result follows.

Let's look at rational domains now. Let $V = X(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0})$ where $f_0, \ldots, f_r \in A$ with no common zeroes in X. By the previous results, we know that U is a Weierstraß domain in a Laurent domain of V and that the intersection of finitely many rational domains is again a rational domain. So it is enough to work with $U = V(g)$ or $U = V(g^{-1})$ where $g \in A\langle \frac{f_1}{f_0} \cdots, \frac{f_r}{f_0} \rangle$, the affinoid algebra of V. By using a subtraction argument as in the previous case, we have $n \in \mathbb{N}$ s.t. $f_0^n g$ extends to a function $g' \in A$. Since f_0 does not have a zero on V,

$$V(g) = V \cap \{x \in X \,:\, |g'(x)| \leq |f_0^n(x)|\},$$
$$V(g^{-1}) = V \cap \{x \in X \,:\, |g'(x)| \geq |f_0^n(x)|\}.$$

Using 2.1.10 on $f_0^{-n}|_V$, we have $c \in K*$ s.t. $|f_0^n(x)| \geq |c|$ for every $x \in V$. But

$$V(g) = V \cap X(\frac{g'}{f_0^n}, \frac{c}{f_0^n}), \quad V(g^{-1}) = V \cap X(\frac{f_0^n}{g'}, \frac{c}{g'}),$$

and hence by 2.3.10, $V(g)$ and $V(g^{-1})$ are rational subdomains in X. $\square$

Proposition 2.3.14. *Let $\varphi : Y = Sp(B) \longrightarrow X = Sp(A)$ be a map of affinoid K*-spaces, and let $x \in X$ correspond to the maximal ideal $\mathfrak{m} \subseteq A$.*
(i) Let φ^* induce a surjection $A/\mathfrak{m} \longrightarrow B/\mathfrak{m}B$. Then we have a special affinoid subdomain $X' \hookrightarrow X$ containing x s.t. the induced map $\varphi' : Y' = \varphi^{-1}(X') \longrightarrow X'$ is a closed immersion, i.e. the corresponding homomorphism of affinoid K-algebras is a surjection.
(ii) Let φ^* induce isomorphisms $A/\mathfrak{m}^n \xrightarrow{\sim} B/\mathfrak{m}^n B$ for every $n \in \mathbb{N}$. We then have a special affinoid subdomain $X' \hookrightarrow X$ that contains x s.t. the induced map $Y' = \varphi^{-1}(X') \longrightarrow X'$ is an isomorphism.

Proof. (i) Note that since $A/\mathfrak{m}$ is a field, either $B/\mathfrak{m}B = 0$ or $A/\mathfrak{m} \cong B/\mathfrak{m}B$. Take the first case where $B/\mathfrak{m} = 0$, i.e. $\mathfrak{m}$ is a unit ideal and hence, by 2.3.6, $\varphi^{-1}(x)$ is empty, i.e. there is no element in Y that corresponds to a maximal ideal in B. Then

$$\varphi^*(m_1)b_1 + \ldots + \varphi^*(m_r)b_r = 1, \quad m_1, \ldots, m_r \in \mathfrak{m},\, b_1, \ldots, b_r \in B.$$

We want to find a special affinoid subdomain $X' \subseteq X$ s.t. $\varphi^{-1}(X')$ is empty. Let $c \in K^*$ s.t. $|c|^{-1} > \max_i \{|b_i|_{\sup}\}$.
We claim that $X' = X(c^{-1}m_1, \ldots, c^{-1}m_r)$ suffices. Indeed, given $x \in X(c^{-1}m_1, \ldots, c^{-1}m_r)$, if $y \in \varphi^{-1}(x)$, then

$$|\varphi^*(m_i)(y)b_i(y)| = |m_i(x)||b_i(y)| \leq |c| \max_i \{|b_i|_{\sup}\} < 1,$$

which contradicts $\sum_i b_i\varphi^*(m_i) = 1$.

Now, let $A/\mathfrak{m} \xrightarrow{\sim} B/\mathfrak{m}B$. Choose power bounded elements $b_i \in B$ for $i = 1, \ldots, r$ s.t. we have a surjection

$$\Phi^* : A\langle T_1, \ldots, T_r \rangle \longrightarrow B, \quad T_i \mapsto b_i$$

extending φ^*. Let $\mathfrak{m} = \langle m_1, \dots, m_s \rangle$. Since φ^* gives an isomorphism $A/\mathfrak{m} \xrightarrow{\sim} B/\mathfrak{m}B$, we have $a_i \in A$ and $c_{i,j} \in B$ s.t.

$$b_i - \varphi^*(a_i) = \sum_{j=1}^{s} c_{ij} \varphi^*(m_j).$$

Note that we have $|b_i| \leq 1$. Also, since we can always multiply the last equation with a constant, we can assume $|c_{ij}| \leq 1$. Put $A' = A\langle c^{-1}m_1, \dots, c^{-1}m_s \rangle$ for $c \in K^*$ with $|c| < 1$. We claim that $X' = X(c^{-1}m_1, \dots, c^{-1}m_s) = \mathrm{Sp}\,(A')$ suffices. Indeed, we have

$$\begin{tikzcd}
 & A\langle T_1, \dots, A_r \rangle \arrow[r, "\Phi^*"] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & A'\langle T_1, \dots, T_r \rangle \arrow[r, "\Phi'^*"] & B\langle c^{-1}\varphi^*(m_1), \dots, c^{-1}\varphi^*(m_s) \rangle = B'
\end{tikzcd}$$

where Φ^* and Φ'^* are surjective. We need to show that the composed map in the lower row is surjective. Since Φ^* is surjective, we have for every element $b' \in B'$

$$b' = \sum_{\mu} \varphi^*(a'_\mu)(\varphi(a_1) + \sum_j c_{1j}\varphi^*(m_j))^{\mu_1} \cdots (\varphi(a_r) + \sum_j c_{rj}\varphi^*(m_j))^{\mu_r}$$

with $|c_{i,j}| \leq |b'|$. On applying the argument to $c_{i,j}$, we can see that the resulting series is convergent.

(ii) By (i), φ^* is surjective. Also we get, $\ker(\varphi^*) \subset \mathfrak{a} := \bigcap_{n \geq 1} \mathfrak{m}^n$. Since A is noetherian, $\mathfrak{a}$ is finitely generated over A and we have $\mathfrak{m} \cdot \mathfrak{a} = \mathfrak{a}$. Hence, by Krull's Intersection theorem, $\mathfrak{a}$ is annihilated by $f = 1 - m$ with $m \in \mathfrak{m}$. Since $A \to A\langle f^{-1} \rangle$ factors through $A[f^{-1}]$, kernel of $A \to A\langle f^{-1} \rangle$ contains $\ker(\varphi^*)$, and we have

$$\begin{tikzcd}
A \arrow[r, "\varphi^*"] \arrow[d] & B \arrow[d] \arrow[dl, dotted, "\alpha^*"'] \\
A\langle f^{-1} \rangle \arrow[r, "\varphi'^*"] & B\langle \varphi^*(f)^{-1} \rangle
\end{tikzcd}$$

s.t. the square and the upper triangle is commutative. Since φ^* is surjective, the lower triangle is commutative as well.

Now all we have to prove is that φ^* is an isomorphism. The surjectivity is clear since the surjectivity of φ^* implies that of φ'^*. To show the injectivity of φ'^*, note that $\alpha^*(\varphi^*(f)) = f$ is invertible and power bounded in $A\langle f^{-1} \rangle$.

By the universal property of $B\langle \varphi^*(f)^{-1} \rangle$, we have a homomorphism $\psi^* : B\langle \varphi^*(f)^{-1} \rangle \to A\langle f^{-1} \rangle$ s.t. α^* coincides with the composition $B \longrightarrow B\langle \varphi^*(f)^{-1} \rangle \xrightarrow{\psi^*} A\langle f^{-1} \rangle$.

Hence, $\psi^* \circ \varphi'^*$ is the identity on $A\langle f^{-1} \rangle$, in particular, φ'^* is injective. $\square$

Corollary 2.3.15. *Consider an affinoid K-space morphism $U \longrightarrow X$ s.t. $U \subset X$ is as an affinoid subdomain. Then U is open in X, and the natural topology of X restricts to that of U.*

Chapter 3

Affinoid Functions

3.1 Germs of Affinoid Functions

Consider an affinoid K-space $X = \mathrm{Sp}\,(A)$. For any affinoid subdomain U in X, $\mathcal{O}_X(U)$ is the affinoid algebra that corresponds to U. For example, if $U = X(f)$, then $\mathcal{O}_X(U) = A\langle f\rangle$. Then, if U, V are affinoid subdomains s.t. $U \subset V$, we have a natural map $\mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U)$. This can be interpreted as the restriction of affinoid functions on V to those on U. In fact, $\mathcal{O}_X$ is a presheaf of affinoid K-algebras on the category of affinoid subdomains of X. We will refer this presheaf as the presheaf of affinoid functions on X.

Take $x \in X$, then

$$\mathcal{O}_{X,x} = \varinjlim_{x\in U} \mathcal{O}_X(U)$$

where the limit is over all affinoid subdomains U in X that contain x, is known as the stalk of $\mathcal{O}_X$ at x. The elements of this stalk are known as the germs of affinoid functions at x. We now discuss an explicit characterisation of $\mathcal{O}_{X,x}$. We represent $f_x \in \mathcal{O}_{X,x}$ by $f \in \mathcal{O}_X(U)$ for $U \subset X$, an affinoid subdomain that contains x. Also, $f_i \in \mathcal{O}_X(U_i)$ where $i = 1, 2$ and $x \in U_1 \cap U_2$ represent the same germ $f_x \in \mathcal{O}_{X,x}$ iff there exists an affinoid subdomain U in X s.t. $x \in U \subset U_1 \cap U_2$ and $\rho^{U_1}_U(f_1) = \rho^{U_2}_U(f_2)$. Here, $\rho^W_U : \mathcal{O}_X(W) \longrightarrow \mathcal{O}_X(U)$ is the map $f \mapsto f|_U$ s.t. $\rho^U_U = \mathrm{id}$ and $\rho^V_U = \rho^W_U \circ \rho^V_W$ for subdomains $U \subset W \subset V$.

Proposition 3.1.1. *Consider an affinoid K-space X and a point that corresponds to a maximal ideal $\mathfrak{m} \subset \mathcal{O}_X(X)$ $x \in X$. Then $\mathcal{O}_{X,x}$ is a local ring where $\mathfrak{m}\mathcal{O}_{X,x}$ is the maximal ideal.*

Proof. Consider an affinoid subdomain $U \subseteq X$. Using 2.3.6, there exists an isomorphism

$$\mathcal{O}_X(X)/\mathfrak{m} \xrightarrow{\sim} \mathcal{O}_X(U)/\mathfrak{m}\mathcal{O}_X(U).$$

Using a limit argument, observe that

$$\mathcal{O}_X(X)/\mathfrak{m} \cong \mathcal{O}_{X,x}/\mathfrak{m}\mathcal{O}_{X,x}$$

which shows that $\mathfrak{m}\mathcal{O}_{X,x} \subset$ is maximal.
Now we show that there is no other maximal ideal in $\mathcal{O}_{X,x}$. Take $f_x = \mathcal{O}_{X,x} \setminus \mathfrak{a}$. Then f_x is represented by, say $f \in \mathcal{O}_X(U)$ for some affinoid subdomain $U \subseteq X$. So, $f(x)$ is non-trivial, and upto multiplication with a scalar, let $|f(x)| \geq 1$. Then $U(f^{-1})$ is an affinoid subdomain containing x, and f is invertible in $\mathcal{O}_X(U(f^{-1}))$, hence f_x is invertible in $\mathcal{O}_{X,x}$. $\square$

Proposition 3.1.2. *For $x \in X$, assume $\mathfrak{m}$ to be the maximal ideal that corresponds to x. Then the natural morphism $A = \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ factors as*

$$A \to A_{\mathfrak{m}} \to \mathcal{O}_{X,x}.$$

Here, $A_{\mathfrak{m}}$ is the localization of A at $\mathfrak{m}$ and the former map is canonical sending A into its localization at $\mathfrak{m}$ and the latter one is an injection. It further induces isomorphisms

$$A/\mathfrak{m}^n \xrightarrow{\sim} A_{\mathfrak{m}}/\mathfrak{m}^n A_{\mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}^n \mathcal{O}_{X,x}$$

for all integers $n \geq 1$. In particular, we have isomorphisms between the $\mathfrak{m}$-adic completion of A and the maximal adic completions of $A_{\mathfrak{m}}$ and $\mathcal{O}_{X,x}$:

$$\varprojlim_n A/\mathfrak{m}^n \cong \hat{A}_{\mathfrak{m}} \cong \hat{\mathcal{O}}_{X,x}.$$

Proof. By 6.1.2, $A_{\mathfrak{m}} \to \mathcal{O}_{X,x}$ is an injection, since the composition $A_{\mathfrak{m}} \to \mathcal{O}_{X,x} \to \hat{\mathcal{O}}_{X,x} = \hat{A_{\mathfrak{m}}}$ is injective. The isomorphism $A/\mathfrak{m}^n \cong A_{\mathfrak{m}}/\mathfrak{m}^n A_{\mathfrak{m}}$ can be seen from the exactness of the localization functor and because $A/\mathfrak{m}^n \cong (A/\mathfrak{m}^n)_{\mathfrak{m}}$. By 2.3.6, we have

$$A/\mathfrak{m}^n \cong \mathcal{O}_X(U)/\mathfrak{m}^n \mathcal{O}_X(U).$$

Now by taking the direct limit on U, we get $A/\mathfrak{m}^n \cong \mathcal{O}_{X,x}/\mathfrak{m}^n \mathcal{O}_X(U)$. $\square$

Corollary 3.1.3. *An affinoid function $f \in A = \mathcal{O}_X(X)$ is trivial iff the image of f at $\mathcal{O}_{X,x}$ is trivial for every $x \in X$.*

Proof. The statement follows from

$$A \hookrightarrow \prod_{\mathfrak{m} \in \mathrm{Sp}(A)} A_{\mathfrak{m}} \hookrightarrow \prod_{x \in X} \mathcal{O}_{X,x}.$$

Here, the first injection is true for arbitrary noetherian rings. The second injection follows from $\hat{A}_{\mathfrak{m}} \cong \hat{\mathcal{O}}_{X,x}$ and $A_{\mathfrak{m}} \hookrightarrow \hat{A}_{\mathfrak{m}}$. $\square$

Corollary 3.1.4. *Consider a covering of affinoid subdomains $X = \cup_{i \in I} X_i$. Then the restriction map*

$$\mathcal{O}_X(X) \to \prod_{i \in I} \mathcal{O}_X(X_i)$$

is injective.

Proof. This follows from the previous corollary and from the fact that $\mathcal{O}_X(X) \hookrightarrow \prod_{x\in X} \mathcal{O}_{X,x}$ factors through $\prod_{i\in I} \mathcal{O}_X(X_i)$. $\square$

Proposition 3.1.5. *Consider an affinoid K-space X. Then for any $x \in X$, $\mathcal{O}_{X,x}$ is noetherian.*

Proof. Take $X = \operatorname{Sp} A$ and $\mathfrak{m} \subset A$ to be the maximal ideal that corresponds to $x \in X$. Claim: $\mathcal{O}_{X,x}$ is $\mathfrak{m}$-adically separated, or, $\bigcap_{n\in\mathbb{N}} \mathfrak{m}^n\mathcal{O}_{X,x} = 0$. If $f \in \cap_{n\geq 0} \mathfrak{m}^n\mathcal{O}_{X,x}$, we have an affinoid subdomain $U \subset X$ that contains x s.t. f_x is represented by $f \in \mathcal{O}_X(U)$ and since $\mathcal{O}_X(U)/\mathfrak{m}^n\mathcal{O}_X U \cong \mathcal{O}_{X,x}/\mathfrak{m}^n\mathcal{O}_{X,x}$ by 3.1.2, it follows that $f \in \mathfrak{m}^n\mathcal{O}_X(U)$ for all $n \geq 1$. Writing $U = \operatorname{Sp}(A')$ and $\mathfrak{m}' = \mathfrak{m}A'$, then the image of f in $\mathcal{O}_{X,x}$ lies in $\cap_{n\geq 1}\mathfrak{m}'^n$ which is zero by Krull's intersection theorem. Hence, we are done.

Similarly, it can be shown that for any finitely generated ideal $\mathfrak{a}_x \subset \mathcal{O}_{X,x}$, $\mathcal{O}_{X,x}/\mathfrak{a}_x$ is $\mathfrak{m}$-adically separated. This means that

$$\mathcal{O}_{X,x}/\mathfrak{a}_x \longrightarrow (\widehat{\mathcal{O}_{X,x}/\mathfrak{a}_x}) \cong \widehat{O}_{X,x}/\hat{\mathfrak{a}}_x.$$

Thus $\hat{\mathfrak{a}}_x \cap \mathcal{O}_{X,x} = \mathfrak{a}_x$.

Now, to show that $\mathcal{O}_{X,x}$ is noetherian, to is enough to prove that any ascending chain of finitely generated ideals in $\mathcal{O}_{X,x}$, $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \ldots \subset \mathcal{O}_{X,x}$ is stable. Since $\widehat{O}_{X,x} = \hat{A}_{\mathfrak{m}}$ is noetherian since so is $A_{\mathfrak{m}}$, the chain $\hat{\mathfrak{a_1}} \subset \hat{\mathfrak{a_2}} \subset \ldots$ is stable in $\widehat{O}_{X,x}$. Then we see that the statement is true from $\mathfrak{a}_i = \hat{\mathfrak{a}}_i \cap \mathcal{O}_{X,x}$. $\square$

3.2 Locally Closed Immersions of Affinoid Spaces

Definition 3.2.1. *An affinoid K-space morphism $\varphi : X' \longrightarrow X$ is said to be a closed immersion if the affinoid K-algebra morphism $\varphi^* : \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X'}(X')$ corresponding to φ is a surjection. Also, we call φ as a locally closed immersion (respectively an open immersion) if it is an injection and, for every $x \in X'$, the induced morphism $\varphi^*_{X,\varphi(x)} \longrightarrow \mathcal{O}_{X',x}$ is a surjection (respectively bijection).*

Definition 3.2.2. *An affinoid K-space morphism $\varphi : X' \longrightarrow X$ is called a Runge immersion if it is the composition of a closed immersion $X' \longrightarrow W$ and an open immersion $W \longrightarrow X$ which defines W as a Weierstraß domain in X.*

Theorem 3.2.1. *(Gerritzen-Grauert). Consider a locally closed immersion of affinoid K-spaces $\varphi : X' \longrightarrow X$. We then have a covering $X = \cup_{i=1}^r X_i$ that conatins finitely many rational subdomains $X_i \subset X$ s.t. φ induces Runge immersions $\varphi_i : \varphi^{-1}(X_i) \longrightarrow X_i$ where $i = 1, \ldots, r$.*

3.3 Tate's Acyclicity Theorem

Consider an affinoid K-space X and the category of affinoid subdomains $\mathfrak{T} = \mathfrak{T}_X$ in X, where the inclusions are morphisms.

Definition 3.3.1. *A presheaf $\mathcal{F}$ on $\mathfrak{T}$ is called a sheaf if for all $U \in \mathfrak{T}$ and all coverings $U = \bigcup_{i\in I} U_i$ for $U_i \in \mathfrak{T}$, we have:*

(S_1) If $f \in \mathcal{F}(U)$ s.t. $f|_{U_i} = 0$ for every $i \in I$, then $f = 0$.
(S_2) For $f_i \in \mathcal{F}(U_i)$ s.t. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every $i, j \in I$, we have $f \in \mathcal{F}(U)$ s.t. $f|_{U_i} = f_i$ for every $i \in I$. This f is unique by (S_1).

These conditions can be rephrased by requiring that the sequence

$$0 \longrightarrow \mathcal{O}_X(U) \longrightarrow \prod_{i \in I} \mathcal{O}_X(U_i) \longrightarrow \prod_{i,j \in I} \mathcal{O}_X(U_i \cap U_j) \qquad (\star)$$

where the first morphism is given by $f \mapsto (f|_{U_i})_{i \in I}$ and the second map by $(f_i)_{i \in I} \mapsto (f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j})_{i,j \in I}$, be exact for every $U \in \mathfrak{T}$ and every covering $\mathfrak{U} = (U_i)_{i \in I}$ of U by sets $U_i \in \mathfrak{T}$. Furthermore, for a presheaf $\mathcal{F}$ on X and a covering $\mathfrak{U} = (U_i)_{i \in I}$ of X by affinoid subdomains $U_i \in X$, $\mathcal{F}$ is called as a $\mathfrak{U}$-sheaf, if for every affinoid subdomain U in X, ($\star$) applied to $\mathfrak{U}|_U = (U_i \cap U_j)_{i \in I}$ turns out to be exact.

By 3.1.4, we know that (S_1) holds for $\mathcal{O}_X$. However that is not the case with (S_2) as the natural topology on X is totally disconnected. So, $\mathcal{O}_X$ does not satisfy the conditions of a sheaf. However, Tate showed that the two conditions hold on $\mathcal{O}_X$ for finite coverings $U = \cup_{i \in I} U_i$. In the rest of the section, we will introduce several intermediate results and finally prove the above theorem due to Tate.

Let us take coverings $\mathfrak{U} = (U_i)_{i \in I}$ and $\mathfrak{V} = (V_j)_{j \in J}$ of X. Then $\mathfrak{V}$ is said to be a refinement of $\mathfrak{U}$ if we have a map $\tau : J \longrightarrow I$ s.t. $V_j \subset U_{\tau(j)}$ for every $j \in J$. Now, let's assume that $\mathcal{F}$ is a presheaf of X.

Lemma 3.3.1. *For coverings $\mathfrak{U} = (U_i)_{i \in I}$ and $\mathfrak{V} = (V_j)_{j \in J}$ of X by affinoid subdomains s.t. $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, if $\mathcal{F}$ is a $\mathfrak{V}$-sheaf, it is a $\mathfrak{U}$-sheaf as well.*
Proof. Let's prove that ($\star$) is exact for $\mathfrak{U}$. Then the same for $\mathfrak{U}|_U$ for affinoid subdomain U in X will follow similarly. Let $f_i \in \mathcal{F}(U_i)$, $i \in I$ s.t. $f_i|_{U_i \cap U_{i'}} = f_{i'}|_{U_i \cap U_{i'}}$ for every $i, i' \in I$. Choose $\tau : J \longrightarrow I$ s.t. $V_j \subset U_{\tau(j)}$. Write $g_j = f_{\tau(j)}|_{V_j}$ for every $j \in J$. Then

$$\begin{aligned} g_j|_{V_j \cap V_{j'}} &= (f_{\tau(j)}|_{U_{\tau(j)} \cap U_{\tau(j')}})|_{V_j \cap V_{j'}} \\ &= (f_{\tau(j')}|_{U_{\tau(j)} \cap U_{\tau(j')}})|_{V_j \cap V_{j'}} = g_{j'}|_{V_j \cap V_{j'}}. \end{aligned}$$

As $\mathcal{F}$ is a $\mathfrak{V}$-sheaf, we have a unique $f \in \mathcal{F}(X)$ s.t. $f|_{V_j} = g_j$ for every $j \in J$. Claim: $f|_{U_i} = f_i$ for every $i \in I$. To see this, take $i \in I$ and observe that

$$(f|_{U_i})|_{U_i \cap V_j} = f|_{U_i \cap V_j} = g_j|_{U_i \cap V_j}$$

for $j \in J$. Also,

$$f_i|_{U_i \cap V_j} = f_i|_{U_i \cap U_{\tau(j)} \cap V_j} = f_{\tau(j)}|_{U_i \cap U_{\tau(j)} \cap V_j} = g_j|_{U_i \cap V_j},$$

wherefrom, $f_i|_{U_i \cap V_j} = (f|_{U_i})|_{U_i \cap V_j}$. Since $\mathcal{F}$ is a $\mathfrak{V}$-sheaf when restricted to U_i, observe that $f|_{U_i} = f_i$ for every $i \in I$. Now f is uniquely determined and we are done. □

Lemma 3.3.2. *For coverings $\mathfrak{U} = (U_i)_{i \in I}$ and $\mathfrak{V} = (V_j)_{j \in J}$ of X by affinoid subdomains. Let*
(i) $\mathcal{F}$ be a $\mathfrak{V}$-sheaf
(ii) $\mathcal{F}|_{V_j}$ be a $\mathfrak{U}|_{V_j}$-sheaf for every $j \in J$.
Then $\mathcal{F}$ is a $\mathfrak{U}$-sheaf as well.

Definition 3.3.2. *We define affinoid covering as a finite covering of X by affinoid subdomains. Taking $f_0, \dots, f_r \in A$ with no common zeroes,*

$$U_i = X\Big(\frac{f_0}{f_i}, \cdots, \frac{f_r}{f_i}\Big), \quad i = 0, \dots, n,$$

and hence we obtain a finite covering $\mathfrak{U} = (U_i)_{i=0,\dots,r}$ of X by rational subdomains. $\mathfrak{U}$ is said to be the rational covering associated to $f_0, \dots, f_r$. We can also refer to it as the rational covering.

Lemma 3.3.3. *Every affinoid covering $\mathfrak{U} = (U_i)_{i \in I}$ of X admits a rational covering as a refinement.*

Proof. WLOG let $\mathfrak{U}$ to be consisting of rational subdomains, or $\mathfrak{U} = (U_i)_{i=1,\dots,n}$ where

$$U_i = X\Big(\frac{f_1^{(i)}}{f_0^{(i)}}, \dots, \frac{f_{r_i}^{(i)}}{f_0^{(i)}}\Big).$$

Now, I be the set of tuples $(v_1, \dots, v_n) \in \mathbb{N}^n$ where $0 \leq v_i \leq r_i$ and for such tuples, let

$$f_{v_1 \dots v_n} = \prod_{i=1}^{n} f_{v_i}^{(i)}.$$

Let I' be the set of all $(v_1, \dots, v_n) \in I$ s.t. at least one of the coordinates of the all elements is zero. Then, claim:

$$f_{v_1 \dots v_n}, \quad (v_1 \dots v_n) \in I',$$

have no common solution on X and, so, generate a rational covering $\mathfrak{V}$ on X. To see that there really is no common zero, let $x \in X$ s.t. all functions vanish at x. Then $x \in U_j$ for some index j, and hence, $f_0^{(j)} \neq 0$. So all products

$$\prod_{i \neq j} f_{v_i}^{(i)}, \quad 0 \leq v_i \leq r_i,$$

evaluate to 0 at x. But this is a contradiction as, for every i, $f_0^{(i)}, \dots, f_{r_i}^{(i)}$ generate the unit ideal in $A = \mathcal{O}_X(X)$. $\mathfrak{V}$ is hence well-defined.

Let's now prove that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. Take $(v_1, \dots, v_n) \in I'$ and let

$$X_{v_1,\dots,v_n} = X\Big(\frac{f_{\mu_1,\dots,\mu_n}}{f_{v_1,\dots,v_n}} \,:\, (\mu_1, \dots, \mu_n) \in I'\Big) \in \mathfrak{V}.$$

Claim: $X_{v_1,\dots,v_n} \subset U_n$. Fixing $x \in X_{v_1,\dots,v_n}$ and $\mu s.t. 0 \leq \mu_n \leq r_n$, claim:

$$|f_{\mu_n}^{(n)}(x)| \leq |f_0^{(n)}(x)| = |f_{v_n}^{(n)}(x)|.$$

For some index j, $x \in U_j$.If $j = n$, we are done. So, let $j \neq n$, say $j = 1$. Then $|f_{\mu_1}^{(1)}(x)| \leq |f_0^{(1)}(x)|$ for $0 \leq \mu_1 \leq r_1$ and since $(0, v_2, \dots, v_{n-1}, \mu_n) \in I'$, we have

$$\Big(\prod_{i=1}^{n-1} |f_{v_i}^{(i)}(x)|\Big) \cdot |f_{\mu_n}^{(n)}(x)| \leq |f_0^{(1)}(x)| \cdot \Big(\prod_{i=2}^{n-1} |f_{v_i}^{(i)}(x)|\Big) \cdot |f_{\mu_n}^{(n)}(x)| \leq \prod_{i=1}^{n} |f_{v_i}^{(i)}(x)|.$$

Now, since $\prod_{i=1}^{n} f_{v_i}^{(i)}(x) \neq 0$, we can divide by $\prod_{i=1}^{n-1} f_{v_i}^{(i)}(x)$ and we are done. □

Definition 3.3.3. *Let A be an affinoid K-space and $X = Sp\,A$. Also, let $f_1, \dots, f_r \in A$. Then*

$$X(f_1^{\alpha_1}, \dots, f_r^{\alpha_r}), \quad \alpha_i \in \{+1, -1\},$$

is a finite covering of X by Laurent domains. This is said to be the Laurent covering associated to $f_1, \dots, f_r$. We can also refer to it loosely as the Laurent covering.

Lemma 3.3.4. *Consider a rational covering $\mathfrak{U}$ of X. We then have a Laurent covering $\mathfrak{V}$ of X s.t., for every $V \in \mathfrak{V}$, $\mathfrak{U}|_V$ is a rational covering of V that is generated by units in $\mathcal{O}_X(V)$.*

Proof. Consider $f_0, \dots, f_r \in \mathcal{O}_X(X)$ s.t. they have no common zeroes on X and they generate the rational covering $\mathfrak{U}$. There exists $c \in K^*$ s.t.

$$|c|^{-1} < \inf_{x \in X} \Big(\max_{i=0,\dots,r} |f_i(x)| \Big).$$

This follows since f_i is invertible on $U_i = X\Big(\frac{f_0}{f_i}, \dots, \frac{f_r}{f_i}\Big)$ and as its inverse assumes maximum on U_i. Denote by $\mathfrak{V}$, the Laurent covering of X that is generated by $cf_0, \dots, cf_r$. Claim: $\mathfrak{V}$ is as required. Let

$$V = X\Big((cf_0)^{\alpha_0}, \dots, (cf_r)^{\alpha_r}\Big) \in \mathfrak{V}$$

s.t. $\alpha_0, \dots, \alpha_r \in \{+1, -1\}$. For some $s \geq -1$, we can have $\alpha_0 = \ldots = \alpha_s = +1$ and $\alpha_{s+1} = \ldots = \alpha_r = -1$. Then

$$X\Big(\frac{f_0}{f_i}, \dots, \frac{f_r}{f_i}\Big) \cap V = \varnothing$$

for $i = 0, \dots, s$, as

$$\max_{i=0,\dots,s} |f_i(x)| \leq |c|^{-1} < \max_{i=0,\dots,r} |f_i(x)|$$

where $x \in V$. Particularly,

$$\max_{i=0,\dots,r} |f_i(x)| = \max_{i=s+1,\dots,r} |f_i(x)|$$

for every $x \in V$. Here $\mathfrak{U}|_V$, generated by $f_{s+1}|_V, \dots, f_r|_V$, is a rational covering. Also, these elements are units in $\mathcal{O}_X(V)$ by construction. □

Lemma 3.3.5. *Let $\mathfrak{U} = \langle f_0, \dots, f_r \rangle$ be a rational covering of $X = Sp\,A$. We then have a Laurent covering $\mathfrak{V}$ of X which is a refinement of $\mathfrak{U}$.*

Proposition 3.3.6. *Consider a presheaf $\mathcal{F}$ on the affinoid K-space X. If $\mathcal{F}$ is a $\mathfrak{U}$-sheaf for every Laurent covering $\mathfrak{U}$ of X, then it is a $\mathfrak{V}$-sheaf for every affinoid covering $\mathfrak{V}$ of X.*

Theorem 3.3.7. *(Tate) Consider an affinoid K-space X. The presheaf $\mathcal{O}_X$ of affinoid functions is a $\mathfrak{U}$-sheaf on X for all finite coverings $\mathfrak{U} = (U_i)_{i \in I}$ of X by affinoid subdomains $U_i \subset X$.*

Proof. Using 3.3.6, it suffices if we show the theorem for Laurent coverings. By induction, it suffices to prove only for Laurent convering that is generated by $f \in \mathcal{O}_X(X)$. Then the proof is immediate using the next result. □

Lemma 3.3.8. *For $f \in A = \mathcal{O}_X(X)$, we have an exact sequence:*

$$0 \longrightarrow A \longrightarrow A\langle f \rangle \oplus A\langle f^{-1} \rangle \xrightarrow{\alpha} A\langle f, f^{-1} \rangle \longrightarrow 0$$

where $\alpha(g,h) = g - h$.

Proof. It's clear that $A \hookrightarrow A\langle f\rangle \oplus A\langle f^{-1}\rangle$ is injective and $A\langle f\rangle \oplus A\langle f^{-1}\rangle \longrightarrow A\langle f, f^{-1}\rangle$ is surjective. We now need to show the exactness at $A\langle f\rangle \oplus A\langle f^{-1}\rangle$. Now, since

$$A\langle f\rangle = A\langle \zeta\rangle/(\zeta - f), \ \ A\langle f^{-1}\rangle = A\langle \eta\rangle/(1 - f\eta)$$

which means $A\langle f, f^{-1}\rangle = A\langle \zeta, \eta\rangle/(\zeta - f, 1 - f\eta) = A\langle \zeta, \eta\rangle/(\zeta - f, 1 - \zeta\eta) = A\langle \zeta, \zeta^{-1}\rangle/(\zeta - f)$.
Let $g = \sum_{i=0}^{\infty} a_i\zeta^i$ and $h = \sum_{j=0}^{\infty} b_j\eta^j$ with $a_i, b_j \in A$. Let there exist $\sum_{i=\infty}^{\infty} c_i\eta^i$ with $c_i \in A$ s.t.

$$\sum_{i=0}^{\infty} a_i\eta^{-i} - \sum_{j=0}^{\infty} b_j\eta^j = (a - f\eta)\sum_{i=-\infty}^{\infty} c_i\eta^i.$$

Claim: There exists $d \in A$ s.t. $d \equiv g \bmod (\zeta - f)$ and $d \equiv h \bmod (1 - f\eta)$. It can be assumed that $a_0 = 0$. We have

$$\begin{cases} a_i = c_{-i} - fc_{-i-1}, \ i \geq 1; \\ -b_j = c_j - fc_{j-1}, \ j \geq 0. \end{cases}$$

Hence, we have

$$g = \sum_{i\geq 1}^{\infty}(c_{-i} - fc_{-i-1})\zeta^i = (\zeta - f)\sum_{i\geq 1}^{\infty} c_{-i}\zeta^{i-1} + fc_{-1},$$

and

$$h = \sum_{j\geq 0}^{\infty}(fc_{j-1} - c_j)\eta^j = fc_{-1} - (1 - f\eta)\sum_{j=0}^{\infty} c_j\eta^j.$$

So, we may take $d = fc_{-1}$. $\square$

Corollary 3.3.9. *Let M be a finite module over A and $f \in A$. Take $M\langle f\rangle := M \otimes_A A\langle f\rangle$, and similarly define $M\langle f^{-1}\rangle$ and $M\langle f, f^{-1}\rangle$. We then have an exact sequence*

$$0 \longrightarrow M \longrightarrow M\langle f\rangle \oplus M\langle f^{-1}\rangle \longrightarrow M\langle f, f^{-1}\rangle \longrightarrow 0.$$

Proof. This follows from 3.3.8 and from the fact that $A\langle f, f^{-1}\rangle$ is flat over A. $\square$

Chapter 4

Rigid Spaces

4.1 Grothendieck Topologies

Definition 4.1.1. *A Grothendieck topology $\mathfrak{T}$ comprises of a category Cat$\mathfrak{T}$ and a set Cov$\mathfrak{T}$ of families $(U_i \longrightarrow U)_{i\in I}$ of maps in Cat$\mathfrak{T}$, known as coverings, s.t. we have:*
(i) If $\Phi : U \longrightarrow V$ is an isomorphism in Cat$\mathfrak{T}$, then $(\Phi) \in$ Cov$\mathfrak{T}$.
(ii) If $(U_i \longrightarrow U)_{i\in I}$ and $(V_{ij} \longrightarrow U_i)_{j\in J_i}$ where $i \in I$ are in Cov$\mathfrak{T}$, then so is the composition $(V_{ij} \longrightarrow U_i \longrightarrow U)_{i\in I, j\in J_i}$.
(iii) If $(U_i \longrightarrow U)_{i\in I}$ is in Cov$\mathfrak{T}$ and if $V \longrightarrow U$ is a map in Cat$\mathfrak{T}$, then the fiber products $U_i \times_U V$ are in Cat$\mathfrak{T}$, and $(U_i \times_U V \longrightarrow V)_{i\in I}$ are in Cov$\mathfrak{T}$.

The elements of Cat $\mathfrak{T}$, called as admissible open subsets of X, can be thought of as open sets of the new topology and the maps in Cat $\mathfrak{T}$ as the inclusions of these open sets. A family $(U_i \longrightarrow U)_{i\in I}$ of Cov $\mathfrak{T}$, called the admissible coverings, can be seen as the covering of U by U_i and a fiber product $U_i \times_U V$ as of $U_i \cap V$. Note that we have not talked about the unions of open sets, and even in situations where that makes sense, we would not require the union of open sets to be open. Now, it can be easily seen that a usual topological space X is naturally endowed with a Grothendieck topology . Note that we only would only consider the admissible coverings $(U_i \longrightarrow U)_{i\in I}$ which really are coverings of U by open sets U_i.

The notion of presheaves and sheaves can be naturally generalized to the setup of Grothendieck topology :

Definition 4.1.2. *Consider a Grothendieck topology $\mathfrak{T}$ and a category $\mathfrak{C}$ that admits cartesian products. A presheaf on $\mathfrak{T}$ taking values in $\mathfrak{C}$ is defined as a contravariant functor $\mathcal{F}$: Cat$\mathfrak{T} \longrightarrow \mathfrak{C}$. A presheaf $\mathcal{F}$ is called a sheaf if the sequence*

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

is exact for any covering $(U_i \longrightarrow U)_{i\in I}$ in Cov$\mathfrak{T}$.

We will next define a Grothendieck topology $\mathfrak{T}$ on X and say that X is a G-topological space. We would take X as an affinoid K-space and assume a topology on X w.r.t. which $\mathcal{O}_X$ is in fact a sheaf.

Definition 4.1.3. *Consider an affinoid K-space X and the category $Cat\mathfrak{T}$ of affinoid subdomains in X s.t. the inclusions are morphisms. Also, let $Cov\mathfrak{T}$ be the set of all finite families $(U_i \longrightarrow U)_{i\in I}$ of inclusions of affinoid subdomains of X s.t. $U = \bigcup_{i\in I} U_i$. Then $\mathfrak{T}$ is said to be the weak Grothendieck topology on X.*

Remark 4.1.1. *It's clear from 2.3.5 and 2.3.15 that every admissible open subset of X is open w.r.t. the canonical topology. Also, by 2.3.14 if $\varphi : Z \longrightarrow X$ is an affinoid K-space morphism, $\varphi^{-1}(U)$ for any $U \subset X$ admissible open, is also admissible open in Z. This will referred by saying φ is continuous w.r.t. the Grothendieck topology in question. Also, note by Tate's result that $\mathcal{O}_X$ is actually a sheaf.*

We will now canonically extend the this weak topology and include more admissible open sets and admissible coverings s.t. affinoid K-space morphisms remain continuous and sheaves extend to sheaves in the new topology.

Definition 4.1.4. *Consider an affinoid K-space X. Then the strong Grothendieck topology on X is defined as:*
(i) U in X is called admissible open if we have a covering $U = \bigcup_{i\in I} U_i$ of U by affinoid subdomains U_i in X s.t. all affinoid K-space morphisms $\varphi : Z \longrightarrow X$ for which $\varphi(Z) \subset U$ the covering $(\varphi^{-1}(U_i))_{i\in I}$ of Z admits a refinement that is a finite covering of Z be affinoid subdomains.
(ii) A covering $V = \bigcup_{j\in J} V_j$ of some admissible open subset V in X with admissible open sets V_j is called admissible if for any map of affinoid K-spaces $\varphi : Z \longrightarrow X$ s.t. $\varphi(Z) \subset V$, the covering $(\varphi^{-1}(V_j))_{j\in J}$ of Z admits a refinement that is a finite covering of Z by affinoid subdomains.

Proposition 4.1.1. *Consider an affinoid K-space X. The strong Grothendieck topology is a Grothendieck topology on X if:*
(G_0) $\varnothing$ and X are admissible open.
(G_1) Consider an admissible covering $(U_i)_{i\in I}$ of an admissible open subset U in X. Also, let $V \subset U$ s.t. $V \cap U_i$ is admissible open for every $i \in I$. Then V is admissible open in X.
(G_2) Consider a covering $(U_i)_{i\in I}$ of admissible open $U \subset X$ by admissible open U_i in X s.t. $(U_i)_{i\in I}$ admits an admissible covering of U as refinement. Then $(U_i)_{i\in I}$ is admissible.

Proposition 4.1.2. *Let $\varphi : Y \longrightarrow X$ be an affinoid K-space morphism. Then φ is continuous w.r.t. the strong Grothendieck topologies on X and Y.*

Proof. Take an admissible open U in X and an admissible covering $\mathfrak{U} = (U_i)_{i\in I}$ of U where U_i's are affinoid subdomains of X. Claim: $V = \varphi^{-1}(U)$ is admissible open in Y. Take an affinoid K-space morphism $\tau : Z \longrightarrow Y$ s.t. $\tau(Z) \subset V$. Then $\varphi \circ \tau$ sends Z into U and $(\tau^{-1}\varphi^{-1}(U_i))_{i\in I}$ of Z is refined by a finite affinoid covering. $\square$

In particular, if U is an affinoid subdomain in X, the strong Grothendieck topology on X restricts to that on U.

Proposition 4.1.3. *Consider an affinoid K-space X. If $f \in \mathcal{O}_X(X)$ consider:*

$$U = \{x \in X \,:\, |f(x)| < 1\},$$
$$U' = \{x \in X \,:\, |f(x)| > 1\},$$
$$U'' = \{x \in X \,:\, |f(x)| > 0\}.$$

Then a finite union of such sets is admissible open and a finite covering by finite unions of such sets is admissible.

Lemma 4.1.4. *Consider an affinoid K-algebra A and*

$$f = (f_1, \dots, f_r), \quad g = (g_1, \dots, g_s), \quad h = (h_1, \dots, h_t)$$

be systems of elements in A s.t. at least one of the following conditions is satisfied by every $x \in Sp\,A$:

$$|f_\rho(x)| < 1, \quad |g_\sigma(x)| > 1, \quad |h_\tau(x)| > 0.$$

Then there are $\alpha, \beta, \gamma \in \sqrt{|K^*|}$ where, $\alpha < 1 < \beta$ s.t. at least one of the following conditions is satisfied by every $x \in Sp\,A$

$$|f_\rho(x)| \leq \alpha, \quad |g_\sigma(x)| \geq \beta, \quad |h_\tau(x)| \geq \gamma.$$

Corollary 4.1.5. *Consider an affinoid K-space X. Then the strong Grothendieck topology on X is finer than the Zariski topology.*

Proof. This is clear as any Zariski open subset of X is a finite union of sets of type U''.

Proposition 4.1.6. *Consider a Grothendieck topology $\mathfrak{T}$ on a set X s.t. the three conditions hold. Consider an admissible covering $(X_i)_{i \in I}$ of X. Then:*
(i) U in X is called admissible open iff every $U \cap X_i$ is admissible open for $i \in I$.
(ii) A covering $(U_j)_{j \in J}$ of an admissible open U in X is admissible iff $(X_i \cap U_j)_{j \in J}$ is an admissible covering of $X_i \cap U$ for every $i \in I$.

Proof. (i) follows from (G_1). Also, (ii) follows from (G_2) as we have admissible coverings $(X_i \cap U)_{i \in I}$ and $(X_i \cap U_j)_{i \in I, j \in J}$ of U. □

Proposition 4.1.7. *Consider a set X and $(X_i)_{i \in I}$ one of its coverings. Consider a Grothendieck topology $\mathfrak{T}_i$ on $X_i, i \in I$ s.t. the three conditions are satisfied. For $i \in I, j \in J$, let $X_i \cap X_j$ be $\mathfrak{T}_i$-open (admissible open w.r.t. $\mathfrak{T}_i$) in X_i and $\mathfrak{T}_i$ and $\mathfrak{T}_j$ restrict in the same Grothendieck topology on $X_i \cap X_j$. Then we have a unique Grothendieck topology $\mathfrak{T}$ on X s.t.:*
(i) X_i is $\mathfrak{T}$-open in X and $\mathfrak{T}$ induces $\mathfrak{T}_i$ on X_i.
(ii) $\mathfrak{T}$ follows the three conditions.
(iii) $(X_i)_{i \in I}$ is a $\mathfrak{T}$-covering of X, or, admissible w.r.t. $\mathfrak{T}$.

Proof. By 4.1.6, we have a unique way of defining $\mathfrak{T}$. $U \subset X$ is called $\mathfrak{T}$-open if $X_i \cap U$ is $\mathfrak{T}_i$-open for $i \in I$. Also, call $\mathfrak{U} = (U_j)_{j \in J}$, where $U_j \subset X$ are $\mathfrak{T}$-open, a $\mathfrak{T}$-covering if $\mathfrak{U}|_{X_i} = (X_i \cap U_j)_{j \in J}$ is a $\mathfrak{T}_i$-covering of $X_i \cap U$ for every $i \in I$. Then it can be checked that $\mathfrak{T}$ is a Grothendieck topology . □

4.2 Sheaves

We will assume that X is a G-topological space in this section. We wish to show that sheaves on the weak Grothendieck topology on X can be canonically extended to the strong Grothendieck topology . For a presheaf $\mathcal{F}$ on X and $x \in X$, define

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)$$

as the stalk of $\mathcal{F}$. Consider a map of presheaves $\sigma : \mathcal{F} \longrightarrow \mathcal{F}'$ on X. Then there is a system of maps $\sigma_U : \mathcal{F}(U) \longrightarrow \mathcal{F}'(U)$ for all U s.t. σ_U are compatible with the restriction maps of $\mathcal{F}$ and $\mathcal{F}'$. This induces $\sigma_x : \mathcal{F}_x \longrightarrow \mathcal{F}'_x$ for every $x \in X$.

Definition 4.2.1. *let $\mathcal{F}$ be a presheaf on X. A sheafification of $\mathcal{F}$ is a map $\mathcal{F} \longrightarrow \mathcal{F}'$ for a sheaf $\mathcal{F}'$ s.t. the next universal property holds:*
Every morphism $\mathcal{F} \longrightarrow \mathcal{G}$ for a sheaf $\mathcal{G}$, factors through $\mathcal{F} \longrightarrow \mathcal{F}'$ via a unique morphism $\mathcal{F}' \longrightarrow \mathcal{G}$.

Here, $\mathcal{F}'$ is said to be the sheaf associated to $\mathcal{F}$. We will now show that this sheafification is always possible. Assume that $\mathcal{F}$ is a presheaf of abelian groups. Thus, methods of Čech cohomology are at our disposal. For admissible open U in X, let

$$\check{\mathrm{H}}^q(U, \mathcal{F}) = \varinjlim H^q(\mathfrak{U}, \mathcal{F}), \quad q \in \mathbb{N}.$$

Here, the limit is over every admissible covering $\mathfrak{U}$ of U. Also, partial ordering here is being a subset. This ordering is directed as, for any such coverings $(U_i)_{i\in I}$, $(V_j)_{j\in J}$ admit a common admissible refinement, like, $(U_i \cap V_j)_{i\in I, j\in J}$. On altering U, one obtains the presheaf $\check{\mathrm{H}}^q(X, \mathcal{F})$ that associates the cohomology group $\check{\mathrm{H}}^q(U, \mathcal{F}|_U)$ to the admissible open $U \subset X$. For an admissible covering $(U_i)_{i\in I}$ of admissible open $U \subset X$, there is a natural map $\mathcal{F}(U) \longrightarrow H^0(\mathfrak{U}, \mathcal{F})$. On altering U, we get $\mathcal{F}(U) \longrightarrow \check{\mathrm{H}}^0(U, \mathcal{F})$. This gives us $\mathcal{F} \longrightarrow \check{\mathcal{H}}^0(X, \mathcal{F})$.

Proposition 4.2.1. *Let $\mathcal{F}$ be a presheaf (not necessarily of commutative groups) on X, a G-topological space.*
(i) The presheaf $\mathcal{F}^+ = \check{\mathcal{H}}^0(X, \mathcal{F})$ satisfies (S_1), i.e. the first property of sheafs. This can be reformulated as: the natural morphism $\mathcal{F}^+(U) \longrightarrow \prod_{i\in I} \mathcal{F}^+(U_i)$ is an injection for every admissible covering $(U_i)_{i\in I}$ of admissible open U in X.
(ii) When $\mathcal{F}$satisfies (S_1), then $\mathcal{F}^+$ satisfies (S_1) and (S_2) which means $\mathcal{F}$ is a sheaf.
(iii) $\mathcal{F}^{++} = \check{\mathcal{H}}^0(X, \check{\mathcal{H}}^0(X, \mathcal{F}))$ is a sheaf. Furthermore, $\mathcal{F} \longrightarrow \mathcal{F}^+ \longrightarrow \mathcal{F}^{++}$ is a sheafification of $\mathcal{F}$.

Definition 4.2.2. *The image of a map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ of abelian sheaves is the sheaf corresponding to the presheaf $U \mapsto \sigma_U(\mathcal{F}(U))$ where U is varied as usual. The quotient $\mathcal{F}/\mathcal{F}_0$ of an abelian sheaf $\mathcal{F}$ by subsheaf $\mathcal{F}_0$ is the sheaf corresponding to the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{F}_0(U)$.*

Proposition 4.2.2. *Consider the Grothendieck topologies $\mathfrak{T}$ and $\mathfrak{T}'$ on X s.t.:*
(i) $\mathfrak{T}'$ is finer than $\mathfrak{T}$.
(ii) Every $\mathfrak{T}'$-open U in X admits a $\mathfrak{T}'$-covering $(U_i)_{i\in I}$ s.t. U_i are $\mathfrak{T}$-open in X.
(iii) Every $\mathfrak{T}'$-covering of a $\mathfrak{T}$-open U in X admits a $\mathfrak{T}$-covering as a refinement.
Then every $\mathfrak{T}$-sheaf $\mathcal{F}$on X admits an extension $\mathcal{F}'$ as a $\mathfrak{T}'$-sheaf on X, which is unique up to a natural isomorphism.

Proof. Consider presheaf $\mathcal{F}'$ w.r.t. $\mathfrak{T}'$ on X s.t.

$$U \mapsto \varinjlim_{\mathfrak{U}} H^0(\mathfrak{U}, \mathcal{F})$$

where the limit varies over all $\mathfrak{T}'$-coverings $\mathfrak{U} = (U_i)_{i\in I}$ of U containing $\mathfrak{T}$-open sets U_i. By (ii), $\mathcal{F}'$ is an extension of $\mathcal{F}$. Since $\mathcal{F}$is a sheaf, $\mathcal{F}'$ is one as well. $\mathcal{F}'$ can be interpreted as the sheaf $\check{\mathcal{H}}^0(X_{\mathfrak{T}'}, \mathcal{F})$. □

Corollary 4.2.3. *Consider an affinoid K-space X. Then any sheaf $\mathcal{F}$ on X w.r.t. the weak Grothendieck topology can be uniquely extended w.r.t. the strong one. This holds particularly on $\mathcal{F} = \mathcal{O}_X$, which we have seen in 3.3.7 that it is a sheaf w.r.t. the weak Grothendieck topology .*

This extended sheaf on $\mathcal{O}_X$ is called the sheaf of rigid analytic functions on X. We denote it by $\mathcal{O}_X$.

4.3 Rigid Spaces

A ringed K-space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ a sheaf of K-algebras on it.

Definition 4.3.1. *A G-ringed K-space is a pair $(X, \mathcal{O}_X)$ where X is a G-topological space. $(X, \mathcal{O}_X)$ is called a locally G-ringed K-space if, additionally, all stalks $\mathcal{O}_{X,x} for x \in X$ are local rings.*
A G-ringed K-space morphism $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a pair (φ, φ^*) s.t. $\varphi : X \longrightarrow Y$ is a morphism, continuous w.r.t. the Grothendieck topologies, and where φ^* is a system of K-homomorphisms $\varphi_V^* : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\varphi^{-1}(V))$ with V varying over the admissible open subsets of Y. Also, φ_V^* needs to be compatible with restrictions, i.e. for V' in V, the diagram

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \xrightarrow{\varphi_V^*} & \mathcal{O}_X(\varphi^{-1}(V)) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y(V') & \xrightarrow{\varphi_{V'}^*} & \mathcal{O}_X(\varphi^{-1}(V')) \end{array}$$

commutes.
Also, if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally G-ringed K-spaces, $(\varphi, \varphi^*) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is said to be a map of locally G-ringed K-spaces if the ring homomorphisms

$$\varphi_x^* : \mathcal{O}_{Y,\varphi(x)} \longrightarrow \mathcal{O}_{X,x}, \quad x \in X,$$

induced from the φ_V^* are local, i.e. the maximal ideal of $\mathcal{O}_{Y,\varphi(x)}$ maps into that of $\mathcal{O}_{X,x}$.

Consider a G-topological space X which has the strong G-topology on it. Also, take the corresponding locally G-ringed K-space $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is the structure sheaf on X. By 3.1.1, we know that all stalks of $\mathcal{O}_X$ are local rings. Hence, $(X, \mathcal{O}_X)$ is a locally G-ringed K-space. Claim: every affinoid K-space morphism $\varphi : X \longrightarrow Y$ induces $(\varphi, \varphi^*) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$. By 4.1.2, φ defines a continuous map of G-topological spaces. By 2.3.9, for an affinoid subdomain V in Y, $\varphi^{-1}(V)$ is an affinoid subdomain in X. Hence, φ induces an affinoid K-algebra morphism $\varphi^*_{(V)} : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\varphi^{-1}(V))$. More generally, if $V \subset Y$ is only admissible open, take an admissible affinoid covering $(V_i)_{i\in I}$ of V to get $\varphi^*_V : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\varphi^{-1}(V))$. This can be seen using:

$$\mathcal{O}_Y(V) \longrightarrow \prod_{i\in I} \mathcal{O}_Y(V_i) \rightrightarrows \prod_{i,j\in I} \mathcal{O}_Y(V_i \cap V_j),$$

$$\mathcal{O}_X(\varphi^{-1}(V)) \longrightarrow \prod_{i\in I} \mathcal{O}_Y(\varphi^{-1}(V_i)) \rightrightarrows \prod_{i,j\in I} \mathcal{O}_Y(\varphi^{-1}(V_i) \cap \varphi^{-1}(V_j)),$$

$$\varphi^*_{V_i} : \mathcal{O}_{(V_i)} \longrightarrow \mathcal{O}_X(\varphi^{-1}(V_i)),$$

$$\varphi^*_{V_i\cap V_j} : \mathcal{O}_{(V_i\cap V_j)} \longrightarrow \mathcal{O}_X(\varphi^{-1}(V_i \cap V_j)).$$

Denoting the system of maps φ^*_V by φ^*, we have a morphism of locally G-ringed spaces $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$.

Proposition 4.3.1. *Consider two affinoid K-spaces X and Y. Then the map from affinoid K-space morphisms $X \longrightarrow Y$ to that of locally G-ringed K-spaces $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$, as described abpve, is a bijection.*

Proof. Associate to any morphism $(\varphi, \varphi^*) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$, the affinoid K-space morphism $X \longrightarrow Y$ that corresponds to $\varphi^*_Y : \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$. Then the result follows. $\square$

Definition 4.3.2. *A rigid (analytic) K-space is a locally G-ringed K-space $(X, \mathcal{O}_X)$ s.t.*
(i) the G-topology of X satisfies the three coniditions (G_0), (G_1) and (G_2)
(ii) There is an admissible covering $(X_i)_{i\in I}$ on X where $(X_i, \mathcal{O}_X|_{X_i})$ is an affinoid K-space for all $i \in I$.
A morphism of rigid K-spaces $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is one in the sense of locally G-ringed K-spaces.

For admissible open $U \subset X$, call $(U, \mathcal{O}_X|_U)$ (also denoted as U) as an open subspace of $(X, \mathcal{O}_X)$. Now, we construct global rigid K-spaces by gluing the local ones.

Proposition 4.3.2. *Let us consider the given information:*
(i) rigid K-spaces X_i, $fori \in I$, and
(ii) open subspaces $X_{ij} \subset X_i$ and ismorphisms $\varphi_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$ for $i, j \in I$, and assume:
(a) $\varphi_{ij} \circ \varphi_{ji} = id$, $X_{ii} = X_i$, for every $i, j \in I$.
(b) φ_{ij} induces isomorphisms $\varphi_{ijk} : X_{ij} \cap X_{ik} \xrightarrow{\sim} X_{ji} \cap X_{jk}$ s.t. $\varphi_{ijk} = \varphi_{kji} \circ \varphi_{ikj}$ for every $i, j, k \in I$.
Then X_i's can be glued by identifying X_{ij} with X_{ji} via φ_{ij} to yield a rigid K-space X admitting $(X_i)_{i\in I}$ as an admissible covering. In other words, there is a rigid K-space X together with an admissible covering $(X'_i)_{i\in I}$ and ismorphisms $\psi_i : X_i \xrightarrow{\sim} X'_i$ restricting to isomorphisms $\psi_{ij} : X_{ij} \xrightarrow{\sim} X'_i \cap X'_j$ s.t. the next diagram

$$\begin{tikzcd}
X_{ij} \arrow[r, "\psi_{ij}"] \arrow[d, "\varphi_{ij}"] & X'_i \cap X'_j \arrow[d, equal] \\
X_{ji} \arrow[r, "\psi_{ji}"] & X'_j \cap X'_i
\end{tikzcd}$$

commutes. Moreover, this X is unique up to natural isomorphism.

Proof. For constructing X, we glue the X_i's by identifying φ_{ij}'s. Or, we say $X' = \coprod_{i\in I} X_i$ and for $x, y \in X'$ if $\varphi_{ij} = y$. This is an equivalence relation. Set X as $X/\sim$. Since X can be covered by X_i, by 4.1.7, we have a unique Grothendieck topology on it s.t. $(X_i)_{i\in I}$ is an admissible covering of X. We then construct $\mathcal{O}_X$ by gluing $\mathcal{O}_{X_i}$. If $U \subset U_j, U_j$, then we identify $\mathcal{O}_{X_i}$ and $\mathcal{O}_{X_j}$ using φ_{ij}. $\square$

Proposition 4.3.3. *Consider rigid K-spaces X, Y and an admissible covering $(X_i)_{i\in I}$ of X. Also, consider maps of rigid K-spaces $\varphi_i : X_i \longrightarrow Y$ s.t. $\varphi_i|_{X_i\cap X_j} : X_i \cap X_j \longrightarrow Y$ coincides with $\varphi_j|_{X_i\cap X_j} \longrightarrow Y$ for $i, j \in I$. Then we have a unique map of rigid K-spaces $\varphi : X \longrightarrow Y$ s.t. $\varphi|_{X_i} = \varphi_i$ for every $i \in I$.*

Proof. This can be seen using the fact that $\mathcal{O}_X$ is a sheaf. □

Corollary 4.3.4. *Let X be a rigid K-space and Y an affinoid K-space. Then the natural morphism*

$$Hom(X, Y) \longrightarrow Hom(\mathcal{O}_Y(Y), \mathcal{O}_X(X)), \quad \varphi \mapsto \varphi_Y^*, \tag{4.1}$$

is bijective.

Definition 4.3.3. *A rigid K-space is said to be connected if there are no non-empty admissible open $X_1, X_2 \subset X$ s.t. $X_1 \cap X_2 = \varnothing$ and (X_1, X_2) is an admissible covering of X.*

Note that by Tate's result, an affinoid K-space $\operatorname{Sp} A$ is connected iff A cannot be written as a non-trivial cartesian product of two K-algebras.

We now give the definition for the connected components of a rigid K-space. For $x, y \in X$, we say $x \sim y$ is there exists connected admissible open $U_0, \ldots, U_n \subset X$ s.t. $s \in U_0, y \in U_n$ and $U_{i-1} \cap U_i \neq \varnothing$ for $i = 1, \ldots, n$.

Proposition 4.3.5. *Consider a rigid K-space X and the relation "$\sim$" on it.*
(i) "$\sim$" is an equivalence.
(ii) For $x \in X$, the equivalence class $Z(x)$ is admissible open in X, and is called the connected component of X containing x.
(iii) The connected components of X form an admissible covering of X.

4.4 The GAGA-Functor

The aim of this section is to associate a rigid K-space Z^{rig} to any K-scheme Z of locally finite type. This is knows as the rigid analytification of Z. We first construct the rigid version of the affine n-space $\mathbb{A}_K^n$. For $r > 0$, let $T_n(r)$ denote the K-algebra of power series $\sum_v a_v \zeta^n$ for $\zeta = (\zeta_1, \ldots, \zeta_n)$ s.t. $\lim_v a_v r^{|v|} = 0$. So, $T_n(r)$ contain power series that converge on a closed n-dimensional ball with radius r. For $c \in K, |c| > 1$, identify $T_n^{(i)} = T_n(|c|^i)$ with Tate algebra $K\langle c^{-i}\zeta_1, \ldots, c^{-i}\zeta_n\rangle$. Now,

$$T_n = T_n^{(0)} \longleftrightarrow T_n^{(1)} \longleftrightarrow T_n^{(2)} \longleftrightarrow \ldots \longleftrightarrow K[\zeta]$$

induces inclusions

$$\mathbb{B}^n = \operatorname{Sp} T_n^{(0)} \hookrightarrow \operatorname{Sp} T_n^{(1)} \hookrightarrow \operatorname{Sp} T_n^{(2)} \hookrightarrow \ldots$$

where $\operatorname{Sp} T_n^{(i)}$ is the n-dimensional ball of radius $|c^i|$. By gluing, the union of these balls can be constructed. The rigid K-space obtained, denoted by $\mathbb{A}_K^{n,\mathrm{rig}}$ has the admissible covering $\mathbb{A}_K^{n,\mathrm{rig}} = \bigcup_{i=0}^{\infty} \operatorname{Sp} T_n^{(i)}$. It is called the rigid analytification of the affine n-space $\mathbb{A}_K^n$.

Lemma 4.4.1. *The inclusions*

$$T_n^{(0)} \supset T_n^{(1)} \supset T_n^{(2)} \supset \ldots \supset K[\zeta]$$

induce inclusions

$$Max\, T_n^{(0)} \subset Max\, T_n^{(1)} \subset Max\, T_n^{(2)} \subset \ldots \subset Max\, K[\zeta]$$

s.t. $Max\, K[\zeta] = \bigcup_{i=0}^{\infty} Max\, T_n^{(i)}$.

Proof. Since $\operatorname{Sp} T_n^{(i)} \hookrightarrow \operatorname{Sp} T_n^{(i+1)}$, the inclusions in the lemma are clear.

Claim: (i) Consider maximal $\mathfrak{m} \subset K\langle\zeta\rangle$. Then $\mathfrak{m}' = \mathfrak{m} \cap K[\zeta]$ is maximal s.t. $\mathfrak{m} = \mathfrak{m}' K\langle\zeta\rangle$.

(ii) For maximal $\mathfrak{m}' \subset K[\zeta]$, we have $i_0 \in \mathbb{N}$ s.t. $\mathfrak{m}' K\langle c^{-i}\zeta\rangle$ is maximal in $K\langle c^{-i}\zeta\rangle = T_n^{(i)}$ for every $i \geq i_0$.

To prove (i), consider the commutative diagram

$$\begin{array}{ccc} K[\zeta] & \longrightarrow & K\langle\zeta\rangle \\ \downarrow & & \downarrow \\ K[\zeta]/\mathfrak{m}' & \longrightarrow & K\langle\zeta\rangle/\mathfrak{m} \end{array}$$

where the horizontal morphisms are injective. Since $K\langle\zeta\rangle/\mathfrak{m}$ is a finite field over K, it must be true for $K[\zeta]/\mathfrak{m}'$ as well. So, $\mathfrak{m}'$ is maximal in $K[\zeta]$. Now consider the commutative diagram:

$$\begin{array}{ccc} K[\zeta]/\mathfrak{m}' & \longrightarrow & K\langle\zeta\rangle/\mathfrak{m}'\langle\zeta\rangle \\ \| & & \downarrow \\ K[\zeta]/\mathfrak{m}' & \longrightarrow & K\langle\zeta\rangle/\mathfrak{m} \end{array}$$

The horizontal maps are surjective since $K[\zeta]$ is dense in $K\langle\zeta\rangle$ and since finite-dimensional vector spaces are complete, and hence closed. The lower one is actually bijective by the definition of $\mathfrak{m}'$. This implies that the upper map is also bijective. This implies (i) as we now have that right vertical map is bijective.

For (ii), consider maximal $\mathfrak{m}' \subset K[\zeta]$. Then $K[\zeta]/\mathfrak{m}'$ is finite over K and hence the absolute value is well-defined. For some $i_0 \in \mathbb{N}$ s.t. $\overline{\zeta}_j \in K[\zeta]/\mathfrak{m}'$ have their absolute values satisfy $|\overline{\zeta}_j| \leq |c|^{i_0}$. Hence, $K[\zeta] \longrightarrow K[\zeta]/\mathfrak{m}'$ factors, for $i \geq i_0$, through $T_n^{(i)} = K\langle c^{-i}\zeta\rangle$ via a unique K-morphism $T_n^{(i)} \longrightarrow K[\zeta]/\mathfrak{m}'$ s.t. $\zeta_j \mapsto \overline{\zeta}_j$. The kernel, deonted by $\mathfrak{m}$, of this latter map is maximal in $T_n^{(i)}$ s.t. $\mathfrak{m} \cap K[\zeta] = \mathfrak{m}'$.

(i) and (ii) imply that $\operatorname{Max} K[\zeta]$ is the union of $\operatorname{Max} T_n^{(i)}$. □

Let's consider the $\operatorname{Sp} K[\zeta]/\mathfrak{a}$, an affine K-scheme of finite type. Here, $\mathfrak{a} \subset K[\zeta]$ is an ideal. We wish to construct its rigid analytification. We have

$$T_n^{(0)}/(\mathfrak{a}) \longleftarrow T_n^{(1)}/(\mathfrak{a}) \longleftarrow T_n^{(2)}/(\mathfrak{a}) \longleftarrow \ldots \longleftarrow K[\zeta]/\mathfrak{a}$$

and the associated sequence of inclusions

$$\operatorname{Max} T_n^{(0)}/(\mathfrak{a}) \hookrightarrow \operatorname{Max} T_n^{(1)}/(\mathfrak{a}) \hookrightarrow \operatorname{Max} T_n^{(2)}/(\mathfrak{a}) \hookrightarrow \ldots \hookrightarrow \operatorname{Max} K[\zeta]/\mathfrak{a}.$$

By the previous lemms, $\operatorname{Max} K[\zeta]/\mathfrak{a} = \bigcup_{i=0}^{\infty} \operatorname{Max} T_n^{(i)}/(\mathfrak{a})$. This union can be made as a rigid K-space by 4.3.2, and is called as the rigid analytification of $\operatorname{Spec} K[\zeta]/\mathfrak{a}$.

Lemma 4.4.2. *Consider an affine K-scheme of finite type Z and a rigid K-space Y. Then the set of maps of locally G-ringed K-spaces $(Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$ corresponds in a bijection to the set of K-homomorphisms $\mathcal{O}_Z(Z) \longrightarrow \mathcal{O}_Y(Y)$.*

Proof. Using 4.3.4 here, first, let's assume that Y is affinoid. Let $B = \mathcal{O}_Y(Y)$ and $C = \mathcal{O}_Z(Z)$ and consider

$\sigma : C \longrightarrow B$, a K-morphism. Taking pre-images of maximal ideals, we get $\operatorname{Max} B \hookrightarrow \operatorname{Max} C \hookrightarrow \operatorname{Spec} C$ and hence, $\varphi : Y \longrightarrow Z$ that is continuous w.r.t. Grothendieck topologies. If $f \in C$ and $\varepsilon \in K^*$, we have

$$\begin{tikzcd} C \arrow[r, "\sigma"] \arrow[d] & B \arrow[d] \\ C[f^{-1}] \arrow[r] & B\langle \varepsilon \cdot \sigma(f)^{-1} \rangle \end{tikzcd}$$

where the bottom map is unique since $\sigma(f)$ is invertible in the range. Varying ε, we get

$$\begin{tikzcd} \mathcal{O}_Z(Z) \arrow[r, "\sigma"] \arrow[d] & \mathcal{O}_Y(Y) \arrow[d] \\ \mathcal{O}_Z(Z_f) \arrow[r] & \mathcal{O}_Y(\varphi^{-1}(Z_f)) \end{tikzcd}$$

where the lower map is unique again. $Z_f \subset Z$ s.t. f is non-trivial. Using the globalization argument with this, we get a map of ringed K-spaces $(\varphi, \varphi^*) : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$ s.t. $\varphi^*_Z = \sigma$. For injectivity and the generalization, the argument is readily seen. $\square$

Now, we want to prove that the rigid analytifications do not depend on the constant $c \in K$.

Definition 4.4.1. *Consider a K-scheme $(Z, \mathcal{O}_Z)$ of locally finite type. A rigid analytification of $(Z, \mathcal{O}_Z)$ is a rigid K-space $(Z^{rig}, \mathcal{O}_{Z^{rig}})$ along with a map of locally G-ringed K-spaces $(i, i^*) : (Z^{rig}, \mathcal{O}_{Z^{rig}}) \longrightarrow (Z, \mathcal{O}_Z)$ s.t. the following universal property holds:*
For a rigid K-space $(Y, \mathcal{O}_Y)$ and a map of locally G-ringed K-spaces $(Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$, the latter factors through a unique map of rigid K-spaces $(Y, \mathcal{O}_Y) \longrightarrow (Z^{rig}, \mathcal{O}_{Z^{rig}})$.

Proposition 4.4.3. *Z^{rig} give rise to analytifications in the sense of the previous definition.*
Proof. Consider an affine K-scheme of fininte type $Z = \operatorname{Spec} K[\zeta]/\mathfrak{a}$ and the corresponding rigid K-space obtained by $Z^{\mathrm{rig}} = \cup_{i=0}^{\infty} \operatorname{Sp} T_n^{(i)}/(\mathfrak{a})$. We have natural morphisms $K[\zeta]/\mathfrak{a} \longrightarrow T_n^{(i)}/(\mathfrak{a})$ which constitute $\mathcal{O}_Z(Z) \longrightarrow \mathcal{O}_{(Z^{\mathrm{rig}})}$. Using the previous lemma, we get a map of locally G-ringed K-spaces

$$(i, i^*) : (Z^{\mathrm{rig}}, \mathcal{O}_{Z^{\mathrm{rig}}}) \longrightarrow (Z, \mathcal{O}_Z).$$

To see that this (i, i^*) satisfies the universal property, look at a map of locally G-ringed K-spaces $(Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$.
By the previous lemma, this morphism corresponds to a K-morphism $\sigma : K[\zeta]/\mathfrak{a} \longrightarrow B$ s.t. $B = \mathcal{O}_Y(Y)$. Claim: for any $i \in \mathbb{N}$ sufficiently large, we have $K[\zeta]/\mathfrak{a} \longrightarrow T_n^{(i)}/\mathfrak{a} \longrightarrow B$. Choose $i \in \mathbb{N}$ s.t. $\overline{\zeta}_j \in K[\zeta]/\mathfrak{a}$ satisfies $|\sigma(\overline{\zeta}_j)|_{\sup} \leq |c|^i$ in B. Then $K[\zeta] \longrightarrow B$ extends uniquely to $T_n^{(i)}$ and the proof follows. $\square$

Proposition 4.4.4. *Any K-scheme Z of locally finite type admits an analytification $Z^{rig} \longrightarrow Z$. Also, the underlying map of sets identifies the points of Z^{rig} with the closed points of Z.*
Proof. This holds when Z is affine. In generality, choose a covering of Z by affine open subschemes $Z_i, i \in J$. These elements admit analytifications, say, i_i. Then, $i_i^{-1}(Z_i \cap Z_j) \longrightarrow Z_i \cap Z_j$ is an analytification of $Z_i \cap Z_j$. We can now glue to get Z^{rig} and a morphism $Z^{\mathrm{rig}} \longrightarrow Z$. Also, the last assertion follows since the same is true for the affine open parts of Z. $\square$

Corollary 4.4.5. *Rigid analytifications defines a functor from the category of K-schemes of locally finite type to the category of rigid K-spaces, called the GAGA-functor.*

Chapter 5

Coherent Sheaves on Rigid Spaces

5.1 Coherent Modules

For $X = \operatorname{Sp} A$ and a module M over A, consider the functor $\mathcal{F}$ from the affinoid subdomains in X to abelian groups that associates $M \otimes_A A'$ to any affinoid subdomain $\operatorname{Sp} A'$ in X. Then $\mathcal{F}$ is a presheaf on X w.r.t. the weak G-topology. By 3.3.7, it is seen as a sheaf. By 4.2.2, $\mathcal{F}$ is, in fact, a sheaf w.r.t. the strong G-topology. We say that $\mathcal{F}$ is an $\mathcal{O}_X$-module, or, for any admissible open U in X, the abelian group $\mathcal{F}(U)$ has an $\mathcal{O}_X(U)$-module structure, s.t. these structures are compatible with restriction maps. Then, $\mathcal{F}$ is called the module over $\mathcal{O}_X$ that is associated to M, the module over A, and $\mathcal{F} = M \otimes_A \mathcal{O}_X$. Also,

$$\mathcal{F}|_{X'} = (M \otimes_A A') \otimes_{A'} \mathcal{O}_{X|X'}$$

for the restriction on any affinoid subdomain $X' = \operatorname{Sp} A'$ in X.

Proposition 5.1.1. *Take an affinoid K-space $X = Sp\, A$. Then:*
(i) The functor

$$* \cdot \otimes_A \mathcal{O}_X : \;\; M \mapsto M \otimes_A \mathcal{O}_X$$

from modules over A to modules over $\mathcal{O}_X$ is fully faithful.
(ii) It commutes with images, tensor products, kernels, and cokernels.
(iii) $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *of modules over A is exact iff the corresponding sequence*

$$0 \longrightarrow M' \otimes_A \mathcal{O}_X \longrightarrow M \otimes_A \mathcal{O}_X \longrightarrow M'' \otimes_A \mathcal{O}_X \longrightarrow 0.$$

of $\mathcal{O}_X$-modules is exact.

Proof. As an maps over $\mathcal{O}_X$, $M \otimes_A \mathcal{O}_X \longrightarrow M' \otimes_A \mathcal{O}_X$ can be determined uniquely by this A-map

$$M = M \otimes_A \mathcal{O}_X(X) \longrightarrow M' \otimes_A \mathcal{O}_X(X) = M',$$

the natural morphism

$$\mathrm{Hom}_A(M, M') \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(M \otimes_A \mathcal{O}_X, M' \otimes_A \mathcal{O}_X)$$

is a bijection.
Hence, the functor $\cdot \otimes \mathcal{O}_X$ is fully faithful and (i) is done. Also, by definition, it commutes with tensor products.
Now, if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence of modules over A, the induced sequence

$$0 \longrightarrow M' \otimes_A A' \longrightarrow M \otimes_A A' \longrightarrow M'' \otimes_A A' \longrightarrow 0$$

is exact for any affinoid subdomain $\mathrm{Sp}\, A'$ in X as the associated morphism from A to A' is flat. Hence, it takes short exact sequences to the short exact ones. Now, since a module M over A is trivial iff $M \otimes_A \mathcal{O}_X$ is trivial, the rest of the parts follow. □

Definition 5.1.1. *Consider a rigid K-space X and a module over $\mathcal{O}_X$ denoted by $\mathcal{F}$. Then:*
(i) $\mathcal{F}$ is said to be of finite type if there is some admissible covering $(X_i)_{i \in I}$ of X along with exact sequences

$$\mathcal{O}_X^{s_i}|_{X_i} \longrightarrow \mathcal{F}|_{X_i} \longrightarrow 0, \quad i \in I.$$

(ii) $\mathcal{F}$ is said to be of finite presentation if we have some admissible covering $(X_i)_{i \in I}$ of X along with exact sequences

$$\mathcal{O}_X^{r_i}|_{X_i} \longrightarrow \mathcal{O}_X^{s_i}|_{X_i} \longrightarrow \mathcal{F}|_{X_i} \longrightarrow 0, \quad i \in I.$$

(iii) $\mathcal{F}$ is said to be coherent if $\mathcal{F}$ is of finite type and if for every admissible open subspace U in X, the kernel of $\mathcal{O}_X^s|_U \longrightarrow \mathcal{F}|_U$ is of finite type.

Remark 5.1.1. *If we have a module $\mathcal{F}$ over $\mathcal{O}_X$ on a rigid K-space X, the it is coherent iff we have an admissible affinoid covering $\mathfrak{U} = (X_i)_{i \in I}$ of X s.t. $\mathcal{F}|_{X_i}$ corresponds to a finite module over $\mathcal{O}_{X_i}(X_i)$ for every $i \in I$. $\mathcal{F}$ is said to be $\mathfrak{U}$-coherent then.*

Theorem 5.1.2. *(Kiehl). Consider an affinoid K-space $X = Sp\, A$ and a module $\mathcal{F}$ over $\mathcal{O}_X$. Then $\mathcal{F}$ is coherent iff $\mathcal{F}$ corresponds to some finite module over A.*

Corollary 5.1.3. *Consider a rigid K-space X and a module $\mathcal{F}$ over $\mathcal{O}_X$ on SX. Then TFAE:*
(i) $\mathcal{F}$ is coherent, or, $\mathcal{F}$ is $\mathfrak{U}$-coherent for an admissible affinoid covering $\mathfrak{U}$ of X.
(ii) $\mathcal{F}$ is $\mathfrak{U}$-coherent for every admissible affinoid coverings $\mathfrak{U}$ of X.
Proof. Let's prove that (i) gives (ii). Let $\mathcal{F}$ be coherent. Also, let X be affinoid, say $X = \mathrm{Sp}\, A$. By the previous theorem, $\mathcal{F}$ corresponds to a finite module over A and we are done. The other way is readily seen. □

For proving 5.1.2, we need the next the couple of lemmas. We will leave them only with the statements here.

Lemma 5.1.4. *If $\mathcal{F}$ is $\mathfrak{U}$-coherent, $H^1(\mathfrak{U}, \mathcal{F}) = 0$.*

Lemma 5.1.5. *Let $H^1(\mathfrak{U}, \mathcal{F}) = 0$ for every $\mathfrak{U}$-coherent $\mathcal{O}_X$-modules $\mathcal{F}$. Then this kind of module corresponds to some module over A which is finite.*

5.2 Grothendieck Cohomology

We talk about modules over $\mathcal{O}_X$ on rigid K-spaces X. Their cohomology is formulated via derived functors. We consider the section functor

$$\Gamma(Z,\cdot) : \mathcal{F} \mapsto \Gamma(X,\mathcal{F}) = \mathcal{F}(X)$$

that for an $\mathcal{O}_X$-module $\mathcal{F}$, associates $\mathcal{F}(X)$ to it, i.e. the group of its global sections, and, for a rigid K-space map $\varphi : X \longrightarrow Y$, the direct image functor

$$\varphi_* : \mathcal{F} \mapsto \varphi_*\mathcal{F}$$

that for an $\mathcal{O}_X$-module $\mathcal{F}$, associates its direct image $\varphi_*\mathcal{F}$. Note that these functors are left exact.

Consider the category of modules over $\mathcal{O}_X$ as $\mathfrak{C}$.

Definition 5.2.1. *$\mathcal{F} \in \mathfrak{C}$ is called an injection if the functor $Hom(\cdot,\mathcal{F})$ is exact, or, given*

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

in $\mathfrak{C}$,

$$0 \longrightarrow Hom(\mathcal{E}'',\mathcal{F}) \longrightarrow Hom(\mathcal{E},\mathcal{F}) \longrightarrow Hom(\mathcal{E}',\mathcal{F}) \longrightarrow 0$$

is exact for all such short exact sequences.

We now give the next proposition without a proof.

Proposition 5.2.1. *The category $\mathfrak{C}$ of modules over $\mathcal{O}_X$ on a rigid K-space X contains enough injectives. In other words, if $\mathcal{F}$ is an onject in the category, we have an injective morphism $\mathcal{F} \hookrightarrow \mathcal{I}$ for some injective $\mathcal{I} \in \mathfrak{C}$.*

Corollary 5.2.2. *Every $\mathcal{F}$ in $\mathfrak{C}$ admits some injective resolution, or, we have an exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots$$

where $\mathcal{I}^i$, $i = 0, 1, \dots$ are injective objects.

Proof. Consider $\mathcal{F} \hookrightarrow \mathcal{I}^0$ where $\mathcal{I}^0$ is an injective. Also, take embedding $\mathcal{I}^0/\mathcal{F} \hookrightarrow \mathcal{I}^1$ into an injective object $\mathcal{I}^1$, then we have embedding $\mathcal{I}^1/\operatorname{im}\mathcal{I}^0 \hookrightarrow \mathcal{I}^2$ into some injective $\mathcal{I}^2$, and similarly. □

We define the right derived functors of $\Gamma = \Gamma(X,\cdot)$, the section functor, and of φ_*, the direct image functor. We choose an injectivce resolution

$$0 \longrightarrow \mathcal{I}^0 \xrightarrow{\alpha^0} \mathcal{I}^1 \xrightarrow{\alpha^1} \mathcal{I}^2 \xrightarrow{\alpha^2} \dots$$

of $\mathcal{O}_X$-module $\mathcal{F}$. We now get a complex of abelian groups on applying the nfunctor Γ:

$$0 \longrightarrow \Gamma(X,\mathcal{I}^0) \xrightarrow{\Gamma(\alpha^0)} \Gamma(X,\mathcal{I}^1) \xrightarrow{\Gamma(\alpha^1)} \Gamma(X,\mathcal{I}^2) \xrightarrow{\Gamma(\alpha^2)} \dots.$$

The cohomology of the complex gives

$$R^q\,\Gamma(X,\mathcal{F}) = H^q(X,\mathcal{F}) = \ker\Gamma(\alpha^q)/\operatorname{im}\Gamma(\alpha^{q-1}).$$

Here, $R^q\,\Gamma(X,\mathcal{F})$ is called the qth cohomology group of X with values in $\mathcal{F}$. It can be shown using homotopies that these cohomologies are not dependent on the particular resolution. Also, $R^q\,\Gamma(X,\cdot) = H^q(X,\cdot)$ is a functor on $\mathfrak{C}$, called as the qth right-derived functor of $\Gamma(X,\cdot)$, the section functor. As the section functor is left-exact, we have $R^0\,\Gamma(X,\cdot) = \Gamma(X,\cdot)$. Also, in case of $\mathcal{F} = \mathcal{O}_X$, $H^q(X,\mathcal{F})$ can be seen as some invariants of the rigid K-space X.

In the same way, we see φ_*. Using φ_* on the given resolution of $\mathcal{F}$, we have

$$0 \longrightarrow \varphi_*\mathcal{I}^0 \xrightarrow{\varphi_*\alpha^0} \varphi_*\mathcal{I}^1 \xrightarrow{\varphi_*\alpha^1} \varphi_*\mathcal{I}^2 \xrightarrow{\varphi_*\alpha^2} \dots$$

and

$$R^q\,\varphi_*(\mathcal{F}) = \ker\varphi_*\alpha^q/\operatorname{im}\varphi_*\alpha^{q-1}$$

is a module over $\mathcal{O}_Y$, called as the qth direct image of $\mathcal{F}$. We have $R^0\,\varphi_*(\mathcal{F}) = \varphi_*(\mathcal{F})$ and $R^q\,\varphi_*(\mathcal{F})$ is the sheaf that corresponds to the presheaf

$$Y \supset V \mapsto H^q(\varphi^{-1}(V), \mathcal{F}|_{\varphi^{-1}(V)}).$$

Theorem 5.2.3. *Consider an exact sequence*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \overset{\beta}{F} \mathcal{F}'' \longrightarrow 0$$

of objects in $\mathfrak{C}$. We then have a corresponding long exact sequence:

$$\begin{aligned}
0 \longrightarrow\ & \Phi(\mathcal{F}') \xrightarrow{\Phi(\alpha)} \Phi(\mathcal{F}) \xrightarrow{\Phi(\beta)} \Phi(\mathcal{F}'') \\
\xrightarrow{\partial}\ & R^1\,\Phi(\mathcal{F}') \xrightarrow{R^1\,\Phi(\alpha)} R^1\,\Phi(\mathcal{F}) \xrightarrow{R^1\,\Phi(\beta)} R^1\,\Phi(\mathcal{F}'') \\
\xrightarrow{\partial}\ & R^2\,\Phi(\mathcal{F}') \xrightarrow{R^2\,\Phi(\alpha)} R^2\,\Phi(\mathcal{F}) \xrightarrow{R^2\,\Phi(\beta)} R^2\,\Phi(\mathcal{F}'') \\
\xrightarrow{\partial}\ & \dots
\end{aligned}$$

When $\mathcal{F}$ is a module over $\mathcal{O}_X$, introduce the Čech cohomology groups $H^q(\mathfrak{U},\mathcal{F})$ for admissible covering $\mathfrak{U}$ of X. Then $\check{\mathrm{H}}^q(X,\mathcal{F})$ is said to be the qth Čech cohomology group of X taking values in $\mathcal{F}$. We then have the following theorems:

Theorem 5.2.4. *Consider an admissible covering $\mathfrak{U}$ of a rigid K-space and a module $\mathcal{F}$ over $\mathcal{O}_X$. Let $H^q(U,\mathcal{F}) = 0$ where $q > 0$ and take a finite intersection of sets in $\mathfrak{U}$ called as U. Then the canonical map*

$$H^q(\mathfrak{U},\mathcal{F}) \longrightarrow H^q(X,\mathcal{F})$$

is bijective for all $q \geq 0$.

Theorem 5.2.5. *Consider an affinoid K-space X. We then get*

$$H^q(X, \mathcal{O}_X) = 0 \text{ where } q > 0.$$

This is the case for any $\mathcal{O}_X$-module $\mathcal{F}$ instead of $\mathcal{O}_X$ that corresponds to a module over $\mathcal{O}_X(X)$.

Chapter 6

Adic Rings and Formal Schemes

6.1 Adic Rings

Definition 6.1.1. *A ring R which has been endowed with a topology is called a topological ring if the addition and multiplication maps are continuous from $R \times R$ to R.*

Consider a commutative ring R which contains identity and take an ideal $\mathfrak{a} \subset R$. Then we have a unique topology on R s.t. it is a topological ring and the ideals $\mathfrak{a}^n$, $n \in \mathbb{N}$ form a basis of nbhds of 0 in R. $U \subset R$ is said to be open if for each $x \in U$, we have $n \in \mathbb{N}$ s.t. $x + \mathfrak{a}^n \subset U$. This is called the $\mathfrak{a}$-adic topology on R. All ideals $\mathfrak{a}^n$ are open as well as closed in R. A topological ring R is called an adic ring if its topology coincides with this for an ideal $\mathfrak{a} \subset R$. Finally, this ideal $\mathfrak{a}$ is called the ideal of definition.

Similarly, an R-module M where R is a topological ring and we have a topology on M is called a topological module over R if the addition and the multiplication maps are continuous. Also, for a module M over R and ideal $\mathfrak{a}$ in R, the $\mathfrak{a}$-adic topology on M is defined as: give the $\mathfrak{a}$-adic topology on R and the unique topology on M so that it is a topological R-module, s.t. $\mathfrak{a}^n M$ for $n \in \mathbb{N}$ form a basis of nbhds. These submodules are also open and closed in M.

Proposition 6.1.1. *Let R be a ring and M a module over R along with $\mathfrak{a}$-adic topologies for $\mathfrak{a} \subset R$.*
(i) R is separated (i.e. Hausdorff) iff $\cap_{n=0}^{\infty} \mathfrak{a}^n = 0$.
(ii) M is separated iff $\cap_{n=0}^{\infty} \mathfrak{a} M = 0$.

Proof. $\cap_{n=0}^{\infty} \mathfrak{a}^n = 0$ iff for each $x \in R - \{0\}$, we have $n \in \mathbb{N}$ s.t. $x \notin \mathfrak{a}^n$. Since $\mathfrak{a}^n$ is both open and closed in R, we have (i). (ii) is shown similarly. □

We have the next theorem that follows from standard commutative algebra.

Theorem 6.1.2. *(Krull's Intersection Theorem). Consider a Noetherian ring R and an ideal $\mathfrak{a}$ in R. Also,* *let M be a finite module over R. We have:*

$$\bigcap_{n=0}^{\infty} \mathfrak{a}^m M = \{x \in M \,:\, \exists\, r \in 1 + \mathfrak{a} \ s.t.\ rx = 0\}$$

Corollary 6.1.3. *Consider a local Noetherian ring R and a maximal ideal $\mathfrak{m}$. Then R is $\mathfrak{m}$-adically separated. This also holds for finitely generated R-modules M.*

Lemma 6.1.4. *(Artin-Rees). Consider a Noetherian ring R, an ideal $\mathfrak{a} \subset R$, a finite R-module M, and an R-submodule M' in M. We then have $n_0 \in \mathbb{N}$ s.t.*

$$(\mathfrak{a}^n M) \cap M' = \mathfrak{a}^{n-n_0}((\mathfrak{a}^{n_0} M) \cap M')$$

for each $n \geq n_0$.

Proof. Let $R_* = \oplus_{n\in\mathbb{N}} \mathfrak{a}^n$ be a graded ring and $M_* = \oplus_{n\in\mathbb{N}} \mathfrak{a}^n M$ a graded R_*-module. As R is Noetherian, $\mathfrak{a} \subset R$ is finitely generated this kind of a system generates R_* as an R-algebra when seen as a system of homogeneous elements with degree as 1. Now, Hilbert's Basis Theorem implies that R_* is Noetherian. Similarly, M_* is Noetherian.
Let $M_n^{'} = \mathfrak{a}^n M \cap M'$ where $n \in \mathbb{N}$. Also, let

$$\bigoplus_{n=0}^{m} M_n^{'} \oplus \bigoplus_{n>m} \mathfrak{a}^{n-m}, \quad m \in \mathbb{N},$$

be an ascending sequence of graded submodules of M_*. The sequence is stationary since M_* is Noetherian. So, we have $m = t \in \mathbb{N}$ s.t.

$$M_n^{'} = \mathfrak{a}^{n-t} M_{n_0}^{'} \tag{6.1}$$

for every $n \geq t$. Hence, $(\mathfrak{a}^n)M \cap M' = \mathfrak{a}^{n-t}((\mathfrak{a}^t M) \cap M')$ for $n \geq t$. □

Corollary 6.1.5. *Taking the assumptions as in the previous lemma, the $\mathfrak{a}$-adic topology of M restricts to that of M'.*

Proof. Since

$$\mathfrak{a}^n M' \subset (\mathfrak{a}^n M) \cap M' \quad \text{and} \quad (\mathfrak{a}^{n+n_0} M) \cap M' \subset \mathfrak{a}^n M',$$

the result follows. □

Let R be an integral domain and K be its field of fractions. Then R is said to be a valuation ring if $x \in R$ or $x^{-1} \in R$ for all $x \in K$.

Proposition 6.1.6. *Consider a valuation ring R.*
(i) Each finitely generated ideal in R is a principal ideal.
(ii) If $\mathfrak{a}, \mathfrak{b}$ are elements in R that are ideals, then $\mathfrak{a}$ is contained in $\mathfrak{b}$ or vice-versa.
Particularly, R is a local ring.

Proof. Take non-zero a, b in R. Then either ab^{-1} or $a^{-1}b$ is in R. So, we get (i). For the next part, let $\mathfrak{a} \not\subset \mathfrak{b}$ and $\mathfrak{b} \not\subset \mathfrak{a}$. We then have some a and b s.t. $a \in \mathfrak{a} - \mathfrak{b}$ and $b \in \mathfrak{b} - \mathfrak{a}$. Using this, we arrive at a contradiction and we are done. □

Let R be a valuation ring. Then R can be seen as a topological ring by having its system of non-trivial ideals for a basis of nbhds of 0. In that case, R becomes separated except for when it is a field. We now discuss the valuation rings that are adic.

Proposition 6.1.7. *Consider a valuation ring R which isn't a field. Then TFAE:*
(i) R is adic with an ideal of definition that is finitely generated.
(ii) There is a minimal non-zero prime ideal $\mathfrak{p}$ in R.
When both (i) and (ii) hold, R's topology coincides with that of the p-adic for any non-zero $p \in \mathfrak{p}$.

Proof. Claim: For every $p \in R$ which is not a unit, $\mathrm{rad}(p)$ is a prime ideal in R.
Let $a, b \in R$ s.t. $ab \in \mathrm{rad}(p)$. By the previous proposition, $\mathrm{rad}(a) \subset \mathrm{rad}(b)$ can be assumed. Then $b|a^n$ for some $n \in \mathbb{N}$. Now, $ab \in \mathrm{rad}(p)$ gives $a \in \mathrm{rad}(p)$. Hence, $\mathrm{rad}(p)$ is a prime ideal.
Let (i) be true. Again by the previous result, R's topology coincides with the p-adic one for non-trivial $p \in R$. Since any non-trivial ideal in R contains a power of p, any non-trivial prime ideal in R contains $\mathrm{rad}(p)$. But the latter is a prime ideal as has been seen, which means it is minmial as required.
Conversely, let's assume (ii). Consider non-trivial $p \in \mathfrak{p}$ and a non-trivial ideal $\mathfrak{a} \subset R$. Claim: Some power of p is contained in $\mathfrak{a}$.
Let $\mathfrak{a}$ be principal, or, $\mathfrak{a} = (a)$. On comparing $\mathrm{rad}(p)$ and $\mathrm{rad}(a)$, we see that both are primes. Hence, $\mathrm{rad}(p) \subset \mathrm{rad}(a)$, and we are done. □

We now deal with general adic rings. Let R be that and let $\mathfrak{a} \subset R$ be the ideal of definition. Since the $\mathfrak{a}$-adic topology on R does not change on translating, we can define convergence naturally. A sequence $x_v \in R$ is said to converge to $x \in R$ if for all $n \in \mathbb{N}$, we have $v_0 \in \mathbb{N}$ s.t. $x_v - x \in \mathfrak{a}^n$ for every $v \geq v_0$. In the same way, x_v is said to be a Cauchy sequence if for all $n \in \mathbb{N}$, then $v_0 \in \mathbb{N}$ s.t. $x_v - x_{v'} \in \mathfrak{a}^n$ for every $v, v' \geq v_0$. We construct a separated completion $\hat{R}$ of R by quotienting the ring of all Cauchy sequences in R by the ideal of all the sequences that converge to zero.

Consider a projective system

$$\ldots \longrightarrow R/\mathfrak{a}^n \longrightarrow \ldots \longrightarrow R/\mathfrak{a}^2 \longrightarrow R/\mathfrak{a}^1 \longrightarrow 0$$

with an ideal of definition $\mathfrak{a}$ of R. Its projective limit

$$\hat{R} = \varprojlim_n R/\mathfrak{a}^n$$

is the (separated) completion of R. Now, the topology on this limit is the coarsest possible one s.t. every natural projection $\pi_n : \hat{R} \longrightarrow R/\mathfrak{a}^n$ is continuous. Here, we have the discrete topology on $R/\mathfrak{a}^n$. So, some subset of $\hat{R}$ is open iff it's a union of some fibers of the π_n's and hence, the ideals $\ker\pi_n \subset \hat{R}$ is a basis of nbhds of 0 in $\hat{R}$. We also have that $\ker\pi_n$ is closed in $\hat{R}$ and that $\mathfrak{a}^n$ is dense in $\ker \pi_n$. We, in fact, have:

Proposition 6.1.8. *If the ideal of definition $\mathfrak{a} \subset R$ is finitely generated, $\mathfrak{a}\hat{R}$ is the closure of $\mathfrak{a}$ in $\hat{R}$ and hence $\hat{R}$ is adic with $\mathfrak{a}\hat{R}$ as the ideal of definition.*

Now, let's take that R is complete and separated under its $\mathfrak{a}$-adic topology, or, the natural morphism

$$R \longrightarrow \varprojlim_n R/\mathfrak{a}^n$$

is isomorphic. If $f \in R$, call

$$R\langle f^{-1}\rangle = \varprojlim_n((R/\mathfrak{a}^n)[f^{-1}])$$

as the complete localization of R by the multiplicative system generated by f. We have a natural morphism $R \longrightarrow R\langle f^{-1}\rangle$. Also,

$$R[f^{-1}] \longrightarrow (R/\mathfrak{a}^n)[f^{-1}]$$

yields a natural morphism $R[f^{-1}] \longrightarrow R\langle f^{-1}\rangle$ which means $\operatorname{img} f$ is invertible in $R\langle f^{-1}\rangle$.

Proposition 6.1.9. *The natural morphism $R[f^{-1}] \longrightarrow R\langle f^{-1}\rangle$ implies that $R\langle f^{-1}\rangle$ is the adic completion of $R[f^{-1}]$ w.r.t. the ideal $\langle\mathfrak{a}\rangle$ in $R[f^{-1}]$. If $\mathfrak{a}$ is finitely generated, $R\langle f^{-1}\rangle$'s topology coincides with the $\mathfrak{a}R\langle f^{-1}\rangle$-adic one.*

Proof. When this exact sequence

$$0 \longrightarrow \mathfrak{a}^n \longrightarrow R \longrightarrow R/\mathfrak{a}^n \longrightarrow 0$$

is tensored with $R[f^{-1}]$ (flat over R), we get this exact sequence:

$$0 \longrightarrow \mathfrak{a}^n R[f^{-1}] \longrightarrow R[f^{-1}] \longrightarrow (R/\mathfrak{a}^n)[f^{-1}] \longrightarrow 0$$

which gives an isomorphism

$$R[f^{-1}]/(\mathfrak{a}^n) \xrightarrow{\sim} (R/\mathfrak{a}^n)[f^{-1}].$$

Hence, $R\langle f^{-1}\rangle = \varprojlim R[f^{-1}]/(\mathfrak{a}^n)$ is the $\mathfrak{a}R[f^{-1}]$-adic completion of $R[f^{-1}]$. But then, the topology on the latter is the $\mathfrak{a}R\langle f^{-1}\rangle$-adic one when $\mathfrak{a}$ is finitely generated. □

We show another way to describe $R\langle f^{-1}\rangle$ now. For the R-algebra $R\langle\zeta\rangle$ of restricted power series, we have a natural continuous map $R\langle\zeta\rangle \longrightarrow R\langle f^{-1}\rangle$ s.t. ζ is mapped to f^{-1}. Finally, we end the section and state the next result.

Proposition 6.1.10. *The natural morphism $R\langle\zeta\rangle \longrightarrow R\langle f^{-1}\rangle$ gives the isomorphism*

$$R\langle\zeta\rangle/(1 - f\zeta) \xrightarrow{\sim} R\langle f^{-1}\rangle.$$

6.2 Formal Schemes

Locally topologically ringed spaces are called Formal schemes where all rings are viewed as elements in the category of topological rings. They are constructed out of local affine parts. We now give their definition. Consider from now onwards that adic rings are both complete and separated. Now, consider some adic ring

A and ideal of definition $\mathfrak{a}$ of A. Let $\operatorname{Spf} A$ be the set of all open prime ideals $\mathfrak{p}$ in A. Since $\mathfrak{p} \in A$, a prime idea, is open iff $\mathfrak{a}^n \in \mathfrak{p}$ where $n \in \mathbb{N}$ which means, $\mathfrak{a} \in \mathfrak{p}$, we have that $\operatorname{Spf} A$ can be naturally associated with the closed subset $\operatorname{Spec} A/\mathfrak{a}$ in $\operatorname{Spec} A$, for any ideal of definition $\mathfrak{a}$.

The Zariski topology on $\operatorname{Spec} A$ induces one on $\operatorname{Spf} A$. So

$$D(f) \mapsto A\langle f^{-1}\rangle = \varprojlim (A/\mathfrak{a}^n[f^{-1}])$$

defines a presheaf $\mathcal{O}$ of topological rings on the category of subsets $D(f)$ in $\operatorname{Spf} A$ for $f \in A$, which is a sheaf. We can extend the sheaf $\mathcal{O}$ to the category of all Zariski open subsets of $\operatorname{Spf} A$.

Definition 6.2.1. *Let A be an adic ring and $\mathfrak{a} \subset A$ is the ideal of definition. Also let $X = Spf A$ and $\mathcal{O}_X$ be the sheaf of topological rings as described in the last paragraph. Then $(X, \mathcal{O}_X)$, the locally ringed space, is said to be the affine formal scheme of A. We still denote it by $Spf A$.*

We run into a subtle issue with this definition. If we have $X = \operatorname{Spf} A$ as an affine formal scheme and $V = D(f) \subset \operatorname{Spf} A$ as basic open for $f \in A$, $(V, \mathcal{O}_X|_V)$ should be interpreted as the affine formal scheme $\operatorname{Spf} A\langle f^{-1}\rangle$. However, it's not necessary that $A\langle f^{-1}\rangle$ is an adic ring again. But there are no issues when $\mathfrak{a}$ is finitely generated because of 6.1.9, since the $A\langle f^{-1}\rangle$'s topology coincides with the $\mathfrak{a}$-adic one then.

We need to construct $\operatorname{Spf} A$ for more general topological rings when we wish to avoid such finiteness conditions. We need that A be admissible in the sense of Grothendieck, or:
(i) A is linearly topologized, i.e. we have a basis of nbhds $(I_\lambda)_{\lambda\in\Lambda}$ of 0 for ideals $I\lambda \in A$. Note that these ideals are open.
(ii) There is an ideal of definition in A, i.e. we have an open ideal $\mathfrak{a}$ in A s.t. $\mathfrak{a}^n \to 0$, i.e., for every nbhd $U \subset A$ of 0, we have $n \in \mathbb{N}$ s.t. $\mathfrak{a}^n \subset U$.
(iii) A is both separated and complete.

If A is an admissible ring along with $(I_\lambda)_{\lambda\in\Lambda}$ as a basis of nbhds of 0, the natural morphism $A \xrightarrow{\sim} \varprojlim_\lambda A/I_\lambda$ is a topological isomorphism. To deal with such rings, we replace $(\mathfrak{a}^n)_{n\in\mathbb{N}}$ with $(I_\lambda)_{\lambda\in\Lambda}$.

Definition 6.2.2. *A formal scheme is a locally topologically ringed space $(X, \mathcal{O}_X)$ s.t. every $x \in X$ admits an open nbhd U where $(U, \mathcal{O}_X|_U)$ is in isomorphism with some affine formal scheme $Spf A$, as we had seen above.*

We construct the global formal schemes by gluing the local ones, as we do usually.

6.3 Algebras of Topologically Finite Type

Consider a (complete and separated) adic ring R and an ideal of definition $I \subset R$ which is finitely generated. Assume R have no I-torsion, or, in other words,

$$(I - \text{torsion})_R = \{r \in R : I^n r = 0 \text{ for } n \in \mathbb{N}\}$$

is the zero set. This condition does not depend on what I is. Fixing generators $g_1, \ldots, g_r \in I$, observe R has no I-torsion iff the canonical map

$$R \longrightarrow \prod_{i=1}^{r} R[g_i^{-1}]$$

is an injection. Now, we will consider only two kinds of rings:

(V) R is an adic valuation ring and has its ideal of definition as finitely generated, that is in fact principal by 6.1.6.

(N) R is a Noetherian adic ring along with an ideal of definition I s.t. R does not have I-torsion.

Proposition 6.3.1. *If R is of class (N), $R\langle\zeta_1, \ldots, \zeta_n\rangle$ is Noetherian.*

Proof. If we have that R is Noetherian, then so is $(R/I)[\zeta_1, \ldots, \zeta_n]$. The result now readily follows. □

Proposition 6.3.2. *$R\langle\zeta_1, \ldots, \zeta_n\rangle$ is flat over R.*

Proof. Flatness holds iff for every ideal $\mathfrak{a}$ in R that is finitely generated, the natural morphism $\mathfrak{a} \otimes_R M \longrightarrow M$ is an injection. If we have that R is an integral domain and also, if each finitely generated ideal is principal in R, the latter is the same as M not admitting any R-torsion. So, if R belong to the class (V), the proof is done by 6.1.6.

When R belongs to class (N), $R \longrightarrow R[\zeta_1, \ldots, \zeta_n]$ is flat since it's module-free. Also, the morphism from $R[\zeta_1, \ldots, \zeta_n]$ into its I-adic completion is flat. This gives the required result. □

We now introduce the analogs of affinoid algebras.

Definition 6.3.1. *Let A be a topological R-algebra. Then it's*

(i) of topologically finite type if it is in isomorphism with an R-algebra $R\langle\zeta_1, \ldots, \zeta_n\rangle/\mathfrak{a}$ which has the I-adic topology, $\mathfrak{a}$ being an ideal in $R\langle\zeta_1, \ldots, \zeta_n\rangle$.

(ii) of topologically finite presentation if $\mathfrak{a}$ is finitely generated as well.

(iii) admissible if A has no I-torsion as well.

Theorem 6.3.3. *(Raynaud-Gruson). Consider an R-algebra A of topologically finite type and a finite A-module M that is flat over R. Then M has a finite presentation, or, M is in isomorphism with the cokernel of an A-linear morphism $A^r \longrightarrow A^s$.*

Proof. Since A is as given, it is a quotient of some algebra of $R\langle\zeta_1, \ldots, \zeta_n\rangle$. Taking M to be a module over this ring, let $A = R\langle\zeta_1, \ldots, \zeta_n\rangle$. In the case (N), we are done since A is Noetherian. In the case (R), choose $t \in R$ that generates an ideal of definition. Then A/tA is an $R/(t)$-algebra of finite presentation and M/tM is a finite module over A/tA, flat over $R/(t)$ and also of finite presentation. Take this exact sequence:

$$0 \longrightarrow N \longrightarrow A^s \longrightarrow M \longrightarrow 0.$$

All these are A-modules. As M is flat over R, when the sequence is tensored with $R/(t)$ over R, the sequence stays exact. Now, finite presentation mentioned above implies N/tN is a finite module over A/tA. Taking N to be a submodule of A^s for $A = R\langle\zeta_1, \ldots, \zeta_n\rangle$, we get that N is a finite module over A, which gives that M is an A-module and has a finite presentation. □

Corollary 6.3.4. *Consider some R-algebra A of topologically finite type. If A has no I-torsion, then A is of topologically finite presentation.*

Definition 6.3.2. *Consider an A-module M. Then it is said to be coherent if M is finitely generated and if each finite submodule of M is of finite presentation. Also, A is said to be a coherent ring if it's coherent as a module over itself.*

Corollary 6.3.5. *Consider some R-algebra A of topologically finite presentation. Then A has to be a coherent ring. Particularly, any module over A of finite presentation is coherent.*

Lemma 6.3.6. *Consider an R-algebra A of topologically finite type, a finite module M over a ring A and a submodule $N \subset M$. Then:*
(i) If N is saturated, i.e.

$$N_{sat} = \{x \in M \,:\, \exists\, n \in \mathbb{N}\, s.t.\, I^n x \subset N\}$$

coincides with N, we have N to be finitely generated.
(ii) The I-adic topology of M restricts to that on N.
Proof. The proof for (i) is straighforward when R is of class (N) and that of (ii) is clear from 6.1.5. Now, consider a ring R of class (V). M/N admits no I-torsion when N is saturated, which implies N is flat over R as R is a valuation ring. So, by 6.3.3, M/N has a finite presentation over A. Also, we have an exact sequence of modules over ring A:

$$0 \longrightarrow K \longrightarrow F \longrightarrow M/N \longrightarrow 0.$$

Here, we have F as finite free and K as finite. It can be assumed that $F \longrightarrow M/N$ is factored through M via a surjection $F \longrightarrow M$ since M is finitely generated. This map then restricts to a surjective morphism $K \longrightarrow N$ which implies N is finitely generated. hence, (i) is done.
For (ii), consider $N_{\mathrm{sat}} \subset M$, a saturation of N. By (i), it's finitely generated. So, we have $m \in \mathbb{N}$ s.t. $I^m N_{\mathrm{sat}} \subset N$ and

$$I^{n+m} M \cap N \subset I^n N \subset I^n M \cap N$$

where $n \in \mathbb{N}$. This concludes (ii). □

Proposition 6.3.7. *Consider an R-algebra of topologically finite type A and also a finite module M over A. Then M is I-adically complete and separated.*
Proof. WLOG substitute A with $R\langle\zeta\rangle$. So, A is I-adically complete and separated. Then, using 6.3.6 and taking M as a quotient of a finite cartesian product of A, M is I-adically complete. Now, let $m \in \cap_{n=0}^{\infty} I^n M$ and let $N = Am \subset M$. By 6.3.6 again, there exists $n \in \mathbb{N}$ s.t. $N = I^n M \cap N \subset IN$. We thus have $(1-s)m = 0$ for some $s \in I$. But then, $1-s$ is unit in R, so, $m = 0$. □

Corollary 6.3.8. *An R-algebra of topologically finite type is I-adically complete and separated.*

Particularly, for A, an R-algebra of topologically finite type, associate it with $\varprojlim_n A/I^n A$. We denote $R_n = R/I^{n+1}$ and $A_n = A/I^{n+1} = A \otimes_R R_n$ where $n \in \mathbb{N}$. We use such concepts for modules over R as well.

Proposition 6.3.9. *Consider A, an R-algebra that is I-adically complete and separated. We have:*
(i) A is of topologically finite type iff A_0 is of finite type over R_0.
(ii) A is of topologically finite presentation iff A_n is of finite presentation over R_n for every $n \in \mathbb{N}$.
Proof. We just show the if ways, as the other ways are trivial. For the rest of the parts, let A_0 of finite type over R_0. We then have a surjective map $\varphi_0 : R_0[\zeta] \longrightarrow A_0$ where $\zeta = (\zeta_1, \ldots, \zeta_m)$. Represent $\varphi_0(\zeta_i)$ by

$a_i \in A$ and define a continuous R-algebra homomorphism $\varphi : R\langle\zeta\rangle \longrightarrow A$ s.t. $\zeta_i \mapsto a_i$. Then $A = \operatorname{im}\varphi + IA$ and φ is a surjection.
Let $\mathfrak{a} = \ker\varphi$, and take an exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow R\langle\zeta\rangle \xrightarrow{\varphi} A \longrightarrow 0$$

where A_n is of finite presentation over R_n. By 6.3.6, there exists $n \in \mathbb{N}$ s.t. $\mathfrak{a} \cap I^{n+1} \subset I\mathfrak{a}$, and hence we have

$$0 \longrightarrow \mathfrak{a}/\mathfrak{a} \cap I^{n+1}R\langle\zeta\rangle \longrightarrow R_n[\zeta] \longrightarrow A_n \longrightarrow 0.$$

We know $\mathfrak{a}/\mathfrak{a} \cap I^{n+1}R\langle\zeta\rangle$ which gives us that $\mathfrak{a}/I\mathfrak{a}$ are finitely generated. So we have finitely generated $\mathfrak{a}'$ in $\mathfrak{a}$ s.t. $\mathfrak{a} = \mathfrak{a}' + I\mathfrak{a}$. Using limits, we get that $\mathfrak{a} = \mathfrak{a}'$, which implies $\mathfrak{a}$ is finitely generated. ☐

Proposition 6.3.10. *Consider a map $\varphi : A \longrightarrow B$ of R-algebras of topologically finite type and let M be a finite module over B. Then M is a flat (respectively faithfully flat) module over A iff M_n is a flat (respectively faithfully flat) module over A_n for every $n \in \mathbb{N}$.*
Proof. We prove the if part, the other way is easy as base change preserves the flatness. Claim: the natural morphism $\mathfrak{a} \otimes_A M \longrightarrow M$ is an injection for every finitely generated $\mathfrak{a}$ in A, which resolves the proof. ☐

Corollary 6.3.11. *Consider an R-algebra A of topologically finite type and $f_1, \ldots, f_r \in A$ be the generators of the unit ideal. Then the natural morphisms $A \longrightarrow A\langle f_i^{-1}\rangle$ are flat and $A \longrightarrow \prod_{i=1}^r A\langle f_i^{-1}\rangle$ is faithfully flat.*
Proof. To prove this, use the previous result along with the results on localization.

Corollary 6.3.12. *Let A be an I-adically complete and separated R-algebra and $f_1, \ldots, f_r \in A$ generate the unit ideal. Then TFAE:*
(i) A is of topologically finite type (respectively finite presentation, respectively admissible).
(ii) $A\langle f_i^{-1}\rangle$ is of topologically finite type (respectively finite presentation, respectively admissible) for every i.

6.4 Admissible Formal Schemes

Consider an I-adically complete and separated R-algebra A.

Definition 6.4.1. *Consider a formal R-scheme X. Then it is said to be locally of topologically finite type (respectively locally of topologically finite presentation, respectively admissible) if we have some open affine covering $(U_i)_{i\in J}$ of X where $U_i = Spf A_i$ and A_i is an R-algebra of topologically finite type (respectively of topologically finite presentation, respectively an admissible R-algebra).*

Proposition 6.4.1. *Consider A an I-adically complete and separated R-algebra. Also, consider $X = Spf A$ as the associated formal R-scheme. Then TFAE:*
(i) X is locally of topologically finite type (respectively locally of topologically finite presentation, respectively admissible).
(ii) A is of topologically finite type (respectively of topologically finite presentation, respectively admissible) as R-algebra.

Proof. This follows directly from 6.3.12.

As in schemes, a formal R-scheme X is said to be of topologically finite type if it is locally of topologically finite type and quasi-compact. It's said to be locally of topologically finite presentation if it is locally of topologically finite presentation, quasi-compact, and quasi-separated.

Consider a formal R-scheme X that is locally of topologically finite type. Also, consider $\mathcal{O}_X$ to be its structure sheaf. Let $\mathcal{J} \subset \mathcal{O}_X$ be the I-torsion of $\mathcal{O}_X$ s.t. for any open $V \subset X$, $\mathcal{J}(V)$ contains every section $f \in \mathcal{O}_X(V)$ s.t. we have an affine open covering $(V_\lambda)_{\lambda \in \Lambda}$ of V s.t. every $f|_{U_\lambda}$ is annihilated by I^n for an n of the ideal of definition I in R. Now, $\mathcal{J}$ is an ideal sheaf in $\mathcal{O}_X$. Also, for an affine open formal subscheme $V \subset X$, say $V = \operatorname{Spf} A$, we have

$$\mathcal{J}(U) = (I - \text{torsion})_A = \{f \in A \,:\, I^n f = 0 \text{ for } n \in \mathbb{N}\}.$$

Actually, we get $(I - \text{torsion})_A \subset \mathcal{J}(U)$, and $A/(I - \text{torsion})_A$ has no I-torsion locally on $\operatorname{Spf} A$ because of 6.3.12. Particularly, we can substitute $\mathcal{O}_X$, the structure sheaf, with $\mathcal{O}_X/\mathcal{J}$ and restrict X to the support X_{ad} of $\mathcal{O}_X/\mathcal{J}$. So we now have a formal R-scheme X_{ad} that is locally of topologically finite type s.t. its structure sheaf has no I-torsion. X_{ad} is locally of topologically finite presentation then, by 6.3.4, it is admissible. X_{ad} is called the admissible formal R-scheme induced from X. When R is consisting of a complete valuation ring of height 1, we have the next result:

Proposition 6.4.2. *Consider a complete valuation ring R of height 1 and its field of fractions K. Then the functor $A \mapsto A \otimes_R K$ on R-algebras A of topologically finite type gives another one: $X \mapsto X_{rig}$ from the category of formal R-schemes that are locally of topologically finite type, to that of rigid K-spaces.*

Here, X_{rig} is called the generic fiber of the formal R-scheme X.

Definition 6.4.2. *For a rigid K-space X_K, any admissible formal R-scheme X s.t. $X_{rig} \xrightarrow{\sim} X_K$ is known as a formal R-model of X_K.*

Chapter 7

Raynaud's View on Rigid Spaces

7.1 Coherent Modules

Here, R is either of type (V) or (N), and it has an ideal of definition I that is finitely generated. This means R is a Noetherian adic ring or an adic valuation ring where the ideal of definition is finitely generated.

Consider an R-algebra of topologically finite type A and the associated formal R-scheme $X = \operatorname{Spf} A$. We have a functor $M \mapsto M^{\Delta}$ associating an $\mathcal{O}_X$-module M^{Δ} to any A-module as: set

$$M^{\Delta}(D_f) = \varprojlim_{n \in \mathbb{N}} M \otimes_A A_n[f^{-}]$$

for $D_f = D(f) \subset X$, a basic open subset. Here $f \in A$ and $A_n = A/I^{n+1}A$. We have a sheaf since $\varprojlim$ is left-exact. We can extend this to every open subset in X. We say that M^{Δ} is the inverse limit of $\widetilde{M_n}$'s, induced on $X_n = \operatorname{Spec} A_n$ from the A_n-modules $M_n = M \otimes_A A_n$. When M is a finite A-module, M^{Δ}, the sheaf has this description:

Proposition 7.1.1. *Let $X = \operatorname{Spf} A$ be a formal R-scheme of topologically finite type. Then, for every finite module M over A, the sheaf M^{Δ} coincides on basic open subsets $D_f \subset X, f \in A$, with the functor*

$$D_f \mapsto M \otimes_A A\langle f^{-1}\rangle.$$

Proof. Since $A\langle f^{-1}\rangle$ is an R-algebra of topologically finite type by 6.4.1, 6.3.7 implies $M \otimes_A A\langle f^{-1}\rangle$, which is a finite $A\langle f^{-1}\rangle$-module, is I-adically complete and separated. We can see $M^{\Delta}(D_f)$ as the I-adic completion of $M \otimes_A A[f^{-1}]$. As the latter is dense in $M \otimes_A A\langle f^{-1}\rangle$, the result follows. □

Corollary 7.1.2. *Consider a formal R-scheme $X = \operatorname{Spf} A$ of topologically finite type.*
(i) $M \mapsto M^{\Delta}$ from the category of finite modules over A to that of modules over $\mathcal{O}_X$ is fully faithful and exact.
(ii) Let X be of topologically finite presentation (which means A is coherent). Then $M \mapsto M^{\Delta}$ commutes

with the formation of images, tensor products, kernels, and cokernels on the category of coherent modules over A. Also, this sequence of coherent modules over A

$$0 \longrightarrow P \longrightarrow M \longrightarrow Q \longrightarrow 0$$

is exact iff the corresponding sequence of modules over $\mathcal{O}_X$

$$0 \longrightarrow P^{\Delta} \longrightarrow M^{\Delta} \longrightarrow Q^{\Delta} \longrightarrow 0$$

is exact.

Definition 7.1.1. *Consider a formal R-scheme X and a module $\mathcal{F}$ over $\mathcal{O}_X$. Then*
(i) $\mathcal{F}$ is said to be of finite type if we have an open covering $(X_j)_{j\in J}$ of X along with exact sequences:

$$\mathcal{O}_X^{s_j}|_{X_j} \longrightarrow \mathcal{F}|_{X_i} \longrightarrow 0, \quad j \in J.$$

(ii) $\mathcal{F}$ is said to be of finite presentation if we have an open covering $(X_j)_{j\in J}$ of X along with exact sequences

$$\mathcal{O}_X^{r_j}|_{X_j} \longrightarrow \mathcal{O}_X^{s_j} \longrightarrow \mathcal{F}|_{X_j} \longrightarrow 0, \quad j \in J.$$

(iii) $\mathcal{F}$ is said to be coherent if $\mathcal{F}$ is of finite type and if for every open subscheme U in X the kernel of any map $\mathcal{O}_X^s|_U \longrightarrow \mathcal{F}|_U$ is of finite type.

In the case of an affine formal R-scheme $X = \operatorname{Spf} A$, the powers $\mathcal{O}_X^r$ can be seen as the module $(A^r)^{\Delta}$ over $\mathcal{O}_X$ corresponding to the module A^r over A. Also, by 6.3.4, A is coherent if it is of topologically finite presentation. We can then conclude from the previous result that kernels and cokernels of maps $\mathcal{O}_X^r \longrightarrow \mathcal{O}_X^s$ correspond to finite A-modules.

Proposition 7.1.3. *Consider a formal R-scheme X that is locally of topologically finite presentation, and consider a module $\mathcal{F}$ over $\mathcal{O}_X$. Then TFAE:*
(i) $\mathcal{F}$ is coherent.
(ii) $\mathcal{F}$ is of finite presentation.
(iii) We have some open affine covering $(X_i)_{i\in J}$ of X s.t. $\mathcal{F}|_{X_i}$ corresponds to a finite module over $\mathcal{O}_{X_i}(X_i)$ for every $i \in J$.
Proof. The first two parts are trivial. Now, for (iii), let $\mathcal{F}$ be of finite presentation as in (ii). We need to consider Xas affine, say $X = \operatorname{Spf} A$ where A is an R-algebra that is of topologically finite presentation. Also take the exact sequence

$$(A^r)^{\Delta} \longrightarrow (A^s)^{\Delta} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Then, by 7.1.2, $(A^r)^{\Delta} \longrightarrow (A^s)^{\Delta}$ corresponds to A-linear morphism $A^r \longrightarrow A^s$ and $\mathcal{F}$ to its cokernel, which being a finite module over A, we are done.
Let $\mathcal{F}$ satisfy (iii). Claim: $\mathcal{F}$ is coherent. To prove this, let $X = \operatorname{Spf} A$ where A is of topologically finite presentation and $\mathcal{F}$ is associated to a finite module M over A. Consider an open subscheme $U \subset X$. Also, let $\varphi : \mathcal{O}_X^s|_U \longrightarrow \mathcal{F}|_U$ be a map of modules over $\mathcal{O}_X$. We let $U = X$. So, φ corresponds to A-linear $A^s \longrightarrow M$. By 6.3.5, as A is coherent, $\ker \varphi$ is of finite type which is the case for the associated module over $\mathcal{O}_X$ as well.

Since the latter coincides with kernel of φ, the result follows. $\square$

We now want to know whether coherent modules on affine formal R-schemes $X = \operatorname{Spf} A$ are associated to coherent modules over A, as in schemes or rigid K-spaces. Towards that goal, we see:

Proposition 7.1.4. *Let $X = \operatorname{Spf} A$ be an affine formal R-scheme of topologically finite presentation. Also, consider a coherent module $\mathcal{F}$ over $\mathcal{O}_X$. Then $\mathcal{F}$ is associated to a coherent module M over A.*

7.2 Admissible Formal Blowing-Up

Before introducing the definitions, we state a couple of lemmas that will be useful.

Lemma 7.2.1. *Consider a module M over A and let $\pi \in A$ be not a zero-divisor. Then TFAE:*
(i) M is flat over A.
(ii) The torsion

$$(\pi - torsion)_M = \{x \in M \,:\, \pi^n x = 0 \text{ for } n \in \mathbb{N}\}$$

of π in M is trivial, $M/\pi M$ is flat over $A/\pi A$, and $M \otimes_A A[\pi^{-1}]$ is flat over $A[\pi^{-1}]$.

Lemma 7.2.2. *(Gabber). Consider R, an adic ring of one of the types (V) and (N). Also, consider an R-algebra A of topologically finite type and an R-algebra C of finite type. Then $\tilde{C}$, the I-adic completion of C, is flat over C.*

This concept of coherent modules also applies to ideals in $\mathcal{O}_X$. An ideal $\mathcal{A} \subset \mathcal{O}_X$ is said to be open, if it consists of powers $I^n\mathcal{O}_X$, locally on X. Now, consider a formal R-scheme X that is locally of topologically finite presentation because a coherent open ideal $\mathcal{A} \subset \mathcal{O}_X$ is associated on $\operatorname{Spf} A \subset X$, an affine open part, to a coherent open ideal $\mathfrak{a} \subset A$.

Definition 7.2.1. *Let X be as above. Then the formal R-scheme*

$$X_{\mathcal{A}} = \varinjlim_{n \in \mathbb{N}} \operatorname{Proj}\,(\bigoplus_{d=0}^{\infty} \mathcal{A}^d \otimes_{\mathcal{O}_X} (\mathcal{O}_X/I^n\mathcal{O}_X))$$

along with the natural projection $X_{\mathcal{A}} \longrightarrow X$ is called the formal blowing-up of $\mathcal{A}$ on X. Such a blowing-up is said to be an admissible formal blowing-up of X.

Proposition 7.2.3. *Admissible formal blowing-up commutes with flat base change.*

Proof. WLOG let $X = \operatorname{Spf} A$ s.t. $\mathcal{A}$ corresponds to a finitely generated open ideal $\mathfrak{a}$ in A. Then, in $X_{\mathcal{A}}$, we can substitute $\mathcal{A}^d$ with $\mathfrak{a}^d$ and R with $\mathcal{O}_X$. Let $\varphi : X' \longrightarrow X$ be a base change morphism s.t. X' is assumed to be affine, say, $X' = \operatorname{Spf} A'$ where A' is an R-algebra of topologically finite presentation. Then

$$X_{\mathcal{A}} \times_X X' = \varinjlim_{n \in \mathbb{N}} \operatorname{Proj}\,(\bigoplus_{d=0}^{\infty} \mathfrak{a}^d \otimes_A A' \otimes_R (R/I^n)).$$

When we have A' flat over A, $\mathfrak{a}^d \otimes_A A'$ and $\mathfrak{a}^d A'$ are canonically isomorphic, which gives

$$X_{\mathcal{A}} \times_X X' = \varinjlim_{n\in\mathbb{N}} \operatorname{Proj}\,(\bigoplus_{d=0}^{\infty}(\mathfrak{a}A')^d \otimes_R (R/I^n))$$

as the admissible blowing-up of the coherent open ideal $\mathcal{A}\mathcal{O}_{X'} \subset \mathcal{O}_{X'}$ on X'. This proof is enough when a complete adic ring R' of the two types are used instead of A'. Then IR' is the ideal of definition of R'. $\square$

Corollary 7.2.4. *Let X be as above. Also, let $\mathcal{A}$ in $\mathcal{O}_X$ be a coherent open ideal. For $U \subset X$, an open formal subscheme, the restriction $X_{\mathcal{A}} \times_X U$ of $X_{\mathcal{A}}$ on X to U coincides with the formal blowing-up of the coherent open ideal $\mathcal{A}|_U \subset \mathcal{O}_U$ on U.*

Let's now establish a relation between admissible formal blowing-up and scheme theoretic blowing-up.

Proposition 7.2.5. *Take $X = Spf A$ as described above. Consider a coherent open ideal $\mathcal{A} = \mathfrak{a}^{\triangle} \subset \mathcal{O}_X$ that corresponds to a coherent open ideal $\mathfrak{a}$ in A. Then $X_{\mathcal{A}}$ is the I-adic completion of the scheme theoretic blowing-up $(Spec\, A)_{\mathfrak{a}}$ of $\mathfrak{a}$ on $Spec\, A$. Or, it is the formal completion of $(Spec\, A)_{\mathfrak{a}}$ along its subscheme defined by $IA \subset A$.*

Proof. The scheme theoretic blowing up of $\mathfrak{a}$ on $\operatorname{Spec} A$ is

$$P = \operatorname{Proj}\,(\bigoplus_{d=0}^{\infty} \mathfrak{a}^d).$$

Also, the I-adic completion of P is

$$\tilde{P} = \varinjlim_{n\in\mathbb{N}}(P \otimes_R R/I^n) = \varinjlim_{n\in\mathbb{N}} \operatorname{Proj}\,(\bigoplus_{d=0}^{\infty} \mathfrak{a}^d \otimes_R R/I^n)$$

as tensoring with R/I^n over R is compatible with localization. This implies, it coincides with $\mathcal{A}$ on X. $\square$

So, when X is admissible, we can give a much precise description of admissible formal blowing-ups.

Proposition 7.2.6. *Take $X = Spf A$ as usual. Also , consider a coherent open ideal $\mathcal{A} = \mathfrak{a}^{\triangle}$ in $\mathcal{O}_X$ that corresponds to a coherent open ideal $\mathfrak{a} = (f_0, \ldots, f_r)$ in A. We have:*
(i) Ideal $\mathcal{A}\mathcal{O}_{X,\mathcal{A}} \subset \mathcal{O}_{X,\mathcal{A}}$ is invertible, or, as modules over $\mathcal{O}_{X,\mathcal{A}}$, it's in local isomorphism with $\mathcal{O}_{X,\mathcal{A}}$.
(ii) Consider the locus U_i in $X_{\mathcal{A}}$ with $\mathcal{A}\mathcal{O}_{X,\mathcal{A}}$ generated by f_i , $i = 0, \ldots, r$. Then the U_i's are an open affine covering of $X_{\mathcal{A}}$.
(iii) Let

$$C_i = A\langle \frac{f_j}{f_i} \,:\, j \neq i\rangle = A\langle \zeta_j \,:\, j \neq i\rangle/(f_i\zeta_j - f_j \,:\, j \neq i).$$

Then the I-torsion of C_i coincides with its f_i-torsion, and $U_i = Spf A_i$ is true for $A_i = C_i/(I - torsion)_{C_i}$.
Proof. We see $S = \bigoplus_{d=0}^{\infty} \mathfrak{a}^d$ as graded rings. Then the scheme theoretic blowing-up of $\mathfrak{a}$ on $\tilde{X} = \operatorname{Spec} A$ is

$$\tilde{X}' = \operatorname{Proj} S = \operatorname{Proj} \bigoplus_{d=0}^{\infty} \mathfrak{a}^d.$$

This admits the canonical open covering $\tilde{X}' = \cup_{i=0}^{r} D_+(f_i)$ where $D_+(f_i)$ is the open set of all homogeneous prime ideals in S. Here, f_i is seen as a homogeneous element with degree as 1 in $\mathfrak{a}^1 \subset S$ and it doesn't vanish. Also, $D_+(f_i) = \operatorname{Spec} S_{(f_i)}$, $S_{(f_i)}$ being the homogeneous localization of S by f_i.
Also, $X_{\mathcal{A}}$ is covered by $\operatorname{Spf} \tilde{S}_{(f_i)}$, the I-adic completions of $D_+(f_i) = \operatorname{Spec} S_{(f_i)}$. By Gabber's lemma, $\tilde{S}_{(f_i)}$ is flat over $S_{(f_i)}$ and hence the ideal $\mathfrak{a}\tilde{S}_{(f_i)}$ in $\tilde{S}_{(f_i)}$ is invertible. $\mathcal{A}\mathcal{O}_{X,\mathcal{A}}$ is hence, an invertible ideal on $X_{\mathcal{A}}$. hence, (i) is done.
Now, the restriction of $D_+(f_i)$ to $X_{\mathcal{A}}$ is U_i. We observe $U_i = \operatorname{Spf} \tilde{S}_{(f_i)}$. Hence, (ii) is done. We now need to verify (iii) for $A_i = \tilde{S}_{(f_i)}$. For that, choose $\zeta_0, \dots, \zeta_r$. We have the natural surjective map

$$A[\zeta_j \,:\, j \neq i] \longrightarrow S_{(f_i)} \subset S_{f_i}, \quad \zeta_j \mapsto \frac{f_j}{f_i}.$$

This factors through

$$\tilde{C}_i = A[\frac{f_j}{f_i} \,:\, j \neq i] = A[\zeta_j \,:\, j \neq i]/(f_i\zeta_j - f_j \,:\, j \neq i),$$

hence we have an isomorphism

$$\tilde{C}_i/(f_i - \text{torsion}) \xrightarrow{\sim} S_{(f_i)},$$

as $S_{(f_i)}$ admits no f_i-torsion. Since $\mathfrak{a}$ is open, it contains a power of I. So, as $\mathfrak{a}\tilde{C}_i$ is generated by f_i, and

$$(f_i - \text{torsion})_{\tilde{C}_i} \subset (I - \text{torsion})_{\tilde{C}_i}.$$

As X is admissible, A, $S = \oplus_{d=0}^{\infty} \mathfrak{a}^d$ and $S_{(f_i)}$ have no I-torsions. Hence, we must have the equality:

$$(f_i - \text{torsion})_{\tilde{C}_i} = (I - \text{torsion})_{\tilde{C}_i}.$$

We consider the I-adic completion C_i of $\tilde{C}_i$ now. We see that

$$C_i = A\langle \frac{f_j}{f_i} \,:\, j \neq i \rangle = A\langle \zeta_j \,:\, j \neq i \rangle/(f_i\zeta_j - f_j \,:\, j \neq i).$$

By Gabber's lemma, C_i, the I-adic completion of $\tilde{C}_i$ is flat over it. Hence,

$$(I - \text{torsion})_{C_i} = (I - \text{torsion})_{\tilde{C}_i} \otimes_{\tilde{C}_i} C_i$$
$$(f_i - \text{torsion})_{C_i} = (f_i - \text{torsion})_{\tilde{C}_i} \otimes_{\tilde{C}_i} C_i,$$

so the torsions coincide. But then

$$A_i = \tilde{S}_{(f_i)} = A\langle \frac{f_j}{f_i} \,:\, j \neq i \rangle / I - \text{torsion},$$

and we are done. □

Corollary 7.2.7. *Consider an admissible formal R-scheme X and a coherent open ideal $\mathcal{A}$ in $\mathcal{O}_X$. Then $X_{\mathcal{A}}$, the formal blowing-up of $\mathcal{A}$ on X admits no I-torsion and due to 6.3.4, $X_{\mathcal{A}}$ is again an admissible formal R-scheme.*

Proposition 7.2.8. *Consider a formal R-scheme X and a coherent open ideal $\mathcal{A}$ in $\mathcal{O}_X$. Then $X_{\mathcal{A}} \longrightarrow X$*

satisfies this universal property:
Every map $\varphi : Y \longrightarrow X$ of formal R-schemes, s.t. $\mathcal{A}\mathcal{O}_Y$ is an invertible ideal in $\mathcal{O}_Y$ uniquely factorizes through $X_{\mathcal{A}}$.

Proof. WLOG let X be affine. Let $X = \operatorname{Spf} A$ and $\mathcal{A}$ correspond to $\mathfrak{a} = (f_0, \dots, f_r) \subset A$. Consider $\varphi : Y \longrightarrow X$, a map of formal schemes, s.t. $\mathcal{A}\mathcal{O}_Y \subset \mathcal{O}_Y$ is invertible. Let $Y = \operatorname{Spf} B$ and $\mathcal{A}\mathcal{O}_Y$ be generated by f_i for an i. Then $\mathcal{A}\mathcal{O}_Y$ corresponds to $f_i B = \mathfrak{a}B \subset B$.
Consider $\varphi^* : A \longrightarrow B$, a map of R-algebras given by $\varphi : Y \longrightarrow X$. As $\mathfrak{a}B$ is invertible, $f_j f_i^{-1} \in B$ are well-defined. So, we have a unique homomorphism

$$A_i = A\langle \frac{f_j}{f_i} : j \neq i \rangle / (f_i - \text{torsion}) \longrightarrow B$$

which extends $\varphi^* : A \longrightarrow B$ s.t. $f_j f_i^{-1} \in A_j$ are mapped to the corresponding fractions in B. Now, the existence is done by $Y \longrightarrow X_{\mathcal{A}}$. For the uniqueness, every factorization $Y \longrightarrow X_{\mathcal{A}}$ of $\varphi : Y \longrightarrow X$ takes Y into $U_i = \operatorname{Spf} A_i$ and we are done. □

Corollary 7.2.9. *Consider an admissible formal R-scheme X and the coherent open ideals $\mathcal{A}, \mathcal{B} \subset \mathcal{O}_X$ on X. Assume $\mathcal{B}' = \mathcal{B}\mathcal{O}_{X_{\mathcal{A}}}$. Then*

$$(X_{\mathcal{A}})_{\mathcal{B}'} \longrightarrow X_{\mathcal{A}} \longrightarrow X$$

or, composition of the formal blowing-up of $\mathcal{B}'$ on $X_{\mathcal{A}}$ with that of $\mathcal{A}$ on X is in natural isomorphism to the formal blowing-up of $\mathcal{A}\mathcal{B}$ on X.

We now state that the formal blowing-ups are transitive in this manner:

Proposition 7.2.10. *Consider an admissible formal R-scheme X that is quasi-compact and quasi-separated. Also let $\varphi : X' \longrightarrow X$ and $\varphi' : X'' \longrightarrow X'$ be formal blowing-ups . Then $\varphi \circ \varphi' : X'' \longrightarrow X$ is again an admissible formal blowing-up.*

Chapter 8

Ramification in Local Fields

8.1 Herbrand's Theorem

If $g \in \mathrm{Gal}(L/K)$, let $i_L(g) = \inf_{a \in \mathcal{O}_L} \mathrm{val}_L(g(a) - a)$.

If $G = \mathrm{Gal}(L/K)$ and $u \geq -1$, let $G_u = \{g \in G : i_L(g) \geq u + 1\}$.

If u is a real number s.t. $u \geq -1$, G_u denotes the ramification group G_i, where i is the minimum integer s.t. $i \geq u$. So

$$s \in G_u \mathrm{iff} i_G(s) \geq u + 1.$$

Take

$$\varphi(u) = \int_0^u \frac{dt}{(G_0 : G_t)}.$$

Proposition 8.1.1. *For all $\sigma \in G/H$,*

$$i_{G/H}(\sigma) = \frac{1}{e'} \sum_{s \to \sigma} i_G(s).$$

Here, $e' = e_{L/K}$.

Proof. If $\sigma = 1$, we have $+\infty$ on both sides, thus we have the result in this case. Assume $\sigma \neq 1$. Let x be an $\mathcal{O}_K$-generator of $\mathcal{O}_L$ and let y be an $\mathcal{O}_K$-generator of $\mathcal{O}'_K$.
So, $e' \cdot i_{G/H}(\sigma) = v_L(\sigma(y) - y)$, and $i_G(s) = v_L(s(x) - x)$. On choosing $s \in G$ a pre-image of σ, the other pre-images are st and t in H. So, we need to prove:

$$a = s(y) - y \text{ and } b = \prod_{t \in H} (st(x) - x)$$

generate the same ideal in $\mathcal{O}_L$.
Consider the minimal polynomial $f \in \mathcal{O}_{K'}[X]$ of x over an intermediate field K'. So, we have $f(X) =$

$\prod_{t\in H}(X-t(x))$. Let $s(f)$ be the polynomial that we get by applying s on the coefficients of f. Then

$$s(f)(X)=\prod_{t\in H}(X-st(x)).$$

Since $s(y)-y$ divides all the coefficients of $s(f)-f$, $s(f)(x)-f(x)=s(f)(x)=\pm b$ is divisible by $a=s(y)-y$.
Claim: $b|a$.
Take a polynomial y in x, with coefficients coming from $\mathcal{O}_K$, i.e. $y=g(x)$. Then x is a root of $g(X)-y$ and the coefficients of $g(X)-y$ lies in $\mathcal{O}_K$. So, the minimal polynomial f divides it:

$$g(X)-y=f(X)\cdot h(X),\ \text{where } h\in\mathcal{O}_K[X].$$

Applying s and substituting x for X, we get

$$y-s(y)=s(f)(x)\cdot s(h)(x),$$

and hence $b=\pm s(f)(x)$ divides a. ☐

Lemma 8.1.2. $\varphi_{L/K}(u)=\frac{1}{|G_0|}\sum_{s\in G}Inf(i_G(s),u+1)-1$.

Proof. Assume that $\theta(u)$ represents the RHS of the above equation. Observe that it is piecewise-linear, continuous and is zero at $u=0$. If $m<u<m+1$, s.t. $m\in\mathbb{Z}$, $\theta'(u)=\frac{1}{|G_0|}|\{s\in G: i_G(s)\geq m+2\}|$. Thus $\theta'(u)=\frac{1}{(G_0:G_{m+1})}$, but this equals $\varphi'(u)$, so θ and φ coincides. ☐

Lemma 8.1.3. *Let $\sigma\in G/H$, and consider the upper bound $j(\sigma)$ of the integers $i_G(s)$ as s runs through the pre-images of σ in G. Then*

$$i_{G/H}(\sigma)-1=\varphi_{L/K}(j(\sigma)-1).$$

Proof. Let $s\in G$ have image σ and $i_G(s)=j(\sigma)$. Also, put $m=i_G(s)$. Two cases arise: (a) If $t\in H$ is in H_{m-1}, $i_G(t)\geq m$ which means $i_G(st)\geq m$, and hence $i_G(st)=m$. (b) If $t\in H$ is not in H_{m-1}, $i_G(t)<m$, and $i_G(st)=i_G(t)$. So, clubbing the two cases, we get $i_G(st)=\inf(i_G(t),m)$. Applying 8.1.1, we get

$$i_{G/H}(\sigma)=\frac{1}{e_{L/K'}}\sum_{t\in H}\inf(i_G(t),m).$$

Now, $i_G(t)$ equals $i_H(t)$, and $e_{L/K'}$ equals the cardinality of H_0. Using 8.1.2 on H,

$$i_{G/H}(\sigma)=1+\varphi_{L/K'}(m-1).$$

☐

Theorem 8.1.4. *(Herbrand's Theorem) If $v=\varphi_{L/K'}(u)$ then $G_uH/H=(G/H)_v$. (Writing in the upper numbering, it implies that upper numbering stays unchanged on taking quotients.)*

Proof. The theorem follows from this observation:
$\sigma\in G_uH/H$ iff $j(\sigma)-1\geq u$ iff $\varphi(j(\sigma)-1)\geq\varphi_{L/K}(u)$ iff $i_{G/H}(\sigma)-1\geq\varphi_{L/K}(u)$ iff $\sigma\in(G/H)_v$. ☐

8.2 Cyclotomic Extensions of $\mathbb{Q}_p$

Proposition 8.2.1. *Let $K = \mathbb{Q}_p$. For $n = p^m$ we adjoin a primitive nth root of unity ζ to K and call it K_n. Then*
(a) $[K_n : K] = (p-1)p^{m-1}$.
(b) We can identify $G(K_n/K)$ with $G(n)$, the group of invertible elements in $\mathbb{Z}/n\mathbb{Z}$.
(c) K_n is a totally ramified extension of K. Also, $\pi = \zeta - 1$ is a uniformizer of K_n, and $\mathcal{O}_{K_n} = \mathcal{O}_K[\zeta]$.

Proof. It can be easily seen that we can identify $G(K_n/K)$ with some subgroup of $G(n)$. Now, since the cardinality of $G(n)$ is $\varphi(n) = (p-1)p^{m-1}$, (a) and (b) are equivalent.
Now, let $u = \zeta^{p^m - 1}$. Since it's a primitive pth root of unity, $u^{p-1} + u^{p-2} + \cdots + 1 = 0$ must be true, whence

$$\zeta^{(p-1)p^{m-1}} + \zeta^{(p-2)p^{m-1}} + \cdots + 1 = 0.$$

Let's denote LHS by F. Then, π is a zero of $F(1+X)$. But this is a degree $\varphi(n)$ Eisenstein equation since $F(1) = p$ is the constant term, and since $X^{\varphi(n)} = 0$ is the reduction modulo p . It can then be easily seen that $[K_n : K] = \varphi(n)$, and that π is a uniformizer of K_n. Also, π generates $\mathcal{O}_{K_n}$ and hence so does ζ. □

If $v \in \mathbb{Z}$ such that $0 \leq v \leq m$, let $G(n)^v$ be the subgroup of $G(n)$ that consists of all elements a s.t. $a \cong 1 \bmod p^v$. Also, we can identify $G(n)/G(n)^v$ with $G(p^v)$, i.e., $\mathrm{Gal}(K_{p^v}/K)$. So $G(n)^v = \mathrm{Gal}(K_n/K_{p^v})$.

Proposition 8.2.2. *The ramification groups G_u of Gal(K_n/K) are:*

$$\begin{aligned}
G_0 &= G, \\
G_u &= G(n)^1, \text{ for } 1 \leq u \leq p-1, \\
G_u &= G(n)^2, \text{ for } p \leq u \leq p^2 - 1, \\
&\vdots \\
G_u &= G(n)^m = \{1\}, \text{ for } p^{m-1} \leq u.
\end{aligned} \tag{8.1}$$

Proof. Consider $a \in G(n)$ that does not equal 1, and the corresponding element s_a of G. Consider the maximum integer v s.t. $a \cong 1 \bmod p^v$. Then we have $a \in G(n)^v$ and $a \notin G(n)^{v+1}$. But,

$$i_G(s_a) = v_{K_n}(s_a(\zeta) - \zeta) = v_{K_n}(\zeta^q - \zeta) = v_{K_n}(\zeta^{q-1} - 1).$$

As $\zeta^{k-1} \leq u \leq p^k - 1$, observe that $s_a \in G_u$ iff $v \geq k$. So, $G_u = G(n)^v$. □

Corollary 8.2.3. *The jumps in the filtration (G^v) are integers. Furthermore,*

$$G^v = G(n)^v \ \ for \ 0 \leq v \leq m,$$

and

$$G^v = \{1\} \ \ for \ v \geq m,$$

Proof. Notice that the jumps happen for $u = p^k - 1$, where $0 \leq k \leq m-1$ (except when $p = 2$ as 0 isn't a jump). Then, it suffices to prove that $\varphi_{L/K}(p^k - 1) = k$ where $k = 0, 1, \ldots, m-1$, which is direct.

8.3 APF Extensions

Consider a local field K and a separable closure $\overline{K}$ of K. Let L be an extension of $K \subset \overline{K}$. If M is an extension of $K \subset \overline{K}$, we notice G_M is the Galois group of $\overline{K}/M$.

Let's first assume L/K is finite. If σ is a K-embedding of L in $\overline{K}$, we say $i_L(\sigma) = \min_{x \in \mathcal{O}_L}(v_L(\sigma x - x) - 1)$ (we agree that if $i_L(\sigma) = +\infty$ if σ is an inclusion). If σ is not an inclusion, we can easily see that for any uniformizer π of L, we have:

$$i_L(\sigma) = v_L(\frac{\sigma\pi}{\pi} - 1) \;\; \text{if}\, \sigma \,\text{acts trivially on}\, k_L, \tag{8.2}$$

$$= -1 \;\; \text{otherwise}. \tag{8.3}$$

If for all $t \geq -1$, we note γ_t as the number of K-embeddings σ of L in $\overline{K}$ which satisfies $i_L(\sigma) \geq t$, we pose, for $u \geq 0$:

$$\varphi_{L/K}(u) = \int_0^u \frac{\gamma_t dt}{\gamma_0}.$$

For $-1 \leq u \leq 0$, we say $\varphi_{L/K}(u) = u$. The function $\varphi_{L/K}$ is an increasing bijection on $[-1, +\infty)$, continuous and piecewise linear; we note that $\psi_{L/K}$ is the inverse function.

Define $G^u = \{\sigma \in G : i_L(\sigma) \geq \psi_{L/K}(u)\}$.

Definition 8.3.1. *The extension L/K is said to be APF (arithmetically profinite), if, for all, $u \geq -1$, the groups $G_K^u G_L$ is open in G_K (it does not depend on what $\overline{K}$ is).*

If L/K is APF, we pose $G_L^0 = G_L \cap G_K^0$ and we define a bijection of $[-1, +\infty)$, increasing, continuous, piecewise linear, such that:

$$\begin{aligned} \psi_{L/K}(u) &= \int_0^u (G_K^0 : G_L^0 G_K^v) dv \;\; \text{if}\, u \geq 0, \\ &= u \;\; \text{if} \; -1 \leq u \leq 0. \end{aligned} \tag{8.4}$$

The extension L/K is said to be strictly APF if:

$$\lim \inf_{u \to +\infty} \frac{\psi_{L/K}(u)}{(G_K^0 : G_L^0 G_K^u)} > 0.$$

We notice $i(L/K)$ is the upper bound of $i \geq -1$ such as $G_K^i G_L = G_K$. If the extension L/K is in fact, a totally ramified p-extension (or, if $i(L/K) > 0$), we pose:

$$c(L/K) = \inf_{u \geq i(L/K)} \frac{\psi_{L/K}(u)}{(G_K^0 : G_L^0 G_K^u)};$$

L/K is therefore strictly APF iff $c(L/K) > 0$. (We observe that $i(L/K)$ and $c(L/K)$ do not depend on what $\overline{K}$ is).

Proposition 8.3.1. *Let M and N be two extensions of K contained in $\overline{K}$ with $M \subset N$. So:*
(a) If M/K is finite, N/K is (strictly) APF iff N/M is;
(b) If N/M is finite, N/K is (strictly) APF iff M/K is;
(c) If N/K is (strictly) APF, M/K is;
(d) If N/K is APF (respectively if N/K is APF and $i(N/K) > 0$ we have $i(M/K) \geq i(N/K)$ (respectively $c(M/K) \geq c(N/K)$)); also if M/K is finite, we have $i(N/M) \geq \psi_{M/K}(i(N/K)) \geq i(N/K)$ (respectively $c(N/M) \geq c(N/K)$)

Proof. (a), (b), (c) can be shown using:

$$(G_K : G_N G_K^u) = (G_K : G_M G_K^u)(G_M : (G_M \cap G_K^u)G_N).$$

(d) If N/K is APF, we have $G_K^{i(N/K)} G_N = G_K$ so $G_K^{i(N/K)} G_M = G_K$ from where $i(M/K) \geq i(N/K)$. If moreover M/K is finite, we have $G_M^{\psi_{M/K}(i(N/K))} G_N = (G_K^{i(N/K)} \cap G_M)G_N = G_M$ and so $i(N/M) \geq \psi_{M/K}(i(N/K))$. Since $\psi_{M/K}(i(N/K)) \geq i(N/K)$, we have $i(N/M) \geq \psi_{M/K}(i(N/K) \geq i(N/K))$.

Suppose that N/K is APF and that $i(N/K) > 0$. For all $u/geq0$, we have:

$$\psi_{N/K}(u) = \int_0^u (G_K^0 : G_N^0 G_K^v)dv$$

and $\psi_{M/K}(u) = \int_0^u (G_K^0 : G_N^0 G_K^v)dv$. For all $v \geq 0$, we see that:

$$(G_K^0 : G_N^0 G_K^v) = (G_K^0 : G_M^0 G_K^v)(G_M^0 : (G_K^v \cap G_M^0)G_N^0)$$

We then deduce that $\frac{\psi_{M/K}(u)}{(G_K^0 : G_M^0 G_K^u)} \leq \frac{\psi_{N/K}(u) \times [M:K]}{(G_K^0 : G_N^0 G_K^u)}$.
It then follws that $c(M/K) \geq c(N/K)$. □

Definition 8.3.2. *Let L/K be finite, and let i be a positive rational number. We say that L/K is elementary of level i if $G_K^i G_L = G_K$ and $G_K^{i+\epsilon} G_L = G_L$ for all $\epsilon > 0$.*

If L/K is elementary of level i, it is totally ramified and since $G_K^i / G_K^{i+\epsilon}$ is a pro-p-group, the degree of L/K is a power of p. Furthermore, if L/K is Galois, $\mathrm{Gal}(L/K)$ is a direct sum of order p cyclic groups.

8.4 Construction of fields $X_K(L)$

Consider a separable algebraic extension L of a field K. Denote by $\mathcal{E}_{L/K}$, the filtration of ordered set of finite extensions of $K \subset L$. Then, we pose:

$$X_K(L)^* = \varprojlim_{E \in \mathcal{E}_{L/K}} E^*,$$

and the image of ${E'}^*$ in E^* (if $E \subset E'$) being $N_{E'/E}$.
We denote $X_K(L) = X_K(L)^* \cup \{0\}$. If $\alpha \in X_K(L)$, it's the same as having a family $(\alpha_E)_{E \in \mathcal{E}_{L/K}}$ with $\alpha_E \in E$ and $N_{E'/E}(\alpha_{E'}) = \alpha_E$ if $E \subset E'$.

We now assume throughout that L is an infinite APF extension of a local field K. Denote K_0 (respectively K_1) for the maximal unramified extension (respectively moderately ramified) of $K \subset L$.

If $\alpha \in X_K(L)$, $v_E(\alpha_E)$ for $E \in \mathcal{E}_{L/K_0}$ does not depend on E: we pose $v(\alpha) = v_E(\alpha_E)$.

Theorem 8.4.1. *(a) Let α and $\beta \in X_K(L)$. Then for all $E \in \mathcal{E}_{L/K}$, the $N_{E'/E}(\alpha_{E'} + \beta_{E'})$ (for $E \supset E$) converges (w.r.t. $\mathcal{E}_{L/E}$) to an element $\gamma_E \in E$ and $\alpha + \beta = (\gamma_E)_{E \in \mathcal{E}_{L/K}}$ to an element of $X_K(L)$.*
(b) Equipped with the addition and multiplication of v as previously defined, $X_K(L)$ is a local field of characteristic p and $v(X_K(L)^*) = \mathbb{Z}$. The map $f_{L/K}$ is an embedding of local fields k_L of L in $X_K(L)$ and it induces an isomorphism of k_L with the residue field of $X_K(L)$.

Let E'' be an extension of $E \subset E'$. Then, if (a) holds for E''/E and E'/E'', it holds for E'/E as well.

The families of $i(L/E)$ and $r(E)$ for $E \in \mathcal{E}_{L/K_1}$ are increasing. If $(K_n)_{n \in (N)}$ is the tower of elementary extensions of L/K, then $i(L/K_n) = i(K_{n+1}/K_n)$.

Let $a = (\overline{a}_E) \in \mathcal{O}_K(L)$. If $a \neq 0$, we have $E \in \mathcal{E}_{L/K_1}$, s.t. $\overline{a}_E \neq 0$. For $E' \in \mathcal{E}_{L/E}$, let $\hat{\overline{a}}_{E'}$ be a pullback of $\overline{a}_{E'}$ in $\mathcal{O}_{E'}$. Then for $E' \in \mathcal{E}_{L/E}$, $v_{E'}(\hat{\overline{a}}_{E'})$ does not depend on E' and neither on the choice of the pullback. We pose $w(a) = v_{E'}(\hat{\overline{a}}_{E'})$. If $a = 0$, we pose $w(a) = +\infty$.

Let $x \in k_L$. For all $E \in \mathcal{E}_{L/K_1}$, let x_E be the $[E : K_1]$-th root of x, $[x_E]$ the multiplicative representative of x_E in $\mathcal{O}_E$ and $\overline{[x_E]}$ the image of $[x_E]$ of $\overline{\mathcal{O}}_E$. Then

$$(\overline{[x_E]})_{E \in \mathcal{E}_{L/K_1} \in \mathcal{O}_K(L)}; \text{ we pose } f(x) = (\overline{[x_E]})_{E \in \mathcal{E}_{L/K_1}}.$$

Proposition 8.4.2. *$\mathcal{O}_K(L)$ is a ring of characteristics p. The function $x : \mathcal{O}_K(L) \to \mathbb{N} \cup \{+\infty\}$ is surjective. It is an valuation for which $\mathcal{O}_K(L)$ is separable and complete. f is an embedding of k_L in the ring $\mathcal{O}_K(L)$ and it induces an isomorphism of k_L with the residue field of $\mathcal{O}_L(K)$.*

8.5 A Characterisation of Strictly APF Extensions

Theorem 8.5.1. *Consider an infinite, totally wildly ramified extension L/K. Then L/K is strictly APF iff we have a tower of finite extensions $\{E_n\}_{n \geq 2}$ of $E_1 := K$ in L where $L = \bigcap E_n$ and a norm-compatible sequence $\{\pi_n\}_{n \geq 1}$ where π_n is a uniformizer of E_n s.t.:*
(a) The degrees $q_n := [E_{n+1} : E_n]$ are bounded above.
(b) If $f_n(x) = x^{q_n} + a_{n,q_{n-1}} x^{q_{n-1}} + \cdots + a_{n,1} x + (-1)^p \pi_n \in E_n[x]$ is the minimal polynomial of π_{n+1} over E_n, then the non-constant, non-leading coefficients $a_{n,i}$ of f_n satisfy $v_K(a_{n,i}) > \epsilon$ for some $\epsilon > 0$, that are not dependent on either n or i.

Furthermore, if L/K is strictly APF, $\{E_n\}$ can be taken as the tower of elementary subextensions and $\{\pi_n\}$ as any norm-compatible sequence of uniformizers.

Proposition 8.5.2. *Consider a tower of finite extensions $E_{n\,n\geq 2}$ of $E_1 := K$ and also consider their rising union $L = \bigcup_{n\geq 1} E_n$. Let $\Phi := \phi_{E_n/K}$ and denote $\alpha_n := sup\{x : \Phi_{n+1}(x) = \Phi_n(x)\}$. Then, L/K will be APF iff the following two conditions hold:*
(a) We have $lim_{n\to\infty}\alpha_n = \infty$. In particular, the pointwise limit $\Phi(x) := lim_{n\to\infty}\Phi_n(x)$ exists, and furthermore, on fixing x_1, we obtain $\Phi(x) = \Phi_n(x)$ for every $x \leq x_1$ and suitable large n.
(b) $\Phi(x)$ as in (a) is continuous and piecewise linear, and has vertices $\{(i_n, b_n)\}_{n\geq 1}$ where $\{i_n\}$ and $\{b_n\}$ are unbounded increasing sequences.
If L/K is APF, we have $\Phi(x) = \phi_{L/K}$ for $\phi_{L/K}$.

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