Study of the Laplacian

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled *"Study of the Laplacian"* towards the partial fulfillment of the BS-MS dual degree program at the Indian Institute of Science Education and Research, Pune represents the study and work carried out by Nishad Bapatdhar at the Department of Mathematics, Indian Institute of Science (IISc) under the supervision of Prof. Manjunath Krishnapur, during the academic year 2019-20.

Manjunath

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To Yosie$^{\pi}$ for always being on my side,
To Amma for whom love I cannot hide,
To Abbu for believing I've got it inside,
$^{\&}$To me, do pardon this smidgin of pride.

Declaration

I hereby declare that the matter embodied in the report entitled *"Study of the Laplacian"*, are the bona fide results of the work carried out by me at the department of Mathematics, Indian Institute of Science (IISc) under the supervision of professor Manjunath Krishnapur, and the same has not been submitted elsewhere for any other degree.

Nishad Bapatdhar
April 2020

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Abstract

The Laplacian on Euclidean space is the unique second order differential operator that commutes with the isometries of the Euclidean space. As such, it features in many laws of physics and is of tremendous significance in mathematics, with connections to other important objects. Of particular interest are eigenvalues and eigenfunctions of the Laplacian on domains in Euclidean space or on manifolds. Connections between these spectral objects (we stick to domains and do not consider the situation of manifolds) and the underlying geometry are the main object of study in this report.

In the first chapter we study the Laplacian on finite graphs. This is just a symmetric, positive semi-definite matrix and is technically much simpler to study than its continuous counterpart. On the other hand, many of the features of the continuous Laplacian can already be seen in this discrete setting. In particular, we see how it helps to count the number of spanning trees of the graph (Kirchhoff's theorem 1.4) and how the well-connectedness of the graph is seen in the second eigenvalue of the Laplacian (Cheeger's inequality 1.5). We also study the discrete analogue of Courant's nodal domain theorem (1.6). The key ingredient in the proofs is the variational characterization of eigenvalues.

In the second chapter, we move to the continuous Laplacian, but in one dimension. These are the Stürm-Liouville operators. The one-dimensionality allows a more precise study of the eigenvalues (Weyl's asymptotics 2.18) and eigenfunctions (oscillation theorem 2.17). Weyl's asymptotics is a feature that is not seen in the setting of finite graphs.

In the third chapter, we study the Laplacian on bounded domains of Euclidean space, with Dirichlet or Neumann boundary conditions. The subject gets much more technical, and one needs the framework of unbounded operators and the theory of Sobolev spaces to make sense of eigenvalues and eigenfunctions. Once that is done and the variational characterization of eigenvalues is proved (3.6 and 3.7), the proofs become quite analogous to the situation of graphs. This is true in particular for Cheeger's inequality (3.17) and Courant's nodal domain theorem (3.14). But we also prove some new theorems such as the Faber-Krahn inequality (3.16) (which asserts

that the ball minimizes the principal Dirichlet eigenvalue among domains with a given volume) and Weyl's asymptotics (3.13) for the eigenvalues of the Laplacian.

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Chapter 1

Graph Laplacian

1.1 Setting

We will use the following notation in the text below.

- Let $G = (V, E)$ be a simple graph, i.e. no loops or multiple edges, with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \{ij \mid i \sim j \text{ in } G\}$, such that G has no isolated vertices.

- The degree of a vertex i is the size of its neighborhood, i.e. $|\{k \in V \mid i \sim k \text{ in } G\}|$ and is represented by d_i.

- The Laplacian of G is defined as the $n \times n$ matrix $L := D - A$, where $D_{n\times n}$ and $A_{n\times n}$ are defined as

$$D_{i,j} = \delta_{ij} d_i \ , \ A_{i,j} = \begin{cases} 0 & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } i \sim j \end{cases}$$

respectively, for $i, j \in V$.

- For a graph G, the normalized Laplacian, $\mathcal{L}$, is defined as

$$\mathcal{L}_{i,j} = \begin{cases} 1 & \text{if } i = j \\ \frac{-1}{\sqrt{d_i d_j}} & \text{if } i \sim j \\ 0 & \text{Otherwise} \end{cases}$$

Note that $\mathcal{L} = D^{-1/2} L D^{-1/2}$.

- Both L and $\mathcal{L}$ are symmetric matrices and hence their eigenvalues are real. They are represented as an increasing sequence $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$. For simplicity, we will use the same notation for both Laplacians and the context will clarify any ambiguity regarding the usage of the same.

- We treat the eigenvectors of $\mathcal{L}$ and L as functions on V i.e. given a $v \in \mathbb{R}^n$, we associate to it the function $f : V \to \mathbb{R}$ defined by $f(i) = v_i$ where v_i is the i^{th} component of v. Note that $f \in \ell^2(V)$ and that $\langle .,. \rangle_{\mathbb{R}^n} = \langle .,. \rangle_{\ell^2(V)}$. It's easy to see that $\mathbb{R}^n \cong \ell^2(V)$.

Examples : We define the line graph and complete graph on n vertices and compute L and $\mathcal{L}$ for both.

Complete graph : The complete graph K_n is a graph on V where $i \sim j$ for all $i \neq j$, $i, j \in V$. So,

$$L_{i,j} = \begin{cases} n-1 & i = j \\ -1 & \text{Otherwise} \end{cases} \qquad \mathcal{L}_{i,j} = \begin{cases} 1 & i = j \\ -\frac{1}{n-1} & \text{Otherwise} \end{cases}$$

Line graph : The line graph is a graph on V where $i \sim i \pm 1$, for $1 < i < n$. The corresponding Laplacians are tri-diagonal matrices given by

$$L_{i,j} = \begin{cases} 2 & i = j \text{ and } 1 < i < n \\ 1 & i = j = 1 \text{ or } i = j = n \\ -1 & i = j \pm 1 \text{ and } 1 \leq i, j \leq n \\ 0 & \text{Otherwise} \end{cases} \qquad \mathcal{L}_{i,j} = \begin{cases} 1 & i = j \\ -\frac{1}{2} & i = j + 1 \text{ and } 1 < i, j < n \\ -\frac{1}{\sqrt{2}} & i = j \pm 1 \text{ and } i, j \in \{1, 2, n-1, n\} \\ 0 & \text{Otherwise} \end{cases}$$

1.2 Properties

1.2.1 Laplacian

From the definition of the graph Laplacian, it may not be immediately clear as to why this object is called the Laplacian. The terminology comes from the continuous version which is the differential operator (in $\mathbb{R}^n$) given by $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$. The similarity is seen by writing the graph Laplacian as

$$Lf(i) = \sum_{k \sim i} f(i) - f(k)$$

In particular, when G is a line graph, as described in the example above, the Laplacian acts just like a discrete second difference operator.

$$Lf(i) = 2f(i) - f(i+1) - f(i-1) = \Big(f(i) - f(i-1)\Big) - \Big(f(i+1) - f(i)\Big)$$

It turns out that the Laplace differential operator is some sort of (scaled)[1] limit of graph Laplacians.

Next, we shall study the quadratic form for L.

Lemma 1.1 *The quadratic form for L has the following property*

$$\langle f, Lf \rangle = \sum_{i \sim j} (f(i) - f(j))^2 \ ; \ \forall f \in \ell^2(V) \tag{1.1}$$

Proof: Let $f \in \ell^2(V)$. Then,

$$\begin{aligned}
\langle f, Lf \rangle &= \sum_{i=1}^{n} f(i) \cdot \sum_{i \sim j} f(i) - f(j) \\
&= \sum_{i=1}^{n} \sum_{i \sim j} f^2(i) - f(i)f(j) \\
&= \sum_{i \sim j} f^2(i) - f(i)f(j) + f^2(j) - f(j)f(i) \\
&= \sum_{i \sim j} (f(i) - f(j))^2
\end{aligned}$$

□

Remark *Note that this result is analogous to that of the Laplacian in the continuous case, where,* $\langle f, -\Delta f \rangle = \int_{\Omega} |\nabla f|^2$ *for all* $f \in C_c^{\infty}(\mathbb{R}^n)$.

Theorem 1.1 shows that L is a positive semi-definite matrix.

Definition (Mean Value Property) *A function* $f \in \ell^2(V)$ *is said to satisfy the mean value property if its value at any vertex i is the average of its value at the neighbors of i.*

Clearly f satisfies the mean value property *if and only if* it is harmonic i.e. if $Lf = 0$, just like its continuous counterpart.

Now that sufficient motivation has been given as to why L can be called the 'Graph Laplacian', we can begin analyzing it's properties. We must first comment on the

[1] We can scale the line graph by adding edge weights/lengths and then take the limit as these lengths go to zero.

nature of the eigenfunctions and eigenvalues of L. By eq. (1.1) we know that L is positive semi-definite. From the spectral theory of symmetric positive semi-definite operators, we get that linear operator L has an orthonormal basis of eigenfunctions with non-negative eigenvalues. Below, we state a special case of the well known min-max formulae for λ_k. These are standard results from the aforementioned spectral theory.

Min-max formula (special case) : A formula for λ_k can obtained by minimizing the Rayleigh quotient $\frac{\langle g, Lg\rangle}{\|g\|^2}$ over the space of all functions in $\ell^2(V)$ that are orthogonal to the first $k-1$ eigenfunctions. i.e. given $k = 1, 2, \ldots, n-1$, let $S = span\{f_0, f_1, \ldots, f_{k-1}\}$ where f_i is the i^{th} eigenfunction of L then,

$$\lambda_k = \min_{g \perp S} \frac{\langle g, Lg\rangle}{\langle g, g\rangle} \tag{1.2}$$

Also note that for λ_{n-1}, the following also holds

$$\lambda_{n-1} = \max_{g \in \ell^2(V)} \frac{\langle g, Lg\rangle}{\langle g, g\rangle} \tag{1.3}$$

We will now study the null space of L.

Lemma 1.2 *1. The multiplicity of the* 0 *eigenvalue is the same for L and $\mathcal{L}$ and is equal to the number of connected components of G. In particular,* λ_0 *is always* 0*.*

2. f_0 is an eigenfunction of L corresponding to the eigenvalue 0 *if and only if $D^{1/2}f_0$ is an eigenfunction of $\mathcal{L}$ with eigenvalue* 0*.*

Proof: For the second part, we note that

$$\mathcal{L}(D^{1/2}f) = D^{-1/2}LD^{-1/2}D^{1/2}f = D^{-1/2}Lf$$

so it is clear that $Lf = 0$ *if and only if* $\mathcal{L}f = 0$. This also shows that the multiplicity of the zero eigenvalue is the same for L and $\mathcal{L}$.

For the first part, if G has k connected components then it's clear that there is a re-ordering of the vertices such that the matrix of L has a block diagonal form with k blocks. Since each block is itself the Laplacian for the subgraph of G containing vertices in that connected component, the columns of each block sum up to 0. In other words, the constant function on that component, extended to a function on V by making it zero outside that component, is an eigenfunction of L with eigenvalue 0. Therefore the multiplicity of the zero eigenvalue of L is at least k.

Further, from eq. (1.1), if $Lf_0 = 0$ then, $f_0(i) = f_0(j)$ for all $i \sim j$. This implies that f_0 has to be constant in each connected component of G. So f_0 is a linear combination of the k-many 0 eigenvalue eigenfunctions obtained earlier. This gives us the reverse bound for the multiplicity of the zero eigenvalue, proving the first part. $\square$

1.2.2 Normalised Laplacian

The normalised laplacian $\mathcal{L}$ is a mathematically nice object to work with as we can use it to compare eigenvalues of different graphs and obtain meaningful information. As we will show below, for any graph on n vertices, all eigenvalues of $\mathcal{L}$ are bounded above by 2.

First, we note that the quadratic form for $\mathcal{L}$ satisfies the following relation. For $g \in \ell^2(V)$, $g \not\equiv 0$ and $h := D^{-1/2}g$ we have,

$$\frac{\langle g, \mathcal{L}g \rangle}{\langle g, g \rangle} = \frac{\langle D^{-1/2}g, LD^{-1/2}g \rangle}{\langle g, g \rangle} = \frac{\langle h, Lh \rangle}{\langle D^{1/2}h, D^{1/2}h \rangle} = \frac{\sum_{i \sim j} \big(h(i) - h(j)\big)^2}{\sum_{i=1}^{n} h^2(i) d_i} \tag{1.4}$$

Lemma 1.3

1. *For a graph which is not a complete graph, we have* $\lambda_1 \leq 1$.
2. *For* $n \geq 2$,
   - $\lambda_1 \leq \frac{n}{n-1}$ *(with equality if and only if G is the complete graph on n vertices).*
   - $\lambda_{n-1} \geq \frac{n}{n-1}$ *(for a graph G without isolated vertices).*
3. $\lambda_{n-1} \leq 2$ *(with equality if and only if a nontrivial connected component of G is bipartite).*

Proof:

1. Since G is not complete, we have vertices i and j such that $i \not\sim j$. Define the function $f \in \ell^2(V)$

$$f(k) = \begin{cases} d_j & \text{if } k = i \\ -d_i & \text{if } k = j \\ 0 & \text{Otherwise} \end{cases}$$

Note that $\langle f, D\mathbb{1}\rangle = 0$. Now, using eq. (1.2) for λ_1 along with eq. (1.4), for $h \in \ell^2(V)$

$$\lambda_1 = \min_{h \perp D\mathbb{1}} \frac{\sum\limits_{i\sim j} \big(h(i)-h(j)\big)^2}{\sum\limits_{i=1}^{n} h^2(i) d_i} \le \frac{\sum\limits_{i\sim j} \big(f(i)-f(j)\big)^2}{\sum\limits_{i=1}^{n} f^2(i) d_i} = 1$$

where $\mathbb{1}$ is the constant vector of 1's and $D\mathbb{1}$ is the zero eigenvalue eigenfunction of $\mathcal{L}$.

2. - Using the fact that the sum of eigenvalues of $\mathcal{L}$ equals $\mathrm{tr}(\mathcal{L}) = n$, we get, $n = \sum\limits_{i=0}^{n-1} \lambda_i \ge \sum\limits_{i=1}^{n-1} \lambda_1 = (n-1)\lambda_1$. For equality observe that a for a complete graph on n vertices, $\mathcal{L}_n = \frac{n}{n-1} I - \frac{1}{n-1}\mathbb{1}$. From this it is clear that $\lambda_1 = \frac{n}{n-1}$. Finally, the first part ensures the *if and only if* condition.
   - Using $\mathrm{tr}(\mathcal{L}) = n$, we get that $n = \sum\limits_{i=0}^{n-1} \lambda_i \le \sum\limits_{i=1}^{n-1} \lambda_{n-1} = (n-1)\lambda_{n-1}$.

3. Using eq. (1.3) for λ_{n-1} and the fact that $\big(h(i)-h(j)\big)^2 \le 2\big(h^2(i)+h^2(j)\big)$, for all $\forall\ 0 \le i \le n-1$ we get,

$$\lambda_i \le \lambda_{n-1} = \max_{h \in \ell^2(V)} \frac{\sum\limits_{i\sim j} \big(h(i)-h(j)\big)^2}{\sum\limits_{i=1}^{n} h^2(i) d_i} \le \max_{h \in \ell^2(V)} \frac{\sum\limits_{i\sim j} 2\big(h^2(i)+h^2(j)\big)}{\sum\limits_{i=1}^{n} h^2(i) d_i} = 2 \tag{1.5}$$

Equality is achieved for non trivial h when $h(i) = -h(j)$, $\forall\ i \sim j$. Thus, there exists a connected component $\widetilde{G}$ of G that cannot have any odd cycle. This implies that $\widetilde{G}$ is bipartite. Finally, for a bipartite graph $G_{A,B}$ where A, B are the two independent vertex sets, the function $f \in \ell^2(V)$ s.t

$$f(i) = \begin{cases} 1 & \text{if } i \in A \\ -1 & \text{if } i \in B \end{cases}$$

is well defined and satisfies $f(i) = -f(j)$ for each $i \sim j$. So $\lambda_{n-1} = 2$.

□

1.3 Relationship between Graph Properties and Laplacian

We have already seen some examples of how the eigenvalues of the Laplacian encode information about G. We wish to delve deeper and exhibit more results of the same type, this time also touching upon the properties of the eigenfunctions of L and $\mathcal{L}$. We will study the counting problem for the number of spanning tree subgraphs of G, Cheeger's inequality and Courant's nodal domain theorem [2].

1.3.1 Number of Spanning Tree Sub graphs

Definition (Spanning tree subgraph) *Given a graph $G = (V, E)$, the subgraph $G_S = (V, E_S)$, $E_S \subseteq E$ is a spanning tree subgraph of G if it is a tree. i.e. if G_S is connected and $|E_S| = n - 1$.*

Theorem 1.4 (Kirchhoff's Theorem) *The number of spanning tree sub graphs of a graph G(V,E) is given by* $\det\left(L(n|n)\right)$ *where $L(n|n)$ is the matrix L without its last row and column.*

Proof: [Proof Sketch] We first define the vertex-edge adjacency matrix.

Definition (Vertex-Edge Adjacency Matrix) *Given an (arbitrary but fixed) ordering of the edges of G we represent the edges by e_k , $k \in \{1, 2, 3, \dots, |E|\}$. The Vertex-Edge Adjacency Matrix, represented by B is a $V \times E$ matrix that satisfies*

$$B_{i,j} = \begin{cases} 1 & \textit{if } e_j = (i,k) \textit{ for some } k \in V\ ; \\ -1 & \textit{if } e_j = (k,i) \textit{ for some } k \in V\ ; \\ 0 & \textit{Otherwise}\ ; \end{cases}$$

It can be shown that $L = BB^T$. Observe then, that $L(n|n) = B(n)B(n)^T$. Where $B(n)$ is the matrix B without its n^{th} row. Applying the Cauchy-Binet formula to $B(n)B(n)^T$ we get,

$$\det\left(B(n)B(n)^T\right) = \sum_{\substack{S \subset \{1,2,\dots,|E|\} \\ |S| = n-1}} \det(B_S)^2$$

[2]The analogues of these two theorems will be studied in the later chapters in both the $\dim(1)$ as well as the $\dim(n)$.

where B_S is the submatrix of $B(n|n)$ whose columns are the columns of $B(n|n)$ at indices from S. The final step is to show that

$$\det(B_S) = \begin{cases} \pm 1 & \text{if } G_S \text{ is a tree} \quad ; \quad \text{where } G_S = G(V, E_S = \{e_k \mid k \in S\}) \\ 0 & \text{Otherwise} \end{cases}$$

A nice way to think about this would be to notice that $L_S = B_S B_S^T = L_{G_S}(n|n)$. If G_S is not a tree then it is not connected since the only connected graph on $n-1$ edges and n vertices is a tree. This means that the zero eigenvalue of the Laplacian of G_S has multiplicity at least two. Upon removing the last column and row of L_{G_S}, we still have at least one zero eigenvalue for $L_{G_S}(n|n)$ since it contains at least one connected component with all vertices present. Finally, if G_S is a tree, note that $\det(B_S) = \sum_{\sigma \in S_{n-1}} \text{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{(n-1),\sigma(n-1)}$ where $a_{i,j} = \left(B_S\right)_{i,j}$. With this expression in mind, observe that since $\left(B_S\right)_{i,j} = 0 \textit{ or } 1$, a nonzero contribution to the summation arises when there is a perfect matching of the $(n-1)$ edges to the $(n-1)$ vertices, where vertices can be matched only with edges that they are part of. Given an arbitrary tree on n vertices we can see that the *only* possible matching for the edges with one endpoint as vertex n is to match them to the neighboring vertices. This process can be repeated by noting that the only possible matching for any unmatched edges with one endpoint being a neighbor of n, is to the other endpoint and so on. Since at each step there is only one choice for a given matching we conclude that the matching is unique. Further, a tree has no cycles, we can be sure that this process will not end up in a contradiction. So, $\det(B_S) = \pm 1$. □

Examples: We explicitly calculate the number of spanning tree subgraphs for two specific examples.

- (Cayley) Complete graph K_n :

  Let the corresponding Laplacian be L. Then, observe that

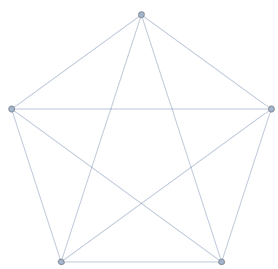
  $$L = nI_{n \times n} - \mathbb{1}_{n \times n}$$

  where I is the identity matrix and $\mathbb{1}$ is the matrix with 1's in all entries. So, we get

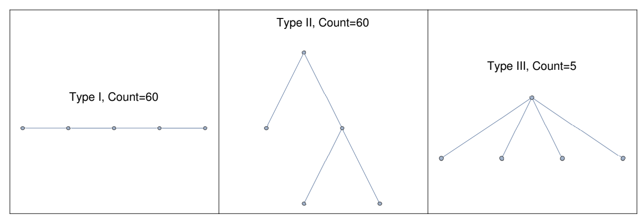
  $$L(n|n) = nI_{(n-1)\times(n-1)} - \mathbb{1}_{(n-1)\times(n-1)} \tag{1.6}$$

  The matrix $\mathbb{1}_{(n-1)\times(n-1)}$ has $n-1$ eigenvalues out of which $n-2$ are 0 and the last one is $n-1$. From eq. (1.6) we can see that the eigenvalues of $L(n|n)$ are of the type $n - \lambda$ where λ is an eigenvalue of $\mathbb{1}$. Using the formula for the determinant in terms of eigenvalues, we see that $\det(L(n|n)) = n^{n-2}$.

Spanning tree subgraphs of K_5 : From Cayley's theorem, we know that the number of spanning tree subgraphs for K_5 is $5^{(5-2)} = 125$. The figure below shows the three types of unlabeled trees that can exist. For each tree type $(I, II$ and $III)$ a simple combinatorial argument gives us the number of labeled spanning tree subgraphs of that type, represented by the variable 'Count'. $(60 + 60 + 5 = 125)$



(a) Diagram of the graph K_5



(b) All unlabelled spanning trees of (K_5) are of these three types

- Complete bipartite graph $K_{n,m}$: A complete bipartite graph $K_{n,m} = (V, E)$ is defined by $V = V_n + V_m$ where $|V_n| = n$, $V_m = m$ and $i \sim j$ only if $i \in V_n,\ j \in V_m$. i.e. the edge set comprises only of all possible edges between two partitions of the vertex set of size m and n respectively. Let the corresponding Laplacian be L. Then, observe that

$$L(n|n) = \begin{bmatrix} nI_{m\times m} & -\mathbb{1}_{m\times(n-1)} \\ -\mathbb{1}_{(n-1)\times m} & mI_{(n-1)\times(n-1)} \end{bmatrix}$$

Using the block matrix formula

$$\begin{bmatrix} A_{m\times m} & B_{m\times(n-1)} \\ C_{(n-1)\times m} & D_{(n-1)\times(n-1)} \end{bmatrix} \begin{bmatrix} I_{m\times m} & 0_{m\times(n-1)} \\ \big(-D^{-1}C\big)_{(n-1)\times m} & I_{(n-1)\times(n-1)} \end{bmatrix} = \begin{bmatrix} \big(A - BD^{-1}C\big)_{m\times m} & B_{m\times(n-1)} \\ 0_{(n-1)\times m} & D_{(n-1)\times(n-1)} \end{bmatrix}$$

we get

$$\det \begin{bmatrix} A_{m\times m} & B_{m\times(n-1)} \\ C_{(n-1)\times m} & D_{(n-1)\times(n-1)} \end{bmatrix} \cdot 1 = \det(D) \cdot \det\Big(A - BD^{-1}C\Big)$$

replacing A, B, C and D with their corresponding counterparts in $L(n|n)$, we get

$$\begin{aligned}\det(L(n|n)) &= \det\Big(mI_{(n-1)\times(n-1)}\Big) \cdot \det\left(nI_{m\times m} - \mathbb{1}_{m\times(n-1)}\tfrac{1}{m}I_{(n-1)\times(n-1)}\mathbb{1}_{(n-1)\times m}\right) \\ &= m^{n-1} \cdot \det\Big(nI_{m\times m} - \tfrac{(n-1)}{m}\mathbb{1}_{m\times m}\Big) \\ &= m^{n-1}(n^{m-1}(n - \tfrac{n-1}{m} \cdot m)) = m^{n-1}n^{m-1}\end{aligned}$$

Remark *For $K_{n,m}$, $m = n$, the number of spanning tree subgraphs is $n^{2n-2} = \left(\frac{1}{2}\right)^{2n-2} 2n^{2n-2}$, where we note that $2n^{2n-2}$ is the number of spanning trees of K_{2n}.*

1.3.2 Cheeger's Inequality

Consider a connected graph $G = (V, E)$ with normalised Laplacian $\mathcal{L}$. Note that for a connected graph, 0 is a simple eigenvalue and so $\lambda_1 > 0$. Intuitively, if we could choose a sequence of graphs $G_k(V_k, E_k)$, $|V_k| = n_k$, $\forall k$ such that $\{\lambda_1^k\}_{k\in\mathbb{N}} \to 0$, then we would expect them to *approach* a disconnected graph. A quantitative measure of this is the Cheeger constant of the graph h_G which is given by

$$h_G = \min_{S,S\subseteq V} \left(\frac{|E(S, \overline{S})|}{\min(\text{vol } S, \text{vol } \overline{S})} \right) \tag{1.7}$$

where, $\overline{S} = V \setminus S$, vol $S = \sum\limits_{v\in S} \deg(v)$ and $E(S, \overline{S}) = \sum\limits_{(v_i,v_j)\in S\times\overline{S}} \mathbf{1}_{[(v_i,v_j)\in E]}$.

Cheeger's constant h_G can be thought of as a measure of the isoperimetric profile of the graph[3].

Remark *If h_G is small, this means that the graph can be split into two nearly disjoint components. From theorem 1.5 we see that this means λ_1 will also be small, confirming our intuition.*

Before we prove this theorem, let us consider an example of a '*nearly*' disconnected graph.

[3] Here, the 'perimeter' can be thought of as the number of edges between S and $\overline{S}$ and the 'volume' can be understood as the degree sum of vertices in S. So, we try and minimize the number of edges between S and $\overline{S}$ while simultaneously maximizing the number of vertices in both.

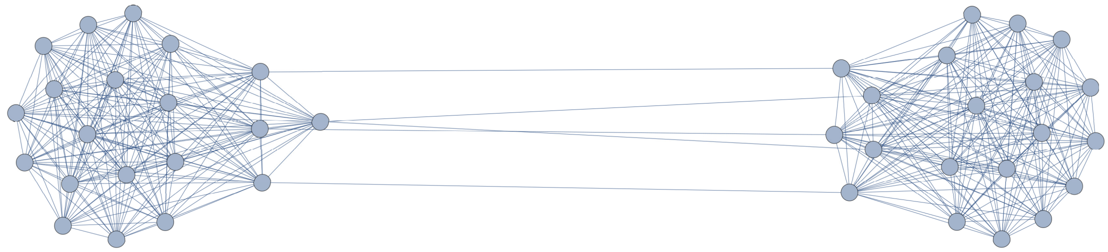


Fig. 1.2 A graph union of two K_{20} graphs by adding 5 edges between them.

A quick calculation using eq. (1.7) yields that $h_G \leq \frac{5}{20\cdot 19} \approx 0.013$. This makes sense because intuitively, h_G tries to find the 'weakest' partition of a graph and the example above has been designed to make it obvious which partition that is.

Theorem theorem 1.5 below, shows the explicit relationship between h_G and λ_1.

Theorem 1.5 (Cheeger Inequality) *For a connected graph G(V,E):* $2h_G \geq \lambda_1 > \frac{h_G^2}{2}$.

Proof: We first prove the upper bound namely $2h_G \geq \lambda_1$. Let $A \subseteq V$ be such that $h_G = \frac{|E(A,\overline{A})|}{\min(\text{vol } A, \text{vol } \overline{A})}$. Then consider the function

$$f(i) = \begin{cases} 1/\text{vol } A & \text{if } i \in A \\ -1/\text{vol } \overline{A} & \text{if } i \in \overline{A} \end{cases}$$

using the special case of the min-max formula, i.e. eq. (1.2), for the first eigenvalue of the normalized Laplacian along with eq. (1.4), we get[4]

$$\begin{aligned} \lambda_1 = \min_{g \perp D\mathbb{1}} \frac{\sum\limits_{i\sim j} \left(g(i) - g(j)\right)^2}{\sum\limits_{i=1}^{n} g^2(i) d_i} &\leq \frac{\sum\limits_{i\sim j} \left(f(i) - f(j)\right)^2}{\sum\limits_{i=1}^{n} f^2(i) d_i} \\ = |E(A,\overline{A})| \left(\frac{1}{\text{vol } A} + \frac{1}{\text{vol } \overline{A}} \right) &\leq \frac{2E(A,\overline{A})}{\min(\text{vol } A,\ \text{vol } \overline{A})} = 2h_G \end{aligned}$$

where $\mathbb{1}$ is the constant vector of 1's (eigenfunction of the $0 = \lambda_0$ eigenvalue.

For the lower bound, we need to do some work. To begin with, let g be an eigenfunction of $\mathcal{L}$ corresponding to λ_1. We re-order the vertices in V such that $g(1) \geq g(2) \geq g(3) \geq \cdots \geq g(n)$. Then, we define $S_i = \{1, 2, , 3, \ldots, i\}$. Let r be the largest index such that vol $S_r \leq (\text{vol } G)/2$. We define $g_\pm : V \to \mathbb{R}$ s.t.

[4] Note that f as defined above is orthogonal to $D\mathbb{1}$

$$g_+ = \begin{cases} g(i) - g(r) & ;\text{if } g(i) \geq g(r) \\ 0 & ;\text{Otherwise} \end{cases} \qquad g_- = \begin{cases} |g(i) - g(r)| & ;\text{if } g(u) \leq g(r) \\ 0 & ;\text{Otherwise} \end{cases}$$

Using eq. (1.2) along with eq. (1.4) for λ_1 we get

$$\lambda_1 = \frac{\sum_{i\sim j} \big(g(i) - g(j)\big)^2}{\sum_{i=1}^{n} g^2(i) d_i} \tag{1.8}$$

Let us deal with the numerator and denominator of eq. (1.8) seperately. Numerator first,

$$\begin{aligned} \sum_{i\sim j} \big(g(i) - g(j)\big)^2 &= \sum_{i\sim j} \big(g(i) - g(r) + g(r) - g(j)\big)^2 = \sum_{i\sim j} \big(g_+(i) - g_-(i) - g_+(j) + g_-(j)\big)^2 \\ &= \sum_{i\sim j} \big(g_+(i) - g_+(j)\big)^2 + \big(g_-(i) + g_-(j)\big)^2 \end{aligned}$$

the last step is because $g_+(i)$ and $g_-(i)$ cannot be nonzero simultaneously. All that remains therefore is to check the inequality holds for the four cases that are $g_\pm(i) = g_\pm(j) = 0$, one for each possible sign pair. For the denominator of eq. (1.8), note that since $g \perp \mathbb{1}$, we see that $\sum_{i=1}^{n} g^2(i) d_i = \min_{c\in\mathbb{R}} \big(g(i) - c\big)^2 d_i \leq \sum_{i=1}^{n} \big(g(i) - g(r)\big)^2 d_i = \sum_{i=1}^{n} \big(g_+^2(i) - g_-^2(i)\big) d_i$.

Combining the inequalities for the numerator and denominator, we get,

$$\lambda_1 \geq \frac{\sum_{i\sim j} \big(g_+(i) - g_+(j)\big)^2 + \sum_{i\sim j} \big(g_-(i) - g_-(j)\big)^2}{\sum_{i=1}^{n} g_+^2(i) d_i + \sum_{i=1}^{n} g_-^2(i) d_i}$$

Since, for $a, b, c, d > 0$ we have $\frac{a+b}{c+d} \geq \min\{\frac{a}{b}, \frac{c}{d}\}$, we can, without loss of generality,[5] assume

$$\lambda_1 \geq \frac{\sum_{i\sim j} \big(g_+(i) - g_+(j)\big)^2}{\sum_{i=1}^{n} g_+^2(i) d_i} \tag{1.9}$$

[5]since the identical process is followed for g_- with some minor obvious sign changes

We make some simplifying definitions at this stage.

$$\begin{aligned} \mathrm{vol}_m\ A &:= \min(\mathrm{vol}\ A, \mathrm{vol}\ G - \mathrm{vol}\ A) \\ \partial(S_i) &:= E(S_i, \overline{S_i}) \\ \alpha_G &:= \min_i h_{S_i} \ \ ; \ \ \text{where } h_{S_i} = \frac{\partial S_i}{\mathrm{vol}_m\ S_i} \end{aligned} \tag{1.10}$$

Next, we multiply and divide eq. (1.9) by $\sum_{i\sim j}\left(g_+(i)+g_+(j)\right)^2$. Again, we deal with the numerator and denominator separately

$$\sum_{i\sim j}\left(g_+(i)-g_+(j)\right)^2 \cdot \sum_{i\sim j}\left(g_+(i)+g_+(j)\right)^2 \underbrace{\geq}_{\text{Cauchy-Schwarz}} \left(\sum_{i\sim j}\left|g_+^2(i)-g_+^2(j)\right|\right)^2$$

note that for $i<j$,

$$\sum_{i\sim j} g_+^2(i)-g_+^2(j) = \sum_{i\sim j}\left(g_+^2(i)-g_+^2(i+1)+g_+^2(i+1)-\cdots+g_+^2(j-1)-g_+^2(j)\right) \tag{1.11}$$

so for a fixed $i=i_0$, the number of terms of the type $\left[g_+^2(i_0)-g_+^2(i_0+1)\right]$ appearing in eq. (1.11) equals[6] $|\partial S_i|$. By definition eq. (1.10), $\partial S_i \geq \alpha_G \cdot \mathrm{vol}_m S_i$. Combining these results gives us

$$\begin{aligned} \left(\sum_{i\sim j}\left|g_+^2(i)-g_+^2(j)\right|\right)^2 &= \left(\sum_{i=1}^{n}\left(g_+^2(i)-g_+^2(i+1)\right)\alpha_G\cdot \mathrm{vol}_m S_i\right)^2 \\ &= \alpha_G^2\left(\sum_{i=1}^{n} g_+^2(i)\left(\mathrm{vol}_m S_i - \mathrm{vol}_m S_{i+1}\right)\right)^2 = \alpha_G^2\left(\sum_{i=1}^{n} g_+^2(i) d_i\right)^2 \end{aligned} \tag{1.12}$$

where the last step is because $(\mathrm{vol}_m S_{i+1} - \mathrm{vol}_m S_i) = d_i$. Combining all these results, we get

$$\alpha_G^2\left(\sum_{i=1}^{n} g_+^2(i) d_i\right)^2 \leq \sum_{i\sim j}\left(g_+(i)-g_+(j)\right)^2 \tag{1.13}$$

Let us now switch our attention to the denominator[7] of eq. (1.9). Note that

$$\sum_{i\sim j}\left(g_+(i)+g_+(j)\right)^2 + \sum_{i\sim j}\left(g_+(i)-g_+(j)\right)^2 = 2\sum_{i=1}^{n} g_+^2(i) d_i \tag{1.14}$$

[6]The number of edges that contain one endpoint in S_i and the other in $\overline{S_i}$, hence $= \partial S_i$

[7]which, as you can recall, we had multiplied and divided by $\sum_{i\sim j}\left(g_+(i)+g_+(j)\right)^2$

Using eq. (1.14), we get

$$\begin{aligned}\sum_{i=1}^{n} g_{+}^{2}(i)d_i \cdot \sum_{i\sim j}\big(g_{+}(i)+g_{+}(j)\big)^2 &= \sum_{i=1}^{n} g_{+}^{2}(i)d_i \cdot \left(2\sum_{i=1}^{n} g_{+}^{2}(i)d_i - \overbrace{\sum_{i\sim j}\big(g_{+}(i)-g_{+}(j)\big)^2}^{\text{positive}}\right)\\ &\leq \sum_{i=1}^{n} g_{+}^{2}(i)d_i \cdot 2\sum_{i=1}^{n} g_{+}^{2}(i)d_i\end{aligned}$$

Combining the results, we get

$$\sum_{i=1}^{n} g_{+}^{2}(i)d_i \cdot \sum_{i\sim j}\big(g_{+}(i)+g_{+}(j)\big)^2 \leq 2\left(\sum_{i=1}^{n} g_{+}^{2}(i)d_i\right)^2 \tag{1.15}$$

Finally, from eqs. (1.9), (1.13) and (1.15), we can conclude that

$$\lambda_1 \geq \frac{\sum_{i\sim j}\big(g_{+}(i)-g_{+}(j)\big)^2}{\sum_{i=1}^{n} g_{+}^{2}(i)d_i} \geq \frac{\alpha_G^2}{2} \geq \frac{h_G^2}{2}$$

Remark (Parallels to the continuous setting) *The discrete equivalent of* $\int |\nabla f|^2$ *is the first difference expression* $\sum_{i=1}^{n}\big(g_{+}(i)-g_{+}(j)\big)^2$ *where the corresponding 'gradient' operator* $\nabla : \ell^2(V) \to \ell^2(E)$ *acts on a function as* $\nabla f(i,j) = f(j) - f(i)$.

$$\begin{aligned}\int_{\Omega} \nabla f^2 &= \int_{0}^{\max f^2} dt \int_{f^2=t} dA\\ \sum_{i\sim j}\left|g_{+}^{2}(i)-g_{+}^{2}(j)\right| &= \alpha_G \sum_{i=1}^{n} g_{+}^{2}(i)\big(vol_m S_i - vol_m S_{i+1}\big)\end{aligned} \tag{1.16}$$

The proof in the continuum setting uses a corollary of the co-area formula. In the proof above, step eq. (1.12) is exactly that except in a discrete setting. They have been placed one below another in eq. (1.16) to highlight the similarities.

□

1.3.3 Courant's Nodal Domain Theorem

Courant's result about nodal domains in the continuum setting can be cast into a discrete form where we study the sign of an eigenfunction of L on V. First, we define nodal domains on graphs as

Definition (Nodal Domain) *For any $f \in \ell^2(V)$, a nodal domain of f is a maximally connected subset of V such that f is either strictly positive or strictly negative on it.*

Now we can formulate the discrete nodal domain theorem.

Theorem 1.6 (Courant's Nodal Domain) *Let L be the Laplacian of a connected graph with n vertices. Then any eigenfunction v_k corresponding to the k^{th} eigenvalue λ_k with multiplicity r has at most $k+r-1$ nodal domains.*

Proof: Let f be the k^{th} eigenfunction with eigenvalue λ_k. Suppose f has m nodal domains $S_1, \ldots, S_m$. We define the functions $\{w_j\}_{j=1,2,\ldots,m}$ as.

$$w_j(i) = \begin{cases} f(i) & i \in S_j \\ 0 & \text{Otherwise} \end{cases}$$

Then, note that $f(i) = \sum_{j=1}^{n} w_j(i)$ for $i \in V$. Since w_i's are mutually orthogonal by design[8], after a normalization, $\{w_i\}_{i=1,2,\ldots,m}$ form an orthonormal basis for an m dimensional subspace. By the usual dimension argument[9], we claim $span\{w_1, w_2, \ldots, w_m\} \cap span\{v_1, \ldots, v_{m-1}\}^{\perp} \neq 0$, where v_i is the i^{th} eigenfunction of L. Let a non-zero function lying in the intersection be g and let it have unit norm. Then, $g(i) = \sum_{j=1}^{n} c_j w_j(i)$ where $c_j \in \mathbb{R}$ for all $j = 1, \ldots, n$ such that $\sum_{j=1}^{n} c_j^2 = 1$. Since $\langle g, v_i \rangle = 0$ for $i = 1, 2, \ldots, m-1$, we have, by the minimax formulae $\langle g, Lg \rangle \geq \lambda_m$.

An elementary calculation shows that

$$\langle g, Lg \rangle - \lambda_k \langle g, g \rangle = -\frac{1}{2} \sum_{i,j=1}^{n} (c_i - c_j)^2 w_i^T L w_j \tag{1.17}$$

Finally, we note that each term in the sum $\sum_{i,j=1}^{n} w_i^T L w_j$ is zero when there is no edge in E with endpoints in S_i and S_j respectively[10]. If on the other hand, S_i and S_j have an edge between them then obviously f has different signs on each subset. In such a case, $w_i^T L w_j$ will have a positive sign because only the off-diagonal terms of L will be used

[8] i.e. w_i's have disjoint domains.

[9] Since $dim(\{v_i\}_i^{\perp}) = n - m + 1$ and $dim(\{w_i\}_i) = m$ if these two sets don't intersect then taking union, we get a $n+1$ sized linearly independent set. This is a contradiction to the assumption $|V| = n$.

[10] In other words, $w_i^T L w_j$ is zero when if the subgraph graph G' with vertex set $V = S_i \cup S_j$ has two connected components.

in the calculation. Thus, $w_i^T L w_j$ is either zero or has a positive sign. This gives us

$$\langle g, Lg \rangle - \lambda_k \leq 0 \implies \lambda_m \leq \langle g, Lg \rangle \leq \lambda_k$$

If λ_k has multiplicity r then $\lambda_m \leq \lambda_{k+r-1}$ since the eigenvalues are arranged in an ascending order, we get $m \leq k + r - 1$, proving the theorem. $\square$

Chapter 2

The Stürm-Liouville Operator

2.1 Introduction

The Stürm-Liouville operator, can be thought of as the continuum analogue of Jacobi matrices[1], as mentioned in the previous section. The operator equation arose from studying the solutions to the Stürm-Liouville differential equation, given below

$$-(p(x)f^{'})^{'} + (q(x) - \lambda r(x))f = 0, \quad \lambda \in \mathbb{R}, \ x \in I = (a,b), \tag{2.1}$$

where,

$$p(x) \in C^1([a,b],\mathbb{R}), \ \ q(x), r(x) \in C^0([a,b],\mathbb{R}), \quad p(x), r(x) > 0, \ x \in [a,b]. \tag{2.2}$$

This form of the 2^{nd} order ODE is of very common occurrence in problems of mathematical physics, which is the origin of it's special importance. A typical example of the occurrence of 2.1 is the wave equation in physics.

The classic wave equation for the motion of a string in one dimension is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$. To solve it one starts with the variable separable ansatz $u(x,t) = f(x)h(t)$. Plugging it into the equation, we get $f''(x) = c^2 \lambda f(x)$ and $h''(t) = \lambda h(t)$. Observe that this is exactly the form for 2.1, for the case of $r(x) = p(x) \equiv 1$ and $q(x) \equiv 0$ i.e. $(f^{''} + \lambda f) = 0$. This equation can be explicitly solved by applying the required boundary conditions. Under Dirichlet boundary conditions (i.e., $f(a) = f(b) = 0$), and taking $[a,b] = [0,h]$, $(h \in \mathbb{R}$, $h > 0)$ for simplicity. The solutions are

$$\begin{cases} u_n(x) = \sin\left(\frac{n\pi}{h}x\right) & ; \text{where } u_n \text{ are eigenfunctions} \\ \lambda_n = \frac{\pi^2 n^2}{h^2} & ; \text{where } \lambda_n \text{ are eigenvalues} \end{cases} \tag{2.3}$$

[1] recall the *Second Difference Property* from the previous section pertaining to the graph Laplacian.

Remark *Note that the solution to the wave equation requires applying different boundary conditions to f and h, solving the respective pde's and finally taking the product of the resultant solutions, which has the general form of the product of sums of sines and cosines. But we will not get into that here.*

2.2 The Stürm-Liouville Operator

More generally, we cannot solve eq. (2.1) precisely like we did with the toy example above. So,we try to extract information about the solutions by treating eq. (2.1) as an operator eigenvalue differential equation. We define the Stürm-Liouville operator as

$$L := \frac{1}{r(x)}\Big(- \frac{d}{dx}p(x)\frac{d}{dx} + q(x)\Big)$$

We define the ambient space to be $C([a,b])$ where $(a, b < \infty)$. We chose the inner product defined by $\langle f, g\rangle_r = \int_a^b f(x)g(x)r(x)dx$. Note that $C([a,b])$ is not a Hilbert space. Since the spectral theorem is valid only for Hilbert spaces we will implicitly consider its completion, $L^2_r([a,b])$.

Remark *$\langle f, g\rangle_r$ is equivalent to $\langle f, g\rangle_{L^2([a,b])}$ since*

$$m\langle f, g\rangle_{L^2([a,b])} \leq \langle f, g\rangle_r \leq M\langle f, g\rangle_{L^2([a,b])} \tag{2.4}$$

where m and M are the minimum and maximum value of $r(x)$ on $[a,b]$. So we can use these two norms interchangeably.

As an operator, $L : \mathfrak{H} \to C([a,b])$ where $\mathfrak{H} := \{f \in C^2([a,b]) \mid f(a) = f(b) = 0\}$, i.e. functions in $\mathfrak{H}$ satisfy Dirichlet boundary conditions. We consider the Dirichlet boundary condition only because it is technically simpler to work with but the steps covered here work for the more general *Robin* boundary condition[2].

Remark *It can be shown that $\mathfrak{H}$ is dense in $C([a,b])$. This shows us that $\mathfrak{H}$ is dense in $L^2([a,b])$ (and therefore dense in $L^2_r([a,b])$ by eq. (2.4)). So the adjoint of L i.e. L^{Adj} is well defined.*

[2]Where f satisfies $0 = \cos(\alpha)f(a) - \sin(\alpha)p(a)f'(a)$ at a and $0 = \cos(\beta)f(b) - \sin(\beta)p(b)f'(b)$ at b. Note that in this type of condition the values of f at separate endpoints satisfy separate boundary conditions. This is an important fact and is called the 'separated endpoint condition'. Note that in cases where the boundary conditions aren't separated, we may not have simplicity of eigenvalues of L like we do here.

One property of L that will be useful is

Lemma 2.1 *L is symmetric $\langle f, Lg\rangle = \langle Lf, g\rangle \ \forall f, g \in \mathfrak{H}$.*

Proof: From the remark above we see that L^{Adj} is well defined. From the definition of L,

$$\begin{aligned}
\langle Lf, g\rangle_r &= \int_a^b g \cdot Lf \cdot r \\
&= \int_a^b g(x) \cdot \Big(-\big(p(x)f'(x)\big)' + q(x)f(x) \Big) \\
&= \big[-g(x)p(x)f'(x)\big]_a^b + \int_a^b g'(x)p(x)f'(x) + \int_a^b q(x)g(x)f(x) \\
&= 0 + \big[-g'(x)p(x)f(x)\big]_a^b - \int_a^b f(x)\big(p(x)g'(x)\big)' + \int_a^b q(x)g(x)f(x) \\
&= \int_a^b f \cdot Lg \cdot r = \langle f, Lg\rangle_r
\end{aligned}$$

□

We want to study the eigenvalues of L, which will give us solutions to the homogeneous problem eq. (2.1). Unfortunately for us, L is unbounded as is easily shown in the case when $L = \frac{d^2}{dx^2}$ (for $r \equiv 1, p \equiv -1$, $q \equiv 0$). For the bounded function sequence $f_n(x) = \sin(nx)$, $n = 1, 2, 3, \ldots$ we have $Lf_n(x) = n^2 \sin(nx)$ so $\|Lf_n(x)\|_{L^2([a,b])} \to \infty$. Another way to get information about the eigenvalues of L is to look at L^{-1}, if it is defined, since the eigenvalues of L and L^{-1} satisfy a simple inverse relation. However, we still don't know if L^{-1} exists, so instead we consider

$$(L - \lambda \mathbb{I})^{-1} \ ; \ \ \lambda \in \mathbb{R} \tag{2.5}$$

In order to talk about such an object, we first have to consider the equation $(L - \lambda\mathbb{I})f = g$. If a unique solution f to this equation exists for every g in $C([a, b])$, then we can talk about the inverse eq. (2.5).

Remark *It might not be clear as to why we are only considering real values for λ. We will see that L has only (countably many) real eigenvalues and the corresponding eigenfunctions are real valued too.*

In the sections below, we will try to find solutions (i.e. $f(x)$) to

$$\begin{aligned}
&\big(L - \lambda\mathbb{I}\big)f = g \\
\iff \ &\frac{d}{dx}p(x)\frac{d}{dx}f'(x) + \big(q(x) - \lambda r(x)\big)f(x) = g(x)r(x) \ \ ; \ \ \text{for } g \in C([a, b])
\end{aligned} \tag{2.6}$$

first as an initial value problem (IVP) with initial value at some x_0 and then as a boundary value problem (BVP) with Dirichlet boundary conditions.

2.3 ODE Theory

We will explore the rich theory of ODE and use important results from it in the context of eq. (2.1) to guarantee existence and uniqueness of solutions. We will apply this theory to show a unique solution $f \in \mathfrak{H}$ exists for each $g \in C([a,b])$ to the equation $(L - \lambda \mathbb{I})f = g$ for certain values of λ.

We first state a corollary of the famous Picard-Lindelöf theorem. (we state without proof as these are standard results in ODE theory.)

Theorem 2.2 (Existence and Uniqueness) *Consider the system of ordinary differential equations (inhomogeneous)*

$$\dot{Y}(x) = A(x)Y(x) + R(x) \tag{2.7}$$

with initial value $Y(x_0) = Y_0$ and for $x \in [a,b]$. Here $Y(x), R(x)$ are $n \times 1$ matrices and $A(x)$ is an $n \times n$ matrix with entries as functions of x. Further, if $A(x)$ and $R(x)$ are continuous on $[a,b]$ then there exists a unique continuous solution of eq. (2.7) for the initial value problem $Y(x_0) = Y_0$ that is defined on the whole interval $[a,b]$.

Lemma 2.3 *The solution to the homogeneous problem*

$$\dot{Y}(x) = A(x)Y(x) \tag{2.8}$$

for the initial condition $Y(x_0) = Y_0$ is given by $\Pi(x, x_0)Y_0$. Where $\Pi(x, x_0)$ is called the principal matrix solution for eq. (2.1). Note that the k^{th} column of Π is the solution to eq. (2.8) with initial condition $Y(x_0) = e_k$ where e_k is the vector that is 1 in the k^{th} position and zero otherwise.

Remark *Due to theorem 2.2 and the fact that for eq. (2.8) linear combinations of solutions are also solutions, we can see that any solution can be constructed as a linear combination of n solutions $Y_n(x)$ such that $Y_n(x_0)$ are linearly independent vectors in $\mathbb{R}^n$.*

An important property satisfied by $\Pi(x, x_0)$ that we will be using is

$$\Pi(x, x_1)\Pi(x_1, x_0) = \Pi(x, x_0) \;\; ; \; x_1, x_0 \in [a,b] \tag{2.9}$$

The proof of eq. (2.9) is relatively easy. Note that both the LHS and the RHS solve eq. (2.8) and are equal at $x = x_1$. From the uniqueness of solutions by theorem 2.2, we are done. A corollary of theorem 2.2 is that $\Pi(x, x_0)^{-1} = \Pi(x_0, x)$.

To solve the inhomogeneous system eq. (2.7) we use the ansatz $Y(x) = \Pi(x, x_0)C(x)$ where $Y(x_0) = \mathbb{I}C(x_0) = Y_0$. Then, from eq. (2.7) we get

$$
\begin{aligned}
& \dot{Y}(x) = A(x)Y(x) + \Pi(x, x_0)\dot{C}(x) \\
\implies & R(x) = \Pi(x, x_0)\dot{C}(x) \\
\implies & \dot{C}(x) = \Pi(x_0, x)R(x) \\
\implies & C(x) = Y_0 + \int_{x_0}^{x} \Pi(x_0, s)R(s) \\
\implies & Y(x) = \Pi(x, x_0)Y_0 + \int_{x_0}^{x} \Pi(x, s)R(s)
\end{aligned}
\tag{2.10}
$$

Finally, we will state Liouville's formula for the Wronski determinant of a matrix of functions.

For a matrix of functions $U(x) = [u_1(x), u_2(x), \ldots, u_n(x)]$, the Wronskian of U, given by $W(U)$ or $W(u_1(x), u_2(x), \ldots, u_n(x))$ equals $u_1(x), u_2(x), \ldots, u_n(x) \det(U)$. There is a very nice result that allows us to talk about the Wronskian of $\Pi(x, x_0)$, or any other linearly independent solution set for eq. (2.8), just as we would with real matrices. That is, we can prove that given a linearly independent solution set for eq. (2.8), it's Wronski determinant is never zero for any $x \in [a, b]$.

Lemma 2.4 *(Liouville's Formula) The Wronski determinant of n solutions $u_1(x), u_2(x), \ldots, u_n(x)$ of eq.* (2.7) *satisfies*

$$
W(x) = W(x_0) \exp\left(\int_{x_0}^{x} \operatorname{tr}(A(s))ds \right) \ ; \ \textit{where } W(x) = W(u_1(x), u_2(x), \ldots, u_n(x))
$$

theorem 2.4 tells us that it is enough to show that the Wronski determinant at any one point in $[a, b]$ is nonzero to ensure that it is non-zero for all $x \in [a, b]$.

2.4 The Resolvent R_L

We now turn our attention to eq. (2.1) with conditions eq. (2.2). Rewriting it in the form eq. (2.8) we get,

$$
\dot{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}} = \begin{bmatrix} 0 & \frac{1}{p(x)} \\ q(x) - \lambda r(x) & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.11}
$$

where $f(x) = y_1(x)$ and $p(x)f'(x) = y_2(x)$. Comparing eq. (2.11) with eq. (2.8), we get that $A(x) = \left(\begin{smallmatrix} 0 & 1/p \\ q-\lambda r & 0 \end{smallmatrix}\right)$ and is continuous. Thus, from section 2.3, specifically theorem 2.2, we can claim existence and uniqueness of a continuous solution defined on the entire interval $[a, b]$ for the eq. (2.1). Further, let the corresponding principle matrix solution for the IVP at x_0 be

$$\Pi(x, x_0) = \begin{bmatrix} c(x, x_0, \lambda) & s(x, x_0, \lambda) \\ p(x)c'(x, x_0, \lambda) & p(x)s'(x, x_0, \lambda) \end{bmatrix}$$

An immediate result is given two linearly independent solutions u, v of eq. (2.11) the Wronskian $W_\lambda(u, v)$ is only a function of the parameter λ, i.e. independent of x. Indeed,

$$W_\lambda(u(x), v(x)) = W_\lambda(u(x_0), v(x_0)) \exp\left(\int_{x_0}^{x} 0\right)) = W_\lambda(u(x_0), v(x_0)) \tag{2.12}$$

by Liouville's formula (theorem 2.4). Note that $W_\lambda(c, s) = W_\lambda(\Pi(x_0, x_0)) = 1$.

Since we want to study the operator equation $(L - \lambda r\mathbb{I})f = gr$, let us first solve the inhomogeneous equation eq. (2.7) where $R(x) = (0, g(x)r(x))^T$. Using eq. (2.10) we get

$$\begin{aligned} y_1(x) &= c(x, x_0, \lambda)y_1(x_0) + s(x, x_0, \lambda)y_2(x_0) + \int_{x_0}^{x} s(x, x_0, \lambda)g(x)r(x) \\ \implies f(x) &= c(x, x_0, \lambda)f(x_0) + s(x, x_0, \lambda)p(x_0)f'(x_0) + \int_{x_0}^{x} s(x, t, \lambda)g(t)r(t)ds \end{aligned} \tag{2.13}$$

2.4.1 Eigenfunction and Eigenvalue Properties of L (Dirichlet)

So far, we have solved the IVP eq. (2.6), We now turn our attention to the corresponding BVP. It is not true that for every λ we can get a solution to eq. (2.6) for all $g \in C([a, b])$. We have to prove that such a λ exists so that $(L - \lambda\mathbb{I})f = g$ is a bijective map. To do that, we first prove some properties about the eigenvalues of L and the Wronskian W_λ.

Lemma 2.5 *For any two functions u, v solving the homogeneous IVP eq. (2.1), if u and v satisfy the same boundary conditions at a (for Dirichlet $u(a) = v(a) = 0$) then $W_\lambda(u, v) = 0$.*

Proof: Since $u(a) = v(a) = 0$, we trivially see that $(u(a), p(a)u'(a)) = k(v(a), p(a)v'(a))$ for $k = \frac{u'(a)}{v'(a)}$. Finally, along with eq. (2.12) we get, $W_\lambda(u, v) = \det(u(a), v(a)) = 0$. □

Corollary 2.5.1 *For two solutions u, v of eq. (2.1), $W_\lambda(u, v) = 0$ if and only if u is a multiple of v.*

Lemma 2.6 *There is at most one linearly independent eigenfunction corresponding to a single eigenvalue λ of L.*

Proof: Suppose not. Let u, v be linearly independent solutions of $(L - \lambda \mathbb{I})h = 0$. Then this means that u and v satisfy the same boundary conditions at a. By theorem 2.5 we get that $W_\lambda(u, v) = 0$ but this contradicts the linear independence of u and v. $\square$

Lemma 2.7 *The eigenvalues of L are real.*

Proof: Let λ be an eigenvalue of L with corresponding eigenfunction u i.e. $Lu(x) = \lambda u(x)$. Then, taking conjugates on both sides, we see $L\bar{u}(x) = \bar{\lambda}\bar{u}(x)$. This shows that $\bar{u}$ is also an eigenfunction of L with eigenvalue $\bar{\lambda}$. Consider the following

$$(\lambda - \bar{\lambda}) \int_a^b |u|^2 = \langle \bar{u}, Lu \rangle_r - \langle u, L\bar{u} \rangle_r = \left[\bar{u}'(x)p(x)u(x) - \bar{u}(x)p(x)u'(x)\right]_a^b = 0$$

$\int_a^b |u|^2 = 0$ implies $u = 0$ since $u \in C^2([a, b])$. But since we are only considering non trivial eigenfunctions, we conclude that $\bar{\lambda} = \lambda$ which means $\lambda \in \mathbb{R}$. $\square$

Corollary 2.7.1 *Up to a complex constant, for any eigenfunction u corresponding to eigenvalue λ, u is a real valued function.*

Proof: Suppose $u(x) = u_1(x) + iu_2(x)$. Since $Lu = \lambda u = L\bar{u} = \lambda\bar{u}$, we see that $u_1 := (u + \bar{u})/2$ is a real valued eigenfunction. By corollary 2.5.1 we see that $u(x) = zu_1(x)$ for some $z \in \mathbb{C}$. $\square$

Lemma 2.8 *Eigenfunctions corresponding to different eigenvalues are orthogonal.*

Proof: Let u, v be eigenfunctions corresponding to eigenvalues λ, μ then,

$$(\lambda - \mu) \int_a^b v(x)u(x)r(x) = \langle v, Lu \rangle_r - \langle u, Lv \rangle_r = \left[v'(x)p(x)u(x) - v(x)p(x)u'(x)\right]_a^b = 0$$

since $\lambda \neq \mu$ we get $\langle v(x), u(x) \rangle_r = 0$. $\square$

2.4.2 Solving The BVP

We will now try to solve the BVP eq. (2.6). First, note that if λ is an eigenvalue of L with corresponding eigenfunction u and if f solves the differential equation $(L - \lambda\mathbb{I})f(x) = g(x)$. Then

$$\langle u, g \rangle_r = \langle u, Lf - \lambda f \rangle_r = \langle Lu - \lambda u, f \rangle_r = \langle 0, f \rangle = 0$$

so we see that a solution f for g s.t. $\langle u, g \rangle \neq 0$ doesn't exist. But what about when λ is not an eigenvalue. Is $L - \lambda\mathbb{I}$ invertible then? The answer is affirmative in this case as we will see.

Let us choose λ such that it is not an eigenvalue[3]. Further, let u_a and u_b be linearly independent solutions to the IVP $Lv = \lambda v$ for some initial conditions at a. Note that the functions $c(x,a,\lambda), s(x,a,\lambda)$ from eq. (2.13), satisfy,

$$c(x,a,\lambda) = \frac{u_a(x)p(a)u_b'(a) - u_a'(a)p(a)u_b(x)}{W_\lambda(u_a,u_b)},\ s(x,a,\lambda) = \frac{u_a(a)u_b(x) - u_a(x)u_b(a)}{W_\lambda(u_a,u_b)}$$

Where,

$$W_\lambda(u_a(x),u_b(x)) = p(x)\det\left(\begin{bmatrix} u_a(x) & u_b(x) \\ u_a'(x) & u_b'(x) \end{bmatrix}\right)$$

Using eq. (2.13), we get

$$\begin{aligned} f(x) &= c(x,a,\lambda)f(a) + s(x,a,\lambda)f'(a) + \int_a^x s(x,t,\lambda)g(t)r(t)dt \\ \implies f(x) &= c_1u_a(x) + c_2u_b(x) + \int_a^x \frac{u_a(t)u_b(x) - u_a(x)u_b(t)}{W_\lambda(u_a,u_b)}g(t)r(t)dt \end{aligned}$$

for some constants c_1 and c_2. Now, let u_a satisfy $u_a = 0$ and u_b satisfy $u_b = 0$.

Remark *Note that this is possible since if this choice makes u_a and u_b linearly dependent then corollary 2.5.1 tells us that they are eigenfunctions of L which is a contradiction to our assumption that λ is not an eigenvalue.*

Applying the boundary conditions to f, we get

$$\begin{aligned} & f(a) = c_2u_b(a) = 0 \quad \implies c_2 = 0 \\ & f(b) = c_1u_a(b) - \int_a^b \frac{u_a(b)u_b(t)}{W_\lambda(u_a,u_b)}g(t)r(t)dt = 0 \implies c_1 = \int_a^b \frac{u_b(t)}{W_\lambda(u_a,u_b)}g(t)r(t)dt \\ \implies & f(x) = \int_a^b \frac{u_a(x)u_b(t)}{W_\lambda(u_a,u_b)}g(t)r(t)dt + \int_a^x \frac{u_a(t)u_b(x) - u_a(x)u_b(t)}{W_\lambda(u_a,u_b)}g(t)r(t)dt \\ & \quad = \int_a^x \frac{u_a(t)u_b(x)}{W_\lambda(u_a,u_b)}g(t)r(t)dt + \int_x^b \frac{u_a(x)u_b(t)}{W_\lambda(u_a,u_b)}g(t)r(t)dt \end{aligned}$$

[3] Some care needs to be taken while making this choice. We need to prove that there exists such a $\lambda \in \mathbb{R}$ that is not an eigenvalue. The proof uses the separability of $L^2([a,b])$. First, we use the fact that eigenfunctions of L are orthogonal and eigenvalues are simple to claim that the set of eigenvalues is in bijection with the set of eigenfunctions which is an orthogonal set in L^2. Due to separability, we conclude that any orthogonal set contains at most countably many elements.

We define the *Green's function* to be

$$G(x,t,\lambda) = \frac{1}{W_\lambda\left(u_a,u_b\right)} \begin{cases} u_a(t)u_b(x) & ;t \leq x \\ u_a(x)u_b(t) & ;t \geq x \end{cases}$$

So, we have solved the BVP eq. (2.6). To prove uniqueness for the solution, let u and v both be solutions. Then $u - v$ solves the homogeneous problem $Lf = \lambda f$ which means that $u - v$ is either a zero solution or is an eigenfunction which contradicts our assumption that λ is not an eigenvalue.

2.4.3 The Resolvent R_L^λ

Let us define the resolvent of $L - \lambda\mathbb{I}$ as the operator R_L^λ, satisfying $R_L^\lambda g(x) := \int_a^b G(x,t,\lambda)g(t)r(t)dt$[4]. We can show that $R_L^\lambda = (L - \lambda\mathbb{I})^{-1}$, ($\lambda \notin \rho(L)$[5]) and extends[6] as a bounded, self adjoint,compact operator over $L_r^2([a,b])$.

We first show that R_L^λ is actually the inverse of $L - \lambda\mathbb{I}$. Indeed $\left(L - \lambda\mathbb{I}\right) \circ R_L^\lambda g = g$ is true by construction and

$$\begin{aligned} R_L^\lambda \circ \left(L - \lambda\mathbb{I}\right) f(x) &= \int_a^b G(x,t,\lambda)\left(L - \lambda\mathbb{I}\right) f(t)r(t)dt \\ &= \frac{\int_a^x u_a(t)u_b(x)\left(L - \lambda\mathbb{I}\right) f(t)r(t)dt + \int_x^b u_a(x)u_b(t)\left(L - \lambda\mathbb{I}\right) f(t)r(t)dt}{W_\lambda\left(u_a,u_b\right)} \\ &= \frac{u_b(x)W_\lambda\left(f(x),u_a(x)\right) - u_a(x)W_\lambda\left(f(x),u_b(x)\right)}{W_\lambda\left(u_a,u_b\right)} = f(x) \end{aligned}$$

Lemma 2.9 *R_L^λ is a compact self-adjoint operator.*

Proof: Note that by definition, $G(x,t,\lambda)$ is a symmetric function in x and t. Moreover, it is continuous in x (as well as t) over a compact interval and hence uniformly continuous. Given a bounded sequence $\{g_n(x)\}_n$ in $L_r^2([a,b])$,

$$\begin{aligned} \left|R_L^\lambda\left(g_n(x) - g_n(y)\right)\right| &= \left|\int_a^b \Big(G(x,t,\lambda) - G(y,t,\lambda)\Big) g_n(t)r(t)dt\right| \\ &\leq \epsilon \cdot \sup_{t\in[a,b]} |r(t)| \cdot \int_a^b |g_n(t)|dt \end{aligned}$$

[4]By theorem 2.2 we see that R_L is well defined over all $g \in C([a,b])$

[5]Where $\rho(L)$ is the set of eigenvalues of L

[6]Using the bounded linear transformation theorem that states existence of a unique extension. In this case, $(L-\lambda)^{-1}$ extends from $C([a,b])$ to $L^2([a,b])$.

$$\leq \epsilon \cdot \sup_{t\in[a,b]} |r(t)| \cdot |b-a| \cdot \|g_n\|_{L^2_r([a,b])} \leq M\epsilon$$

where $|G(x,t,\lambda) - G(y,t,\lambda)| \leq \epsilon,\ \forall |x-y| \leq \delta$. Similarly we can see that $R^\lambda_L g_n$ is uniformly bounded. So we see that $\{R^\lambda_L g_n\}_n$ is an equicontinuous set of functions. By the well known theorem due to Arzelá-Ascholi, we can find a uniformly convergent sub sequence that is continuous on $[a,b]$. Finally, uniform convergence implies convergence in $L^2_r([a,b])$ norm, proving that R^λ_L is compact.

For self-adjoint, consider

$$\begin{aligned} = \langle R^\lambda_L g(x), f(x)\rangle_r &= \int_a^b f(x)\left(\int_a^b G(x,t,\lambda)g(t)r(t)dt\right)r(x)dx \\ &= \underbrace{\int_a^b g(t)\left(\int_a^b G(t,x,\lambda)f(x)r(x)dx\right)r(t)dt}_{\text{by Fubini-Tonelli}} \\ &= \langle g(x), R^\lambda_L f(x)\rangle_r \end{aligned}$$

Note that it suffices that $g, f \in L^2_r([a,b])$. So we see that R^λ_L is indeed self adjoint as it is symmetric and $\mathrm{Dom}(R^\lambda_L) = \mathrm{Dom}\big(\mathrm{Adjoint}(R^\lambda_L)\big)$. □

Remark *Finally, we remark that $R^\lambda_L g = 0$ implies $g \equiv 0$. This is true since for $g \in C([a,b])$ we have $0 = \big(L - \lambda\mathbb{I}\big)R^\lambda_L g = g$. For $g \in L^2([a,b])$ we take a sequence $\{g_n\}_n$ in $C([a,b])$ converging to g in L^2_r. Then, for any $f \in \mathfrak{H}$, we have*

$$\begin{aligned} \left|\langle f, g_n\rangle_r\right| = \left|\int_a^b (L-\lambda\mathbb{I})R^\lambda_L g_n(x)f(x)r(x)dx\right| &\leq \int_a^b \left|R^\lambda_L g_n(x)\big(L-\lambda\mathbb{I}\big)f(x)r(x)\right|dx \\ \leq \left\|R^\lambda_L g_n\right\|_{L^\infty}\left\|\big(L-\lambda\mathbb{I}\big)f(x)r(x)\right\|_{L^1} \to 0 \end{aligned}$$

since from lemma theorem 2.9, we can see $\left\|R^\lambda_L g_n - 0\right\|_{L^\infty} \leq K\|g_n - g\|_{L^2} \to 0$. Further, $\mathfrak{H}$ is dense in $L^2_r([a,b])$ so $g \equiv 0$. This result is important because it guarantees that R^λ_L has infinitely many non-zero eigenvalues.

2.4.4 Result

Applying the spectral theorem to R^λ_L, we get a sequence of real eigenvalues for R^λ_L that converge to zero. Moreover, the eigenfunctions form an infinite basis for the space $L^2_r([a,b])$. For an eigenvalue μ and corresponding eigenvector v of R^λ_L, we see $v = \big(L-\lambda\big)R^\lambda_L v = \mu\big(L - \lambda\mathbb{I}\big)v$ which gives us $Lv = \frac{(\lambda+1)}{\mu}v$. So the eigenvalues of L converge to ∞. More precisely, one can say this.

L has a countable number of discrete, real and simple eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ which accumulate only at ∞ and are bounded below. The corresponding eigenfunctions u_n $(n \in \mathbb{N})$ can be chosen real valued and form an orthonormal basis for $L^2_r([a,b])$.

Remark *The existence of a lower bound on the eigenvalues of L is shown by finding a lower bound on the quadratic form for L as follows,* $\langle u, Lu \rangle_r = \int_a^b -u(pu')' + qu^2 = \int_a^b pu'^2 + qu^2 \geq \int_a^b qu^2 \geq M\|u\|^2_{L^2_r}$ *where* $M = \sup_{x\in[a,b]} q(x)$*, which can take negative values too.*

Another important result about the eigenvalues of L is

Lemma 2.10 (Rayleigh-Ritz Principle)

$$\lambda_1 = \min_{f\in\mathfrak{H}:||f||=1} \langle f, Lf \rangle$$

Remark *This is a Min-Max*[7] *type result.*

Proof: Let u_k be the eigenfunction of L corresponding to the eigenvalue λ_k. For any $v \in \mathfrak{H}$, $\|v\|_{L^2_r} = 1$, we have $v = \sum_{i=1}^{\infty} \alpha_i u_i$ since $\{u_i\}_i$ forms a basis for $L^2_r([a,b])$. So, $\langle v, Lv \rangle_r = \sum_{i=1}^{\infty} \lambda_i \alpha_i^2 \geq \lambda_1$ with equality *if and only if* $v = u_1$. $\square$

2.5 Oscillation Properties

In this section, we will study the background material needed to understand two important theorems i.e. Weyl's asymptotics and Courants nodal domain theorem. The former studies the asymptotic properties of the eigenvalues of L and the latter studies the zero set of the eigenfunctions.

To begin with, we will perform a coordinate change which will help us study the eigenfunctions and eigenvalues of L better. Next, we will study Stürm's comparison theorem which will allow us to prove Courant's nodal domain theorem. Finally, we will work out some initial steps that will allow us to prove Weyl's asymptotic theorem.

[7] Similar to the Min-Max theorems for linear operators on finite dimensional vector spaces.

2.5.1 Prüfer Variables

We want to study the zeros of the eigenfunctions of L. A change of coordinates to the polar form will help in this case. Recall that a solution u of the IVP eq. (2.1) with $u(a) = 0$ and $p(a)u'(a) = 1$, has the vector form $[u(x), p(x)u'(x)]$. Changing to polar coordinates, let

$$\begin{aligned} u(x) &= \rho(x,\lambda)\sin(\theta(x,\lambda)) \\ p(x)u'(x) &= \rho(x,\lambda)\cos(\theta(x,\lambda)) \end{aligned} \tag{2.14}$$

Rewriting the equation $Lu = \lambda u$ using eq. (2.14) we get

$$\theta'(x,\lambda) = \frac{\cos^2(\theta(x,\lambda))}{p(x)} + (\lambda r(x) - q(x))\sin^2(\theta(x,\lambda)) \tag{2.15}$$

$$\rho'(x,\lambda) = \rho(x,\lambda)\left(\frac{1}{p(x)} + q(x) - \lambda r(x)\right)\cos(\theta(x,\lambda))\sin(\theta(x,\lambda)) \tag{2.16}$$

Here, we call θ and ρ as *Prüfer Variables*.

Remark *Under polar coordinates eq. (2.1) is reduced to two first order ODE's. Unfortunately we see that this simplification has resulted in the equations being non-linear. However, if we know θ then we can solve for ρ explicitly and get $\rho = c \cdot \exp\left(\frac{1}{2}\int_{x_0}^{x}(q - \lambda r + \frac{1}{p})\sin(2\theta)\right)$. So we can focus on studying θ since ρ is easily found once we know θ*

Note that since the *RHS* for both equations is clearly bounded as well as Lipshitz continuous in θ and ρ respectively, the *Picard-Lindelöf* theorem[8]guarantees us a unique solution for eq. (2.15) and eq. (2.16). Further, note that if θ is a solution to the IVP eq. (2.15) where $\theta(a,\lambda) = 0$ then so is $n\pi + \theta(x,\lambda)$. Finally we note that $\rho = \sqrt{u^2 + (pu')^2} > 0$ since if $\rho(x,\lambda) = 0$ for any $x \in [a,b]$, then $u \equiv 0$ as for that x we have $u(x) = 0$ and $u'(x) = 0$.

2.5.2 Stürm Comparison Theorems

In this section we will write $\theta(x,\lambda)$ as $\theta(x)$ for brevity because we are interested in properties of θ as a function of x for a fixed value of λ.

Before we take a look at the comparison theorems, we state the following lemma

Lemma 2.11 *$\theta(x) = m\pi$ has at most one solution for $x \in [a,b]$. Moreover $\forall x_3 < x_1 < x_2$ where $\theta(x_1) = m\pi$, we have $\theta(x_3) < \theta(x_1) < \theta(x_2)$.*

[8] The *Picard-Lindelöf* theorem states that for the differential equation $y'(x) = F(x,y)$ if F is bounded, Lipshitz in y, continuous in x and defined on a compact domain in $\mathbb{R}^2$, a unique solution y^* exists for some initial conditions.

Proof: Note that for an eigenfunction u, $u(x) = 0$ whenever $\theta(x, \lambda) = m\pi$. Let x_0 be such that $u(x_0) = 0$. Then, $\theta'(x_0) = p(x_0)^{-1} > 0$ by the definition of $p(x)$[9]. Now, suppose $\theta(x_1) = m\pi$ and $\theta(x_2) \leq m\pi$ where $x_1 < x_2$. Observe that $\theta(x^*) = m\pi$ where $x^* = inf\{x \mid x \in (x_1, x_2],\ \theta(x) \leq m\pi\}$. Note that because $\theta'(x_1) > 0$ we have $x^* \neq x_1$. By definition, $\theta(x) > m\pi$ for $x \in (x_1, x^*)$. This means that $\theta'(x^*) < 0$ which is a contradiction meaning that $\theta(x_2) > m\pi$.

Finally, if $\exists x'$ such that $\theta(x') = m\pi$ then by the above argument, $x' < x_1$. Applying the above argument using x' gives us the contradiction $\theta'(x_1) < 0$ leading to the conclusion that $\theta(x_3) < m\pi$ for $x_3 < x_1$. □

Corollary 2.11.1 *The number of zeros of u equals the number of multiples of π traversed by θ.*

For λ large enough, note that $\theta'(x) > 0 \ \ \forall x \in [a, b]$. So we see that for large enough eigenvalues, the eigenfunction of the larger eigenvalue oscillates much more. Meaning that we should expect u_k[10] to have more zeros as $k \to \infty$. This is exactly what we see from the theorems below. We also show interlacing properties of two neighboring eigenfunctions as we had mentioned in section **??** on Jacobi matrices.

Theorem 2.12 *Suppose y and z are the respective solutions to the differential equations*

$$\begin{aligned} y'(x) &= F(x, y) \\ z'(x) &= G(x, z) \end{aligned} \tag{2.17}$$

where $x \in [a, b]$. Further, suppose that $F(x, s) \geq G(x, s)$ where $F(x, s)$ and $G(x, s)$ are continuous with respect to x and Lipshitz continuous with respect to s with common Lipshitz constant M.

Then, if $y(a) \geq z(a)$ we have $y(x) \geq z(x)$, $\forall x \in [a, b]$.

Proof: Let $g(x) = z(a) - y(a)$. We want to show that $g(x) \leq 0$. By contradiction, let there exist $c \in (a, b)$ such that $g(c) > 0$. Then, let $x^* = \sup\{x \mid x \in [a, c],\ g(x) \leq 0\}$. By continuity of g, we get $g(x^*) = 0$. Moreover, by definition, $g(x) > 0$ for $x \in (x^*, c)$. From eq. (2.17), we get

$$z(x) - y(x) = \int_{x^*}^{x} \Big(G(s, z(s)) - F(s, y(s)) \Big) ds \ \ ; \ \ x \in (x^*, c)$$

$$\implies \qquad g(x) \leq \int_{x^*}^{x} \Big(G(s, z(s)) - G(s, y(s)) + \overbrace{G(s, y(s)) - F(s, y(s))}^{\text{always nonzero by assumption}} \Big) ds$$

[9] See eq. (2.1).

[10] Where u_k is that k^{th} eigenvalue of L.

$$\implies \qquad g(x) \leq \int_{x^*}^{x} \Big(G(s,z(s)) - G(s,y(s))\Big) ds$$
$$\implies \qquad g(x) \leq \int_{x^*}^{x} L|z(s)-y(s)|ds = L\int_{x^*}^{x} g(s)ds$$
$$\implies \qquad g'(x) \leq Lg(x) \implies \left(g(x)e^{-Lx}\right)' \leq 0 \implies g(x) \leq 0\,,\ x \in (x^*, c)$$

which is a contradiction. So we get $y(x) \geq z(x)$ for $x \in [a,b]$ □

Corollary 2.12.1 *Suppose for some $x^* \in [a,b]$ we have $y(x^*) = z(x^*)$. Then $y(x) = z(x)$ for all $x \in [a, x^*]$.*

Proof: Consider $\hat{y}(x) = y(x^* - x)$ and $\hat{z}(x) = z(x^* - x)$. Then the differential equations for $\hat{y}$ and $\hat{z}$ become

$$\hat{y}'(x) = -F(x^* - x, \hat{y}) \ ; \ \hat{z}'(x) = -G(x^* - x, \hat{z})$$

Note that $-G(x,s) > -F(x,s)$ and that $\hat{z}(0) \geq \hat{y}(0)$. Applying theorem 2.12 in this setting gives us $\hat{z}(x) \geq \hat{y}(x)$ for $x \in [0, x^* - a]$ but we already have $y(x) \geq z(x)$ for all $x \in [a,b]$. Hence, $z(x) = y(x)$ on $[a, x^*]$. □

Corollary 2.12.2 *If the inequality $F(x,s) \geq G(x,s)$ in theorem 2.12 was strict, i.e. $F(x,s) > G(x,s)$ then applying the theorem, we get $y(x) > z(x)$ for $x \in (a,b]$*

Let u and $\hat{u}$ be the non-trivial solutions to $(Pu')' + Qu = 0$ and $(Pu')' + \widehat{Q}u = 0$, where the corresponding solutions are $\hat{\theta}$ and θ are

$$\frac{d}{dx}\theta = \frac{\cos^2(\theta)}{P} + Q\sin^2(\theta) \tag{2.18}$$
$$\frac{d}{dx}\hat{\theta} = \frac{\cos^2(\hat{\theta})}{P} + \widehat{Q}\sin^2(\hat{\theta}) \tag{2.19}$$

We can finally state the Stürm comparison theorem.

Theorem 2.13 (Stürm Comparison Theorem) *Let u and $\hat{u}$ be as given above. Further, suppose $\hat{u}$ vanishes at two points x_1 and x_2 in $[a,b]$ and $Q \geq \hat{Q}$ on $[x_1, x_2]$. Then u vanishes at least once on this interval.*

Proof: Let $G(x,\hat{\theta}) = \frac{d}{dx}\hat{\theta}(x)$ and $F(x,\theta) = \frac{d}{dx}\theta(x)$. Then note that $F(x,s) \geq G(x,s)$ on $[x_1, x_2]$ and both F and G are Lipshitz in the second coordinate. Also note that $\hat{u}(x_1) = 0$ which means that $\hat{\theta}$ can be chosen such that $\hat{\theta}(x_1) = 0$ (since $m\pi + \hat{\theta}$ also solves eq. (2.19)). Similarly, $\theta(x_1)$ can be chosen to lie in $[0,\pi]$. So we get $\hat{\theta}(x_1) \leq \theta(x_1)$. Moreover note that $\hat{\theta}(x_2) = k\pi$ for some $k \in \mathbb{N} \setminus \{0\}$. Applying theorem 2.12 with

F and G as defined above, we get $\hat{\theta}(x) \le \theta(x)$ for all $x \in [x_1, x_2]$, which implies that $k\pi \le \theta(x_2)$. Since k is at least 1, we get $\pi \le \theta(x_2)$ meaning that for some $x \in [x_1, x_2]$ we have $u(x) = 0$. □

Corollary 2.13.1 *If $\hat{u}$ has k zeros on $[x_1, x_2]$ then u has at least $k-1$ zeros there.*

2.6 Courant's Nodal Domain Theorem

Before we state and prove Courant's theorem, we need to show certain properties of θ as a function of λ. Note that we can make equations eq. (2.18) and eq. (2.19) the same as eq. (2.15) by choosing

$$P = p\ ; \quad Q = \lambda r - q\ ; \quad \widehat{Q} = \hat{\lambda} r - q\ ; \tag{2.20}$$

Suppose u satisfies the IVP eq. (2.1) with $u(a) = 0$. Then from eq. (2.15) the corresponding $\theta(x, \lambda)$ is the solution to the differential equation eq. (2.18) with $Q = \lambda r - q$ and $\theta(a, \lambda) = 0$. We state the properties satisfied by θ as a function of λ in these three lemmas.

Lemma 2.14 *$\theta(x, \lambda)$ is a continuous and strictly monotonically increasing function of λ for each fixed $x \in (a, b]$*

Proof: Note that $\theta'(x, \lambda_1) < \theta'(x, \lambda_2)$ for $\lambda_1 < \lambda_2$. Also, $\theta(a, \lambda_1) = \theta(a, \lambda_2) = 0$. By theorem 2.12 and corollary 2.12.2, we get that $\theta(x, \lambda_1) < \theta(x, \lambda_2)$ for $x \in (a, b]$. Continuity comes from the fact that

$$\begin{aligned} \left|\theta(x, \lambda + \tfrac{\epsilon}{2M(b-a)}) - \theta(x, \lambda)\right| &\le \int_a^x \frac{\epsilon}{2M(b-a)} r \left|\sin^2\left(\theta(s, \lambda + \tfrac{\epsilon}{2M})\right) - \sin^2\left(\theta(s, \lambda)\right)\right| ds \\ &\le \epsilon \end{aligned}$$

where $M = \sup_{x \in [a,b]} r(x)$. □

Let p_m, q_m, r_m be the minimum value of p, q and r in $[a, b]$ and let p_M, q_M, r_M be the corresponding maximum value in $[a, b]$. Consider the differential equation $(p_M u')' + (q_M - \lambda r_m)u = 0$, with corresponding polar form for $\theta = \theta_0(x, \lambda)$. Rearranging the terms, we get $u'' = \frac{(\lambda r_m - q_M)}{p_M} u$. We know the solutions to this with $u(a) = 0$ are $u(x) = \sin(\kappa(x-a))$ where $\kappa = \sqrt{\frac{\lambda r_m - q_M}{p_M}}$. Let us choose λ large enough such that $\theta_0(b, \lambda) = \kappa(b-a) > (n+1)\pi$. Note that $\theta_0'(x, \lambda) \le \theta'(x, \lambda)$ for any $\theta(x, \lambda)$ solving eq. (2.18) with $P(x), Q(x)$ taken from eq. (2.20) and $\theta(a, \lambda) = 0$. Applying theorem 2.13, we see that the $u(x)$ corresponding to $\theta(x, \lambda)$ has at least n zeros. So, we have proved

Lemma 2.15 *For large enough λ, there exists a solution u of the IVP eq. (2.1) with $u(a) = 0$ that has at least n zeros.*

Two important relations about the behavior of θ with λ are

Lemma 2.16 *For a fixed $x \in (a, b]$,*

$$\theta(x, \lambda) \to \infty \text{ as } \lambda \to \infty \tag{2.21}$$

$$\theta(x, \lambda) \to 0 \text{ as } \lambda \to -\infty \tag{2.22}$$

Proof: In the proof lemma theorem 2.15, it is clear that for any $n \in \mathbb{N}$ and $x \in (a, b]$, we can find λ large enough (call it λ_S) so that $\theta_0(x, \lambda_S) > n\pi$. Also, by the same lemma, $\theta_0(x, \lambda_S) \leq \theta(x, \lambda_S)$. Combining these two results, we get that $\theta(x, \lambda) > n\pi$ for all $\lambda > \lambda_S$, λ_S depending on n and x. This proves eq. (2.21).

Let $\theta_\infty(x, \lambda)$ be a solution of the IVP eq. (2.18) with $P = p_m$ and $Q = \lambda r_M + |q|_M$ where $|q|_M$ is the maximum value of $|q|$ on $[a, b]$. Then note that

$$\theta'(x, \lambda) \leq \theta'_\infty(x, \lambda) = \frac{\cos^2(\theta_\infty)}{p_m} + (\lambda r_M + |q|_M \sin^2(\theta_\infty) \leq K + \lambda r_M \, ; \quad K = p_m^{-1} + |q|_M$$

Choosing λ (call it λ_s) negative such that $K + \lambda_s r_M < 0$ gives us $\theta'(x, \lambda_s) < 0$. This implies $\theta(x, \lambda_s) \leq \pi$[11] for $x \in [a, b]$. Given some $x_1 \in (a, b)$ and $\epsilon > 0$ we define the linear function $w(x) : [a, x_1] \to [\epsilon, \pi]$ as

$$w(x) = \pi - \frac{(x - a)}{(x_1 - a)}(\pi - \epsilon); \text{ where } w'(x) = -\frac{\pi - \epsilon}{x_1 - a}$$

Next we choose λ (call it λ_p) negative enough such that $K + \lambda r_M < -\frac{\pi - \epsilon}{x_1 - a}$. Let $\bar{\lambda} = \min\{\lambda_s, \lambda_p\}$. Then, $\theta'(x, \bar{\lambda}) < w'(x)$ for $x \in (a, x_1]$ and $0 = \theta(a, \bar{\lambda}) < \pi = w(a)$. By theorem 2.12 and corollary 2.12.2 we get $\theta(x_1, \bar{\lambda}) < \epsilon$, proving eq. (2.22). Also note that $\theta(b, \bar{\lambda}) \leq \epsilon$. $\square$

We can now state and prove the following theorem.

Theorem 2.17 (Courant's Nodal Domain theorem) *Let the set eigenfunctions of L with Dirichlet boundary conditions be $\{u_n\}_n$ where the corresponding eigenvalues satisfy $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Then, the zeros of the n^{th} eigenfunction u_n partition the interval (a, b) into precisely n disjoint connected sets.*

Proof: The above statement is equivalent to saying that u_n has $n - 1$ distinct zeros in (a, b). To prove this, let us start with a solution $u(x)$ to the IVP eq. (2.1) with

[11] Since if not then there is some $x_0 \in (a, b)$ such that $\theta(x_0, \lambda_s) > \pi$ but then $\theta'(x_0, \lambda_s) > 0$ which is a contradiction.

$u(a) = 0$ and corresponding *Prüfer variable* $\theta(x, \lambda)$ with $\theta(a, \lambda) = 0$. We know that if $\theta(b, \lambda) = n\pi$, $n \in \mathbb{N} \setminus \{0\}$ then u is an eigenfunction of L with eigenvalue λ.

Now, by lemma theorem 2.16, we can choose λ (call it λ') such that $\theta(b, \lambda') = \pi$. Note that the corresponding $u(x, \lambda')$ is the first eigenfunction (i.e. u_1) since $\theta(b, \lambda') > 0$ and lemma theorem 2.14 tell us that the first instance of the Dirichlet boundary condition being satisfied for u, i.e. $u(b) = 0$ is when $\theta(b, \lambda') = \pi$. So we know that for u_1, the corresponding *Prüfer variable* θ is $\theta(b, \lambda')$. But this means that u_1 doesn't change sign over (a, b). So, u_1 satisfies Courant's theorem. Similarly, for u_n, we choose λ (call it λ') such that $\theta(b, \lambda') = n\pi$. Then the corresponding u is u_n. Moreover, from $\theta(x, \lambda')$ we get that u_n has $n - 1$ zeros in (a, b). Finally, from corollary 2.11.1 we conclude that the zeros of $\theta(x, \lambda')$ correspond to distinct zeros of u_n. The statement of the theorem follows from this. $\square$

Corollary 2.17.1 *For neighboring eigenfunctions u_n and u_{n+1}, their zeros interlace. That is, exactly one zero of u_{n+1} is contained between any two consecutive zeros of u_n.*

Proof: From theorems 2.13 and 2.17 we get the required result. $\square$

2.7 Weyl's Asymptotics

Having studied some properties of the eigenfunctions of L (Dirichlet), we turn our attention to its eigenvalues. We know from sub-section section 2.4.4 that they are simple, bounded below and can be arranged in an increasing sequence tending to ∞. In this section, we make and prove a precise statement, due to Weyl, about the asymptotic properties of these eigenvalues. Namely,

Theorem 2.18 (Weyl's Asymptotics) *Consider p, q and r as defined in eq. (2.1). Further, we also assume $r(x) \in C^1([a, b])$. Then, for large n, the eigenvalues λ_n of L satisfy*

$$\lambda_n = n^2\pi^2 \left(\int_a^b \sqrt{\frac{r(t)}{p(t)}} dt \right)^{-2} + \mathcal{O}(n)$$

Remark *Recall the example problem $f'' + \lambda f = 0$ that was worked out in eq. (2.3). There, it is clear that $\lambda_n \propto n^2$. More precisely, we have $p \equiv 1, r \equiv 1$. So, Weyl's theorem tells us that $\lambda_n = \frac{\pi^2}{(b-a)^2} n^2 + \mathcal{O}(n)$ which is what we got in eq. (2.3).*

Proof: We prove Weyl's asymptotics by considering the *Prüfer* variables for $\big(h(x)u(x)$ $, h^{-1}(x)p(x)u'(x)\big)$ where $h(x) = \sqrt{\lambda r(x) p(x)}$ and $\lambda > 0$. Let the corresponding θ be $\theta_h(x)$ and let the θ corresponding to (u, pu') be $\theta(x)$. Note that $\theta(x) = m\pi$ if and only if $\theta_h(x) = m\pi$. It can be shown by calculation, that the differential equation satisfied by θ_h when $Lu = \lambda u$ is

$$\theta_h' = \sqrt{\lambda}\sqrt{\frac{r}{p}} - \frac{q}{\sqrt{\lambda pr}}\sin^2(\theta_h) + \sin(2\theta_h)\frac{(pr)'}{pr}$$

then, for large λ, $\frac{q}{\sqrt{\lambda pr}}\sin^2(\theta_h) \to 0$. Also note that since $|\sin(2\theta_h)| \leq 1$ and $\sup_{x\in[a,b]} (pr)'/pr = M < \infty$. Using this and integrating both sides, we have

$$\theta_h(x) - \theta_h(a) = \sqrt{\lambda}\int_a^x \sqrt{\frac{r(t)}{p(t)}}dt + \mathcal{O}(1)$$

Since $\theta(x) = \theta_h(x)$ at any multiple of π, if $u = u_n$ is the n^{th} eigenfunction of L with Dirichlet boundary conditions then $\theta(b) = \theta_h(b) = n\pi$ and,

$$\begin{aligned}
n\pi &= \sqrt{\lambda_n}\int_a^x \sqrt{\frac{r(t)}{p(t)}}dt + \mathcal{O}(1) \\
\implies \big(n\pi + \mathcal{O}(1)\big)^2 &= \lambda_n\bigg(\int_a^x \sqrt{\frac{r(t)}{p(t)}}dt\bigg)^2 \\
\implies \qquad \lambda_n &= n^2\pi^2\bigg(\int_a^x \sqrt{\frac{r(t)}{p(t)}}dt\bigg)^{-2} + \mathcal{O}(n)
\end{aligned}$$

which proves Weyl's theorem. $\square$

Chapter 3

Laplacian on bounded domains in $\mathbb{R}^n$

The Laplacian, as defined on bounded domains in $\mathbb{R}^n$, belongs to the general class of uniformly elliptic differential operators. First, we define the general setting in which we will work and then study some theorems about solutions to the Dirichlet and Neumann boundary value problems involving these elliptic operators. We will then focus entirely on the Laplacian and study its eigenvalues and eigenfunctions. Finally, we will state and prove the four main theorems in this section, namely, Weyl's eigenvalue asymptotics, Courant's nodal domain theorem, Cheeger's inequality and the Faber-Krahn inequality.

3.1 Setting

Let Ω be an bounded open set in $\mathbb{R}^n$.

Let L be a partial differential operator, given by two equivalent forms,

- Divergence form :

$$Lu = -\sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u(x) \tag{3.1}$$

- Non divergence form :

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u(x) \tag{3.2}$$

where $a_{ij}(x), b_i(x), c(x) \in C^\infty(\Omega) \ \ \forall i, j$.

Note that both forms are equivalent in the sense that using the chain rule on the first summation in the *divergence* form, yields the *nondivergence* form where the corresponding coefficient $(\widetilde{b_i})$ of u_{x_i} is $\widetilde{b_i}(x) = \sum\limits_{j=1}^{n} \left(a_{ij}(x)\right)_{x_j} + b_i(x)$.

Definition *We say that L is uniformly elliptic if the $n \times n$ matrix $A(x) = (a(x)_{ij})_{i,j \le n}$, is uniformly positive definite. i.e. for some $\theta > 0$, we have $\langle \zeta, A(x)\zeta \rangle_{\mathbb{R}^n} \ge \theta \|\zeta\|^2_{\mathbb{R}^n}$ for all $x, \zeta \in \mathbb{R}^n$.*

Having defined the elliptic operator L, we will study the solutions to two specific boundary value problems. They are the Dirichlet and Neumann boundary value problems, which are given below.

$$\text{(Dirichlet)} \quad \begin{cases} Lu = f & \text{in } \Omega \\ \;\; u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

$$\text{(Neumann)} \quad \begin{cases} Lu = f & \text{in } \Omega \\ \sum\limits_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \eta_j = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

where $f \in L^2(\Omega)$, $u \in C^2(\Omega)$ and η_j is the j^{th} component of the outward normal vector to the surface $\partial\Omega$ at that point. Note that for the Neumann boundary value problem, for Ω to be bounded is not enough. We need to impose some additional smoothness conditions on its boundary $\partial\Omega$. Hence, we will consider the Neumann problem only for those Ω that have a nice enough boundary.

Remark *The Laplacian $-\Delta u = -\sum\limits_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ is obtained by taking $a_{ij}(x) = \delta_{ij}, b_i(x), c(x) = 0$.*

We want to further extend the class of solutions of the boundary value problem to 'weak' solutions in either of the Sobolev function spaces $H^1(\Omega)$ or $H^1_0(\Omega)$ as defined in 13 and 15 respectively. Here, 'weak' is understood in the sense of weak derivatives as defined in 12. One of the reasons we want to do this is that unlike in the case of the Stürm-Liouville operator, where we were able to obtain existence, uniqueness and regularity conditions for the solutions from the theory of ODE's, we can't guarantee the smoothness of solutions in this setting. To get around this, we use the Hilbert nature of $H^1(\Omega)$ and re-define our boundary value problem in the form of a bi-linear operator equation $B[u,v] = \langle f, v \rangle_{L^2}$. Using the nice properties satisfied by this operator, we will study these weak solutions and the eigenfunctions of L. We will also mention the conditions under which the eigenfunctions enjoy additional regularity such as strong/classical derivatives up to second order.

Definition *We define the bilinear operator $B[u,v]$ for some functions $u,v \in H^1(\Omega)$ as*

$$B[u,v] = \sum_{i,j=1}^{n} \int\limits_{\Omega} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{n} \int\limits_{\Omega} b_i(x)\frac{\partial u}{\partial x_i}v(x) + \int\limits_{\Omega} c(x)u(x)v(x)$$

We state here, the new settings in which we will analyze weak solutions to eqs. (3.3) and (3.4).

- (*Dirichlet*): A function $u \in H_0^1(\Omega)$ is called a *weak* solution to 3.3[1] if,

$$B[u,v] = \langle f, v\rangle_{L^2(\Omega)}\ , \ \forall v \in H_0^1(\Omega)$$

- (*Neumann*): A function $u \in H^1(\Omega)$ is called a *weak* solution to 3.4 if,

$$B[u,v] = \langle f, v\rangle_{L^2(\Omega)}\ , \ \forall v \in H^1(\Omega)$$

We now state and prove the three existence theorems for the weak solutions to the Dirichlet[2] boundary value problem.

3.2 Existence Theorems

Theorem 3.1 (First Existence Theorem) *There is a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each function $f \in L^2(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ that satisfies*

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

Proof: From the energy estimates in theorem B.2 we get that there exists β and γ such that

$$\beta\|u\|^2_{H_0^1(\Omega)} \leq B[u,u] + \gamma\|u\|^2_{L^2(\Omega)}\ , \quad \forall u \in H_0^1(\Omega) \tag{3.6}$$

[1] $H_0^1(\Omega)$ is the completion of $C_c^\infty(\Omega)$ under the norm : $\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{1/2}$. So, the trace sense, as understood from appendix A.3, we can think of $H_0^1(\Omega)$ as the generalization of the Dirichlet boundary condition to the space of weakly once-differentiable functions.

[2] The corresponding theorems for the Neumann can also be proved in a similar manner. One important distinction is that the Hilbert space taken in this case is the closed subspace of $H^1(\Omega)$ that is perpendicular to the constant function $u_c(x) = c$, i.e. $\{u | \int\limits_{\Omega} u(x) = 0, u \in H^1(\Omega)\}$ or $\{u | u(x) = c, u \in H^1(\Omega), c \in \mathbb{R}\}^\perp$. Otherwise, $B[u,v]$ doesn't satisfy eq. (3.6).

we define $B_\mu[u,v] := B[u,v] + \mu \int\limits_\Omega u(x)v(x)$ for any $\mu \geq \gamma$. Then, $B_\mu[u,v]$ satisfies the conditions for the Lax-Milgram theorem B.1, which means that for each f in $L^2(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ that is a weak solution to 3.5. $\square$

Theorem 3.2 (Second Existence Theorem)

1. *Precisely one of the following statements holds:*

   (a) *For each f in $L^2(\Omega)$, there exists a weak solution u for the boundary value problem*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.7}$$

   Or else

   (b) *There exists a weak solution $u \not\equiv 0$ for the homogeneous boundary value problem*

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.8}$$

2. *Let L^* be the adjoint operator for L. Then, should assertion (b) hold, the dimension of the subspace $N \subset H_0^1(\Omega)$ of weak solutions of 3.8 is finite and equals the dimension of the subspace $N^* \subset H_0^1(\Omega)$ of weak solutions of*

$$\begin{cases} L^*u = 0 & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega \end{cases}$$

3. *Finally, the boundary value problem 3.7 has a weak solution if and only if*

$$\langle f, v \rangle_{L^2(\Omega)} = 0, \ \forall\, v \in N^*.$$

Proof: For part one, observe that u is a weak solution to the boundary value problem 3.7 *if and only if* it satisfies

$$\begin{cases} Lu + \mu u = \mu u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.9}$$

From theorem 3.1, we know that for $\mu \geq \gamma$ we have a bijective linear map $L_\mu : H_0^1(\Omega) \to L^2(\Omega)$ where $g := L_\mu u$ satisfies $B_\mu[u,v] = \langle g, v \rangle_{L^2(\Omega)} (\forall v \in H_0^1(\Omega))$. Let the inverse map be given by $L_\mu^{-1} : L^2(\Omega) \to H_0^1(\Omega)$. Then, from 3.9 we see that $B_\mu[u,v] = \langle f + \mu u, v\rangle_{L^2(\Omega)}$. In terms of L_μ^{-1},

$$\begin{aligned} & L_\mu^{-1}(f+\mu u) = u \\ \Longrightarrow \quad & u - \mu L_\mu^{-1} u = L_\mu^{-1} f \\ \Longrightarrow \quad & u - Ku = h \, ; \;\; K := \mu L_\mu^{-1} \, , \; h := L_\mu^{-1} f \end{aligned}$$

We also observe that the linear operator K is bounded and compact. Indeed,

$$\begin{aligned} & \beta \|u\|^2_{H_0^1(\Omega)} \le B_\mu[u,u] = \langle g, u\rangle_{L^2(\Omega)} \le \|g\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \le \|g\|_{L^2(\Omega)} \|u\|_{H_0^1(\Omega)} \\ \Longrightarrow \; & \beta \left\| L_\mu^{-1} g \right\|_{H_0^1(\Omega)} \le \|g\|_{L^2(\Omega)} \\ \Longrightarrow \quad & \|Kg\|_{H_0^1(\Omega)} \le C \|g\|_{L^2(\Omega)} \quad ; \;\; C = \tfrac{\mu}{\beta} \end{aligned}$$

To prove compactness, we refer to the Rellich-Kondrachov theorem A.10 which shows that $H_0^1(\Omega) \subset\subset L^2(\Omega)$. Note that theorem A.10 proves a stronger statement, i.e. precompactness of $H^1(\Omega)$ in $L^2(\Omega)$. However, it is only true when $\partial\Omega$ is at least C^1. For general open bounded Ω, it can be shown that precompactess in $L^2(\Omega)$ still holds, but only for $H_0^1(\Omega)$ (for further details, see Evans' book[7], pg.289, section 5.7, remark after thm1). So, we have that K is a linear, bounded, compact operator. We can now use the Fredholm alternative theorem C.1 to conclude that the following holds.[3]

$$\begin{cases} \text{for each } h \in L^2(\Omega), \text{ the equation} \\ \quad u - Ku = h \\ \text{has a unique solution } u \in L^2(\Omega) \end{cases}$$

or else

$$\begin{cases} \text{the equation} \\ \; u - Ku = 0 \\ \text{has non-trivial solutions } u \not\equiv 0 \in L^2(\Omega) \end{cases}$$

Parts two and three follow in a similar manner from the results of theorem C.1. □

Theorem 3.3 (Third Existance Theorem)

[3] In order to apply Fredholm theory, implicitly consider the extension of K to $L^2(\Omega)$ using the well known Bounded Linear Transformation theorem.

1. *There exists a countable set (possibly finite) $\Sigma \subset \mathbb{R}$ such that the boundary value problem*

$$\begin{cases} Lu = \lambda u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.10}$$

   has a unique weak solution for each $f \in L^2(\Omega)$ if and only if $\lambda \notin \Sigma$.

2. *If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k\in\mathbb{N}}$, which is a non-decreasing sequence with $\lambda_k \to \infty$.*

Proof: Let $\widetilde{L}u = Lu - \lambda u$. Theorem 3.2 tells us that 3.10 has a unique weak solution for each $f \in L^2(\Omega)$ if the only weak solution to

$$\begin{cases} \widetilde{L}u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is $u \equiv 0$. This is equivalent to requiring that the only weak solution to

$$\begin{cases} Lu + \gamma u = (\lambda + \gamma)u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.11}$$

is $u \equiv 0$. Where γ is chosen as in theorem 3.1. Now, 3.11 implies that the only solution to

$$\begin{cases} u = \frac{(\lambda+\gamma)}{\gamma} Ku & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is $u \equiv 0$. This means that

$$\frac{\gamma}{(\lambda+\gamma)} \text{ is } \textbf{not} \text{ an eigenvalue of the operator } K \tag{3.12}$$

From eq. (3.6) we see that if λ is an eigenvalue of L then $\lambda \geq -\gamma$. Also, 3.10 admits a unique weak solution *if and only if* 3.12 holds. Since K is a compact operator, we know that the spectrum of K equals it's eigenvalues (except not necessarily zero). Moreover, the eigenvalues of K are either finite or are a sequence converging to zero (from the theory of compact operators over Hilbert spaces) and so, 3.10 has a unique weak solution for all $\lambda \in \Sigma$ except a countable sequence of real values, i.e. Σ. Finally, if Σ is infinite, then, because $\lambda \geq -\gamma$ and the eigenvalues of $L - \lambda\mathbb{I}$ are the inverses of that of K, we have that $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$ where λ_k is a non decreasing sequence with $\lambda_k \to \infty$ as $k \to \infty$. □

Definition (Eigenvalues and Eigenfunctions of L) *A complex number λ is called a Dirichlet or Neumann eigenvalue if there is a u in $H^1(\Omega)$ or $H_0^1(\Omega)$ respectively such that u is a weak solution to the problem $Lu = \lambda u$ in Ω, along with the respective Dirichlet or Neumann boundary condition. In such a case, u is called an eigenfunction of L corresponding to the eigenvalue λ*

Note that each element of Σ, defined in theorem 3.3, is a Dirichlet eigenvalue of L. Also note that theorem 3.2 shows that the nullspace (called eigenspace) of $L - \lambda\mathbb{I}$ is finite dimensional meaning that the eigenvalue λ has finite multiplicity. Finally, observe that theorem 3.3 only tells us that the real eigenvalues of L are at most countable in number. It can however be shown that the same fact holds for the complex eigenvalues of L.

3.3 Eigenvalues and Eigenfunctions of the Laplace Operator and Min-max Principles

So far, we have dealt with the form of the elliptic operator L as defined in eq. (3.1) and eq. (3.2) . The purpose of this was to show that the existence theorems in section 3.2 hold for such a general class of operators. From now on however, we will only deal with the Laplace operator given by

$$-\Delta u := -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \tag{3.13}$$

We note that the bi-linear operator in this case is $B[u, v] = \int_\Omega \nabla u \cdot \nabla v$. Also it is clear that $B[u, v] = B[v, u]$. Hence, B satisfies all the properties of an inner product on $H_0^1(\Omega)$.

Remark *Note that the following theorems are true for any of the elliptic operators mentioned above which satisfy the additional property of being self-adjoint. Some additional remarks have been made in section section 3.4 regarding self-adjoint operators.*

Why study $-\Delta$? It is worthwhile, at this stage, to mention why special attention is being paid to the operator Δ. One important reason why it is a useful object in mathematics (as well as physics) is that it is rotationally and translationally invariant. We prove this fact and also show the relationships between eigenvalues and eigenfunctions and their transformed counterparts.

Lemma 3.4 *The Laplacian is invariant under an isometric transformation, i.e. given an orthogonal matrix M and given some $x_0 \in \mathbb{R}^n$ we have, assuming u is smooth enough,*

$$\Delta\big(u(x-x_0)\big) = \big(\Delta u\big)(x-x_0) \tag{3.14}$$
$$\Delta\big(u(Mx)\big) = \big(\Delta u\big)(Mx) \tag{3.15}$$

Furthermore, under a scale change by some $k \neq 0$, we have

$$\Delta\big(u(ky)\big) = k^2(\Delta u)(ky) \tag{3.16}$$

Proof: eq. (3.14) is a result of a straightforward use of the chain rule of differentiation.

For eq. (3.15), we use the chain rule in the same way as previously but we need to be a bit careful.

$$\begin{aligned} D\big(u(Mx)\big) &= \big(Du\big)(Mx) \circ M \\ \Longrightarrow D^2\big(u(Mx)\big) &= \Big(\big(D^2u\big)(Mx) \circ M^T\Big) \circ M = \big(D^2u\big)(Mx) \\ \Longrightarrow \quad \Delta\big(u(Mx)\big) &= \big(\Delta u\big)(Mx) \end{aligned}$$

The M^T in the second step comes from the way the terms in the matrix D^2u are arranged. The last step comes from the fact that the Laplacian is the sum of the diagonal entries of the second derivative matrix.

Finally, for eq. (3.16) we can easily see that $\boldsymbol{\nabla}\big(u(ky)\big) = k(\boldsymbol{\nabla} u)(ky)$. Applying the $\boldsymbol{\nabla}$ operator once more gives us the required result. $\square$

As a result of theorem 3.4 and the regularity properties of the eigenfunctions of $-\Delta$ given in appendix B.2 that guarantee their smoothness in Ω, we obtain the corollary

Corollary 3.4.1 *Let u be an eigenfunction of $-\Delta$ with eigenvalue λ. Let Ω be an open bounded domain. Let $\widetilde{\Omega} := \{Mx + x_0 \mid x \in \Omega,\ x_0 \in \mathbb{R}^n\}$ be an isometric transformation of Ω for some orthogonal matrix M and $x_0 \in \mathbb{R}$. Also define $\bar{\Omega} := \{kx \mid x \in \Omega\}$ for some $k > 0$. Then,*

1. *The function $u\big(M^T(y-x_0)\big)$ where $y \in \widetilde{\Omega}$ is an eigenfunction of $-\Delta$ on $\widetilde{\Omega}$ with eigenvalue λ (i.e. eigenvalues are preserved under isometric transformations of Ω).*

2. *The function $u(\frac{y}{k})$ where $y \in \bar{\Omega}$ is an eigenfunction of $-\Delta$ on $\bar{\Omega}$ with eigenvalue $\frac{\lambda}{k^2}$ (i.e. doubling the domain size shrinks the eigenvalues by a factor of one quarter)*

Having provided sufficient motivation for the study of $-\Delta$, we study the properties of its eigenvalues and eigenfunctions.

Theorem 3.5 (Eigenvalues)

1. *Each eigenvalue of $-\Delta$ is real and positive.*
2. *Furthermore, if we enumerate the eigenvalues, counting their (finite) multiplicities separately, then we get an increasing sequence of eigenvalues with an infinite limit i.e $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$.*
3. *Finally, there exists an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$, where $w_k \in H_0^1(\Omega)$ is an eigenfunction of $-\Delta$ corresponding to λ_k.*

Remark *Since Ω is open and bounded, we automatically have that ω_k are $C^\infty(\Omega)$, from the regularity property given in theorem B.3. Furthermore, if $\partial\Omega$ is smooth, then ω_k are $C^\infty(\bar{\Omega})$ (refer to theorem B.4).*

Proof: Note that for $-\Delta$ given in eq. (3.13), we see that the corresponding γ from the first existence theorem 3.1 is 0. This is a result of Poincare's inequality, mentioned in theorem A.9, which bounds the L^2 norm of u by the L^2 norm of its weak gradient, for $u \in H_0^1(\Omega)$. So, we can find a unique weak solution u for the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for each $f \in L^2(\Omega)$. Then, as defined in the proof of the second existence theorem 3.2, $-\Delta$ has an inverse, given by the bounded, linear and compact operator $(-\Delta)^{-1}$. Let us represent it as $S := (-\Delta)^{-1}$. We claim that S is also self-adjoint. Note that since $S : L^2(\Omega) \to L^2(\Omega)$ it is enough to show that S is symmetric. Let $f, g \in L^2(\Omega)$. Then, $Sf, Sg \in H_0^1(\Omega)$. Let $Sf = u$ and $Sg = v$ then by definition, u, v are solutions to the PDE's

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

respectively. Since $B[u, v] = B[v, u]$ as mentioned above, we get

$$\langle g, Sf \rangle = \langle g, u \rangle = B[v, u] = B[u, v] = \langle f, v \rangle = \langle f, Sg \rangle$$

hence S is self-adjoint.

Finally, note that $Sg = 0$ implies $g = 0$ trivially as S is invertible. Applying the spectral theorem to S, we get eigenvalues $\{\mu_n\}_n$ and corresponding eigenfunctions $\{w_n\}_n$ which form an orthonormal basis of L^2. Moreover, since $Sw_k = \mu_k w_k$ we see that $w_k \in H_0^1(\Omega)$. We also see that $\mu_k \neq 0$ for any $k \in \mathbb{N}$ since $\mu_k = \langle w_k, Sw_k \rangle =$

$B[w_k, w_k] >= 0$, since equality is *if and only if* $w_k \equiv 0$ as B is an inner product on $H_0^1(\Omega)$. Also, $\mu_k \to 0$ by compactness of S. Using these facts, we get

$$Sw_k = \mu_k w_k \implies -\Delta S w_k = -\mu_k \Delta w_k \implies -\Delta w_k = \frac{1}{\mu_k} w_k$$

where the eigenvalues of $-\Delta$ are positive and form an increasing sequence with an infinite limit. This proves the theorem. $\square$

Theorem 3.6 (Variational Principle for the Principal Eigenvalue) *Let λ_k and w_k be the k^{th} eigenvalue and eigenfunction of $-\Delta$ respectively where the eigenvalues are ordered in an increasing divergent sequence starting with λ_1. Then,*

1. *λ_1 is simple and satisfies*

$$\lambda_1 = \min\{B[u,u] \mid u \in H_0^1(\Omega),\ \|u\|_{L^2(\Omega)} = 1\}$$

2. *The above minimum is attained by w_1, which doesn't change sign within Ω*

Proof: $\{w_k\}_k$ forming an orthonormal basis for $L^2(\Omega)$. Let the corresponding eigenvalues be $\{\lambda_k\}_k$. Then, $B[w_k, w_k] = \lambda_k \|w_k\|^2_{L^2(\Omega)} = \lambda_k$ and $B[w_k, w_l] = \lambda_k \langle w_k, w_l \rangle_{L^2(\Omega)} = 0$ for $k \neq l$. Since $H_0^1(\Omega) \subset L^2(\Omega)$, for any $u \in H_0^1(\Omega)$ with $\|u\|_{L^2(\Omega)} = 1$, we can write

$$u = \sum_{i=1}^{\infty} d_i w_i, \quad \sum_{i=1}^{\infty} d_i^2 = 1.$$

Next, viewing $B[u, v]$ as an inner product over $H_0^1(\Omega)$, we claim that $\{y_k\}_k$ is an orthonormal basis for $H_0^1(\Omega)$ where $y_k = \frac{w_k}{\sqrt{\lambda_k}}$. Since $B[y_k, y_j] = \delta_{kj}$, it is enough to show that if $u \in H_0^1(\Omega)$ and $B[u, y_k] = 0$ for all k, then $u \equiv 0$. Suppose $B[u, y_k] = \langle u, \sqrt{\lambda_k} w_k \rangle = 0$ for all k. This means that u is orthogonal to w_k for each k. Since $\{w_k\}_k$ form an ONB of $L^2(\Omega)$ we get $u \equiv 0$. Finally, computing $B[u, u]$ we get

$$\|u\|_B^2 = B[u,u] = \sum_{i=1}^{\infty} d_i^2 \lambda_i \geq \lambda_1 \sum_{i=1}^{\infty} d_i^2 = \lambda_1$$

where $\|\cdot\|_B$ is the norm induced by the inner product $B[u, v]$ over $H_0^1(\Omega)$. For equality, let $u = w_1$. We have proved (1).

For (2), we claim the following. Let $u \in H_0^1(\Omega)$, $\|u\|_{L^2(\Omega)} = 1$ then, $B[u, u] = \lambda_1$ *if and only if* u is a weak solution for the differential equation $-\Delta u = \lambda_1 u$ for $u \in \Omega$, with Dirichlet boundary conditions. Indeed, the converse is obvious. For the forward implication, let $u = \sum_{i=1}^{\infty} d_i w_i$. Then

$$0 = B[u,u] - \lambda_1 = \sum_{i=1}^{\infty} d_i^2 (\lambda_i - \lambda_1)$$

so we get $u = \sum_{i=1}^{p} d_i w_i$ where λ_1 has multiplicity p and $-\Delta w_i = \lambda_1 w_i$ for $i = 1, 2, \ldots, p$. So u is an eigenfunction of $-\Delta$. This proves our claim.

For (3), let u be an eigenfunction of $-\Delta$ with eigenvalue λ_1. We define

$$u^+(x) = \begin{cases} u(x) & \text{if } u(x) > 0 \\ 0 & \text{Otherwise} \end{cases}$$

and define $u^- := u^+ - u$. It can be shown that both u^+ and u^- lie in $H_0^1(\Omega)$ and that $\int_\Omega \nabla u^+ \cdot \nabla u^- = 0$. Using these facts and part (1), we see that

$$\begin{aligned} \lambda_1 = B[u,u] = B[u^+ - u^-, u^+ - u^-] &= B[u^+, u^+] + B[u^-, u^-] \\ &\geq \lambda_1 \|u^+\|^2_{L^2(\Omega)} + \lambda_1 \|u^-\|^2_{L^2(\Omega)} = \lambda_1 \end{aligned}$$

Since the inequality is actually an equality, we see that $B[u^+, u^+] = \lambda_1 u^+$ and the same is satisfied by u^-. By part (1), we see that u^+, u^- are eigenfunctions of $-\Delta$ with eigenvalue λ_1. Now, if $\partial\Omega$ is smooth then we see that $u^+, u^- \in C^\infty(\bar{\Omega})$. Using the maximum principle given in theorem B.5 on u^+ and noting that $-\Delta u^+ = \lambda_1 u^+ \geq 0$, we get $u^+ > 0$ in Ω or $u^+ \equiv 0$. Similarly, we get $u^- < 0$ in Ω or $u^- \equiv 0$. This means that exactly one of u^+ or u^- is identically zero. Thus proving that u doesn't change sign in Ω if it is an eigenfunction with associated eigenvalue λ_1. Finally, if $\tilde{u}, u$ are distinct eigenfunctions of $-\Delta$ with eigenvalue λ_1, then noting that $\int_\Omega \tilde{u} = \alpha \int_\Omega u \neq 0$ for some real α, we get $\int_\Omega \tilde{u} - \alpha u = 0$ which implies[4] $\tilde{u} = \alpha u$. We have proved that λ_1 is a simple eigenvalue. □

Using a similar argument and considering the restriction of $-\Delta$ to the space orthogonal to $w_1, w_2, \ldots, w_k$ we get

Theorem 3.7 (Rayleigh's Theorem) *We define $S := span\{w_1, w_2, \ldots, w_{k-1}\}^\perp$, where w_k is the k^{th} Dirichlet eigenfunction of $-\Delta$ with unit norm. Then for any $g \in S$ with unit L^2 norm, $B[g,g] \geq \lambda_k$ with equality if and only if g is an eigenfunction of $-\Delta$ with eigenvalue λ_k.*

[4]Since $\tilde{u} - \alpha u$ is a nonzero eigenfunction of $-\Delta$ with eigenvalue λ_1 we have a contradiction because $\int_\Omega \tilde{u} - \alpha u$ cannot be zero.

Lemma 3.8 (Courant's Minimax Principle) *Assume the operator $-\Delta$ with Dirichlet boundary conditions has eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$. Then,*

$$\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{\substack{u \in S^\perp \\ \|u\|_{L^2}=1}} B[u,u] \;\; ; \;\; (k = 1, 2, \dots) \tag{3.17}$$

where Σ_{k-1} is the set of all subspaces S of dimension $k-1$

Proof: Let $\mathcal{B} := \{S \mid S \subset H_0^1(\Omega),\ \dim(S) = k-1\}$ and $W_k := \operatorname{span}\{w_1, w_2, \dots, w_k\}$ where w_k is the eigenfunction of $-\Delta$ with eigenvalue λ_k. We first prove this fact.

Lemma 3.9 *For any $S \in \mathcal{B}$, $S^\perp \cap W_k \neq \{0\}$.*

Proof: We prove the claim by contradiction. Suppose that there exists $S \in \mathcal{B}$ such that $S^\perp \cap W_k = \{0\}$. Then for all $u \in W_k$, $u|_S \neq 0$ where $u|_S$ is the projection of u onto the subspace S. Since $\dim(S) = k-1$ we see that the set $\{w_i|_S \mid i = 1, 2, \dots, k\}$ is linearly dependent. So we get β_i , $i = 1, 2, \dots, k$ not all zero such that $\sum_{i=1}^{k} \beta_i w_i|_S = 0$. But this then gives us a nonzero function $u = \sum_{i=1}^{k} \beta_i w_i \in W_k$ such that $u|_S = 0$ which is a contradiction. □

Using lemma 3.9, let u be a nonzero function in $S^\perp \cap W_k$ such that $\|u\|_{L^2(\Omega)} = 1$. Then, $u = \sum_{i=1}^{k} d_i w_i$ and $B[u,u] = \sum_{i=1}^{k} d_i^2 \lambda_i \leq \lambda_k$. So, minimizing over all functions in S, we get

$$\min_{\substack{u \in S^\perp \\ \|u\|_{L^2}=1}} B[u,u] \leq \lambda_k \implies \max_{S \in \mathcal{B}} \min_{\substack{u \in S^\perp \\ \|u\|_{L^2}=1}} B[u,u] \leq \lambda_k$$

Finally, if we choose $S = W_{k-1}$ then $w_k \in S^\perp$ and $B[w_k, w_k] = \lambda_k$ which gives us eq. (3.17). □

Corollary 3.9.1 *One can also show, using similar methods, that*

$$\lambda_k = \min_{S \in \Sigma_k} \max_{\substack{u \in S \\ \|u\|_{L^2}=1}} B[u,u] \;\; ; \;\; (k = 1, 2, \dots) \tag{3.18}$$

3.4 Some Important Remarks for Sections 3.1 to 3.3

Some important points to be mentioned are :

1. Generally, it is the ellipticity condition that allows us to prove the three existence theorems for the eigenvalues and eigenfunctions.

2. Since the compact operator K mentioned in theorem 3.2 can have finitely many nonzero eigenvalues, we are not immediately guaranteed an increasing sequence of eigenvalues for the general elliptic operator, like we get for the Laplace operator. Interestingly though, one can still show that show that the principal eigenvalue λ_1 that is real and simple and that the corresponding eigenfunction doesn't change sign in Ω.

3. For self-adjoint operators L, we see that the corresponding bi-linear function $B[u,v]$ becomes symmetric and thus a valid inner product over $H_0^1(\Omega)$. Moreover, the inverse of L (or $L - \mu\mathbb{I}$) becomes self-adjoint because of this, allowing us to apply the spectral theorem and obtain a basis for $L^2(\Omega)$. The property that eigenvalues are real also comes from the self-adjoint nature of L.

4. The proof of the three existence theorems and the analogous eigenvalue theorems for the Laplacian with Neumann boundary conditions follows a similar structure as that of the Dirichlet boundary case and hence will not be mentioned in this report. Hence we shall only state that for the Neumann boundary case, the main theorem 3.5 holds. Moreover, it should be mentioned that in theorems 3.1 to 3.3 and 3.5, we consider the subspace of $H^1(\Omega)$ perpendicular to the constant function $f(x) = c,\ f \in H^1(\Omega)$ since otherwise the ellipticity property fails to hold.

3.5 Basic Properties of the Eigenvalues of $-\Delta$

We now move on to study the interesting relationships between the eigenvalues of the Dirichlet Laplacian and the domain Ω on which the problem is defined. Initially, we will study gross aspects like the monotonicity principle which relates the eigenvalues of subdomains to the parent domain. This will lead us into Weyl's asymptotics, which uses these results to tell us the rate at which $\lambda_n \to \infty$ for large n.

Lemma 3.10 (Simple Domain Monotonicity) *Let Ω_1 and Ω be bounded open sets such that $\Omega_1 \subseteq \Omega$. Let $\lambda_k^{\Omega_1}$ and λ_k^{Ω} be the k^{th} Dirichlet eigenvalue for Ω_1 and Ω respectively. Then, $\lambda_k^{\Omega_1} \geq \lambda_k^{\Omega}$.*

Proof: Since $\Omega_1 \subseteq \Omega$, we see that each $u \in H_0^1(\Omega_1)$ can be extended to a $H_0^1(\Omega)$ function by setting it's value outside Ω_1 to zero. We can do this because u can be approximated in $H_0^1(\Omega_1)$ norm by $C_c^\infty(\Omega_1)$ functions u_k that can be extended as zero to Ω and thus lie in $C_c^\infty(\Omega)$. Also note that $B_{\Omega_1}[u,v] = B_\Omega[\tilde{u},\tilde{v}]$ for all $u,v \in H_0^1(\Omega_1)$, where $\tilde{u}$ and $\tilde{v}$ are the aforementioned extentions of u and v. Then, using the min-max formula eq. (3.18), we get

$$\lambda_k^{\Omega_1} = \min_{\substack{S \subseteq H_0^1(\Omega_1) \\ \dim(S)=k}} \max_{\substack{u \in S \\ \|u\|_{L^2}=1}} B_{\Omega_1}[u,u] \;\; ; \;\; (k = 1, 2, \dots) \tag{3.19}$$

Since, $S \subseteq H_0^1(\Omega_1)$ can be extended to a subset of $H_0^1(\Omega)$ we can see that if we further minimize the RHS of eq. (3.19) by letting $S \subseteq H_0^1(\Omega)$, which is a larger class of functions, we then get λ_k^{Ω}. So $\lambda_k^{\Omega_1} \geq \lambda_k^{\Omega}$. □

Remark *For the Neumann eigenvalue, this form of simple domain monotonicity is not true. As a counter example, let Ω be the open square $[0,1] \times [0,1]$in $\mathbb{R}^2$. Let $\Omega_1 \subset \Omega$ be a thin rectangle of length l and width ϵ such that the diagonals of the square Ω are its axes of symmetry. We can clearly see that l can be made arbitrarily close to $\sqrt{2}$ by choosing an appropriately small ϵ. It can be shown*[5] *that the 1^{st} positive Neumann eigenvalue (μ_1) for any rectangle of length l and breadth b is $\pi^2 \min\{\frac{1}{l^2}, \frac{1}{b^2}\}$. Choosing ϵ such that $l > 1$, we see that $\mu_1^{\Omega_1} \leq \mu_1^{\Omega}$. Finally, consider a subsquare $\Omega_2 := \{(1/2)x \mid x \in \Omega\}$ i.e. a scaling of Omega by half. We know from corollary 3.4.1 that $\mu_1^{\Omega_2} = 4\mu_1^{\Omega}$. This tells us that in general, μ_i's need not exhibit simple domain monotonicity.*

Lemma 3.11 (Domain Monotonicity) *Let $\Omega_1, \Omega_2 \dots, \Omega_m$ be disjoint disjoint subdomains of Ω . Let λ_k be the k^{th} Dirichlet eigenvalue for the domain Ω and let μ_k be the k^{th} term in the increasing sequence obtained by combining and re-ordering the Dirichlet eigenvalues for $\Omega_1, \Omega_2, \dots \Omega_m$. Then, $\lambda_k \leq \mu_k$.*

Proof: Let $\{\phi_k\}_k$ be the corresponding eigenfunctions for the Dirichlet Laplace operator $-\Delta$ on Ω and let $\{\psi_k\}_k$ be the normalized eigenfunctions corresponding to the eigenvalues μ_k. Therefore, ψ_k is an eigenfunction over the domain Ω_i for some i. We can trivially extend ψ_k to the boundary of Ω as an $H_0^1(\Omega)$ function. Let $f \in span\{\phi_1, \phi_2, \dots, \phi_{k-1}\}^{\perp} \cap span\{\psi_1, \psi_2, \dots, \psi_k\}$, $f \not\equiv 0$. We can get such an f by applying lemma 3.9 here. We claim $\psi_k \perp \psi_l$. Indeed, we see this is the case, since if ψ_k and ψ_l belong to the same domain Ω_i then they are orthogonal by theorem 3.5. If not then their supports are disjoint and so they are trivially orthogonal. Further, we note that $B_{\Omega}[u,u] = B_{\Omega_i}[u,u]$ for $u \in H_0^1(\Omega_i)$ and that f can be normalized and written as

[5] Using the standard variable separable technique to solve $-\Delta u(x,y) = \lambda u(x,y)$ and then applying the isometric invariance for $-\Delta$.

the sum $\sum_{i=1}^{k} d_i\psi_i$ for some real d_i. Using these facts, we get,

$$\begin{aligned}\lambda_k \leq B_\Omega[f,f] &= \sum_{i,j=1}^{k} d_i d_j B_\Omega[\psi_i,\psi_j] \\ &= \sum_{i=1}^{k} d_i^2 B_{\Omega_i}[\psi_i,\psi_i] = \sum_{i=1}^{k} d_i^2 \mu_i \leq \mu_k \ ;\end{aligned}$$

where ψ_i is a normalized eigenfunction for Ω_i. □

We now relate the Neumann eigenvalues of the subdomains to the Neumann eigenvalues of the parent domain.

Lemma 3.12 *Let Ω be an open bounded domain with piecewise smooth boundary such that $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_m}$. Where $\Omega_1, \Omega_2, \ldots, \Omega_m$ are disjoint subdomains with piecewise smooth boundary. Suppose that λ_k is the k^{th} Dirichlet eigenvalue of Ω and μ_k is the k^{th} term of the increasing sequence obtained by combining and re-ordering the Neumann eigenvalues for $\Omega_1, \ldots, \Omega_n$. Then, $\mu_k \leq \lambda_k$.*

Proof: As in the proof of lemma 3.11, let $\{\phi_k\}_k$ be the corresponding Dirichlet eigenfunctions for $-\Delta$ on Ω and let $\{\psi_k\}_k$ be the eigenfunctions corresponding to the Neumann eigenvalues μ_k. Therefore, ψ_k is an eigenfunction over the domain Ω_i for some i. Extending ψ_k as zero outside the subdomain Ω_i, we see that it (we call it ψ_k and context will be clear as to which meaning of ψ_k we use) lies in $L^2(\Omega)$. Note that $\{\psi_1, \psi_2, \ldots, \psi_{k-1}\}$ is an orthogonal set in $L^2(\Omega)$. Let $f \in span\{\psi_1, \psi_2, \ldots, \psi_{k-1}\}^\perp \cap span\{\phi_1, \phi_2, \ldots, \phi_k\}$ such that $\|f\|_{L^2(\Omega)} = 1$. Since $f|_{\Omega_i} \in H^1(\Omega_i)$ we have $B_{\Omega_i}[f,f] \geq \mu_k \|f|_{\Omega_i}\|^2_{L^2(\Omega_i)}$ for all $i = 1, 2, \ldots, m$. Using all these facts, we get

$$\begin{aligned}\lambda_k \geq B_\Omega[f,f] = \int_\Omega |\nabla f|^2 &= \sum_{i=1}^{m} \int_{\Omega_i} |\nabla f|^2 \\ \sum_{i=1}^{m} B_{\Omega_i}[f,f] &\geq \mu_k \sum_{i=1}^{m} \|f|_{\Omega_i}\|^2_{L^2(\Omega_i)} = \mu_k\end{aligned}$$

□

Having proved these elementary properties satisfied by the Dirichlet and Neumann Laplace operator on different domains, we can move on to our first main result for this section on the Laplacian in $\mathbb{R}^n$.

3.6 Weyl's Asymptotics

Weyl's Asymptotics tells us the rate at which the eigenvalues (λ_n) of the Laplacian grow for large values of n. Alternatively, we can also use these asymptotics to count the number of eigenvalues that are less than some given $\alpha > 0$. Again, this estimate becomes more and more precise as we choose larger and larger values of α.

Definition ($\sim$) *For two real functions $a(k)$ and $b(k)$, the notation $a(k) \sim b(k)$ means that the ratio $\frac{a(k)}{b(k)}$ goes to 1 as $k \to \infty$.*

We now state Weyl's precise result,

Theorem 3.13 (Weyl's Asymptotics) *For an open and bounded domain $\Omega \subseteq \mathbb{R}^n$, let the Dirichlet eigenvalues of the Laplace operator be given by $\{\lambda_k\}_k$. Then,*

$$\lambda_k \sim 4\pi^2 \left(\frac{k}{\omega_n Vol(\Omega)} \right)^{2/n} \tag{3.20}$$

$$\lim_{\alpha \to \infty} \frac{N(\alpha)}{\alpha^{n/2}} = \frac{\omega_n}{(2\pi)^n} Vol(\Omega) \tag{3.21}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $N(\alpha)$ counts the number of eigenvalues less than or equal to α.

Remark *Note that we only consider the case of $n = 2$ here. The proof for higher dimensions follows the exact same logic and proof structure but is notationally tedious. Hence, for the sake of brevity and clarity, we will only prove the two dimensional case here.*

The proof of Weyl's theorem is done in three steps. First, we consider a bounded rectangle in $\mathbb{R}^2$. We explicitly solve for the Dirichlet and Neumann eigenvalues here and prove Weyl's asymptotic results in this case. Next, we take a domain that is a disjoint union of rectangles and we use the monotonicity results obtained earlier to obtain the required asymptotic results. Finally, using the simple monotonicity property, we bound an arbitrary domain both inside and outside by disjoint unions of rectangles and take the limit, as the internal and external domains converge in volume to the given one, thus proving Weyl's theorem.
Proof:

1. Let $\Omega = (0, a) \times (0, b)$. We want to solve the eigenvalue equation

$$\begin{cases} \Delta\omega = \lambda\omega & \text{in } \Omega \\ \omega = 0 & \text{on } \partial\Omega \end{cases} \tag{3.22}$$

The eigenvalues and eigenfunctions for both the Dirichlet and Neumann boundary conditions, are

$$\text{Dirichlet :} \qquad w(x)_{m,n} = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \; ; \; \lambda_{m,n} = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) \; ; \; m,n \geq 1 \tag{3.23}$$

$$\text{Neumann :} \quad w(x)_{m,n} = \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right) \; ; \; \lambda_{m,n} = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) \; ; \; m,n \geq 0 \tag{3.24}$$

Now, notice that the formula for $\lambda_{m,n}$ is actually the equation of an ellipse, if we allow $m, n \in \mathbb{R}$. This tells us that $N(\alpha)$ for $\alpha > 0$ is exactly the number of lattice points (m, n) in the first quadrant of the ellipse $\alpha = \pi^2(\frac{x^2}{a^2} + \frac{y^2}{b^2})$. If we consider all unit squares such that the coordinates of their top right edge is a lattice point (m, n), then, vol(1^{st} quadrant of ellipse)$\sim N(\alpha)$. Note that the discrepancy arises when unit squares of the aforementioned type are cut by the ellipse. Since the perimeter of the ellipse is $\propto \sqrt{\alpha}$ the error is $\mathcal{O}(\sqrt{\alpha})$. So, we get

$$N(\alpha) = \frac{\alpha \cdot ab}{4\pi} + \mathcal{O}(\sqrt{\alpha})$$

dividing by α and taking $\alpha \to \infty$, we recover Weyl's estimate for $N(\alpha)$ in dimension 2.

Remark *We claim that the same result holds for $\hat{N}(\alpha)$, the number of Neumann eigenvalues less than α. Indeed, the only difference from $N(\alpha)$, as we see from eqs.* (3.23) *and* (3.24)*, is that for the Neumann case, we count the eigenvalues lying on the x and y axes. This extra contribution to $\hat{N}(\alpha)$ is, however, not significant, since the number of such eigenvalues is $\propto \sqrt{\alpha}$. It is easy to see this since the number of eigenvalues on the x axis is given by all $m \in \mathbb{N}$, $m > 0$ such that $m^2 \leq \frac{\alpha}{\pi^2}a^2$ which implies $m \propto (\sqrt{\alpha})$.*

Finally, it is good to remark that since $-\Delta$ is invariant under isometric transformations, as seen in lemma 3.4, the results mentioned here hold for all rectangles in $\mathbb{R}^n$ regardless of position or orientation.

2. Let Ω be a domain such that $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_m}$, where Ω_i's are pairwise disjoint open rectangles. Let the Dirichlet eigenvalues on Ω be $\{\lambda_k\}_k$. Let λ'_k be the k^{th} eigenvalue in the increasing sequence obtained by combining and reordering the Dirichlet eigenvalues for $\Omega_1, \Omega_2, \ldots, \Omega_m$. Let μ'_k be the k^{th} eigenvalue in the corresponding combined Neumann eigenvalue sequence. Also, let $\lambda_k^{\Omega_i}$ and

$\mu_k^{\Omega_i}$ be the k^{th} Dirichlet and Neumann eigenvalues of Ω_i. Furthermore, let $N(\alpha)$, $N'(\alpha)$, $\hat{N}'(\alpha)$, $N_i(\alpha)$ and $\hat{N}_i(\alpha)$ be the largest k such that $\lambda_k \leq \alpha$, $\lambda'_k \leq \alpha$, $\mu'_k \leq \alpha$, $\lambda_k^{\Omega_i} \leq \alpha$ and $\mu_k^{\Omega_i} \leq \alpha$ respectively.

Then, by lemmas 3.11 and 3.12, we get $\mu'_k \leq \lambda_k \leq \lambda'_k$. Therefore

$$N'(\alpha) \leq N(\alpha) \leq \hat{N}'(\alpha)$$

Note that $\mu'_k \leq \lambda_k$ implies that $N(\alpha) \leq \hat{N}'(\alpha)$. It is also easy to see that $N'(\alpha) = \sum_{i=1}^{m} N_i(\alpha)$ and similarly, $\hat{N}'(\alpha) = \sum_{i=1}^{m} \hat{N}_i(\alpha)$. From the first part, we know that

$$\hat{N}_i(\alpha) = N_i(\alpha) = \sum_{i=1}^{m} \frac{\alpha \cdot Vol(\Omega_i)}{4\pi} + \mathcal{O}(\sqrt{\alpha})$$

Since the volume of Ω is the sum of volumes of Ω_i, we get

$$\frac{\alpha \cdot Vol(\Omega)}{4\pi} + \mathcal{O}(\sqrt{\alpha}) \leq N(\alpha) \leq \frac{\alpha \cdot Vol(\Omega)}{4\pi} + \mathcal{O}(\sqrt{\alpha})$$

Dividing by α and taking the limit as $\alpha \to \infty$ gives us the desired result.

3. Fix $\epsilon > 0$. Then, there exist domains Ω_1 and Ω_2 of the type considered in step 2 (essentially finite unions of disjoint rectangles), such that $\Omega_1 \subseteq \Omega \subseteq \Omega_2$ and $\text{vol}(\Omega_2) - \text{vol}(\Omega_1) < \epsilon$.

   By lemma 3.10, we can conclude that $\lambda_k^{\Omega_1} \geq \lambda_k^{\Omega} \geq \lambda_k^{\Omega_2}$ where each entry is the k^{th} Dirichlet eigenvalue of the Laplace operator on the domain indicated in the superscript. Then, $N_{\Omega_1}(\alpha) \leq N_{\Omega}(\alpha) \leq N_{\Omega_2}(\alpha)$. Since we have already shown the asymptotic results for Ω_1 and Ω_2, it follows that

$$\frac{\alpha \cdot Vol(\Omega_1)}{4\pi} + \mathcal{O}(\sqrt{\alpha}) \leq N_{\Omega}(\alpha) \leq \frac{\alpha \cdot Vol(\Omega_2)}{4\pi} + \mathcal{O}(\sqrt{\alpha})$$

   Finally, dividing by alpha, and taking the limit as $\alpha \to \infty$, we get

$$\frac{Vol(\Omega_1)}{4\pi} \leq \frac{N_{\Omega}(\alpha)}{\alpha} \leq \frac{Vol(\Omega_2)}{4\pi} \leq \frac{Vol(\Omega_1) + \epsilon}{4\pi}$$

   Finally, taking the limit as $\epsilon \to 0$, we get the required asymptotics.

It is easy to see that eqs. (3.20) and (3.21) are equivalent statements, so we can easily prove one given the other.

□

3.7 Courant's Nodal Domain Theorem

Courant's nodal domain theorem is a statement about the sign of eigenfunctions of the Laplace operator. We have seen proofs of the corresponding theorem in the context of graphs (theorem 1.6) and the Stürm-Liouville operator (theorem 2.17). The result we obtain here in the case of the operator $-\Delta$ is exactly the same as that on graphs, i.e an upper bound of $(k+r-1)$ on the number of nodal domains for the k^{th} eigenfunction where the corresponding eigenvalue has multiplicity r. In fact, the proof structure and ideas are very similar in the two cases as we will see. The case for the Stürm-Liouville operator is more special. Here, the reason that the upper bound of k is exactly achieved is because each eigenvalue is simple and that we can count the number of partitions of a given domain $[a, b]$ by simply counting the zeros of the corresponding eigenfunction. This fact immediately fails for dimension 2 onward.

We have seen from the proof of theorem 3.6 that the sign of w_1, the eigenfunction corresponding to eigenvalue λ_1 doesn't change sign inside the domain Ω. We can make similar statements for w_k, $k \geq 2$.

Remark *Since w_1 doesn't change sign and $\langle w_1, w_k \rangle_{L^2} = 0$ for all $k \geq 2$, we can easily conclude that w_k has to change sign in Ω. This is another way to prove that λ_1 is simple.*

Before we state the main theorem, let us define what a nodal domain is

Definition (Nodal Domain) *Given an eigenfunction w_k of the Laplace operator with Dirichlet or Neumann boundary conditions, a nodal domain S is a connected component[6] of one of the two disjoint open sets $\Omega_+ := \{x \mid w_k(x) > 0\}$ and $\Omega_- := \{x \mid w_k(x) < 0\}$*

Remark *$\Omega_- := \{x \mid w_k(x) < 0\}$ and it's positive counterpart are open because w_k is a continuous function since, if Ω is open and bounded then the eigenvalues of the Dirichlet or Neumann Laplace operator $-\Delta$ are smooth inside Ω by the regularity properties mentioned in appendix B.2.*

Theorem 3.14 (Courant's Nodal Domain Theorem) *Let w_k be the k^{th} eigenfunction of the Laplace operator $-\Delta$ with Dirichlet or Neumann boundary conditions for an open, connected and bounded domain Ω. Then, w_k has at most $k+r-1$ nodal domains. Where r is the multiplicity of the corresponding eigenvalue λ_k*

Proof: We proceed with a proof by contradiction. Suppose w_k has m nodal domains, $\Omega_1, \Omega_2, \ldots, \Omega_m$ where $m > k$. Further, suppose that the corresponding eigenvalue λ_k

[6] Where a connected component of a set is any maximally connected subset.

satisfies $\lambda_k < \lambda_{k+1}$. Define functions $f_i : \Omega \to \mathbb{R}$, $i = 1, 2, \dots, m$ such that

$$f_i(x) = \begin{cases} w_k(x) & \text{if } x \in \Omega_i \\ 0 & \text{Otherwise} \end{cases}$$

Note that all f_i's lie in $H_0^1(\Omega)$. This is true since w_k vanishes on the boundaries of Ω_i. Further, we claim that f_i is the first Dirichlet eigenfunction for the domain Ω_i with eigenvalue λ_k. Indeed, since f_i satisfies $-\Delta w_k(x) = -\Delta f_i$ for all x in Ω_i and by definition f_i doesn't change sign in Ω_i we are done by theorem 3.6. Also note that f_i's are pairwise orthogonal[7] to each other and that $w_k(x) = \sum_{i=1}^{m} f_i(x)$. Now, by the argument given in lemma 3.9, we can find $g \in span\{w_1, w_2, \dots, w_k\}^{\perp} \cap span\{f_1, f_2, \dots, f_m\}$ such that $g \not\equiv 0$. Let $g(x) = \sum_{i=1}^{m} d_i f_i(x)$. We get

$$\begin{aligned} B[g,g] = \|\nabla g\|^2_{L^2(\Omega)} &= \sum_{i=1}^{m} d_i^2 \|\nabla f_i\|^2_{L^2(\Omega)} \\ &= \lambda_k \sum_{i=1}^{m} d_i^2 \|f_i\|^2_{L^2(\Omega_i)} = \lambda_k \|g\|^2_{L^2(\Omega)} \end{aligned} \tag{3.25}$$

Since g is orthogonal to $w_1, \dots, w_k$ we use Rayleigh's theorem (3.7) to conclude that $B[g,g] \geq \lambda_{k+1} \|g\|^2_{L^2(\Omega)}$. However, eq. (3.25) tells us that $B[g,g] = \lambda_k \|g\|^2_{L^2(\Omega)}$. So we get the inequality $\lambda_k \geq \lambda_{k+1}$ which contradicts our assumption that $\lambda_k < \lambda_{k+1}$. This implies that $m \leq k$. But note that our assumption forces us to choose the largest possible index for sets of repeated eigenvalues. This means that given an ordering of the eigenvalues $\lambda_1 < \lambda_2 \leq \dots$, a choice of k gives us λ_k that is at most $r - 1$ positions away from the last repeated eigenvalue. Hence, we say that w_k has at most $k + r - 1$ nodal domains.

□

3.8 Cheeger's Constant and the Faber-Krahn Inequality

We will now move on to study how the eigenvalues of the Laplace operator encode information about the domain Ω on which the PDE is defined. Specifically, we will look at what information the first Dirichlet eigenvalue λ_1 contains about Ω. The two main results that we will see here are Cheeger's inequality and the Faber-Krahn inequality.

[7] Since their domains are disjoint.

Before we prove them, we need to mention an important mathematical formula that we will use to analyze them.

3.8.1 The Co-Area Formula

The co-area formula is given in the lemma below

Lemma 3.15 (Co-Area Formula) *Let Ω be an open bounded domain in $\mathbb{R}^n$. Let $f \in C^1(\Omega)$ be positive in Ω and extend to zero on $\partial\Omega$. Denote by $\Omega_t = \{f > t\} \subset \Omega$ and $V(t) = Vol(\Omega_t)$. Then,*

1. *V is continuous everywhere and differentiable a.e on the set $\{t \mid \nabla f(x) \neq 0, \forall f(x) = t\}$, and*
$$V'(t) = -\int_{\{f=t\}} \frac{1}{|\nabla f|} dA \tag{3.26}$$

2. *For any $g \in L^1(\Omega)$, we have*
$$\int_\Omega g\, dV = \int_0^\infty dt \int_{\{f=t\}} \frac{g}{|\nabla f|}\, dA \tag{3.27}$$

Remark *An interesting example to see the application of the co-area formula is on the disk. Let $f(x,y) = 1 - \sqrt{x^2+y^2}$, where x, y lie in the unit disk D and let $g \in L^1(D)$. Applying the co-area formula and noting that $|\nabla f(x,y)| = \left|\frac{(x,y)}{\sqrt{x^2+y^2}}\right| = 1$, we get*

$$\int_\Omega g\, dV = \int_0^1 dt \int_{\{f=t\}} g\, dA = \int_0^1 r\, dr \int_0^{2\pi} g\, d\theta$$

which, as we can see, is equivalent to a polar coordinate change.

From lemma 3.15, we immediately see that the first Dirichlet eigenfunction w_1 defined on a domain Ω with a smooth boundary satisfies the required conditions, allowing us to use the co-area formula eq. (3.27) with $f = w_1$ for any $g \in L^1(\Omega)$.

Remark *Note that the $V(t)$, from lemma 3.15, corresponding to $f = w_1$ is continuous for all $t \in [0, \infty]$. To prove this we start by considering all those $x' \in \Omega$ such that $\nabla w_1(x') = 0$. Since we can choose w_1 to be positive in Ω, we know that $-\Delta w_1 > 0$ in Ω. So, at any such x', we have $\Delta w_1(x') < 0$. This means that $\frac{\partial^2 w_1(x')}{\partial x_i^2} < 0$ for some i, where x_i is the i^{th} coordinate direction of x. Since $w_1 \in C^\infty(\Omega)$ we can choose an open neighborhood U around x' such that $\frac{\partial^2 w_1(x')}{\partial x_i^2} < 0$, $\forall x \in U$. Finally, using the implicit function theorem on $\frac{\partial w_1(x')}{\partial x_i}$, we conclude that the set of all $x \in U$ s.t. $\frac{\partial w_1(x)}{\partial x_i} = 0$ is measure zero since it is the graph of a C^∞ function.*

So, for any t, $V(t) - V(t+\delta) \to 0$ as $\delta \to 0$ since the contribution from the set of x s.t. $\nabla w_1(x) = 0,\ w_1(x) = t$ is zero.

3.8.2 Faber-Krahn Inequality

The Faber-Krahn inequality is similar to the isoperimetric inequality in that here, we want to find the shape of a domain that minimizes the value of the first Dirichlet eigenvalue. Interestingly, just like in the isoperimetric case, we find that the domain that minimizes λ_1 is the ball in $\mathbb{R}^n$.

Theorem 3.16 (Faber (1923), Krahn (1924)) *Let Ω be an open bounded domain with smooth boundary and B be a ball with the same volume as Ω. Then $\lambda_1(\Omega) \geq \lambda_1(B)$ with equality if and only if Ω is a ball.*

Proof: Let w_1 be the principal eigenfunction of $-\Delta$ on Ω, such that it is positive in Ω and $\sup_\Omega w_1 = 1$. We define

Definition (Schwarz rearrangement) *For any measurable set A in $\mathbb{R}$, let A^* denote the ball centered at the origin with $vol(A) = vol(A^*)$. Then, if u is a non-negative measurable function defined on Ω and vanishing on the boundary $\partial\Omega$, we denote by $\Omega_c = \{x \in \Omega \mid u(x) \geq c\}$ its super level sets. The Schwarz rearrangement of u is the function u^* defined on Ω^* as,*

$$u^*(x) = \sup\{c \mid x \in \Omega_c^*\} \tag{3.28}$$

Let us introduce the notation, $B := \Omega^*$. Let w_1^* be the Schwarz rearrangement of w_1. Note that by definition, $w_1^*(0) = 1$. Also, we can see that w_1^* is spherically symmetric and is monotonically decreasing in $|x|$. We see this since by eq. (3.28), $x \in B_c$ implies that $x' \in B_c$ only if $|x'| \leq |x|$. So, we see that for eq. (3.28), the RHS for x is contained inside the RHS for x', proving the monotonically decreasing property. Spherical symmetry can be seen in the case $|x'| = |x|$, where we conclude that the sets in the RHS are identical.

Since $\mathrm{Vol}(\Omega_t) = \mathrm{Vol}(B_t)$, by differentiating both sides with respect to t, and using eq. (3.26), we have that[8]

$$\int_{\{w_1=t\}} (\nabla w_1)^{-1} dS = \int_{\{w_1^*=t\}} (\nabla w_1^*)^{-1} dS \tag{3.29}$$

[8] Here we are assuming that w_1^* is $C^1(B)$. To make this proof more universal, it is possible to invoke the Pólya-Szegö theorem, which ensures that w_1^* is at least $H_0^1(B)$, and then use a more general form of the co-area formula that allows us to conclude 3.29 and the rest of the results. See [10], pg.85, theorem 4.1.1)

then using 3.27, we get

$$\int_{\Omega} w_1^2(x)\,dx = \int_0^1 dt \int_{\{w_1=t\}} \frac{w_1^2(x)}{|\nabla w_1(x)|} dS = \int_0^1 t^2 dt \int_{\{w_1=t\}} \frac{1}{|\nabla w_1(x)|} dS$$
$$= \int_0^1 t^2 dt \int_{\{w_1^*=t\}} \frac{1}{|\nabla w_1^*(x)|} dS = \int_{\Omega^*} (w_1^*(x))^2\,dx$$

meaning that $\|w_1\|^2_{L^2(\Omega)} = \|w_1^*\|^2_{L^2(B)}$.

Now, we recall that the isoperimetric inequality in our context states that for Ω_t and B_t such that $\mathrm{Vol}(\Omega_t) = \mathrm{Vol}(B_t)$, we have $\mathrm{Area}(\partial\Omega_t) \geq \mathrm{Area}(\partial B_t)$. Equivalently,

$$\int_{\{w_1=t\}} dS \geq \int_{\{w_1^*=t\}} dS$$

Squaring both sides and examining the LHS, we get

$$\left(\int_{\{w_1=t\}} \sqrt{\frac{|\nabla w_1|}{|\nabla w_1|}} dS\right)^2 \underbrace{\leq}_{\text{Cauchy-Schwarz}} \left(\int_{\{w_1=t\}} |\nabla w_1| dS\right)\left(\int_{\{w_1=t\}} |\nabla w_1|^{-1} dS\right) \tag{3.30}$$

For the RHS, note that w_1^* is spherically symmetric so ∇w_1^* is constant on the surface $\{w_1^* = t\}$. So, we get,

$$\left(\int_{\{w_1^*=t\}} dS\right)^2 = \left(\int_{\{w_1^*=t\}} |\nabla w_1^*| dS\right)\left(\int_{\{w_1^*=t\}} |\nabla w_1^*|^{-1} dS\right)$$

Putting the LHS and RHS together and using eq. (3.29), we get

$$\int_{\{w_1=t\}} |\nabla w_1| dS \geq \int_{\{w_1^*=t\}} |\nabla w_1^*| dS \tag{3.31}$$

Next, we define the functions, $\psi(t) := \|\nabla w_1\|^2_{L^2(\Omega_t)}$ and $\psi^*(t) := \|\nabla w_1^*\|^2_{L^2(B_t)}$. Then, note that

$$\psi(t) = \int_t^1 dt \int_{\{w_1=t\}} |\nabla w_1| dS \implies \psi'(t) = -\int_{\{w_1=t\}} |\nabla w_1| dS$$

similarly, we obtain the corresponding result for $(\psi^*)'(t)$. Finally, note that eq. (3.31) tells us that $(\psi^*)'(t) \geq \psi'(t)$. Also, $\psi(1) = \psi^*(1) = 0$. By integrating both sides for $t = 0$ and using this initial condition, we end up with $\|\nabla w_1^*\|^2_{L^2(B_t)} \leq \|\nabla w_1\|^2_{L^2(\Omega_t)} =$

$\lambda_1(\Omega)\|w_1\|_{L^2(\Omega)}$. Finally, by theorem 3.6, we get that $\lambda_1(B)\|w_1^*\|_{L^2(B_t)} \leq \|\nabla w_1^*\|_{L^2(B)}$.
$\square$

Remark *In the case where $\lambda_1(\Omega) = \lambda_1(B)$ we are forced to need equality to hold both in the isoperimetric inequality, which immediately tells us that Ω is a ball, and in eq. (3.30), which tells us that ∇w_1 is constant on level curves, implying that the first eigenfunction of $-\Delta$ on the ball domain is spherically symmetric.*

3.8.3 Cheeger's Inequality

Definition *For a open bounded domain Ω with smooth boundary, we define the Cheeger's constant $\mathfrak{h}(M)$ as*

$$\mathfrak{h} = \inf_{\Omega'} \frac{Area(\partial\Omega')}{Vol(\Omega')}$$

where, Ω' ranges over all open subsets of Ω with smooth boundary (so that the 'area' function is well defined on $\partial\Omega'$).

Theorem 3.17 (Cheeger) *Let λ_1 be the principal eigenvalue of $-\Delta$ for the domain Ω as defined above. Then, we have*

$$\lambda_1 \geq \frac{\mathfrak{h}^2}{4}$$

Proof: The proof of this theorem is fairly straightforward once we have the co-area formula. Let w_1 be the principal eigenfunction of $-\Delta$ corresponding to the eigenvalue λ_1, then by B.4, we get $w_1 \in C^\infty(\overline{\Omega})$. Moreover, we can choose w_1 to be positive in all of Ω. Next, note that $\nabla w_1^2 = 2w_1\nabla w_1$. Using Hölder's inequality, we get

$$\int_\Omega \left|\nabla w_1^2\right| dx \leq 2\|w_1\|_{L^2(\Omega)}\|\nabla w_1\|_{L^2(\Omega)}$$

We return to this result later. First, let us try to find bounds for $\int_\Omega |\nabla w_1^2|$. Note that $B[w_1, w_1] = \lambda_1$. Using this, we get

$$\lambda_1 = \frac{\|\nabla w_1\|^2_{L^2(\Omega)}}{\|w_1\|^2_{L^2(\Omega)}} \geq \frac{1}{4}\frac{\left(\int_\Omega |\nabla w_1^2| dx\right)^2}{\|w_1\|^2_{L^2(\Omega)}} \tag{3.32}$$

Using the co-area eq. (3.27) for $f = w_1^2$, we get

$$\int_\Omega |\nabla f| dx = \int_0^\infty dt \int_{\{f=t\}} 1 \cdot dS$$

but note that $\int_{\{f=t\}} 1 \cdot dS = \text{Area}(\partial\Omega_t)$ where $\Omega_t = \{x \mid f(x) > t\}$. Then, by the definition of Cheeger's constant,

$$\begin{aligned}
\int_\Omega |\nabla f| dx = \mathfrak{h} \int_0^\infty \text{Vol}(\Omega_t) dt = \mathfrak{h} \int_0^\infty \mu\{f > t\} dt \\
= \mathfrak{h} \overbrace{\int_0^\infty \int_\Omega \chi_{(f(x)>t)} dx \cdot dt = \mathfrak{h} \int_\Omega \int_0^\infty \chi_{(f(x)>t)} dt \cdot dx}^{\text{Fubini-Tonelli}} \\
= \mathfrak{h} \int_\Omega \int_0^{f(x)} dt \cdot dx = \mathfrak{h} \int_\Omega f(x) dx
\end{aligned}$$

Finally, replacing $f = w_1^2$ and using eq. (3.32), we get the required result. □

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Appendix A

Sobolev Spaces

This section contains all the properties related to Sobolev Spaces that were studied in order to make sense of the Laplacian as defined on domains in $\mathbb{R}^n$. We start with the definition of weak derivatives.

Definition (Weak Derivative) *Let Ω be an open subset of $\mathbb{R}^n$. Suppose $u, v \in L^1_{loc}(\Omega)$ and α is a multi index. Then, v is called the α^{th} weak partial derivative of u or $D^\alpha u = v$ if,*

$$\int_\Omega u(x) D^\alpha \phi(x) = (-1)^{|\alpha|} \int_\Omega v(x)\phi(x) \;, \;\; \forall \phi \in C_c^\infty(\Omega) \tag{A.1}$$

Here, ϕ is called a test function.

Remark *Sobolev spaces are a large class of function spaces that allow us to 'extend' the notion of differentiability (to weak differentiability) to include a wider, more general class of functions. We need them primarily because the spaces are Banach so we need not worry about functional limits 'exiting' the space. This is useful when trying to prove existence and uniqueness theorems about functions in such spaces.*

For our purposes, we only need the Sobolev spaces $H^1(\Omega)$ and $H^1_0(\Omega)$. We define them below.

Definition ($H^1(\Omega)$) *For a domain Ω, the Sobolev space $H^1(\Omega)$ consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each $i = 1, 2, \ldots, n$, $\frac{\partial u}{\partial x_i}$ exists in weak sense and belongs to $L^2(\Omega)$.*

Definition ($H^1(\Omega)$ norm) *The Sobolev space $H^1(\Omega)$ is a normed linear space under the norm*

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right)^{1/2} \tag{A.2}$$

Definition ($H_0^1(\Omega)$) *For a domain Ω, the Sobolev space $H_0^1(\Omega)$ is the completion of $C_c^\infty(\Omega)$ in $H^1(\Omega)$ under the norm $\|\cdot\|_{H^1(\Omega)}$.*

As stated above, the following theorem is true for all Sobolev spaces in general and for H^1 and H_0^1 in particular.

Theorem A.1 (Sobolev Spaces are Banach) *$H^1(\Omega)$ and $H_0^1(\Omega)$ are Banach spaces.*

A.1 Approximations

We now move on to the approximation theorems that allow us to prove many properties about Sobolev functions by first using $C^\infty(\Omega)$ functions and then generalizing to Sobolev functions by taking the limit of an approximating sequence.

Theorem A.2 (Local Approximation) *Assume $u \in H^1(\Omega)$ for and set*

$$u^\epsilon = \eta_\epsilon * u \quad in\ \Omega_\epsilon \tag{A.3}$$

Where $\Omega_\epsilon = \{x \in \Omega \mid dist(x, \partial\Omega) > \epsilon\}$. Then,

1. *$u^\epsilon \in C^\infty(\Omega)$ for each $\epsilon > 0$*
2. *$u^\epsilon \to u$ in $H^1_{loc}(\Omega)$, as $\epsilon \to 0$*

Theorem A.3 (Global Approximation by Smooth Functions) *Assume Ω is bounded, and suppose as well that $u \in H^1(\Omega)$. Then, there exist functions $u_m \in C^\infty(\Omega) \cap H^1(\Omega)$ such that*

$$u_m \to u \quad , \quad in\ H^1(\Omega) \tag{A.4}$$

Note that boundedness is not enough to guarantee an approximating sequence that is $C^\infty(\bar{\Omega})$. For that, we need an additional constraint on the smoothness of the boundary. More precisely,

Theorem A.4 (Global Approximation by Functions Smooth up to the Boundary) *Assume Ω is bounded and $\partial\Omega$ is C^1. Suppose $u \in H^1(\Omega)$ then there exist functions $u_m \in C^\infty(\bar{\Omega})$ such that $u_m \to u$ in $H^1(\Omega)$.*

A.2 Extension Theorem

The extension theorem allows us to 'extend' functions in $H^1(\Omega)$ to the whole of $\mathbb{R}^n$. It is useful because the extension preserves the weak derivative whilst simultaneously keeping the norm of the extended function comparable to its norm inside Ω. In fact, the change in norm is uniform across all $u \in H^1(\Omega)$. Moreover, the extension is given by a linear operator. Because of the aforementioned uniformity, it is bounded.

Theorem A.5 *Assume Ω is bounded and $\partial\Omega$ is C^1. Select a bounded open set V such that $\Omega \subset\subset V$. Then, there exists a bounded linear operator $E : H^1(\Omega) \to H^1(\mathbb{R}^n)$ such that for each $u \in H^1(\Omega)$,*

1. *$Eu = u$ a.e in Ω.*
2. *Eu has support within V.*
3. *$\|Eu\|_{H^1(\mathbb{R}^n)} \leq k\|u\|_{H^1(\Omega)}$ where k depends only on Ω and V.*

A.3 Trace Theorem

Since $L^2(\Omega)$ functions are defined a.e, it doesn't make sense to talk about their restriction to zero measure sets and in particular, to sets such as $\partial\Omega$ (assuming it is at least piecewise C^1). So, to assign boundary values to u is meaningless. Things become more complicated in $H^1(\Omega)$ because the existence of weak derivatives is sensitive to value changes in such sets. One way to make sense of what it means to 'extend' an $H^1(\Omega)$ function to the boundary is to use an approximating sequence of continuous functions defined on $\bar{\Omega}$ and find the limit of their restriction to $\partial\Omega$ in the surface norm. Since there can be many different convergent sequences of functions to a given $u \in H^1(\Omega)$ we get an $L^2(\Omega)$ function in the restriction. Interestingly, this operation just like the extension theorem, is carried out by a bounded linear operator. The details are given below

Theorem A.6 *Assume Ω is bounded and $\partial\Omega$ is C^1. Then, there exists a bounded linear operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ such that,*

1. *$Tu = u|_{\partial\Omega}$ if $u \in H^1(\Omega) \cap C(\bar{\Omega})$.*
2. *$\|Tu\|_{L^2(\partial\Omega)} \leq k\|u\|_{H^1(\Omega)}$ for each $u \in H^1(\Omega)$, where k depends only on Ω.*

Theorem A.7 (Trace zero functions in $H^1(\Omega)$) *Assume Ω is bounded and $\partial\Omega$ is C^1. Suppose u in $H^1(\Omega)$. Then,*

$$u \in H_0^1(\Omega) \quad \text{if and only if} \quad Tu = 0 \;\text{ on }\; \partial\Omega. \tag{A.5}$$

A.4 Sobolev Inequalities

Theorem A.8 *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n > 2$ and suppose $\partial\Omega$ is C^1. Assume $u \in H^1(\Omega)$. Then for $p^* = \frac{2n}{n-2}$, we have $u \in L^{p^*}(\Omega)$ with the estimate*

$$\|u\|_{L^{p^*}(\Omega)} \leq k\|u\|_{H^1(\Omega)} \tag{A.6}$$

where the constant k depends only on n and Ω.

Theorem A.9 *Assume Ω is a bounded open subset of $\mathbb{R}^n$, $n > 2$. Suppose $u \in H_0^1(\Omega)$. Then, we have the estimate*

$$\|u\|_{L^q(\Omega)} \leq k\|Du\|_{L^2(\Omega)} \tag{A.7}$$

where, $q \in [1, \frac{2n}{n-2}]$ and k depends only on q, n and Ω

A.5 Compactness

Definition *Let X, Y be Banach spaces such that $X \subset Y$. We say that X is compactly embedded in Y, written $X \subset\subset Y$, provided*

1. *$\|u\|_Y \leq k\|u\|_X$ where $u \in X$ and for some constant k.*
2. *Each bounded sequence in X is precompact in Y.*

Theorem A.10 (Rellich-Kondrachov compactness theorem) *Assume Ω is a bounded open subset of $\mathbb{R}^n$ and $\partial\Omega$ is C^1. Then, $H^1(\Omega) \subset\subset L^q(\Omega)$ for each $q \in [1, \frac{2n}{n-2})$.*

Remark *This theorem is important in that it gives us the compactness of the inverse of elliptic operators.*

A.6 The dual of $H_0^1(\Omega)$ $\big($ *i.e.* $H^{-1}(\Omega)\big)$

Definition *We denote by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$. Moreover, for $f \in H^{-1}(\Omega)$, we define the norm,*

$$\|f\|_{H^{-1}(\Omega)} := \sup\{\langle f, u\rangle \mid u \in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)} \leq 1\} \tag{A.8}$$

Remark *It can be shown that $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$*

Appendix B

Second Order Elliptic Equations

B.1 Background theorems for Existence and Uniqueness

Let H be a Hilbert space. Then,

Theorem B.1 (Lax-Milgram) *Assume that $B : H \times H \to \mathbb{R}$ is a bilinear mapping for which there exist constants $\alpha, \beta > 0$ such that*

$$|B[u,v]| \leq \alpha \|u\| \|v\| \ \ ; \ u,v \in H \tag{B.1}$$

and

$$\beta \|u\|^2 \leq B[u,u] \ \ ; \ u \in H \tag{B.2}$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H. The there exists a unique element $u \in H$ such that $B[u,v] = \langle f, v \rangle$ for all $v \in H$.

Remark

Theorem B.2 (Energy Estimates) *There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that*

$$|B[u,v]| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \tag{B.3}$$

and

$$\beta \|u\|^2_{H_0^1(\Omega)} \leq B[u,u] + \gamma \|u\|^2_{L^2(\Omega)} \tag{B.4}$$

for all $u, v \in H_0^1(\Omega)$.

B.2 Regularity Results

This section contains two important theorems that justifies our study of weak solutions. It states the setting in which weak solutions and the classical (i.e. solutions that are at least $C^2(\Omega)$) solutions coincide. This gives us a way to guarantee the existence of solutions and eigenfunctions that are differentiable in the usual sense.

Theorem B.3 (Infinite differentiability in the interior) *Assume $a_{ij}, b_i, c \in C^\infty(\Omega)$ where $i, j = 1, 2, \ldots, n$ and $f \in C^\infty(\Omega)$. Suppose $u \in H^1(\Omega)$ is a weak solution of the elliptic PDE $Lu = f$ in Ω. Then,*

$$u \in C^\infty(\Omega) \tag{B.5}$$

Theorem B.4 (Infinite differentiability up to the boundary) *Assume $a_{ij}, b_i, c \in C^\infty(\bar{\Omega})$ where $i, j = 1, 2, \ldots, n$ and $f \in C^\infty(\bar{\Omega})$. Suppose $u \in H_0^1(\Omega)$ is a weak solution of the boundary value problem*

$$\begin{cases} Lu = f & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{B.6}$$

Assume also that $\partial\Omega$ is C^∞. Then,

$$u \in C^\infty(\bar{\Omega}) \tag{B.7}$$

B.3 Maximum Principles

For the specific case of $L = -\Delta$, we use the regularity properties in B.2 to conclude that for Ω open and bounded, with $\partial\Omega$ smooth, the solutions to the Dirichlet/Neumann problem lie in $C^\infty(\Omega)$. This allows us to study the action of the Laplacian on these solutions in the usual way. The

Theorem B.5 (Strong maximum principle) *Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$ where Ω is a connected open and bounded set in $\mathbb{R}^n$.*

1. *If $-\Delta u \leq 0$ in Ω and u attains its maximum over $\bar{\Omega}$ at an interior point then u is constant within Ω.*

2. *Similarly, if $-\Delta u \geq 0$ in Ω and u attains its minimum over $\bar{\Omega}$ at an interior point then u is constant within Ω.*

Appendix C

Hilbert Spaces

Let H be a Hilbert space.

C.1 Fredholm Theory

Theorem C.1 (Fredholm Alternative) *Let $K : H \to H$ be a compact linear operator. Then,*

1. *The dimension of the null space of $I - K$, written as $N(I - K)$ is finite,*
2. *$Range(I - K)$ is closed,*
3. *$Range(I - K) = N(I - K^*)^\perp$, where K^* is the adjoint of K,*
4. *$N(I - K) = \{0\}$ if and only if $R(I - K) = H$,*
5. *The dimension of $N(I - K)$ equals the dimension of $N(I - K^*)$*