Fundamental Aspects of Quantum Mechanics and Their Interplay with Quantum Information

Study of Modern Variations of the Wigner's Friend Thought Experiment

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by

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under the guidance of

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Certificate

This is to certify that this dissertation entitled **Fundamental Aspects of Quantum Mechanics and Their Interplay with Quantum Information** towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Vighnesh Digamber Vernekar at Bose Institute, Kolkata under the supervision of Prof. Dipankar Home, Department of Physics, Centre for Astroparticle Physics and Space Science, Bose Institute, Kolkata during the academic year 2019-20.

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Declaration

I hereby declare that the matter embodied in the report entitled 'Fundamental Aspects of Quantum Mechanics and Their Interplay with Quantum Information' are the results of the work carried out by me at the Centre for Astroparticle Physics and Space Science, Department of Physics, Bose Institute, Kolkata, under the supervision of Prof. Dipankar Home, and the same has not been submitted elsewhere for any other degree.

Vighnesh Digamber Vernekar.

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Abstract

Quantum Mechanics has been a spectacularly successful theory. It correctly predicts and explains the phenomena of subatomic particles, nuclei, atoms, molecules et cetera. Whether quantum mechanics applies to observers has been a long thought problem. Wigner's Friend is a thought experiment that illustrates this. Of late, there has been a surge of interest in the Wigner's Friend thought experiment. Extended Wigner Friend Scenarios (EWFS) involving 'entangled friends' have been proposed to test whether quantum theory is applicable to observers. In this work, we systematically analyze the recently proposed 'Local Friendliness' inequalities in the context of bipartite EWFS with respect to pure states and the Werner states. Further, we formulate various Extended Wigner Friend Scenarios, and identify the trivial scenarios. For each of these EWFS, we attempt to specify the structure of the Local Friendliness polytope, thereby characterizing the correlations allowed by the Local Friendliness assumptions. In the scenarios where 'Genuine Local Friendliness' inequalities have been found, we provide their quantum bounds.

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Chapter 1

Introduction

1.1 Preliminaries

In this section we review some concepts and terms commonly used. We use the material primarily from the text [1].

1.1.1 Postulates of Quantum Mechanics

1. *Postulate 1.* Every closed physical system has an associated Hilbert space $\mathcal{H}$, known as the state space. The system is described completely by a unit vector $|\psi\rangle \in \mathcal{H}$ called the state vector.

2. *Postulate 2.* The time evolution of a closed system is unitary. A state $|\psi_1\rangle$ at time t_1 is related to the state $|\psi_2\rangle$ at time t_2 by a unitary operator U which depends only on t_1 and t_2 by
$$|\psi_2\rangle = U(t_1, t_2)|\psi_1\rangle$$

3. *Postulate 3.* Measurement in quantum mechanics is described by measurement operators acting on the system. We denote by $\{M_m\}$ a collection of measurement operators. The index m refers to the possible outcomes of the measurement. The measurement operators satisfy the *completeness relation*, $\sum_m M_m^\dagger M_m = \mathbb{I}$. If a quantum mechanical system is in the state $|\psi\rangle$ just before the measurement, the probability of getting the outcome m is given by the Born rule
$$P(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$
The state of the system post-measurement is given by
$$|\psi'\rangle = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

A *positive operator-valued measure* (POVM) is a collection $\{E_m\}$ of positive operators E_m where $E_m := M_m^\dagger M_m$. It is clearly evident that

$$\sum_m E_m = \mathbb{I} \quad \text{and} \quad P(m) = \langle\psi|E_m|\psi\rangle$$

The set $\{E_m\}$ is sufficient to determine the probability of an outcome when a measurement is made.

A *projective measurement* is described by a Hermitian operator M—called an observable—acting on the state space of the system being observed. The observable M can be written as

$$M = \sum_m \lambda_m |m\rangle\langle m|$$

where $\{|m\rangle\}$ is an orthonormal basis of eigenvectors of M and λ_m are the corresponding eigenvalues. The term $|m\rangle\langle m|$ represents the projection onto the eigenspace of M with the eigenvalue λ_m. If a system is in the state $|\psi\rangle$ just before the measurement, the probability of getting the outcome λ_m is given by

$$P(m) = \langle\psi|m\rangle\langle m|\psi\rangle = |\langle m|\psi\rangle|^2$$

The post-measurement state is given by

$$|\psi'\rangle = \frac{|m\rangle\langle m||\psi\rangle}{\sqrt{P(m)}} = |m\rangle$$

Projective measurements—also called *von Neumann Measurements*—are a class of POVMs. We will mostly use projective measurements in this work.

4. *Postulate 4.* The state space of a composite system is given by the tensor product of the state spaces of the component physical systems. Consider two systems in states $|\psi_1\rangle$ and $|\psi_2\rangle$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ as their respective state spaces. The state space of the joint system is given by $\mathcal{H}_1 \otimes \mathcal{H}_2$. If we have operators A_1 and A_2 acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, the action of the joint operator $A_1 \otimes A_2$ is given by

$$A_1 \otimes A_2(|\psi_1\rangle \otimes |\psi_2\rangle) = A_1|\psi_1\rangle \otimes A_2|\psi_2\rangle$$

1.1.2 Miscellany

- A quantum mechanical system whose state $|\psi\rangle$ is known exactly is said to be in a *pure state*. A pure state cannot be represented as a mixture of other states. The *density matrix* or the *density operator* for a pure state $|\psi\rangle$ is defined by

$$\rho = |\psi\rangle\langle\psi|$$

- A *mixed state* is a statistical mixture of pure states. The *density matrix* for a mixed state, where the quantum mechanical system is in the state $|\psi_i\rangle$ with probability p_i is given by

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

- The density matrix ρ is a positive operator and its trace is equal to 1. For a pure states we have $\mathrm{tr}(\rho^2) = 1$, while for mixed states we have $\mathrm{tr}(\rho^2) < 1$.

- Every closed system can be described by a density operator acting on the state space of the system. The unitary time evolution of a density operator from ρ_1 to ρ_2 is given by

$$\rho_2 = U(t_1, t_2)\ \rho_1\ U(t_1, t_2)^\dagger$$

- For a quantum mechanical system in the state ρ just before the measurement M_m, the probability of getting the outcome m is given by

$$P(m) = \mathrm{tr}(\rho M_m)$$

- A *qubit* is a unit of quantum information. The information is described by the state of a two-level quantum mechanical system whose state space is a two-dimensional vector space over complex numbers, i.e. $\mathbb{C}^2$. The orthonormal basis states of this state space are conventionally represented as $|0\rangle$ and $|1\rangle$. An arbitrary state can be represented as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \text{with } \alpha, \beta \in \mathbb{C}^2 \quad \text{and } \alpha^2 + \beta^2 = 1$$

  The two-level quantum mechanical system is usually taken to be a spin-$\frac{1}{2}$ particle. The $|0\rangle$ and $|1\rangle$ basis states are then the spin up and spin down states along some chosen axis, usually the z-axis.

- A state $|\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is called *separable* if it can be written as the product $|\psi_1\rangle \otimes |\psi_2\rangle$ for $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$.

- If a state is not separable, it is called *entangled*. For example the following two qubit state is an entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

  where we write $|ab\rangle$ for $|a\rangle \otimes |b\rangle$.

1.2 The Measurement Problem in Quantum Mechanics

Time evolution of an isolated system in quantum mechanics is unitary, reversible and deterministic. The measurement of a dynamical variable of the system results in a 'collapse of the wavefunction', and the state of the system after the measurement is one of the eigenstates of the corresponding measurement operator. The probability of obtaining this eigenstate is given by Born rule. The post-measurement state update is non-unitary, irreversible and probabilistic. The measurement problem in quantum mechanics refers to the incompatibility of unitary, deterministic time evolution with the non-unitary, probabilistic state update after measurement.

1.3 Wigner's Friend Thought Experiment

Wigner's Friend, a thought experiment proposed by Wigner [2] in 1961, illustrates the measurement problem. The experiment involves a 'friend' in an isolated lab measuring a dynamical variable of a quantum mechanical system. For brevity, we consider the quantum mechanical system to be a spin-$\frac{1}{2}$ particle and the dynamical variable to be z-spin. Further, let's say that the initial state of the particle is

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|z_+\rangle + |z_-\rangle \right) .$$

Upon measurement, the friend obtains one of the outcomes, $|z_+\rangle$ or $|z_-\rangle$ with equal probabilities. There is a collapse of the wavefunction and the state of the particle, depending on the observed outcome, is $|z_+\rangle$ or $|z_-\rangle$. Meanwhile, Wigner—the 'Superobserver', outside the lab, considers the entire lab to be a quantum mechanical system evolving unitarily. Wigner assigns the joint system of friend and the particle the state

$$|\phi\rangle = \frac{1}{\sqrt{2}} \left(|z_+\rangle \otimes |F_+\rangle \right) + \frac{1}{\sqrt{2}} \left(|z_-\rangle \otimes |F_-\rangle \right)$$

where $|F_\pm\rangle$ refer to Friend's state. Wigner opens the laboratory and asks the friend the result of his measurement. Depending on the friend's answer, Wigner assigns his system the state $|\phi\rangle = |z_+\rangle \otimes |F_+\rangle$ or $|\phi\rangle = |z_-\rangle \otimes |F_-\rangle$. From Wigner's perspective, this is when the collapse of the wavefunction of the system occurs. According to the friend however, the measurement result was determined long before Wigner inquires about it. When we consider the perspectives of Wigner and his friend as equally valid, we have a paradox.

1.4 Recent Interest in Wigner's Friend Thought Experiment

Of late, there has been a surge of interest in this longstanding paradox [3–6]. In 2018, Frauchiger and Renner [3] used a modified Wigner's Friend scenario to investigate whether quantum mechanics is universally applicable. Further, Brukner [4] proposed an Extended Wigner's

Friend Scenario (EWFS) involving spatially separated 'entangled friends' and derived a No-Go Theorem. The no-go theorem states that if quantum mechanics is applicable on the scale of observers, then, one of the following:

(a). Observer Independent Facts

(b). Freedom of Choice

(c). Locality

should be violated. Brukner claimed that the assumptions (a), (b) and (c) lead to a deterministic Local Hidden Variable model, and hence to to Bell inequalities. Using an optimal state and measurement operators, Brukner showed that Bell Inequality is violated for the Extended Wigner Friend Scenario.

Some loopholes in Brukner's argument were pointed out by Healey [5]: Brukner's argument depends on a *postulate*: There is a matter of fact about the results of all the measurements, even unperformed ones. This postulate is equivalent to the assumption that all possible measurement outcomes are predetermined by hidden variables [6]. Brukner claimed that this postulate follows from the assumption of Observer Independent Facts (OIF), but this claim was not justified in Brukner's paper.

Subsequently, the authors in [6] addressed Healey's concerns and formalized Brukner's arguments in [6]. Further, they also showed that the assumptions (a) OIF, (b) Freedom of Choice, (c) Locality (the conjunction of these three assumptions is named 'Local Friendliness') do not lead to a deterministic LHV model in the general case.

We build upon the work in [6], and formulate various Extended Wigner Friend Scenarios. In each of these scenarios, we characterize the correlations implied by Local Friendliness assumptions by specifying the structure of the Local Friendliness polytope. Where we find new inequalities, we compute the maximum violation of the inequalities allowed by quantum theory using semidefinite programming. We also analyze the inequalities found in [6] for pure and mixed states.

Chapter 2

Theory

2.1 Bell Scenario

Alice and Bob are space-like separated and share a system on which they can make measurements. Let $\mathbf{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathbf{B} = \{B_1, B_2, \ldots, B_n\}$ denote their respective set of possible measurements. Each of the measurements results in one of the two possible outcomes, 0 or 1. Alice's measurement outcome is denoted a, Bob's b. Each run of the experiment involves Alice and Bob choosing and performing one of the measurements and obtaining an outcome. We will call this setup a *Bell scenario*. The Bell scenario with m number of measurements for Alice and n measurements for Bob will be referred to as an mn Bell scenario. It is not necessary that a Bell scenario be limited to two outcome measurements. In this thesis however, we consider only the scenarios where measurements result in one of two possible outcomes.

We denote by $P(ab|A_xB_y)$ the joint probability of getting the outcome pair 'ab' when measurements 'A_xB_y' are performed. For example, let us say Alice chooses the measurement A_1 and Bob chooses the measurement B_2. Then the probability that Alice obtains the outcome '0' *and* Bob obtains the outcome '1' is denoted $P(01|A_1B_2)$. We denote by $P(a|A_xB_y)$ and $P(b|A_xB_y)$ the marginal probabilities of Alice and Bob respectively. The marginal probabilities are defined by

$$P(a|A_xB_y) = \sum_{b=0}^{1} P(ab|A_xB_y)$$
$$P(b|A_xB_y) = \sum_{a=0}^{1} P(ab|A_xB_y)$$

The no-signaling constraints eq. (2.2) introduced in section 2.3 imply that $P(a|A_xB_y) \equiv P(a|A_x)$ and $P(b|A_xB_y) \equiv P(b|A_xB_y)$. In the dichotomic outcome case we are interested in, the Bell scenario is completely characterized by the $2m \times 2n = 4mn$ joint probabilities. The set of $4mn$ joint probabilities is called a *behavior*. Every behavior can be thought of as a point in a subspace

$\mathbb{P}$ of $\mathbb{R}^{4mn}$ [7]. The subspace $\mathbb{P}$ is the probability space defined by the constraints

$$P(ab|A_xB_y) \geq 0 \quad \text{(Positivity)}$$
$$\sum_{a,b} P(ab|A_xB_y) = 1 \quad \text{(Normalization)}$$

Using the notation introduced in [8], a behavior $\mathbf{p} \in \mathbb{P}$ can be conveniently represented in the form of a $2m \times 2n$ matrix:

$$\mathbf{p} = \left(\begin{array}{cc|ccc|cc} P(00|A_1B_1) & P(01|A_1B_1) & \dots & \dots & \dots & P(00|A_1B_n) & P(01|A_1B_n) \\ P(10|A_1B_1) & P(11|A_1B_1) & \dots & \dots & \dots & P(10|A_1B_n) & P(11|A_1B_n) \\ \hline \vdots & \vdots & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ \hline P(00|A_mB_1) & P(01|A_mB_1) & \dots & \dots & \dots & P(00|A_mB_n) & P(01|A_mB_n) \\ P(10|A_mB_1) & P(11|A_mB_1) & \dots & \dots & \dots & P(10|A_mB_n) & P(11|A_mB_n) \end{array}\right) \tag{2.1}$$

The correlators $\langle A_x \rangle, \langle B_y \rangle, \langle A_xB_y \rangle$ are defined by

$$\langle A_x \rangle = \sum_a (-1)^a P(a|A_x)$$
$$\langle B_y \rangle = \sum_b (-1)^b P(b|B_y)$$
$$\langle A_xB_y \rangle = \sum_{a,b} (-1)^{a+b} P(ab|A_xB_y)$$

We may at times choose to denote the outcomes of Alice and Bob by $\{-1, 1\}$ instead of $\{0, 1\}$. In such a case the correlators are defined by

$$\langle A_x \rangle = \sum_a aP(a|A_x)$$
$$\langle B_y \rangle = \sum_b bP(b|B_y)$$
$$\langle A_xB_y \rangle = \sum_{a,b} abP(ab|A_xB_y)$$

2.2 Elements of Polytope Theory

Definition 1. A set $C \subseteq \mathbb{R}^n$ is called *convex* if for all $a, b \in C$, the line joining a and b lies entirely within C.

$$\lambda x + (1 - \lambda)y \in C \quad \forall x, y \in C \text{ and } \forall \lambda \in [0, 1].$$

Definition 2. The *convex hull* of a set of points **S** is the smallest convex set containing **S**.

Definition 3. A *polyhedron* is a set

$$P = \{x \in \mathbb{R}^n : Ax \leq b, Cx = d\}, \quad A, C \in \mathbb{R}^{m\times n}, m \geq n$$

Definition 4. A polyhedron is *bounded* if there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in P$.

Definition 5. A *polytope* is a bounded polyhedron.

Definition 6. The representation of the polytope in terms of halfspaces and hyperplanes is called the *H-representation*.

Definition 7. A polytope can be represented as the convex hull of the extreme points of the polytope. Such a representation in is called the *V-representation*.

Theorem 1 (*Minkowsky-Weyl*)**.** Every polytope has an H-representation and a V-representation. The H-representation and V-representation of a polytope are equivalent.

Definition 8. The conversion from the V-representation to the H-representation is called *facet enumeration*.

Definition 9. The conversion from the H-representation to the V-representation is called *vertex enumeration*.

A detailed introduction to polytope theory can be found in [9]. There are algorithms to convert between the H and V representations, as well as software implementations of the algorithms. PANDA [10], PORTA [11], polymake [12], `lrs` [13] are some examples of the software implementations.

2.3 No-Signaling Behavior

A natural constraint on the behaviors $\mathbf{p} \in \mathbb{P}$ apart from the positivity and normalization conditions is the no-signaling principle—Alice cannot signal to Bob by her measurement choice and vice-versa. The no-signaling principle implies that Alice's marginal probability distribution, $P(a|x)$ be independent of Bob's measurement choice, and Bob's marginal probability distribution, $P(b|y)$ be independent of Alice's measurement choice.

$$\begin{aligned}
\sum_b P(ab|A_xB_y) &= \sum_b P(ab|A_xB_{y'}) &&= P(a|A_x) &&\forall\, a, A_x, B_y, B_{y'} \\
\sum_a P(ab|A_xB_y) &= \sum_a P(ab|A_{x'}B_y) &&= P(b|B_y) &&\forall\, b, A_x, A_{x'}, B_y
\end{aligned} \tag{2.2}$$

Let $\mathcal{NS} = \{P(ab|xy)\}$ be the set of all joint probabilities that satisfy the positivity, normalization and the no-signaling constraints. The number of constraints is finite and the constraints are all linear. The set $\mathcal{NS}$ hence, is a polyhedron. It should be evident from the normalization and positivity conditions that the set $\mathcal{NS}$ is bounded. Nevertheless, as $P(ab|xy)$ are all probabilities, we have $P(ab|xy) \leq 1$ for all $P(ab|xy)$. So the set $\mathcal{NS}$ is bounded, which implies that $\mathcal{NS}$ is a polytope. We will refer to it as the no-signaling polytope.

2.3.1 The 22 No-signaling polytope

Alice and Bob have two measurements each to choose from and each of the measurement results in one of the two possible outcomes, 0 or 1.

$$\mathbf{A} = \{A_1, A_2\}, \quad \mathbf{B} = \{B_1, B_2\}$$

The constraints on the joint probabilities $P(ab|A_iB_j)$ are:

- Positivity:

$$P(ab|A_iB_j) \geq 0 \quad \forall\, a, b, A_i, B_j$$

- Normalization:

$$\begin{aligned}
P(00|A_1B_1) + P(01|A_1B_1) + P(10|A_1B_1) + P(11|A_1B_1) &= 1\\
P(00|A_1B_2) + P(01|A_1B_2) + P(10|A_1B_2) + P(11|A_1B_2) &= 1\\
P(00|A_2B_1) + P(01|A_2B_1) + P(10|A_2B_1) + P(11|A_2B_1) &= 1\\
P(00|A_2B_2) + P(01|A_2B_2) + P(10|A_2B_2) + P(11|A_2B_2) &= 1
\end{aligned}$$

- No-signaling:

$$\begin{aligned}
P(00|A_1B_1) + P(01|A_1B_1) &= P(00|A_1B_2) + P(01|A_1B_2)\\
P(00|A_2B_1) + P(01|A_2B_1) &= P(00|A_2B_2) + P(01|A_2B_2)\\
P(00|A_1B_1) + P(10|A_1B_1) &= P(00|A_2B_1) + P(10|A_2B_1)\\
P(00|A_1B_2) + P(10|A_1B_2) &= P(00|A_2B_2) + P(10|A_2B_2)\\
P(10|A_1B_1) + P(11|A_1B_1) &= P(10|A_1B_2) + P(11|A_1B_2)\\
P(10|A_2B_1) + P(11|A_2B_1) &= P(10|A_2B_2) + P(11|A_2B_2)\\
P(01|A_1B_1) + P(11|A_1B_1) &= P(01|A_2B_1) + P(11|A_2B_1)\\
P(01|A_1B_2) + P(11|A_1B_2) &= P(01|A_2B_2) + P(11|A_2B_2)
\end{aligned}$$

The $\mathcal{NS}$ polytope for the bipartite two-input two-output is the set of all $P(ab|A_iB_j)$ which satisfy the above constraints. The constraints on the joint probabilities give the **H**-representation of the $\mathcal{NS}$ polytope. We can write these constraints in a text file and feed it to PANDA, which gives us the extremal vertices of the $\mathcal{NS}$ polytope. There are 24 extremal vertices and every extremal vertex is a probability distribution that represents a behavior. A complete list of the vertices is given in table 2.1.

Measurements	A_1B_1				A_1B_2				A_2B_1				A_2B_2			
Outcome-Pairs	00	01	10	11	00	01	10	11	00	01	10	11	00	01	10	11
1.	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
2.	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0
3.	0	0	0	1	0	0	1	0	0	0	0	1	0	0	1	0
4.	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0
5.	0	0	1	0	0	0	0	1	0	0	1	0	0	0	0	1
6.	0	0	1	0	0	0	0	1	1	0	0	0	0	1	0	0
7.	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0
8.	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0
9.	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	1
10.	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
11.	0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0
12.	0	1	0	0	1	0	0	0	0	1	0	0	1	0	0	0
13.	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
14.	1	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0
15.	1	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0
16.	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
17.	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
18.	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0
19.	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
20.	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$
21.	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
22.	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$
23.	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
24.	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 2.1: Vertices of the 2222 no-signaling polytope

Of the 24 vertices, 16 are 'local' and the rest 8 are 'nonlocal'. The local vertices can be concisely represented as [14]

$$P(ab|xy) = \begin{cases} 1, & a = \alpha X \oplus \beta,\ b = \gamma Y \oplus \delta \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha, \beta, \gamma, \delta \in \{0,1\}$ and $\oplus$ denotes addition modulo 2. And the nonlocal vertices can be concisely represented

$$P(ab|xy) = \begin{cases} \frac{1}{2} & a \oplus b = XY \oplus \alpha X \oplus \beta Y \oplus \gamma \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha, \beta, \gamma \in \{0,1\}$ and $\oplus$ denotes addition modulo 2. For better readability, the vertices can be written in the form of a matrix. For example the vertex 17 in the above table can be written as:

$$C = \begin{array}{c} \\ A_1B_1 \\ A_1B_2 \\ A_2B_1 \\ A_2B_2 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

The elements in the matrix correspond to the probability of getting the particular outcome when the joint measurements are made. For example, the matrix element $C_{44} = \frac{1}{2}$ represents the probability of getting the outcome '11' when the joint measurement A_2B_2 is made.

2.4 Local Behavior

The set of Local behaviours $\mathcal{L}$ is the set containing elements $P(ab|A_xB_y)$ which can be represented in the form [7]

$$P(ab|A_xB_y) = \int_\Lambda d\lambda\ q(\lambda)\ P(a|A_x,\lambda)\ P(b|B_y,\lambda)$$

The variables λ here are some 'hidden' variables that belong to a space Λ and determine the outcomes a and b. The $q(\lambda)$ is the probability density that governs the distribution of the values of λ. A *local deterministic distribution* [15] is a probability distribution which satisfies

$$\begin{aligned} &P(ab|A_xB_y) = P(a|A_x)P(b|B_y) \\ \text{with} \quad &P(a|A_x),\ P(b|B_y) \in \{0,1\} \quad \forall \quad a, b, A_x, B_y \end{aligned}$$

For an mn Bell scenario, there are $2^m 2^n$ different local deterministic distributions. A probability distribution which can be written as the convex combination of local deterministic distributions is called a *local distribution*. The set of all local distributions forms a polytope with the local deterministic distributions as its vertices. We will refer to it as the Local polytope and denote it by L_{mn} for an mn Bell scenario. The local polytope is a subset of the no-signaling polytope. Moreover, the vertices of the Local polytope are always the vertices of the no-signaling polytope.

A *Bell Inequality* is a linear inequality which is satisfied by every local distribution [16]. On the other hand, a distribution is local only if it satisfies all the Bell inequalities. The Bell Inequalities form the nontrivial facets of the Local polytope. The trivial facets of the Local polytope are the positivity and normalization constraints. In principle, the Bell inequalities for any mn Bell scenario can be found by facet enumeration of the local polytope. An example of a Bell Inequality is the CHSH inequality [17]. Written in terms of correlators, the CHSH inequality reads

$$\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \leq 2$$

2.5 Bell Violation in Quantum Mechanics

Consider the singlet state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$$

Alice and Bob perform measurement of spin along some chosen axis. In particular, say their measurements are given by [1]

$$\begin{aligned} A_1 &= Z & A_2 &= X \\ B_1 &= \frac{-Z - X}{\sqrt{2}} & B_2 &= \frac{Z - X}{\sqrt{2}} \end{aligned}$$

where Z and X denote the spin measurements along the Z and X axes respectively. The correlators when used in quantum mechanical calculations are the expectation values of the observables, and are defined by

$$\begin{aligned} \langle A_x \rangle &= \mathrm{Tr}[(A_x \otimes \mathbb{I})\rho] \\ \langle B_y \rangle &= \mathrm{Tr}[(\mathbb{I} \otimes B_y)\rho] \\ \langle A_x B_y \rangle &= \mathrm{Tr}[(A_x \otimes B_y)\rho] \end{aligned}$$

where 'Tr' stands for trace and ρ is the density matrix of the system shared by Alice and Bob. It is a straightforward calculation to see that the singlet state along with the measurement operators above violates the CHSH inequality. The value of the LHS of the inequality is seen to be $2\sqrt{2}$.

This points out that quantum mechanics is not a local theory.

2.5.1 Optimal Measurements for Bell Violation and the Quantum Bound

It is pertinent to note that the violation above occurs for optimal state and measurement operators. Separable state do not violate the Bell Inequality. All pure entangled states violate the Bell Inequality. Given a state, we are interested in finding the optimal measurement operators such that a Bell violation occurs. This can be done by implementing a *see-saw iteration*, the details of which can be found in [18]. We shall use this algorithm to find optimal measurement operators for the violation of Local Friendliness inequalities in Chapter 3.

It turns out that the value $2\sqrt{2}$ is the maximum that can be achieved in quantum theory. The algebraic maximum is of course 4, which is also the no-signaling bound. Quantum theory does not violate the Bell inequality beyond a quantum bound. The quantum bound can be found by solving a hierarchy of semidefinite program as described in [19]. We will use this method to find the quantum bound of the new inequalities we find in Chapter 3. To model the semidefinite programs, one can use the MATLAB modeling environment YALMIP [20] and solve them using SeDuMi [21]. One can also use QETLAB [22] for solving the semidefinite programs.

2.6 Extended Wigner Friend Scenario

The modified EWFS considered in [6] involves a bipartite version of Wigner's Friend: Alice and Bob take the role of 'Superobservers' with Charlie and Debbie as their respective friends. Charlie and Debbie are spatially separated with each of their system being a spin-$\frac{1}{2}$ particle. Further, the systems of Charlie and Debbie are entangled. The measurements of Alice and Bob are labelled $x \in \{1, 2, 3\}$ and $y \in \{1, 2, 3\}$ with a and b as their corresponding outcomes. Each run of the experiment involves

a. Charlie and Debbie measuring the z-spin of their particles, with their measurement outcomes labelled c and d respectively.

b. Alice and Bob randomly choosing and performing one of the three measurements.

For $x = 2$ and $x = 3$, Alice performs a measurement on the joint system. For $x = 1$, Alice just opens Charlie's lab, asks him for his outcomes c, and assigns her own outcome a the value of c. Unless $x = 1$, all records for the value of c are erased when Alice performs her measurement. i.e. for $x \neq 1$, Charlie's outcome does not matter. Bob and Debbie operate similarly. Therefore, in general, at the end of the experiment, the only information available is the values of a, b, x, and y; the values of c and d can not be accessed.

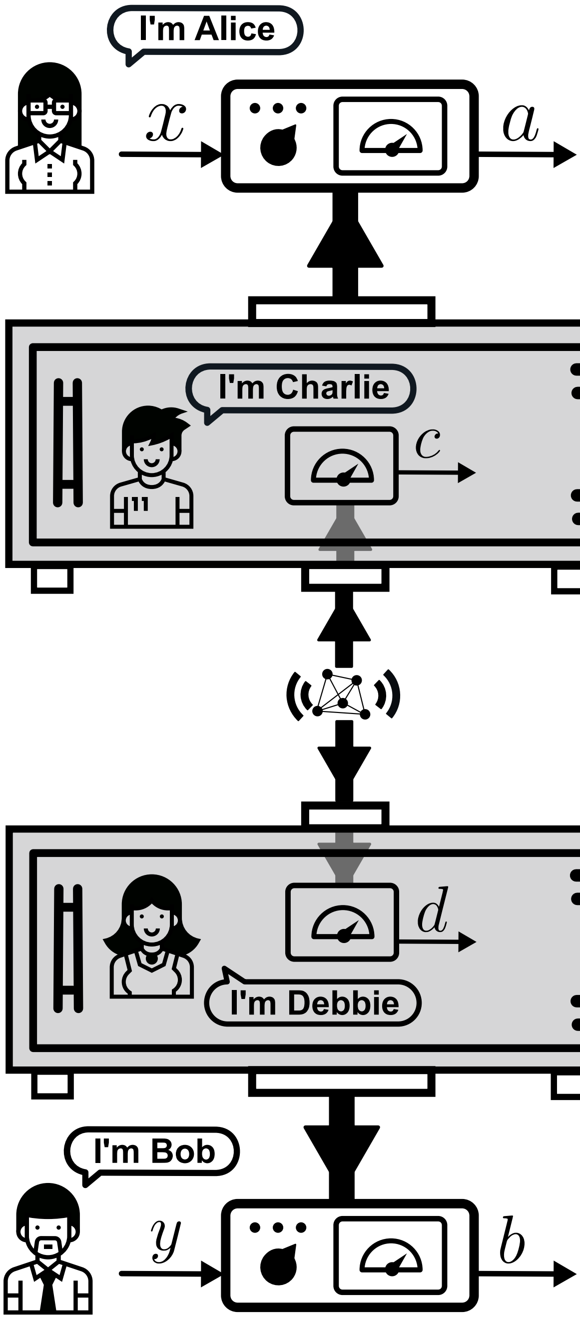

Figure 2.1: The extended Wigner's friend scenario. Figure taken from [6].

2.7 Local Friendliness

We refer to the conjunction of the assumptions—Observer Independent Facts, Freedom of Choice, Locality—as Local Friendliness. In the context of the extended Wigner's friend scenario, we reproduce the following definitions from [6].

- *Observer Independent Facts*: An observed event is a real single event, and not 'relative' to anything or anyone. This means that the results of a *performed* experiment are observer-independent (i.e. absolute). For example, Alice's outcome a corresponding to the measurement $x = 2$ has a value only when Alice performs the measurement $x = 2$. We do not say anything about the outcomes of the unperformed measurements. In the EWFS, this assumption implies that once Alice, Bob, Charlie and Debbie perform their respective measurements, there are well-defined values for the observed outcomes a, b, c and d.
  Formally, this implies the existence of a theoretical joint probability distribution $P(abcd|xy)$ from which the empirical probability distribution $\wp(ab|xy) = \sum_{c,d} P(abcd|xy)$ can be obtained, while also ensuring that the observed outcomes for $x, y = 1$ are consistent between the superobservers and the friends, i.e. $\exists\, P(abcd|xy)$ such that:

$$
\begin{aligned}
\text{i)} \quad & \wp(ab|xy) = \Sigma_{c,d} P(abcd|xy) \quad \forall\, a, b, x, y, \\
\text{ii)} \quad & P(a|cd, x = 1, y) = \delta_{a,c} \quad \forall\, a, c, d, y, \\
\text{iii)} \quad & P(b|cd, x, y = 1) = \delta_{b,d} \quad \forall\, b, c, d, x.
\end{aligned}
$$

- *Locality*: The choice of measurement settings has no influence on the outcomes of distant measurements.

$$
\begin{aligned}
P(a|cdxy) &= P(a|cdx) \quad \forall\, a, c, d, x, y. \\
P(b|cdxy) &= P(b|cdy) \quad \forall\, b, c, d, x, y.
\end{aligned}
$$

- *Freedom of Choice*: Experimental settings of Alice and Bob can be chosen freely, i.e. they are uncorrelated with any relevant variables prior to that choice.

$$
P(cd|xy) = P(cd) \quad \forall\, c, d, x, y.
$$

The values c and d respectively determine the outcomes a and b when Alice and Bob choose the measurements $x = 1$ and $y = 1$ i.e. they play the role of hidden variables λ. The hidden variables here correspond to observed events, and there are no assumptions made about the hidden variables predetermining all measurement outcomes.

2.8 LF Inequalities

The LF assumptions impose constraints on the correlations that obey the LF assumptions. We now derive the constraints on on the correlations for the EWFS described in Section 2.6. From observer independent facts, we have,

$$\begin{aligned}\wp(ab|xy) &= \sum_{c,d} P(abcd|xy) \\ &= \sum_{c,d} P(ab|cdxy)P(cd|xy)\end{aligned}$$

From freedom of choice,

$$\wp(ab|xy) = \sum_{c,d} P(ab|cdxy)P(cd)$$

Now, $P(ab|cdxy)$ can be decomposed in two ways, and then using locality it can be further reduced:

$$\begin{aligned}P(ab|cdxy) &= P(a|bcdxy)P(b|cdxy) \\ &= P(a|bcdxy)P(b|cdy) \\ &\text{or} \\ P(ab|cdxy) &= P(a|cdxy)P(b|acdxy) \\ &= P(a|cdx)P(b|acdxy)\end{aligned}$$

By construction, we have $P(a|cd, x = 1, y) = \delta_{a,c}$ and $P(b|cd, x, y = 1) = \delta_{b,d}$. So,

$$\wp(ab|xy) = \begin{cases} \sum_{c,d} \delta_{a,c} P(b|cdy)P(cd) & \text{if } x = 1 \\ \sum_{c,d} \delta_{b,d} P(a|cdx)P(cd) & \text{if } y = 1 \\ \sum_{c,d} P_{NS}(ab|cdxy)P(cd) & \text{if } x \neq 1, y \neq 1 \end{cases} \tag{2.3}$$

where P_{NS} denotes some probability distribution that satisfies the condition of locality. We are looking for the most general form of $\wp(ab|xy)$. Once we fix the values of c and d, the most general distribution that satisfies the locality condition is given by the no-signaling polytope. Therefore, P_{NS} is the no-signaling polytope with one measurement setting less for Alice and Bob.

Now let us consider the scenario where both Alice and Bob have three measurements to choose from and each measurement results in one of the two possible outcomes, 0 or 1. The no-signaling polytope for one less measurement setting on each side is the 2222 no-signaling polytope. It has 24 vertices. There are 4 different combinations of 'cd' viz., $00, 01, 10, 11$. Hence there are $24 \times 4 = 96$ vertices of the LF polytope.

We detail how to find the vertices of the LF polytope. The no-signaling polytope has 16 local vertices and 8 nonlocal vertices. Therefore the Local Friendliness polytope has $16\times 4 = 64$ local vertices and $8 \times 4 = 32$ nonlocal vertices. Finding the local vertices is a trivial task. The local vertices of the LF polytope are the same as the ones that characterize the L_{33} polytope. How do we find the nonlocal vertices? Consider the vertex 17 of the NS polytope from the table 2.1.

$$C = \begin{array}{c} \\ A_2B_2 \\ A_2B_3 \\ A_3B_2 \\ A_3B_3 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

Now consider the deterministic strategy $c = 0, d = 0$. The above matrix is now extended as

$$C = \begin{array}{c} \\ A_1B_1 \\ A_2B_2 \\ A_2B_3 \\ A_3B_2 \\ A_3B_3 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

Now how do we get the $A_1B_2, A_1B_3, A_2B_1, A_3B_1$ distributions? From $c = 0$ and $d = 0$, we have $a_1 = 0$ and $b_1 = 0$. The values of a_2, b_2, a_3, b_3 are either 0 or 1 with probabilities $\frac{1}{2}$ each. So all we do now is fill in the matrix by taking into account the values of a_1 and b_1. For example, when Alice chooses the measurement A_2 and Bob chooses the measurement B_1, the probability of getting $b_1 = 0$ is 1 as is determined by d. The probability of getting $a_2 = 0$ is $\frac{1}{2}$ and the probability of getting $a_2 = 1$ is $\frac{1}{2}$ as well. So,

$$\begin{aligned} P(00|A_2B_1) &= \frac{1}{2} \\ P(01|A_2B_1) &= 0 \\ P(10|A_2B_1) &= \frac{1}{2} \\ P(11|A_2B_1) &= 0 \end{aligned}$$

In a similar manner all the other elements of the matrix can be filled up. Therefore the vertex of the Local Friendliness polytope corresponding to the vertex 17 of the no-signaling polytope and the deterministic strategy $c = 0$ and $d = 0$ is

$$C = \begin{array}{c} \\ A_1B_1 \\ A_1B_2 \\ A_1B_3 \\ A_2B_1 \\ A_3B_1 \\ A_2B_2 \\ A_2B_3 \\ A_3B_2 \\ A_3B_3 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

The complete list of vertices can be found in table 2.2 at the end of this chapter. Once we have the vertices of the LF polytope, we write them down in a text file, feed it to PANDA to obtain the constraints on the probabilities in terms of inequalities. We see that there are 932 different inequalities. However, the inequalities are not all inequivalent. Many of the inequalities can be transformed from one to another by relabelling the measurement choices, measurement outcomes and the parties making the measurements. It turns out that there are 9 inequivalent classes of inequalities, which we list below. Of the inequalities, there are facets of the polytope that are not the facet of the local polytope. We call such facets *Genuine LF* inequalities.

1. Genuine LF facet 1 (appearing 256 times among the 932 facets):

$$\begin{aligned} -\langle A_1\rangle - \langle A_2\rangle - \langle B_1\rangle - \langle B_2\rangle \\ -\langle A_1B_1\rangle - 2\langle A_1B_2\rangle - 2\langle A_2B_1\rangle + 2\langle A_2B_2\rangle \\ -\langle A_2B_3\rangle - \langle A_3B_2\rangle - \langle A_3B_3\rangle \leq 6 \end{aligned} \tag{2.4}$$

2. Genuine LF facet 2 (appearing 256 times):

$$\begin{aligned} -\langle A_1\rangle - \langle A_2\rangle - \langle A_3\rangle - \langle B_1\rangle \\ -\langle A_1B_1\rangle - \langle A_2B_1\rangle - \langle A_3B_1\rangle - 2\langle A_1B_2\rangle \\ +\langle A_2B_2\rangle + \langle A_3B_2\rangle - \langle A_2B_3\rangle + \langle A_3B_3\rangle \leq 5 \end{aligned} \tag{2.5}$$

3. I_{3322} with marginals over input 1 and 2 (appearing 256 times):

$$\begin{aligned} -\langle A_1\rangle + \langle A_2\rangle + \langle B_1\rangle - \langle B_2\rangle \\ +\langle A_1B_1\rangle - \langle A_1B_2\rangle - \langle A_1B_3\rangle - \langle A_2B_1\rangle \\ +\langle A_2B_2\rangle - \langle A_2B_3\rangle - \langle A_3B_1\rangle - \langle A_3B_2\rangle \leq 4 \end{aligned}$$

4. I_{3322} with marginals over input 2 and 3 (appearing 64 times):

$$\begin{aligned} -\langle A_2\rangle - \langle A_3\rangle - \langle B_2\rangle - \langle B_3\rangle \\ -\langle A_1B_2\rangle + \langle A_1B_3\rangle - \langle A_2B_1\rangle - \langle A_2B_2\rangle \\ -\langle A_2B_3\rangle + \langle A_3B_1\rangle - \langle A_3B_2\rangle - \langle A_3B_3\rangle \leq 4 \end{aligned}$$

5. "Brukner inequality": CHSH for input 1 and 2 (appearing 32 times):

$$\langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle \leq 2$$

6. "Semi-Brukner" inequality: CHSH for input 2, 3 of Alice, and input 1, 2 of Bob (appearing 32 times):

$$\langle A_2B_1\rangle + \langle A_2B_2\rangle + \langle A_3B_1\rangle - \langle A_3B_2\rangle \leq 2$$

7. Positivity for input 1 of Alice and input 1 of Bob (appearing 4 times):

$$1 + \langle A_1\rangle + \langle B_1\rangle + \langle A_1B_1\rangle \geq 0$$

8. Positivity for input 1 of Alice and input 2 of Bob (appearing 16 times):

$$1 + \langle A_1\rangle + \langle B_2\rangle + \langle A_1B_2\rangle \geq 0$$

9. Positivity for input 2 of Alice and input 2 of Bob (appearing 16 times):

$$1 + \langle A_2\rangle + \langle B_2\rangle + \langle A_2B_2\rangle \geq 0$$

2.9 Quantum Violation of LF inequality

Consider the two-qubit photon polarization state of the form

$$\rho_\mu = \mu\,|\Phi^-\rangle\langle\Phi^-| + \frac{1-\mu}{2}(|HV\rangle\langle HV| + |VH\rangle\langle VH|) \tag{2.6}$$
$$\text{with} \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle), \quad 0 \le \mu \le 1.$$

where H and V denote horizontal and vertical polarizations respectively.

- Alice's measurement operators are represented by $A_x = 2\Pi_x - |H\rangle\langle H| - |V\rangle\langle V|$, where $\Pi_x = |\phi_x\rangle\langle\phi_x|$ is the projector on to the state $|\phi_x\rangle = \frac{1}{\sqrt{2}}(|H\rangle + e^{i\phi_x}|V\rangle)$.
- Bob's measurement operators are represented by $B_y = 2\Pi_y - |H\rangle\langle H| - |V\rangle\langle V|$, where $\Pi_y = |\beta_y\rangle\langle\beta_y|$ is the projector on to the state $|\beta_y\rangle = \frac{1}{\sqrt{2}}(|H\rangle + e^{i(\beta-\phi_y)}|V\rangle)$.

It was shown by the authors of [6] that the state (2.6) with the above measurement operators violates the LF inequality for $\phi_1 = 168°$, $\phi_2 = 0°$, $\phi_3 = 118°$, $\beta = 175°$.

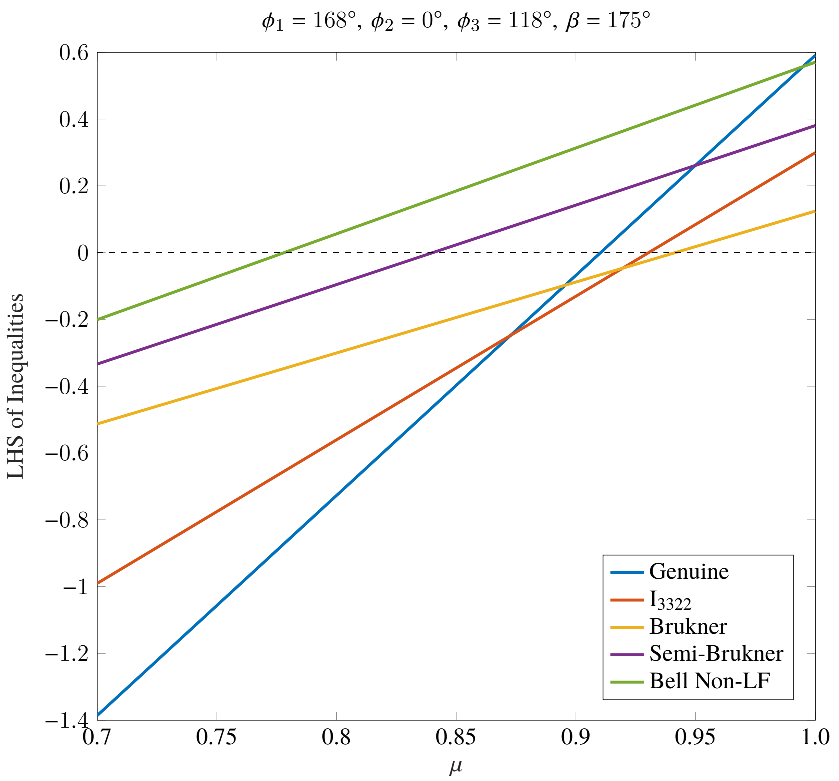

Figure 2.2: Graph for the values used in [6]

A_1B_1				A_1B_2				A_1B_3				A_2B_1				A_2B_2				A_2B_3				A_3B_1				A_3B_2				A_3B_3			
1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
1	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0
1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0
1	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1
1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0
1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0
1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0
1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	1
1	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	1	0	0	0
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1	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	1	0	0	0	0	1	0	0	1	0	0	0
1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	1	1	0	0	0	0	1	0	0	0	1	0	0
1	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	0
1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	1
0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0
0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0
0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0
0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	1
0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0
0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1
0	1	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0	0	0
0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0	0	1	0	0
0	1	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0
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0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	1	0	0
0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	1	0	0	1	0
0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1

A_1B_1	A_1B_2	A_1B_3	A_2B_1	A_2B_2	A_2B_3	A_3B_1	A_3B_2	A_3B_3
0 0 1 0	0 0 1 0	0 0 1 0	1 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0
0 0 1 0	0 0 1 0	0 0 0 1	1 0 0 0	1 0 0 0	0 1 0 0	1 0 0 0	1 0 0 0	0 1 0 0
0 0 1 0	0 0 1 0	0 0 1 0	1 0 0 0	1 0 0 0	1 0 0 0	0 0 1 0	0 0 1 0	0 0 1 0
0 0 1 0	0 0 1 0	0 0 0 1	1 0 0 0	1 0 0 0	0 1 0 0	0 0 1 0	0 0 1 0	0 0 0 1
0 0 1 0	0 0 0 1	0 0 1 0	1 0 0 0	0 1 0 0	1 0 0 0	1 0 0 0	0 1 0 0	1 0 0 0
0 0 1 0	0 0 0 1	0 0 0 1	1 0 0 0	0 1 0 0	0 1 0 0	1 0 0 0	0 1 0 0	0 1 0 0
0 0 1 0	0 0 0 1	0 0 1 0	1 0 0 0	0 1 0 0	1 0 0 0	0 0 1 0	0 0 0 1	0 0 1 0
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0 0 1 0	0 0 0 1	0 0 0 1	0 0 1 0	0 0 0 1	0 0 0 1	1 0 0 0	0 1 0 0	0 1 0 0
0 0 1 0	0 0 0 1	0 0 1 0	0 0 1 0	0 0 0 1	0 0 1 0	0 0 1 0	0 0 0 1	0 0 1 0
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0 0 0 1	0 0 0 1	0 0 1 0	0 1 0 0	0 1 0 0	1 0 0 0	0 1 0 0	0 1 0 0	1 0 0 0
0 0 0 1	0 0 0 1	0 0 0 1	0 1 0 0	0 1 0 0	0 1 0 0	0 1 0 0	0 1 0 0	0 1 0 0
0 0 0 1	0 0 0 1	0 0 1 0	0 1 0 0	0 1 0 0	1 0 0 0	0 0 0 1	0 0 0 1	0 0 1 0
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0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1	0 1 0 0	0 1 0 0	0 1 0 0
0 0 0 1	0 0 0 1	0 0 1 0	0 0 0 1	0 0 0 1	0 0 1 0	0 0 0 1	0 0 0 1	0 0 1 0
0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1

A_1B_1				A_1B_2				A_1B_3				A_2B_1				A_2B_2				A_2B_3				A_3B_1				A_3B_2				A_3B_3			
1	0	0	0	½	½	0	0	½	½	0	0	½	0	½	0	½	0	0	½	½	0	0	½	½	0	½	0	½	0	0	½	0	½	½	0
0	1	0	0	½	½	0	0	½	½	0	0	0	½	0	½	½	0	0	½	½	0	0	½	0	½	0	½	½	0	0	½	0	½	½	0
0	0	1	0	0	0	½	½	0	0	½	½	½	0	½	0	½	0	0	½	½	0	0	½	½	0	½	0	½	0	0	½	0	½	½	0
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0	1	0	0	½	½	0	0	½	½	0	0	0	½	0	½	0	½	½	0	0	½	½	0	0	½	0	½	0	½	½	0	½	0	0	½
0	0	1	0	0	0	½	½	0	0	½	½	½	0	½	0	0	½	½	0	0	½	½	0	½	0	½	0	0	½	½	0	½	0	0	½
0	0	0	1	0	0	½	½	0	0	½	½	0	½	0	½	0	½	½	0	0	½	½	0	0	½	0	½	0	½	½	0	½	0	0	½
1	0	0	0	½	½	0	0	½	½	0	0	½	0	½	0	½	0	0	½	0	½	½	0	½	0	½	0	½	0	0	½	½	0	0	½
0	1	0	0	½	½	0	0	½	½	0	0	0	½	0	½	½	0	0	½	0	½	½	0	0	½	0	½	½	0	0	½	½	0	0	½
0	0	1	0	0	0	½	½	0	0	½	½	½	0	½	0	½	0	0	½	0	½	½	0	½	0	½	0	½	0	0	½	½	0	0	½
0	0	0	1	0	0	½	½	0	0	½	½	0	½	0	½	½	0	0	½	0	½	½	0	0	½	0	½	½	0	0	½	½	0	0	½
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0	1	0	0	½	½	0	0	½	½	0	0	0	½	0	½	0	½	½	0	½	0	0	½	0	½	0	½	0	½	½	0	0	½	½	0
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0	0	0	1	0	0	½	½	0	0	½	½	0	½	0	½	0	½	½	0	½	0	0	½	0	½	0	½	0	½	½	0	0	½	½	0
1	0	0	0	½	½	0	0	½	½	0	0	½	0	½	0	½	0	0	½	½	0	0	½	½	0	½	0	0	½	½	0	½	0	0	½
0	1	0	0	½	½	0	0	½	½	0	0	0	½	0	½	½	0	0	½	½	0	0	½	0	½	0	½	0	½	½	0	½	0	0	½
0	0	1	0	0	0	½	½	0	0	½	½	½	0	½	0	½	0	0	½	½	0	0	½	½	0	½	0	0	½	½	0	½	0	0	½
0	0	0	1	0	0	½	½	0	0	½	½	0	½	0	½	½	0	0	½	½	0	0	½	0	½	0	½	0	½	½	0	½	0	0	½
1	0	0	0	½	½	0	0	½	½	0	0	½	0	½	0	0	½	½	0	0	½	½	0	½	0	½	0	½	0	0	½	0	½	½	0
0	1	0	0	½	½	0	0	½	½	0	0	0	½	0	½	0	½	½	0	0	½	½	0	0	½	0	½	½	0	0	½	0	½	½	0
0	0	1	0	0	0	½	½	0	0	½	½	½	0	½	0	0	½	½	0	0	½	½	0	½	0	½	0	½	0	0	½	0	½	½	0
0	0	0	1	0	0	½	½	0	0	½	½	0	½	0	½	0	½	½	0	0	½	½	0	0	½	0	½	½	0	0	½	0	½	½	0
1	0	0	0	½	½	0	0	½	½	0	0	½	0	½	0	½	0	0	½	0	½	½	0	½	0	½	0	0	½	½	0	0	½	½	0
0	1	0	0	½	½	0	0	½	½	0	0	0	½	0	½	½	0	0	½	0	½	½	0	0	½	0	½	0	½	½	0	0	½	½	0
0	0	1	0	0	0	½	½	0	0	½	½	½	0	½	0	½	0	0	½	0	½	½	0	½	0	½	0	0	½	½	0	0	½	½	0
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1	0	0	0	½	½	0	0	½	½	0	0	½	0	½	0	0	½	½	0	½	0	0	½	½	0	½	0	½	0	0	½	½	0	0	½
0	1	0	0	½	½	0	0	½	½	0	0	0	½	0	½	0	½	½	0	½	0	0	½	0	½	0	½	½	0	0	½	½	0	0	½
0	0	1	0	0	0	½	½	0	0	½	½	½	0	½	0	0	½	½	0	½	0	0	½	½	0	½	0	½	0	0	½	½	0	0	½
0	0	0	1	0	0	½	½	0	0	½	½	0	½	0	½	0	½	½	0	½	0	0	½	0	½	0	½	½	0	0	½	½	0	0	½

Table 2.2: Vertices of the Local Friendliness Polytope

Chapter 3

Results

We first analyze the LF inequalities presented in the previous chapter with respect to a couple of states. Then we consider various Extended Wigner Friend Scenarios. Where new inequalities are found, we give their quantum bounds. The quantum bounds are calculated up to the second level of the NPA hierarchy [19].

3.1 Analysis of the LF inequalities

3.1.1 Pure State

Consider the pure state given by

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$$
$$\text{with} \quad \beta = \sqrt{1-\alpha^2}, \quad 0 \leq \alpha \leq 1$$

The state is entangled for all $\alpha \neq 0$, and $\alpha \neq 1$. The violation of the different inequalities presented in previous chapter can be seen in Figure 3.1. All pure entangled states violate the Genuine LF inequalities. From the figure we see that

- The behaviour of the violation with respect to the Bell inequalities and the Genuine LF inequalities is different.
  - The maximal violation for the Bell inequalities—CHSH and I_{3322}—occurs for the maximally entangled state with $\alpha = \frac{1}{\sqrt{2}}$.
  - The maximal violation for the Genuine LF inequalities occurs for the non-maximally entangled states. That is, while the Bell violation is an increasing function of entanglement, the violation of Genuine LF inequalities is not. This is in agreement with the fact that entanglement and nonlocality are different resources [23].
  - In particular, we see that the violation of the Genuine LF inequalities has a local minimum where the Bell-CHSH violation is maximum.

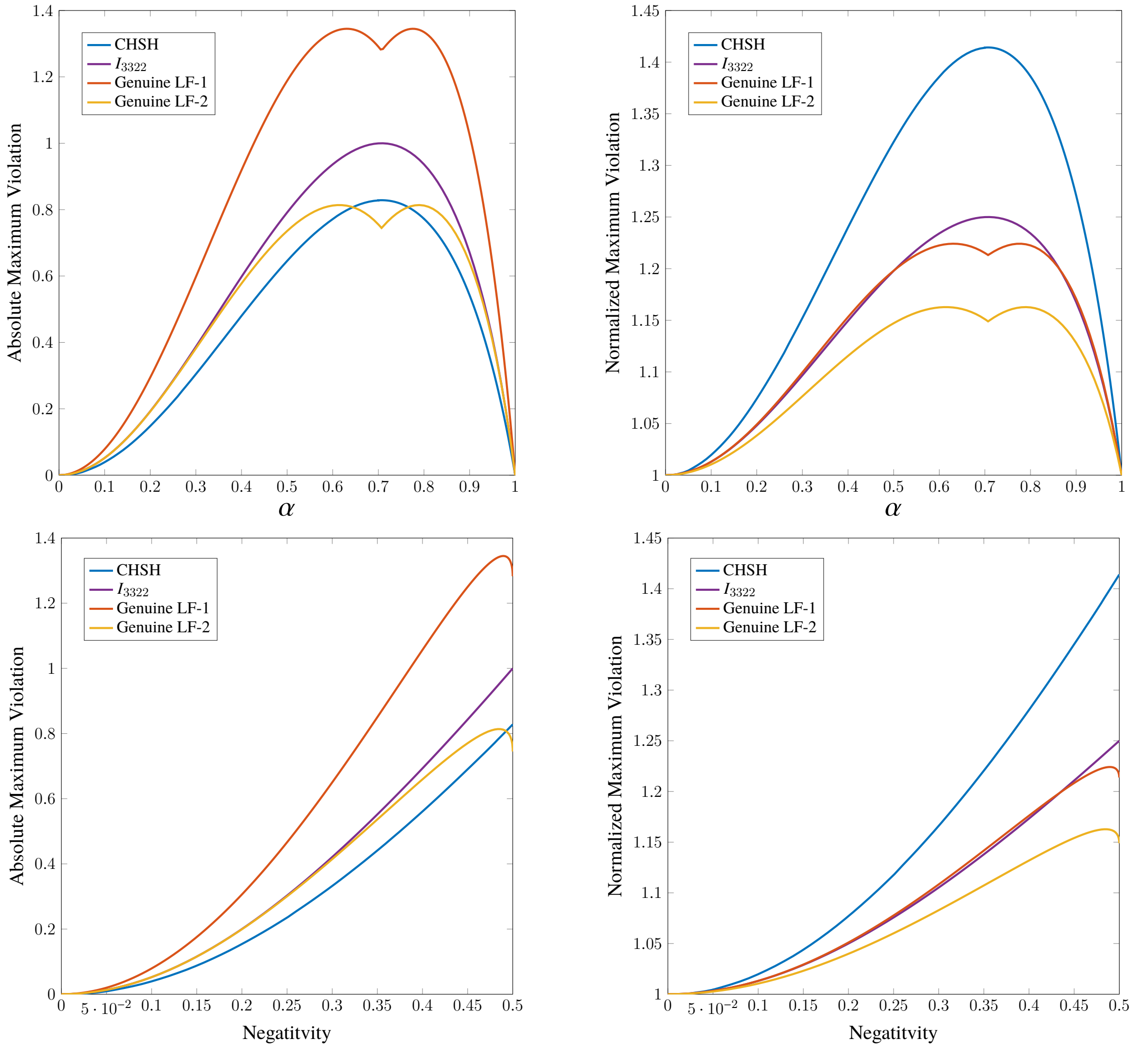

Figure 3.1: Violation of the LF Inequalities - Pure State

- The Genuine LF Inequality-1 (equation 2.4) is maximally violated for $\alpha = 0.6311$. The maximal violation allowed in quantum theory was found in [6] to be 1.345.

- For the Genuine LF Inequality-2 (equation 2.5) the maximum violation by the qubit state is found to be 0.8138 for $\alpha = 0.6145$. The maximal violation allowed in quantum theory is 0.880 and can be achieved by using a qutrit state [6] with Schmidt coefficients 0.509, 0.570, 0.645.

3.1.2 Werner State

Werner states, ρ_W are the bipartite states which are invariant when the two parties—say Alice and Bob—apply the same unitary operation [24].

$$(U \otimes U)\rho_W(U^\dagger \otimes U^\dagger) = \rho_W$$

For two qubits, the Werner State is given by

$$\rho_{\mathrm{W}} = p|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2} \tag{3.1}$$
$$\text{where} \quad \psi = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

This state is entangled whenever $p > 1/3$, and for $1/3 < p \leq 1/2$ it has a local description while being entangled [25]. Figure 3.2 shows the violation of the inequalities in the case of Werner State (3.1).

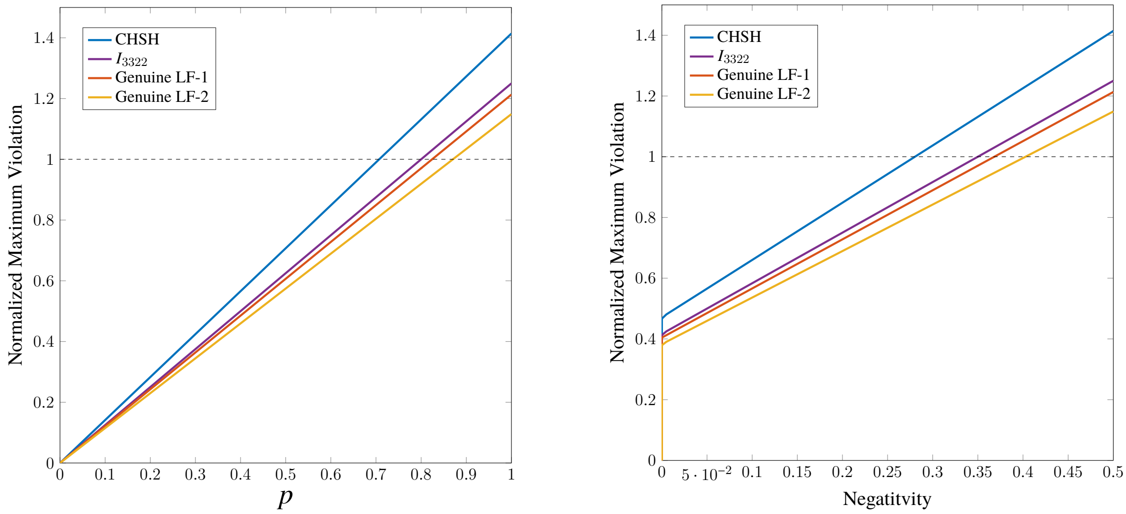

Figure 3.2: Violation of the LF Inequalities - Werner State

- The minimal values of p such that the inequalities are violated are

$$\begin{aligned} \text{CHSH} &\longrightarrow 0.7071 \\ I_{3322} &\longrightarrow 0.8000 \\ \text{Genuine LF-1} &\longrightarrow 0.8243 \\ \text{Genuine LF-2} &\longrightarrow 0.8704 \end{aligned}$$

- This shows that for values of p where the state ρ_{W} admits a local hidden variable description, the Genuine LF inequalities are not violated.

- There are also values of p where the state does not admit a local hidden variable description, and does not violate the Genuine LF inequalities.

- One can have a Bell violation in the domain $0.7071 \leq p \leq 0.8243$ without an LF violation.

- The maximal violation of LF inequalities occurs for $p = 1$, the pure state $|\psi\rangle$. The value of maximal violation is seen to be 1.2788 and 0.7446 for LF-1 and LF-2 respectively.

3.2 More Extended Wigner Friend Scenarios

3.2.1 Notation

Consider the EWFS where Alice and Bob choose from p and q number of measurements respectively. Let r and s denote the number of measurements made by Alice and Bob wherein they ask their friends Charlie and Debbie the outcome of their measurements. We necessarily have $r \leq p$ and $s \leq q$. Let us denote this scenario by W_{pq}^{rs}. All the measurements result in one of the two possible outcomes +1 or −1. The outcomes of Alice, Bob, Charlie and Debbie are labelled $a, b, c,$ and d respectively. The scenario considered in [6] for example, where Alice and Bob have 3 measurements each, of which 1 measurements on each side corresponds to asking their friend's outcome is denoted W_{33}^{11}.

3.2.2 W_{22}^{rs}

We have the following possible scenarios: W_{22}^{00}, W_{22}^{01}, W_{22}^{10}, W_{22}^{11}, W_{22}^{20}, W_{22}^{02}, W_{22}^{12}, W_{22}^{21}, and W_{22}^{22}.

The W_{22}^{00} polytope is same as the $\mathcal{NS}_{22}$ polytope. There are no measurements which correspond to inferring the result of the friends' measurement. The OIF assumption therefore plays no role. So, the only assumptions that now go into the derivation of structure of the W_{22}^{00} polytope are Locality and Freedom of Choice. The most general form of correlations obeying these assumptions are the nosignalling correlations. Hence, $W_{22}^{00} \equiv \mathcal{NS}_{22}$.

W_{22}^{11}, considered by Brukner in [4], is the Local polytope as pointed out in [6]. W_{22}^{22} is a trivial scenario where all the outcomes of Alice and Bob are determined by the outcomes of Charlie and Debbie respectively. It is same as the Local polytope.

The cases W_{22}^{10} and W_{22}^{01} are symmetric up to relabelling of the parties, Alice ↔ Bob and Charlie ↔ Debbie. So it suffices to consider only one of them. Let us consider the case W_{22}^{01}. Let Alice's measurements be given by $\mathbf{X} = \{1, 2\}$ and Bob's by $\mathbf{Y} = \{1, 2\}$. For $Y = 1$, Bob opens the lab, asks Debbie her outcome d, and assigns his own outcome $b = d$. For $Y = 2$, Bob performs a measurement directly on the joint system of his friend and the particle. For $X = 1$ and $X = 2$ Alice performs measurements directly on the joint system of her friend and the particle.

Let $\wp(ab|xy)$ denote the joint probability of getting the outcome pair 'ab' when measurements 'xy' are performed. From Observer Independent Facts, we have

$$\wp(ab|xy) = \sum_{d} P(abd|xy)$$

where $P(abd|xy)$ denotes a theoretical probability distribution from which the observed $\wp(ab|xy)$ can be obtained. Using Locality and Freedom of Choice, we know that $\wp(ab|xy)$ can be written as

$$\wp(ab|xy) = \begin{cases} \sum_d \delta_{b,d} P(a|dx) P(d) & \text{if } y = 1 \\ \sum_d P_{\mathcal{NS}}(ab|dxy) P(d) & \text{if } y \neq 1 \end{cases}$$

where for fixed values of d, $P_{\mathcal{NS}}(ab|xy)$ denotes the 21 no-signaling probability distribution; i.e. the corresponding $\mathcal{NS}$ polytope here is $\mathcal{NS}_{21}$. The vertices of this polytope are all local deterministic. Hence the scenario W_{22}^{10} reduces to the local polytope.

W_{22}^{20} and W_{22}^{02} are symmetric up to relabelling of parties, so are W_{22}^{21} and W_{22}^{12}. Following a similar line of reasoning, it is clearly evident that all of these reduce to the Local polytope. Hence in the W_{22}^{rs} scenario there are no Genuine LF inequalities.

3.2.3 Trivial Scenarios

The above results can be generalized to arbitrary p and q. We will consider the following scenarios to be trivial.

- W_{pq}^{00}: This is the $\mathcal{NS}_{pq}$ polytope for all p, q.
- W_{pq}^{rs} with either $r = p$ or $s = q$: This is the Local polytope for p and q number of measurements for Alice and Bob respectively.
- W_{pq}^{rs} with either $r = p - 1$ or $s = q - 1$: This is again the Local polytope for p and q number of measurements for Alice and Bob respectively, because the vertices of the corresponding no-signaling polytope here are all deterministic.

3.2.4 W_{32}^{rs}

The only nontrivial scenario here is W_{32}^{10}. The $\mathcal{NS}_{22}$ polytope as noted earlier, has 24 vertices. There are two different possible values of c, +1 and −1. The LF polytope in this case therefore has 48 vertices. We write down these vertices into a text file and feed it to PANDA to get the inequalities. It turns out that there are 40 facets of this polytope. Of these 24 are positivity facets, and the rest are all Bell-CHSH inequalities. All these inequalities include the Alice's measurement setting 1; i.e. they are of the form what is called "Semi-Brukner" inequality in [6]. Effectively, up to symmetry transformations we have only one class of nontrivial inequality in this scenario.

$$\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \leq 2$$

There are no Genuine LF inequalities here. Looking at the inequality may appear that the LF polytope in this case is the same as the Local polytope. It turns out however that it is not the case. The local polytope in fact, is a proper subset of the LF polytope.

3.2.5 W_{42}^{rs}

There are two nontrivial scenarios: W_{42}^{10} and W_{42}^{20}.

W_{42}^{20}**:**

There are two measurements of Alice which correspond to asking Charlie his outcome. Each of these measurements has two possible outcomes. Therefore we have 4 deterministic strategies. The LF polytope has $24 \times 4 = 96$ vertices. There are 72 facets, of which 32 are positivity facets. The rest are again, Bell-CHSH inequalities which include either Alice's measurement setting 1 or 2. Therefore up to symmetry transformations, we have two classes of nontrivial inequalities: the ones which include only one of Alice's measurement settings 1 and 2, and the ones which include both of them.

$$\langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle \leq 2$$
$$\langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_3B_1\rangle - \langle A_3B_2\rangle \leq 2$$

W_{42}^{10}**:**

The corresponding no-signaling polytope here is $\mathcal{NS}_{32}$.
The $\mathcal{NS}$ polytope for p measurement settings for Alice and q measurement settings for Bob with 2 outcomes for each measurements was characterized in [26]. The vertices of the corresponding local deterministic polytope are all vertices of this $\mathcal{NS}_{pq}$ polytope. Using the notation introduced in eq. (2.1), up to relabelling, the nonlocal extremal distributions of this $\mathcal{NS}_{pq}$ polytope have the form [15]:

$$\begin{array}{c}
\begin{array}{cc} \overbrace{S \;\; \dots \;\; S}^{q-2-g} & \overbrace{L \;\dots\; L}^{g} \end{array} \\
\begin{pmatrix}
S & S & S & \dots & S & L & \dots & L \\
S & A & S/A & \dots & S/A & L & \dots & L \\
S & S/A & S/A & \dots & S/A & L & \dots & L \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
S & S/A & S/A & \dots & S/A & L & \dots & L \\
K & K & K & \dots & K & M & \dots & M \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
K & K & K & \dots & K & M & \dots & M
\end{pmatrix}
\begin{array}{l} \\ \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} p-2-h \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} h \end{array}
\end{array} \tag{3.2}$$

where $g \in \{0, 1, \dots, q-2\}$, $h \in \{0, 1, \dots, p-2\}$ and

$$S = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad K = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad L = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We first find the vertices of the $\mathcal{NS}_{32}$ polytope. This polytope has 128 vertices, of which 32 are local deterministic and the rest 96 are nonlocal. The LF polytope here therefore has 256 vertices. We find that there are 56 inequalities, of which 32 are positivity facets. The rest are again, "Semi-Brukner" inequalities which include Alice's measurement setting 1. Hence, the only non trivial inequality in this scenario is:

$$\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \leq 2$$

The inclusion relations of the different polytopes are $L_{42} \subset W_{42}^{20} \subset W_{42}^{10}$.

It was shown in [27] that in the bipartite Bell scenario where one party has p measurements to choose from and the other party has 2 measurements to choose from, all with 2 outcomes each, there are no new Bell inequalities inequivalent to the Bell-CHSH. While we do not have a formal proof, looking at the above cases of W_{42}^{rs} and W_{32}^{rs}, we suspect that "Semi-Brukner" type of inequalities are the only nontrivial inequalities in nontrivial scenarios of W_{p2}^{rs}.

3.2.6 W_{33}^{rs}

The nontrivial scenarios here are W_{33}^{01} and W_{33}^{11}. The scenario W_{33}^{11} was studied and characterized in [6]. We consider the W_{33}^{01} case. $\mathcal{NS}_{32}$ has 128 vertices as noted in the previous subsection. There are two possible values of d viz., +1 and −1. Therefore the Local-Friendliness polytope in this case has $128 \times 2 = 256$ vertices. There are 84 facets of this polytope, of which 36 are positivity facets. The rest 48 are all the "Semi-Brukner" inequalities, which include Bob's measurement setting 1.

$$\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \leq 2$$

The inclusion relations of the different polytopes are $L_{33} \subset W_{33}^{01} \subset W_{33}^{11}$.

3.2.7 W_{43}^{rs}

The nontrivial scenarios here are W_{43}^{11}, W_{43}^{21}, W_{43}^{10}, W_{43}^{20} and W_{43}^{01}. We have studied only W_{43}^{21} and W_{43}^{11}.

W_{43}^{21}:

$\mathcal{NS}_{22}$ polytope is the relevant polytope. There are $2 \times 2 \times 2 = 8$ deterministic strategies, and hence 192 vertices of the LF polytope. Here we find a number of Genuine LF inequalities. That is there are new inequalities which are not the facets of the local polytope. The local polytope

L_{43} has 12480 facets and three inequalities not equivalent to either CHSH or the I_{3322}. We have found 8 new inequalities. That is there are at least 8 classes of inequalities which are not the facets of the L_{43} polytope. The list may not be complete. It is tedious to write them down in the usual format. We therefore use the following notation adapted from [28].

$$W := \sum_{i=1}^{p} \alpha_i \langle A_i \rangle + \sum_{j=1}^{q} \beta_j \langle B_j \rangle + \sum_{i,j=1}^{p,q} \gamma_{ij} \langle A_i B_j \rangle \leq \delta \tag{3.3}$$

where the α_i and β_j give the coefficients in front of the respective correlators. This can be represented in the form of a matrix.

$$W := \left(\begin{array}{c|ccc} & \beta_1 & \beta_2 & \beta_3 \\ \hline \alpha_1 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \alpha_2 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \alpha_3 & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \alpha_4 & \gamma_{41} & \gamma_{42} & \gamma_{43} \end{array} \right) \leq \delta \tag{3.4}$$

The inequalities can then be conveniently represented in the form of a table from which they can be read off. For each inequality, we also give the quantum bound, which we find using semidefinite programming, up to second level of the NPA hierarchy. The last column in the table gives the quantum bound.

#	α_1	α_2	α_3	α_4	β_1	β_2	β_3	γ_{11}	γ_{12}	γ_{13}	γ_{21}	γ_{22}	γ_{23}	γ_{31}	γ_{32}	γ_{33}	γ_{41}	γ_{42}	γ_{43}	δ	Q
1.	1	1	0	-1	0	0	-1	-1	-1	1	1	2	0	-1	2	1	-1	1	-1	7	8.6026
2.	1	1	0	-1	-2	0	1	1	-1	-1	1	1	-1	0	1	1	-2	1	-2	7	8.5191
3.	1	1	0	0	0	1	1	-1	0	-2	1	-2	0	1	2	1	1	1	-2	8	9.7476
4.	1	1	0	2	2	0	0	-1	-2	0	-1	1	1	0	1	1	-2	2	-2	8	9.6569
5.	1	2	-1	1	-1	1	3	-1	0	-2	2	-2	-2	1	1	1	1	2	-2	9	10.7778
6.	2	1	0	1	1	-3	0	-2	2	2	1	2	0	0	1	1	-2	2	-3	10	12.1417
7.	1	2	0	0	-1	-2	-2	-1	2	0	2	1	3	-2	1	3	0	2	-2	11	13.3299
8.	1	2	-1	-1	-1	2	-2	-1	-2	0	2	0	4	-3	2	4	-1	2	-2	13	15.7027

Table 3.1: Inequalities for the W_{43}^{21} Scenario

W_{43}^{11}:

$\mathcal{NS}_{32}$ is the relevant polytope. The LF polytope has $128 \times 4 = 512$ vertices. We find some new Genuine LF inequalities. They are listed in the table below. The list may not be complete.

#	α_1	α_2	α_3	α_4	β_1	β_2	β_3	γ_{11}	γ_{12}	γ_{13}	γ_{21}	γ_{22}	γ_{23}	γ_{31}	γ_{32}	γ_{33}	γ_{41}	γ_{42}	γ_{43}	δ	Q
1.	-1	1	0	-1	1	0	-2	1	0	-2	-1	1	1	-2	0	-2	1	1	-1	7	8.6780
2.	-1	0	0	1	1	-1	2	1	0	2	0	1	1	3	1	-2	-1	1	-1	8	9.4638
3.	-1	0	-1	2	-1	0	-1	-1	-1	-3	-1	2	1	1	1	-1	2	2	-2	10	12.2955

Table 3.2: Inequalities for the W_{43}^{11} Scenario

3.2.8 W_{44}^{rs}

The nontrivial scenarios here are W_{44}^{10}, W_{44}^{20}, W_{44}^{11}, W_{44}^{21}, and W_{44}^{22}. The problem of computing the inequalities becomes harder as the number of measurements increases. Table 3.3 [15] gives an idea of the scale of the problem. The Local polytope L_{44} has more than 36 Million facets. The computation of the facets of the LF polytope in this scenario is quite prohibitive.

p, q	Number of inequivalent inequalities	Number of Facets
2, 2	2	24
3, 3	3	684
4, 3	6	12480
4, 4	175	36391264

Table 3.3: Bell Inequality Classes for Bipartite Scenarios

However, we tried computing the facets of the W_{44}^{22} polytope. $\mathcal{NS}_{22}$ is the relevant polytope. There are $2 \times 2 \times 2 \times 2 = 16$ deterministic strategies and hence 384 vertices of the LF polytope. It turns out that PANDA is not suitable for this computation. We used `mplrs` [29] to enumerate the facets. `mplrs` is a C wrapper for `lrs` that allows parallelization. Further, it supports a 'checkpoint' file: if `mplrs` is terminated in the midst of calculation, it can be restarted using the checkpoint file and the calculation continues from the checkpoint. We found 187 new Genuine LF inequalities for the W_{44}^{22} scenario. They are listed at the end of this chapter, in the table 3.6. For each inequality we have also computed the quantum mechanical bound using semidefinite programming, up to second level of the NPA hierarchy.

3.3 Analysis of the newfound inequalities

We have randomly chosen one inequality from each of the W_{43}^{11}, W_{43}^{21}, and W_{44}^{22} scenarios. The chosen inequalities are:

- W_{43}^{11}:

$$\begin{aligned}
-\langle A_1\rangle + \langle A_4\rangle + \langle B_1\rangle - \langle B_2\rangle + 2\langle B_3\rangle \\
+\langle A_1B_1\rangle + 2\langle A_1B_3\rangle + \langle A_2B_2\rangle + \langle A_2B_3\rangle + 3\langle A_3B_1\rangle \\
+\langle A_3B_2\rangle - 2\langle A_3B_3\rangle - \langle A_4B_1\rangle + \langle A_4B_2\rangle - \langle A_4B_3\rangle \le 8
\end{aligned}$$

- W_{43}^{21}:

$$\begin{aligned}
\langle A_1\rangle + \langle A_2\rangle + \langle B_2\rangle + \langle B_3\rangle - \langle A_1B_1\rangle - 2\langle A_1B_3\rangle \\
+\langle A_2B_1\rangle - 2\langle A_2B_2\rangle + \langle A_3B_1\rangle + 2\langle A_3B_2\rangle + \langle A_3B_3\rangle \\
+\langle A_4B_1\rangle + \langle A_4B_2\rangle - 2\langle A_4B_3\rangle \le 8
\end{aligned}$$

- W_{44}^{22}:

$$\begin{aligned}
-\langle A_1\rangle - 3\langle A_2\rangle + \langle A_3\rangle - \langle A_4\rangle + \langle B_1\rangle + \langle B_2\rangle + 3\langle B_3\rangle + 3\langle B_4\rangle \\
+2\langle A_1B_1\rangle - 4\langle A_1B_2\rangle + 2\langle A_1B_3\rangle + 5\langle A_1B_4\rangle + 3\langle A_2B_1\rangle + 4\langle A_2B_2\rangle \\
+3\langle A_2B_3\rangle + 7\langle A_2B_4\rangle + 2\langle A_3B_1\rangle + \langle A_3B_2\rangle - 3\langle A_3B_3\rangle - \langle A_3B_4\rangle \\
+4\langle A_4B_1\rangle + 5\langle A_4B_3\rangle - 8\langle A_4B_4\rangle \le 26
\end{aligned}$$

Then an analysis of these inequalities is performed *à la* section 3.1 with respect to pure states and the Werner states. The violation of the inequalities is presented in figure 3.3 and figure 3.4. Table 3.4 summarizes inferences from the figure.

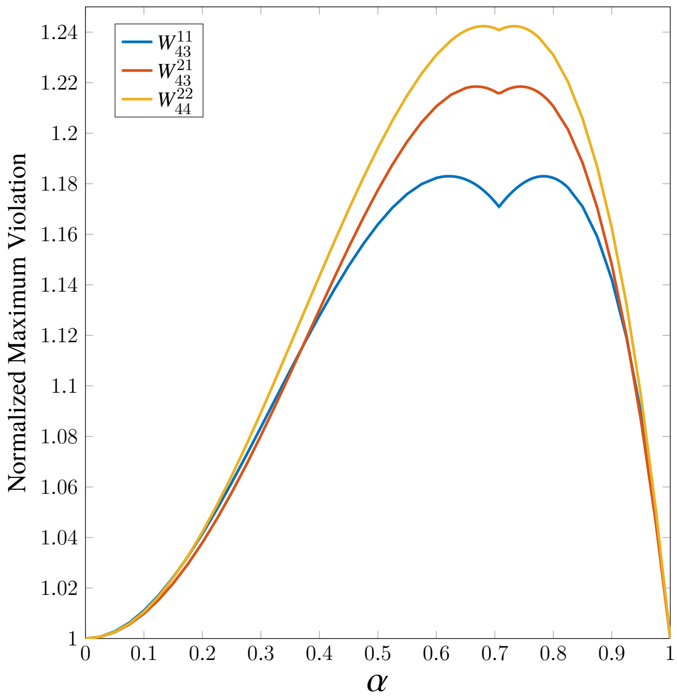

Figure 3.3: Violation of the new LF inequalities - Pure State

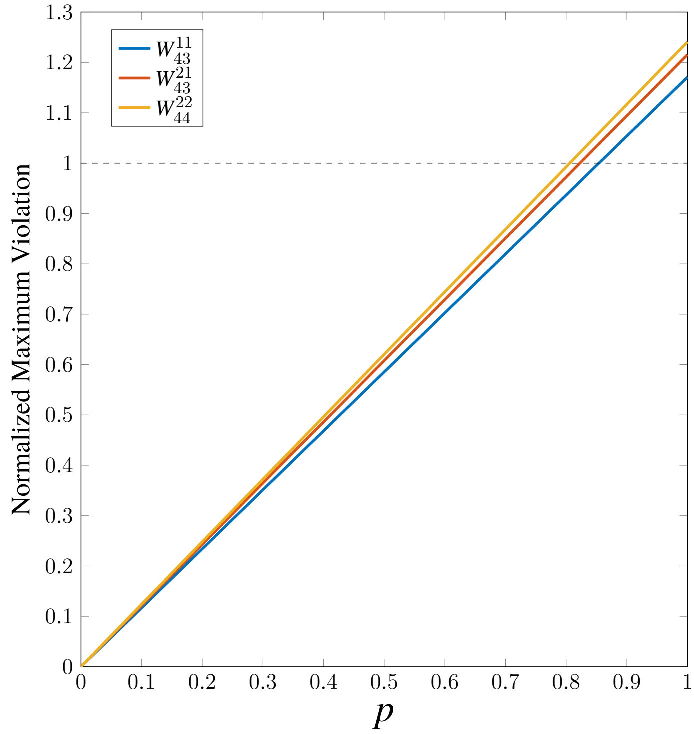

Figure 3.4: Violation of the new LF inequalities - Werner State

Inequality	α	**p**
CHSH	0.7071	0.7071
I_{3322}	0.7071	0.8000
W_{43}^{11}	0.6215	0.8544
W_{43}^{21}	0.6683	0.8227
W_{44}^{22}	0.6805	0.8059

Table 3.4: Violation of the new LF inequalities for the pure state and the Werner state. The second column gives the value of 'α' at which the maximal violation of the inequality occurs for the pure state. The third column gives the minimal value of 'p' at which the violation of the inequalities occurs for the Werner state. The CHSH and I_{3322} inequalities are included for comparison with the newfound inequalities.

Apart from that, for all the newfound inequalities, we have computed the value of α at which the maximum violation of the inequalities occurs for the pure state. The maximal violation of *every* Genuine LF inequality occurs for some non-maximally entangled state. That is, all the 3 inequalities in the scenario W_{43}^{11}, all the 6 inequalities in W_{43}^{21}, and all the 187 inequalities in W_{44}^{22} are violated maximally by some non-maximally entangled state. The value of α for these maximal violations is different for different inequalities. This is a notable contrast with the Bell scenarios, where in each scenario there exist inequalities which are maximally violated by the maximally entangled state. This behavior summarized in Table 3.5.

Similarly, for all the newfound inequalities, the minimal value of p at which the violation of the inequalities occurs is computed for the Werner state. We find that the none of the inequalities are violated where the Werner state has a local description; however there are also domains where Werner states do not have a local description and the inequalities are not violated.

Scenario	**Number of Inequalities**	**No. of inequalities maximally violated by the maximally entangled state**
L_{22}	1	1
L_{33}	2	2
W_{33}^{11}	2	None
L_{43}	5	3
W_{43}^{11}	≥ 3	None
W_{43}^{21}	≥ 6	None
L_{44}	174	21
W_{44}^{22}	≥ 187	None

Table 3.5: Violation of the inequalities for the pure state $\psi = \alpha|00\rangle + \beta|11\rangle$. The number of inequalities in case of the Extended Wigner Friend Scenarios W_{pq}^{rs} includes only to the Genuine LF inequalities. For the Bell Scenarios L_{pq}, the number of inequalities does not include the positivity inequalities.

α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	γ_{11}	γ_{12}	γ_{13}	γ_{14}	γ_{21}	γ_{22}	γ_{23}	γ_{24}	γ_{31}	γ_{32}	γ_{33}	γ_{34}	γ_{41}	γ_{42}	γ_{43}	γ_{44}	δ	Q
0	-2	1	0	0	-2	4	1	-4	4	4	4	7	2	6	1	1	1	-2	-1	-2	3	4	-5	25	31.2935
-1	-4	2	-2	-1	2	6	2	2	-3	4	2	1	5	5	5	2	0	-2	-2	0	4	5	-7	25	30.8491
0	-2	1	0	0	-2	1	4	-4	4	4	4	5	2	3	6	1	1	-1	-2	0	-3	7	-4	25	32.1794
6	2	1	-2	-2	2	0	-5	4	-6	2	6	2	7	2	5	1	1	-2	-1	3	2	2	-5	25	31.4062
-4	4	2	1	-1	1	6	5	1	4	4	5	2	2	-4	0	-1	6	-2	-5	1	-1	4	-5	25	30.0677
-2	-4	2	-1	-1	1	7	2	2	-4	6	2	1	5	7	3	2	1	-3	-2	0	1	5	-5	25	30.2005
-2	-3	2	2	2	-1	2	6	-3	1	1	5	5	2	6	4	0	2	-2	-2	-4	0	7	-5	25	30.2721
-1	-3	1	-2	1	1	3	2	2	-3	2	4	3	4	3	7	2	1	-3	-1	4	1	5	-8	25	31.7736
0	-5	3	1	1	2	3	5	3	-3	-2	2	4	5	1	5	0	3	-3	-3	6	-1	3	-5	25	29.3244
4	-2	2	3	-3	3	2	1	5	-4	1	6	2	4	3	1	2	2	-3	-1	4	-3	3	-7	25	30.6373
1	-4	1	-3	2	3	2	0	3	-4	1	1	5	5	3	7	2	1	-3	-1	4	3	3	-7	25	31.5860
-2	3	-4	2	-1	4	-2	0	1	4	3	2	6	-3	3	3	7	4	-5	-2	3	-1	3	-3	25	31.2140
0	-2	1	4	0	-2	0	1	-4	4	4	4	5	2	4	1	1	3	-4	-1	0	5	4	-5	25	32.2896
3	-1	4	1	2	-4	2	1	-3	6	0	6	5	4	2	0	-2	5	-3	-4	-2	3	3	-3	25	30.4561
0	-1	2	2	0	-1	0	2	-7	4	4	1	4	3	4	2	2	6	-4	-2	1	2	4	-3	25	32.1122
3	-2	1	3	-2	3	1	1	5	-4	2	6	3	5	3	1	2	1	-3	-1	4	-3	3	-7	25	30.9804
-2	-4	1	-2	3	3	3	2	3	5	3	-3	3	4	1	4	3	2	-4	-2	6	-6	3	-1	25	31.0613
-3	-6	-1	3	1	2	8	4	-1	-2	5	1	2	5	5	4	-2	2	5	-4	0	3	-3	-3	25	29.4318
3	-2	3	1	-2	1	0	4	4	-1	4	-2	2	5	5	4	0	4	-3	-4	-4	-1	6	-2	25	30.8573
2	-2	1	4	-2	2	0	1	4	-3	3	6	2	4	3	1	3	1	-4	-1	5	-2	4	-7	25	31.0908
-2	-3	1	-1	1	2	5	1	-3	3	5	3	4	3	5	1	0	6	-6	-1	-2	2	5	-4	25	31.1880
-3	-2	1	-1	1	2	3	3	-3	4	4	6	4	2	2	2	0	6	-4	-3	-2	2	5	-4	25	31.5756
-3	-2	1	-1	1	2	4	2	-3	4	5	5	4	2	3	1	0	6	-5	-2	-2	2	5	-4	25	31.2950
0	-2	0	1	1	3	2	1	4	-4	4	4	2	4	3	1	6	2	-5	-1	3	-3	4	-5	25	31.5914
0	-2	0	1	1	3	2	1	4	-4	4	4	3	5	3	1	5	1	-5	-1	3	-3	4	-5	25	31.5236
0	-2	1	2	1	-1	0	3	-4	4	4	4	5	3	3	3	0	5	-5	-1	-2	3	4	-5	25	32.3348
0	-1	2	-2	0	-1	2	0	-4	4	4	4	4	3	2	4	2	6	-2	-4	2	-2	6	-4	25	32.4894
-1	-1	1	2	1	-1	0	3	-4	3	3	5	5	4	4	2	0	5	-5	-1	-2	3	4	-5	25	31.0834
-1	-4	4	0	0	1	4	4	-2	1	3	1	6	2	4	4	5	-4	-1	-4	1	2	4	-5	25	31.6904
-1	-6	2	-2	1	4	6	2	-3	2	2	0	4	6	6	2	0	6	-6	-2	-2	2	4	-2	25	30.6707
1	-2	2	0	-2	1	2	4	4	-3	4	2	2	4	4	4	0	2	-2	-2	-4	0	8	-4	25	31.3915
0	-3	1	-2	-1	2	4	1	3	-3	3	3	2	5	4	6	1	1	-2	-1	3	3	5	-9	26	33.3554
-1	0	2	-1	2	-1	4	1	-2	8	6	1	4	4	-4	4	3	4	-4	-5	1	-1	2	-1	26	33.4451
2	-7	2	-3	2	3	7	2	-2	3	-4	1	3	6	4	6	2	1	-3	-2	-1	5	4	-5	26	30.9290
4	-1	4	3	-3	0	1	8	2	2	2	-6	6	2	1	6	3	1	-4	-4	4	-3	2	-4	26	30.8476
2	-6	2	4	1	3	1	7	-2	3	1	-4	2	6	6	4	2	1	-2	-3	1	-5	6	-4	26	30.7546
-3	-1	5	-1	-2	0	3	3	-5	7	2	3	3	2	1	3	5	8	-3	-5	1	-1	3	-2	26	32.5463
-1	-4	1	-2	1	4	4	1	-3	4	2	2	4	6	4	2	0	8	-6	-3	-2	2	4	-2	26	32.1790
1	-5	3	3	0	2	2	6	-1	2	1	-3	3	4	7	5	4	1	-2	-4	0	-5	6	-4	26	31.3708
3	-2	0	5	-1	4	-2	1	3	-4	3	7	2	4	3	1	2	0	-3	-1	4	-4	5	-8	26	32.5368
2	1	5	0	2	-5	2	3	-3	5	1	5	5	6	2	0	-4	6	-2	-5	0	0	3	-3	26	31.3450
-1	-3	1	-1	1	1	3	3	2	-4	2	5	3	4	3	7	2	1	-3	-1	4	0	5	-8	26	32.3010

Table 3.6: Inequalities for the W_{44}^{22} Scenario

α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	γ_{11}	γ_{12}	γ_{13}	γ_{14}	γ_{21}	γ_{22}	γ_{23}	γ_{24}	γ_{31}	γ_{32}	γ_{33}	γ_{34}	γ_{41}	γ_{42}	γ_{43}	γ_{44}	δ	Q
-1	-7	2	2	1	4	4	5	-3	2	3	-1	4	6	5	4	0	6	-4	-4	2	-2	2	-4	26	30.4322
0	-3	2	1	-2	1	4	3	4	-4	4	4	2	5	7	1	3	1	-3	-3	1	-1	4	-5	26	31.9557
1	-4	3	4	-1	2	1	6	-2	3	0	-4	1	5	6	4	3	1	-2	-3	1	-5	7	-5	26	31.0592
3	-5	3	-1	-1	3	3	5	6	-3	2	-4	5	6	1	5	0	3	-3	-3	-2	3	3	-3	26	31.1655
3	-5	3	-1	-1	3	3	5	6	-3	2	-4	5	6	1	5	0	4	-3	-4	-2	2	3	-2	26	31.1701
1	-3	-5	3	2	6	-2	0	5	-6	1	3	3	6	3	3	5	4	-3	-3	1	-2	3	-3	26	31.3009
3	-2	1	0	-2	3	1	4	4	-3	4	0	2	4	3	5	1	4	-1	-5	-3	0	7	-4	26	31.7754
4	-3	4	-1	2	-3	2	5	-2	6	6	-2	4	3	0	4	1	3	-4	-4	3	-3	4	-3	26	30.8489
2	-4	2	-2	3	-1	3	3	-3	6	3	-2	6	4	0	6	1	2	-3	-2	5	-1	3	-5	26	31.3636
2	-3	2	3	-3	2	3	2	5	-4	2	5	2	5	5	1	3	2	-3	-2	3	-3	3	-6	26	31.1888
-3	-2	2	1	1	2	2	5	-3	4	4	6	4	2	0	4	0	6	-2	-6	2	-2	4	-5	26	32.3498
3	-2	1	4	-2	3	0	1	4	-3	2	6	2	4	3	1	3	2	-5	-1	5	-2	4	-7	26	31.9059
3	-2	1	4	-2	3	0	1	4	-4	3	6	2	4	3	1	4	2	-4	-1	4	-3	4	-7	26	32.4043
-3	-2	0	-1	2	1	4	1	4	-3	6	4	2	5	4	1	6	1	-5	-2	2	-2	5	-4	26	32.6919
2	-1	3	0	-3	2	1	2	5	-3	5	1	4	5	2	4	6	-2	-5	-2	0	2	3	-5	26	32.8184
-1	-2	-2	1	2	3	0	1	4	-3	4	4	2	6	4	2	6	3	-4	-3	2	-3	4	-4	26	33.2626
-3	-2	1	0	1	2	2	3	-4	5	4	6	5	3	2	2	0	6	-4	-3	-2	2	4	-4	26	32.7266
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α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	γ_{11}	γ_{12}	γ_{13}	γ_{14}	γ_{21}	γ_{22}	γ_{23}	γ_{24}	γ_{31}	γ_{32}	γ_{33}	γ_{34}	γ_{41}	γ_{42}	γ_{43}	γ_{44}	δ	Q
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α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	γ_{11}	γ_{12}	γ_{13}	γ_{14}	γ_{21}	γ_{22}	γ_{23}	γ_{24}	γ_{31}	γ_{32}	γ_{33}	γ_{34}	γ_{41}	γ_{42}	γ_{43}	γ_{44}	δ	Q
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α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	γ_{11}	γ_{12}	γ_{13}	γ_{14}	γ_{21}	γ_{22}	γ_{23}	γ_{24}	γ_{31}	γ_{32}	γ_{33}	γ_{34}	γ_{41}	γ_{42}	γ_{43}	γ_{44}	δ	Q
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-2	-5	2	-2	2	1	5	5	3	-5	3	7	5	6	5	11	3	2	-5	-2	7	0	8	-13	41	51.2704
-3	-6	3	0	-2	1	11	4	4	-7	10	4	2	8	12	4	4	2	-5	-4	0	0	8	-8	42	50.8159
1	-2	3	-3	-1	2	3	1	6	-7	7	7	5	7	3	7	10	2	-3	-8	-4	4	10	-7	43	55.1319
2	-3	2	0	-4	1	1	7	8	-6	8	4	4	7	6	8	2	2	-2	-4	-6	0	13	-7	43	54.0813
1	-6	5	8	-3	4	1	10	-3	4	1	-7	2	8	10	6	5	1	-4	-5	3	-9	12	-8	44	52.1675
-4	-1	10	-2	-4	-1	6	4	-8	12	4	4	4	3	2	4	10	14	-6	-8	2	-2	6	-4	45	56.2957
2	-14	3	-1	4	7	8	9	4	-2	1	-5	8	9	9	6	1	8	-8	-4	-7	6	8	-6	46	54.1039
1	-4	2	0	-4	1	2	8	8	-7	8	6	4	8	6	10	2	2	-2	-4	-6	0	14	-8	47	59.5710
8	-1	8	5	-6	-1	2	15	4	4	4	-12	10	3	2	12	6	2	-8	-8	6	-4	4	-7	48	56.9389

Chapter 4

Conclusion

Wigner's Friend is a longstanding thought experiment which illustrates the measurement problem in quantum mechanics. This thesis is primarily concerned with the recently formulated Extended Wigner's Friend Scenario (EWFS) in the context of spatially separated bipartite qubits. The EWFS consists of a bipartite setup of Wigner's Friend thought experiment where the friends share a correlated system. With respect to the EWFS, under the assumptions of Observer Independent Facts, Freedom of Choice, and Locality a new class of testable inequalities namely Local Friendliness (LF) Inequality was derived in [6]. It was shown in [6] that quantum mechanical correlations violate the Local Friendliness inequalities. The authors in [6] analysed the inequality for a restricted type of bipartite mixed entangled states. We first reproduce their results, and verify them. Then, the LF inequalities are analysed with respect to pure states and Werner States. It is seen that the Bell inequality and LF inequality are not equivalent. Further, in both the cases, it is seen that a quantum violation of LF inequality implies the quantum violation of Bell inequality; but not the other way round. That is, the set of LF correlations satisfying the LF inequality is larger than the set of Bell-Local correlations.

We then formulated various Extended Wigner Friend Scenarios. The LF polytope in each of these cases was specified, thereby characterizing the LF correlations implied by the LF assumptions. We saw that as we add more measurements that correspond to inferring friend's outcome, the LF polytope converges to the Local polytope. In some scenarios, a number of Genuine LF inequalities were found. For each of the new inequalities, the quantum bound up to the second level of the NPA hierarchy was computed. A randomly chosen inequality from each scenario was analysed with respect to pure states and the Werner states. For the pure state, in all these cases, the maximal violation of the inequalities was found to occur for some non-maximally entangled state.

Building upon these results, we are considering a reformulation of the EWFS, where we reinterpret the assumption of Observer Independent Facts as Objective Value Assignment. The reformulation of the EWFS will make use of weak measurement along with the usual joint measurement of the friend and the system by Alice and Bob. Apart from testing the Local Friendliness assumptions, this reformulation is expected to reveal novel form of quantum nonlocality.

We intend to derive the explicit form of the inequalities for the scenario where each measurement results in one of the three possible outcomes. The characterization of the LF inequalities in multipartite EWFS is being taken up as well. Nonlocal correlations are known to be an information theoretic resource [14]. There may be potential applications of LF correlations in quantum information processing tasks, like random number generation, quantification and certification, cryptography etc. These will be considered in a future work.

Bibliography

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, USA, 10th edition, 2011).

[2] E. P. Wigner, *Remarks on the mind-body question*, In I. J. Good, editor, *The Scientist Speculates*, pages 284–302 (Heinemann, London, 1961).

[3] D. Frauchiger and R. Renner, *Quantum theory cannot consistently describe the use of itself*, Nature Communications, **9**, 3711 (2018).

[4] Č. Brukner, *A no-go theorem for observer-independent facts*, Entropy, **20**, 350 (2018).

[5] R. Healey, *Quantum theory and the limits of objectivity*, Foundations of Physics, **48**(11), 1568 (2018).

[6] K. W. Bong, A. Utreras-Alarcón, F. Ghafari, Y. C. Liang, N. Tischler, E. G. Cavalcanti, G. J. Pryde, and H. M. Wiseman, *Testing the reality of Wigner's friend's experience*, arXiv:1907.05607 (2019).

[7] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, *Bell nonlocality*, Rev. Mod. Phys., **86**(2), 419 (2014).

[8] B. Tsirelson, *Some results and problems on quantum Bell-type inequalities*, Hadronic Journal Supplement, **8**, 329 (1993).

[9] G. M. Ziegler, *Lectures on polytopes* (Springer-Verlag, New York, 1995).

[10] S. Lörwald and G. Reinelt, `PANDA`: *A software for polyhedral transformations*, EURO Journal on Computational Optimization, **3**, 297 (2015).

[11] T. Christof, A. Löbel, and S. Schenker, `PORTA`: *Polyhedron representation transformation algorithm*, `http://porta.zib.de` (1997)

[12] E. Gawrilow and M. Joswig, `polymake`: *A framework for analyzing convex polytopes*, In *Polytopes—Combinatorics and Computation,* volume 29 of *DMV Sem*, pages 43–73. (Birkhäuser, Basel, 2000).

[13] D. Avis. `lrs`: *A revised implementation of the reverse search vertex enumeration algorithm*, In *Polytopes—Combinatorics and Computation*, volume 29 of *DMV Sem*, pages 177–198. (Birkhäuser, Basel, 2000).

[14] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, *Nonlocal correlations as an information-theoretic resource*, Phys. Rev. A, **71**, 022101 (2005).

[15] T. Cope and R. Colbeck, *Bell inequalities from no-signaling distributions*, Phys. Rev. A, **100**, 022114 (2019).

[16] J. S. Bell, *On the Einstein-Podolsky-Rosen paradox*, Physics Physique Fizika, **1**(3), 195 (1964).

[17] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, *Proposed experiment to test local hidden-variable theories*, Phys. Rev. Lett., **23**(15), 880 (1969).

[18] R. F. Werner and M. M. Wolf, *Bell inequalities and entanglement*, Quantum Information & Computation, **1**(3), 1 (2001).

[19] M. Navascués, S. Pironio, and A. Acín, *A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations*, New Journal of Physics, **10**(7), 073013 (2008).

[20] J. Lofberg, `YALMIP`: *A toolbox for modeling and optimization in Matlab*, In 2004 IEEE international conference on robotics and automation (IEEE Cat. No. 04CH37508), 284 (2004).

[21] J. F. Sturm, `SeDuMi`: *A Matlab toolbox for optimization over symmetric cones*, Optimization methods and software, **11**(1-4), 625 (1999).

[22] N. Johnston, QETLAB: *A Matlab toolbox for quantum entanglement, version 0.9*, `http://qetlab.com`, (2016)

[23] N. Brunner, N. Gisin, and V. Scarani, *Entanglement and non-locality are different resources*, New Journal of Physics, **7**, 88 (2005).

[24] R. F. Werner, *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*, Phys. Rev. A, **40**, 4277 (1989).

[25] A. Acín, *Werner states*, In D. Greenberger, K. Hentschel, and F. Weinert, editors, *Compendium of Quantum Physics*, pages 843–845 (Springer Berlin-Heidelberg, Berlin, Heidelberg, 2009).

[26] N. S. Jones and L. Masanes, *Interconversion of nonlocal correlations*, Phys. Rev. A, **72**, 052312 (2005).

[27] D. Collins and N. Gisin, *A relevant two qubit Bell inequality inequivalent to the CHSH inequality*, Journal of Physics A: Mathematical and General, **37**(5), 1775 (2004).

[28] E. Oudot, J. D. Bancal, P. Sekatski, and N. Sangouard, *Bipartite nonlocality with a many-body system*, New Journal of Physics, **21**(10), 103043 (2019).

[29] D. Avis and C. Jordan, `mplrs`: *A scalable parallel vertex/facet enumeration code*, Mathematical Programming Computation, **10**(2), 267 (2018).

[30] S. Pironio, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, *Random numbers certified by Bell's theorem*, Nature, **464**, 1021 (2010).