An analysis of Page Curve and the Information Paradox

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Certificate

This is to certify that this dissertation entitled *An analysis of Page Curve and the Information Paradox* towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Vaishnavi Patil at Indian Institute of Science, Bengaluru under the supervision of Dr. Chethan Krishnan, Professor, Department of Physics, IISc, during the academic year 2019-2020.

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This thesis is dedicated to those who love physics

Declaration

I hereby declare that the matter embodied in the report entitled *An analysis of Page Curve and the Information Paradox*, is the result of the work carried out by me at the Department of Physics, IISc, under the supervision of Dr. Chethan Krishnan and the same has not been submitted elsewhere for any other degree.

Dr. Chethan Krishnan

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Abstract

We study the various versions of holographic entanglement entropy formula that has come out over time. This starts with the Ryu-Takayangi formula which later modified to the HRT prescription and then the FLM prescription which gave quantum corrections and finally the method of finding a quantum extremal surface to calculate the entanglement entropy. Then we look at entanglement wedge reconstruction and its application to an evaporating black hole in AdS as done in the work of Penington. Using there was an attempt to find a Page curve for it and this calculation implies that the graph of entropy versus time will turn around as expected in the case of information being preserved during the black hole evaporation. At the end we also see that these results match with the work done by Hayden and Preskill to find the time required for any information to come out of the black hole in an information preserving model.

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Chapter 1

Basic Ideas of Holographic Entanglement Entropy

1.1 Introduction to the Information Paradox

The similarity between Black Holes and thermodynamic systems has been observed for some time now. The first law of black hole mechanics makes a statement similar to the first law of thermodynamics, which is

$$dE = TdS_{BH} + \Omega dJ + \Phi dQ \tag{1.1}$$

where T : temperature, Ω : Angular velocity, Φ : electric potential

The second law of black hole mechanics states that the Bekenstein-Hawking entropy S_{BH} is non-decreasing with time [1]. The Bekenstein-Hawking entropy is defined as:

$$S_{BH} = \frac{A}{4\hbar G} \tag{1.2}$$

where A : area of the event horizon of the black hole.

Classical general relativity tells us that black holes are objects which absorb all particles and no emission can occur. The famous semi-classical calculation by Hawking [2] of quantum field theory in curved spacetime showed that black holes radiate as if they were hot bodies. This would lead to the shrinking and eventual disappearance of the black holes. This seems to be violating the classical area law due to the decreasing event horizon. However the Generalized Second law is proposed which says the Generalized Entropy S_{gen} is non-decreasing with time.

$$S_{gen} = \frac{Area(Horizon)}{4G\hbar} + S_{out} + counterterms \tag{1.3}$$

where, S_{out} is the entanglement (von Neumann) entropy of the degrees of freedom outside the event horizon. The counterterms are present to remove the divergences present in quantum field theories. The area term is the leading order (in $\hbar$) or the classical component here.

For a Schwarzschild black hole in 3+1 dimensions having a radius R, the Hawking temperature is:

$$T = \frac{\hbar c}{4\pi R} \tag{1.4}$$

Since the temperature is inversely proportional to the radius, the evaporation becomes faster as the black hole becomes smaller.

The null worldlines along the event horizon are extremely unstable as any displacement along the radial direction can change the trajectories vastly. The Hawking radiation occurs due to this stretching effect at the event horizon. We know that in a quantum field theory vacuum (which can be approximately assumed at the short scale on the horizon) spacelike separated points have correlations. These correlations are stretched over large distances due to the radiation and hence points inside and outside the black hole are entangled. Precise calculations show that this state is completely thermal.

The problem in this phenomenon is that if we started with the universe (containing a black hole) in a pure state, then after the complete evaporation we are simply left with the radiation whose quantum state is mixed. This violates the fundamental principle of unitary evolution in quantum mechanics. It can be rephrased that final mixed state of the Universe (a closed system) will not contain all the information about the initial pure state we started with and hence the information about the initial state is lost.

So the information paradox is simply the question of whether the process of formation and evaporation of black holes can be described in a unitary way. Unitarity would imply that the von Neumann entropy of the radiation should first increase with time and then start to decrease after a particular time (called the "Page time"). This rise and fall of the entropy with time is called the Page curve and if this curve can be shown it will be equivalent to the resolution of the paradox.

With the discovery of the AdS/CFT correspondence [3], we understood that information can actually come out of the black hole. The AdS/CFT correspondence states that a theory of quantum gravity living in a bulk (d+1)-dimensional Anti-de Sitter (AdS_{d+1}) spacetime is equivalent to a certain Conformal Field Theory (CFT_d) living at the d-dimensional asymptotic boundary of the spacetime. The boundary CFT is a unitary theory in which information is preserved. Due to the equivalence, we can say that the information is always preserved but we want to show this through a bulk calculation or a phenomenon. We also want to answer why Hawking's calculation gives a thermal final state of radiation and how it can be corrected or re-interpreted. Finally we don't want to restrict our attention to the AdS case and want to explain the information paradox in the real Universe which is an asymptotically flat spacetime.

1.2 Holography, Ryu-Takayanagi and HRT Formula

The idea of holography introduced by 't Hooft [4] (which is subsequently seen in AdS/CFT), is that a $(d+1)$-dimensional theory of quantum gravity has degrees of freedom similar to a d-dimensional quantum theory. He argued that models of quantum gravity in (3+1)-dimension will have constraints such that their degrees of freedom and observables can be defined on a (2+1)-dimensional lattice model with Boolean variables. This can be explicitly seen in the formula for Bekenstein-Hawking entropy where the information content (i.e entropy) of a black hole which is a 3-dimensional object scales with the 2-dimensional area of the event horizon. Even in quantum field theories, we know that the entropy of a region scales with the boundary of that region which is one dimension lower than the bulk state (the entropy always diverges but we regulate it with a UV cutoff).

The Bekenstein-Hawking formula was generalized by Ryu and Takayanagi [5] using the AdS/CFT correspondence. The entanglement entropy of a system is the von Neumann entropy which basically measures how closely entangled the state of the system is. The von Neumann entropy is given by

$$S_A = -Tr_A \rho_A \log \rho_A \tag{1.5}$$

where A is a subsystem and ρ_A is the reduced density matrix for it.

Ryu and Takayanagi proposed an "area law" for the entanglement entropy in conformal field theories living on the boundary of AdS. Here they consider a subsystem (or a subregion of a spatial slice) A in a d-dimensional CFT which has a boundary ∂A. The Ryu-Takayanagi (RT) formula is as follows:

$$S_A = \frac{Area(\gamma_A)}{4G_N^{d+1}} \tag{1.6}$$

where G_N^{d+1} is Newton constant of (d+1)-dimensional bulk and γ_A is a surface in the bulk anchored at ∂A which has the minimal area. This surface called the RT surface is similar to a holographic screen for the subregion A. This formula reproduces the standard von Neumann formula in a CFT and also follows the basic properties of an entanglement entropy such as:

1. Strong subadditivity:

$$S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C} \tag{1.7}$$

$$S_A + S_C \leq S_{A+B} + S_{B+C} \tag{1.8}$$

2. Equality of complementary regions:

$$S_A = S_B \tag{1.9}$$

where A is the complement of B

The density matrix is defined as:

$$\rho = \sum_i p_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right| \tag{1.10}$$

where p_i is the probability of the system being in the pure quantum state $|\Psi_i\rangle$. At finite temperature we use the thermal density matrix which is defined at inverse temperature β as:

$$\rho_{thermal} = e^{-\beta\mathcal{H}} \tag{1.11}$$

The RT prescription was later modified by Hubeny, Rangamani and Takayanagi [6] to give a time-dependent covariant version of the entanglement entropy in holographic systems. This prescription called the HRT formula is based on light sheet construction of extremal surfaces. Given any subregion A on the holographic boundary, we find an *extremal surface* χ in the bulk which is again anchored at the boundary ∂A with a zero expansion coefficient. It is a codimenion-2 hypersurface which is homologous to R and has extremal area. In case of multiple such surfaces found, we choose the one with minimum area. Then the holographic entanglement entropy is given by:

$$S(A) = \frac{Area(\chi)}{4G_N^{d+1}} \tag{1.12}$$

This formula easily reduces to the Ryu-Takayanagu formula for static spacetimes.

1.2.1 Causal wedge and causal surface

Definition: A *Domain of Dependence* D_A for a boundary region A are the set of all points on the boundary which can be completely determined by the information given on A. If A belongs to a Cauchy slice Σ then:

$$\begin{aligned} &x \in D_A^+ \iff \mathcal{I}^-(x) \cap \Sigma \subseteq A \\ &x \in D_A^- \iff \mathcal{I}^+(x) \cap \Sigma \subseteq A \\ &D_A = D_A^- \cup D_A^+ \end{aligned} \tag{1.13}$$

where $\mathcal{I}^\pm$ is the causal future/past on the boundary.

Definition: A *Causal Wedge* W_A of a boundary subregion A is the set of all points in the bulk spacetime which lie at the intersection of the causal future and causal past (in the bulk) of the domain of dependence D_A.

$$W_A = \mathcal{I}^-(D_A) \cap \mathcal{I}^+(D_A) \tag{1.14}$$

where $\mathcal{I}^\pm$ is the causal future/past in the bulk.

Definition: A *Causal Surface* C_A is the intersection of the past horizon and future horizon of the boundary domain of dependence.

$$C_A = \partial\mathcal{I}^-(D_A) \cap \partial\mathcal{I}^+(D_A) \tag{1.15}$$

where $\mathcal{I}^\pm$ is the causal future/past in the bulk. C_A is the spcelike boundary of W_A.

The Causal Wedge is the region of the bulk which can be reconstructed using the data on the boundary region A. This is possible classically.

Definition: A given type of matter obeys the *Null Energy Condtion* (NEC) if $T_{ab}k^ak^b \geq 0$ for any null vector k^a. Similarly a spacetime manifold obeys the *Null Convergence Condition* (NCC) if $R_{ab}k^ak^b \geq 0$ for any null vector k^a [9].

Definition: An endless null geodesic is said to be *generic* if there exists a point x with a tangent vector k on the geodesic such that $R_{ab}k^ak^b \neq 0$. When this is satisfied for every null ray, we say *generic condition* holds.

The Null Convergence Condition means that energy density is non-negative in every local frame and the Generic Condition means that the null rays will encounter some matter density. Both these conditions are purely physical and will generally be assumed for classical matter.

1.2.2 The HRT surface lies deeper than the Causal Surface

Theorem: An extremal (HRT) surface χ lies spacelike separated from C_A and on the side away from the boundary subregion A.

Proof: Wall [10] proved this by contradiction. Assume that χ lies inside C_A (i.e. on the side of A). We also assume the null convergence condition and the generic condition.

1. Since χ is an extremal surface we can shoot null congruence $N(A)$ (codimension 1) from it towards A which has expansion coefficient $\theta < 0$. This can be seen by noting that the extremal surface χ has expansion coefficient $\theta = 0$ and the Raychoudhari equation[1] along with NCC and generic condition will make $\theta < 0$ everywhere else on

[1] $\frac{d\theta}{dt} = -\frac{1}{2}\theta^2 - 2\sigma^2 - R_{ab}k^ak^b$ where θ is the expansion of congruences of null geodesics, t is an affine

the null congruence. $\partial I^-(D_A)$ is a causal horizon, so according to the Second Law of Horizons its area must increase as we move in the future away from the causal surface and similarly area of $\partial I^+(D_A)$ will increase as we move in the past away from the causal surface. If $\theta < 0$ then by the Raychoudhari equation, the rays would have to focus which is not possible since they are shot back from the null infinity. Hence on the cuasal surface $\theta > 0$.

2. Now we can deform the boundary subregion A continuously such that the new causal surface $C_{A'}$ is nowhere in the exterior of $N(A)$ and touches it at a point x. According to Theorem 4 of Wall [10], if two surfaces (N_1, N_2) touch at a point and N_2 lies nowhere in the past of N_1 then in the neighbourhood of the point of contact either they coincide or N_2 expands faster than N_1 ($\theta[N_2] > \theta[N_1]$). Therefore we get $\theta[N(A)] > \theta[C_{A'}]$.

Points 1. and 2. give a contradiction hence we conclude that χ lies in the exterior of C_A.

A corollary of the above theorem is that the causal surfaces of complementary subregions on the boundary will never overlap. This can be understood by noting the fact that HRT surfaces for complementary subregions will be equivalent as A and A^c have common boundary ∂A and hence the extremal surface anchored at that will be the same from both sides. Since the causal surface is separated from the HRT surface towards its corresponding subregion, the two surfaces C_A and C_{A^c} cannot overlap.

1.3 Maximin prescription

Wall in [10] formulates another type of bulk surface called the "Maximin Surface". Maximin surface is proved to be equivalent to the standard HRT surface and it follows certain properties such as Strong Subadditivity, Monogamy of Mutual Information etc. The maximin surface is easier to calculate and is found useful when we apply it in the context of the information paradox.

The bulk spacetime here is AlAdS (asymptotically locally Anti de Sitter). It is also assumed to satisfy the NCC, generic condition and global hyperbolicity.

parameter, σ is the shear and R_{ab} is the curvature tensor

Definition: Consider a complete achronal (nowhere timelike) slice which passes through the boundary ∂A of the subregion A and find the minimal surface anchored there. Then maximize the area of that surface by varying the achronal slice itself. This will give us the *Maximin Surface* which is homologous to the boundary subregion A and has a codimension-2.

Theorem: The maximin surface exists.

Outline of Proof: (i) First we consider all possible surfaces on an achronal slice anchored at ∂A.

- We can consider the achronal slice to be a compact metric space.
- The space of all the surfaces anchored at ∂A has a natural topology and is compact.
- The area of the surface is lower semicontinuous[2].

Therefore, we can say that a minimum for this codimension-2 surface exists according to the extreme value theorem.

(ii) Now we consider the space X of all possible complete achronal slices in our spacetime. Here we assume that the spacetime has no horizons.

- Each achronal slice in X can be thought of as a function which takes each spatial point and associates to it a time of the slice at that point. The space X is equibounded which means that at each spatial point there is an upper and lower bound to the associated time for all achronal slices.
- X is equicontinuous which means that at each spatial point there is an upper and lower bound on the derivative of the time function for all achronal slices.
- X is closed and compact.
- The area of the minimum area surface of each achronal slice is upper semicontinous[3].

[2] A function f is lower semicontinous at x_i if the value of $f(x)$ is not much lower than the value of $f(x_i)$ when x is close to x_i

[3] A function f is upper semicontinous at x_i if the value of $f(x)$ is not much higher than the value of $f(x_i)$ when x is close to x_i

Therefore, we can say that the maximal value along the variation of the achronal slices for the minimum area surface exists according to the extreme value theorem.

Hence the maximin surface exists.

1.3.1 Equivalence of HRT and maximin surfaces

Theorem: The HRT surface is equivalent to the maximin surface.

Proof: Following the arguments of [10], we first show that the maximin surface is extremal and then that it is the HRT surface. The proof is more heuristic and not mathematically completely rigorous.

1. In the case that the first derivative of the achronal slice which contains the maximin surface is continuous, we can see that the maximin surface is extremal. Since maximin surface is codimension-2, we can vary it in two normal directions. We know that the surface is minimall when varied along the achronal slice and maximal when the slice itself is varied which ensures that it is maximal along one more direction. Hence these two together ensure that the maximin surface is extremal along any direction of variation. If the achronal slice has a discontinuous first derivative, the direction vector along which the surface is maximal will be a linear combination of the tangent vectors of the achronal slice which are normal to the maximin surface. The linearly independent directions along which the surface can be varied is only two hence it will be extremal.

2. On the achronal slice on which the maximin surface lies, it has the least area (by construction). If there exists another extremal surface outside of this slice, then we consider its representative[4] on this slice and we know that the representative has lesser area than its associated extremal surface (Theorem 3 in [10]) and the maximin surface will have lesser area than the representative (or any surface) on that achronal slice. Hence we find the the maximin surface has the minimum area of all extremal surfaces therefore it is the HRT surface.

[4] From any extremal surface if we shoot null surfaces (which will have codeimension-1), then the intersection of it with any achronal slice which passes through ∂A is called the "representative" of the extremal surface on that slice. This representative will have codimension-2.

1.3.2 Properties of HRT/Maximin surface

1. Strong Subadditivity: Consider three disjoint boundary regions A,B,C. Then the following inequality is satisfied:

$$Area[\chi(AB)] + Area[\chi(BC)] \geq Area[\chi(ABC)] + Area[\chi(B)] \tag{1.16}$$

   Here, χ is the HRT or maximin surface corresponding to each of the given unions of subregions.

2. Monogamy of Mutual Information: Consider three disjoint boundary regions A,B,C. Then the following inequality is satisfied:

$$\begin{aligned} Area[\chi(AB)] + Area[\chi(BC)] + Area[\chi(AC)] \geq & Area[\chi(A)] + Area[\chi(B)] \\ & + Area[\chi(C)] + Area[\chi(ABC)] \end{aligned} \tag{1.17}$$

3. The area of the HRT/maximin surface $\chi(A)$ is less than the area of the Causal surface C_A.

$$Area[\chi(A)] > Area[C_A] \tag{1.18}$$

4. For a bigger boundary subregion which contains another one, the HRT/maximin surface is further in the bulk than the HRT/maximin surface corresponding to the contained subregion with the two surfaces spacelike separated.

$$A \subset B \Rightarrow r(A) \subset r(B) \tag{1.19}$$

   where $r(A)$ is the region between the HRT/maximin surface $\chi(A)$ and the domain of dependence D_A which is spacelike separated from $\chi(A)$.

1.4 FLM formula and towards a Quantum Extremal Surface

The above mentioned results (Bekenstein-Hawking entropy as well as HRT formula) are obtained at $\mathcal{O}(\hbar^{-1})$. At higher orders we have to take into account the quantum effects. In the semiclassical approximation we find the entropy of the black hole to be the generalized entropy (Eq 1.3) which follows the Generalized Second Law (GSL). Similarly the entanglement entropy in AdS/CFT (HRT formula) has higher order corrections, the first of which was given by Faulkner, Lewkowycz and Maldacena in [8]. The FLM formula is calculated at $\mathcal{O}(\hbar^0)$ and gives the generalized entropy for a region A in the boundary CFT as:

$$S_{gen}(A) = \frac{Area(\chi)}{4G\hbar} + S_{bulk} + counterterms \tag{1.20}$$

where χ is the regular HRT surface (or extremal surface). If we consider any Cauchy slice (a non timelike surface) which is divided into two halves by χ then S_{bulk} is the von Neumann entropy of the modes contained on the half which has the boundary region A. Note that if the state of the entire system is pure then both sides of the HRT surface will have the same von Neumann entropy. So we find that in both the cases (that is black holes and CFT) the area term gives the classical piece of the entropy and we have to add the quantum terms for higher order calculations.

To calculate the entropy at all orders of $\hbar$, Engelhardt and Wall [7] suggested another prescription of finding what they called a "Quantum Extremal Surface". In FLM formula, we extremized the area to find χ and then added the bulk entropy term. Instead we first define the generalized entropy as the sum of the area term and the bulk term and then extremize the total. The surface which achives this is then defined as the quantum extremal surface (QES) which is a deformation from the HRT surface. [7] showed that at $\mathcal{O}(\hbar^0)$, the QES prescription matches the FLM formula and at $\mathcal{O}(\hbar^{-1})$ it matches the classical HRT formula. They conjectured that at all orders of $\hbar$, the entanglement entropy of a boundary subregion A of a quantum field theory dual to a bulk theory is given by the generalized entropy of the quantum extremal surface χ_q corresponding to the subregion A.

$$S(A) = S_{gen}(\chi_q) = \frac{Area(\chi_q)}{4G_N} + S_{bulk}(\chi_q) \tag{1.21}$$

1.4.1 QES lies deeper in the bulk than Causal Surfaces

Theorem: A Quantum Extremal Surface χ_q never cuts the Causal Wedge W_A of a boundary subregion A. Also when generic condition holds, χ_q is spacelike separated from the Causal Surface C_A.

Proof: Engelhardt and Wall [7] prove this by contradiction. Let χ_q divide a Cauchy slice of the bulk AdS into two halves $Int(\chi_q))$ and $Ext(\chi_q)$ with the exterior $Ext(\chi_q))$ being the side which contains the boundary subregion A and interior is the side deeper in the bulk. Assume that $C_A \cap Int(\chi_q) \neq \phi$.

We start with the boundary domain of dependence D_A of the subregion A and continuously shrink it by deforming A appropriately so as to obtain a region $A\prime$ such that the causal surface of A' ($D_{A}\prime$) lies in $Ext(\chi_q)$.

Define $\mathcal{H}^+$ as the future causal horizon and $\mathcal{H}^-$ as the past causal horizon of $D_{A}\prime$. We find $D_{A}\prime$ such that either $\mathcal{H}^+$ or $\mathcal{H}^-$ touches χ_q at points $\{p\}$.Without loss of generality, we use $\mathcal{H}^+$ as the surface which touches χ_q. At any one of the points $\{p\}$, differentiating S_{gen} for χ_q and $\mathcal{H}^+$ with respect to their normal direction N^a, we get:

$$\frac{\partial S_{gen}(N(\chi_q))}{\partial N^a} k^a \geq \frac{\partial S_{gen}(\mathcal{H}^+)}{\partial N^a} k^a \tag{1.22}$$

where $N(\chi_q)$ is the null surface generated by shooting light rays out from χ_q. By definition, $\frac{\partial S_{gen}(N(\chi_q))}{\partial N^a} k^a = 0$ so we get,

$$0 \geq \frac{\partial S_{gen}(\mathcal{H}^+)}{\partial N^a} k^a$$

which contradicts the GSL, which is

$$0 < \frac{\partial S_{gen}(\mathcal{H}^+)}{\partial N^a} k^a$$

Hence the assumption is wrong, therefore $C_A \cap Int(\chi_q) = \phi$, and hence a QES χ_q never cuts the Causal Wedge of a subregion A.

An important point to note here is that unlike the classical case, the bulk here is quantum and hence we no longer assume the Null Energy Condition. The Generalized Second Law, however, is assumed to hold even when there are quantum corrections to the spacetime.

Chapter 2

Page Curve of an evaporating Black Hole in AdS

2.1 Entanglement Wedge Reconstruction

It has been conjectured that the operators living in the bulk of AdS spacetime with a quantum gravity theory can be reconstructed by the states of the boundary CFT. The question for a long time was knowing which region of the bulk can be reconstructed from a given subregion of the boundary A whose state is known. It was first argued that this region should at least contain the Causal Wedge W_A of the boundary subregion A [14] and that the extremal surface χ contains the information content of A. Hubeny and Rangamani [14] refer to the area of χ as the "causal holographic information" and show that it agrees with the entanglement entropy. However, later developments suggested that reconstruction of more region in the bulk is possible. This was first developed in [11] with subsequent progress in [10] and [13]. Consequently, the region of the bulk which can be reconstructed using a given boundary region (or is encoded in the boundary region) was called its Entanglement Wedge.

Definition: The *Entanglement Wedge* is the domain of dependence of the spacelike surface bounded by the HRT surface[1] and the boundary subregion under consideration.

[1]Note that from now on we will refer to the Quantum extremal surface as the Ryu-Takayanagi surface or the HRT surface. If we wish to refer to the classical version of the entropy formula we will call it the classical HRT surface instead. This is seen to be the general convention.

Similarly, the domain of dependence of the complementary region for a given spatial slice in the bulk gives the entanglement wedge of the complementary subregion of the boundary. This is called Entanglement wedge complementarity.

Note that the extremal surface (and the entanglement entropy) is sensitive to the degrees of freedom on the boundary subregion. Also the facts that the extremal surface lies outside the causal surface and that it moves deeper in the bulk as the boundary subregion grows provide some consistency checks to the claim of entanglement wedge reconstruction. So an operator in the bulk is reconstructible from a boundary subregion if and only if the operator acts only in the entanglement wedge of that subregion.

2.2 Evaporating Black Hole

To study the information paradox, Penington in [12] looks at an evaporating black hole (one sided) which was created by a collapse. The set-up consists of a black hole in AdS (asymptotically) which is emitting the Hawking radiation. The radiation when it reaches the boundary of the AdS bulk, is collected in an auxiliary system or a reservoir (bath). This is done by setting the boundary of AdS to be absorbing rather than reflecting as is usually the case.

Once the radiation is extracted in the reservoir with a Hilbert space which is denoted as $\mathcal{H}_{rad}$, the entire system inside the AdS bulk (and the CFT state correspondingly) consists of just the Black Hole. Hence the entanglement entropy of the black hole will be given by the holographic entanglement entropy (or the generalised entropy) of the entire boundary CFT instead of just one subregion on it. This is the entanglement entropy between the black hole and the radiation. To know this, we must find the quantum HRT surface for the entire boundary and then calculate its generalised entropy.

If we consider only the classical holography, then the HRT surface corresponding to the entire boundary is a surface without a boundary since it has to be homologous to the AdS boundary. It will thus have a topology of S^{d-2} in d-dimensional AdS bulk and with extremal area. It can be seen trivially that this will thus simply be a null surface (i.e., area is zero). The Entanglement Wedge for the entire boundary CFT is then all of the AdS spacetime containing the black hole. Hence the interior of the black hole is reconstructible from the

CFT i.e., it is encoded in the Hilbert space of the CFT. This means that the interior of the black hole can never be decrypted from the radiation hence the information is lost.

However, now we consider the quantum extremal surfaces which extremize the generalised entropy. The generalized entropy has two components which are the area term and the bulk entropy term. Extremizing this is involves a balance of the two. We can see that the classical HRT surface mentioned above will still be a quantum extremal surface also. There is another quantum extremal surface we can find which will be close to (and just slightly inside) the black hole event horizon. In both cases the topology of the surface is still equivalent to S^{d-2} with it being homologous to the AdS boundary. Note that here we consider the black hole radius to be much larger than the Planck scales which is why Hawking's calculation (which was semi-classical) is valid. The two cases are as follows:

1. The QES is empty i.e., $\chi_q = \phi$:

$$\begin{aligned} S_{gen}(\chi_q = \phi) =& \frac{Area(\phi)}{4G_N} + S_{bulk}(\chi_q = \phi) \\ =& S_{rad} \end{aligned} \tag{2.1}$$

   The area of the of the empty surface is clearly zero. The entanglement wedge for it will be the entire AdS bulk. Since we started out with a pure state of the initial black hole, at any point in the evaporation the remaining state of the black hole (or the CFT) and the radiation form a bipartite system which purifies each other i.e., they are complementary and together form a pure state. Hence the von Neumann entropy of the entire bulk will be equal to the entropy of the radiation.

2. The QES is non empty (located close to the event horizon):

$$\begin{aligned} S_{gen}(\chi_q \neq \phi) =& \frac{Area(\chi_q)}{4G_N} + S_{bulk} \\ \approx& \frac{A_{hor}}{4G_N} = S_{BH} \end{aligned} \tag{2.2}$$

   Here, the entanglement wedge (of the entire boundary CFT) is the region between the QES (approximately at the black hole event horizon) and the boundary of the AdS. Note that in this case the interior of the black hole in now in the entanglement wedge of the radiation (due to complementarity of the entanglement wedges). The bulk entropy term for this will be small compared to the area term hence we find the generalised entropy to be approximately equivalent to the Bekenstein-Hawking entropy at the leading order.

At each epoch of time in the process of black hole evaporation, the quantum HRT surface will be one of the above two QES which has lesser generalized entropy. So the entanglement entropy between the boundary CFT and the radiation bath is given by:

$$S = min(S_{rad}, S_{BH}) \tag{2.3}$$

2.3 The Page Curve

Hawking's original semi-classical calculation said that as the black hole originally in a pure state completely evaporated, the radiation left over would be in a mixed or a thermal state. This means that an S matrix which describes the evolution from the initial black hole to the final radiation does not exist. The information that went inside the black hole is thus permanently lost to any observer outside. One of the suggested solutions to this problem is that the S matrix exists and the information that went inside the black hole does actually come out encoded in the radiation. Another suggestion was the left over remnants having all the information that went inside. But this option is more or less discarded now because it seems implausible that arbitrarily high density of information can be contained in the remnants. Don Page in [15] suggested that in the beginning of the evaporation if the information from the black hole comes out, it will be at a very slow rate to be measured. Assuming that the information is conserved during the evaporation (as well as formation), he plotted the average entanglement entropy of the radiation against its thermodynamic entropy. By starting with a pure initial state, this curve graph first increases and then decreases. In the beginning the emitted radiation has a smaller Hilbert space than the black hole Hilbert space, so the information content in the radiation is quite small. Towards the end of evaporation, Hilbert space of radiation would be large then the total information will be encoded in the correlations between the subparts of the total radiation. The expected information encoded in the radiation increases as the evaporation proceeds.

The Page curve has been derived for various models but Penington [12] attempts to derive it from a bulk calculation instead of the boundary CFT. Here, we will follow the work done in [12]. The objective is to show that the plot of the entanglement entropy with respect to time of evaporation of the black hole turns around i.e, it first increases and then goes down.

The time at which it is maximum is called the Page time (commonly the halfway point). The Page time is defined as the point at which the entanglement entropy of radiation becomes equal to the Bekenstein-Hawking entropy.

$$S_{rad}(t = t_{Pg}) = \frac{A_{hor}(t = t_{Pg})}{4G_N} \tag{2.4}$$

2.4 Phase transition of the HRT surface

In the previous section we saw that there are two possible quantum extremal surfaces[2].

1. Before the Page time: $S_{rad} < S_{BH}$
   As the HRT surface is that QES which has lesser generalized entropy, the empty surface becomes the HRT surface and the generalized entropy of the black hole is the von Neumann entropy of the radiation. Initially ($t \approx 0$) there is a pure state black hole and no radiation. At this point the entropy will be almost zero. As the evaporation proceeds and more radiation is collected then the Hilbert space of the reservoir $\mathcal{H}_{rad}$ becomes larger and hence the von Neumann entropy (given as $-Tr\rho \log \rho$) increases with time. Here the interior of the black hole is encoded in the CFT and the reduced density matrix of $\mathcal{H}_{rad}$ is thermal. So far the information in the black hole has not escaped.

2. At the Page time: $S_{rad} = S_{BH}$
   Both the QES have the same generalized entropy and at this point there happens a phase transition in the HRT surface.

   Note that even though the Page time is sometimes called the halfway point, the area of the event horizon at Page time is more than half the original area of the black hole. This is because the evaporation of a black hole is a thermodynamically irreversible process hence the entanglement entropy of radiation S_{rad} will be greater than the loss

[2] we will calcualate them explicitly in the later sections

of Bekenstein-Hawking entropy.

$$S_{rad}(t_{Pg}) > \frac{A_{hor}(t=0) - A_{hor}(t=t_{Pg})}{4G_N} \Rightarrow A_{hor}(t=t_{Pg}) > \frac{A_{hor}(t=0)}{2} \tag{2.5}$$

3. After the Page time: $S_{rad} > S_{BH}$
   Now the surface with lesser generalized entropy is the non-empty QES, hence that becomes the HRT surface. Now the generalized entropy of the black hole is approximately the Bekenstein-Hawking entropy which is proportional to the area of the event horizon. So as the black hole evaporates, it loses its mass and shrinks in radius. This causes the area of the black hole to decrease and consequently the entropy decreases with time.

Thus we find that the Page curve can be reproduced from a bulk perspective by showing a phase transition of the HRT surface.

2.5 Calculating the HRT surfaces

In this section, we will formally calculate the spacetime position of the HRT surfaces (both classical and quantum) for the evaporating black hole geometry which is asymptotically AdS. We assume the rotational symmetry of the spacetime and the surface which is topologically equivalent to S^{d-2} can be specified with just the radial and time coordinates.

As already mentioned previously, we consider the boundary of AdS to be absorbing instead of reflecting. This means that the radiation which reached the boundary and then entered the bath can never come back to the black hole. This makes the black hole system Markovian (irreversible), but considering the reservoir Hilbert space $\mathcal{H}_{rad}$ which stores the radiation makes the entire system unitary. This is equivalent to solving the black hole information paradox as long as we know which parts of the bulk are encoded in the radiation and which in the boundary CFT at each epoch of time.

An important point here is how we define the entanglement wedge of the radiation bath $\mathcal{H}_{rad}$. Since the CFT is just a boundary region (the entire boundary in this case), the entanglement wedge for it can be found in the traditional way as the domain of dependence of the spacelike region joining the HRT surface and the AdS boundary. We already know that if we have two states which purify each other then their entanglement wedges will be complementary to each other. This means that *all* the degrees of freedom of the total system must be in one of the two entanglement wedges. So we define the entanglement wedge for the radiation to contain any bulk degrees of freedom which are bounded just by the HRT surface as well as all the radiation which reached the boundary and hence left the bulk system i.e, the modes contained in $\mathcal{H}_{rad}$. Hence both the entanglement wedges depend on the same HRT surface and the bulk von Neumann entropy of both will also be equal as expected.

2.5.1 Classical HRT surface

We have already seen that the classical HRT surface is equivalent to the maximin surface hence we will use the maximin construction to find the location of it.

The classical HRT/maximin surface for an evaporating black hole will be empty surface as it has the least area i.e, zero. But we know that this will not represent the true dynamics of the evaporation. In the quantum extremal surfaces, there exist both empty and non-empty surfaces. Hence we will try to get the approximate location of the non-empty QES using the classical maximin prescription. For this purpose we can assume that we have a two-sided black hole and only one side of it is evaporating. Note that trying to find the non-empty surface is an attempt to find the HRT surface post Page time

To find the maximin surface, we want to find a minimal area surface on a Cauchy (achronal) slice and then the slice which maximizes the area. We consider an AdS Schwarzschild black hole geometry (in 3+1 dimensions) whose metric is given as:

$$ds^2 = -f(r)dv^2 + 2drdv + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.6}$$

$$f(r) = 1 + \frac{r^2}{l^2} - \frac{16\pi G_N M}{(d-2)\Omega_{d-2} r^{d-3}} \tag{2.7}$$

where, l is the lengthscale of AdS in d-dimensions and M is the mass of the black hole.

The coordinates here are more suited to describe the motion of an infalling observer and v-coordinate can be thought of as a time coordinate on an infalling lightray. Since the evaporation is slow, we are approximating a static black hole. So the area of the infalling lightcone is found to be decreasing with time. Hence the Cauchy slice we want is the past lightcone of the given time slice of the entire boundary.

To find null rays: $ds^2 = 0 \Rightarrow 2dvdr = f(r)dv^2$

1. $dv = 0 \Rightarrow v = constant$
   This is the ingoing lightray.

2. $dr/dv = f(r)/2$
   This is the outgoing lightray.

The Schwarzschild radius r_s is given by $f(r_s) = 0$ which is slowly decreasing with time. We define inverse temperature as $\beta = 4\pi/f'(r_s)$. Therefore we get the following differential equation for the outgoing lightcone:

$$\frac{dr}{dv} \approx \frac{2\pi}{\beta}(r - r_s) \tag{2.8}$$

where we have taken r to be close to r_s.

To solve the above equation, we change the integration variable to $r' = r - r_s$. Then we get:

$$\frac{dr'}{dv} = \frac{dr}{dv} - \frac{dr_s}{dv} = \frac{2\pi}{\beta}r' - \frac{dr_s}{dv} \tag{2.9}$$

We assume that the inverse temperature β and the black hole evaporation rate (given by dr_s/dv) does not change much in a given epoch of time where we are doing this calculation. Hence we can take them to be constant in the leading order semiclassical calculation. By integrating the above equation, we get:

$$v = \frac{\beta}{2\pi} \ln c \left(\frac{2\pi}{\beta}r' - \frac{dr_s}{dv} \right) \tag{2.10}$$

$$\Rightarrow r' = ce^{2\pi v/\beta} + \frac{\beta}{2\pi}\frac{dr_s}{dv} \tag{2.11}$$

where c is the constant of integration. Reinserting the form of r' and we get the form of r. From here on we refer to this r as $r_{lightcone}$ as it gives the radial coordinate of null rays.

$$r_{lightcone} = r_s + ce^{2\pi v/\beta} + \frac{\beta}{2\pi}\frac{dr_s}{dv} \tag{2.12}$$

By taking c to be positive, we get the lightcone which escapes the black hole and by taking it negative we get the lightcone which falls inside. The event horizon of the black hole is largest radius inside which objects cannot escape the black hole. Thus $c = 0$ will give the location of the event (or causal) horizon.

$$r_{hor} = r_s + \frac{\beta}{2\pi}\frac{dr_s}{dv} \tag{2.13}$$

This is inside the Schwarzschild radius r_s as r_s is decreasing with v.

We make the choice of $c = r_s$ here to find a null geodesic that escapes the black hole near the horizon. This gives us:

$$r_{lightcone} = r_s + r_s e^{2\pi v/\beta} + \frac{\beta}{2\pi}\frac{dr_s}{dv} \tag{2.14}$$

This is our achronal slice and we minimize eq (2.14) by differentiating it with respect to v and then equating it with zero. This gives the value of v at which the minimum of r occurs. Here we are ignoring $O(\beta)$ terms. Then we insert this value of v in the above equation to find the minimum value of r (and hence the minimum area of the surface) and this gives us the location of the classical maximin surface. This is found to be:

$$r = r_s \quad ; \quad v = -\frac{\beta}{2\pi}\log\frac{r_s}{\beta|dr_s/dv|} + O(\beta) \tag{2.15}$$

However, we must keep in mind that this is not the actual HRT surface because it is extremal only on the chosen Cauchy slice (the past lightcone of the boundary) but not with respect to variations outside of it. Another contradiction which this surface leads to is that the causal wedge of the boundary will be outside of its entanglement as defined using this surface. The null energy condition (NEC) was used to prove that the maximin surface is equivalent to the HRT surface classicaly. But the evaporating black hole we consider here does not satisfy the NEC. Therefore we now look at the quantum version of this surface as the bulk entropy terms can resolve these problems.

2.5.2 Quantum HRT surface

The quantum maximin prescription which came up very recently in [16] is also expected to be equivalent to the quantum HRT surface. However here we still use the procedure of extremizing the generalized entropy to calculate the quantum extremal surface.

If we had reflecting boundary conditions in AdS then the bulk outgoing modes and the reflected modes reach an equilibrium such that the degrees of freedom at any point in the bulk will not change with time. In that case, the QES will also become static and independent of boundary time. But with reflecting boundary conditions, the outgoing modes which are in the bulk at some point of time will not be so at a later time. Hence the QES which depends on the bulk entropy will evolve with time.

To find the quantum extremal surface we need the form of the bulk von Neumann entropy. We know that the surface must lie close to the horizon so we make approximations based on that. Here the spacetime can be considered to be given by $\mathbb{R}^{1,1} \times S^{d-1}$ and bulk modes are all free fields with the ingoing and outgoing being decoupled. So the entropy term will contain the sum of all modes and the Minkowski vacuum formula can be applied here:

$$S_{bulk} = \frac{c_{evap}}{6} \log\left(\frac{r_{lightcone}(v) - r)}{\sqrt{\epsilon_1 \epsilon_2}}\right) \tag{2.16}$$

where, c_{evap} is an evaporation parameter depending on the number of bosonic modes N_b and the number of fermionic modes N_f in 2-dimensions as $c_{evap} = N_b + N_f/2$. $r_{lightcone}$ is the radial coordinate of the outgoing lightcone which was calculated in the previous subsection and ϵ_1, ϵ_2 are cutoffs on the QES and the lightcone whose dependence we can find to get the following expression for the bulk entropy:

$$S_{bulk} = \frac{c_{evap}}{6} \log(r_{lightcone}(v) - r) - \frac{c_{evap}\pi v}{6\beta} \tag{2.17}$$

To proceed further it is useful to find an expression for dr_s/dv which can be done by using the laws of black hole thermodynamics. The first law connects the rate of change of energy (mass) to the rate of change of entropy (Bekenstein-Hawking).

$$\beta dM = \frac{dA_{hor}}{4G_N} \tag{2.18}$$

where, $A_{hor} = \Omega_{d-2} r_{hor}^{d-2}$ for a d-dimensional AdS. Also if we ignore $O\beta$ terms then $r_{hor} \approx r_s$.

The Stefan-Boltzmann law gives the rate of change of mass as:

$$\frac{dM}{dv} = \frac{c_{evap}\pi}{12\beta^2} \tag{2.19}$$

Comparing 2.18 and 2.19 and substituting the value of A_{hor} we get:

$$\frac{dr_s}{dv} = \frac{c_{evap}\pi G_N}{3\beta\Omega_{d-2}(d-2)r_s^{d-3}} \tag{2.20}$$

Now we proceed to extremize the generalized entropy ($S_{gen} = S_{bulk} + A/4G_N$) with respect to r and v since we have assumed spherical symmetry.

1. Keeping r constant and varying with v:

$$\begin{aligned}
&\frac{\partial S_{gen}}{\partial v} = 0 \\
&\Rightarrow \frac{\partial S_{bulk}}{\partial v} + \frac{\partial}{\partial v}\left(\frac{A}{4G_N}\right) = 0 \\
&\Rightarrow \frac{1}{6(r_{lightcone} - r)}\frac{dr_{lightcone}}{dv} - \frac{\pi}{6\beta} = 0 \\
&\Rightarrow r_{lightcone} - r = 2(r_{lightcone} - r_s)
\end{aligned} \tag{2.21}$$

In the last line we have used eq (2.14)

2. Keeping v constant and varying with r:

$$\begin{aligned}
&\frac{\partial S_{gen}}{\partial r} = 0 \\
&\Rightarrow \frac{\partial S_{bulk}}{\partial r} + \frac{\partial}{\partial r}\left(\frac{A}{4G_N}\right) = 0 \\
&\Rightarrow -\frac{c_{evap}}{6(r_{lightcone} - r)} + \frac{(d-2)\Omega_{d-2}r_s^{d-3}}{4G_N} \\
&\Rightarrow r_{lightcone} - r = 4(r_s - r_{hor})
\end{aligned} \tag{2.22}$$

In the last line we have used eq (2.20)

Solving (2.21) and (2.22), we find the location of the QES to be at:

$$v = -\frac{\beta}{2\pi} \log \frac{S_{BH}}{c_{evap}} \tag{2.23}$$

$$r = r_s - \frac{\beta}{\pi} \frac{\partial r_s}{\partial v} \tag{2.24}$$

Here we are ignoring the terms of the $O(\beta)$ and we assume S_{BH} using dimensional analysis and taking $dr_s/dv \sim O(G_N)$.

From this it also follows that $r = r_{hor} - (r_s - r_{hor})$ which means that the QES is inside the Schwarzschild radius (or the apparent horizon) double the distance that the causal horizon is.

From this we know the entanglement wedge of the CFT. It will not contain the region outside of the past lightcone of the boundary. Because this region is not in the past domain of dependence of the subregion between the boundary point (an epoch of time) and the HRT surface. The future domain of dependence however, will extend all the way till the AdS boundary because of absorbing boundary conditions.

The causal wedge of the boundary contains the bulk causal future as well as the bulk causal past of the entire boundary's domain of dependence. But due to absorbing boundary conditions the past of the boundary will not lie in its domain of dependence. Therefore the causal wedge will simply be the part of the causal future of the boundary in the bulk which is outside of the black hole event horizon. This is obviously contained in the entanglement wedge.

2.6 The Hayden Preskill decoding criterion

Hayden and Preskill [17] modeled the dynamics of an evaporating black hole as a fast thermalizing system such that the internal information is processed instantly by a random unitary transformation. Using quantum information theory they answer the time and rate at which the information inside the black hole can come out encoded in the radiation.

They showed that past the halfway point when about half the black hole entropy is radiated and when the interior of black hole and the emitted radiation is close to maximally entangled, then any amount of information which goes inside the black hole will come out with just a little more information than that being radiated.

There is the classic story of a person named Alice who wants to hide the contents of her diary by destroying it and hence throws it into a black hole. Bob, who wants to uncover the secrets stored in the diary has access to the Hawking radiation and tries to decode the diary using that. The information paradox is rephrased in the question of whether Bob will succeed or not.

If we only consider the classical case and assume that Bob who has been studying the black hole from its origin knows the exact internal state of the black hole before the diary entered it. Let the diary thrown inside be of size k-bits where the black hole size is much greater than k. Bob can decode the diary with a probability of failure less than 2^{-c} by collecting $k + c$ bits of the radiation.

$$P_{fail} \leq 2^{k-s} = 2^{-c} \tag{2.25}$$

where, $s = k + c$.

In the case that the information is quantum i.e, a k-qubit string, we consider a third person Charlie who has a reference qubit system with the same size as that of the diary. We start with Charlie's and Alice's systems being maximally entangled and together form a pure state. Then each state separately is maximally mixed. So if Bob can recover a state from the radiation that is maximally entangled with the state Charlie holds, then we can conclude that he has decoded the message in the quantum version. Here we assume that since Bob has been collecting the radiation since the beginning, he has access to a quantum state that is maximally entangled with the state of the black hole interior. We consider a bipartite system of black hole interior B and the Hawking radiation R. The Bekenstein-Hawking entropy will be approximately $\log B$. During the initial stage of radiation $|B| >> |R|$ and hence the radiation is almost maximally entangled with the black hole interior. After the halfway point, $|R| >> |B|$ and the interior will be almost maximally entangled with the radiation. Since Bob already has a system maximally entangled with the interior, after the black hole has absorbed the diary, it thoroughly mixes the information and the new black hole interior is now maximally entangled with the radiation plus the reference system with Charlie. Then after Bob has collected $s = k + c$ qubits of radiation, he can purify the reference state that Charlie holds with the fidelity deviating from 1 by less than 2^{-c} which means he has decoded the information Alice wanted to hide. Hence we see that in both classical and quantum cases, Bob has found the information in the diary quite fast.

A note here that the above calculations were done for the black hole past the halfway point and if the diary was thrown into the black hole in the initial stages then the radiation will not leak the contents of the diary for a long time. Once the size of the black hole interior becomes equal to the size of total radiation plus the reference, then the information will start to appear in the radiation. The conclusion by Hayden and Preskill [17] is somewhat similar to what Don Page [15] said that the information comes out at a steady rate only after the halfway point.

The *Hayden-Preskill decoding criterion* says that if a diary is thrown in the black hole before the Page time, then it can be decoded from the Hawking radiation at the Page time and if it is thrown in after the Page time then it can be decoded in one scrambling time. The scrambling time is given by:

$$t_{scr} = \frac{\beta}{2\pi} \log S_{BH} \tag{2.26}$$

where, β is the inverse temperature and S_{BH} is the Bekenstein-Hawking entropy. Note that this is based on the assumption that the size of the state of diary is small compared to the size of black hole and also that the state of black hole is known by the outside observer.

Now we can see that this criterion is realised by the calculations of the HRT surfaces. Before the Page time the null surface is the HRT surface which means that the black hole interior is not in the entanglement wedge of the radiation hence it cant be decoded from the radiation.

After the Page time, the HRT surface has a phase transition and suddenly the black hole interior lies in the entanglement wedge of the radiation hence the information inside the black hole can be reconstructed using the radiation. The exact calculation found the QES to lie close to the event horizon with the infalling time (2.23) to be approximately equivalent to the scrambling time (2.26) in the past assuming $c_{evap} \sim O(1)$. Therefore, the information that went inside the horizon before this time can be reconstructed from $\mathcal{H}_{rad}$. Here the backreation of the diary on the black hole geometry will be of a subleading order to cause delay in the reconstruction. Thus the Hayden-Preskill criterion gives more confidence in the location of the QES and the entanglement wedges.

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